The resonance amplitude associated with the Gamow states

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Abstract

The Gamow states describe the quasinormal modes of quantum systems. It is shown that the resonance amplitude associated with the Gamow states is given by the complex delta function. It is also shown that under the near-resonance approximation of neglecting the lower bound of the energy, such resonance amplitude becomes the Breit-Wigner amplitude. This result establishes the precise connection between the Gamow states, Nakanishi's complex delta function and the Breit-Wigner amplitude. In addition, this result provides another theoretical basis for the phenomenological fact that the almost-Lorentzian peaks in cross sections are produced by intermediate, unstable particles.

Keywords: Gamow states; resonances; rigged Hilbert space; complex delta function; Breit-Wigner lineshape

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1 Introduction

Resonances appear in all areas of quantum physics, in both the relativistic and non-relativistic regimes. Resonances are intrinsic properties of a quantum system, and they describe the system's preferred ways of decaying. Experimentally, resonances appear as sharp peaks in the cross section that resemble the Breit-Wigner (Lorentzian) lineshape.

The Gamow states are the natural wave functions of resonances, and they were introduced by Gamow in his paper on α-decay of radioactive nuclei [1]. Since then, they have been used by a number of authors, see e.g. [2–29]. Likewise the bound states, the Gamow states are properties of the Hamiltonian, and they are associated with the natural frequencies of the system. The usefulness of the Gamow states is attested by the remarkable success of the Gamow Shell Model [18, 22, 24–26, 28] and similar nuclear-structure formalisms [16, 17].

Because resonances leave a quasi-Lorentzian fingerprint in the cross section, and because the Gamow states are the natural wave functions of resonances, the resonance amplitude associated with the Gamow states must be related to the Breit-Wigner amplitude. The purpose of this paper is to show that the resonance amplitude associated with the Gamow states is proportional to the complex delta function, $\delta(E - z_{\rm R})$, and that such amplitude can be approximated in the near-resonance region by the Breit-Wigner amplitude. More precisely, we will show that the transition amplitude from a resonance state of energy $z_{\rm R}$ to a scattering state of energy $E \geq 0$, $\mathcal{A}(z_{\rm R} \to E)$, is given by

$$\mathcal{A}(z_{\rm R} \to E) = {\rm i}\sqrt{2\pi}\mathcal{N}_{\rm R}\delta(E - z_{\rm R}) \simeq -\frac{\mathcal{N}_{\rm R}}{\sqrt{2\pi}}\frac{1}{E - z_{\rm R}}\,, \quad E \geq 0\,, \tag{1.1}$$

where

$$\mathcal{N}_{\rm R}^2 \equiv {\rm i}\,{\rm res}[S(E)]_{E=z_{\rm R}} \equiv {\rm i}\,{\rm r}_{\rm R}\,, \tag{1.2}$$

S denotes the S matrix, and ${\rm r}_{\rm R}$ denotes the residue of S at $z_{\rm R}$. In addition, we will see that the lower bound of the energy (threshold) is the reason why this amplitude is not exactly but only approximately given by the Breit-Wigner amplitude.

Section 2 provides a quick summary of the most important properties of the Gamow states, along with some basic phenomenological properties of resonances. The proof of (1.1) is provided in Sec. 3. The conclusions are included in Sec. 4.

For the sake of clarity, we shall prove Eq. (1.1) using the example of the spherical shell potential for zero angular momentum. However, as explained in Appendix A, the result is valid for any partial wave and for spherically symmetric potentials that fall off faster than exponentials. Finally, in Appendix B, we provide a thorough characterization of the complex delta function and its associated functional, since they have rarely appeared in the literature.

2 Basics of resonances and Gamow states

Resonance peaks are characterized by the energy $E_{\rm R}$ at which they occur and by their width $\Gamma_{\rm R}$. The resonance peak is related to a pole of the S matrix at the complex number $z_{\rm R} = E_{\rm R} - {\rm i}\,\Gamma_{\rm R}/2$, because the theoretical expression of the cross section in terms of the S matrix fits the experimental cross section in the neighborhood of $E_{\rm R}$, see Eqs. (2.2) and (2.3) below.

When the peak is too narrow and its width cannot be measured, one measures the lifetime $\tau_{\rm R}$ of the decaying particle. Decaying systems follow the exponential decay law, except for short- and long-term deviations.

Although a decaying particle has a finite lifetime, it is otherwise assigned all the properties that are attributed to stable particles, like angular momentum, charge, spin and parity. For example, a radioactive nucleus has a finite lifetime, but otherwise it possesses all the properties of stable nuclei; in fact, it is included in the periodic table of the elements along with the stable nuclei. Similarly, most elementary particles are unstable, and they are listed along with the stable ones in the Particle Data Table [30] and attributed values for the mass, spin and width (or lifetime). Thus, stable particles differ from unstable ones by the value of their width, which is zero in the case of stable particles and different from zero in the case of unstable ones. Hence, phenomenologically, unstable particles are not less fundamental than the stable ones.

A priori, resonances and decaying particles are different entities. A resonance refers to the energy distribution of the outgoing particles in a scattering process, and it is characterized by its energy and width. A decaying state is described in a time-dependent setting by its energy and lifetime. Yet the difference is quantitative rather than qualitative, and both concepts are related by

$$\Gamma_{\mathrm{R}} = \frac{\hbar}{\tau_{\mathrm{R}}} \,, \tag{2.1}$$

though in most systems one can measure either τ_{R} or Γ_{R}, but not both.

Theoretically, however, the relation (2.1) is usually justified as an approximation, $\tau_{\mathrm{R}}\Gamma_{\mathrm{R}} \sim \hbar$, as a kind of time-energy uncertainty relation. For a long time, it was not possible to experimentally check whether the relation (2.1) is exact or approximate, since the lifetime and width could not be measured in the same system. This changed with the measurements of the width [31] and lifetime [32] of the $3p\ ^2P_{3/2}$ state of Na, which provide a firm experimental basis that Eq. (2.1) holds exactly, not just approximately. Thus, resonances and decaying systems are two sides of the same phenomenon.

Although the resonance peaks in the cross section resemble the Lorentzian, the resonance lineshape does not coincide exactly with the Lorentzian. Two features of the cross section reveal so. First, the maximum of the resonance peak never occurs at $E = E_{\mathrm{R}}$, whereas the maximum of the Lorentzian occurs exactly at $E = E_{\mathrm{R}}$. And second, the Laurent expansion of the S matrix around the resonance pole,

$$S(E) = \frac{\mathrm{r}_{\mathrm{R}}}{E - z_{\mathrm{R}}} + B(E) \,, \tag{2.2}$$

which produces the Lorentzian peak in the cross section [33],

$$\sigma \sim \frac{1}{(E - E_{\mathrm{R}})^2 + (\Gamma_{\mathrm{R}}/2)^2} \,, \tag{2.3}$$

is valid only in the vicinity of the resonance pole. Because (2.2) and (2.3) are valid only in the vicinity of the resonance energy, the Lorentzian lineshape is just a near-resonance approximation to the exact resonance lineshape.

Because the Lorentzian does not coincide exactly with the resonance lineshape, the Breit-Wigner amplitude cannot coincide exactly with the resonance amplitude. One can reach the same conclusion by using the point of view of decaying states as follows.

The Breit-Wigner amplitude yields the exponential decay law only when it is defined over the whole of the energy real line $(-\infty, \infty)$ rather than just over the scattering spectrum (see e.g. [34]). Because in quantum mechanics the scattering spectrum has a lower bound, the Breit-Wigner amplitude would yield the exponential decay law only if it was defined also at energies that do not belong to the scattering spectrum. Thus, the Breit-Wigner amplitude is incompatible with the exponential decay law, and therefore cannot coincide with the exact resonance/decay amplitude.

Mathematically, the Gamow states are eigenvectors of the Hamiltonian with a complex eigenvalue $z_{\mathrm{R}} = E_{\mathrm{R}} - \mathrm{i}\,\Gamma_{\mathrm{R}}/2$,

$$H|z_{\mathrm{R}}\rangle = z_{\mathrm{R}}|z_{\mathrm{R}}\rangle \,, \tag{2.4}$$

and, in the radial position representation, they satisfy a "purely outgoing boundary condition" (POBC) at infinity:

$$\langle r|z_{\mathrm{R}}\rangle \sim \mathrm{e}^{\mathrm{i}\sqrt{(2m/\hbar^2)z_{\mathrm{R}}}\,r} \,, \quad \text{as } r \to \infty \,. \tag{2.5}$$

The time-independent Schrödinger equation (2.4) subject to the POBC (2.5) is equivalent to the following integral equation of the Lippmann-Schwinger type:

$$|z_{\mathrm{R}}\rangle = \frac{1}{z_{\mathrm{R}} - H_0 + \mathrm{i}0} V|z_{\mathrm{R}}\rangle \,, \tag{2.6}$$

where H_0 is the free Hamiltonian and V is the potential. Since Eq. (2.6) also yields the bound states, the Gamow states are a natural generalization to resonances of the wave functions of bound states. The bound and resonance energies obtained by solving (2.6) coincide with the poles of the S matrix.

The time evolution of a Gamow state is given by

$$\mathrm{e}^{-\mathrm{i}Ht/\hbar}|z_{\mathrm{R}}\rangle = \mathrm{e}^{-\mathrm{i}E_{\mathrm{R}}t/\hbar}\mathrm{e}^{-\Gamma_{\mathrm{R}}t/(2\hbar)}|z_{\mathrm{R}}\rangle \,, \tag{2.7}$$

and therefore the Gamow states abide by the exponential decay law. Because the eigenvalue of Eq. (2.4) is also a pole of the S matrix, Eq. (2.7) implies that Eq. (2.1) holds. In this way, the Gamow states unify the concepts of resonance and decaying particle, and they provide a "particle status" for them.

Furthermore, since one can obtain both the bound and the resonance energies from Eq. (2.6), or from the poles of the S matrix, resonances are qualitatively the same as bound states. The only difference is quantitative: The Gamow states have a non-zero width (i.e., finite lifetime), whereas the bound states have a zero width (i.e., infinite lifetime).

An important feature of the Gamow states is that they form a basis that expands any wave packet φ^+, see e.g. review [35]. The basis formed by the Gamow states is not complete though, and one has to add an additional set of kets to complete the basis. In a system with several resonances, we have that

$$\varphi^+(t) = \sum_n \mathrm{e}^{-\mathrm{i}z_n t/\hbar}|z_n\rangle\langle z_n|\varphi^+\rangle + \int_0^{-\infty} \mathrm{d}E\, \mathrm{e}^{-\mathrm{i}Et/\hbar}|E^+\rangle\langle^+E|\varphi^+\rangle \,, \tag{2.8}$$

where $z_n = E_n - \mathrm{i}\,\Gamma_n/2$ denotes the nth resonance energy. In this equation, the sum contains the resonance contribution, whereas the integral contains the background. For simplicity, we have omitted the contribution from the bound states. The main virtue of resonance expansions is to isolate each resonance's contribution to the wave packet.

Resonance expansions allow us to understand the deviations from exponential decay [36]. In the energy region where one resonance R is dominant, the expansion (2.8) can be written as

$$\varphi^+(t) = \mathrm{e}^{-\mathrm{i}z_\mathrm{R}t/\hbar}|z_\mathrm{R}\rangle\langle z_\mathrm{R}|\varphi^+\rangle + \text{background(R)}\,, \tag{2.9}$$

where the term "background(R)" contains all contributions not associated with the resonance R, including those from other resonances. Because "background(R)" will always be nonzero, there will always be deviations from exponential decay. The magnitude of these deviations depends on how well we tune the system around the resonance energy: The better we tune the system around the Gamow state $|z_\mathrm{R}\rangle$, the smaller "background(R)" will be. Note that "background(R)" is the analog to the background $B(E)$ of the expansion (2.2).

3 Proof

3.1 Preliminaries

The proof of (1.1) presented below is a straightforward application of the theory of distributions. Rather than working in a general setting, we will use the example of the spherical shell potential,

$$V(\mathbf{x}) = V(r) = \begin{cases} 0 & 0<r<a \\ V_0 & a<r<b \\ 0 & b<r<\infty\,, \end{cases} \tag{3.1}$$

and restrict ourselves to the s partial wave.

In order to prove (1.1), we need to recall that in quantum mechanics, the transition amplitude from one state to another is given by the scalar product of those states:

$$\mathcal{A}(z_\mathrm{R} \to E) = \langle^- E|z_\mathrm{R}\rangle\,, \quad E \geq 0\,, \tag{3.2}$$

where $\langle^- E|$ is the "out" bra solution of the Lippmann-Schwinger equation.

For the potential (3.1), the "out" Lippmann-Schwinger eigenfunction reads, in the radial position representation, as

$$\langle r|E^-\rangle \equiv \chi^-(r;E) = N(E)\,\frac{\chi(r;E)}{\mathcal{J}_-(E)}\,, \qquad E \in [0,\infty)\,, \tag{3.3}$$

where $N(E)$ is a delta-normalization factor,

$$N(E) = \sqrt{\frac{1}{\pi}\frac{2m/\hbar^2}{\sqrt{2m/\hbar^2\, E}}} \, , \tag{3.4}$$

$\chi(r;E)$ is the regular solution,

$$\chi(r;E) = \begin{cases} \sin(kr) & 0<r<a \\ \mathcal{J}_1(E)\mathrm{e}^{\mathrm{i}Qr} + \mathcal{J}_2(E)\mathrm{e}^{-\mathrm{i}Qr} & a<r<b \\ \mathcal{J}_3(E)\mathrm{e}^{\mathrm{i}kr} + \mathcal{J}_4(E)\mathrm{e}^{-\mathrm{i}kr} & b<r<\infty \, , \end{cases} \tag{3.5}$$

the wave numbers k and Q are given by

$$k = \sqrt{\frac{2m}{\hbar^2}E} \, , \quad Q = \sqrt{\frac{2m}{\hbar^2}(E-V_0)} \, , \tag{3.6}$$

and $\mathcal{J}_{\pm}(E)$ are the Jost functions,

$$\mathcal{J}_+(E) = -2\mathrm{i}\mathcal{J}_4(E) \, , \quad \mathcal{J}_-(E) = 2\mathrm{i}\mathcal{J}_3(E) \, . \tag{3.7}$$

The resonance energies z_{R} produced by the potential (3.1) coincide with the zeros of $\mathcal{J}_+$. With each resonance energy z_{R}, we associate a Gamow eigenfunction $u(r;z_{\mathrm{R}})$:

$$\langle r|z_{\mathrm{R}}\rangle = u(r;z_{\mathrm{R}}) = \sqrt{\frac{m}{\hbar^2 k_{\mathrm{R}}}}\mathcal{N}_{\mathrm{R}} \times \begin{cases} \frac{1}{\mathcal{J}_3(k_{\mathrm{R}})}\sin(k_{\mathrm{R}}r) & 0<r<a \\ \frac{\mathcal{J}_1(k_{\mathrm{R}})}{\mathcal{J}_3(k_{\mathrm{R}})}\,\mathrm{e}^{\mathrm{i}Q_{\mathrm{R}}r} + \frac{\mathcal{J}_2(k_{\mathrm{R}})}{\mathcal{J}_3(k_{\mathrm{R}})}\,\mathrm{e}^{-\mathrm{i}Q_{\mathrm{R}}r} & a<r<b \\ \mathrm{e}^{\mathrm{i}k_{\mathrm{R}}r} & b<r<\infty \, , \end{cases} \tag{3.8}$$

where

$$k_{\mathrm{R}} = \sqrt{\frac{2m}{\hbar^2}z_{\mathrm{R}}} \, , \quad Q_{\mathrm{R}} = \sqrt{\frac{2m}{\hbar^2}(z_{\mathrm{R}}-V_0)} \, , \tag{3.9}$$

and where $\mathcal{N}_{\mathrm{R}}$ is given by Eq. (1.2). From Eqs. (3.3) and (3.8) we obtain

$$\chi^-(r;z_{\mathrm{R}}) = \frac{1}{\mathrm{i}\sqrt{2\pi}\mathcal{N}_{\mathrm{R}}}u(r;z_{\mathrm{R}}) \, . \tag{3.10}$$

As shown in [37], the analytic continuations of the Lippmann-Schwinger kets –and therefore also the Gamow kets– are well defined as antilinear functionals over the space of test functions ψ^- for which the following quantities are finite:

$$\|\psi^-\|_{n,n'} := \sqrt{\int_0^\infty \mathrm{d}r\, \left|\frac{nr}{1+nr}\,\mathrm{e}^{nr^2/2}(1+H)^{n'}\psi^-(r)\right|^2} \, , \quad n,n' = 0,1,2,\ldots \tag{3.11}$$

The action of the Gamow ket $|z_{\mathrm{R}}\rangle$ on the test functions ψ^- is explicitly given by

$$\langle\psi^-|z_{\mathrm{R}}\rangle = \int_0^\infty \mathrm{d}r\,\langle\psi^-|r\rangle\langle r|z_{\mathrm{R}}\rangle = \int_0^\infty \mathrm{d}r\,\psi^-(r)^* u(r;z_{\mathrm{R}}) \, . \tag{3.12}$$

For the sake of clarity, we need to introduce a special notation that will specify when we are working in the energy representation: Whenever we work in such representation, we will add a hat to the corresponding quantity. For example, the energy representation of a wave function $\psi^-(r)$ will be denoted by $\widehat{\psi}^-(E)$.

By using the operator U_- of [38], one can obtain the energy representation of $\psi^-(r)$,

$$\widehat{\psi}^-(E) = (U_-\psi^-)(E) = \int_0^\infty \mathrm{d}r\, \psi^-(r)\chi^-(r;E)^* \, . \tag{3.13}$$

As shown in [37], when $\psi^-(r)$ satisfies (3.11), the analytic continuation of $[\widehat{\psi}^-(E)]^*$ exists. We shall denote such analytic continuation by $\widehat{\psi}^-(z^*)^*$:

$$\widehat{\psi}^-(z^*)^* = \int_0^\infty \mathrm{d}r\, \psi^-(r)^*\chi^-(r;z) \, . \tag{3.14}$$

Not only $\widehat{\psi}^-(z^*)^*$ exist, all the test functions $[\widehat{\psi}^-(z^*)]^*$ are analytic at the resonance energies.[1] As explained in Appendix B, this means that the antilinear complex delta functional at the resonance energies z_{R} can be defined as

$$\begin{array}{rcl} \widehat{\delta}_{z_{\mathrm{R}}} : \widehat{\Phi}_{-\mathrm{exp}} & \longmapsto & \mathbb{C} \\ \widehat{\psi}^- & \longmapsto & \langle \widehat{\psi}^- | \widehat{\delta}_{z_{\mathrm{R}}} \rangle \equiv \widehat{\delta}_{z_{\mathrm{R}}}(\widehat{\psi}^-) := \widehat{\psi}^-(z_{\mathrm{R}}^*)^* \, , \end{array} \tag{3.15}$$

where $\widehat{\Phi}_{-\mathrm{exp}}$ is the space of test functions $\widehat{\psi}^-$. That is, $\widehat{\delta}_{z_{\mathrm{R}}}$ associates a test function $\widehat{\psi}^-$ with the value that the analytic continuation of $[\widehat{\psi}^-]^*$ takes at z_{R}.

3.2 The Gamow state and the complex delta function

In order to obtain the equality of Eq. (1.1), we are going first to denote the energy representation of the Gamow ket as

$$|\widehat{z_{\mathrm{R}}^-}\rangle \equiv U_-|z_{\mathrm{R}}\rangle \, . \tag{3.16}$$

Then, it follows that

$$\begin{aligned} \langle \widehat{\psi}^- | \widehat{z_{\mathrm{R}}^-} \rangle &= \langle \widehat{\psi}^- | U_- | z_{\mathrm{R}} \rangle \\ &= \langle U_-^\dagger \widehat{\psi}^- | z_{\mathrm{R}} \rangle \\ &= \langle \psi^- | z_{\mathrm{R}} \rangle \\ &= \int_0^\infty \mathrm{d}r\, [\psi^-(r)]^* u(r;z_{\mathrm{R}}) && \text{by (3.12)} \\ &= \mathrm{i}\sqrt{2\pi}\, \mathcal{N}_{\mathrm{R}} \int_0^\infty \mathrm{d}r\, \psi^-(r)^* \chi^-(r;z_{\mathrm{R}}) && \text{by (3.10)} \\ &= \mathrm{i}\sqrt{2\pi}\, \mathcal{N}_{\mathrm{R}}\, \widehat{\psi}^-(z_{\mathrm{R}}^*)^* && \text{by (3.14)} \\ &= \mathrm{i}\sqrt{2\pi}\, \mathcal{N}_{\mathrm{R}} \langle \widehat{\psi}^- | \widehat{\delta}_{z_{\mathrm{R}}} \rangle && \text{by (3.15)} \, . \end{aligned} \tag{3.17}$$

[1] The poles of $[\widehat{\psi}^-(z^*)]^*$ are located on the first sheet of the Riemann surface, that is, on the upper half of the wave-number plane.

This equation proves that the energy representation of the Gamow functional, $|\widehat{z_{\rm R}^-}\rangle$, is proportional to the antilinear complex delta functional, $|\widehat{\delta}_{z_{\rm R}}\rangle$.

In the energy representation, the identity $\widehat{\mathbb{I}}$ can be written as

$$\int_0^{\infty} {\rm d}E \, |\widehat{E}^-\rangle\langle^-\widehat{E}| = \widehat{\mathbb{I}} \, , \tag{3.18}$$

where $|\widehat{E}^-\rangle$ denotes the energy representation of $|E^-\rangle$. By inserting (3.18) into the first and the last terms of (3.17), we obtain

$$\int_0^{\infty} {\rm d}E \, \langle\widehat{\psi}^-|\widehat{E}^-\rangle\langle^-\widehat{E}|\widehat{z_{\rm R}^-}\rangle = \int_0^{\infty} {\rm d}E \, {\rm i}\sqrt{2\pi}\,\mathcal{N}_{\rm R}\langle\widehat{\psi}^-|\widehat{E}^-\rangle\langle^-\widehat{E}|\widehat{\delta}_{z_{\rm R}}\rangle \, . \tag{3.19}$$

Because Eq. (3.19) holds for any $\widehat{\psi}^-$, it follows that

$$\langle^-\widehat{E}|\widehat{z_{\rm R}^-}\rangle = {\rm i}\sqrt{2\pi}\,\mathcal{N}_{\rm R}\langle^-\widehat{E}|\widehat{\delta}_{z_{\rm R}}\rangle \equiv {\rm i}\sqrt{2\pi}\,\mathcal{N}_{\rm R}\,\delta(E - z_{\rm R}) \, , \tag{3.20}$$

which, after dropping the hat notation and using Eq. (3.2), becomes the equality in Eq. (1.1).

3.3 The Gamow state and the Breit-Wigner amplitude

In order to obtain the approximation of Eq. (1.1), we are going to obtain first the transition amplitude $\widetilde{\mathcal{A}}(z_{\rm R} \to E)$ from a resonance state of energy $z_{\rm R}$ to a scattering state of energy $-\infty < E < \infty$,

$$\widetilde{\mathcal{A}}(z_{\rm R} \to E) = \langle^-E|z_{\rm R}\rangle \, , \quad E \in (-\infty, \infty) \, . \tag{3.21}$$

Even though a quantum system can only decay to a scattering state of energy $E \geq 0$, we are going to ask the system to pretend that it can also decay to negative energies. Mathematically, this is equivalent to ask the system to pretend that its scattering spectrum runs from $-\infty$ to $+\infty$. Physically, it is equivalent to ignore the effect of the lower bound of energy $E = 0$. Calculating (3.21), i.e., forcing the system to decay also to negative energies, needs a regulator. The regulator we will use is ${\rm e}^{-{\rm i}\alpha E}$, where $\alpha > 0$ and E has zero or negative imaginary part. The reason why we use this regulator is that for complex z, the wave functions $\widehat{\psi}^-(z^*)^*$ grow slower than ${\rm e}^{|{\rm Im}(\sqrt{2m/\hbar^2\, z})|^2}$ in the lower half plane of the second sheet [37]. More precisely, Proposition 3 in [37] shows that for each $n = 1, 2, \ldots$ and for each $\beta > 0$, there is a $C > 0$ such that in the lower half plane of the second sheet, $\widehat{\psi}^-(z^*)^*$ is bounded by

$$|\widehat{\psi}^-(z^*)^*| \leq C \, \frac{1}{|z|^{1/4}|1+z|^n} \, {\rm e}^{\frac{|{\rm Im}(\sqrt{2m/\hbar^2\, z})|^2}{2\beta}} \, . \tag{3.22}$$

This estimate implies that ${\rm e}^{-{\rm i}\alpha z}\widehat{\psi}^-(z^*)^*$ tends to zero in the infinite arc of the lower half of the second sheet,

$$\lim_{z\to\infty} {\rm e}^{-{\rm i}z\alpha}\widehat{\psi}^-(z^*)^* = 0 \, , \quad \alpha > 0 \, . \tag{3.23}$$

In its turn, the limit (3.23) enables us to apply Cauchy's theorem to obtain

$$\widehat{\psi}^{-}(z_{\rm R}^{*})^{*} = \lim_{\alpha\to 0} -\frac{1}{2\pi{\rm i}}\int_{-\infty}^{\infty} {\rm d}E\, {\rm e}^{-{\rm i}\alpha E}\widehat{\psi}^{-}(E)^{*}\frac{1}{E-z_{\rm R}}\, , \tag{3.24}$$

where the integral is performed infinitesimally below the real axis of the second sheet. By multiplying both sides of (3.24) by ${\rm i}\sqrt{2\pi}\mathcal{N}_{\rm R}$, and by recalling (3.17), we obtain

$$\langle\widehat{\psi}^{-}|\widehat{z_{\rm R}}\rangle = \lim_{\alpha\to 0}\int_{-\infty}^{\infty} {\rm d}E\, {\rm e}^{-{\rm i}\alpha E}\psi^{-}(E)^{*}(-1)\frac{\mathcal{N}_{\rm R}}{\sqrt{2\pi}}\frac{1}{E-z_{\rm R}}\, . \tag{3.25}$$

In the bra-ket notation, Eq. (3.25) reads as

$$\langle\widehat{\psi}^{-}|\widehat{z}_{\rm R}\rangle = \lim_{\alpha\to 0}\int_{-\infty}^{\infty} {\rm d}E\, {\rm e}^{-{\rm i}\alpha E}\langle\widehat{\psi}^{-}|\widehat{E}^{-}\rangle\langle^{-}\widehat{E}|\widehat{z}_{\rm R}\rangle\, . \tag{3.26}$$

Comparison of (3.25) with (3.26) yields the following expression for the amplitude (3.21):

$$\widetilde{\mathcal{A}}(z_{\rm R}\to E) = -\frac{\mathcal{N}_{\rm R}}{\sqrt{2\pi}}\frac{1}{E-z_{\rm R}}\, ,\ \ E\in(-\infty,\infty)\, . \tag{3.27}$$

Thus, if the scattering spectrum was the whole real line, the resonance amplitude would be exactly the Breit-Wigner amplitude. However, because the scattering spectrum has a lower bound, the resonance amplitude is not exactly the Breit-Wigner amplitude. Only when we can neglect the effect of the threshold, the resonance amplitude coincides with the Breit-Wigner amplitude:

$$\mathcal{A}(z_{\rm R}\to E) \simeq \widetilde{\mathcal{A}}(z_{\rm R}\to E)\, , \tag{3.28}$$

which is the approximation on the right-hand side of (1.1). In particular, when the threshold can be ignored, the complex delta function becomes for all intends and purposes the Breit-Wigner amplitude.

It should be stressed that the amplitude (3.21) is not physical, because in (3.21) the energy E runs over the whole real line rather than over the scattering spectrum. However, such unphysical amplitude helps us understand what the physical amplitude –the complex delta function– is, by allowing us to see how the resonance would decay if the scattering spectrum was the whole real line.

3.4 Further remarks

Aside from phase space factors, cross sections are determined by the transition amplitude from an "in" to an "out" state, $\mathcal{A}(E_i\to E_f)$. If $|E^{\pm}\rangle$ denote the "in" and "out" solutions of the Lippmann-Schwinger equation, then

$$\mathcal{A}(E_i\to E_f) = \langle^{-}E_f|E_i^{+}\rangle = S(E_i)\delta(E_f-E_i)\, . \tag{3.29}$$

If we imagine now that instead of an initial state $|E_i^+\rangle$ we had an unstable particle $|z_{\rm R}\rangle$, the transition (decay) amplitude $\mathcal{A}(z_{\rm R} \to E_f)$ would be given by (1.1). Using the approximate decay amplitude of (1.1), one obtains the following approximate decay probability:

$$|\mathcal{A}(z_{\rm R} \to E_f)|^2 \simeq \frac{|\mathcal{N}_{\rm R}|^2}{2\pi} \frac{1}{(E_f - E_{\rm R})^2 + (\Gamma_{\rm R}/2)^2} \, ; \tag{3.30}$$

that is, the decay probability of a resonance is given by the Lorentzian when the effect of the threshold can be ignored.[2] Because the almost-Lorentzian decay probability (3.30) coincides with the almost-Lorentzian peaks in cross sections, resonances can be interpreted as intermediate, unstable particles.

Finally, it is worthwhile to compare the Gamow states with the states introduced by Kapur and Peierls [39]. As mentioned above, the Gamow states are eigenfunctions of the Hamiltonian that satisfy the POBC (2.5) at infinity; the wave numbers involved in the POBC (2.5) are complex and proportional to the square root of the complex eigenenergies of the Gamow states; such complex eigenenergies are the same as the poles of the S matrix, and they do not depend on any external parameter or energy. By contrast, the Kapur-Peierls states are eigenfunctions of the Hamiltonian that satisfy a POBC at a finite radial distance r_0, where r_0 is such that the potential vanishes for $r > r_0$; the wave numbers involved in the POBC satisfied by the Kapur-Peierls states are real and proportional to the square root of the real energy of the incoming particle; the POBC satisfied by the Kapur-Peierls states makes them and their associated complex eigenenergies depend on r_0 and on the real energy of the incoming particle; also, the complex eigenenergies of the Kapur-Peierls states are not the same as the poles of the S matrix. Thus, the Kapur-Peierls states do not seem to be related to the standard Breit-Wigner amplitude, because such amplitude does not depend on r_0 and its complex energy does not depend on the energy of the incoming particle.

4 Conclusions

Since resonances leave an almost-Lorentzian fingerprint in the cross section, and since the Gamow states are the wave functions of resonances, the decay amplitude provided by a Gamow state should be linked to the Breit-Wigner amplitude. In this paper, we have found that the precise link is given by Eq. (1.1), and we have interpreted this result by saying that the resonance amplitude associated with a Gamow state is exactly given by the complex delta function, and that the Breit-Wigner amplitude is an approximation to such resonance amplitude, which approximation is valid when we can neglect the effect of the threshold. Thus, Eq. (1.1) establishes the precise relation between the Gamow state, Nakanishi's complex delta function and the Breit-Wigner amplitude. In addition, Eq. (1.1) affords another theoretical argument in favor of

[2] A much more detailed study of the dependence of the cross section (and expectation values of observables) on the Breit-Wigner amplitude can be found in e.g. [2, 8, 14].

interpreting the almost-Lorentzian peaks in cross sections as intermediate, unstable particles—resonances are real (as opposed to virtual) particles, in accordance with resonance phenomenology.

As is well known, the actual resonance lineshape of cross sections can be very different from a quasi-Lorentzian one, due to the effect of thresholds, other resonances, or extra channels. The usefulness of (1.1) does not lie in predicting the exact shape of the cross section, but rather in identifying what contribution to the cross section comes from each pole of the S matrix. In particular, although the equality in Eq. (1.1) is always exact, for practical purposes the approximation in Eq. (1.1) is useful only for narrow resonances.

When we add Eq. (1.1) to the other known properties of the Gamow states, we see that such states have all the necessary properties to describe resonance/unstable particles:

- They are associated with poles of the S matrix.
- They exhibit the correct phenomenological signatures of both resonances (almost-Lorentzian lineshape) and unstable particles (exponential decay), and they provide a firm theoretical basis for (2.1).
- They are basis vectors that isolate each resonance's contribution to a wave packet.

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A Generalizations

Equation (1.1) is not valid only for the spherical shell potential (3.1) but actually holds for a quite large class of potentials. The reason can be found in well-known results of scattering theory [40, 41]. As explained in [41], page 191, partial wave analysis is valid whenever the spherically symmetric potential satisfies the following requirements:

Ĩ. $V(r) = O(r^{-3-\epsilon})$ as $r \to \infty$.

II. $V(r) = O(r^{-3/2+\epsilon})$ as $r \to 0$.

III. $V(r)$ is continuous for $0 < r < \infty$, except perhaps at a finite number of finite discontinuities.

These conditions are, however, not sufficient to guarantee that the S matrix $S(E)$, the Jost functions $\mathcal{J}_{\pm}(E)$ and the Lippmann-Schwinger eigenfunction $\chi^{-}(r;E)$ can be analytically continued into the whole complex plane. Such analytic continuation is guaranteed when we replace condition $\widetilde{\mathrm{I}}$ by the more stringent

I. $V(r)$ falls off faster than exponentials as $r \to \infty$,

as stated throughout Chapters 11 and 12 of [41], and in Chapter 5 of [40], especially in Theorem 5.3.2. Thus, when $V(r)$ satisfies I-III, even though we may not know their exact analytic expressions, we know that $S(E)$, $\mathcal{J}_{\pm}(E)$ and $\chi^{-}(r;E)$ can be analytically continued into the whole complex plane and that the Gamow eigenfunction $u(r;z_{\rm R})$ is well defined. Moreover, since in the asymptotic region $r \to \infty$ the expressions of $u(r;z_{\rm R})$ and $\chi^{-}(r;E)$ for any potential satisfying I-III are the same as the expressions of $u(r;z_{\rm R})$ and $\chi^{-}(r;E)$ for the spherical shell potential in the region $r > b$ (with different expressions for the Jost functions), the general proof goes through exactly the same lines as the proof for the spherical shell potential. Finally, the argument extends without difficulty to higher angular momentum.

B The complex delta functional

In quantum mechanics, the complex delta function was originally introduced by Nakanishi [42] to describe resonances in the Lee model [43]. In mathematics, the complex delta function was introduced by Gelfand and Shilov [44]. The purpose of this appendix is to introduce the precise mathematical definition of the complex delta function and to show that, when the test functions are analytic, such definition coincides with the one given by Nakanishi.

B.1 Three definitions of the (linear) complex delta functional

The complex delta functional has different forms depending on the properties of the test functions on which it acts. We shall review the three most important forms, namely when the complex delta functional acts on analytic functions (this form is used in this paper and and was introduced in [44]), when it acts on meromorphic functions (this is the form used by Nakanishi [42]), and when it acts on non-meromorphic functions (this form was introduced in [44]). When the space of test functions are analytic, as is our case, these three forms coincide (as they should) and can be written as in Eq. (3.15).

B.1.1 First definition—the test functions are analytic

According to page 1 of Volume I of Ref. [44], a distribution is a function that associates a complex number with each function belonging to a vector space:

$$\begin{array}{rcl} \text{distribution} : \{\text{Space of functions}\} & \longmapsto & \mathbb{C} \\ \text{function} & \longmapsto & \text{complex number}\,. \end{array} \tag{B.1}$$

The functions in the "{Space of functions}" are usually called test functions. Because a distribution maps functions into complex numbers, they are usually called functionals. Such functionals can be linear or antilinear.

A more precise definition is the following. If Φ is a vector space of test functions endowed with a topology, a linear (antilinear) distribution F is a function from Φ to $\mathbb{C}$

$$\begin{array}{rcl} F : \Phi & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & F(\phi) \end{array} \tag{B.2}$$

such that

(*i*) F is well defined,

(*ii*) F is linear (antilinear),

(*iii*) F is continuous.

A very important example of distribution is the (linear) Schwartz delta functional at a real number E. Such functional associates with each test function ϕ the value that ϕ takes at E:

$$\begin{array}{rcl} \delta_E : \Phi_{\rm Schw} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \delta_E(\phi) = \phi(E)\,, \end{array} \tag{B.3}$$

where the test functions of $\Phi_{\rm Schw}$ are infinitely differentiable and of polynomial falloff. It is straightforward to show that definition (B.3) satisfies the above requirements (*i*)-(*iii*).

The (linear) complex delta functional is defined in a completely analogous way. As stated by Gelfand and Shilov [44, Vol. 2, page 85], the point E in Eq. (B.3) may be complex in the spaces of analytic functions. If $\Phi_{\rm anal}$ denotes a vector space of *analytic* functions at the complex point z_0, then the linear complex delta functional at z_0 is defined as a function that associates with each test function ϕ the value that the analytic continuation of ϕ takes at z_0:

$$\begin{array}{rcl} \delta_{z_0} : \Phi_{\rm anal} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \delta_{z_0}(\phi) = \phi(z_0)\,. \end{array} \tag{B.4}$$

Two important comments are in order here. First, the test functions of $\Phi_{\rm anal}$ must be analytic at z_0; that is, z_0 is not a singularity (e.g., a pole) of any ϕ, otherwise definition (B.4) makes no sense. And second, the complex delta functional is completely specified by Eq. (B.4) because the test functions are analytic at z_0, and therefore one does not need to introduce any contour in the definition of (B.4), even though one could use such a contour, as in Eq. (B.5) below.

Definition (B.4) actually fulfills the requirements (*i*)-(*iii*).[3] The only property that is conceptually challenging is (*i*). Because we are assuming that the test functions are analytic at z_0, $\phi(z_0)$ exists and is unique, which grants requirement (*i*). Thus, definition (B.4) completely, rigorously and unambiguously defines the complex delta functional.

B.1.2 Second definition—the test functions are meromorphic

Many functions are not analytic but just meromorphic. That is, when we analytically continue them, they have isolated singularities ("poles") in the complex plane. At such poles, definition (B.4) makes no sense, and one has to extend it. If $\Phi_{\rm mero}$ is a vector space of meromorphic functions at z_0, the (linear) complex delta functional at z_0 is defined as

$$\begin{array}{rcl} \delta_{z_0} : \Phi_{\rm mero} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \delta_{z_0}(\phi) = \frac{1}{2\pi {\rm i}} \oint {\rm d}z \, \frac{\phi(z)}{z-z_0} \, . \end{array} \tag{B.5}$$

One can again check very easily that definition (B.5) satisfies requirements (*i*)-(*iii*). Note that because in definition (B.5) the test functions are meromorphic, such definition depends on Cauchy's theorem and on the contour used.[4]

If we denote by a_0 the zeroth term of the Laurent expansion of $\phi(z)$ around z_0, then definition (B.5) associates a_0 with each test function ϕ, since

$$a_0 = \frac{1}{2\pi {\rm i}} \oint {\rm d}z \, \frac{\phi(z)}{z-z_0} \, . \tag{B.6}$$

Thus, we may write definition (B.5) as

$$\begin{array}{rcl} \delta_{z_0} : \Phi_{\rm mero} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \delta_{z_0}(\phi) = a_0 \, . \end{array} \tag{B.7}$$

Obviously, both (B.5) and (B.7) define the same functional, because both associate the same complex number with the same function, even though in (B.7) no contour integral has been explicitly used.

Now, when $\phi(z)$ is analytic at z_0, a_0 is simply $\phi(z_0)$. Thus, when the test functions are not just meromorphic but also *analytic* at z_0, definitions (B.5) and (B.7) become

[3] In this paper, we omit any explicit discussion on the continuity requirement (*iii*). The reason is that first, the continuity of the complex delta function is guaranteed by the results of [37], and second, continuity is not essential to our main discussion.

[4] The contour used in Eq. (B.5) is assumed to be a circle around z_0 such that the test function ϕ is analytic inside such circle except perhaps at z_0.

definition (B.4), because in such case all these definitions associate each function ϕ with one and the same complex number $\phi(z_0)$. This is why, when the test functions ϕ are all *analytic* at z_0, one can define the complex delta functional by way of Eq. (B.4), as Gelfand and Shilov do in page 85, Vol. II of [44].

B.1.3 Third definition—the test functions are not meromorphic

When the test functions are not meromorphic, definitions (B.4), (B.5) and (B.7) make no sense. One can still define a complex delta functional at the origin following the prescription of Gelfand and Shilov [44, Vol. I, Appendix B]. When the functions are meromorphic, such definition of the complex delta functional at the origin becomes (B.5) and (B.7).

However, because in this paper we use test functions that are analytic at the resonance energies, we do not need to use this general definition or definition (B.5), because all these definitions actually become (B.4).

B.2 Three definitions of the (antilinear) complex delta functional

In this paper, we have used antilinear (rather than linear) functionals. We will therefore briefly explain how one defines such functionals for the cases considered in the previous section.

The (antilinear) Schwartz delta functional at a real number E associates with each test function ϕ, the complex conjugate of the value that ϕ takes at E:

$$\begin{array}{rcl} \widehat{\delta}_E : \Phi_{\rm Schw} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \widehat{\delta}_E(\phi) = \phi(E)^* \, . \end{array} \tag{B.8}$$

When we write the action of $\widehat{\delta}_E$ as an integral operator, the kernel of such integral operator is Dirac's delta function:

$$\widehat{\delta}_E(\phi) = \int_0^{\infty} {\rm d}E' \, \delta(E'-E)\phi(E')^* = \phi(E)^* \, . \tag{B.9}$$

If $\Phi_{\rm anal}$ denotes a vector space of test functions ϕ such that ϕ^* are all *analytic* at z_0, then the antilinear complex delta functional at z_0 is a function that associates with each test function ϕ, the value that the analytic continuation of ϕ^* takes at z_0:

$$\begin{array}{rcl} \widehat{\delta}_{z_0} : \Phi_{\rm anal} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \widehat{\delta}_{z_0}(\phi) = \phi(z_0^*)^* \, . \end{array} \tag{B.10}$$

When we write the expression for $\widehat{\delta}_{z_0}$ as an integral operator, the kernel of such integral operator is the complex delta function:

$$\widehat{\delta}_{z_0}(\phi) = \int_0^{\infty} {\rm d}E' \, \delta(E'-z_0)\phi(E')^* = \phi(z_0^*)^* \, . \tag{B.11}$$

When the test functions are only meromorphic and z_0 is one of their poles, definition (B.10) needs to be changed to

$$\begin{array}{rcl} \widehat{\delta}_{z_0} : \Phi_{\rm mero} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \widehat{\delta}_{z_0}(\phi) = \frac{1}{2\pi {\rm i}} \oint {\rm d}z \, \frac{\phi(z^*)^*}{z-z_0} \, . \end{array} \tag{B.12}$$

If we denote by a_0^* the zeroth term of the Laurent expansion of $\phi(z^*)^*$ around z_0, then definition (B.12) associates a_0^* with each test function ϕ, and therefore we can write

$$\begin{array}{rcl} \widehat{\delta}_{z_0} : \Phi_{\rm mero} & \longmapsto & \mathbb{C} \\ \phi & \longmapsto & \widehat{\delta}_{z_0}(\phi) = a_0^* \, . \end{array} \tag{B.13}$$

If the functions are not even meromorphic, we need to use the prescription of Gelfand and Shilov [44, Vol. I, Appendix B].

The same conclusions as in the previous section apply to the antilinear complex delta functional. When $\phi(z^*)^*$ are all *analytic* at z_0, a_0^* is simply $\phi(z_0^*)^*$. Thus, when the test functions are all *analytic* at z_0, definition (B.12) becomes definition (B.10), and we are allowed to use (B.10).

B.3 Nakanishi's definition

Nakanishi [42] uses a slightly different version of the complex delta function. When he writes $\delta_{\rm N}(\phi)$ as an integral operator, Nakanishi uses the following expression:

$$\delta_{\rm N}(\phi) = \int_\gamma {\rm d}E \, \phi(E^*)^* \delta_{\rm N}(E - z_{\rm R}) \, , \tag{B.14}$$

where

$$\delta_{\rm N}(E - z_{\rm R}) = \frac{1}{2\pi{\rm i}} \left(\frac{1}{E^{(-)} - z_{\rm R}} - \frac{1}{E^{(+)} - z_{\rm R}} \right) \, , \tag{B.15}$$

and where the contour γ is such that the integral in Eq. (B.14) decomposes into two terms. The end points of the integration paths are the same for the two terms, namely, 0 and $+\infty$. The integration path for the first term, $\frac{1}{E^{(-)}-z_{\rm R}}$, passes below $z_{\rm R}$, whereas the integration path for the second term, $\frac{1}{E^{(+)}-z_{\rm R}}$, passes above $z_{\rm R}$. Adding the two terms we obtain

$$\int_\gamma {\rm d}E \, \phi(E^*)^* \delta_{\rm N}(E - z_{\rm R}) = \frac{1}{2\pi{\rm i}} \oint {\rm d}E \, \frac{\phi(E^*)^*}{E - z_{\rm R}} = \phi(z_{\rm R}^*)^* \, . \tag{B.16}$$

Thus, the distributional definition (B.10) is equivalent to Nakanishi's definition (B.14)-(B.16), because both approaches associate the same complex number, $\phi(z_{\rm R}^*)^*$, with the same test function, ϕ.

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