On some bound and scattering states associated with the cosine kernel

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January 3, 2008

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Abstract

It is explained how to provide self-adjoint operators having scattering states forming a multiplicity one continuum and bound states whose corresponding eigenvalues have an asymptotic density equivalent to the one of the zeros of the Riemann zeta function. It is shown how this can be put into an integro-differential form of a type recently considered by Sierra.

1 Introduction

The present publication was motivated by a recent investigation of Sierra [23] (also [21, 22]). It will be explained how the differential system and isometric expansion we obtained in [7], and the associated self-adjoint operators, provide, when moved back to the "x-picture" a particular example of a type of differential-integral equation considered in [23, section IV].

In [6] we determined "explicitely" certain objects depending on a parameter $a > 0$ and associated to the cosine (or sine) kernel. In [7] we obtained the differential system (which turned out to be of the Dirac type with coefficients involving some Fredholm determinants of the cosine (or sine) kernel) obeyed by the objects from [6] as functions of $\log(a)$. We mentioned the associated isometric spectral expansion and also explained how this investigation had led to the realization of the Fourier transform as a scattering, an objective which had resulted from our definition and study of the "$\log|x| + \log|p|$" operator ([4]).

In [10] we gave a detailed exposition of these results. Some general aspects could be predicted in advance as they involve some Hilbert spaces of entire functions in the sense of [3]. The specific spaces had actually already be defined in [2, 20], among spaces associated to general Hankel transforms. The results from [2, 20] regarded mainly the spaces associated to the Bessel function J_0. In [6] we obtained for the cosine and sine kernels the "explicit" form taken by some functions whose existence was a consequence of the general theory of [3]: in particular certain entire functions $\mathcal{A}_a(s)$ and $\mathcal{B}_a(s)$ having all their zeros ρ on a line, and such that the quotients $\frac{\mathcal{A}_a(s)}{s-\rho}$ (resp. $\frac{\mathcal{B}_a(s)}{s-\rho}$) give orthogonal bases of some Hilbert space K_a. It was then shown in [7] that these objects obey differential equations whose coefficients are expressed in terms of certain Fredholm determinants. This observation also applied to the J_0 kernel and we gave the detailed exposition of these results in [10], where the chapters V and VI with minor adaptations apply to all Hankel kernels, in particular to the cosine and sine kernels. A further advance had been realized in the theory of the J_0 kernel and associated spaces whose nature was revealed to be special relativistic [9]. Other Hilbert spaces of entire functions, directly associated with the Riemann zeta function and which had not been considered before, were defined by the author in [5] and shortly thereafter understood to be in connection with the "co-Poisson" formula.

In [23], Sierra considers integro-differential operators which are singular rank two perturbations of $-i(x\frac{d}{dx} + \frac{1}{2})$ on $L^2(0, +\infty; dx)$. We will explain how the objects, differential equations, and spectral expansion from [6, 7, 10], when seen in the "x picture" lead to the kind of equations considered by Sierra. As was mentioned in a number of our other papers (such as [8]) the theory leads to "bound states" associated to zeros on the critical line which have an asymptotic density equivalent to the one applying to the zeros of the Riemann zeta function. These bound states are created on an interval $(\log(a), +\infty)$ by a Dirichlet condition at $\log(a)$ and a potential function which has an exponential increase at $+\infty$. The Dirac system seen on $(-\infty, \log(a))$ leads to a purely continuous spectrum, because the potential vanishes exponentially quickly at $-\infty$. Hence we do have here the combination of scattering and bound states mentioned in section IV of [23] as a possible property of the integro-differential equations considered there. It is not clear if the particular example which is considered in section VI of [23] truly leads to bound states, or rather only to resonances. On the other hand, the Dirac and Schrödinger equations from [7] do have, as we mentioned in our publications, this property of leading to "quantum zeros" lying on the critical line with the expected density (a general density result has been given in [8], see Theorem 7.7 and Remark 19).

We shall not enter here into other topics considered by Sierra, such as obtaining operators leading exactly to the zeros of Riemann.

2 Some notations

For the sake of facilitating comparisons with [23] we shall often write $s = \frac{1}{2} + iE$ when using complex numbers although the letter E is also used with a completely different meaning. Also, our scalar products $(\cdot\,|\,\cdot)$ will be conjugate linear in the left entry (bra) and complex linear in the right entry (ket). As we refer to somewhat lengthy developments from our previous papers we did not modify our notations too much, in particular with respect to the use of the letters A and B. We will not here attempt to analyse our final equations in the spirit of [23, IV] but only want to show how they emerge from our framework, so the incompatibility of notations will remain virtual.

The Mellin transform is given by the formula $\widehat{f}(\frac{1}{2} + iE) = \int_0^\infty f(x) \frac{x^{-iE}}{\sqrt{x}}\,dx = \int_0^\infty f(x) x^{-s}\,dx$ (and not with $+iE$) because this represents the scalar product $(\psi_E^0|f)$, of f with the generalized eigenvector $\psi_E^0(x) = \frac{x^{iE}}{\sqrt{x}}$ of the "free" operator $H_0 = -i(x\frac{d}{dx} + \frac{1}{2})$, $H_0(\psi_E^0) = E\psi_E^0$. The inverse Mellin transform is given by the formula $f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{f}(\frac{1}{2} + iE) \frac{x^{iE}}{\sqrt{x}}\,dE = \frac{1}{2\pi} \int_{\Re(s)=\frac{1}{2}} \widehat{f}(s) x^{s-1}\,|ds|$.

Regarding the Fourier transform, it will be used in the form of the cosine transform $\mathcal{C}(f)(y) = \int_0^{+\infty} 2\cos(2\pi xy) f(x) dx$, for functions or distributions supported in $[0, +\infty)$. The cosine kernel $2\cos(2\pi xy)$ has only the two eigenvalues $+1$ and -1 on $(0, \infty)$. When restricted to a finite interval $(0, a)$ it has a discrete spectrum $1 > |\lambda_1(a)| \geq |\lambda_2(a)| \geq \dots$ which accumulates at zero. This restriction is conjugated to, hence has the same spectrum as, the kernel $C_a(x, y) = 2a\cos(2\pi a^2 xy)$ acting on the interval $(0, 1)$. We will make use of the Fredholm determinants $\det(1 \pm C_a)$. Some of their properties (in particular the asymptotic as $a \to +\infty$) can be derived from knowledge of the corresponding facts for $\det(1 - C_a^2)$. As the square C_a^2 is the Dirichlet kernel $\frac{\sin(2\pi a^2(x-y))}{\pi(x-y)}$ acting on the even subspace of $L^2(-1, +1; dx)$, the question is reduced to the finite Dirichlet kernel, whose properties have been extensively studied in the literature in connection with random matrices: see [17] and the further references included in our papers [7, 10].

3 A support condition and associated Hilbert spaces

To each square integrable function f on $(0, +\infty)$, one can associate its Mellin transform $\widehat{f}(s) = \int_0^\infty f(x) x^{-s} dx$ on the critical line $\Re(s) = \frac{1}{2}$. This Mellin-Plancherel transform is a unitary identification of $L^2(0, +\infty; dx)$ with $L^2(\Re(s) = \frac{1}{2}; \frac{|ds|}{2\pi})$. The closed subspace $L^2(1, +\infty; dx)$ (resp. $L^2(0, 1; dx)$) becomes identified with (boundary values of the functions of) the Hardy space of the half-plane $\Re(s) > \frac{1}{2}$ (resp. $\Re(s) < \frac{1}{2}$). Let us recall how the cosine transform $g(y) = \mathcal{C}(f)(y) = \int_0^{+\infty} 2\cos(2\pi xy) f(x) dx$ is represented on the Mellin side. The formula is:

$$\widehat{\mathcal{C}(f)}(s) = \chi(s)\widehat{f}(1 - s) \qquad \text{where} \qquad \chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \tag{1}$$

This is the same $\chi(s)$,[1] of modulus one on the critical line, which appears in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ of the Riemann zeta function. Indeed this functional equation can be seen to be a manifestation of the self-invariance under the cosine transform of the distribution $\sum_{n\geq 1}\delta(x-n) - \mathbf{1}_{x>0}(x)$, whose Mellin transform is nothing else but $\zeta(s)$ itself.

Of course $\zeta(s)$ is not square integrable on the critical line but the functions $s \mapsto \frac{\zeta(s)}{s-\rho}$ are, with ρ a zero of the zeta function. Let us consider the vectors $\zeta_{\rho,k}(s) = \frac{\zeta(s)}{(s-\rho)^k}$ where ρ is a non-trivial zero of zeta, and k is an integer between 1 and the multiplicity of ρ. Taking inverse Mellin transforms we also tacitly consider the $\zeta_{\rho,k}$ to be vectors in $L^2(0,+\infty;dx)$. The following holds ([8]):

Theorem 1. *The vectors $\zeta_{\rho,k}$ are linearly independent in the sense that none is in the closure of the linear combinations of the others. Furthermore the smallest closed subspace of $L^2(0,+\infty;dx)$ containing these vectors consists exactly of those functions f in L^2 which are constant on $(0,1)$ and whose cosine transforms are also constant on $(0,1)$.*

We mentioned this theorem as motivation to study the space K_1 of functions f in L^2 which vanish identically on $(0,1)$ and whose cosine transform also vanish identically on that same interval (the condition of vanishing is slightly easier than the one of being constant). More generally we can consider the space K_a where the interval $(0,1)$ is replaced with the interval $(0,a)$ (we would have been led directly to these spaces if we had considered the functions $\frac{L(s,\chi)}{s-\rho}$ where χ is an even Dirichlet character):

Definition 2. We let, for each $a > 0$, K_a be the subspace of $L^2(0,+\infty;dx)$ consisting of functions which vanish identically on $(0,a)$ and whose cosine transforms also vanish identically on that same interval.

The subspace K_a is closed and its orthogonal complement is the (non-orthogonal, but direct and closed) sum $L^2(0,a;dx) + \mathcal{C}(L^2(0,a;dx))$.

Proposition 3. *If $f \in K_a$ then $\widehat{f}(s)$ is an entire function which has trivial zeros at $s = 0, -2, -4, -6, \ldots$*

Indeed $\widehat{f}(s)$ as Mellin transform of f which is supported in $[a,+\infty)$ is analytic in the half-plane $\Re(s) > \frac{1}{2}$, and its boundary values on the line are the Mellin-Plancherel transform of f (taken a priori in the L^2-sense). The same applies to $g = \mathcal{C}(f)$ and on the critical line one has the almost everywhere identity $\widehat{g}(s) = \chi(s)\widehat{f}(1-s)$. So the function $k(s) = \chi(s)\widehat{g}(1-s)$ is analytic in the half-plane $\Re(s) < \frac{1}{2}$, and its boundary values from the left coincides with the boundary values of $\widehat{f}$ from the right. From general principles of complex analysis ("edge-of-the-wedge" theorems) the two

[1] let us observe in passing $\chi(s)\chi(1-s) = 1$ and using formally $f(x) = \delta(x-1)$ that $\chi(s) = \int_0^\infty 2\cos(2\pi x)x^{-s}\,dx$. This is valid as a semi-convergent integral, and also as an identity of tempered distributions in $\log(x)$ and $\Im(s)$ ($\Re(s) = \frac{1}{2}$).

analytic functions "glue" to give an entire function. It would be easy to give in this one-dimensional case more details for this glueing, but anyhow various simple elementary constructive proofs of the Proposition are available, such as the one given in [5].[2] The trivial zeros are a corollary of the identity $\widehat{\mathcal{C}(f)}(s) = \chi(s)\widehat{f}(1-s)$, because $\widehat{f}(1-s)$ must have the zeros to compensate the poles of $\chi(s)$ at 1, 3, 5, ...

The spaces K_a, mainly as spaces of the entire functions $\mathcal{F}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\widehat{f}(s)$ (but seen as functions of the variable z such that $s = \frac{1}{2} - iz$), had appeared earlier in the literature: in de Branges investigation [2] where they are studied as a special instance of spaces verifying the three axiomatic properties[3] whose general consequences are the subject of the book [3]. The well-known Paley-Wiener spaces (of Fourier transforms of L^2 functions of compact support) are another, simpler, instance of the general theory of [3]. A particularly interesting (and challenging) aspect of the spaces K_a is that the associated entire functions are not of finite exponential type. The reader interested generally speaking in [3] is also referred to [14] and to [13] and [19], especially for the rather intimate connection with standard aspects of spectral expansions associated to differential equations of the Sturm-Liouville or Schrödinger types.

In the process in [6] and [7] of studying the spaces K_a directly we have in particular recovered by other means the consequences predicted by the general theory of [3], in an "explicit" form. In this section and the next we briefly review some of these results.

The evaluation maps $f \mapsto \widehat{f}(s)$ are continuous linear forms on K_a, they thus correspond to certain vectors in K_a indexed by the complex numbers. The space K_a is "elucidated" once the mutual scalar products between these evaluators are computed. Let $\mathcal{Z}_z$ be the entire function such that $\mathcal{F}(z) = (\mathcal{Z}_z|\mathcal{F})$ for all completed Mellin transforms $\mathcal{F}$ of elements of K_a (we may consider $\mathcal{Z}_z$ either as an element of $K_a \subset L^2(0,\infty;dx)$ or in its incarnation as an entire function $\mathcal{Z}_z(s)$; the dependency on z is anti-analytic). Then, from [3] it is known that there must exist an entire function $\mathcal{E}_a$ (not quite but almost unique) such that:[4]

$$\mathcal{Z}_z(w) = (\mathcal{Z}_w|\mathcal{Z}_z) = \frac{\overline{\mathcal{E}_a(z)}\mathcal{E}_a(w) - \mathcal{E}_a(\overline{1-z})\overline{\mathcal{E}_a(\overline{1-w})}}{\overline{z} + w - 1} \tag{2}$$

Such an $\mathcal{E}_a$ once found also characterizes the space of the entire functions $\mathcal{F}(s)$ as follows: the ratio $\frac{\mathcal{F}}{\mathcal{E}_a}$ must belong to the Hardy space of the half-plane $\Re(s) > \frac{1}{2}$ and also the ratio $\frac{\mathcal{G}}{\mathcal{E}_a}$ where $\mathcal{G}(s) = \overline{\mathcal{F}(\overline{1-s})}$. The formula $\|f\|_2 = \|\frac{\mathcal{F}}{\mathcal{E}_a}\|_{\mathbb{H}^2}$ holds, that is, somewhat surprisingly:

$$\frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\frac{1}{2}+iE)|^2\,dE = \frac{1}{2\pi}\int_{\mathbb{R}} |\mathcal{F}(\frac{1}{2}+iE)|^2\,\frac{dE}{|\mathcal{E}_a(\frac{1}{2}+iE)|^2} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|\widehat{f}(\frac{1}{2}+iE)|^2}{|\widehat{E_a}(\frac{1}{2}+iE)|^2}\,dE \tag{3}$$

[2] in [5] we use $\int_0^\infty f(x)x^{s-1}\,dx$ whereas systematically in later references we always use $\int_0^\infty f(x)x^{-s}\,dx$. Hence the s from [5] is the $1-s$ from here.

[3] not needed here, see [3], and for the K_a spaces [2, 20] and [6, 8, 10].

[4] we replace here the horizontal line always used in [3] by the critical line $\Re(s) = \frac{1}{2}$. Hence formula (2) does not look as in [3].

We have introduced here the notation $\widehat{E_a}(s) = \mathcal{E}_a(s)/\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and indeed this proves useful when studying the spaces K_a.

The reader is referred to [3] and [14] for the general proof (which is short) of existence of such "$\mathcal{E}$-functions" when the axioms of [3] are verified. From the evaluator identity one deduces that $|\mathcal{E}_a(s)|^2 > |\mathcal{E}_a(\overline{1-s})|^2$ when $\Re(s) > \frac{1}{2}$, so all zeros of $\mathcal{E}_a(s)$ verify $\Re(s) \leq \frac{1}{2}$. A zero on the symmetry line would mean that all functions $\mathcal{F}(s)$ vanish there. It can be shown that this is not the case, so the zeros of $\mathcal{E}_a(s)$ in fact verify $\Re(s) < \frac{1}{2}$. [5]

The author has contributed the "explicit" determination of a suitable function $\mathcal{E}_a$ [6]. It is realized as $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\widehat{E_a}(s)$ where $E_a(x)$ is in fact a distribution supported on $[a, +\infty)$ and whose cosine transform is also supported on $[a, +\infty)$. To obtain E_a, we studied the vectors $X_s(x)$ which are such that $\int_a^\infty f(x)X_s(x)\,dx = \widehat{f}(s)$ for $f \in K_a$. For $\Re(s) > \frac{1}{2}$ the vector X_s is simply the orthogonal projection to K_a of $x^{-s}\mathbf{1}_{x>a}(x)$. We determined directly the integrals $\int_a^\infty X_z(x)X_w(x)\,dx$. A completely detailed exposition with all proofs has been given in our manuscript [10]. As explained there, the reader will only need to replace all occurences of the kernel $J_0(2\sqrt{xy})$ by the kernel $2\cos(2\pi xy)$, and also to replace all occurences of $\Gamma(s)$ by $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. The chapters of [10] which are related to the present publication are the fifth and the sixth. The simplifications allowed by the kernel $J_0(2\sqrt{xy})$ are made use of only in other chapters of [10].

The formula obtained in [6] is:

Theorem 4. *Let $\phi_a^\pm$ be the even entire functions solving:*

$$\phi_a^+(x) + \int_0^a 2\cos(2\pi xy)\phi_a^+(y)\,dy = 2\cos(2\pi ax) \tag{4a}$$

$$\phi_a^-(x) - \int_0^a 2\cos(2\pi xy)\phi_a^-(y)\,dy = 2\cos(2\pi ax) \tag{4b}$$

The function $\widehat{E_a}(s)$ given as

$$\widehat{E_a}(s) = \sqrt{a}\left(a^{-s} + \frac{1}{2}\int_a^\infty (\phi_a^+(x) - \phi_a^-(x))x^{-s}\,dx\right) \tag{5}$$

is an entire function and is such that for all $z, w \in \mathbb{C}$

$$\widehat{X_z^a}(w) = \widehat{X_w^a}(z) = \int_a^\infty X_z^a(x)X_w^a(x)\,dx = \frac{\widehat{E_a}(z)\widehat{E_a}(w) - \widehat{\mathcal{C}(E_a)}(z)\widehat{\mathcal{C}(E_a)}(w)}{z+w-1} \tag{6}$$

Equation (6) looks different from (2) because we dropped Gamma factors and also because we are not using the Hilbert scalar product to avoid anti-analyticity. Equations (4a), (4b) are first considered in $L^2(0, a; dx)$. They have solutions because ± 1 are not in the spectrum of the finite

[5] Lagarias has considered in [15] an entire function which has the properties of an $\mathcal{E}$-function à la de Branges if and only if the Riemann hypothesis holds. Lagarias $\mathcal{E}$ function would vanish at the locations on the critical line of the non-simple zeros of zeta, if some exist.

cosine kernels. They imply that $\phi_a^+(x)$ and $\phi_a^-(x)$ are even entire functions, in particular they have meaning for $x > a$. The integral for $\widehat{E_a}(s)$ is absolutely convergent for $\Re(s) > 0$.

The function $\mathcal{E}_a(s)$ is real for s real. Writing $\mathcal{E}_a(s) = \mathcal{A}_a(s) - i\mathcal{B}_a(s)$ on the critical line, with $\mathcal{A}_a(s)$ and $\mathcal{B}_a(s)$ real there, we thus have for s on the critical line: $2\mathcal{A}_a(s) = \mathcal{E}_a(s) + \overline{\mathcal{E}_a(s)} = \mathcal{E}_a(s) + \mathcal{E}_a(\overline{s}) = \mathcal{E}_a(s) + \mathcal{E}_a(1-s)$. The last equation, for general s, defines $\mathcal{A}_a$ as an entire function, which is even under $s \mapsto 1-s$, real for s real, and real for s on the critical line. Also $2\mathcal{B}_a(s) = i(\mathcal{E}_a(s) - \mathcal{E}_a(1-s))$ is an entire function, which is odd under $s \mapsto 1-s$, imaginary for s real, and real for s on the critical line. We shall write $\mathcal{A}_a(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\widehat{A_a}(s)$, $\mathcal{B}_a(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\widehat{B_a}(s)$, so that $A_a(x)$ (resp. $B_a(x)$) is a cosine invariant (resp. skew-invariant) distribution supported in $[a, +\infty)$. With $\delta_a(x) = \delta(x-a)$ one has indeed:

$$A_a(x) = \frac{\sqrt{a}}{2}\left(\delta_a(x) + \phi_a^+(x)\mathbf{1}_{x>a}(x)\right) \tag{7a}$$

$$-iB_a(x) = \frac{\sqrt{a}}{2}\left(\delta_a(x) - \phi_a^-(x)\mathbf{1}_{x>a}(x)\right) \tag{7b}$$

The evaluator identity (2) can also be written as

$$(\mathcal{Z}_w|\mathcal{Z}_z) = 2\frac{\overline{\mathcal{A}_a(z)}\mathcal{B}_a(w) - \overline{\mathcal{B}_a(z)}\mathcal{A}_a(w)}{i(\overline{z} + w - 1)} \tag{8}$$

In particular, for $z = w$ one obtains

$$\|\mathcal{Z}_z\|^2 = 2\frac{\Im(\overline{\mathcal{A}_a(z)}\mathcal{B}_a(z))}{\Re(z) - \frac{1}{2}} \qquad (\Re(z) \neq \frac{1}{2}) \tag{9}$$

$$\|\mathcal{Z}_{\frac{1}{2}+iE}\|^2 = -2\mathcal{A}_a(\frac{1}{2}+iE)\, i\mathcal{B}_a'(\frac{1}{2}+iE) + 2\mathcal{B}_a(\frac{1}{2}+iE)\, i\mathcal{A}_a'(\frac{1}{2}+iE) \tag{10}$$

Derivatives in (10) are taken with respect to $s = \frac{1}{2} + iE$ (not E). Equation (9) implies that the entire functions $\mathcal{A}_a$ and $\mathcal{B}_a$ have all their zeros on the critical line. It is indeed a fact that all vectors $\mathcal{Z}_z$ are non-zero (it is easy to see that "$\mathcal{F}(z) = 0$ for all $\mathcal{F}$" is impossible). From (9) we deduce that the ratio $\frac{\mathcal{B}_a(s)}{\mathcal{A}_a(s)}$ is of positive imaginary part when $\Re(s) > \frac{1}{2}$. It is elementary, as $\frac{\mathcal{B}_a(s)}{\mathcal{A}_a(s)}$ is meromorphic, that this implies that its poles and zeros (all on the critical line) are simple. Next, if $\mathcal{A}_a$ had a multiple zero ρ, then $\mathcal{B}_a$ would have to vanish there, but then taking a limit in (9) we would obtain $\|\mathcal{Z}_\rho\| = 0$, which we have already mentioned is false. So the zeros of $\mathcal{A}_a$ and $\mathcal{B}_a$ are simple; using (10) one argues that the respective zeros interlace. These arguments apply in the general situation of [3]. A simplification which is valid here and that we have made use of is that the evaluators $\mathcal{Z}_z$ are non-zero also when z is on the symmetry line.

From (8) two vectors $\mathcal{Z}_\rho$ and $\mathcal{Z}_{\rho'}$ associated to distinct zeros of $\mathcal{A}_a$ are always orthogonal. Furthermore from (8) if $\rho_n = \frac{1}{2} + iE_n$ is a zero of $\mathcal{A}_a$ then $\mathcal{Z}_{\rho_n}(s) = 2i\mathcal{B}_a(\rho_n)\frac{\mathcal{A}_a(s)}{s-\rho_n}$ (we used $\rho_n = 1 - \overline{\rho_n}$). So the vectors $\frac{\mathcal{A}_a(s)}{s-\rho_n}$ are mutually orthogonal. From (10):

$$\left\|\frac{\mathcal{A}_a(s)}{s-\rho_n}\right\|_{K_a}^2 = \int_{\Re(s)=\frac{1}{2}} \left|\frac{\widehat{A_a}(s)}{s-\rho_n}\right|^2 \frac{|ds|}{2\pi} = \frac{1}{2}\,\frac{d}{dE}\bigg|_{E=E_n} \frac{\mathcal{A}_a(\frac{1}{2}+iE)}{\mathcal{B}_a(\frac{1}{2}+iE)} \tag{11}$$

We can write this also as

$$\left\|\frac{\mathcal{A}_a(s)}{s-\rho_n}\right\|_{K_a}^2 == \frac{1}{2}\frac{d}{dE}\bigg|_{E=E_n}\frac{\widehat{A_a}(\frac{1}{2}+iE)}{\widehat{B_a}(\frac{1}{2}+iE)} = \frac{1}{2}\frac{d}{i\,dE}\bigg|_{E=E_n}\frac{\widehat{A_a}(\frac{1}{2}+iE)}{\widehat{E_a}(\frac{1}{2}+iE)} \tag{12}$$

The vectors $\frac{\mathcal{B}_a(s)}{s-\rho}$ corresponding to the zeros of $\mathcal{B}_a$ provide another orthogonal system. These vectors corresponding to the zeros of $\mathcal{A}_a$ (or $\mathcal{B}_a$) span all of K_a. This orthogonal basis theorem is a general fact from [3] (under circumstances which are verified here).

In the next sections we prove this orthogonal basis theorem from the standard material provided by the Weyl-Stone-Titchmarsh-Kodaira theory of second order differential equations ([11, 16]), as an outcome of studying the deformation with respect to $a > 0$ of the spaces K_a ([7]). Anyhow, obviously one of the main origins of the axioms of [3] is precisely the behavior of the objects from the Weyl-Stieltjes spectral theory of differential or difference equations (see Remling [19] for the spectral problems of Schrödinger equations from the point of view of these axioms).

4 Dirac system and associated isometric expansion

Definition 5. We let C_a be the kernel $2\cos(2\pi xy)$ acting on $L^2(0, a; dx)$. And we define

$$\mu(u) = a\frac{d}{da}\log\frac{\det(1+C_a)}{\det(1-C_a)} \qquad \text{with } u = \log(a) \tag{13}$$

Definition 6. We let $\mathcal{A}(u,s) = \mathcal{A}_a(s)$ and $\mathcal{B}(u,s) = \mathcal{B}_a(s)$ for $a = \exp(u)$, $-\infty < u < \infty$.

Theorem 7 ([7])**.** *The following differential system holds, where $s = \frac{1}{2} + iE$, $E \in \mathbb{C}$:*

$$(\frac{d}{du} + \mu(u))\mathcal{A}(u,s) = -E\mathcal{B}(u,s) \tag{14a}$$
$$(\frac{d}{du} - \mu(u))\mathcal{B}(u,s) = +E\mathcal{A}(u,s) \tag{14b}$$

Equivalently:

$$\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{d}{du} - \begin{bmatrix}0 & \mu(u)\\ \mu(u) & 0\end{bmatrix}\right)\begin{bmatrix}\mathcal{A}(u,s)\\ \mathcal{B}(u,s)\end{bmatrix} = E\begin{bmatrix}\mathcal{A}(u,s)\\ \mathcal{B}(u,s)\end{bmatrix} \tag{15}$$

Combining the differential system (15) and the evaluator formula (8) one obtains the following identity

$$(\mathcal{Z}_w^a|\mathcal{Z}_z^a) = \int_{\log(a)}^{\infty} \overline{2\mathcal{A}(u,z)}\,2\mathcal{A}(u,w) + \overline{2\mathcal{B}(u,z)}\,2\mathcal{B}(u,w)\,\frac{du}{2} \tag{16}$$

and in particular we see that for each $z \in \mathbb{C}$ the two-component vector $\begin{bmatrix}\mathcal{A}(u,z)\\ \mathcal{B}(u,z)\end{bmatrix}$ is square-integrable at $+\infty$.

Let us recall that according to a general theorem [16, §13, Thm 7.1] for Dirac systems with continuous coefficients we are automatically in the limit-point case at infinity. Being in the limit point case at $+\infty$ means that for each $E \notin \mathbb{R}$ there is exactly one (up to a scalar multiple) non-vanishing solution which is square-integrable at $+\infty$. Thus we have identified this solution, and we have seen that it also exists for E real (the solutions which are square-integrable at $-\infty$ are also known [7]; we will mention them in a later section). There can't be other linearly independent square-integrable solutions as this would characterize the limit-circle case, which never happens for Dirac type systems with continuous coefficients.[6]

Let us consider the Dirac-type differential operator $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dfrac{d}{du} - \begin{bmatrix} 0 & \mu(u) \\ \mu(u) & 0 \end{bmatrix}$ acting on pairs $\begin{bmatrix} \alpha(u) \\ \beta(u) \end{bmatrix}$, $-\infty < u < +\infty$. As we are in the limit-point case at infinity (plus or minus), from Hermann Weyl's theory (adapted to first-order two-component systems) we know that the symmetric operator defined by this differential operator with domain given by the functions of class C^1 and with compact support is essentially self-adjoint ([16, 11, 18]). In Theorem 15 of [10] we determined the associated isometric spectral expansion and showed that this self-adjoint operator H_0 is $-i(x\frac{d}{dx} + \frac{1}{2})$ acting on $L^2(0, \infty; dx)$ or equivalently the operator of multiplication by E on the critical line:

Theorem 8 ([10], Theorem 15)**.** *There are unitary identifications between $L^2(0, \infty; dx)$, $L^2(\Re(s) = \frac{1}{2}; \frac{|ds|}{2\pi|\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})|^2})$ and $L^2(\mathbb{R} \to \mathbb{C}^2; \frac{du}{2})$ given by the following formulas (to be understood as suitable L^2-limits, see [10]), where always $s = \frac{1}{2} + iE$, $E \in \mathbb{R}$:*

$$\mathcal{F}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \int_{0<x<\infty} f(x) x^{-s}\, dx \tag{17}$$

$$\begin{bmatrix} \alpha(u) \\ \beta(u) \end{bmatrix} = \int_{E\in\mathbb{R}} \mathcal{F}(s) \begin{bmatrix} 2\mathcal{A}(u,s) \\ 2\mathcal{B}(u,s) \end{bmatrix} \frac{dE}{2\pi\, |\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})|^2} \tag{18}$$

$$\mathcal{F}(s) = \int_{u\in\mathbb{R}} \alpha(u)\, 2\mathcal{A}(u,s) + \beta(u)\, 2\mathcal{B}(u,s)\, \frac{du}{2} \tag{19}$$

$$\int_0^\infty |f(x)|^2\, dx = \int_{-\infty}^\infty |\mathcal{F}(s)|^2\, \frac{dE}{2\pi\, |\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})|^2} = \int_{-\infty}^\infty (|\alpha(u)|^2 + |\beta(u)|^2) \frac{du}{2} \tag{20}$$

Under these identifications:

- *The orthogonal projection of $f(x)$ to K_{a_0} corresponds to $\begin{bmatrix} \alpha(u) \\ \beta(u) \end{bmatrix} \mapsto \begin{bmatrix} \alpha(u) \\ \beta(u) \end{bmatrix} \mathbf{1}_{u>\log(a_0)}(u)$,*
- *the Cosine Transform $f \mapsto \mathcal{C}(f)$ corresponds to $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \alpha \\ -\beta \end{smallmatrix}\right]$ (and $\mathcal{F}(s) \mapsto \mathcal{F}(1-s)$),*
- *the self-adjoint operator $-i(x\frac{d}{dx} + \frac{1}{2})$ on $L^2(0, +\infty; dx)$, which multiplies by E with maximal domain on the critical line, corresponds to the unique self-adjoint completion of the differential operator on the u-line:*

$$H_0 = \begin{bmatrix} 0 & \frac{d}{du} \\ -\frac{d}{du} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mu(u) \\ \mu(u) & 0 \end{bmatrix} \tag{21}$$

  with initial domain the functions of class C^1 (or even C^∞) with compact support.

[6] in [10] we give an eigensolution for $E = 0$ which we prove to be not square-integrable at infinity. This explicit argument confirms that we are not in the limit-circle case.

This Theorem is an elaboration deduced mainly from the evaluator identity (16). Let us also mention that the integrals in (18) are absolutely convergent when f is in the domain of H_0, that is, when $s \cdot \widehat{f}(s)$ is square integrable on the critical line. The functions $\alpha(u)$ and $\beta(u)$ are then continuous; actually we know that they are absolutely continuous as they must be in the domain of H_0 in the u-picture, so their derivatives must be locally square integrable.

The operator $xp = -ix\frac{d}{dx}$, or $\frac{1}{2}(xp + px) = -i(x\frac{d}{dx} + \frac{1}{2})$ was considered "semi-classically" by Berry and Keating [1] in a discussion where it was accompanied with the conditions $|x| > l_x$, $|p| > l_p$ (the operator $-i(x\frac{d}{dx} + \frac{1}{2})$ with conditions $|x|, |p| < \Lambda$ arose in the work of Connes [12]). Certainly it is natural to consider that the conditions $|x| > l_x$, $|p| > l_p$, treated "semi-classically" in [1], could at the "quantum" (that is, operator-theoretical level) be related with the condition of support which defines the spaces K_a (and if we were considering functions odd in x, we would have the analogues of the spaces K_a associated with the sine kernel). Recently, German Sierra in [23] (see also [21, 22]) has generalized the conditions $|x| > l_x$, $|p| > l_p$ through the consideration of differential-integral equations obtained from the differential operator xp by the addition of (singular) finite rank perturbations. Sierra presents general aspects of the corresponding eigenfunctions. This leads to scattering states (deformations of the "free" states $x^{iE}x^{-\frac{1}{2}}$), and possibly also to bound states in some fine-tuned situations. In the next section we see how we have natural bound states in K_a and then we will show that in the x-picture these states are eigensolutions in the form considered in [23, Section IV]. The scattering states will span the perpendicular complement of K_a in L^2.

5 Bound states

We consider the differential system (15) on the half-interval $(u_0, +\infty)$, $u_0 = \log(a_0)$ with boundary condition $\alpha(u_0) = 0$. From general theorems [18, 16, 11] we know that this leads to a self-adjoint operator and associated isometric expansion (with discrete spectrum from the behavior of $\mu(u)$ at $+\infty$, but we will confirm this directly). To obtain the spectrum we apply Hermann Weyl's method ([11, §9], [16, §3]). For each $s \in \mathbb{C}$ we let $\psi(u, s)$ be the unique solution of the system (15) for the eigenvalue E, $s = \frac{1}{2} + iE$, and with the initial condition $\psi(u_0, s) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and let $\phi(u, s)$ be the unique solution for the eigenvalue E with the initial condition $\phi(u_0, s) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Let, for $\Im(E) > 0$, the quantity $m(E)$ be chosen such that $\psi(u, s) - m(E)\phi(u, s)$ is the unique linear combination which has the property of being square-integrable at $+\infty$.

To check up on signs, let us see what would happen if the potential $\mu(u)$ were identically vanishing. Then we would have $\psi(u, s) = \left[\begin{smallmatrix}\cos(E(u-u_0))\\ \sin(E(u-u_0))\end{smallmatrix}\right]$ and $\phi(u, s) = \left[\begin{smallmatrix}-\sin(E(u-u_0))\\ \cos(E(u-u_0))\end{smallmatrix}\right]$, and the m function would be the constant $+i$. In the general case the m function is known to be an analytic function mapping the upper half-plane to itself. Here, as the one-dimensional space of square-integrable solutions has basis $\left[\begin{smallmatrix}\mathcal{A}(u,s)\\ \mathcal{B}(u,s)\end{smallmatrix}\right]$ we have $m(E) = \frac{-\mathcal{B}_{a_0}(s)}{\mathcal{A}_{a_0}(s)}$. We had already argued from the evaluator formula (8) that this meromorphic function has positive imaginary part on the half-plane $\Im(E) > 0$. As our m-function is meromorphic and real-valued for E real, the spectral measure ν, which is given via the formula $\nu(a,b) = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_a^b \Im(m(E+i\epsilon))\,dE$ (under the condition $\nu\{a\}+\nu\{b\}=0$) is a sum of Dirac measures and the spectrum is thus purely discrete, having its support on the zeros E_n of $\mathcal{A}_{a_0}(\frac{1}{2}+iE)$. We obtain:[7]

$$\nu(dE) = \sum_n \frac{\mathcal{B}_{a_0}(\frac{1}{2}+iE_n)}{i\mathcal{A}'_{a_0}(\frac{1}{2}+iE_n)}\delta(E-E_n)\,dE \tag{22}$$

From the general theory the Hilbert space $L^2((u_0,+\infty)\to\mathbb{C}^2;du)$ admits an orthogonal basis (T_n) given by the two-component vectors $T_n(u) = \phi(u,\frac{1}{2}+iE_n)$ which have, according to the spectral measure formula (22), norms $\int_{u_0}^\infty |T_n(u)|^2\,du = \frac{i\mathcal{A}'_{a_0}(\frac{1}{2}+iE_n)}{\mathcal{B}_{a_0}(\frac{1}{2}+iE_n)}$. The spectral measure is $\nu(dE) = \sum_n \frac{\delta(E-E_n)}{\|T_n\|^2}\,dE$ and the Plancherel formula is

$$\int_{u_0}^\infty |T(u)|^2\,du = \sum_n \frac{|\widetilde{T}(E_n)|^2}{\int_{u_0}^\infty |T_n(u)|^2\,du} \qquad \widetilde{T}(E_n) = \int_{u_0}^\infty T(u)\cdot\phi(u,\frac{1}{2}+iE_n)\,du \tag{23}$$

Let us consider the vectors

$$Z_n(u) = \begin{bmatrix} 2\mathcal{A}(u,\frac{1}{2}+iE_n) \\ 2\mathcal{B}(u,\frac{1}{2}+iE_n)\end{bmatrix}\mathbf{1}_{u>u_0}(u) = 2\mathcal{B}(u_0,\frac{1}{2}+iE_n)\,T_n(u) \tag{24}$$

We deduce from what has just been stated regarding the T_n's that the Z_n's are an orthogonal basis of $L^2((u_0,+\infty)\to\mathbb{C}^2;\frac{1}{2}du)$ with Hilbert norms in this space:

$$(Z_n|Z_n) = 2i\mathcal{A}'_{a_0}(\frac{1}{2}+iE_n)\mathcal{B}_{a_0}(\frac{1}{2}+iE_n) \tag{25}$$

From Theorem 8 we know that Z_n in the "$\mathcal{F}(s)$-picture" corresponds precisely to the evaluator $\mathcal{Z}^{a_0}_{\frac{1}{2}+iE_n}$. Equations (25) and (10) do indeed match. Furthermore the two-component vector $Z_n(u)$ solves the differential system on $(u_0,+\infty)$ for the eigenvalue E_n. In conclusion the differential operator $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{d}{du} - \begin{bmatrix}0 & \mu(u)\\ \mu(u) & 0\end{bmatrix}$ acting on pairs $\begin{bmatrix}\alpha(u)\\ \beta(u)\end{bmatrix}$ on (u_0,∞) with the boundary condition $\alpha(u_0)=0$ is self-adjoint with a purely discrete spectrum given by the imaginary parts E_n of the zeros of $\mathcal{A}_{a_0}$ ($a_0 = \exp(u_0)$). We notice, as $\mathcal{Z}^{a_0}_{\frac{1}{2}+iE_n}(\frac{1}{2}+iE) = 2\mathcal{B}_{a_0}(\frac{1}{2}+iE_n)\frac{\mathcal{A}_{a_0}(\frac{1}{2}+iE)}{E-E_n}$, that the eigenvectors $T_n(u)$ which are such that $T_n(u_0) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ correspond exactly in the $\mathcal{F}(s)$-picture to the functions $s\mapsto \frac{\mathcal{A}_{a_0}(\frac{1}{2}+iE)}{E-E_n}$.

Let us denote by $H_0^{a_0}$ the self-adjoint operator given by the differential system (15) on $(u_0,+\infty)$ with boundary condition $\left[\begin{smallmatrix}0\\ *\end{smallmatrix}\right]$ at $u_0 = \log(a_0)$. With reference to the comment after Theorem 8, we observe that its eigenvectors (in the $\mathcal{F}(s)$ picture) $s\mapsto\frac{\mathcal{A}_{a_0}(s)}{s-\rho}$ ($\mathcal{A}_{a_0}(\rho)=0$) are not located in the domain of H_0 (which multiplies by E), and indeed $\widehat{A_{a_0}}(s)$ is not square-integrable on the critical line, as can be seen directly or from the presence of the Dirac at a in $A_0(x)$ or the behavior of $A_{a_0}(x)$ as $x\to+\infty$. To discuss this further let us mention:

[7] derivatives are computed with respect to $s=\frac{1}{2}+iE$ so $i\mathcal{A}'_{a_0}(\frac{1}{2}+iE_n)$ is real.

Proposition 9. *If f is in K_a and is in the domain of H_0 in L^2, then $H_0(f)$ is in K_a. The restriction of H_0 to K_a has deficiency indices $(1,1)$. The self-adjoint operator H_0^a is an extension of this restriction.*

Let f in K_a be such that $H_0(f)$ is in L^2. In the u-picture the two-component vector T representing f must have its components absolutely continuous, and the action of H_0 on T is the one given as a differential operator. As f is in K_a, the components of T vanish identically for $u < \log(a)$ so their derivatives also. Hence $H_0(f) \in K_a$. The statement is also easy to establish directly in the x-picture using the formula $\mathcal{C}(x\frac{d}{dx} + \frac{1}{2})f = -(x\frac{d}{dx} + \frac{1}{2})\mathcal{C}(f)$ (in the distribution sense). Let us note though that by necessity the function $T(u)$ being continuous will be such that $\alpha(\log(a)) = 0$ and $\beta(\log(a)) = 0$. The domain of the restriction of H_0 to K_a, in the u-picture is thus given by the absolutely continuous square integrable functions, with both components vanishing at $\log(a)$, and whose images under the differential operator are square-integrable. This shows that the operator H_0^a is indeed an extension of this restriction. From well-known theorems [18, 16], the deficiency indices of such a differential operator on a semi-infinite interval, in the limit-point case at infinity, are $(1,1)$ with the one-parameter family of possible boundary conditions at $\log(a)$ giving the self-adjoint extensions. We can confirm directly in the $\mathcal{F}(s)$-picture the computation of the deficiency indices. Indeed $i(H_0 - \frac{i}{2})$ is multiplication by s. As $s\mathcal{F}(s)$ vanishes at 0 (here $\mathcal{F}$ is taken in a K_a, so is also defined away from critical line) it is perpendicular (if $\mathcal{F}$ is in the domain of H_0) to the evaluator $\mathcal{Z}_0^a$. Conversely if $\mathcal{G}$ in K_a is perpendicular to $\mathcal{Z}_0^a$, the quotient $\frac{1}{s}\mathcal{G}(s)$ is an entire function and it is easy to prove that it corresponds to an element of K_a (see [8] for details). So the range of $H_0|_{K_a} - \frac{i}{2}$ has codimension 1, and similarly for the range of $H_0|_{K_a} + \frac{i}{2}$. This confirms that the deficiency indices are $(1,1)$.

6 Continuous spectrum

Definition 10. We let L_a be the perpendicular complement of K_a in L^2. In the x picture one has $L_a = L^2(0,a;dx) + \mathcal{C}(L^2(0,a;dx))$. In the u picture L_a is the space of the square integrable two-component vectors supported on $(-\infty, \log(a))$.

Again, from the u-picture we see that if f is in L_a and is in the domain of H_0 in L^2 then $H_0(f)$ is again in L_a. And we again have a one-parameter family of self-adjoint extensions of the restriction of H_0 to L_a. But all these extensions will have continuous spectra, as the potential function $\mu(u)$ vanishes exponentially as $u \to -\infty$. Summing up:

Proposition 11. *If f is in L_a and is in the domain of H_0 in L^2, then $H_0(f)$ is in L_a. The restriction of H_0 to L_a has deficiency indices $(1,1)$. Its self-adjoint extensions correspond to a one-parameter family of boundary conditions at $\log(a)$ and each has a purely continuous spectrum.*

To obtain a self-adjoint operator on all of L^2 we thus need to choose the boundary condition at u_0. A specific choice will emerge from the following procedure. The explicit spectral expansion will be given in the next section, after having explained how the results obtained in the present section fit into the framework considered by Sierra in [23], once we go back to the x-picture.

We consider H_0 as a differential operator in the u picture. The bound states $\frac{\mathcal{A}_a(\frac{1}{2}+iE)}{E-E_n}$ are there the vectors denoted $T_n(u)$, which have support in $u \geq u_0$, and verify $\lim_{u\to u_0} T_n(u) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. They obey for $-\infty < u < +\infty$ the differential equation, say as distributions:

$$\begin{bmatrix} 0 & \frac{d}{du} \\ -\frac{d}{du} & 0 \end{bmatrix} T_n(u) - \begin{bmatrix} 0 & \mu(u) \\ \mu(u) & 0 \end{bmatrix} T_n(u) = E_n T_n(u) + \begin{bmatrix} \delta(u-u_0) \\ 0 \end{bmatrix} \tag{26}$$

We will write (recalling the use of $\frac{du}{2}$ in the u picture):

$$\delta_0(u) = \begin{bmatrix} 2\delta(u-u_0) \\ 0 \end{bmatrix} \qquad \delta_1(u) = \begin{bmatrix} 0 \\ 2\delta(u-u_0) \end{bmatrix} \tag{27}$$

and shall convene that $\left[\begin{smallmatrix}(\delta_0|T)\\(\delta_1|T)\end{smallmatrix}\right] = \frac{1}{2}(\lim_{u\to u_0^+} + \lim_{u\to u_0^-})T(u)$ when the limits exist. So here $(\delta_0|T_n) = 0$ and $(\delta_1|T_n) = \frac{1}{2}$. This transforms the inhomogeneous equation into a linear homogeneous equation:

$$\begin{bmatrix} 0 & \frac{d}{du} \\ -\frac{d}{du} & 0 \end{bmatrix} T_n(u) - \begin{bmatrix} 0 & \mu(u) \\ \mu(u) & 0 \end{bmatrix} T_n(u) = E_n T_n(u) + (\delta_1|T_n)\delta_0(u) \tag{28}$$

In order to obtain a formally symmetric expression we add one more term, obtaining in the end the following perturbation of the differential operator H_0:

$$H(T) = H_0(T) - (\delta_1|T)\delta_0 - (\delta_0|T)\delta_1 \tag{29}$$

According to the remark made after Theorem 8, as a corollary to the fact that the functions $\frac{\widehat{A_a}(s)}{s}$ and $\frac{\widehat{B_a}(s)}{s}$ are square-integrable on the critical line, when T is in the domain of H_0 its components are continuous functions and evaluation at u_0 is a continuous linear form (given by equation (18)). As a bilinear form, the perturbation term is thus bounded relative to $(T|T) + (H_0(T)|H_0(T))$.

Let us consider for a given E the equation $H(T) = ET$ on the full line. So T must be of the form $T(u) = L(u)\mathbf{1}_{u<u_0}(u) + R(u)\mathbf{1}_{u>u_0}(u)$ where L and R verify on the full line (as differential equations) $H_0(L) = EL$, $H_0(R) = ER$. Writing $L(u_0) = \left[\begin{smallmatrix}l_0\\l_1\end{smallmatrix}\right]$ and $R(u_0) = \left[\begin{smallmatrix}r_0\\r_1\end{smallmatrix}\right]$ we obtain from (29) the matching conditions:

$$-l_1\delta_0 + l_0\delta_1 + r_1\delta_0 - r_0\delta_1 = (l_1+r_1)\delta_0 + (l_0+r_0)\delta_1 \quad \iff l_1 = r_0 = 0 \tag{30}$$

We thus have for each E a two dimensional space of eigensolutions, which are of the form:

$$T_E(u) = \lambda L_E(u)\mathbf{1}_{u<u_0}(u) + \mu R_E(u)\mathbf{1}_{u>u_0}(u) \tag{31}$$

where L_E verifies the differential equation $H_0(L_E) = EL_E$ and the condition $L_E(u_0) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and R_E verifies the differential equation $H_0(R_E) = ER_E$ and the condition $R_E(u_0) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

As we saw in the last section the vectors $R_E(u)\mathbf{1}_{u>u_0}(u)$ lead to a well-defined spectral expansion of K_a where in fact only those among the R_E which are square-integrable contribute, the suitable E's being the (imaginary parts of the) zeros of the entire function $\mathcal{A}_{a_0}(s)$. On the other hand, on the basis of standard theorems of spectral expansions [18, 16, 11] the vectors $L_E(u)\mathbf{1}_{u<u_0}(u)$ for E real are the generalized eigenvectors of the self-adjoint extension of the restriction of H_0 to $(-\infty, \log(a))$, given by the boundary condition $\left[\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right]$ at u_0.

We now extend H_0^a to all of L^2 defining it on L_a to be this self-adjoint extension of $H_0|_{L_a}$. In the end we have thus been led by this procedure to a well-defined self-adjoint operator on L^2 whose spectrum has on one hand a discrete part corresponding to K_a and on the other hand a continuous part corresponding to L_a. Its eigenvectors are the solutions of the equations:

$$H_0(T) - (\delta_1|T)\delta_0 - (\delta_0|T)\delta_1 = ET \tag{32}$$

But only those solutions that are square-integrable as $u \to +\infty$ have to be retained, those solutions being: on one hand the vectors previously denoted $T_n(u)$ on $(u_0, +\infty)$, on the other hand a continuous spectrum of generalized elements living on $(-\infty, u_0)$. The spectral density will be determined in the next section. Here we now explain how the x picture looks like.

According to Theorem 8, equation (18), $\delta_0(u)$ corresponds in the $\mathcal{F}(s)$ picture to $2\mathcal{A}_{a_0}(s)$ and $\delta_1(u)$ corresponds to $2\mathcal{B}_{a_0}(s)$ (here we are strictly on the critical line, these functions are real-valued). In the $\widehat{f}(s)$ picture, $\delta_0(u)$ corresponds thus to $2\widehat{A_a}(s)$ and $\delta_1(u)$ to $2\widehat{B_a}(s)$. In the x picture δ_0 and δ_1 correspond to the distribution $2A_a(x)$ and $2B_a(x)$. The operator H takes the form:

$$H(f) = -i(x\frac{d}{dx} + \frac{1}{2})f - (2B_a|f)2A_a - (2A_a|f)2B_a\,, \tag{33}$$

where the "scalar products" $(2A_a|f)$ and $(2B_a|f)$ have to be defined in a suitable manner if f does not belong to the domain of H_0. We will switch to another linear combination allowing an easier discussion and having the format considered by Sierra in [23].

Using $E_a = A_a - iB_a$ and $F_a = \mathcal{C}(E_a) = A_a + iB_a$ we rewrite equation (33) as

$$H(f) = -i(x\frac{d}{dx} + \frac{1}{2})f + 2i(E_a|f)F_a - 2i(F_a|f)E_a \tag{34}$$

The (real valued) distribution E_a differs from a Dirac $\sqrt{a}\,\delta_a(x)$ by a square integrable function supported on $[a, +\infty)$, and its Cosine transform $F_a = \mathcal{C}(E_a) = A_a + iB_a$ is a function supported on $[a, +\infty)$ which differs from $\sqrt{a}\,2\cos(2\pi ax)$ by $O(\frac{1}{x})$ at $+\infty$. When we try, directly in the x picture, to solve the eigen-equation $H(f) = Ef$ we have to give a meaning to $(E_a|f)$ where f may have a jump at a (we may give an imaginary part to E and look for a square-integrable f to simplify the discussion). The correct choice is to take $(\delta_a|f)$ for those f to be the average between the left and right values at $x = a$. And $(F_a|f)$ needs to be defined suitably as well. The discussion in the u-picture was technically easier and we shall defer to another publication the more detailed direct treatment of (34) in the x-picture.

Let us mention that if we repeat the whole procedure but starting from the self-adjoint operator on K_a corresponding to the zeros of the $\mathcal{B}_a$ function we end up with the following operator

$$H(f) = -i(x\frac{d}{dx} + \frac{1}{2})f + 2i(F_a|f)E_a - 2i(E_a|f)F_a \tag{35}$$

which differs from the previous one only by the sign of the perturbation.

The general equation considered by Sierra in [23, IV] is

$$H(f) = -i(x\frac{d}{dx} + \frac{1}{2})f + i(\psi_2|f)\psi_1 - i(\psi_1|f)\psi_2 \tag{36}$$

Thus (34) and (35) are indeed special cases. Furthermore Sierra considers the "duality" condition that ψ_1 and ψ_2 are a Cosine (or Sine) transform pair. Our special instances do have this property. Finally in [23, VI] the case $\psi_2(x) = \sqrt{2}\sqrt{a}\ \delta_a(x)$ and $\psi_1 = \mathcal{C}(\psi_2)$ is devoted special attention. Whether $\psi_2(x) = \sqrt{2}\sqrt{a}\ \delta_a(x)$ leads to bound states is not clear.

7 More on the scattering spectrum

We want to provide more information on the "scattering" part of the spectrum. When E is not real the differential system (15) has (up to a multiple) exactly one solution which is square-integrable at $-\infty$. Fortunately, this solution is known explicitely. It was determined in [7], as well as the way $\begin{bmatrix}\mathcal{A}(u,s)\\ \mathcal{B}(u,s)\end{bmatrix}$ for E real is expressed as a linear combination of the boundary values of such solutions and their complex conjugates.

Theorem 12 ([7]; Theorem 16 of [10])**.** *The eigensolution* $\begin{bmatrix}\alpha_E(u)\\ \beta_E(u)\end{bmatrix}$ *of the differential system which is square-integrable at* $-\infty$ *for* $\Im(E) > 0$ *(*$\Re(s) < \frac{1}{2}$*) is provided by the following formulas (where* $a = \exp(u)$*):*

$$\alpha_E(u) = J(u, \frac{1}{2} + iE) = e^{-iEu} - e^{\frac{1}{2}u}\int_0^a \phi_a^+(x)x^{-s}\,dx \tag{37}$$

$$\beta_E(u) = K(u, \frac{1}{2} + iE) = ie^{-iEu} + ie^{\frac{1}{2}u}\int_0^a \phi_a^-(x)x^{-s}\,dx \tag{38}$$

The following identity holds:

$$\begin{bmatrix}2\mathcal{A}(u, \frac{1}{2} + iE)\\ 2\mathcal{B}(u, \frac{1}{2} + iE)\end{bmatrix} = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\begin{bmatrix}J(u, \frac{1}{2} + iE)\\ K(u, \frac{1}{2} + iE)\end{bmatrix} + \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\begin{bmatrix}J(u, \frac{1}{2} - iE)\\ -K(u, \frac{1}{2} - iE)\end{bmatrix} \tag{39}$$

We will also use the notation $\widehat{j_a}(s) = J(u, s)$ and $\widehat{k_a}(s) = K(u, s)$, for $u = \log(a)$:

$$J(u, s) = \widehat{j_a}(s) = a^{\frac{1}{2}-s} - \sqrt{a}\int_0^a \phi_a^+(x)x^{-s}\,dx \tag{40a}$$

$$K(u, s) = \widehat{k_a}(s) = ia^{\frac{1}{2}-s} + i\sqrt{a}\int_0^a \phi_a^-(x)x^{-s}\,dx \tag{40b}$$

The functions $\phi_a^+(x)$ and $\phi_a^-(x)$ being entire and even, we see that $\widehat{j_a}$ and $\widehat{k_a}$ are meromorphic in the entire complex plane with possible poles at $s = 1$, $s = 3$, $s = 5$, ... Of course they still provide after analytic continuation a solution of the differential system with respect to $\log(a)$. We have

$$2\mathcal{A}_a(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\widehat{j_a}(s) + \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\widehat{j_a}(1-s) \tag{41}$$

As $\mathcal{A}_a(s)$ is entire, if $\widehat{j_a}$ does not have a pole at $1 + 2m$ ($m \in \mathbb{N}$), necessarily the second term on the right-hand side is forbidden to have a pole there, which means that necessarily one must have a zero of $\widehat{j_a}$ at $-2m$. We will now argue that neither $\widehat{j_a}$ nor $\widehat{k_a}$ can have zeros in the half-plane $\Re(s) < \frac{1}{2}$ (which corresponds to $\Im(E) > 0$). This will prove that $\widehat{j_a}$ (and similarly $\widehat{k_a}$) does have poles at $s = 1$, $s = 3$, $s = 5$, ...

Indeed as in the previous section we apply Hermann Weyl's theory ([11, §9], [16, §3]), but on the interval $(-\infty, u_0)$. We maintain the same notations $\psi(u, s)$ with boundary condition $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\phi(u, s)$ with boundary condition $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ but convene that the m-function for $\Im(E) > 0$ is determined by the condition that $-m(E)\psi(u, s) + \phi(u, s)$ is square integrable at $-\infty$. Thus, we have the identity:

$$\Im(E) > 0 \implies m(E) = \frac{-J(u_0, \frac{1}{2} + iE)}{K(u_0, \frac{1}{2} + iE)} = \frac{\widehat{-j_{a_0}}(s)}{\widehat{k_{a_0}}(s)} \tag{42}$$

It is a general theorem that the m function is an analytic function mapping the upper half-plane to itself. So if $\widehat{k_{a_0}}(s)$ has a zero for $\Re(s) < \frac{1}{2}$, then $\widehat{j_{a_0}}(s)$ has to vanish also for the same s_0. But $\left[\begin{smallmatrix}J(u,s_0)\\K(u,s_0)\end{smallmatrix}\right]$ is solution of a first order differential equation with respect to u. If it vanishes for some u it has to vanish for all u's. From (40a), (40b) however it is easily argued that $\widehat{j_a}(s) \sim a^{\frac{1}{2}-s}$ as $a \to 0^+$, and $\widehat{k_a}(s) \sim_{a\to 0} ia^{\frac{1}{2}-s}$ ($\Re(s) < 1$).[8]

The Wronskian $W(s) = \begin{vmatrix}\mathcal{A}(u,s) & J(u,s)\\ \mathcal{B}(u,s) & K(u,s)\end{vmatrix}$ is independent of u. We can evaluate it at $u \to -\infty$, perhaps first for $0 < \Re(s) < \frac{1}{2}$, using the asymptotic of J and K as $a \to 0^+$ and the identity (39). This gives:

$$W(s) = i\,\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2}) \tag{43}$$

Analytic continuation removes the restriction on s. This confirms that J or K (in fact both as we saw) must have poles at 1, 3, ...

The Wronskian $W_1(u, s) = \frac{i}{2}\begin{vmatrix}J(u,s) & \overline{J(u,s)}\\ K(u,s) & \overline{K(u,s)}\end{vmatrix} = \Im(-J(u, s)\overline{K(u, s)})$ verifies the differential equation

$$\frac{d}{du}W_1(u, s) = \Im(E)\,(|J(u, s)|^2 + |K(u, s)|^2) \tag{44}$$

For $\Re(s) < 1$, we have $W_1(u, s) \sim_{a\to 0} a^{1-2\Re(s)}$. For $\Re(s) < \frac{1}{2}$, we have $\Im(E) > 0$ and deduce from (44) that $\Im(-J(u, s)\overline{K(u, s)}) > 0$ which confirms the m-property of the ratio $\frac{-J(u,s)}{K(u,s)}$ (and the fact

[8] the equations (40a) (40b) also give easily for a given a asymptotic expansions of $\widehat{j_a}(s)$, $\widehat{k_a}(s)$ in powers of $\frac{1}{s}$ in the critical strip, and this gives thus corresponding information on $\widehat{A_a}$, $\widehat{B_a}$ and $\widehat{E_a}$.

that none of J and K can vanish in this half-plane, whatever the value of u). Additionally, for $\Re(s) = \frac{1}{2}$ we obtain:

Proposition 13. *For $s = \frac{1}{2} + iE$, $E \in \mathbb{R}$ and $-\infty < u < +\infty$ there holds*

$$\Im\left(-J(u,s)\,\overline{K(u,s)}\right) = 1 \tag{45}$$

Hence the functions $J(u,s)$ and $K(u,s)$ do not vanish on the line $\Re(s) = \frac{1}{2}$.

Alternatively, we could also have deduced (not only for E real but for all $E \in \mathbb{C}$) the constant value $-2i$ of the Wronskian $\begin{vmatrix} J(u,\frac{1}{2}+iE) & J(u,\frac{1}{2}-iE) \\ K(u,\frac{1}{2}+iE) & -K(u,\frac{1}{2}-iE)\end{vmatrix}$ from the behavior at $-\infty$. The spectral measure $\nu(a,b) = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_a^b \Im(m(E+i\epsilon))\,dE$, $m(E) = \frac{-J(u_0,s)}{K(u_0,s)}$ is thus purely absolutely continuous:

$$\nu(dE) = \frac{1}{\pi\,|K(u_0,\frac{1}{2}+iE)|^2}\,dE \tag{46}$$

The spectral expansion takes the following form: we send $T(u) = \left[\begin{smallmatrix}\alpha(u)\\ \beta(u)\end{smallmatrix}\right]$ to

$$\widetilde{T}(E) = \int_{-\infty}^{u_0} \alpha(u)\psi_0(u,\frac{1}{2}+iE) + \beta(u)\psi_1(u,\frac{1}{2}+iE)\,du \tag{47}$$

where $\psi(u,s) = \left[\begin{smallmatrix}\psi_0(u,s)\\ \psi_1(u,s)\end{smallmatrix}\right]$ is the unique solution of the differential system verifying the boundary condition $\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$ at u_0, [9] and this transform is isometric with the following Plancherel identity:

$$\int_{-\infty}^{u_0} |\alpha(u)|^2 + |\beta(u)|^2\,\frac{du}{2} = \frac{1}{2\pi}\int_{\mathbb{R}} |\widetilde{T}(E)|^2 \frac{dE}{|K(u_0,\frac{1}{2}+iE)|^2} \tag{48}$$

The two linearly independent eigensolutions of the differential system are $\begin{bmatrix} J(u,\frac{1}{2}+iE)\\ K(u,\frac{1}{2}+iE)\end{bmatrix}$ and $\begin{bmatrix} J(u,\frac{1}{2}-iE)\\ -K(u,\frac{1}{2}-iE)\end{bmatrix}$ with Wronskian $-2i$. Hence:

$$\psi(u,s) = \frac{-i\,K(u_0,\frac{1}{2}-iE)}{2}\begin{bmatrix} J(u,\frac{1}{2}+iE)\\ K(u,\frac{1}{2}+iE)\end{bmatrix} + \frac{-i\,K(u_0,\frac{1}{2}+iE)}{2}\begin{bmatrix} J(u,\frac{1}{2}-iE)\\ -K(u,\frac{1}{2}-iE)\end{bmatrix} \tag{49}$$

$$\psi_0(u,\frac{1}{2}+iE) = \Im\left(K(u_0,\frac{1}{2}-iE)J(u,\frac{1}{2}+iE)\right) \tag{50}$$

$$\psi_1(u,\frac{1}{2}+iE) = \Im\left(K(u_0,\frac{1}{2}-iE)K(u,\frac{1}{2}+iE)\right) \tag{51}$$

[9] as $s = \frac{1}{2} + iE$, $E \in \mathbb{R}$, this solution is real valued.

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