Dimer λ_d Expansion, A Contour Integral Stationary Point Argument

Paul Federbush
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043
(pfed@umich.edu)

Abstract

In the development of a presumed asymptotic expansion for λ_d in previous work, a basic step involved extracting the asymptotic behavior of a sum as being dominated by the largest term in the sum. But this argument only is valid when the terms in the sum are positive. In the case terms in the sum are not all positive (some of the $\bar{J}_n$ are negative) we replace the 'largest term' argument by the route of this paper. The sum is replaced by a contour integral. We then find a stationary point of the integrand that we conjecture dominates the asymptotic behavior of the integral.

An asymptotic expansion for λ_d in powers of $1/d$ has been studied in a series of papers, [1] - [4]. In [1] we extracted the $N \to \infty$ asymptotic behavior of the sum in equation (33) therein

$$Z^* = \sum_{\alpha_i} \beta(N, \sum\ i\ \alpha_i\ )\bar{J}_1^{\alpha_1} \cdots \bar{J}_{s+1}^{\alpha_{s+1}} \cdot \frac{N^{\sum\ \alpha_i}}{\alpha_1! \cdots \alpha_{s+1}!} \tag{1}$$

(with very slight change in notation) by seeking the largest term in this sum. This argument depended on the terms in the sum being positive. For a while we feared the expansion of λ_d only applied where only positive $\bar{J}_n$ were involved. But, as promised after equation (35) of [1], an alternate argument will be given herein yielding the same expressions for the λ_d asymptotic series.

We replace (1) by the asymptotically equivalent formula:

$$Z^* = \left(\frac{1}{2\pi i}\right)^{s+1} \oint dz_1 \cdots \oint dz_{s+1} \frac{1}{z_1 \cdots z_{s+1}} \left[\sum_{\alpha_1 = -\mu N}^{\mu N} \cdot\cdot \sum_{\alpha_{s+1} = -\mu N}^{\mu N} \beta(N, \sum\ i\ \alpha_i) \cdot \right. \tag{2}$$

$$\left. \cdot\ z_1^{\alpha_1} \cdots z_{s+1}^{\alpha_{s+1}} \right] e^{N\ \sum_{i=1}^{s+1}\ \bar{J}_i\ \frac{1}{z_i}}$$

In both expressions we are seeking contributions from $\bar{J}_1$, ..., $\bar{J}_{s+1}$, and μ and the $\bar{J}_i$ should satisfy

$$|\bar{J}_i|\ << \mu << 1\ . \tag{3}$$

We make a formal change of variables (in the $N \to \infty$ limit)

$$\begin{aligned} \alpha_i &\longrightarrow N\rho_i \\ \Delta\ \alpha_i &\longrightarrow Nd\rho_i \end{aligned} \tag{4}$$

to arrive at

$$Z^* \cong \left(\frac{1}{2\pi i}\right)^{s+1} \oint dz_1 \cdots \oint dz_{s+1} \frac{1}{z_1 \cdots z_{s+1}} \left[\int_{-\mu}^{\mu} Nd\rho_1 \cdot\cdot \int_{-\mu}^{\mu} Nd\rho_{s+1} \beta(N, \sum_i Ni\rho_i) \cdot \right. \tag{5}$$

$$\left. \cdot\ e^{N\ \sum_{i=1}^{s+1} \rho_i ln(z_i)} \right] e^{N\ \sum_{i=1}^{s+1}\ \bar{J}_i\ \frac{1}{z_i}}$$

and substituting from the unnumbered equation between (33) and (34) in [1], (5) becomes

$$Z^* \cong \left(\frac{1}{2\pi i}\right)^{s+1} \oint dz_1 \cdots \oint dz_{s+1} \frac{1}{z_1 \cdots z_{s+1}} \int_{-\mu}^{\mu} N d\rho_1 \cdots \int_{-\mu}^{\mu} N d\rho_{s+1} \cdot \tag{6}$$

$$\cdot \; e^{N\left\{\left[\left(\frac{1-2\sum_i \rho_i}{2}\right) ln(1-2\sum_i \rho_i)+\sum_i i\rho_i\right]+\sum_i \rho_i ln(z_i)+\sum_i \bar{J}_i \frac{1}{z_i}\right\}} \; .$$

We conjecture that the $N \to \infty$ behavior of (1), or (2), may be obtained from the behavior of the integrand of (6) at its stationary point. We expect there will soon be a proof of this, perhaps it follows from a known result. Applying the conjecture and differentiating the quantity in braces, $\{ \cdot \}$, with respect to ρ_i and z_i and setting all derivatives equal zero one gets exactly equations (34) - (36) of [1].

Our 'contour integral stationary point' technique is easily seen to pass the following two tests.

<u>Test 1</u> If all the $\bar{J}_i$ are non-negative then the result obtained agrees with the result of the 'largest term' computation.

<u>Test 2</u> If the function β is set equal 1, this method yields the correct result

$$Z^* \;=\; e^{N\sum \bar{J}_i}. \tag{7}$$

Proving the conjecture above may seem peremptory, but what one really needs is the result in the limit $s \to \infty$, for the bounds actually holding for $\bar{J}_i$ (for large d). We do not yet know what bounds hold for the $\bar{J}_i$, a seemingly very difficult task to find such. Much work remains.

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References

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