On Gevrey functions solutions of partial differential equations with fuchsian and irregular singularities

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Abstract

We construct formal power series solutions of non linear partial integro-differential equations with fuchsian and irregular singularities at the origin in $\mathbb{C}^2$ for given initial conditions being formal power series. We give sufficient conditions under which there exist actual sectorial holomorphic solutions which are Gevrey asymptotic to the given formal series solutions for given $1-$summable formal series initial conditions. A phenomenon of small divisors is observed for the appearance of singularities of the Borel transform of the constructed formal series due to the presence of the fuchsian singularity. This property has an effect on the Gevrey order asymptotic for the constructed holomorphic solutions which becomes larger than the Gevrey order of the initial conditions.

Key words: Cauchy problem, partial integro-differential equation, Gevrey estimates, fuchsian singularity, irregular singularity, small divisor. 2000 MSC: 35C10, 35C20.

1 Introduction

We consider a family of non linear partial integro-differential equations of the following form

$$(1)\quad ((z\partial_z+1)^{r_1}(4t^2\partial_t+2t)^{r_2}+1)\partial_z^S\hat{X}(t,z) = \sum_{(k_0,k_1)\in\mathcal{S}} b_{k_0k_1}(z)\partial_t^{k_0}\partial_z^{k_1}\hat{X}(t,z) + b(z)\partial_t\int_0^t \hat{X}(t-\tau,z)\hat{X}(\tau,z)d\tau,$$

where $S,r_1,r_2\geq 1$ are positive integers, $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ and the coefficients $b_{k_0k_1}(z),b(z)$ are holomorphic functions on a neighborhood of the origin in $\mathbb{C}$. These equations (1) belong to a class of partial differential equations with irregular singularity at $t=0$ (meaning that $r_2\geq 1$) and with Fuchsian singularity at $z=0$ (meaning that $r_1\geq 1$).

We are looking for formal series solutions of (1)

$$\hat{X}(t,z)=\sum_{m\geq 0}\hat{X}_m(t)\frac{z^m}{m!}$$

in the variable z with coefficients $\hat{X}_m(t)$ be formal series for given initial conditions $(\partial_z^j \hat{X})(t,0) = \hat{X}_j(t)$, $0 \leq j \leq S-1$ be formal series in the variable t. Our goal is to construct actual holomorphic solutions of (1) which are Gevrey asymptotic to the given formal series solution $\hat{X}(t,z)$ with respect to t in some sector with finite radius centered at 0 in $\mathbb{C}$, uniformly in z on some neighborhood of 0 in $\mathbb{C}$.

In the case of non linear partial differential equations with Fuchsian singularity, the existence of formal power series solutions, their convergence and divergence properties have been studied in great details by several authors, see [10],[13]. The study of non linear partial differential equations with irregular singularity is an active subject of research and several results concerning divergence properties of formal series solutions and resummation procedures have been obtained, see [5], [6], [8], [14]. But there are very few results concerning the convergence and Gevrey asymptotics of formal series solutions of non linear partial differential equations having simultaneously a Fuchsian and an irregular singularity.

The case $r_1 = 0$ for which only a irregular singularity appears has been already studied in a previous paper, see [12]. In this situation, we have shown that the formal series solutions $\hat{X}(t,z)$ are $1-$summable with respect to t in some direction $d \in \mathbb{R}$ as series with coefficients in the Banach space of holomorphic functions on some neighborhood of 0, for well chosen initial formal series conditions $(\partial_z^j \hat{X})(t,0)$, $0 \leq j \leq S-1$, which are $1-$summable in direction d.

When $r_1 \neq 0$, the picture looks different. Indeed, we will show that the formal series solutions $\hat{X}(t,z)$ are in general no longer $1-$summable with respect to t in direction d, uniformly in z, for given initial $1-$summable formal series conditions $(\partial_z^j \hat{X})(t,0)$ in direction d. But we show that there exist actual solutions $X(t,z)$ which are Gevrey asymptotic of order $2(1+\frac{r_1}{2r_2})$ to $\hat{X}(t,z)$ with respect to t on some sector $S_{d,R}$ centered at 0, with radius $R > 0$, for well chosen directions $d \in \mathbb{R}$, uniformly in z on some neighborhood D_ρ of 0 (see Definition 5).

In the first part of this paper, we will study a Cauchy problem for non linear partial differential equations in a class of holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$ (see Theorem 1). This result will be useful to study some Borel transform of the formal series $\hat{X}(t,z)$ with respect to t in the second part.

In a second part, we observe a phenomenon of *small divisors* introduced by the fuchsian operator $(z\partial_z + 1)^{r_1}$ which will have an effect on the Gevrey asymptotic of the actual solutions $X(t,z)$ of (1) we will construct in the third part. This phenomenon is similar to the one described in the paper of B. Braaksma and L. Stolovitch, see [4], in which these authors investigate Gevrey properties of holomorphic diffeomorphisms in sectorial domains transforming singular holomorphic vector fields at the origin in $\mathbb{C}^n$ into their normal forms.

More precisely, we show that the singularities of some Borel transform of $\hat{X}_m(t)$ with respect to t form a set which accumulates to the origin in $\mathbb{C}$ at a distance not larger than $1/(m+1)^c$, for $c > 0$, as m tends to infinity. Moreover, we can estimate the rate of growth of a weighted supremum norm of all the $l-$derivative of these Borel transforms with respect to t on a small disc centered at 0 with radius $1/(2(m+1)^c)$ and on sectors S_d with infinite radius in some well chosen directions $d \in \mathbb{R}$ in terms of $l, m \geq 0$ (see Theorem 2).

In a third part, we will construct actual holomorphic solutions $X(t,z)$ of (1) defined on a product $S_{d,R} \times D_\rho$ from the given formal series $\hat{X}(t,z)$ using a Borel-Laplace transform procedure (see Definition 4). But, we will show that this small divisor property observed in the second part induces a large multiplier effect on the $l-$derivative of $X(t,z)$ with respect to $t \in S_{d,R}$, for $l \geq 0$, which grows like $CK^l l!\Gamma(1+\gamma l)$, for some $C, K > 0$, $\gamma > 2$. Moreover, we will show that the constructed holomorphic solutions $X(t,z)$ are Gevrey asymptotic of order $2(1+\frac{r_1}{2r_2})$ to $\hat{X}(t,z)$ with respect to t on $S_{d,R}$, uniformly in z near 0 (Theorem 3).

2 A non-linear Cauchy problem

In this section, we describe a version of the Cauchy-Kowalevski theorem for non-linear partial differential equations for holomorphic functions.

Definition 1 *Let T, X be real numbers such that $T, X > 0$. We define a vector space $G(T,X)$ of holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$. A formal series $U(t,x) \in \mathbb{C}[[t,x]]$,*

$$U(t,x) = \sum_{\beta,l \geq 0} u_{l,\beta} \frac{t^l}{l!} \frac{x^\beta}{\beta!}$$

belongs to $G(T,X)$, if the series

$$\sum_{l,\beta \geq 0} \frac{|u_{l,\beta}|}{(l+\beta)!} T^l X^\beta,$$

converges. We also define a norm on $G(T,X)$ as

$$||U(t,x)||_{(T,X)} = \sum_{l,\beta \geq 0} \frac{|u_{l,\beta}|}{(l+\beta)!} T^l X^\beta.$$

One can easily show that $(G(T,X), ||.||_{(T,X)})$ is a Banach space.

Remark: Let $U(t,x)$ be in $G(T_0,X_0)$ for given $T_0, X_0 > 0$. Then, $U(t,x)$ also belong to the spaces $G(T,X)$ for all $T \leq T_0$ and $X \leq X_0$. Moreover, the maps $T \to ||U(t,x)||_{(T,X)}$ and $X \to ||U(t,x)||_{(T,X)}$ are increasing functions from $[0,T_0]$ (resp. $[0,X_0]$) into $\mathbb{R}_+$.

We can now state a version of the Cauchy problem that will be needed in the next sections.

Theorem 1 *Consider a partial differential equation of the form*

$$(2) \quad \partial_x^S u(t,x) = \sum_{(k_0,k_1) \in \mathcal{S}_1} a_{k_0 k_1}(t,x) \partial_t^{k_0} \partial_x^{k_1} u(t,x) + u(0,x) \sum_{(\kappa_0,\kappa_1) \in \mathcal{S}_2} b_{\kappa_0 \kappa_1}(t,x) \partial_t^{\kappa_0} \partial_x^{\kappa_1} u(t,x) + c(t,x)(u(t,x))^2$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are finite subsets of $\mathbb{N}^2$. We assume that

i) *There exist $T_0, X_0 > 0$ such that the coefficients $a_{k_0 k_1}(t,x)$, $b_{\kappa_0 \kappa_1}(t,x)$ and $c(t,x)$ belong to $G(T_0,X_0)$, for all $(k_0,k_1) \in \mathcal{S}_1$ and $(\kappa_0,\kappa_1) \in \mathcal{S}_2$. Moreover, there exists $C_1 > 0$ such that*

$$||a_{k_0 k_1}(t,x)||_{(T,X)} \leq C_1 X \quad , \quad ||b_{\kappa_0 \kappa_1}(t,x)||_{(T,X)} \leq C_1 X \quad , \quad ||c(t,x)||_{(T,X)} \leq C_1 X$$

for all $0 < T \leq T_0$, $0 < X \leq X_0$, for all $(k_0,k_1) \in \mathcal{S}_1$ and $(\kappa_0,\kappa_1) \in \mathcal{S}_2$.

ii) *For all $(k_0,k_1) \in \mathcal{S}_1$, all $(\kappa_0,\kappa_1) \in \mathcal{S}_2$, we have $S \geq k_0 + k_1$, $S \geq \kappa_0 + \kappa_1$ and $S > k_1$, $S > \kappa_1$.*

iii) *The initial conditions*

$$(3) \quad (\partial_x^k u)(t,0) = \varphi_k(t) \quad , \quad 0 \leq k \leq S-1,$$

belong to $G(T_0,X_0)$.

Then, there exist $T, X > 0$ small enough such that the problem (2), (3) has a unique solution $u(t,x)$ in $G(T,X)$.

We start with some preliminary lemma from [12].

Lemma 1 *Let $U_1(t,x)$, $U_2(t,x)$ be in $G(T,X)$. Then, the product $U_1(t,x)U_2(t,x)$ belongs to $G(T,X)$. Moreover,*

$$||U_1(t,x)U_2(t,x)||_{(T,X)} \leq ||U_1(t,x)||_{(T,X)}||U_2(t,x)||_{(T,X)}. \tag{4}$$

In other words, the normed vector space $(G(T,X), ||.||_{(T,X)})$ is in fact a Banach algebra.

In the following, for $u(t,x) \in \mathbb{C}[[t,x]]$, we denote by $\partial_x^{-1}u(t,x)$ the formal series $\int_0^x u(t,\tau)d\tau$.

Lemma 2 *Let $h_0, h_1 \in \mathbb{N}$ such that $h_0 \leq h_1$. The operator $\partial_t^{h_0}\partial_x^{-h_1}$ is a bounded linear operator from $(G(T,X), ||.||_{(T,X)})$ into itself. Moreover, there exists a constant $C > 0$ such that the estimates hold*

$$||\partial_t^{h_0}\partial_x^{-h_1}U(t,x)||_{(T,X)} \leq CT^{-h_0}X^{h_1}||U(t,x)||_{(T,X)}, \tag{5}$$

for all $U(t,x) \in G(T,X)$.

Lemma 3 *Let $h_1, h_2 \in \mathbb{N}$ and let $U(t,x)$ be in $G(T_0,X_0)$. Then, there exist $T, X > 0$ small enough (depending on h_1, h_2) such that the formal series $\partial_t^{h_1}\partial_x^{h_2}U(t,x)$ belongs to $G(T,X)$. Moreover, there exists a constant $K > 0$ such that*

$$||(\partial_t^{h_1}\partial_x^{h_2}U)(t,x)||_{(T,X)} \leq T^{-h_1}X^{-h_2}K||U(t,x)||_{(T_0,X_0)}, \tag{6}$$

for all $U(t,x) \in G(T_0,X_0)$.

In the following, we put

$$w(t,x) = \sum_{j=0}^{S-1} \varphi_j(t)\frac{x^j}{j!},$$

where $\varphi_j(t)$ belong to $G(T_0,X_0)$.

Definition 2 *Consider the map $A : \mathbb{C}[[t,x]] \to \mathbb{C}[[t,x]]$ defined as follows*

$$\begin{aligned}A(v(t,x)) = &\sum_{(k_0,k_1)\in\mathcal{S}_1} a_{k_0k_1}(t,x)(\partial_t^{k_0}\partial_x^{k_1-S}v(t,x) + \partial_t^{k_0}\partial_x^{k_1}w(t,x)) \\ &+ (\partial_x^{-S}v(0,x) + w(0,x)) \sum_{(\kappa_0,\kappa_1)\in\mathcal{S}_2} b_{\kappa_0\kappa_1}(t,x)(\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v(t,x) + \partial_t^{\kappa_0}\partial_x^{\kappa_1}w(t,x)) \\ &+ c(t,x)(\partial_x^{-S}v(t,x) + w(t,x))^2.\end{aligned} \tag{7}$$

By construction, if $v(t,x) \in \mathbb{C}[[t,x]]$ is a solution of $A(v(t,x)) = v(t,x)$, then

$$u(t,x) = \partial_x^{-S}v(t,x) + w(t,x)$$

is a solution of (2) and (3).

Proposition 1 *Assume that the hypotheses* **i)**, **ii)**, **iii)** *of Theorem 1 are fulfiled. Then, there exist $T, X > 0$ small enough, such that the following properties are satisfied. There exists a closed ball $B_R = \{v \in G(T,X) : ||v||_{(T,X)} \leq R\}$, $R > 0$, such that the map $A : \mathbb{C}[[t,x]] \to \mathbb{C}[[t,x]]$ satisfy*

1) *$A(B_R) \subset B_R$,*

2) *For all $v_1, v_2 \in B_R$, we have $||A(v_2) - A(v_1)||_{(T,X)} \leq (1/2)||v_2 - v_1||_{(T,X)}$.*

Proof 1) Due to Lemma 1,2,3, there exists a constant $C > 0$ such that

$$(8)\quad ||a_{k_0k_1}(t,x)(\partial_t^{k_0}\partial_x^{k_1-S}v(t,x)+\partial_t^{k_0}\varphi_j(t)\frac{x^q}{q!})||_{(T,X)} \leq CC_1X(T^{-k_0}X^{S-k_1}R+T^{-k_0}||\varphi_j(t)\frac{x^q}{q!}||_{(T_0,X_0)}),$$

$$(9)\quad ||(\partial_x^{-S}v(0,x)+w(0,x))b_{\kappa_0\kappa_1}(t,x)(\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v(t,x)+\partial_t^{\kappa_0}\varphi_j(t)\frac{x^q}{q!})||_{(T,X)} \leq CC_1X(X^SR+||w(t,x)||_{(T,X)})(T^{-\kappa_0}X^{S-\kappa_1}R+T^{-\kappa_0}||\varphi_j(t)\frac{x^q}{q!}||_{(T_0,X_0)})$$

$$(10)\quad ||c(t,x)(\partial_x^{-S}v(t,x)+w(t,x))^2||_{(T,X)} \leq CC_1X(X^{2S}R^2+2X^SR||w(t,x)||_{(T,X)}+||w(t,x)||^2_{(T,X)})$$

for all $v(t,x)\in B_R$, all $(k_0,k_1)\in S_1$, all $(\kappa_0,\kappa_1)\in S_2$, and all $0\leq j,q\leq S-1$.

Conclusion: Let $R>0$. If X,T are chosen small enough, the sum over $(k_0,k_1)\in\mathcal{S}_1$, $(\kappa_0,\kappa_1)\in\mathcal{S}_2$ and $0\leq j,q\leq S-1$, of the right hand side of the inequalities (8), (9), (10), can be made lower than R for all $v(t,x)\in B_R$. So that if $v\in B_R$ then $A(v)\in B_R$.

2) Due to Lemma 1,2, there exist a constant $C>0$ such that

$$(11)\quad ||a_{k_0k_1}(t,x)\partial_t^{k_0}\partial_x^{k_1-S}(v_2(t,x)-v_1(t,x))||_{(T,X)} \leq CC_1XT^{-k_0}X^{S-k_1}||v_2(t,x)-v_1(t,x)||_{(T,X)}$$

for all $v_1,v_2\in B_R$, all $(k_0,k_1)\in\mathcal{S}_1$. We define

$$\mathcal{A}_{2,1}(t,x)=b_{\kappa_0,\kappa_1}(t,x)\left\{(\partial_x^{-S}v_2(0,x)+w(0,x))(\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v_2(t,x)+\partial_t^{\kappa_0}\partial_x^{\kappa_1}w(t,x)) - (\partial_x^{-S}v_1(0,x)+w(0,x))(\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v_1(t,x)+\partial_t^{\kappa_0}\partial_x^{\kappa_1}w(t,x))\right\},$$

that can be rewritten in the form

$$\mathcal{A}_{2,1}(t,x)=b_{\kappa_0,\kappa_1}(t,x)\left\{(\partial_x^{-S}v_2(0,x)-\partial_x^{-S}v_1(0,x))(\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v_2(t,x)+\partial_t^{\kappa_0}\partial_x^{\kappa_1}w(t,x)) + (\partial_x^{-S}v_1(0,x)+w(0,x))(\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v_2(t,x)-\partial_t^{\kappa_0}\partial_x^{\kappa_1-S}v_1(t,x))\right\}.$$

From the lemma 1,2,3, we get a constant $C>0$ such that

$$(12)\quad ||\mathcal{A}_{2,1}(t,x)||_{(T,X)} \leq CC_1X||v_2(t,x)-v_1(t,x)||_{(T,X)}\left\{X^S(T^{-\kappa_0}X^{S-\kappa_1}R+T^{-\kappa_0}X^{-\kappa_1}||w(t,x)||_{(T_0,X_0)}) \times (X^SR+||w(t,x)||_{(T_0,X_0)})T^{-\kappa_0}X^{S-\kappa_1}\right\}$$

for all $v_1,v_2\in B_R$, all $(\kappa_0,\kappa_1)\in\mathcal{S}_2$. Moreover, we define

$$\mathcal{B}_{2,1}(t,x)=c(t,x)(\partial_x^{-S}v_2(t,x)+w(t,x))^2-c(t,x)(\partial_x^{-S}v_1(t,x)+w(t,x))^2,$$

that can be rewritten in the form

$$\mathcal{B}_{2,1}(t,x)=c(t,x)\left\{(\partial_x^{-S}v_2(t,x)-\partial_x^{-S}v_1(t,x))(\partial_x^{-S}v_2(t,x)+\partial_x^{-S}v_1(t,x)+2w(t,x))\right\}.$$

From Lemma 1,2, there exists a constant $C > 0$ such that

$$||\mathcal{B}_{2,1}(t,x)||_{(T,X)} \leq CC_1X||v_2(t,x) - v_1(t,x)||_{(T,X)}X^S(2X^SR + 2||w(t,x)||_{(T,X)}) \tag{13}$$

for all $v_1, v_2 \in B_R$.

Conclusion: If X, T are chosen small enough, the sum over $(k_0, k_1) \in \mathcal{S}_1$, $(\kappa_0, \kappa_1) \in \mathcal{S}_2$, of the right hand side of the inequalities (11), (12), (13) can be made lower than $(1/2)||v_2(t,x) - v_1(t,x)||_{(T,X)}$. So that if $v_1, v_2 \in B_R$, then $||A(v_2) - A(v_1)||_{(T,X)} \leq (1/2)||v_2(t,x) - v_1(t,x)||_{(T,X)}$.
□

We can now give the proof of Theorem 1.

Proof Let the assumptions **i)**, **ii)**, **iii)** of Theorem 1 hold. From the fact that $(G(T,X), ||\ ||_{(T,X)})$ is a Banach space, the closed ball (B_R, d) equipped with the metric $d(x,y) = ||y - x||_{(T,X)}$ is a complete metric space. From the proposition 1, the map A is a shrinking map from (B_R, d) into itself. From the classical fixed point theorem, we deduce that there exists a unique $v \in B_R$ such that $A(v) = v$. Then, by construction, the formal series $u(t,x) = \partial_x^{-S}v(t,x) + w(t,x)$ is the solution of the problem (2), (3) and $u(t,x)$ belongs to $G(T,X)$. □

3 Borel transform of formal series solutions of partial differential equations

We consider non linear partial integro-differential equations of the following form

$$((z\partial_z + 1)^{r_1}(4t^2\partial_t + 2t)^{r_2} + 1)\partial_z^S \hat{X}(t,z) = \sum_{(k_0,k_1)\in\mathcal{S}} b_{k_0k_1}(z)\partial_t^{k_0}\partial_z^{k_1}\hat{X}(t,z) + b(z)\partial_t \int_0^t \hat{X}(t-\tau,z)\hat{X}(\tau,z)d\tau, \tag{14}$$

where $S, r_1, r_2 \geq 1$ are positive integers, where $(z\partial_z + 1)^{r_1}$ denotes the operator $z\partial_z + 1$ composed r_1 times and where $(4t^2\partial_t + 2t)^{r_2}$ denotes the operator $4t^2\partial_t + 2t$ composed r_2 times. We make the following assumptions on the integers S, r_1, r_2, the set $\mathcal{S}$ and the coefficients $b_{k_0k_1}(z), b(z)$.

(H1) The functions $b_{k_0k_1}(z), b(z)$, $(k_0, k_1) \in \mathcal{S}$ are holomorphic on a neighborhood of the origin in $\mathbb{C}$ and satisfy

$$b_{k_0k_1}(0) = 0 \quad , \quad b(0) = 0,$$

for all $(k_0, k_1) \in \mathcal{S}$.

(H2) We have $S \geq 2 + [\frac{r_1}{2r_2}]$. The set $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ such that for all $(k_0, k_1) \in \mathcal{S}$,

$$S \geq (3 + [\frac{r_1}{r_2}])k_0 + k_1 \quad , \quad S > k_1,$$

where $[x]$ denotes the integer part of a real number x.

In the next lemma, we show the existence and unicity of formal power series solutions of (14).

Lemma 4 *For given initial conditions $(\partial_z^j \hat{X})(t,0) = \hat{X}_j(t) \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, be formal series with coefficients in $\mathbb{C}$, there exists a unique formal series*

$$\hat{X}(t,z) = \sum_{m\geq 0} \hat{X}_m(t)\frac{z^m}{m!},$$

where $\hat{X}_m(t) \in \mathbb{C}[[t]]$, $m \geq 0$, which satisfies the equation (14).

Proof Let the expansions

$$b_{k_0k_1}(z) = \sum_{k\geq 0} b_{k_0k_1k} z^k \quad , \quad b(z) = \sum_{k\geq 0} b_k z^k$$

be on a neighborhood of 0, for all $(k_0,k_1) \in \mathcal{S}$. The formal series $\hat{X}(t,z)$ satisfies the equation (14), if and only if the formal series $\hat{X}_m(t)$, $m \geq 0$, satisfy the following recursion

$$(15) \quad ((m+1)^{r_1}(4t^2\partial_t + 2t)^{r_2} + 1)\frac{\hat{X}_{m+S}(t)}{m!} = \sum_{(k_0,k_1)\in\mathcal{S}} \sum_{q_1+q_2=m} b_{k_0k_1q_1} \frac{\partial_t^{k_0}\hat{X}_{q_2+k_1}(t)}{q_2!} + \sum_{q_1+q_2+q_3=m} b_{q_3} \frac{\partial_t \int_0^t \hat{X}_{q_1}(t-\tau)\hat{X}_{q_2}(\tau)d\tau}{q_1!q_2!}$$

for all $m \geq 0$. From the fact that the operator $(m+1)^{r_1}(4t^2\partial_t + 2t)^{r_2} + 1$ is invertible on $\mathbb{C}[[t]]$ for all $m \geq 0$, we get from **(H2)** that all $\hat{X}_m(t)$, for $m \geq S$ can be expressed uniquely in terms of the formal series $\hat{X}_k(t)$, $0 \leq k \leq m-1$. This gives the existence and unicity of $\hat{X}(t,z)$ formal series solution of (14). □

In the following definition, we introduce a modified Borel transform of order 1 of formal series.

Definition 3 *Let*

$$\hat{X}(t) = \sum_{j\geq 0} a_j \frac{t^j}{j!}$$

be a formal series with coefficients $a_j \in \mathbb{C}$. We define the formal series

$$(\mathcal{B}_1\hat{X})(t) = \sum_{j\geq 0} a_j \frac{t^j}{(2j)!}$$

as the Borel transform of order 1 of $\hat{X}(t)$ constructed from the moment function $m(u) = \Gamma(1+2u)/\Gamma(1+u)$ for $\mathrm{Re}(u) \geq 0$ (see section 5.5 of [1] for a reference).

Lemma 5 *A formal series*

$$\hat{X}(t,z) = \sum_{m\geq 0} \hat{X}_m(t) \frac{z^m}{m!}$$

where $\hat{X}_m(t) \in \mathbb{C}[[t]]$, $m \geq 0$, satisfies the problem (14), with given initial conditions $(\partial_z^j \hat{X})(t,0) = \hat{X}_j(t)$, $0 \leq j \leq S-1$, if and only if the formal series

$$V(t,z) = \sum_{m\geq 0} (\mathcal{B}_1\hat{X}_m)(t^2) \frac{z^m}{m!}$$

satisfies the following non linear partial differential equation

$$(16) \quad (t^{2r_2}(z\partial_z + 1)^{r_1} + 1)\partial_z^S V(t,z) = \sum_{(k_0,k_1)\in\mathcal{S}} b_{k_0k_1}(z)\partial_t^{2k_0}\partial_z^{k_1} V(t,z) + b(z)\partial_t \int_0^t V(t-\tau,z)V(\tau,z)d\tau,$$

with given initial conditions

$$(17) \qquad (\partial_z^j V)(t,0) = (\mathcal{B}_1\hat{X}_j)(t^2) \in \mathbb{C}[[t]] \quad , \quad 0 \leq j \leq S-1.$$

Proof Let for all $m \geq 0$,

$$\hat{X}_m(t) = \sum_{j \geq 0} a_{j,m} \frac{t^j}{j!}.$$

So that

$$(\mathcal{B}_1 \hat{X}_m)(t^2) = \sum_{j \geq 0} a_{j,m} \frac{t^{2j}}{(2j)!}.$$

From the classical identity

$$\int_0^t (t-\tau)^{j_1} \tau^{j_2} d\tau = \frac{j_1! j_2!}{(1+j_1+j_2)!} t^{j_1+j_2+1},$$

we get that

$$\partial_t \int_0^t \hat{X}_{m_1}(t-\tau) \hat{X}_{m_2}(\tau) d\tau = \sum_{j \geq 0} (\sum_{j_1+j_2=j} a_{j_1,m_1} a_{j_2,m_2}) \frac{t^j}{j!}$$

and that

$$\partial_t \int_0^t (\mathcal{B}_1 \hat{X}_{m_1})((t-\tau)^2)(\mathcal{B}_1 \hat{X}_{m_2})(\tau^2) d\tau = \sum_{j \geq 0} (\sum_{j_1+j_2=j} a_{j_1,m_1} a_{j_2,m_2}) \frac{t^{2j}}{(2j)!},$$

for all $m_1, m_2 \geq 0$. Moreover, we have that

$$(4t^2 \partial_t + 2t) \hat{X}_m(t) = \sum_{j \geq 1} (2j)(2j-1) a_{j-1,m} \frac{t^j}{j!}$$

and

$$t^2 (\mathcal{B}_1 \hat{X}_m)(t^2) = \sum_{j \geq 1} (2j)(2j-1) a_{j-1,m} \frac{t^{2j}}{(2j)!},$$

for all $m \geq 0$. The Lemma follows. □

Let $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$ be an open sector centered at 0 with bisecting direction $d \in \mathbb{R}$ and with opening $\delta > 0$. For all integers $m \geq 0$, let D_m be the open disc centered at 0 with radius $1/(2(m+1)^{r_1/2r_2})$. We denote by $\mathcal{A}_{\nu,d,\delta,m}$ the vector space of holomorphic functions G on $S_{d,\delta} \cup D_m$, continuous on the closure $\bar{S}_{d,\delta} \cup \bar{D}_m$ such that

$$||G(t)||_{d,m} := M_0 \sup_{t \in \bar{S}_{d,\delta} \cup \bar{D}_m} (1+|t|^2) e^{-\nu|t|} |G(t)| < +\infty,$$

where $\nu > 0$ and $M_0 > 1$ is a constant such that

$$\int_0^{|t|} \frac{ds}{(1+s^2)(1+(|t|-s)^2)} \leq \frac{M_0}{(1+|t|^2)},$$

for all $t \in \bar{S}_{d,\delta} \cup \bar{D}_1$. In the following, we will need a lemma from [7], Corollary 4.9.

Lemma 6 *The following inequality hold*

$$|| \int_0^t F(t-\tau) G(\tau) d\tau ||_{d,m} \leq ||F(t)||_{d,m} ||G(t)||_{d,m},$$

for all $F, G \in \mathcal{A}_{\nu,d,\delta,m}$.

In the following theorem, we estimate the rate of growth of the coefficients in $|| \ ||_{d,m}-$norms of formal series solutions of the problem (16) for given initial conditions (17) in some vector space $\mathcal{A}_{\nu,d,\delta,j}$, $0 \leq j \leq S-1$.

Theorem 2 *Let a formal series*

$$V(t,z) = \sum_{m \geq 0} V_m(t) \frac{z^m}{m!}$$

be a solution of (16) with initial conditions

$$(\partial_z^j V)(t,0) = V_j(t) \in \mathcal{A}_{\nu,d,\delta,j} \quad , \quad 0 \leq j \leq S-1,$$

for $\nu > 0$, where $d \in \mathbb{R}$ and $\delta > 0$ are such that the sector $\bar{S}_{d,\delta}$ does not contain any of the complex numbers

$$r_{k,m} = \frac{1}{2(m+1)^{\frac{r_1}{2r_2}}} \exp(\frac{i\pi}{2r_2}(2k+1)) \quad , \quad k \in \{0,\ldots,2r_2-1\},$$

for all $m \geq 0$. Note that $2r_{k,m}$ are the roots of the polynomial $(m+1)^{r_1} t^{2r_2} + 1$. Assume that there exist $C_1 > 0, T > 0$ such that

$$||\partial_t^{n_0} V_j(t)||_{d,j} \leq C_1 (\frac{1}{T})^{n_0} n_0! \tag{18}$$

for all $0 \leq j \leq S-1$, all $n_0 \geq 0$.

Then, there exist $C_2, T, X > 0$ such that

$$||\partial_t^{n_0} V_m(t)||_{d,m} \leq C_2 (\frac{1}{X})^m (\frac{1}{T})^{n_0} (m+1)^{\frac{r_1}{2r_2} n_0} n_0! m! \tag{19}$$

for all $n_0, m \geq 0$.

Proof We expand the functions

$$b_{k_0 k_1}(z) = \sum_{k \geq 0} b_{k_0 k_1 k} z^k \quad , \quad b(z) = \sum_{k \geq 0} b_k z^k$$

in power series near 0. Let $V(t,z) = \sum_{m \geq 0} V_m(t) \frac{z^m}{m!}$ be a formal series satisfying (16). We have the following recursion relation for the formal series $V_m(t)$,

$$\begin{aligned} \frac{V_{m+S}(t)}{m!} = & \sum_{(k_0,k_1) \in \mathcal{S}} \sum_{q_1+q_2=m} \frac{b_{k_0 k_1 q_1}}{(m+1)^{r_1} t^{2r_2} + 1} \frac{\partial_t^{2k_0} V_{q_2+k_1}(t)}{q_2!} \\ & + \sum_{q_1+q_2+q_3=m} \frac{b_{q_3}}{(m+1)^{r_1} t^{2r_2} + 1} \frac{\partial_t \int_0^t V_{q_1}(t-\tau) V_{q_2}(\tau) d\tau}{q_1! q_2!} \end{aligned} \tag{20}$$

for all $t \in \bar{S}_{d,\delta} \cup \bar{D}_m$.

From the Leibniz formula, we get that

$$\begin{aligned} & \partial_t^{n_0} (\frac{b_{k_0 k_1 q_1}}{(m+1)^{r_1} t^{2r_2} + 1} \frac{\partial_t^{2k_0} V_{q_2+k_1}(t)}{q_2!}) \\ & = \sum_{h_1+h_2=n_0} \frac{n_0!}{h_1! h_2!} b_{k_0 k_1 q_1} \partial_t^{h_2} (\frac{1}{(m+1)^{r_1} t^{2r_2} + 1}) \frac{\partial_t^{h_1+2k_0} V_{q_2+k_1}(t)}{q_2!} \end{aligned} \tag{21}$$

and

$$(22)\quad \partial_t^{n_0}\Big(\frac{b_{q_3}}{(m+1)^{r_1}t^{2r_2}+1}\frac{\partial_t\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau}{q_1!q_2!}\Big) = \sum_{h_1+h_2=n_0}\frac{n_0!}{h_1!h_2!}\frac{b_{q_3}}{q_1!q_2!}\partial_t^{h_2}\Big(\frac{1}{(m+1)^{r_1}t^{2r_2}+1}\Big)\partial_t^{h_1+1}\Big(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau\Big)$$

for all $n_0 \geq 0$. In the following, we will need the following estimates

Lemma 7 *There exists a constant $C > 0$ (which depends on the sector $\bar{S}_{d,\delta}$) such that*

$$(23)\quad |\partial_t^{h_2}\Big(\frac{1}{(m+1)^{r_1}t^{2r_2}+1}\Big)| \leq h_2!\Big(\frac{1}{C}\Big)^{h_2+1}(m+1)^{\frac{r_1}{2r_2}h_2}$$

for all $t \in \bar{S}_{d,\delta} \cup \bar{D}_m$, for all $m \geq 0$.

Proof We start from the partial fraction decomposition over $\mathbb{C}$

$$\frac{1}{(m+1)^{r_1}t^{2r_2}+1} = \sum_{k=0}^{2r_2-1}\frac{A_{k,m}}{t-(e^{\frac{i\pi}{2r_2}(2k+1)}/(m+1)^{\frac{r_1}{2r_2}})},$$

for all $t \in \bar{S}_{d,\delta} \cup \bar{D}_m$, where

$$A_{k,m} = \frac{e^{-i\pi\frac{2r_2-1}{2r_2}(2k+1)}}{2r_2(m+1)^{\frac{r_1}{2r_2}}}$$

So that for all $h_2 \geq 0$, we have

$$(24)\quad \partial_t^{h_2}\Big(\frac{1}{(m+1)^{r_1}t^{2r_2}+1}\Big) = (-1)^{h_2}h_2!\sum_{k=0}^{2r_2-1}\frac{A_{k,m}}{(t-(e^{\frac{i\pi}{2r_2}(2k+1)}/(m+1)^{\frac{r_1}{2r_2}}))^{h_2+1}}$$

On the other hand, we have the following estimates. There exists $C > 0$ such that

$$|t-\frac{e^{\frac{i\pi}{2r_2}(2k+1)}}{(m+1)^{\frac{r_1}{2r_2}}}| \geq \frac{C}{(m+1)^{\frac{r_1}{2r_2}}},$$

for all $t \in \bar{S}_{d,\delta} \cup \bar{D}_m$, for all $k \in \{0,\ldots,2r_2-1\}$. Indeed, let $t \in \bar{D}_m$. We write t in the form

$$t = \frac{\rho}{2(m+1)^{\frac{r_1}{2r_2}}}e^{i\theta}$$

for $\rho \in [0,1]$ and $\theta \in [0,2\pi)$. So that

$$(25)\quad |t-\frac{e^{\frac{i\pi}{2r_2}(2k+1)}}{(m+1)^{\frac{r_1}{2r_2}}}| = \frac{1}{(m+1)^{\frac{r_1}{2r_2}}}|\frac{1}{2}\rho e^{i\theta}-e^{\frac{i\pi}{2r_2}(2k+1)}| \geq \frac{1}{2(m+1)^{\frac{r_1}{2r_2}}},$$

for all $\rho \in [0,1]$, all $\theta \in [0,2\pi)$, all $0 \leq k \leq 2r_2-1$. Let $t \in \bar{S}_{d,\delta}$. For all $0 \leq k \leq 2r_2-1$, we can write t in the form

$$t = \frac{\rho}{(m+1)^{\frac{r_1}{2r_2}}}e^{i(\frac{\pi}{2r_2}(2k+1)+\tau_k)}$$

where $\rho \in [0,+\infty)$ and $\tau_k \in \mathbb{R}$. By construction, the direction d and the opening δ of the sector $\bar{S}_{d,\delta}$ are chosen in such a way that for all $k \in \{0,\ldots,2r_2-1\}$, we have $e^{i\tau_k} \neq 1$. So that there exists $C > 0$ such that

$$|\rho e^{i\tau_k} - 1| \geq C,$$

for all $\rho \geq 0$, for all τ_k with $k \in \{0,\ldots,2r_2-1\}$. We deduce that

$$|t - \frac{e^{\frac{i\pi}{2r_2}(2k+1)}}{(m+1)^{\frac{r_1}{2r_2}}}| = \frac{1}{(m+1)^{\frac{r_1}{2r_2}}}|\rho e^{i\tau_k} - 1| \geq \frac{C}{(m+1)^{\frac{r_1}{2r_2}}}, \tag{26}$$

for all $t \in \bar{S}_{d,\delta}$, for all $k \in \{0,\ldots,2r_2-1\}$. Finally, one can check that the equality (24) with the estimates (25) and (26) yields the estimates (23). □

Lemma 8 *We have the estimates*

$$\begin{aligned}(27)\quad & ||\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau)||_{d,m+S} \\ & \leq \left(\sum_{l_1+l_2=h_1} \frac{h_1!}{l_1!l_2!}||\partial_t^{l_1}V_{q_1}(t)||_{d,m+S}||\partial_t^{l_2}V_{q_2}(t)||_{d,m+S}\right) + ||\partial_t^{h_1+1}V_{q_1}(t)||_{d,m+S}||V_{q_2}(t)||_{d,m+S}\end{aligned}$$

for all $h_1,q_1,q_2,m \geq 0$.

Proof We have

$$\begin{aligned}(28)\quad & \partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau) \\ & = \left(\sum_{l_1+l_2=h_1} (\partial_t^{l_1}V_{q_1})(0)\partial_t^{l_2}V_{q_2}(t)\right) + \int_0^t (\partial_t^{h_1+1}V_{q_1})(t-\tau)V_{q_2}(\tau)d\tau\end{aligned}$$

By construction of the norm $||\ ||_{d,m+S}$, we have that

$$|(\partial_t^{l_1}V_{q_1})(0)| \leq ||\partial_t^{l_1}V_{q_1}(t)||_{d,m+S}. \tag{29}$$

On the other hand, we have that

$$1 \leq \frac{h_1!}{l_1!l_2!}, \tag{30}$$

for all $l_1,l_2 \geq 0$ such that $l_1+l_2=h_1$. Finally, taking the $||\ ||_{d,m+S}$ norm of the equality (28), using the inequalities (29), (30) and applying the Lemma 6 give the estimates (27). □

Now, we differentiate n_0 times the equality (20) and we take the $||\ ||_{d,m+S}$ norm. Using the equalities (21) and (22) and the estimates of Lemma 7,8, we obtain the inequalities

$$\frac{||\partial_t^{n_0}V_{m+S}(t)||_{d,m+S}}{m!} \leq \sum_{(k_0,k_1)\in\mathcal{S}} (\mathcal{M}_1(n_0,m)+\mathcal{M}_2(n_0,m)+\mathcal{M}_3(n_0,m)) \tag{31}$$

for all $m,n_0 \geq 0$, where

$$\mathcal{M}_1(n_0,m) = \sum_{q_1+q_2=m}\sum_{h_1+h_2=n_0} n_0!|b_{k_0k_1q_1}|(\frac{1}{C})^{h_2+1}(m+1)^{\frac{r_1}{2r_2}h_2}\frac{||\partial_t^{h_1+2k_0}V_{q_2+k_1}(t)||_{d,m+S}}{h_1!q_2!},$$

$$\mathcal{M}_2(n_0,m) = \sum_{q_1+q_2+q_3=m}\sum_{h_1+h_2=n_0}\sum_{l_1+l_2=h_1} n_0!|b_{q_3}|(\frac{1}{C})^{h_2+1}(m+1)^{\frac{r_1}{2r_2}h_2}\frac{||\partial_t^{l_1}V_{q_1}(t)||_{d,m+S}}{l_1!q_1!}\frac{||\partial_t^{l_2}V_{q_2}(t)||_{d,m+S}}{l_2!q_2!}$$

and

$$\mathcal{M}_3(n_0,m) = \sum_{q_1+q_2+q_3=m}\sum_{h_1+h_2=n_0} n_0!|b_{q_3}|(\frac{1}{C})^{h_2+1}(m+1)^{\frac{r_1}{2r_2}h_2}\frac{||\partial_t^{h_1+1}V_{q_1}(t)||_{d,m+S}}{h_1!q_1!}\frac{||V_{q_2}(t)||_{d,m+S}}{q_2!}.$$

Lemma 9 *The positive numbers*

$$w_{n_0,m} = \frac{||\partial_t^{n_0}V_m(t)||_{d,m}}{(m+1)^{\frac{r_1}{2r_2}n_0}}$$

satisfy the following inequalities

$$\frac{w_{n_0,m+S}}{n_0!m!} \leq \sum_{(k_0,k_1)\in\mathcal{S}}(\mathcal{N}_1(n_0,m)+\mathcal{N}_2(n_0,m)+\mathcal{N}_3(n_0,m)) \tag{32}$$

for all $n_0,m \geq 0$, *where*

$$\mathcal{N}_1(n_0,m) = \sum_{q_1+q_2=m}\sum_{h_1+h_2=n_0}|b_{k_0k_1q_1}|(\frac{1}{C})^{h_2+1}(q_2+k_1+1)^{\tilde{r}_1k_0}\frac{w_{h_1+2k_0,q_2+k_1}}{h_1!q_2!},$$

$$\mathcal{N}_2(n_0,m) = \sum_{q_1+q_2+q_3=m}\sum_{h_1+h_2=n_0}\sum_{l_1+l_2=h_1}|b_{q_3}|(\frac{1}{C})^{h_2+1}\frac{w_{l_1,q_1}}{l_1!q_1!}\frac{w_{l_2,q_2}}{l_2!q_2!}$$

and

$$\mathcal{N}_3(n_0,m) = \sum_{q_1+q_2+q_3=m}\sum_{h_1+h_2=n_0}|b_{q_3}|(\frac{1}{C})^{h_2+1}(q_1+1)^{\check{r}_1}\frac{w_{h_1+1,q_1}}{h_1!q_1!}\frac{w_{0,q_2}}{q_2!},$$

where $\tilde{r}_1 = [\frac{r_1}{r_2}]+1$ *and* $\check{r}_1 = [\frac{r_1}{2r_2}]+1$.

Proof We rewrite the expressions $\mathcal{M}_j(n_0,m)$, $j=1,2,3$, in the following form

$$\frac{\mathcal{M}_1(n_0,m)}{(m+1+S)^{\frac{r_1}{2r_2}n_0}n_0!} = \sum_{q_1+q_2=m}\sum_{h_1+h_2=n_0}|b_{k_0k_1q_1}|(\frac{1}{C})^{h_2+1}(\frac{m+1}{m+1+S})^{\frac{r_1}{2r_2}n_0} \times \frac{(q_2+k_1+1)^{\frac{r_1}{2r_2}(h_1+2k_0)}}{(m+1)^{\frac{r_1}{2r_2}h_1}}\frac{||\partial_t^{h_1+2k_0}V_{q_2+k_1}(t)||_{d,m+S}}{(q_2+k_1+1)^{\frac{r_1}{2r_2}(h_1+2k_0)}h_1!q_2!}, \tag{33}$$

$$\frac{\mathcal{M}_2(n_0,m)}{(m+1+S)^{\frac{r_1}{2r_2}n_0}n_0!} = \sum_{q_1+q_2+q_3=m}\sum_{h_1+h_2=n_0}\sum_{l_1+l_2=h_1}|b_{q_3}|(\frac{1}{C})^{h_2+1}(\frac{m+1}{m+1+S})^{\frac{r_1}{2r_2}n_0} \times \frac{(q_1+1)^{\frac{r_1}{2r_2}l_1}(q_2+1)^{\frac{r_1}{2r_2}l_2}}{(m+1)^{\frac{r_1}{2r_2}l_1}(m+1)^{\frac{r_1}{2r_2}l_2}}\frac{||\partial_t^{l_1}V_{q_1}(t)||_{d,m+S}}{(q_1+1)^{\frac{r_1}{2r_2}l_1}l_1!q_1!}\frac{||\partial_t^{l_2}V_{q_2}(t)||_{d,m+S}}{(q_2+1)^{\frac{r_1}{2r_2}l_2}l_2!q_2!} \tag{34}$$

and

$$\frac{\mathcal{M}_3(n_0,m)}{(m+1+S)^{\frac{r_1}{2r_2}n_0}n_0!} = \sum_{q_1+q_2+q_3=m}\sum_{h_1+h_2=n_0} |b_{q_3}|(\frac{1}{C})^{h_2+1}(\frac{m+1}{m+1+S})^{\frac{r_1}{2r_2}n_0} \times \frac{(q_1+1)^{\frac{r_1}{2r_2}(h_1+1)}}{(m+1)^{\frac{r_1}{2r_2}h_1}}\frac{||\partial_t^{h_1+1}V_{q_1}(t)||_{d,m+S}}{(q_1+1)^{\frac{r_1}{2r_2}(h_1+1)}h_1!q_1!}\frac{||V_{q_2}(t)||_{d,m+S}}{q_2!}, \tag{35}$$

for all $n_0,m \geq 0$. On the other hand, we have the inequalities

$$||F(t)||_{d,m_2} \leq ||F(t)||_{d,m_1} \tag{36}$$

for all $m_1 \leq m_2$, all functions $F(t) \in \mathcal{A}_{\nu,d,\delta,m_1}$, which results from the inclusion $\bar{D}_{m_2} \subset \bar{D}_{m_1}$. Moreover, we have also the inequalities

$$(\frac{q_2+k_1+1}{m+1})^{\frac{r_1}{2r_2}h_1} \leq (\frac{m+S+1}{m+1})^{\frac{r_1}{2r_2}n_0} \tag{37}$$

for $0 \leq h_1 \leq n_0$, $0 \leq q_2 \leq m$ and $0 \leq k_1 \leq S$. Finally, starting from the inequality (31) and using the equalities (33), (34), (35) and the inequalities (36) and (37), we get the inequalities (32). □

We consider now the following auxiliary Cauchy problem

$$\partial_x^S U(t,x) = \sum_{(k_0,k_1)\in\mathcal{S}} R(t)|b|_{k_0k_1}(x)\partial_t^{2k_0}\partial_x^{k_1}(x\partial_x+1)^{\tilde{r}_1k_0}U(t,x) + R(t)|b|(x)((U(t,x))^2 + U(0,x)\partial_t(x\partial_x+1)^{\check{r}_1}U(t,x)) \tag{38}$$

with initial conditions

$$(\partial_x^j U)(t,0) = \sum_{n_0\geq 0} w_{n_0,j}\frac{t^{n_0}}{n_0!} = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1, \tag{39}$$

where $\tilde{r}_1 = [\frac{r_1}{r_2}]+1$, $\check{r}_1 = [\frac{r_1}{2r_2}]+1$,

$$R(t) = \sum_{n_0\geq 0}(\frac{1}{C})^{n_0+1}t^{n_0} = \frac{1}{C-t}$$

and

$$|b|_{k_0k_1}(x) = \sum_{m\geq 0}|b_{k_0k_1m}|x^m \quad , \quad |b|(x) = \sum_{m\geq 0}|b_m|x^m.$$

From the assumptions **(H1)**, **(H2)**, we get that the hypotheses **i)**, **ii)** of Theorem 1 are satisfied for the equation (38). From the assumption (18), we deduce that the hypothesis **iii)** of Theorem 1 are satisfied for the problem (38), (39). We apply Theorem 1 to the problem (38), (39). We get that the unique formal solution

$$U(t,x) = \sum_{n_0,m\geq 0} U_{n_0,m}\frac{t^{n_0}}{n_0!}\frac{x^m}{m!}$$

of the problem (38), (39) belongs to the space $G(T,X)$ for $T,X > 0$ small enough. In particular, there exists $C_2 > 0$ such that

$$|U_{n_0,m}| \leq C_2(\frac{1}{T})^{n_0}(\frac{1}{X})^m(n_0+m)! \tag{40}$$

for all $n_0,m \geq 0$.

Lemma 10 *The positive numbers $U_{n_0,m}$ satisfy the following recursion relation*

$$\frac{U_{n_0,m+S}}{n_0!m!} = \sum_{(k_0,k_1)\in\mathcal{S}} (\mathcal{U}_1(n_0,m) + \mathcal{U}_2(n_0,m) + \mathcal{U}_3(n_0,m)) \tag{41}$$

for all $n_0, m \geq 0$, where

$$\mathcal{U}_1(n_0,m) = \sum_{q_1+q_2=m} \sum_{h_1+h_2=n_0} |b_{k_0k_1q_1}|(\frac{1}{C})^{h_2+1}(q_2+k_1+1)^{\tilde{r}_1k_0}\frac{U_{h_1+2k_0,q_2+k_1}}{h_1!q_2!},$$

$$\mathcal{U}_2(n_0,m) = \sum_{q_1+q_2+q_3=m} \sum_{h_1+h_2=n_0} \sum_{l_1+l_2=h_1} |b_{q_3}|(\frac{1}{C})^{h_2+1}\frac{U_{l_1,q_1}}{l_1!q_1!}\frac{U_{l_2,q_2}}{l_2!q_2!}$$

and

$$\mathcal{U}_3(n_0,m) = \sum_{q_1+q_2+q_3=m} \sum_{h_1+h_2=n_0} |b_{q_3}|(\frac{1}{C})^{h_2+1}(q_1+1)^{\check{r}_1}\frac{U_{h_1+1,q_1}}{h_1!q_1!}\frac{U_{0,q_2}}{q_2!},$$

with initial conditions

$$U_{n_0,j} = w_{n_0,j} \quad , \quad 0 \leq j \leq S-1$$

for all $n_0 \geq 0$.

Proof A direct calculation shows that

$$R(t)|b|_{k_0k_1}(x)\partial_t^{2k_0}\partial_x^{k_1}(x\partial_x+1)^{\tilde{r}_1k_0}U(t,x) = \sum_{n_0,m\geq 0} \mathcal{U}_1(n_0,m)t^{n_0}x^m,$$

$$\partial_x^S U(t,x) = \sum_{n_0,m\geq 0} \frac{U_{n_0,m+S}}{n_0!m!}t^{n_0}x^m \quad , \quad R(t)|b|(x)(U(t,x))^2 = \sum_{n_0,m\geq 0} \mathcal{U}_2(n_0,m)t^{n_0}x^m$$

and

$$R(t)|b|(x)U(0,x)\partial_t(x\partial_x+1)^{\check{r}_1}U(t,x) = \sum_{n_0,m\geq 0} \mathcal{U}_3(n_0,m)t^{n_0}x^m,$$

which gives the lemma. □

From the inequalities (32) and (41), we deduce that

$$w_{n_0,m} \leq U_{n_0,m} \tag{42}$$

for all $n_0, m \geq 0$. Finally, from this latter inequality (42) and the estimates (40), we get that the estimates (19) hold. This yields Theorem 2. □

4 Borel-Laplace transform of formal series solutions of partial differential equations

We recall the classical definition of $k-$summability of formal series with complex coefficients, see [1].

Definition 4 *A formal series*

$$\hat{X}(t) = \sum_{j=0}^{\infty} a_j t^j$$

with coefficients $a_j \in \mathbb{C}$, is said to be $k-$summable with respect to t in the direction $d \in \mathbb{R}$, for $k > 0$, if

i) *There exist $\rho \in \mathbb{R}_+$ such that the following formal series, called formal Borel transform of order k, with respect to the variable t,*

$$W(t) = \sum_{j=0}^{\infty} \frac{a_j t^j}{\Gamma(1 + j/k)},$$

is absolutely convergent for $|t| < \rho$,

ii) *There exists $\delta > 0$ such that the series $W(t)$ can be analytically continued with respect to t in a sector $S_{d,\delta} = \{t \in \mathbb{C}^* : |d - \arg(t)| < \delta\}$. Moreover, there exist $\delta_1 < \delta$, $C > 0$, and $K > 0$ such that*

$$|W(t)| \leq Ce^{K|t|^k}$$

for all $t \in S_{d,\delta_1}$.

If this is so, the vector Laplace transform of order k of $W(t)$,

$$X(t) = t^{-k} \int_0^{\infty e^{i\gamma}} W(\tau) e^{-(\tau/t)^k} d\tau^k,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the $k-$sum of the formal series $\hat{X}(t)$. In particular, the function $X(t)$ is asymptotic to $\hat{X}(t)$ with Gevrey filtration of order $1/k$, with respect to t on a finite sector

$$S_{d,\theta,R} = \{t \in \mathbb{C}^* : |t| \leq R \quad , \quad |d - \arg(t)| < \theta/2\},$$

where $\theta > \pi/k$ and $R > 0$. This means that for all $\theta_1 < \theta$, there exist $C, M > 0$, such that

$$|X(t) - \sum_{p=0}^{n-1} a_p t^p| \leq CM^n (n!)^{1/k} |t|^n$$

for all $n \geq 1$ and $t \in S_{d,\theta_1,R}$.

In the next proposition, we deduce the $1-$summability property from the modified Borel transform of order 1 introduced in section 3. This proposition is an application of a general moment summability result (see Theorem 38, page 108 in [1]).

Proposition 2 *A formal series*

$$\hat{X}(t) = \sum_{j=0}^{\infty} a_j \frac{t^j}{j!}$$

with coefficients $a_j \in \mathbb{C}$, is $1-$summable with respect to t in direction $d \in \mathbb{R}$ if

i) *There exists $\rho \in \mathbb{R}_+$ such that the modified Borel transform of order 1,*

$$(\mathcal{B}_1 \hat{X})(t) = \sum_{j=0}^{\infty} a_j \frac{t^j}{(2j)!}$$

is absolutely convergent for all $t \in D_\rho$.

ii) *There exists* $\delta > 0$ *such that the series* $(\mathcal{B}_1\hat{X})(t)$ *can be analytically continued with respect to* t *in a sector* $S_{d,\delta} = \{t \in \mathbb{C}^* : |d - \arg(t)| < \delta\}$. *Moreover, there exist* $\delta_1 < \delta$, $C > 0$, *and* $K > 0$ *such that*

$$|(\mathcal{B}_1\hat{X})(t)| \leq Ce^{K|t|}$$

for all $t \in S_{d,\delta_1}$.

Remark 1 : From Theorem 38 in [1], the $1-$sum $X(t)$ of $\hat{X}(t)$ defined on some finite sector $S_{d,\theta,R}$ where $\theta > \pi$ and $R > 0$ can be expressed in terms of the modified Borel transform of order 1 using an integral operator

$$X(t) = \int_0^{\infty e^{id}} e(u/t)\frac{(\mathcal{B}_1\hat{X})(u)}{u}du = \int_0^{\infty e^{i(d-\arg(t))}} e(u)\frac{(\mathcal{B}_1\hat{X})(ut)}{u}du$$

where the kernel function $e(z)$ satisfies the following properties :
i) The function $e(z)$ is holomorphic on $S_{0,\pi/2}$.
ii) The integral $\int_0^{x_0} x^{-1}|e(xe^{i\tau})|dx$ exists for all $x_0 > 0$ and $2|\tau| < \pi$.
iii) For all $\delta > 0$, there exist $C, K > 0$ such that $|e(z)| \leq C\exp(-|z|/K)$ for $2|\arg(z)| \leq \pi - \delta$.
iv) We have $\Gamma(1 + 2u)/\Gamma(1 + u) = m(u) = \int_0^\infty x^{u-1}e(x)dx$ for $\mathrm{Re}(u) \geq 0$.

Lemma 11 *Let*

$$\hat{X}(t,z) = \sum_{m\geq 0} \hat{X}_m(t)\frac{z^m}{m!}$$

be a formal series with coefficients $\hat{X}_m(t) \in \mathbb{C}[[t]]$, *solution of (14), for initial conditions* $(\partial_z^j\hat{X})(t,0) = \hat{X}_j(t)$, $0 \leq j \leq S-1$. *We make the following assumptions.*
1) The Borel transforms $(\mathcal{B}_1\hat{X}_j)(t)$, $0 \leq j \leq S-1$, *are holomorphic functions on the discs* $D(0, 1/(4(j+1)^{r_1/r_2}))$.
2) The functions $t \mapsto V_j(t) = (\mathcal{B}_1\hat{X}_j)(t^2)$, $0 \leq j \leq S-1$, *belong to the space* $\mathcal{A}_{\nu,d,\delta,j}$ *for* $\nu > 0$ *and* $d \in \mathbb{R}$ *satisfying the hypotheses of Theorem 2. Let the estimates (18) be satisfied for the functions* $V_j(t)$, $0 \leq j \leq S-1$.
Then, for all $m \geq 0$, *the formal series* $\hat{X}_m(t)$ *are* $1-$*summable in the direction* $2d$.

Proof From Lemma 5 and the recursion (20), we get from the assumption 1) that all the Borel transforms $(\mathcal{B}_1\hat{X}_m)(t)$, $m \geq 0$, are holomorphic functions on $D(0, 1/(4(m+1)^{r_1/r_2}))$. From Theorem 2, we apply the estimates (19) for $n_0 = 0$. So that there exist $C_2, X > 0$ such that

$$||V_m(t)||_{d,m} \leq C_2(\frac{1}{X})^m m!$$

for all $m \geq 0$. We deduce that, for all $m \geq 0$, the modified Borel transform $(\mathcal{B}_1\hat{X}_m)(t)$ is holomorphic on $D(0, 1/(4(m+1)^{r_1/r_2}))$ and can be analytically continued on a sector $S_{2d,\delta}$ where $\delta > 0$ such that there exist $C, X > 0$ with

$$|(\mathcal{B}_1\hat{X}_m)(t)| \leq C(\frac{1}{X})^m m!\frac{e^{\nu|t|^{1/2}}}{1+|t|} \tag{43}$$

for all $t \in \bar{S}_{2d,\delta}$. From Proposition 2, we deduce that for all $m \geq 0$, the formal series $\hat{X}_m(t)$ are $1-$summable in the direction $2d$. □

Lemma 12 *Under the hypotheses of Lemma 11, let $X_m(t)$ be the $1-$sum of $\hat{X}_m(t)$ in the direction $2d$. There exist $C, X, T > 0$ and a sector $S_{d,\theta,R}$ in direction d, with opening $0 < \theta \leq (\pi - \delta)/2$, for $\delta > 0$ and radius $R > 0$ such that*

$$\sup_{t \in S_{d,\theta,R}} |\partial_t^{n_0}(X_m(t^2))| \leq C(\frac{1}{X})^m (\frac{1}{T})^{n_0} (m+1)^{\frac{r_1 n_0}{2r_2}} (n_0!)^2 m! \tag{44}$$

for all $m \geq 0$, all $n_0 \geq 0$.

Proof From Lemma 11 and remark 1, we get that the $1-$sum $X_m(t)$ of $\hat{X}_m(t)$ can be expressed as an integral representation

$$X_m(t) = \int_0^{\infty e^{i(2d - \arg(t))}} e(u) \frac{\mathcal{B}_1 \hat{X}_m(ut)}{u} du \tag{45}$$

where $e(u)$ is a kernel function satisfying the conditions i),ii),iii),iv) of Remark 1, for all $t \in \bar{S}_{2d,\zeta,R^2}$ where $0 < \zeta \leq \pi - \delta$, for $\delta > 0$ and $R > 0$. The fact that the radius R^2 does not depend on m is a consequence of the estimates ii) and iii) of Remark 1 and (43). Indeed, there exist $C, X > 0$ such that

$$|e(u) \frac{\mathcal{B}_1 \hat{X}_m(ut)}{u}| \leq C(\frac{1}{X})^m m! \frac{|e(re^{i(2d - \arg(t))})|}{r} \frac{e^{\nu r^{1/2} |t|^{1/2}}}{1 + r|t|},$$

for $u = re^{i(2d - \arg(t))}$, all $r > 0$ and $t \in \bar{S}_{2d,\zeta,R^2}$. So that the integral (45) exists for all $t \in \bar{S}_{2d,\zeta,R^2}$, for all $R > 0$.

We make the change of variable $u = v^2$ in the integral (45) to get

$$X_m(t^2) = 2 \int_0^{\infty e^{i(d - \arg(t))}} e(v^2) \frac{V_m(vt)}{v} dv \tag{46}$$

for all $t \in \bar{S}_{d,\theta,R}$ where $0 < \theta \leq (\pi - \delta)/2$, for $\delta > 0$ and $R > 0$.

From the estimates (19) with $n_0 = 0$, we get constants $C_2, X > 0$ such that

$$|X_m(t^2)| \leq C_2 (\frac{1}{X})^m m! \int_0^{+\infty} \frac{e(r^2 e^{i(2d - 2\arg(t))})}{r} \frac{e^{\nu r|t|}}{1 + r^2 |t|^2} dr$$

for all $t \in \bar{S}_{d,\theta,R}$, all $m \geq 0$. So that, from the estimates ii) and iii) of Remark 1, we get constants $C, X > 0$ such that

$$|X_m(t^2)| \leq C(\frac{1}{X})^m m!,$$

for all $t \in \bar{S}_{d,\theta,R}$, all $m \geq 0$. The estimates (44) then hold for $n_0 = 0$.

From the Cauchy formula, this integral (46) does not depend on small changes of the ray of integration $\{v \in \mathbb{C}^* / \arg v = d - \arg(t)\}$. We deduce that the n_0-derivative of $X_m(t^2)$ writes

$$\partial_t^{n_0}(X_m(t^2)) = 2 \int_0^{\infty e^{i(d - \arg(t))}} e(v^2) v^{n_0 - 1} (\partial_t^{n_0} V_m)(vt) dv$$

for all $n_0 \geq 1$. Using the estimates (19) and iii) of Remark 1, we get constants $C_2, K, X, T > 0$ such that

$$|\partial_t^{n_0}(X_m(t^2))| \leq C_2 (\frac{1}{X})^m (\frac{1}{T})^{n_0} (m+1)^{\frac{r_1 n_0}{2r_2}} n_0! m! \int_0^{+\infty} e^{-r^2/K} r^{n_0 - 1} \frac{e^{\nu r|t|}}{1 + r^2 |t|^2} dr \tag{47}$$

for all $n_0 \geq 1$, for all $t \in S_{d,\theta,R}$.

Let $M_1 > 0$ be such that $M_1 \geq \eta + \nu R$, for a given $\eta > 0$. We choose $M_2 > 0$ which satisfies $KM_1^2 < 4M_2$. We get that $\frac{r^2}{K} \geq M_1 r - M_2$ for all $r \geq 0$. We deduce the estimates

$$\int_0^{+\infty} e^{-r^2/K} r^{n_0-1} \frac{e^{\nu r|t|}}{1+r^2|t|^2} dr \leq e^{M_2} \int_0^{+\infty} e^{-(M_1-\nu|t|)r} r^{n_0-1} dr \leq e^{M_2} (\frac{1}{\eta})^{n_0} (n_0-1)! \tag{48}$$

for all $t \in S_{d,\theta,R}$. Finally, the estimates (44) follows from (47), (48), for all $n_0 \geq 1$. □

Definition 5 *Let $X(t,z)$ be a holomorphic function on a domain $S_{d,\theta,R} \times D_K$, where $S_{d,\theta,R}$ is an open sector centered at 0 with direction $d \in \mathbb{R}$, opening $\theta \in (0, 2\pi)$, radius R and D_K is a disc centered at 0 with radius $K > 0$. Let $\hat{X}(t,z)$ be a formal series*

$$\hat{X}(t,z) = \sum_{l,m \geq 0} a_{lm} t^l z^m$$

in the variables t, z. We make the assumption that there exists $0 < \rho \leq K$ such that, for all $N \geq 1$, the formal series

$$\hat{X}^{N-1}(t,z) = \sum_{m \geq 0} \sum_{l=0}^{N-1} a_{lm} t^l z^m$$

is absolutely convergent for $z \in \bar{D}_\rho$ and is a polynomial in t. We say that the function $X(t,z)$ is Gevrey asymptotic of order γ to the formal series $\hat{X}(t,z)$ with respect to $t \in S_{d,\theta,R}$, uniformly in z on $\bar{D}_\rho$, if there exist $C, K_1 > 0$ such that

$$\sup_{|z| \leq \rho} |X(t,z) - \hat{X}^{N-1}(t,z)| \leq C K_1^N |t|^N \Gamma(1 + \gamma N)$$

for all $t \in S_{d,\theta,R}$, for all $N \geq 1$.

In the next theorem, we give sufficient conditions under which for a given formal series $\hat{X}(t,z)$ solution of (14) there exists an actual solution $X(t,z)$ of (14) which is Gevrey asymptotic to $\hat{X}(t,z)$ with respect to t in a sector, uniformly in z in a disc.

Theorem 3 *Let*

$$\hat{X}(t,z) = \sum_{m \geq 0} \hat{X}_m(t) \frac{z^m}{m!}$$

be a formal series where the coefficients $\hat{X}_m(t) \in \mathbb{C}[[t]]$, solution of (14) with initial conditions $(\partial_z^j \hat{X})(t,0) = \hat{X}_j(t) \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, constructed from Lemma 4. We make the following assumptions.
1) For $0 \leq j \leq S-1$, the Borel transforms $(\mathcal{B}_1 \hat{X}_j)(t)$ are holomorphic functions on the disc $D(0, 1/(4(j+1)^{r_1/r_2}))$.
2) For $0 \leq j \leq S-1$, the functions $t \mapsto V_j(t) = (\mathcal{B}_1 \hat{X}_j)(t^2)$ belong to the space $\mathcal{A}_{\nu,d,\delta,j}$ for $\nu > 0$ and $d \in \mathbb{R}$ such that the sector $\bar{S}_{d,\delta}$ does not contain any of the complex numbers

$$r_{k,m} = \frac{1}{2(m+1)^{\frac{r_1}{2r_2}}} \exp(\frac{i\pi}{2r_2}(2k+1)) \quad , \quad k \in \{0, \ldots, 2r_2 - 1\},$$

for all $m \geq 0$. Moreover, assume that there exists $C_1 > 0, T > 0$ such that

$$||\partial_t^{n_0} V_j(t)||_{d,j} \leq C_1 (\frac{1}{T})^{n_0} n_0! \tag{49}$$

for all $0 \leq j \leq S-1$*, all* $n_0 \geq 0$*.*

Then, the function

$$X(t,z) = \sum_{m \geq 0} X_m(t) \frac{z^m}{m!} \tag{50}$$

where $X_m(t)$*,* $m \geq 0$*, are the* $1-$*sums of the formal series* $\hat{X}_m(t)$*, is holomorphic on the domain* $S_{2d,\zeta,R^2} \times D_X$ *where* S_{2d,ζ,R^2} *is a sector with opening* $0 < \zeta \leq \pi - \delta$*, for* $\delta > 0$ *and radius* $R^2 > 0$ *and* D_X *is a disc centered at 0 with radius* $X > 0$*. The function* $X(t,z)$ *is a solution of (14) on* $S_{2d,\zeta,R^2} \times D_X$*. Moreoever, the function* $X(t,z)$ *is Gevrey asymptotic of order* $2(1+\frac{r_1}{2r_2})$ *to* $\hat{X}(t,z)$ *on* S_{2d,ζ,R^2}*, uniformly in* z *on* $\bar{D}_\rho$ *for* ρ *small enough.*

Proof From the estimates (44) for $n_0 = 0$, we get that the series (50) converge uniformly on every compact set $K \times \bar{D}_\rho$ of $S_{2d,\zeta,R^2} \times D_X$ where K is a compact set of the sector S_{2d,ζ,R^2} and $0 < \rho < X$, for $X > 0$ from (44). So that $X(t,z)$ defines a holomorphic function on the domain $S_{2d,\zeta,R^2} \times D_X$.

Moreover from the general results on derivation, product, convolution of $1-$summable formal series, see [1], section 6.3, the $1-$sums $X_m(t)$ satisfy the same recursion relation (15) as $\hat{X}_m(t)$, for all $m \geq 0$, in Lemma 4. So that $X(t,z)$ is a solution of (14) on the domain $S_{2d,\zeta,R^2} \times D_X$. The rest of the proof will be divided in three parts.

Part 1. In this step we give uniform estimates on the n_0-derivative with respect to t of the function $X(t^2,z)$.

We consider the holomorphic function $X(t^2,z)$ for all $(t,z) \in S_{d,\theta,R} \times D_X$, where $S_{d,\theta,R}$ is a sector with direction d, opening θ with $0 < \theta \leq (\pi-\delta)/2$, for $\delta > 0$, and radius $R > 0$. From the estimates (44), we get that there exist $C, X, T > 0$ such that

$$\sup_{(t,z) \in S_{d,\theta,R} \times \bar{D}_\rho} |\partial_t^{n_0}(X(t^2,z))| \leq \frac{CX}{\rho}(\frac{1}{T})^{n_0}(n_0!)^2 \sum_{m \geq 0} (m+1)^{\frac{r_1 n_0}{2r_2}} (\frac{\rho}{X})^{m+1} \tag{51}$$

for all $n_0 \geq 1$, all $0 < \rho < X$. From the estimates (6.9) in [4] page 625, we know that there exist constants $C_3, C_4 > 0$ depending on ρ, X such that

$$|\sum_{m=1}^{+\infty} m^{\frac{r_1 n_0}{2r_2}} (\frac{\rho}{X})^m| \leq C_3 C_4^{\frac{r_1 n_0}{2r_2}} \Gamma(\frac{r_1 n_0}{2r_2}) \tag{52}$$

for all $n_0 \geq 1$. Using classical estimates on the Gamma function, we deduce from (51) and (52) that there exists $0 < \rho < X$, there exist $C, K > 0$ such that

$$\sup_{(t,z) \in S_{d,\theta,R} \times \bar{D}_\rho} |\partial_t^{n_0}(X(t^2,z))| \leq CK^{n_0} n_0! \Gamma(1 + (1+\frac{r_1}{2r_2})n_0) \tag{53}$$

for all $n_0 \geq 1$.

Part 2. In this step, we get uniform estimates on the n_0-derivative with respect to t of the function $X(t,z)$.

We will follow a method introduced in the paper [16]. We consider the differential operator $\mathbf{Y} = \frac{1}{2t}\partial_t$ defined on $\mathbb{C}^*$. We have that

$$(\partial_t^l X)(t^2,z) = \mathbf{Y}^{(l)}(X(t^2,z)) \tag{54}$$

where $\mathbf{Y}^{(l)}$ denotes the operator $\mathbf{Y}$ composed l times, for all $l \geq 0$, all $(t,z) \in S_{d,\theta,R} \times \bar{D}_{\rho}$.

From the estimates (45) in [16], page 183, we know that

$$\mathbf{Y}^{(l)} = \sum_{k=1}^{l} Y_{l,k}(t)\partial_t^k \quad \text{with} \quad |Y_{l,k}(t)| \leq 2^{2l-k}(l-k)!|t|^{k-2l} \tag{55}$$

for all $l \geq 1$, $t \in \mathbb{C}^*$. We consider the function

$$P_l(t,z) = \sum_{p=0}^{2l-1} (\partial_t^p (X(t^2,z))_{|t=0} \frac{t^p}{p!}$$

which is a polynomial in t, with holomorphic coefficients on D_{ρ}, for all $l \geq 1$. From the Taylor formula with integral remainder,

$$X(t^2,z) = P_l(t,z) + \int_0^t \frac{(t-s)^{2l-1}}{(2l-1)!} \partial_s^{2l}(X(s^2,z))ds,$$

we get that

$$\sup_{z \in \bar{D}_{\rho}} |\partial_t^k (X(t^2,z) - P_l(t,z))| \leq \sup_{z \in \bar{D}_{\rho}, \xi \in (0,t)} |\partial_{\xi}^{2l}(X(\xi^2,z))| \frac{|t|^{2l-k}}{(2l-k)!}$$

for all $1 \leq k \leq l$, all $t \in S_{d,\theta,R}$. From the estimates (53) we get that there exist $C,K > 0$ such that

$$\sup_{z \in \bar{D}_{\rho}} |\partial_t^k (X(t^2,z) - P_l(t,z))| \leq CK^{2l} \frac{(2l)!}{(2l-k)!} \Gamma(1 + 2(1 + \frac{r_1}{2r_2})l)|t|^{2l-k}$$

for all $1 \leq k \leq l$, all $t \in S_{d,\theta,R}$. From the estimates $(2l)! \leq 2^{2l}(2l-k)!k!$, for $1 \leq k \leq l$, we deduce that

$$\sup_{z \in \bar{D}_{\rho}} |\partial_t^k (X(t^2,z) - P_l(t,z))| \leq C(2K)^{2l} k! \Gamma(1 + 2(1 + \frac{r_1}{2r_2})l)|t|^{2l-k} \tag{56}$$

for all $1 \leq k \leq l$, all $t \in S_{d,\theta,R}$. Using the estimates $(l-k)!k! \leq l!$, for $1 \leq k \leq l$, we get from the estimates (55) and (56) that there exist $C,K > 0$ such that

$$\sup_{z \in \bar{D}_{\rho}} |\mathbf{Y}^{(l)}(X(t^2,z) - P_l(t,z))| \leq C(4K)^{2l} \Gamma(1 + 2(1 + \frac{r_1}{2r_2})l)l! \tag{57}$$

for all $l \geq 1$, for all $t \in S_{d,\theta,R}$.

On the other hand, from the Faà di Bruno formula, we get that

$$\partial_t^p (X(t^2,z)) = \sum_{m_1 + 2m_2 = p} \frac{p!}{m_1!m_2!} (\partial_t^{m_1+m_2} X)(t^2,z)(2t)^{m_1}$$

for all $p \geq 0$, all $t \in S_{d,\theta,R}$, all $z \in \bar{D}_{\rho}$. From this formula, we deduce that $(\partial_t^p (X(t^2,z))_{|t=0} = 0$ for all $z \in \bar{D}_{\rho}$ if p is odd. So that the polynomial $P_l(t,z)$ in t can be written $P_l(t,z) = Q_l(t^2,z)$ where

$$Q_l(t,z) = \sum_{q=0}^{l-1} (\partial_t^{2q}(X(t^2,z))_{|t=0} \frac{t^q}{(2q)!}.$$

By construction of the operator $\mathbf{Y}$, we have the identity $\mathbf{Y}^{(l)}(P_l(t,z)) = (\partial_t^l Q_l)(t^2,z)$ for all $t \in \mathbb{C}^*$, all $z \in \bar{D}_\rho$. From the fact that the degree of $Q_l(t,z)$ in t is lower than $l-1$, we deduce that

$$\mathbf{Y}^{(l)}(P_l(t,z)) = 0 \tag{58}$$

for all $l \geq 1$, for all $t \in \mathbb{C}^*$, all $z \in \bar{D}_\rho$.

Finally, from the equalities (54), (58) and the inequalities (57), we deduce that there exists $0 < \rho < X$, there exist $C, K > 0$ such that

$$\sup_{(t,z) \in S_{2d,\zeta,R^2} \times \bar{D}_\rho} |(\partial_t^{n_0} X)(t,z)| \leq C K^{n_0} n_0! \Gamma(1 + 2(1 + \frac{r_1}{2r_2})n_0) \tag{59}$$

for all $n_0 \geq 1$.

Part 3. In this step we show that the constructed holomorphic solution $X(t,z)$ of (14) is Gevrey asymptotic of order $2(1+\frac{r_1}{2r_2})$ to the formal series solution $\hat{X}(t,z)$ of (14), with respect to $t \in S_{2d,\zeta,R^2}$, uniformly in z on a neighborhood of 0.

From the Cauchy formula applied to $(\partial_t^{n_0} X)(t,z)$ with respect to z on D_ρ, for all $n_0 \geq 0$, all $t \in S_{2d,\zeta,R^2}$, using the uniform estimates (59), we deduce that there exist $C, K_1, K_2 > 0$ such that

$$\sup_{t \in S_{2d,\zeta,R^2}} |\partial_t^{n_0} X_m(t)| \leq C K_1^{n_0} K_2^m m! n_0! \Gamma(1 + 2(1 + \frac{r_1}{2r_2})n_0) \tag{60}$$

for all $n_0, m \geq 0$.

From the construction of $X_m(t)$ as $1-$sum of $\hat{X}_m(t)$, $m \geq 0$, we know that $(\partial_t^{n_0} \hat{X}_m)(0) = (\partial_t^{n_0} X_m)(0)$, for all $m \geq 0$. So that the formal series $\hat{X}(t,z)$ can be rewritten in the form

$$\hat{X}(t,z) = \sum_{n_0,m \geq 0} \frac{\partial_t^{n_0} X_m(0)}{n_0! m!} t^{n_0} z^m.$$

For all $N \geq 1$, we define the polynomial

$$\hat{X}_m^{N-1}(t) = \sum_{n_0=0}^{N-1} \frac{\partial_t^{n_0} X_m(0)}{n_0!} t^{n_0}.$$

From the estimates (60), we deduce that the formal series

$$\hat{X}^{N-1}(t,z) = \sum_{m \geq 0} \frac{\hat{X}_m^{N-1}(t)}{m!} z^m$$

is absolutely convergent for all $z \in \mathbb{C}$ such that $|z| < 1/K_2$ and is polynomial in t.

From the Taylor formula with integral remainder, we have that

$$|X_m(t) - \hat{X}_m^{N-1}(t)| \leq \frac{|t|^N}{N!} \sup_{s \in S_{2d,\zeta,R^2}} |\partial_s^N X_m(s)|$$

for all $t \in S_{2d,\zeta,R^2}$, all $m \geq 0$, all $N \geq 1$. Due to the estimates (60), we deduce that there exist $C, K_1, K_2 > 0$ such that

$$|X_m(t) - \hat{X}_m^{N-1}(t)| \leq C K_1^N |t|^N \Gamma(1 + 2(1 + \frac{r_1}{2r_2})N) K_2^m m! \tag{61}$$

for all $t \in S_{2d,\zeta,R^2}$, all $m \geq 0$, all $N \geq 1$. From the estimates (61), we finally get that there exist $C, K_1, K_2 > 0$ such that

$$\sup_{|z| \leq \rho} |X(t,z) - \hat{X}^{N-1}(t,z)| \leq \frac{C}{1 - K_2\rho} K_1^N |t|^N \Gamma(1 + 2(1 + \frac{r_1}{2r_2})N)$$

for all $0 < \rho < K_2$, for all $t \in S_{2d,\zeta,R^2}$, all $N \geq 1$. This means that $X(t,z)$ is Gevrey asymptotic of order $2(1 + \frac{r_1}{2r_2})$ to the formal series $\hat{X}(t,z)$ with respect to t on S_{2d,ζ,R^2}, uniformly in z on $\bar{D}_\rho$. □

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