QUANTUM DYNAMICS VIA COMPLEX ANALYSIS METHODS: GENERAL UPPER BOUNDS WITHOUT TIME-AVERAGING AND TIGHT LOWER BOUNDS FOR THE STRONGLY COUPLED FIBONACCI HAMILTONIAN

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Abstract. We develop further the approach to upper and lower bounds in quantum dynamics via complex analysis methods which was introduced by us in a sequence of earlier papers. Here we derive upper bounds for non-time averaged outside probabilities and moments of the position operator from lower bounds for transfer matrices at complex energies. Moreover, for the time-averaged transport exponents, we present improved lower bounds in the special case of the Fibonacci Hamiltonian. These bounds lead to an optimal description of the time-averaged spreading rate of the fast part of the wavepacket in the large coupling limit. This provides the first example which demonstrates that the time-averaged spreading rates may exceed the upper box-counting dimension of the spectrum.

1. Introduction

This paper studies the long-time behavior of the solution to the time-dependent Schrödinger equation, $i\partial_t \psi = H\psi$, in the Hilbert space $\ell^2(\mathbb{Z})$ with an initially localized state, say $\psi(0) = \delta_0$. More precisely, the Schrödinger operator is of the form

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n), \tag{1}$$

where $V : \mathbb{Z} \to \mathbb{R}$ is the potential, and the solution is given by

$$\psi(t) = e^{-itH}\delta_0. \tag{2}$$

The probability of finding the state in $\{n \in \mathbb{Z} : n \geq N\}$ at time t is given by

$$P_r(N,t) = \sum_{n \geq N} |\langle e^{-itH}\delta_0, \delta_n \rangle|^2.$$

Similarly,

$$P_l(N,t) = \sum_{n \leq -N} |\langle e^{-itH}\delta_0, \delta_n \rangle|^2$$

is equal to the probability of finding the state in $\{n \in \mathbb{Z} : n \leq -N\}$ at time t.

It is in general a hard problem to bound these so-called outside probabilities from above. Our recent paper [13] introduced a way of estimating their time-averages from above in terms of the norms of transfer matrices at complex energies. In this paper we show how similar estimates can be obtained without the need to take time-averages.

Date: January 22, 2008.

D. D. was supported in part by NSF grant DMS–0653720.

To state these results, let us recall the definition of the transfer matrices. For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$, define the transfer matrix $\Phi(n,z)$ by

$$\Phi(n,z) = \begin{cases} T(n,z)\cdots T(1,z) & n \ge 1, \\ \mathrm{Id} & n = 0, \\ [T(n+1,z)]^{-1}\cdots[T(0,z)]^{-1} & n \le -1, \end{cases} \tag{3}$$

where

$$T(m,z) = \begin{pmatrix} z - V(m) & -1 \\ 1 & 0 \end{pmatrix}.$$

The definition is such that $u : \mathbb{Z} \to \mathbb{C}$ solves

$$u(n+1) + u(n-1) + V(n)u(n) = zu(n) \tag{4}$$

if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \Phi(n,z) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

for every $n \in \mathbb{Z}$.

Theorem 1. *Suppose H is given by* (1)*, where V is a bounded real-valued function, and $K \ge 4$ is such that $\sigma(H) \subseteq [-K+1, K-1]$. Then, the outside probabilities can be bounded from above in terms of transfer matrix norms as follows:*

$$P_r(N,t) \lesssim \exp(-cN) + t^4 \int_{-K}^{K} \left(\max_{0 \le n \le N-1} \left\| \Phi\left(n, E + it^{-1}\right) \right\|^2 \right)^{-1} dE,$$

$$P_l(N,t) \lesssim \exp(-cN) + t^4 \int_{-K}^{K} \left(\max_{-N+1 \le n \le 0} \left\| \Phi\left(n, E + it^{-1}\right) \right\|^2 \right)^{-1} dE.$$

The implicit constants depend only on K.

This result, which is the non-time averaged analogue of [13, Theorem 7], has a number of consequences akin to those discussed in [13]. For $p > 0$, consider the p-th moment of the position operator,

$$\langle |X|^p_{\delta_0} \rangle(t) = \sum_{n \in \mathbb{Z}} |n|^p |\langle e^{-itH}\delta_0, \delta_n \rangle|^2$$

and the upper and lower transport exponents $\beta^+_{\delta_0}(p)$ and $\beta^-_{\delta_0}(p)$, given, respectively, by

$$\beta^+_{\delta_0}(p) = \limsup_{t \to \infty} \frac{\log \langle |X|^p_{\delta_0} \rangle(t)}{p \log t}.$$

and

$$\beta^-_{\delta_0}(p) = \liminf_{t \to \infty} \frac{\log \langle |X|^p_{\delta_0} \rangle(t)}{p \log t}.$$

Let us briefly discuss a connection with the outside probabilities. Set

$$P(N,t) = P_l(N,t) + P_r(N,t).$$

Following [16], for $0 \le \alpha \le \infty$, define

$$S^-(\alpha) = -\liminf_{t \to \infty} \frac{\log P(t^\alpha - 1, t)}{\log t} \tag{5}$$

and

$$S^+(\alpha) = -\limsup_{t \to \infty} \frac{\log P(t^\alpha - 1, t)}{\log t}. \tag{6}$$

For every α, $0 \le S^+(\alpha) \le S^-(\alpha) \le \infty$.

These numbers control the power decaying tails of the wavepacket. In particular, the following critical exponents are of interest:

$$\alpha_l^\pm = \sup\{\alpha \ge 0 : S^\pm(\alpha) = 0\}, \tag{7}$$

$$\alpha_u^\pm = \sup\{\alpha \ge 0 : S^\pm(\alpha) < \infty\}. \tag{8}$$

We have that $0 \le \alpha_l^- \le \alpha_u^- \le 1$, $0 \le \alpha_l^+ \le \alpha_u^+ \le 1$, and also that $\alpha_l^- \le \alpha_l^+$, $\alpha_u^- \le \alpha_u^+$. One can interpret $\alpha_l^\pm$ as the (lower and upper) rates of propagation of the essential part of the wavepacket, and $\alpha_u^\pm$ as the rates of propagation of the fastest (polynomially small) part of the wavepacket; compare [16]. In particular, if $\alpha > \alpha_u^+$, then $P(t^\alpha, t)$ goes to 0 faster than any inverse power of t. Since a ballistic upper bound holds in our case (for any potential V), a slight modification of [16, Theorem 4.1] yields

$$\lim_{p \to 0} \beta_{\delta_0}^\pm(p) = \alpha_l^\pm$$

and

$$\lim_{p \to \infty} \beta_{\delta_0}^\pm(p) = \alpha_u^\pm.$$

In particular, since $\beta_{\delta_0}^\pm(p)$ are nondecreasing, we have that

$$\beta_{\delta_0}^\pm(p) \le \alpha_u^\pm \quad \text{for every } p > 0. \tag{9}$$

Corollary 1. *Suppose H is given by* (1)*, where V is a bounded real-valued function, and $K \ge 4$ is such that $\sigma(H) \subseteq [-K+1, K-1]$. Suppose that, for some $C \in (0,\infty)$ and $\alpha \in (0,1)$, we have*

$$\int_{-K}^{K} \left(\max_{1 \le n \le Ct^\alpha} \left\| \Phi\left(n, E + it^{-1}\right) \right\|^2 \right)^{-1} dE = O(t^{-m}) \tag{10}$$

and

$$\int_{-K}^{K} \left(\max_{1 \le -n \le Ct^\alpha} \left\| \Phi\left(n, E + it^{-1}\right) \right\|^2 \right)^{-1} dE = O(t^{-m}) \tag{11}$$

for every $m \ge 1$. Then

$$\alpha_u^+ \le \alpha. \tag{12}$$

In particular,

$$\beta_{\delta_0}^+(p) \le \alpha \quad \textit{for every } p > 0. \tag{13}$$

Theorem 1 and Corollary 1 provide the analogues of the central general results from [13] for quantities that are not time-averaged. For comparison purposes, let us introduce the following notation. If $f(t)$ is a function of $t > 0$ and $T > 0$ is given, we denote the time-averaged function at T by $\langle f \rangle(T)$:

$$\langle f \rangle(T) = \frac{2}{T} \int_0^\infty e^{-2t/T} f(t)\, dt.$$

Thus, we consider, for example, $\langle P_l(N,\cdot)\rangle(T)$, $\langle P_r(N,\cdot)\rangle(T)$, and $\langle\langle |X|_{\delta_0}^p\rangle\rangle(T)$. For these time-averaged quantities, we can then define the transport exponents $\langle \beta_{\delta_0}^\pm(p)\rangle$ and their limiting values $\langle \alpha_l^\pm \rangle$ and $\langle \alpha_l^\pm \rangle$ in the same way as above. For example, if in the formulation of Corollary 1, we replace α_u^+ by $\langle \alpha_u^+ \rangle$ and $\beta_{\delta_0}^+(p)$ by $\langle \beta_{\delta_0}^+(p)\rangle$, we obtain the assertion of [13, Theorem 1].

Let us now turn to a discussion of a special case. The Fibonacci Hamiltonian is the discrete one-dimensional Schrödinger operator in $\ell^2(\mathbb{Z})$ as in (1) with potential $V : \mathbb{Z} \to \mathbb{R}$ is given by

$$V(n) = \lambda \chi_{[1-\phi^{-1},1)}(n\phi^{-1} + \theta \mod 1). \tag{14}$$

Here, $\lambda > 0$ is the coupling constant, ϕ is the golden mean,

$$\phi = \frac{\sqrt{5}+1}{2}$$

and $\theta \in [0, 1)$ is the phase. This is the most prominent model of a one-dimensional quasicrystal; compare the survey articles [4, 24]. It is known that the spectrum of H is independent of θ [2]; let us denote it by Σ_λ. Moreover, the Lebesgue measure of Σ_λ is zero [23] and all spectral measures are purely singular continuous [8].

The quantum evolution with H given by the Fibonacci Hamiltonian has been studied in many papers. It had long been expected to be anomalous in the sense that it is markedly different from the behavior in the periodic case (leading to ballistic transport) and the random case (leading to dynamical localization); see, for example, papers in the physics literature by Abe and Hiramoto [1, 18]. Lower bounds, showing in particular the absence of dynamical localization, were shown in [3, 7, 10, 11, 12, 19, 20]. There are far fewer paper establishing upper bounds for this model, especially for quantities like the moments of the position operator. Killip et al. showed for $\theta = 0$ and $\lambda \geq 8$ that the slow part of the wavepacket does not move ballistically [20]. Their result was extended to general θ in [5]. The first result establishing for $\lambda \geq 8$ bounds from above for the whole wavepacket, and hence quantities like the moments of the position operator, is contained in [13]. In that paper only the case $\theta = 0$ is studied but it is remarked that the ideas from [5] will allow one to treat general θ's.

The paper [13] introduced a new tool, the complex trace map, that allows one to use complex analysis methods in a context where real analysis methods were used earlier. This was important in our proof of non-trivial upper bounds for the (time-averaged) transport exponents. Given the analysis of [13] and Corollary 1 above, which strengthens [13, Theorem 1], we obtain the following strengthening of [13, Theorem 3].

Theorem 2. *Consider the Fibonacci Hamiltonian, that is, the operator* (1) *with potential* (14)*. Assume that* $\lambda \geq 8$ *and let*

$$\alpha(\lambda) = \frac{2\log\phi}{\log S_l(\lambda)}.$$

with

$$S_l(\lambda) = \frac{1}{2}\left((\lambda-4) + \sqrt{(\lambda-4)^2 - 12}\right). \tag{15}$$

Then, $\alpha_u^+ \leq \alpha(\lambda)$*, and hence* $\beta^+(p) \leq \alpha(\lambda)$ *for every* $p > 0$.

As a byproduct of our study of the complex trace map in [13], we established a distortion result that is useful to bound the transport exponents from either side. Since [13] focused on quantum dynamical upper bounds, we present the application of the distortion result to quantum dynamical lower bounds here. For this result, we still need to consider time-averaged quantities!

Theorem 3. *Consider the Fibonacci Hamiltonian, that is, the operator* (1) *with potential* (14)*. Suppose* $\lambda > \sqrt{24}$ *and let*

$$S_u(\lambda) = 2\lambda + 22.$$

We have

$$\langle \beta^-_{\delta_0}(p) \rangle \geq \frac{2\log\phi}{\log S_u(\lambda)} - \frac{2}{p}\left(1 + \frac{C\log\lambda}{\log S_u(\lambda)}\right) \tag{16}$$

for a suitable constant C*, and therefore*

$$\langle \alpha^-_u \rangle \geq \frac{2\log\phi}{\log S_u(\lambda)}. \tag{17}$$

Remark. In the special case $\theta = 0$, the lower bound for $\langle \beta^\pm_{\delta_0}(p) \rangle$ can be improved; see Theorem 4 at the end of this paper.

The best previously known lower bound for $\langle \alpha^-_u \rangle$ in the large coupling regime was obtained in [6]. It reads

$$\langle \alpha^-_u \rangle \geq \dim^\pm_B(\Sigma_\lambda). \tag{18}$$

Here, $\dim^\pm_B(\Sigma_\lambda)$ denotes the upper/lower box counting dimension of Σ_λ. Such a bound holds whenever the transfer matrices are polynomially bounded on the spectrum, which in particular holds in the Fibonacci case.

The authors of [6] performed a detailed study of these dimensions. A particular consequence of their study is the following asymptotic statement,

$$\lim_{\lambda\to\infty} \dim^\pm_B(\Sigma_\lambda) \cdot \log\lambda = f^\# \log\phi, \tag{19}$$

where $f^\#$ is an explicit constant (the unique maximum of an explicit function) that is roughly given by

$$f^\# \approx 1.83156.$$

On the other hand, for $\lambda \geq 8$, [13, Theorem 3] (see also Theorem 2 above) gives the following upper bound for $\langle \alpha^+_u \rangle$,

$$\langle \alpha^+_u \rangle \leq \frac{2\log\phi}{\log S_l(\lambda)}. \tag{20}$$

Thus, combining (17) and (20), we find the following exact asymptotic result.

Corollary 2. *For the Fibonacci Hamiltonian, we have*

$$\lim_{\lambda\to\infty} \langle \alpha^\pm_u \rangle \cdot \log\lambda = 2\log\phi. \tag{21}$$

A particular consequence is that, as $\lambda \to \infty$, the limit behavior is the same for both $\langle \alpha^+_u \rangle$ and $\langle \alpha^-_u \rangle$. It would be of interest to show that these quantities are in fact equal for finite λ.

Comparing the asymptotic results (19) and (21), we see that the Fibonacci Hamiltonian at large coupling may serve as an example where the inequality in (18) is strict. This result is also relevant to a question raised by Last in [21, Section 9] who asked whether the upper box counting dimension of the spectrum could serve as an upper bound for dynamics and suggested that the expected answer is negative in general. See [17, 25] for numerical results providing evidence supporting this expectation.

Let us put the recent results for the Fibonacci Hamiltonian in perspective. The present paper and [6] result from an attempt to describe dimensions and transport exponents *exactly*. This is certainly a challenging problem and we have precise results only in an asymptotic regime; compare (19) and (21). It would certainly be of interest to prove exact results also for fixed finite λ. Moreover, the behavior in the small coupling regime is poorly understood. Many results that hold for λ above some critical coupling do not obviously extend to smaller values since parts of the proofs break down. There is a definite need for new insights in order to prove results at small coupling.

2. Upper Bounds for Outside Probabilities Without Time-Averaging

Our goal in this section is to prove Theorem 1 and Corollary 1. In particular, we will see how a specific consequence of the Dunford functional calculus allows us to replace the Parseval formula, which was a crucial ingredient in several earlier papers on quantum dynamical bounds [10, 11, 12, 13, 16, 20]. This observation is the key to obtaining upper bounds for non-averaged quantities.

Lemma 1. *We have*

$$\langle e^{-itH}\delta_0, \delta_n\rangle = -\frac{1}{2\pi i}\int_\Gamma e^{-itz}\langle (H-z)^{-1}\delta_0, \delta_n\rangle\, dz$$

for every $n \in \mathbb{Z}$, $t \in \mathbb{R}$, and positively oriented simple closed contour Γ in $\mathbb{C}$ that is such that the spectrum of H lies inside Γ.

Proof. This is a consequence of the so-called Dunford functional calculus; see [14] and also [15, 22]. □

Lemma 2. *Suppose H is given by* (1)*, where V is a bounded real-valued function, and $K \geq 4$ is such that $\sigma(H) \subseteq [-K+1, K-1]$. Then,*

$$P_r(N,t) \lesssim \exp(-cN) + \int_{-K}^{K} \sum_{n \geq N} |\langle (H - E - it^{-1})^{-1}\delta_0, \delta_n\rangle|^2 dE,$$

$$P_l(N,t) \lesssim \exp(-cN) + \int_{-K}^{K} \sum_{n \leq -N} |\langle (H - E - it^{-1})^{-1}\delta_0, \delta_n\rangle|^2 dE.$$

Proof. Given $t > 0$, we will consider the following contour Γ: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\begin{aligned}
\Gamma_1 &= \left\{ z = E + iy : E \in [-K, K],\ y = t^{-1} \right\} \\
\Gamma_2 &= \left\{ z = E + iy : E = -K,\ y \in [-1, t^{-1}] \right\} \\
\Gamma_3 &= \{ z = E + iy : E \in [-K, K],\ y = -1 \} \\
\Gamma_4 &= \left\{ z = E + iy : E = K,\ y \in [-1, t^{-1}] \right\}.
\end{aligned}$$

Notice that for $z \in \Gamma$, we have $\Im z \leq t^{-1}$ and hence $|e^{-itz}| = e^{t\Im z} \leq e$. Thus, by Lemma 1,

$$|\langle e^{-itH}\delta_0, \delta_n\rangle| \lesssim \sum_{j=1}^{4} \int_{\Gamma_j} \left|\langle (H-z)^{-1}\delta_0, \delta_n\rangle\right| |dz|.$$

If $z \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, then clearly $\operatorname{dist}(z, \sigma(H)) \geq 1$ and hence the well-known Combes-Thomas estimate allows us to bound the contributions from $\Gamma_2, \Gamma_3, \Gamma_4$ to $P_r(N,t)$ and $P_l(N,t)$ by $C\exp(-cN)$.

The integral over Γ_1 can be estimated using the Cauchy-Schwarz inequality:

$$\left(\int_{\Gamma_1} \left|\langle (H-z)^{-1}\delta_0, \delta_n\rangle\right| \, |dz|\right)^2 \le C(K) \int_{-K}^{K} \left|\langle (H-E-it^{-1})^{-1}\delta_0, \delta_n\rangle\right|^2 dE.$$

Combining these estimates, we obtain the assertion of the lemma. □

Proof of Theorem 1. We have the following estimates [13, p. 811]:

$$\langle P_r(N,\cdot)\rangle(T) \lesssim \exp(-cN) + T^{-1}\int_{-K}^{K} \sum_{n\ge N} |\langle (H-E-iT^{-1})^{-1}\delta_0, \delta_n\rangle|^2 dE, \tag{22}$$

$$\langle P_l(N,\cdot)\rangle(T) \lesssim \exp(-cN) + T^{-1}\int_{-K}^{K} \sum_{n\le -N} |\langle (H-E-iT^{-1})^{-1}\delta_0, \delta_n\rangle|^2 dE. \tag{23}$$

It was then shown in [13] how to derive

$$\langle P_r(N,\cdot)\rangle(T) \lesssim \exp(-cN) + T^3\int_{-K}^{K} \left(\max_{0\le n\le N-1} \left\|\Phi\left(n, E+iT^{-1}\right)\right\|^2\right)^{-1} dE,$$

$$\langle P_l(N,\cdot)\rangle(T) \lesssim \exp(-cN) + T^3\int_{-K}^{K} \left(\max_{-N+1\le n\le 0} \left\|\Phi\left(n, E+iT^{-1}\right)\right\|^2\right)^{-1} dE.$$

from (22) and (23). If we use Lemma 2 instead of (22) and (23) and then follow the very same steps, we obtain the desired bounds

$$P_r(N,t) \lesssim \exp(-cN) + t^4\int_{-K}^{K} \left(\max_{0\le n\le N-1} \left\|\Phi\left(n, E+it^{-1}\right)\right\|^2\right)^{-1} dE,$$

$$P_l(N,t) \lesssim \exp(-cN) + t^4\int_{-K}^{K} \left(\max_{-N+1\le n\le 0} \left\|\Phi\left(n, E+it^{-1}\right)\right\|^2\right)^{-1} dE.$$

This concludes the proof. □

Proof of Corollary 1. Let us choose $N(t) = \lfloor Ct^\alpha \rfloor$, where $C \in (0,\infty)$ and $\alpha \in (0,1)$ are chosen such that (10) and (11) hold. Observe that $P(N(t),t) = P(\lceil N(t)\rceil, t)$. Then Theorem 1 shows that $P_r(N(t),t)$ and $P_l(N(t),t)$ go to 0 faster than any inverse power of t. By definition of $S^+(\alpha)$ and α_u^+ (cf. (6) and (8)), it follows that $\alpha_u^+ \le \alpha$, which is (12). Finally, (13) follows from (9). □

3. Tight Lower Bounds for Strongly Coupled Fibonacci

3.1. **The Complex Trace Map and the Distortion of Balls.** In this subsection we recall some ideas from [13] and present an improvement of the key distortion result contained in that paper.

For $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, consider the transfer matrices $\Phi(n,z)$ associated with the difference equation (4) where V is the Fibonacci potential given by (14). Notice that these matrices depend on both λ and θ.

Define the matrices $M_k(z)$ by

$$\Phi_{\theta=0}(F_k, z) = M_k(z), \quad k \ge 1, \tag{24}$$

where F_k is the k-th Fibonacci number, that is, $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for $k \ge 1$.

It is well-known that

$$M_{k+1}(z) = M_{k-1}(z)M_k(z), \quad k \ge 2.$$

For the variables $x_k(z) = \frac{1}{2}\mathrm{Tr}\, M_k(z))$, $k \ge 1$, we have the recursion

$$x_{k+1}(z) = 2x_k(z)x_{k-1}(z) - x_{k-2}(z) \tag{25}$$

and the invariant

$$x_{k+1}(z)^2 + x_k(z)^2 + x_{k-1}(z)^2 - 2x_{k+1}(z)x_k(z)x_{k-1}(z) - 1 \equiv \frac{\lambda^2}{4}. \tag{26}$$

Letting $x_{-1}(z) = 1$ and $x_0(z) = z/2$, the recursion (25) holds for all $k \ge 0$.

For $\delta \ge 0$, consider the sets

$$\sigma_k^\delta = \{z \in \mathbb{C} : |x_k(z)| \le 1 + \delta\}.$$

We have

$$\sigma_k^\delta \cup \sigma_{k-1}^\delta \supseteq \sigma_{k+1}^\delta \cup \sigma_k^\delta \to \Sigma_\lambda.$$

Assume $\lambda > \lambda_0(\delta)$, where

$$\lambda_0(\delta) = [12(1+\delta)^2 + 8(1+\delta)^3 + 4]^{1/2}. \tag{27}$$

The invariant (26) implies

$$\sigma_k^\delta \cap \sigma_{k+1}^\delta \cap \sigma_{k+2}^\delta = \emptyset. \tag{28}$$

Moreover, the set σ_k^δ has exactly F_k connected components. Each of them is a topological disk that is symmetric about the real axis.

It is known that all roots of x_k are real. Consider such a root z, that is, $x_k(z) = 0$ and define

$$m(z) = \#\{0 \le l \le k-1 : |x_l(z)| \le 1\}.$$

Let

$$c_{k,m} = \#\{\text{roots of } x_k \text{ with } m(z) = m\}.$$

An explicit formula for $c_{k,m}$ was found in [6, Lemma 5] (noting that our $c_{k,m}$ equals $a_{k,m} + b_{k,m}$ in the notation of that paper). All we need here is the following consequence of this result.

Lemma 3. *For every $k \ge 2$, $c_{k,m}$ is non-zero if and only if $\frac{k}{2} \le m \le \frac{2k}{3}$.*

Let $\{z_k^{(j)}\}_{1 \le j \le F_k}$ be the zeros of x_k and write $m_k^{(j)} = m(z_k^{(j)})$ for $1 \le j \le F_k$. Denote $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$. Then, we have the following distortion result.

Proposition 1. *Fix $k \ge 3$, $\delta > 0$, and $\lambda > \max\{\lambda_0(2\delta), 8\}$. Then, there are constants $c_\delta, d_\delta > 0$ such that*

$$\bigcup_{j=1}^{F_k} B\left(z_k^{(j)}, r_k^{(j)}\right) \subseteq \sigma_k^\delta \subseteq \bigcup_{j=1}^{F_k} B\left(z_k^{(j)}, R_k^{(j)}\right), \tag{29}$$

where $r_k^{(j)} = c_\delta S_u(\lambda)^{-m_k^{(j)}}$, and $R_k^{(j)} = d_\delta S_l(\lambda)^{-m_k^{(j)}}$. The first inclusion in (29) only needs the assumption $\lambda > \lambda_0(2\delta)$.

Proof. The proof of this result is analogous to the proof of [13, Proposition 3]. As explained there, $m_k^{(j)}$ governs the size of $|x_k'(z_k^{(j)})|$, which in turn is closely related to the size and shape of the connected component of σ_k^δ that contains $z_k^{(j)}$ by Koebe's Distortion Theorem. $\square$

3.2. **Power-Law Upper Bounds for Transfer Matrices.** In this subsection we prove power-law upper bounds for the norm of the transfer matrices for suitable complex energies and suitable maximal scales.

Proposition 2. *For every $\lambda, \delta > 0$, there are constant C, γ such that for every k, every $z \in \sigma_k^\delta$, and every N with $1 \le N \le F_k$, we have*

$$\|\Phi(N, z)\| \le CN^\gamma. \tag{30}$$

The constant γ behaves like $O(\log \lambda)$ as $\lambda \to \infty$ for each fixed δ.

Proof. For $\theta = 0$, a modification of the proof of [11, Proposition 3.2] gives the result. The extension to arbitrary θ then follows using ideas from [9]. For the convenience of the reader, we sketch the main parts of the argument.

Using the invariant (26) and [13, Lemma 4], it is readily seen that $z \in \sigma_k^\delta$ implies the uniform bound

$$|x_j(z)| \le C_{\lambda,\delta} \qquad \text{for } 0 \le j \le k-1, \tag{31}$$

where $C_{\lambda,\delta}$ is an explicit constant that behaves like $O(\lambda)$ as $\lambda \to \infty$; compare [11, Lemma 3.1].

Next, the uniform trace bound (31) yields an upper bound for the matrix norm,

$$\|M_j(z)\| \le (C_{\lambda,\delta})^j \qquad \text{for } 0 \le j \le k-1, \tag{32}$$

compare [11, Eq. (48)].

The final step is an interpolation of (32). Given any transfer matrix, $\Phi(N, z)$, it is possible to write it as a product of matrices of type $M_j(z)$; compare [8, Section 3]. A careful analysis of the factors that occur, together with the estimate (32), then gives (30). The bound on γ follows as in [11, Proposition 3.2]. □

3.3. **Lower Bounds for the Transport Exponents.** In this subsection we prove Theorem 3 using the results from the previous two sections.

Proof of Theorem 3. By assumption, $\lambda > \sqrt{24}$. Thus, it is possible to choose $\delta > 0$ so that $\lambda > \lambda_0(2\delta)$. Recall that $\sigma_k^\delta = \{z \in \mathbb{C} \ : \ |x_k(z)| \le 1 + \delta\}$. It follows from Lemma 3 and Proposition 1 that σ_k^δ has a connected component D_k such that

$$B(z_k, r_k) \subset D_k,$$

where $z_k \in \mathbb{R}$ and

$$r_k = c_\delta S_u(\lambda)^{-\lceil \frac{k}{2} \rceil}.$$

It follows from Proposition 2 that for $z \in D_k$ and $1 \le N \le F_k$, we have

$$\|\Phi(N, z)\| \le CN^\gamma$$

with suitable constants C and γ.

Let us fix k and take $N = F_k \sim \phi^k$ and $\varepsilon = \frac{1}{T} \le \frac{r_k}{2}$. Thus,

$$T \ge \frac{2}{r_k} = \frac{2}{c_\delta} S_u(\lambda)^{\lceil \frac{k}{2} \rceil} \gtrsim N^s,$$

where

$$s = \frac{\log S_u(\lambda)}{2 \log \phi}.$$

Due to the Parseval formula (see, e.g., [20, Lemma 3.2]), we can bound the time-averaged outside probabilities from below as follows,

$$\langle P(N,\cdot)\rangle(T) \gtrsim \varepsilon \int_{\mathbb{R}} \|\Phi(N,E+i\varepsilon)\|^{-2}\, dE. \tag{33}$$

To bound the integral from below, we integrate only over those $E \in (z_k - r_k, z_k + r_k)$ for which $E + i\varepsilon \in B(z_k, r_k) \subset D_k$. The length of such an interval I_k is larger than cr_k for some suitable $c > 0$. For $E \in I_k$, we have

$$\|\Phi(N, E + i\varepsilon)\| \lesssim N^\gamma \lesssim T^{\frac{\gamma}{s}}.$$

Therefore, (33) gives

$$\langle P(N,\cdot)\rangle(T) \gtrsim \frac{r_k}{T} T^{-\frac{2\gamma}{s}} \gtrsim T^{-2-\frac{2\gamma}{s}}, \tag{34}$$

where $N = F_k$, $T \geq N^s$, for any k.

Now let us take any $T \geq 1$ and choose k maximal with $F_k^s \leq T$. Then,

$$F_k^s \leq T < F_{k+1}^s \leq 2^s F_k^s.$$

It follows from (34) that

$$\langle P(\tfrac{1}{2}T^{\frac{1}{s}},\cdot)\rangle(T) \geq \langle P(N,\cdot)\rangle(T) \gtrsim T^{-2-\frac{2\gamma}{s}}$$

for all $T \geq 1$. It follows from the definition of $\langle \beta^-(p)\rangle$ and $\langle \alpha_u^-\rangle$ that

$$\langle \beta_{\delta_0}^-(p)\rangle \geq \frac{1}{s} - \frac{2}{p}\left(1 + \frac{\gamma}{s}\right)$$

and

$$\langle \alpha_u^-\rangle \geq \frac{1}{s}.$$

Since

$$\frac{1}{s} = \frac{2\log\phi}{\log S_u(\lambda)},$$

and $\gamma = O(\log\lambda)$, this completes the proof of Theorem 3. □

3.4. **Improved Lower Bounds for Zero Phase.** We conclude with a discussion of the special case $\theta = 0$. For this particular value of the phase, the result above can be improved. The key change is that in this case, one has "all the possible squares adjacent to the origin." This then enables one to directly estimate $\langle P(N,\cdot)\rangle(T)$ from below without employing the general upper bounds for the transfer matrices.

Theorem 4. *Consider the operator* (1) *with potential* (14) *and phase* $\theta = 0$. *Suppose* $\lambda > \sqrt{24}$ *and let*

$$S_u(\lambda) = 2\lambda + 22.$$

We have

$$\langle \beta_{\delta_0}^-(p)\rangle \geq \frac{2\log\phi}{\log S_u(\lambda)} - \frac{2}{p}\left(1 - \frac{2\log\frac{\sqrt{17}}{4}}{5\log S_u(\lambda)}\right) \tag{35}$$

for every $p > 0$.

Proof. Starting again with the Parseval formula [20, Lemma 3.2], we have

$$\langle P_r(N,\cdot)\rangle(T) = \frac{1}{\pi T}\sum_{n\geq N}\int_{\mathbb{R}} |\langle (H-E-iT^{-1})^{-1}\delta_0,\delta_n\rangle|^2\,dE. \tag{36}$$

Using the fact that

$$u(n) = \langle (H-E-iT^{-1}\delta_0,\delta_n\rangle$$

solves the difference equation $Hu = (E+iT^{-1})u$ away from zero, one can bound the right-hand side of (36) from below by the Gordon-type mass reproduction technique used, for example, in [3, 7, 12, 19]. We refer the reader to [12, Lemmas 2 and 3] for a formulation of this technique that is particularly well suited to our situation here. It is important to point out that while in those papers, only real energies are considered, the statements extend to complex energies as soon as one can arrange for the same transfer matrix trace estimates to hold.

As in the proof of Theorem 3, since $\lambda > \sqrt{24}$, it is possible to choose $\delta > 0$ so that $\lambda > \lambda_0(2\delta)$. Fixing this value of δ, we consider the set $\sigma_k^\delta = \{z \in \mathbb{C} : |x_k(z)| \leq 1+\delta\}$. Notice that $x_k(z)$ is one-half the trace of the transfer matrix of the potential under consideration from the origin to F_k. This is why the case $\theta = 0$ is special! It follows from [13, Lemma 4] that for $0 \leq j \leq k-1$, $\min\{|x_j(z)|, |x_{j+1}(z)|\} \leq 1+\delta$ for every $z \in \sigma_k^\delta$. Since, for every j, the potential repeats its values from 1 to F_j once, we can apply [12, Lemma 2] to bound the right-hand side of (36) from below. Of course we use either j or $j+1$, depending on which of $|x_j(z)$ and $|x_{j+1}(z)|$ is bounded by $1+\delta$. Carrying this out inductively, we find that

$$\sum_{n\geq N} |u(n)|^2 \geq CN^{2\kappa}(|u(1)|^2 + |u(0)|^2)$$

for a δ-dependent constant

$$\kappa < \frac{\log\frac{\sqrt{17}}{4}}{5\log\phi}$$

and a corresponding suitable constant $C > 0$. One checks that κ can be made arbitrarily close to $\frac{\log\frac{\sqrt{17}}{4}}{5\log\phi}$ by making δ sufficiently small. The same reasoning can be applied to $\langle P_l(N,\cdot)\rangle(T)$. In the end, one only needs to observe that

$$|u(1)|^2 + |u(0)|^2 + |u(-1)|^2 \geq c(1 + (\Im F(E+\tfrac{i}{T}))^2),$$

uniformly in the energy, where F is the Borel transform of the spectral measure; compare [10, 11].

From this point on, one can proceed as in the proof of Theorem 3. It then follows, with the same notation as above, that

$$\begin{aligned} \langle P(cT^{\frac{1}{s}},\cdot)\rangle(T) &\geq cT^{-1+2\kappa/s}\int_{I_k}(1+(\Im F(E+\tfrac{i}{T}))^2)\,dE \\ &\geq cT^{-1+2\kappa/s}|I_k| \\ &\geq cT^{-2+2\kappa/s} \end{aligned} \tag{37}$$

and hence

$$\langle \beta_{\delta_0}^-(p)\rangle \geq \frac{1}{s} - \frac{2}{p}\left(1 - \frac{\kappa}{s}\right)$$

from which the result follows. □

Remark. It is possible to improve the estimate (35) if one revisits the inequality (37) and observes that the spectral measure is singular and the integral over I_k in question is larger than just $|I_k|$. This can be made quantitative using ideas from [12].

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