SPECTRAL THEORY OF ELLIPTIC OPERATORS IN EXTERIOR DOMAINS

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Abstract. We consider various closed (and self-adjoint) extensions of elliptic differential expressions of the type $\mathcal{A} = \sum_{0 \leqslant |\alpha|, |\beta| \leqslant m} (-1)^\alpha D^\alpha a_{\alpha,\beta}(x) D^\beta$, $a_{\alpha,\beta}(\cdot) \in C^\infty(\overline{\Omega})$, on smooth (bounded or unbounded) domains Ω in $\mathbb{R}^n$ with compact boundary $\partial\Omega$. Using the concept of boundary triples and operator-valued Weyl–Titchmarsh functions, we prove various trace ideal properties of powers of resolvent differences of these closed realizations of $\mathcal{A}$ and derive estimates on eigenvalues of certain self-adjoint realizations in spectral gaps of the Dirichlet realization.

Our results extend classical theorems due to Višik, Povzner, Birman, and Grubb.

1. Introduction

Let Ω be an open domain in $\mathbb{R}^n$ (bounded or unbounded) with compact boundary $\partial\Omega$. Throughout we assume that $\partial\Omega$ is an $(n-1)$-dimensional (not necessarily connected) C^∞-manifold. Let $\mathcal{A}$ be the differential expression

$$\mathcal{A} = \sum_{0 \leqslant |\alpha|, |\beta| \leqslant m} (-1)^\alpha D^\alpha a_{\alpha,\beta}(x) D^\beta, \quad a_{\alpha,\beta}(\cdot) \in C^\infty(\overline{\Omega}), \tag{1.1}$$

$\operatorname{ord}(\mathcal{A}) = 2m$, which is elliptic in $\overline{\Omega}$. Moreover, we assume that $\mathcal{A}$ is properly elliptic in $\overline{\Omega}$ (which is automatically satisfied if either $n > 2$ or the symbol of $\mathcal{A}$ is real, cf. [11]). In addition to (1.1) we consider its formal adjoint $\mathcal{A}^\top = \sum_{0 \leqslant |\alpha|, |\beta| \leqslant m} (-1)^\alpha D^\beta \overline{a_{\beta,\alpha}(x)} D^\beta$, which is also properly elliptic in $\overline{\Omega}$ (cf. [11]).

Denote by $A = A_{\min}$ ($A^\top = A^\top_{\min}$) the minimal operator associated in $L^2(\Omega)$ with the differential expression $\mathcal{A}$ (resp., $\mathcal{A}^\top$), that is, the closure of $\mathcal{A}$ defined on $C_0^\infty(\Omega)$. The maximal operators $A_{\max}$ and $A^\top_{\max}$ are then defined by $A_{\max} = (A^\top_{\min})^* = (A^\top)^*$ and $A^\top_{\max} = (A_{\min})^* = A^*$, respectively. We emphasize that $H^{2m}(\Omega) \subset \operatorname{dom}(A_{\max}) \subset H^{2m}_{\mathrm{loc}}(\Omega)$, while $\operatorname{dom}(A_{\max}) \neq H^{2m}(\Omega)$.

After the pioneering work by Vishik [15], nonlocal boundary value problems of the form $A_{\max} u = f$, $(\partial u/\partial n - Ku) \restriction \partial\Omega = 0$ for elliptic operators (1.1) (with $m = 1$) in bounded domains were considered by numerous authors (see, e.g., [3, 9] and the references therein). Vishik was the first to consider these problems in the framework of extension theory of dual pairs of operators. Starting with a formula for the domain $\operatorname{dom}(A_{\max})$ of $A_{\max}$, he applied it to an appropriate regularization of the classical Green's formula, using the Calderon operator. The latter allowed him to extend the Green's formula from $H^{2m}(\Omega)$ to $\operatorname{dom}(A_{\max})$. The next fundamental contribution to the subject was made by Grubb [9]. Using the theory of Lions and Magenes [11], Grubb substantially extended and completed the results of [15]. In particular, Grubb obtained the (regularized) Green's formula which (in the special case $m = 1$) reads as follows:

$$(A_{\max} u, v)_{L^2(\Omega)} - (u, A^\top_{\max} v)_{L^2(\Omega)} = \big(\widetilde{\Gamma}_{\Omega,1} u, \widetilde{\Gamma}^\top_{\Omega,0} v\big)_{1/2,-1/2} - \big(\widetilde{\Gamma}_{\Omega,0} u, \widetilde{\Gamma}^\top_{\Omega,1} v\big)_{-1/2,1/2}.$$

Here $(\cdot,\cdot)_{s,-s}$ denotes the duality pairing between $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$, $u \in \operatorname{dom}(A_{\max})$, $v \in \operatorname{dom}(A^\top_{\max})$, and $\widetilde{\Gamma}_{\Omega,0}$, $\widetilde{\Gamma}_{\Omega,1}$, $\Gamma^\top_{\Omega,0}$, and $\widetilde{\Gamma}^\top_{\Omega,1}$ are regularized trace operators, having the properties

$$\widetilde{\Gamma}_{\Omega,1} \colon \mathcal{D}(A_{\max}) \to H^{1/2}(\partial\Omega), \quad \widetilde{\Gamma}^\top_{\Omega,0} \colon \mathcal{D}(A_{\max}) \to H^{-1/2}(\partial\Omega), \quad \operatorname{ran}\big((\widetilde{\Gamma}_{\Omega,1}, \widetilde{\Gamma}^\top_{\Omega,0})\big) = H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega).$$

Later, we will use a somewhat different approach (cf. Proposition 2.7).

1991 *Mathematics Subject Classification.* Primary 35P20, 35J40, 47F05; Secondary 47A10.

Key words and phrases. Elliptic Partial Differential Equations, Spectral Theory, Boundary Triples, Weyl–Titchmarsh Operators.

On the other hand, during the past three decades a new approach to the extension theory, based on the concept of a boundary triple and the corresponding operator-valued Weyl–Titchmarsh function, was developed in [7] (cf. the references therein for the symmetric case) and in [13] (in the case of dual pairs). In this paper we apply some results and technique from [7] and [13] to elliptic operators on unbounded domains. The most important ingredients from the elliptic theory we need are the regularized Green's formula and *a priori* coercivity-type estimates for the elliptic realizations $\widehat{A}_B$ of $\mathcal{A}$ (see (2.2) below). To obtain the latter on unbounded domains one needs additional restrictions on the coefficients of $\mathcal{A}$, since an elliptic realization is not necessarily coercive. Here we restrict ourselves to the case of bounded coefficients $a_{\alpha,\beta}(\cdot)$. Using the formalism of boundary triples and the corresponding operator-valued Weyl–Titchmarsh functions in [7, 13], we investigate the resolvent difference of two realizations and complement the results of Povzner [14], Birman [5], and Grubb [10] in this direction. In addition, assuming $A_{\min} > 0$, we compute the number of negative eigenvalues of a realization A_K and the number of eigenvalues of A_K within spectral gaps of the Dirichlet realization $\widehat{A}_{\gamma_D}$, where $\gamma_D = \{\gamma_j\}_0^{m-1}$.

Notations. $\mathfrak{H}$ and $\mathcal{H}$ represent complex, separable Hilbert spaces; $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_\infty(\mathcal{H})$, and $\mathcal{C}(\mathcal{H})$ denote the sets of bounded, compact, and closed linear operators in $\mathcal{H}$; $\mathrm{dom}(\cdot)$, $\mathrm{ran}(\cdot)$, and $\ker(\cdot)$ denote the domain, range, and kernel of a linear operator, $\rho(\cdot)$ and $\sigma(\cdot)$ stand for the resolvent set and spectrum of a linear operator. As usual, $C^\infty(\Omega)$ denotes the set of infinity differentiable functions in the domain Ω, $C_0^\infty(\Omega)$ the subset of $C^\infty(\Omega)$-functions of compact support in Ω; $C_b(\Omega) = C(\Omega) \cap L^\infty(\Omega)$, $C_u(\Omega)$ the set of uniformly continuous functions in Ω, $C_{ub}(\Omega) = C_u(\Omega) \cap C_b(\Omega)$, and $H^s(\Omega)$ the usual Sobolev spaces.

2. Dual pairs, boundary triples, and operator-valued Weyl–Titchmarsh functions

Definition 2.1. Let A and $A^\top$ be densely defined (not necessarily closed) linear operators in $\mathfrak{H}$. Then A and $A^\top$ form a *dual pair* (*DP*) $\{A, A^\top\}$ in $\mathfrak{H}$ if $(Af, g) = (f, A^\top g)$ for all $f \in \mathrm{dom}(A)$, $g \in \mathrm{dom}(A^\top)$. An operator $\widetilde{A}$ is called a *proper extension of the DP* $\{A, A^\top\}$, and we write $\widetilde{A} \in Ext\{A, A^\top\}$, if $A \subsetneqq \widetilde{A} \subsetneqq (A^\top)^*$.

Definition 2.2. (cf. [12], [13]) (i) Let $\mathfrak{H}$, $\mathcal{H}_0$, and $\mathcal{H}_1$ be complex, separable Hilbert spaces and

$$\Gamma^\top = \begin{pmatrix} \Gamma_0^\top \\ \Gamma_1^\top \end{pmatrix} : \mathrm{dom}((A^\top)^*) \to \mathcal{H}_0 \oplus \mathcal{H}_1 \ \text{ and } \ \Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \mathrm{dom}(A^*) \to \mathcal{H}_1 \oplus \mathcal{H}_0$$

be linear mappings. Then $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^\top, \Gamma\}$ is called a *boundary triple* for the dual pair $\{A, A^\top\}$ if $\Gamma^\top$ and Γ are surjective and the Green's identity holds,

$$\big((A^\top)^* f, g\big)_{\mathfrak{H}} - (f, A^* g)_{\mathfrak{H}} = \big(\Gamma_1^\top f, \Gamma_0 g\big)_{\mathcal{H}_1} - \big(\Gamma_0^\top f, \Gamma_1 g\big)_{\mathcal{H}_0}, \quad f \in \mathrm{dom}\big((A^\top)^*\big), \, g \in \mathrm{dom}(A^*).$$

We set $A_0 = (A^\top)^* \restriction \ker(\Gamma_0^\top)$ and $A_0^\top = A^* \restriction \ker(\Gamma_0)$.
(ii) The operator-valued function $M_\Pi(z)$ defined by

$$\Gamma_1^\top f_z = M_\Pi(z) \Gamma_0^\top f_z, \quad f_z \in \ker\big((A^\top)^* - z\big), \ \ z \in \rho(A_0),$$

is called the *Weyl–Titchmarsh function* corresponding to the boundary triple Π.

Due to Green's identity, $(A_{\min} u, v)_{L^2(\Omega)} = (u, A_{\min}^\top v)_{L^2(\Omega)}$, $u, v \in C_0^\infty(\Omega)$, the operators A and $A^\top$ form a dual pair $\{A, A^\top\}$ of elliptic operators in $L^2(\Omega)$. Any proper extension $\widetilde{A} \in \mathrm{Ext}\{A, A^\top\}$ of $\{A, A^\top\}$ is called a realization of $\mathcal{A}$. Clearly, any realization $\widetilde{A}$ of $\mathcal{A}$ is closable. We equip $\mathrm{dom}(A_{\max})$ and $\mathrm{dom}(A_{\max}^\top)$ with the corresponding graph norms. It is known (cf. [4, 11]) that if a domain Ω is bounded, then $\mathrm{dom}(A_{\min}) = \mathrm{dom}(A_{\min}^\top) = H_0^{2m}(\Omega)$, where the norms in $\mathrm{dom}(A_{\min})$ and $H_0^{2m}(\Omega)$ are equivalent. Denote by γ_j the mappings $\gamma_j \colon C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$, $\gamma_j u = \gamma_0(\partial^j u/\partial n^j) = \partial^j u/\partial n^j \restriction \partial\Omega$, $1 \leqslant j \leqslant m-1$, $\gamma_0 u = u \restriction \partial\Omega$, where n stands for the interior normal to $\partial\Omega$. Next we introduce the boundary operators B_j as

$$B_j u = \sum_{0 \leqslant |\beta| \leqslant m_j} b_{j\beta} \gamma_0(D^\beta u), \quad b_{j\beta}(\cdot) \in C^\infty(\partial\Omega), \quad \mathrm{ord}(B_j) = m_j \leqslant 2m-1. \tag{2.1}$$

Here $B_j : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ will eventually be extended to appropriate Sobolev spaces $H^s(\Omega)$ and in some cases to $\mathcal{D}(A_{\max})$. B_j in (2.1) can also be rewritten as $B_j u = b_j \gamma_{m_j} u \ + \ \sum_{0 \leqslant k \leqslant m_j - 1} T_{j,k} \gamma_k u$, where $b_j(\cdot) \in C^\infty(\partial\Omega)$ and $T_{j,k}$ are tangential differential operators in $\partial\Omega$ of orders $\mathrm{ord}(T_{j,k}) \leqslant m_j - k$ with $C^\infty(\partial\Omega)$-coefficients.

With any elliptic operator A (1.1) and a system $B = \{B_j\}_{j=1}^{m-1}$ we associate the operator $\widehat{A}_B$ defined by

$$\widehat{A}_B = A_{\max} \upharpoonright \operatorname{dom}(\widehat{A}_B), \quad \operatorname{dom}(\widehat{A}_B) = H^{2m}_B(\Omega) = \{u \in H^{2m}(\Omega) \,|\, Bu = 0\}. \tag{2.2}$$

Our considerations are based on [11, Thm. 2.2.1]. According to this result, for any elliptic differential expression $\mathcal{A}$ in (1.1) and any normal system $\{B_j\}_{j=0}^{m-1}$ on $\partial\Omega$ given by (2.1), there exists a system of boundary operators $\{C_j\}_{j=0}^{m-1}$, $\operatorname{ord}(C_j) = \mu_j \leqslant 2m-1$, such that the system $\{B_0, \dots, B_{m-1}, C_0, \dots, C_{m-1}\}$ is a Dirichlet system of order $2m$ and another Dirichlet system of boundary operators $\{B_j^\top\}_{j=0}^{m-1} \cup \{C_j^\top\}_{j=0}^{m-1}$, such that the following Green's formula hold

$$\big(\mathcal{A}u, v\big)_{L^2(\Omega)} - \big(u, \mathcal{A}^\top v\big)_{L^2(\Omega)} = \sum_{0 \leqslant l \leqslant m-1} \big[\big(C_j u, B_j^\top v\big)_{L^2(\partial\Omega)} - \big(B_j u, C_j^\top v\big)_{L^2(\partial\Omega)}\big], \quad u, v \in H^{2m}(\Omega). \tag{2.3}$$

Next, following [9] and [11], we introduce the spaces $D^s_A(\Omega) = \{u \in H^s(\Omega) \,|\, Au \in H^0(\Omega)\}$, $s \in \mathbb{R}$, provided with the graph norm $\|u\|_{D^s_A(\Omega)} = (\|u\|_s^2 + \|Au\|_0^2)^{1/2}$. Clearly, $D^0_A(\Omega) = \operatorname{dom}(A)$ and $D^s_A(\Omega) \hookrightarrow H^s(\Omega)$.

Definition 2.3. (i) The operator $\widehat{A}_B$ defined by (2.2) is called *elliptic* and is put in the class $\operatorname{Ell}(A)$ if A is properly elliptic on Ω and the system $\{B_j\}_{j=0}^{m-1}$ is normal and satisfies the covering condition (cf. [11, Sects 2.1.1–2.1.4]) at any point of the boundary $\partial\Omega$.
(ii) The operator $\widehat{A}_B$ is called *coercive* in $H^s(\Omega)$ with $s \geqslant 2m$ if the *a priori* estimate (2.25) in [2] (cf. also [11, Sect. 2.9.6]) holds.

We note that $\widehat{A}_B$ is a closed operator if B satisfies the covering condition (cf. [2, Sect. 6.5], [11, Thm. 2.8.4]).

If Ω is bounded, then any elliptic differential expression $\mathcal{A}$ with $C(\overline{\Omega})$-coefficients is uniformly elliptic in $\overline{\Omega}$. In this case, $\widehat{A}_B \in \operatorname{Ell}(A)$ if and only if $\widehat{A}_B$ is coercive in $H^{2m}(\Omega)$ (see [1], [11, Sect. 2.9.6]). If Ω is unbounded, then the condition $\widehat{A}_B \in \operatorname{Ell}(A)$ is still necessary for coerciveness in $H^{2m}(\Omega)$, though, it is no longer sufficient without additional assumptions on A.

Hypothesis 2.4. *Assume that $\mathcal{A}$ is a uniformly elliptic operator, $a_{\alpha,\beta}(\cdot) \in C_b(\Omega)$ for $|\alpha| + |\beta| \leqslant 2m$ and $a_{\alpha,\beta}(\cdot) \in C_{ub}(\Omega)$ for $|\alpha| + |\beta| = 2m$.*

Proposition 2.5. *Assume Hypothesis* 2.4*, $\widehat{A}_B \in \operatorname{Ell}(A)$, and $0 \in \rho(\widehat{A}_B)$. Then for any $s \in \mathbb{R}$, the mappings B and $B^\top$ isomorphically map $Z^s_A(\Omega) = \{u \in D^s_A(\Omega) \,|\, A_{\max}u = 0\}$ and $Z^s_{A^\top}(\Omega) = \{u \in D^s_{A^\top}(\Omega) \,|\, A^\top_{\max}u = 0\}$ isomorphically onto $\Pi_{j=1}^m H^{s-m_j-(1/2)}(\partial\Omega)$ and onto $\Pi_{j=1}^m H^{s-2m+\mu_j+(1/2)}(\partial\Omega)$, respectively.*

Definition 2.6. ([9, 15]) (i) Under the assumptions of Proposition 2.5, let $\varphi \in \Pi_{j=0}^{m-1} H^{s-m_j-(1/2)}(\partial\Omega)$, $s \in \mathbb{R}$. Then one defines $P(z)\varphi$ to be the unique $u \in Z^s_{A-zI_{L^2(\Omega)}}(\Omega)$ satisfying $Bu = \varphi$.
(ii) The *Calderon* operator $\Lambda(z)$ is defined by

$$\Lambda(z)\colon \Pi_{j=0}^{m-1} H^{s-m_j-(1/2)}(\partial\Omega) \to \Pi_{j=0}^{m-1} H^{s-\mu_j-(1/2)}(\partial\Omega), \quad \Lambda(z)\varphi = CP(z)\varphi. \tag{2.4}$$

(iii) Similarly, let $\psi \in \Pi_{j=0}^{m-1} H^{s-2m+\mu_j+1/2}(\partial\Omega)$. Then $P(z)^\top\psi$ is defined to be the unique solution in $Z^s_{A^\top - zI_{L^2(\partial\Omega)}}(\Omega)$ of $B^\top u = \psi$ and the *Calderon* operator $\Lambda(z)^\top$ is defined as $\Lambda(z)^\top\psi = C^\top P_z^\top \psi$.

Let $\Delta_{\partial\Omega}$ be the Laplace-Beltrami operator in $L^2(\partial\Omega)$, $-\Delta_{\partial\Omega,1} = -\Delta_{\partial\Omega} + I_{L^2(\partial\Omega)}$. Then $-\Delta_{\partial\Omega,1} = -\Delta^*_{\partial\Omega,1} \geq I_{L^2(\partial\Omega)}$. Moreover, $(-\Delta_{\partial\Omega,1})^{-s/2}$ isomorphically maps $H^0(\partial\Omega)$ onto $H^s(\partial\Omega)$, $s \in \mathbb{R}$. Next, we introduce the diagonal $m \times m$ operator matrices $-\Delta_{\partial\Omega,1,m}$ and $-\Delta_{\partial\Omega,1,\mu}$ with the (j,j)-th entry $(-\Delta_{\partial\Omega,1})^{(m_j/2)+(1/4)}$ (resp., $(-\Delta_{\partial\Omega,1})^{m-(\mu_j/2)-(1/4)}$).

Proposition 2.7. *Assume Hypothesis* 2.4*, $\widehat{A}_B \in \operatorname{Ell}(A)$, and $0 \in \rho(\widehat{A}_B)$. Set*

$$\Gamma_{\Omega,0}u = (-\Delta_{\partial\Omega,1,m})^{-1}Bu, \quad \Gamma_{\Omega,1}u = (-\Delta_{\partial\Omega,1,\mu})(Cu - \Lambda(0)Bu), \quad u \in \operatorname{dom}(A_{\max}), \tag{2.5}$$

$$\Gamma^\top_{\Omega,0}v = (-\Delta_{\partial\Omega,1,\mu})^{-1}B^\top v, \quad \Gamma^\top_{\Omega,1}v = \big(-\Delta_{\partial\Omega,1,m})(C^\top v - \Lambda(0)^\top B^\top v\big), \quad v \in \operatorname{dom}(A^\top_{\max}). \tag{2.6}$$

Then the following holds:
(i) $\Pi = \{\mathcal{H}_{\partial\Omega} \oplus \mathcal{H}_{\partial\Omega}, \Gamma_\Omega, \Gamma_\Omega^\top\}$, *with*

$$\mathcal{H}_{\partial\Omega} = \Pi_{j=0}^{m-1} H^0(\partial\Omega) = \Pi_{j=0}^{m-1} L^2(\partial\Omega), \quad \Gamma_\Omega = (\Gamma_{\Omega,0}, \Gamma_{\Omega,1}), \quad \Gamma_\Omega^\top = (\Gamma_{\Omega,0}^\top, \Gamma_{\Omega,1}^\top),$$

forms a boundary triple for the dual pair $\{A, A^\top\}$ *of elliptic operators in* $L^2(\Omega)$*. In particular, the following Green's formula holds*

$$(A_{\max}u, v)_{L^2(\Omega)} - (u, A_{\max}^\top v)_{L^2(\Omega)} = (\Gamma_{\Omega,1}u, \Gamma_{\Omega,0}^\top v)_{\mathcal{H}_{\partial\Omega}} - (\Gamma_{\Omega,0}u, \Gamma_{\Omega,1}^\top v)_{\mathcal{H}_{\partial\Omega}}, \quad u \in \operatorname{dom}(A_{\max}),\, v \in \operatorname{dom}(A_{\max}^\top).$$

(ii) The corresponding operator-valued Weyl–Titchmarsh function is given by

$$M_{\Omega,\Pi}(z) = (-\Delta_{\partial\Omega,1,\mu})\big(\Lambda(z) - \Lambda(0)\big)(-\Delta_{\partial\Omega,1,m}), \quad z \in \rho(\widehat{A}_B).$$

In the context of operator-valued Weyl–Titchmarsh functions and elliptic partial differential operators we also refer to the recent preprint [6] (and the references cited therein).

Definition 2.8. For any operator $K\colon \operatorname{dom}(K) \to \Pi_{j=0}^{m-1} H^{-\mu_j-(1/2)}(\partial\Omega)$, $\operatorname{dom}(K) \subseteq \Pi_{j=0}^{m-1} H^{-m_j-(1/2)}(\partial\Omega)$, we set

$$A_K = A_{\max} \restriction \operatorname{dom}(A_K), \quad \operatorname{dom}(A_K) = \{u \in \operatorname{dom}(A_{\max}) \,|\, Bu \in \operatorname{dom}(K),\ Cu = KBu\}. \tag{2.7}$$

Definition 2.9. Define $\mathcal{S}_p(\mathfrak{H}) = \{T \in \mathcal{B}_\infty(\mathfrak{H}) \,|\, s_j(T) = O(j^{-1/p}) \text{ as } j \to \infty\}$, $p > 0$, where $s_j(T)$, $j \in \mathbb{N}$, denote the singular values of T (i.e., the eigenvalues of $(T^*T)^{1/2}$ ordered in decreasing magnitude, counting multiplicity).

Theorem 2.10. *Assume the conditions of Proposition* 2.7 *and suppose that* $0 \in \rho(\widehat{A}_C)$ *and* $K \in \mathcal{C}(\mathcal{H}_{\partial\Omega})$*. Then:*
(i) For any realization $A_K \in \mathcal{C}(L^2(\Omega))$ *of the form* (2.7)*, satisfying* $\rho(A_K) \cap \rho(\widehat{A}_B) \neq \emptyset$*, the following holds,*

$$\big[(A_K - zI_{L^2(\Omega)})^{-l} - (\widehat{A}_B - zI_{L^2(\Omega)})^{-l}\big] \in \mathcal{S}_{\frac{n-1}{2ml-1/2}}\big(L^2(\Omega)\big), \quad z \in \rho(A_K) \cap \rho(\widehat{A}_B), \quad \ell \in \mathbb{N}. \tag{2.8}$$

(ii) If $B = \{B_j\}_{j=0}^{m-1}$ *is a Dirichlet system,* $K \in \mathcal{B}(\mathcal{H}_{\partial\Omega})$*, and* $\rho(A_K) \cap \rho(\widehat{A}_B) \neq \emptyset$*, one has*

$$\big[(A_K - zI_{L^2(\Omega)})^{-1} - (\widehat{A}_B - zI_{L^2(\Omega)})^{-1}\big] \in \mathcal{S}_{\frac{n-1}{2m}}\big(L^2(\Omega)\big), \quad z \in \rho(A_K) \cap \rho(\widehat{A}_B).$$

Combining Weyl's theorem with Theorem 2.10 one obtains the following result:

Corollary 2.11. *Assume the conditions of Theorem* 2.10*. Then,* $\sigma_{\rm ess}(A_K) = \sigma_{\rm ess}(\widehat{A}_B)$*.*

In the case of elliptic realizations $\widehat{A}_G \in \operatorname{Ell}(A)$, we have the following stronger result:

Theorem 2.12. *Suppose that the conditions of Proposition* 2.7 *are satisfied and* $\widehat{A}_G \in \operatorname{Ell}(A)$*, that is,* $\widehat{A}_G$ *is the elliptic realization of* A *with* $G = \{G_j\}_{j=0}^{m-1}$*. Then for any* $\ell \in \mathbb{N}$*,*

$$\big[(\widehat{A}_G - zI_{L^2(\Omega)})^{-\ell} - (\widehat{A}_B - zI_{L^2(\Omega)})^{-\ell}\big] \in \mathcal{S}_{\frac{n-1}{2m\ell}}(L^2(\Omega)), \quad z \in \rho(\widehat{A}_G) \cap \rho(\widehat{A}_B). \tag{2.9}$$

3. The formally self-adjoint case, nonnegative elliptic operators, and eigenvalues in gaps

Let $\mathcal{A}$ be a formally self-adjoint elliptic differential expression of the form (1.1), that is, $\mathcal{A} = \mathcal{A}^\top$ or equivalently, $a_{p,q} = \overline{a_{q,p}} \in C^\infty(\overline{\Omega})$. In this case $A = A_{\min} = A_{\min}^\top = A^\top$, that is, A is symmetric, and $A_{\max} = (A_{\min}^\top)^* = A^*$. If a normal system $\{B_j\}_{j=0}^{m-1}$ is chosen to be formally self-adjoint, that is, $\hat{A}_B = (\widehat{A}_B)^*$, then a system $\{C_j\}_{j=0}^{m-1}$ can be chosen formally self-adjoint too. In this case $B_j^\top = B_j$ and $C_j^\top = C_j$, and the Green's formula (2.3) holds with $B_j^\top = B_j$ and $C_j^\top = C_j$. Moreover, in this case, $\mu_j = \operatorname{ord}(C_j) = \operatorname{ord}(C_j^\top) = 2m-1-m_j$. It follows that $\Delta_{\partial\Omega,1,\mu} = \Delta_{\partial\Omega,1,m}$. Hence, Proposition 2.7 yields the following result:

Proposition 3.1. *Let* $\mathcal{A}$ *be a formally symmetric elliptic differential expression and assume that* $\widehat{A}_B$ *and* $\widehat{A}_C$ *are self-adjoint. In addition, assume the conditions of Propositon* 2.7 *are satisfied. Then:*
(i) $\Pi = \{\mathcal{H}_{\partial\Omega}, \Gamma_0, \Gamma_1\}$ *with* $\mathcal{H}_{\partial\Omega} = \Pi_{j=0}^{m-1} L^2(\partial\Omega)$*, and* $\Gamma_{\Omega,0}$, $\Gamma_{\Omega,1}$ *defined by* (2.5)*, forms a boundary triple for the operator* A^**. In particular, the following Green's formula holds*

$$(A_{\max}u, v)_{L^2(\Omega)} - (u, A_{\max}v)_{L^2(\Omega)} = (\Gamma_{\Omega,1}u, \Gamma_{\Omega,0}v)_{\mathcal{H}_{\partial\Omega}} - (\Gamma_{\Omega,0}u, \Gamma_{\Omega,1}v)_{\mathcal{H}_{\partial\Omega}}, \quad u, v \in D(A_{\max}).$$

(ii) The corresponding Weyl–Titchmarsh operator is given by $M_{\Omega,\Pi}(z) = (-\Delta_{\partial\Omega,1,m})\big(\Lambda(z) - \Lambda(0)\big)(-\Delta_{\partial\Omega,1,m})$.

For any self-adjoint operator $T = T^* \in \mathcal{C}(\mathfrak{H})$ with associated family of spectral projections $E_T(\cdot)$, we set $\kappa_{(\alpha,\beta)}(T) = \dim(E_T((\alpha,\beta))\mathfrak{H})$, $-\infty \le \alpha < \beta$ (these numbers may of course be infinite).

Theorem 3.2. *Suppose that $A > 0$ is a positive definite elliptic operator, and $\Pi = \{\mathcal{H}_{\partial\Omega}, \Gamma_0, \Gamma_1\}$ is the boundary triple for A^* in Proposition 3.1 with $A_0 := A^* \upharpoonright \ker(\Gamma_0) = \widehat{A}_{\gamma_D}$, the Dirichlet realization of A. Assume also that the operator $\widehat{A}_C > 0$ is positive definite, $0 \in \rho(\widehat{A}_C)$. Let K be a densely defined (not necessarily closed) symmetric operator in $\mathcal{H}_{\partial\Omega}$ and A_K the corresponding extension defined by (2.7). Then:*
(i) The Calderon operator $\Lambda(0)$ is self-adjoint and negative definite, $\Lambda(0) < 0$.
(ii) If K is $\Lambda(0)$-bounded with bound strictly less than one, then A_K is symmetric (but not necessarily closed). If in addition, $\operatorname{ran}(K - \Lambda(0))$ is closed, then so is A_K, that is, $A_K \in \mathcal{C}(L^2(\Omega))$.
(iii) If K is $\Lambda(0)$-compact and self-adjoint, then A_K is self-adjoint $A_K = (A_K)^*$, $\kappa_{(-\infty,0)}(A_K) < \infty$, and

$$\kappa_{(-\infty,0)}(A_K) = \kappa_{(-\infty,0)}\big(I_{L^2(\partial\Omega)} + (-\Lambda(0))^{-1/2} K (-\Lambda(0))^{-1/2}\big). \tag{3.1}$$

(iv) If K is $\Lambda(0)$-compact and sectorial (resp., m-sectorial) with vertex ζ and semi-angle $\omega \in [0, \pi/2)$, then A_K is sectorial (resp., m-sectorial) with vertex ζ and semi-angle ω too.

Proposition 3.3. *Let $\mathcal{A}$ be formally self-adjoint and assume the conditions of Proposition 3.1. Assume also that $A_K = (A_K)^*$ is a self-adjoint extension of the form (2.7) with $K \in \mathcal{C}(\mathcal{H})$. Then the absolutely continuous parts $A_{K,\mathrm{ac}}$ and $\widehat{A}_{B,\mathrm{ac}}$ of A_K and $\widehat{A}_B$, respectively, are unitarily equivalent. In particular, $\sigma_{\mathrm{ac}}(A_K) = \sigma_{\mathrm{ac}}(\widehat{A}_B)$.*

Proposition 3.4. *Suppose that $A = A_{\min}$ is symmetric, and let $\widehat{A}_B = \widehat{A}_{\gamma_D}$ be the Dirichlet realization of A. Assume the conditions of Theorem 2.12 to be satisfied and that $\widehat{A}_G = (\widehat{A}_G)^* \in \mathrm{Ell}(A)$ is an elliptic realization of A with $G = \{G_j\}_0^{m-1}$. Then the absolutely continuous parts $\widehat{A}_{G,\mathrm{ac}}$ and $\widehat{A}_{\gamma_D,\mathrm{ac}}$ of $\widehat{A}_G$ and $\widehat{A}_{\gamma_D}$, respectively, are unitarily equivalent. In particular, $\sigma_{\mathrm{ac}}(\widehat{A}_G) = \sigma_{\mathrm{ac}}(\widehat{A}_{\gamma_D})$.*

Finally, we turn to eigenvalues in spectral gaps:

Definition 3.5. Let A be a symmetric operator in $\mathcal{H}$. Then (α,β), $-\infty < \alpha < \beta < \infty$, is called a *gap of A* if $\|(2A - (\alpha+\beta)I_{\mathcal{H}})f\|_{\mathcal{H}} \geqslant (\beta - \alpha)\|f\|_{\mathcal{H}}$ for all $f \in \mathrm{dom}(A)$.

By Corollary 2.11, $\sigma_{\mathrm{ess}}(A_K) = \sigma_{\mathrm{ess}}(\widehat{A}_B)$. Therefore, in the gaps of $\widehat{A}_B$, the point spectrum of A_K can possibly accumulate at most at the endpoints of the gaps. Next, we actually show that $\sigma_{\mathrm{p}}(A_K)$ cannot accumulate at the left end point of any gap:

Theorem 3.6. *Suppose that the conditions of Theorem 3.2 are satisfied, and that K is a symmetric $\Lambda(0)$-compact operator in $\mathcal{H}_{\partial\Omega}$. In addition, let (α,β) be a finite gap of $A_0 = \widehat{A}_{\gamma_D}$ and introduce $T_0(z) = \Lambda(z) - \Lambda(0)$. Then:*
(i) $T(z) = \overline{T_0(z)} \in \mathcal{B}_\infty(\mathcal{H}_{\partial\Omega})$ for all $z \in \rho(\widehat{A}_{\gamma_D})$.
(ii) There exists $\varepsilon_0 \in (0, (\beta-\alpha)/2)$ such that $E_{A_K}((\alpha, \alpha+\varepsilon_0)) = 0$, hence $\kappa_{(\alpha,\beta-\varepsilon)}(A_K) = \dim(E_{A_K}((\alpha, \beta - \varepsilon))) < \infty$ for any $\varepsilon \in (0, \beta-\alpha)$. Moreover, for any $\varepsilon \in (0,\varepsilon_0)$ the following equality holds (with $\Lambda := \Lambda(0)$):

$$\kappa_{(\alpha,\beta-\varepsilon)}(A_K) = \kappa_{(-\infty,0)}(I_{\mathcal{H}_{\partial\Omega}} + \Lambda^{-1/2}(K + T(\beta-\varepsilon))\Lambda^{-1/2}) - \kappa_{(-\infty,0)}(I_{\mathcal{H}_{\partial\Omega}} + \Lambda^{-1/2}(K + T(\alpha+\varepsilon))\Lambda^{-1/2}).$$

Remark 3.7. For Robin-type realizations $[\partial u/\partial n - \sigma u] \upharpoonright \partial\Omega = 0$, $\sigma \in L^\infty(\partial\Omega)$, of Schrödinger operators $-\Delta + q$ on exterior domains $\Omega \subset \mathbb{R}^3$, the estimate (2.9) (with $\ell = 1$) goes back to the pioneering work by Povzner [14]. For Robin realizations A_σ of a second-order elliptic operator $\mathcal{A} = -\sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{j,k}(x) \frac{\partial}{\partial x_j} + q(x)$, with $q \ge 1$, and $\sum_{j,k=1}^n \xi_j a_{j,k}(x)\xi_k > 0$ for all $\{x,\xi\} \in \Omega \times (\mathbb{R}^n \setminus \{0\})$,

$$A_\sigma = A_{\max} \upharpoonright \mathrm{dom}(A_\sigma),\ \mathrm{dom}(A_\sigma) = \{u \in H^2(\Omega) \,|\, (\partial u/\partial\nu - \sigma u) \upharpoonright \partial\Omega = 0\},\ \partial/\partial\nu = \sum_{j,k=1}^n a_{j,k}(x)\cos(n, x_j)\frac{\partial}{\partial x_k},$$

$\sigma \in L^\infty(\partial\Omega)$, the estimate (2.9) was obtained by Birman [5]. Moreover, in [5, Thm. 6.6] it is also proved that $\kappa_{(-\infty,0)}(A_\sigma) < \infty$. Thus, for $m = 1$ and $A_K = A_\sigma$, equality (3.1) with K being a multiplication operator, $K\colon u \mapsto \sigma u$, yields a stronger result as it describes the actual number of eigenvalues in the gap $(-\infty, 0)$.

For positive elliptic realizations $\widehat{A}_G$ of a nonnegative elliptic operator A of order $2m$ in a bounded domain $\Omega \subset \mathbb{R}^n$, the estimate (2.9) is implied by a sharp estimate due to Grubb [10, eq. (3.22)].

Detailed proofs of these results will appear in [8].

Acknowledgments. Mark Malamud is indebted to the Department of Mathematics of the University of Missouri, Columbia, MO, USA, for the hospitality extended to him during his month long stay in April/May of 2007 in connection with a Miller Family Scholarship, which enabled this collaboration.

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