# Extended scaling relations for planar lattice models


G. Benfatto[1] P. Falco[2]
V. Mastropietro[1]

[1] Dipartimento di Matematica, Università di Roma "Tor Vergata"
via della Ricerca Scientifica, I-00133, Roma

[2] Mathematics Department, University of British Columbia,
Vancouver, BC Canada, V6T 1Z2



**Abstract**

It is widely believed that the critical properties of several planar lattice models, like the Eight Vertex or the Ashkin-Teller models, are well described by an effective Quantum Field Theory obtained as formal scaling limit. On the basis of this assumption several extended scaling relations among their indices were conjectured. We prove the validity of some of them, among which the ones by Kadanoff, [13], and by Luther and Peschel, [16].


## 1 Introduction and main results

Integrable models in statistical mechanics, like the Ising or the Eight vertex (8V) models in two dimensions, provide conceptual laboratories for the understanding of phase transitions. Integrability is however a rather delicate property requiring very special features, and it is usually lost in more realistic models.

The principle of *universality*, phenomenologically quite well verified, says that the singularities for second order phase transitions should be insensitive to the specific details of the model, provided that symmetry and some form of locality are retained. From the theoretical side, a mathematical justification of universality in planar lattice models is rather complex to provide. Only very recently Pinson and Spencer established, see [27, 24], a form of universality for the 2D Ising model; they added to the Ising Hamiltonian a perturbation breaking the integrability and showed that the indices they can compute were *exactly the same* as the Ising model ones.

While the critical indices of the Ising model are expressed by *pure numbers*, there are other lattice models in which some of the critical exponents vary continuously with the parameters appearing in the Hamiltonian. A celebrated example is provided by the Eight vertex model, solved by Baxter in [2]; even if it can be mapped in two Ising models coupled by a quartic interaction, its critical indices are different from the Ising ones.

Several authors, starting from Kadanoff and collaborators [13, 14, 15] and Luther and Peschel [16] , have argued that many models, like the *Askhin-Teller* (AT) model and several others, belongs to the class of universality of the 8V model. The notion of universality in this case is much more subtle; it does not mean that the indices are the same for all the models in the same class (on the contrary, the indices depend on all details of the Hamiltonian), but that there are *scaling relations* between them, such that all indices can be expressed in terms of any one of them.

The notion of universality for models with continuously varying indices has been deeply investigated over the years, see for instance [15, 22, 23, 28]; it has been pointed out that such models are well described in the scaling limit by an effective Quantum Field Theory, and on the basis of this assumption several extended scaling relations between their indices were derived. While the assumption of continuum scaling limit description of planar lattice models is very powerful, it is well known that a mathematical justification of it is very difficult, see *e.g.* [26].

The aim of this paper is to provide a mathematical proof of some of the exact scaling relations derived in the literature for planar lattice models. We will focus mainly on the 8V and AT models, but, as we will explain after the main theorem below, our result can be extend to several other models.

We start from the well known (see [3]) Ising formulation of the 8V and the AT models. Let $\Lambda$ be a square subset of $\mathbb{Z}^2$ of side $L$; if $\mathbf{x} = (x_0, x) \in \Lambda$ and $\mathbf{e}_0 = (1, 0)$, $\mathbf{e}_1 = (0, 1)$, we consider two independent configurations of spins, $\{\sigma_{\mathbf{x}} = \pm 1\}_{\mathbf{x}\in\Lambda}$ and $\{\sigma'_{\mathbf{x}} = \pm 1\}_{\mathbf{x}\in\Lambda}$ and the Hamiltonian

$$H(\sigma, \sigma') = H_J(\sigma) + H_{J'}(\sigma') - J_4 V(\sigma, \sigma') \; , \tag{1}$$

where $J > 0$ and $J' > 0$ are two parameters, $H_J$ is the (ferromagnetic) Ising Hamiltonian in the lattice $\Lambda$,

$$H_J(\sigma) = -J \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \; , \tag{2}$$

$V$ is the quartic interaction and $-J_4$ is the coupling. In the the AT model, $J$ and $J'$ can be different (that case is called *anisotropic*) and $V = V_{AT}$, with

$$V_{AT}(\sigma, \sigma') = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_{\mathbf{x}} \sigma'_{\mathbf{x}+\mathbf{e}_j} \; . \tag{3}$$

In the 8V model $J = J'$ and $V = V_{8V}$, with

$$V_{8V}(\sigma, \sigma') = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)} \sigma_{\mathbf{x}+\mathbf{e}_0} \sigma'_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)} \sigma'_{\mathbf{x}+\mathbf{e}_1} \; . \tag{4}$$

In this paper we will focus our attention on two observables,

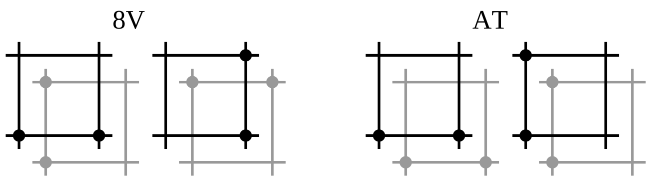


Figure 1: : The quartic interaction in the 8V and in the AT case. The gray and the black square are the same square of the lattice.

$$O^{\varepsilon}_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} + \varepsilon \sum_{j=0,1} \sigma'_{\mathbf{x}} \sigma'_{\mathbf{x}+\mathbf{e}_j} \quad , \quad \varepsilon = \pm \; , \tag{5}$$

and their truncated correlations in the *thermodynamic limit*

$$G^{\varepsilon}(\mathbf{x} - \mathbf{y}) = \lim_{\Lambda\to\infty} \langle O^{\varepsilon}_{\mathbf{x}} O^{\varepsilon}_{\mathbf{y}} \rangle_{\Lambda} - \langle O^{\varepsilon}_{\mathbf{x}} \rangle_{\Lambda} \langle O^{\varepsilon}_{\mathbf{y}} \rangle_{\Lambda} \quad , \quad \varepsilon = \pm \; , \tag{6}$$

where $\langle \cdot \rangle_\Lambda$ is the average over all configurations of the spins with statistical weight $e^{-\beta H(\sigma,\sigma')}$. In the AT model, $\langle O^+_{\mathbf{x}} \rangle$ is called the *energy*, while $\langle O^-_{\mathbf{x}} \rangle$ is called the *crossover*; in the 8V model is the opposite, see *e.g.* [22].

Despite their similarity, an exact solution exists for the 8V model but *not* for the AT model. In recent times the methods of *constructive* fermionic Renormalization (see *e.g.* [21] for an updated introduction) has been applied to such models, using the well known representation of such correlations in terms of *Grassmann integrals*, see *e.g.* [25]. It was proved in [17, 18] that both the 8V and the isotropic AT systems have a nonzero *critical temperature*, $T_c$, such that, if $T \neq T_c$, $G^\varepsilon(\mathbf{x}-\mathbf{y})$ decays faster than any power of $\xi|\mathbf{x}-\mathbf{y}|$, with

$$\xi \sim C\,|T-T_c|^\alpha \text{ , as } T \to T_c \; . \tag{7}$$

Moreover, at criticality, there are two constants $C_\varepsilon$, $\varepsilon = \pm$, such that

$$G^\varepsilon(\mathbf{x}-\mathbf{y}) \sim \frac{C_\varepsilon}{|\mathbf{x}-\mathbf{y}|^{2x_\varepsilon}} \text{ , as } |\mathbf{x}-\mathbf{y}| \to \infty \; , \tag{8}$$

where $x_\pm$ are critical indices expressed by *convergent series* in $J_4$. The analysis in [18] allows to compute the indices $\alpha, x_\pm$ with arbitrary precision (by an explicit computation of the lowest orders and a rigorous bound on the rest); the complexity of such expansions makes however essentially impossible to see directly from them the extended scaling relations.

In the case of the anisotropic AT model, it was proven in [11] that there are two critical temperatures, $T_{1,c}$ and $T_{2,c}$, and the corresponding critical indices are the same as those of the Ising model. However as $J - J' \to 0$:

$$|T_{1,c} - T_{2,c}| \sim |J-J'|^{x_T} \; , \tag{9}$$

with a *transition index*, $x_T$, different form 1 if $J_4 \neq 0$.

In this paper we will prove the following Theorem.

**Theorem 1.1** *If the coupling is small enough, the critical indices of the 8V or AT verify*

$$x_- = \frac{1}{x_+} \; , \tag{10}$$

$$\alpha = \frac{1}{2-x_+} \; ; \tag{11}$$

*and, in the case of the anisotropic AT model,*

$$x_T = \frac{2-x_+}{2-x_-} \; . \tag{12}$$

*Moreover, if $-J_4^{AT}$ and $-J_4^{8V}$ denote the coupling in the two models, there exists a choice of $J_4^{AT}$ as function of $J_4^{8V}$ such that the above critical indices coincide.*

**Remarks**

1. Equation (10) is the *extended scaling law* first conjectured by Kadanoff for the AT and 8V models, mainly on the basis of numerical evidence (see eq.(13b) and (15b) of [13]). The scaling relation (12) was never conjectured before. Note that all the critical indices we consider can be expressed as simple functions of one of them, in agreement with the general belief.

2. A similar theorem can be proved for a number of other models in the same class of universality. An example is provided by the $XYZ$ model, describing the nearest-neighbor interaction of quantum spins on a chain with couplings $J_1, J_2, J_3$. In [7], by a rigorous Renormalization Group analysis valid for small values of $J_3$, it was possible to write two critical indices as a convergent series in $J_3$; there were the index $1+\eta_1$, appearing in the oscillating part of the spin-spin correlation along the $z$ direction (see (1.20) of [7]), and $1+\eta_2$, the index appearing in the decay rate (see (1.19) of [7]). In such a case the analogue of the second of (1.10) can be written as

$$1+\eta_2 = \frac{1}{2-2(1+\eta_1)^{-1}} \tag{13}$$

   The above relation for the $XYZ$ indices has been conjectured by Luther and Peschel in [16] (see eq.(16) and table I of that paper).

3. Our results could be easily extended to any Hamiltonian of the form (1), if the quartic interactions verifies some symmetry conditions, listed in App. O of [18].

4. Several other relations are conjectured in the literature, concerning critical indices which are much more difficult to study with our methods, like the indices of the polarization correlations. New ideas seems to be required to treat such cases.

The paper is organized in the following way. In §2 we summarize the analysis given in [18,19], in which the correlations of the AT or 8V models are written in terms of Grassmann integrals and are analyzed using constructive Renormalization Group methods. The outcome of such analysis is that the critical indices $x_+$, $x_-$, $\alpha$ and $x_T$ can be written, in the small coupling region, as *model independent* convergent series of a single parameter, $\lambda_{-\infty}$, the asymptotic limit of the effective coupling on large scale. Note that $\lambda_{-\infty}$ is in turn a convergent series (that does depend on all the details of the lattice model) of the coupling $J_4$. Such expansions allow in principle to compute the indices with arbitrary precision, but this is not needed to prove (11) and (12), which simply follow from dimensional arguments. On the contrary, dimensional arguments are not sufficient to prove (10); and it is apparently impossible to check it directly in terms of the series representing $x_+$ and $x_-$, as functions of $\lambda_{-\infty}$.

In §3 we show that such indices are *equal* to the indices of the Quantum Field Theory coinciding with the formal scaling limit of the spin models, provided the *bare parameters* of such a theory are chosen properly as suitable functions of the parameters of the 8V or AT models; such functions are expressed in terms of convergent expansions depending on all details of the spin models. On the other hand, the QFT verifies extra quantum symmetries with respect to the original spin Hamiltonian (1), implying a set of *Ward Identities* and closed equations allowing to get simple exact expressions for the critical indices in terms of the coupling of the QFT; (10) follows from such expressions.

# 2 RG analysis of spin models

## 2.1 Fermionic representation of the partition function

We begin with considering the partition function of the Ising model with a quadratic interaction, external sources $A_{j,\mathbf{x}}$, and periodic conditions at the boundary of $\Lambda$:

$$Z(I) = \sum_\sigma \exp\Big[\sum_{\substack{j=0,1\\ \mathbf{x}\in\Lambda}} I_{j,\mathbf{x}}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\Big] \tag{14}$$

where $I_{j,\mathbf{x}} = A_{j,\mathbf{x}} + \beta J$. The purpose of adding the external source is twofold: by taking derivatives w.r.t. $A$, either we can write the partition function for (1) in terms of two non-interacting Ising models, or we can generate the correlations of the quadratic observables.

Indeed, since $\sigma_{\mathbf{x}}, \sigma'_{\mathbf{x}} = \pm 1$,

$$\exp\left(\alpha \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_{\mathbf{y}} \sigma'_{\mathbf{y}+\mathbf{e}_{j'}}\right) = \cosh(\alpha) + \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_{\mathbf{y}} \sigma'_{\mathbf{y}+\mathbf{e}_{j'}} \sinh(\alpha) \;,$$

so that the partition function of the two models with external fields is given by:

$$\begin{aligned} Z(J_4, I, I') &= [\cosh(\beta J_4)]^{2|\Lambda|} \cdot \\ &\cdot \prod_{\substack{j=0,1\\ \mathbf{x}\in\Lambda}} \left[1 + \tanh(\beta J_4) \frac{\partial^2}{\partial \widetilde{A}_{j,\mathbf{x}} \widetilde{A}'_{j,\mathbf{x}}}\right] Z(I) Z(I') \;, \end{aligned} \tag{15}$$

where $I'_{j,\mathbf{x}} = A'_{j,\mathbf{x}} + \beta J'$; and, in the AT case, $\widetilde{A}_{j,\mathbf{x}} = A_{j,\mathbf{x}}$ and $\widetilde{A}'_{j,\mathbf{x}} = A'_{j,\mathbf{x}}$, while, in the 8V case, $\widetilde{A}_{0,\mathbf{x}} = A_{0,\mathbf{x}}$, $\widetilde{A}'_{0,\mathbf{x}} = A'_{1,\mathbf{x}}$, $\widetilde{A}_{1,\mathbf{x}} = A_{1,\mathbf{x}+\mathbf{e}_0}$, $\widetilde{A}'_{1,\mathbf{x}} = A'_{0,\mathbf{x}+\mathbf{e}_1}$.

For $Z(I)$, the partition function of the Ising model with periodic boundary condition, a fermionic representation is known since a long time, see [25].

The result is the following. Let $\gamma = (\varepsilon_0, \varepsilon_1)$, with $\varepsilon_0, \varepsilon_1 = \pm$ and let $\{H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ be a family of Grassmann variables verifying the $\gamma$-boundary conditions, namely

$$\begin{aligned} \bar{H}_{\mathbf{x}+(L,0)} &= \varepsilon_0 \bar{H}_{\mathbf{x}} \quad , \quad \bar{H}_{\mathbf{x}+(0,L)} = \varepsilon_1 \bar{H}_{\mathbf{x}} \;, \\ H_{\mathbf{x}+(L,0)} &= \varepsilon_0 H_{\mathbf{x}} \quad , \quad H_{\mathbf{x}+(0,L)} = \varepsilon_1 H_{\mathbf{x}} \;, \end{aligned} \tag{16}$$

and similar relations for $V, \bar{V}$ (we are skipping the $\gamma$ dependence in $H$'s and $V$'s). Then we consider the *Grassmann functional integral*

$$Z_\gamma = \int dH dV \; e^{S(t)} \;, \tag{17}$$

where the action $S(t)$ is the following function of the parameters $t = \{t_{j,\mathbf{x}}\}_{\substack{\mathbf{x}\in\Lambda\\ j=0,1}}$ and of the Grassmann variables with $\gamma-$boundary condition:

$$\begin{aligned} S(t) = &\sum_{\mathbf{x}\in\Lambda} \Big[t_{0,\mathbf{x}} \bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} + t_{1,\mathbf{x}} \bar{V}_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1}\Big] + \\ &+ \sum_{\mathbf{x}\in\Lambda} \Big[\bar{H}_{\mathbf{x}} H_{\mathbf{x}} + \bar{V}_{\mathbf{x}} V_{\mathbf{x}} + \bar{V}_{\mathbf{x}} \bar{H}_{\mathbf{x}} + V_{\mathbf{x}} H_{\mathbf{x}} + V_{\mathbf{x}} \bar{H}_{\mathbf{x}} + H_{\mathbf{x}} \bar{V}_{\mathbf{x}}\Big] \;. \end{aligned} \tag{18}$$

Choosing $t_{j,\mathbf{x}} = \tanh I_{j,\mathbf{x}}$, and for $c_{j,\mathbf{x}} = \cosh I_{j,\mathbf{x}}$, the partition function (14) can be written in the following way:

$$Z(I) = (-1)^{|\Lambda|} 2^{|\Lambda|} \left(\prod_{j,\mathbf{x}} c_{j,x}\right) \sum_\gamma \frac{(-1)^{\delta_\gamma}}{2} Z_\gamma \tag{19}$$

where $\delta_\gamma = 1$ for $\gamma = (+,+)$, and $\delta_\gamma = 0$ otherwise.

By (15), $Z(J_4, I, I')$ can be written by doubling the above representation and explicitly taking the derivatives w.r.t. $\widetilde{A}_{j,\mathbf{x}}$ and $\widetilde{A}'_{j,\mathbf{x}}$. After some trivial algebra, we get the following result.

Let us call $\widetilde{t}_{j,\mathbf{x}}$, $\widetilde{c}_{j,\mathbf{x}}$ the expressions obtained from $t_{j,\mathbf{x}}$, $c_{j,\mathbf{x}}$ by substituting $A_{j,\mathbf{x}}$ with $\widetilde{A}_{j,\mathbf{x}}$; in a similar way we define $\widetilde{t}'_{j,\mathbf{x}}$, $\widetilde{c}'_{j,\mathbf{x}}$. Let us now define:

$$f_{j,\mathbf{x}} = 1 + \tanh(\beta J_4) \widetilde{t}_{j,\mathbf{x}} \widetilde{t}'_{j,\mathbf{x}} \;,$$

$$g_{j,\mathbf{x}} = \frac{\widetilde{t}'_{j,\mathbf{x}}}{(\widetilde{c}_{j,\mathbf{x}})^2}\frac{\tanh(\beta J_4)}{f_{j,\mathbf{x}}} \quad , \quad g'_{j,\mathbf{x}} = \frac{\widetilde{t}_{j,\mathbf{x}}}{(\widetilde{c}'_{j,\mathbf{x}})^2}\frac{\tanh(\beta J_4)}{f_{j,\mathbf{x}}} \ ,$$

$$h_{j,\mathbf{x}} = \frac{1}{(\widetilde{c}'_{j,\mathbf{x}})^2(\widetilde{c}_{j,\mathbf{x}})^2}\frac{\tanh(\beta J_4)}{f_{j,\mathbf{x}}} - g_{j,\mathbf{x}}g'_{j,\mathbf{x}} \ . \tag{20}$$

Then we can write the partition function of the interacting models as

$$Z(J_4, I, I') = 4^{|\Lambda|}\left[\cosh(\beta J_4)\right]^{2|\Lambda|}\left(\prod_{j,\mathbf{x}} f_{j,\mathbf{x}}c_{j,\mathbf{x}}c'_{j,\mathbf{x}}\right)\cdot$$

$$\cdot\sum_{\gamma,\gamma'}\frac{(-1)^{\delta_\gamma+\delta_{\gamma'}}}{4}Z_{\gamma,\gamma'}(J_4) \ , \tag{21}$$

where $Z_{\gamma,\gamma'}(J_4)$ is the Grassmannian functional integral

$$Z_{\gamma,\gamma'}(J_4) = \int dHdVdH'dV' \ e^{\widetilde{S}(\widetilde{t}+g)+\widetilde{S}'(\widetilde{t}'+g')+V(h)} \ , \tag{22}$$

with boundary conditions $\gamma = (\varepsilon_0, \varepsilon_1)$ and $\gamma' = (\varepsilon'_0, \varepsilon'_1)$ on the variables $H$, $V$ and $H'$, $V'$, respectively. Moreover $\widetilde{S}(t)$ and $\widetilde{S}'(t)$ have a definition which depends on the model. $\widetilde{S}(t)$ is equal to $S(t)$ in the AT model, while, in the 8V model, it is the function which is obtained from $S(t)$, by substituting, in the first line of (18), $\bar{V}_{\mathbf{x}}V_{\mathbf{x}+\mathbf{e}_1}$ with $\bar{V}_{\mathbf{x}+\mathbf{e}_0}V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1}$. $\widetilde{S}'(t)$, in the AT case, is obtained from $S(t)$, by simply replacing $H, V$ with $H', V'$, while, in the 8V case, we also have to substitute $\bar{H}'_{\mathbf{x}}H'_{\mathbf{x}+\mathbf{e}_0}$ with $\bar{V}'_{\mathbf{x}}V'_{\mathbf{x}+\mathbf{e}_1}$ and $\bar{V}'_{\mathbf{x}}V'_{\mathbf{x}+\mathbf{e}_1}$ with $\bar{H}'_{\mathbf{x}+\mathbf{e}_1}H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0}$. Finally, $V(h)$ is a quartic interaction that, in the AT case, is given by

$$V_{AT}(h) = \sum_{\mathbf{x}\in\Lambda}\left[h_{0,\mathbf{x}}\bar{H}_{\mathbf{x}}H_{\mathbf{x}+\mathbf{e}_0}\bar{H}'_{\mathbf{x}}H'_{\mathbf{x}+\mathbf{e}_0} + h_{1,\mathbf{x}}\bar{V}_{\mathbf{x}}V_{\mathbf{x}+\mathbf{e}_1}\bar{V}'_{\mathbf{x}}V'_{\mathbf{x}+\mathbf{e}_1}\right] \ , \tag{23}$$

while, in the 8V case, is given by

$$V_{8V}(h) = \sum_{\mathbf{x}\in\Lambda}\left[h_{0,\mathbf{x}}\bar{H}_{\mathbf{x}}H_{\mathbf{x}+\mathbf{e}_0}\bar{V}'_{\mathbf{x}}V'_{\mathbf{x}+\mathbf{e}_1} + h_{1,\mathbf{x}}\bar{V}_{\mathbf{x}+\mathbf{e}_0}V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1}\bar{H}'_{\mathbf{x}+\mathbf{e}_1}H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0}\right] \ . \tag{24}$$

We remark that

$$g_{j,\mathbf{x}}, \ g'_{j,\mathbf{x}}, \ h_{j,\mathbf{x}} = \mathrm{O}(\beta J_4) \ . \tag{25}$$

## 2.2 Fermionic representation of the correlations

The truncated correlations of the quadratic observables are obtained by taking two derivatives of $\ln Z(J_4, I, I')$ w.r.t. the external sources in two different points, and putting such external sources to zero. The addends $2|\Lambda|\ln[2\cosh(\beta J_4)]$ and $\sum_{j,\mathbf{x}}(\ln f_{j,\mathbf{x}} + \ln c_{j,\mathbf{x}} + \ln c'_{j,\mathbf{x}})$ do not contribute when we take two derivatives in the $A$ variables of two different points. Moreover, it has been proved in [18] that all and 16 partition functions $Z_{\gamma,\gamma'}$ have the same thermodynamic limit; hence, from now on we will substitute them with the same one, that with $\gamma = \gamma' = (-,-)$. If we define $\partial^\varepsilon_{j,\mathbf{x}} = \partial/\partial A_{j,\mathbf{x}} + \varepsilon\partial/\partial A'_{j,\mathbf{x}}$, we get:

$$\langle O^\varepsilon_{\mathbf{x}}; O^\varepsilon_{\mathbf{y}}\rangle^T_\Lambda = \sum_{i,j}\partial^\varepsilon_{i,\mathbf{x}}\partial^\varepsilon_{j,\mathbf{y}}\ln Z_{\gamma,\gamma}\Bigg|_{A\equiv 0} = \frac{\partial^2\ln\bar{Z}(\bar{A})}{\partial\bar{A}^\varepsilon_{\mathbf{x}}\partial\bar{A}^\varepsilon_{\mathbf{y}}}\Bigg|_{\bar{A}\equiv 0} \tag{26}$$

where

$$\bar{Z}(\bar{A}) = \int dHdVdH'dV' \ e^{S(s)+S(s')+2\lambda V+B(\bar{A})} \ , \tag{27}$$

$s$, $s'$ and $h$ are $j, \mathbf{x}-$independent parameters, defined as

$$
\begin{aligned}
s &= t_{j,\mathbf{x}} + g_{j,\mathbf{x}}\big|_{A\equiv 0} = \tanh(\beta J) + \mathrm{O}(\beta J_4) \\
s' &= t'_{j,\mathbf{x}} + g'_{j,\mathbf{x}}\big|_{A\equiv 0} = \tanh(\beta J') + \mathrm{O}(\beta J_4) \\
2\lambda &= h_{j,\mathbf{x}}\big|_{A\equiv 0} = \mathrm{O}(\beta J_4) \; ; \qquad (28)
\end{aligned}
$$

$B(\bar{A})$ is an interaction with external sources $\bar{A}^\varepsilon_{\mathbf{x}}$, given, in the AT case, by

$$
\begin{aligned}
B(\bar{A}) &= \sum_{\substack{\mathbf{x}\in\Lambda \\ \varepsilon=\pm}} \bar{A}^\varepsilon_{\mathbf{x}} \left[ q_\varepsilon \left( \bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1} \right) + q'_\varepsilon \left( \bar{H}'_{\mathbf{x}} H'_{\mathbf{x}+\mathbf{e}_0} + \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right) \right] + \\
&+ \sum_{\substack{\mathbf{x}\in\Lambda \\ \varepsilon=\pm}} \bar{A}^\varepsilon_{\mathbf{x}} p_\varepsilon \left( \bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} \bar{H}'_{\mathbf{x}} H'_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1} \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right) \; , \qquad (29)
\end{aligned}
$$

while, in the 8V case, it is given by

$$
\begin{aligned}
& B(\bar{A}) = \sum_{\mathbf{x}\in\Lambda, \varepsilon=\pm} \bar{A}^\varepsilon_{\mathbf{x}} \left[ q_\varepsilon \left( \bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}+\mathbf{e}_0} V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1} \right) + \right. \\
& \left. + q'_\varepsilon \left( \bar{H}'_{\mathbf{x}+\mathbf{e}_1} H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0} + \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right) \right] + \qquad (30) \\
& + \sum_{\substack{\mathbf{x}\in\Lambda \\ \varepsilon=\pm}} \bar{A}^\varepsilon_{\mathbf{x}} p_\varepsilon \left( \bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} + \bar{V}_{\mathbf{x}+\mathbf{e}_0} V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1} \bar{H}'_{\mathbf{x}+\mathbf{e}_1} H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0} \right) \; ;
\end{aligned}
$$

finally, $q_\varepsilon$, $q'_\varepsilon$ and $p_\varepsilon$ are given by the $j, \mathbf{x}-$independent parameters

$$
\begin{aligned}
q_\varepsilon &= \sum_i \left( \frac{\partial}{\partial A_{j,\mathbf{x}}} + \varepsilon \frac{\partial}{\partial A'_{j,\mathbf{x}}} \right) (\widetilde{t}_{i,\mathbf{x}} + g_{i,\mathbf{x}}) \Bigg|_{A\equiv 0} \quad , \quad q'_\varepsilon = \{ \widetilde{t}, g \to \widetilde{t}', g' \} \; , \\
p_\varepsilon &= \sum_i \left( \frac{\partial h_{i,\mathbf{x}}}{\partial A_{j,\mathbf{x}}} + \varepsilon \frac{\partial h_{j,\mathbf{x}}}{\partial A'_{j,\mathbf{x}}} \right) \Bigg|_{A\equiv 0} \; . \qquad (31)
\end{aligned}
$$

Note that $q_\varepsilon = 1 - \tanh(\beta J) + O(\beta J_4)$, $q'_\varepsilon = \varepsilon[1 - \tanh(\beta J')] + O(\beta J_4)$ and $p_\varepsilon = O(\beta J_4)$.

## 2.3 Dirac and Majorana fermions

In order to make more evident the analogy of the above functional integral with the action of a fermionic (Euclidean) Quantum Field Model, it is convenient to make a change of variables in the Grassmann algebra. This change of variables is the analogous in the euclidean theories of the transformation from *Dirac fermions* to *Majorana fermions* in real time QFT.

The new Grassmannian variables will be denoted by $\psi_{\mathbf{x}}$, $\bar{\psi}_{\mathbf{x}}$, $\chi_{\mathbf{x}}$ and $\bar{\chi}_{\mathbf{x}}$ and are related to the old ones by the equations:

$$
\begin{aligned}
\bar{H}_{\mathbf{x}} + iH_{\mathbf{x}} &= e^{i\frac{\pi}{4}} \left( \psi_{\mathbf{x}} - \chi_{\mathbf{x}} \right) \quad , \quad \bar{V}_{\mathbf{x}} + iV_{\mathbf{x}} = \psi_{\mathbf{x}} + \chi_{\mathbf{x}} \; , \\
\bar{H}_{\mathbf{x}} - iH_{\mathbf{x}} &= e^{-i\frac{\pi}{4}} \left( \bar{\psi}_{\mathbf{x}} - \bar{\chi}_{\mathbf{x}} \right) \quad , \quad \bar{V}_{\mathbf{x}} - iV_{\mathbf{x}} = \bar{\psi}_{\mathbf{x}} + \bar{\chi}_{\mathbf{x}} \; . \qquad (32)
\end{aligned}
$$

A similar transformation is done for the primed variables. After a straightforward computation, we see that the action (18), calculated at $t_{j,\mathbf{x}} = s$, $\forall j, \mathbf{x}$, can be written in terms of the Majorana fields as

$$
S(s) = A(\psi, m_s) + A(\chi, M_s) + Q(\psi, \chi) \; , \qquad (33)
$$

where $m_s = 1-\sqrt{2}+s$, $M_s = 1+\sqrt{2}+s$ and, if we define $\partial^i\psi_{\mathbf{x}} = \psi_{\mathbf{x}+\mathbf{e}_i} - \psi_{\mathbf{x}}$,

$$\begin{aligned} A(\psi,m) &= \frac{s}{4}\sum_{\mathbf{x}\in\Lambda}\left[\psi_{\mathbf{x}}\left(\partial^0 - i\partial^1\right)\psi_{\mathbf{x}} + \text{c.c.}\right] - im\sum_{\mathbf{x}\in\Lambda}\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} + \\ &+ \frac{s}{4}\sum_{\mathbf{x}\in\Lambda}\left[\bar\psi_{\mathbf{x}}\left(-i\partial^0 - i\partial^1\right)\psi_{\mathbf{x}} + \text{c.c.}\right]\ , \end{aligned} \tag{34}$$

$$\begin{aligned} Q(\psi,\chi) = &- \frac{s}{4}\sum_{\mathbf{x}\in\Lambda}\left[\psi_{\mathbf{x}}\left(\partial^0 + i\partial^1\right)\chi_{\mathbf{x}} + \left\{\psi\leftrightarrow\chi\right\} + \text{c.c.}\right] - \\ &- \frac{s}{4}\sum_{\mathbf{x}\in\Lambda}\left[\bar\chi_{\mathbf{x}}\left(-i\partial^0 + i\partial^1\right)\psi_{\mathbf{x}} + \left\{\psi\leftrightarrow\chi\right\} + \text{c.c.}\right]\ , \end{aligned} \tag{35}$$

where, in agreement with (32), we are calling complex conjugation (c.c.) the operation on the Grassmann algebra which amounts to exchange $\psi_{\mathbf{x}}$ with $\bar\psi_{\mathbf{x}}$, $\chi_{\mathbf{x}}$ with $\bar\chi_{\mathbf{x}}$ and $i$ with $-i$.

The quartic interaction of the AT model becomes:

$$\begin{aligned} V_{AT} = -\lambda\sum_{\mathbf{x}\in\Lambda}\left[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\psi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\chi'_{\mathbf{x}}\chi'_{\mathbf{x}} + \{\psi\leftrightarrow\chi\}\right] - \\ -\lambda\sum_{\mathbf{x}\in\Lambda}\left[\bar\chi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\chi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\chi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\psi'_{\mathbf{x}}\chi'_{\mathbf{x}} + \{\psi\leftrightarrow\chi\}\right] + \text{irr.}\ , \end{aligned} \tag{36}$$

where the irrelevant part (irr.) is made of quartic terms with at least one (discrete) derivative; we will discuss later on why these term are less important. In the case of the 8V model, the second square bracket has $+\lambda$ in front, rather than $-\lambda$.

If we set $b_\varepsilon = (q_\varepsilon + \varepsilon q'_\varepsilon)/2$ and $d_\varepsilon = (q_\varepsilon - \varepsilon q'_\varepsilon)/2$, the interaction with the external field is given by

$$\begin{aligned} B(\bar A) = -i\sum_{\substack{\mathbf{x}\in\Lambda\\ \varepsilon=\pm}} b_\varepsilon \bar A^\varepsilon_{\mathbf{x}}\left[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} + \varepsilon\bar\psi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\chi_{\mathbf{x}}\chi_{\mathbf{x}} + \varepsilon\bar\chi'_{\mathbf{x}}\chi'_{\mathbf{x}}\right] - \\ -i\sum_{\substack{\mathbf{x}\in\Lambda\\ \varepsilon=\pm}} d_\varepsilon \bar A^\varepsilon_{\mathbf{x}}\left[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} - \varepsilon\bar\psi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\chi_{\mathbf{x}}\chi_{\mathbf{x}} - \varepsilon\bar\chi'_{\mathbf{x}}\chi'_{\mathbf{x}}\right] + \text{irr.}, \end{aligned}$$

where the irrelevant terms are, in this case, either quartic in the fields or quadratic with derivatives. We remark that, if $J = J'$, then $d_\varepsilon = 0$, while $b_\varepsilon = 1 - \tanh(\beta J) + O(\beta J_4)$.

We now make another change of variables, defined by the relations

$$\psi^\varepsilon_{\mathbf{x},+} = \frac{\psi_{\mathbf{x}} - \varepsilon i\psi'_{\mathbf{x}}}{\sqrt{2}}\quad,\quad \psi^\varepsilon_{\mathbf{x},-} = \frac{\bar\psi_{\mathbf{x}} - \varepsilon i\bar\psi'_{\mathbf{x}}}{\sqrt{2}}\quad,\quad \varepsilon=\pm\ , \tag{37}$$

and the similar ones for the $\chi$-variables. If we put $u = (s+s')/2$, $v = (s-s')/2$ and $m_\varepsilon = (m_s + \varepsilon m_{s'})/2$, we get

$$\begin{aligned} &A(\psi,m_s) + A(\psi',m_{s'}) = \\ &= \sum_{\mathbf{x}\in\Lambda}\Big\{\frac{u}{4}\left[\psi^+_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^-_{\mathbf{x},+} + \psi^-_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^+_{\mathbf{x},+} + \text{c.c.}\right] + \\ &+\frac{u}{4}\left[\psi^-_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^+_{\mathbf{x},-} + \psi^+_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^-_{\mathbf{x},-} + \text{c.c.}\right] + \\ &+\frac{v}{4}\left[\psi^+_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^+_{\mathbf{x},+} + \psi^-_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^-_{\mathbf{x},+} + \text{c.c.}\right] + \\ &+\frac{v}{4}\left[\psi^-_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^-_{\mathbf{x},-} + \psi^+_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^+_{\mathbf{x},-} + \text{c.c.}\right] - \\ &-im_+\left[\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+} - \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-}\right] + im_-\left[\psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\right]\Big\}\ , \end{aligned} \tag{38}$$

where now the c.c. operation amounts to exchange $\psi^{\varepsilon}_{\mathbf{x},\omega}$ with $\psi^{-\varepsilon}_{\mathbf{x},-\omega}$ and $i$ with $-i$.

The interaction with the external source is

$$\begin{aligned}B(\bar{A}) = i\sum_{\mathbf{x}\in\Lambda}(b_+\bar{A}^+_{\mathbf{x}} + d_-\bar{A}^-_{\mathbf{x}})[\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-} - \psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+} + \\ +\chi^+_{\mathbf{x},+}\chi^-_{\mathbf{x},-} - \chi^+_{\mathbf{x},-}\chi^-_{\mathbf{x},+}] + i\sum_{\mathbf{x}\in\Lambda}(b_-\bar{A}^-_{\mathbf{x}} + d_+\bar{A}^+_{\mathbf{x}})\cdot \\ \cdot[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-} + \psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \chi^+_{\mathbf{x},+}\chi^+_{\mathbf{x},-} + \chi^-_{\mathbf{x},+}\chi^-_{\mathbf{x},-}] + \mathrm{irr.}\ .\end{aligned} \tag{39}$$

Finally the quartic self interaction is given by

$$\begin{aligned}\mathcal{V}(\psi,\chi) = \lambda\sum_{\mathbf{x}\in\Lambda}\left[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \chi^+_{\mathbf{x},+}\chi^+_{\mathbf{x},-}\chi^-_{\mathbf{x},+}\chi^-_{\mathbf{x},-}\right] + \\ +v(\psi,\chi) + \text{irrel. terms}\ ,\end{aligned} \tag{40}$$

where $v(\psi,\chi)$ is a quartic interaction depending both on $\psi$ and $\chi$, which has a different expression in the AT and 8V models, as well as the irrelevant terms.

## 2.4 Multiscale integration

Let $\mathcal{D}$ be the set of $\mathbf{k}$'s such that $k_0 = \frac{2\pi}{L}(n_0+\frac{1}{2})$ and $k_1 = \frac{2\pi}{L}(n_1+\frac{1}{2})$, for $n_0, n_1 = -\frac{L}{2},\ldots,\frac{L}{2}-1$, and $L$ and even integer. Then, the Fourier transform for the fermions with antiperiodic boundary condition is defined by

$$\psi^{\varepsilon}_{\mathbf{x},\omega} \stackrel{def}{=} \frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}} e^{i\varepsilon\mathbf{k}\mathbf{x}}\widehat{\psi}^{\varepsilon}_{\mathbf{k},\omega}\ . \tag{41}$$

Therefore (38) can be written as

$$A(\psi, m_s) + A(\psi', m_{s'}) = \frac{u}{2|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}}\Phi^+_{\mathbf{k}}S(\mathbf{k})\Phi_{\mathbf{k}}\ , \tag{42}$$

where

$$\begin{aligned}\Phi_{\mathbf{k}} &= (\widehat{\psi}^-_{\mathbf{k},+}, \widehat{\psi}^-_{\mathbf{k},-}, \widehat{\psi}^+_{-\mathbf{k},+}, \widehat{\psi}^+_{-\mathbf{k},-})\ ,\\ \Phi^+_{\mathbf{k}} &= (\widehat{\psi}^+_{\mathbf{k},+}, \widehat{\psi}^+_{\mathbf{k},-}, \widehat{\psi}^-_{-\mathbf{k},+}, \widehat{\psi}^-_{-\mathbf{k},-})\ ,\end{aligned} \tag{43}$$

and , if we define

$$\begin{aligned}\widehat{D}_\omega(\mathbf{k}) &= -i\sin k_0 + \omega\sin k_1\ ,\\ \mu(\mathbf{k}) &= (\cos k_0 + \cos k_1 - 2) + 2\frac{1-\sqrt{2}+u}{u}\ ,\\ \sigma(\mathbf{k}) &= \frac{v}{u}(\cos k_0 + \cos k_1 - 2) + 2\frac{v}{u}\ ,\end{aligned} \tag{44}$$

the matrix $S(\mathbf{k})$ is given by

$$S(\mathbf{k}) = \begin{pmatrix} \widehat{D}_-(\mathbf{k}) & i\mu(\mathbf{k}) & \frac{v}{u}\widehat{D}_-(\mathbf{k}) & i\sigma(\mathbf{k}) \\ -i\mu(\mathbf{k}) & \widehat{D}_+(\mathbf{k}) & -i\sigma(\mathbf{k}) & \frac{v}{u}\widehat{D}_+(\mathbf{k}) \\ \frac{v}{u}\widehat{D}_-(\mathbf{k}) & +i\mu(\mathbf{k}) & \widehat{D}_-(\mathbf{k}) & i\sigma(\mathbf{k}) \\ -i\mu(\mathbf{k}) & \frac{v}{u}\widehat{D}_+(\mathbf{k}) & -i\sigma(\mathbf{k}) & \widehat{D}_+(\mathbf{k}) \end{pmatrix}. \tag{45}$$

From now until the end of the section we will only consider the case $J = J'$; some details about the anisotropic AT model are deferred to the appendix.

Hence we have $v = 0$ and $\sigma(\mathbf{k}) \equiv 0$, so that we get the much simpler equation

$$A(\psi, m_s) + A(\psi', m_{s'}) = -\frac{1}{|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}} \sum_{\omega,\omega'} \widehat{\psi}^+_{\mathbf{k},\omega} \widehat{\psi}^-_{\mathbf{k},\omega'} T_{\omega,\omega'}(\mathbf{k}) \;, \tag{46}$$

with

$$T(\mathbf{k}) = u \begin{pmatrix} i\sin k_0 + \sin k_1 & -i\mu(\mathbf{k}) \\ & \\ i\mu(\mathbf{k}) & i\sin k_0 - \sin k_1 \end{pmatrix} \;. \tag{47}$$

In the same way and with similar definitions, we get also

$$A(\chi, M_s) + A(\chi', M_{s'}) = -\frac{1}{|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}} \sum_{\omega,\omega'} \widehat{\chi}^+_{\mathbf{k},\omega} \widehat{\chi}^-_{\mathbf{k},\omega'} T^\chi_{\omega,\omega'}(\mathbf{k}) \;, \tag{48}$$

where $T^\chi(\mathbf{k})$ is the matrix obtained from $T(\mathbf{k})$ by substituting $\mu(\mathbf{k})$ with

$$\mu^\chi(\mathbf{k}) = (\cos k_0 + \cos k_1 - 2) + 2\frac{1+\sqrt{2}+u}{u} \;. \tag{49}$$

Hence, we can write the functional integral (27) as

$$\bar{Z}(\bar{A}) = \frac{1}{\mathcal{N}} \int P(d\psi) P_\chi(d\chi)\; e^{\mathcal{Q}(\psi,\chi)+\mathcal{V}(\psi,\chi)+B(\bar{A})} \;, \tag{50}$$

where $\mathcal{N}$ is a normalization constant and $P(d\psi)$ is the (Grassmannian) Gaussian measure with propagator

$$g(\mathbf{x}) = \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}\mathbf{x}} T^{-1}(\mathbf{k}) \;, \tag{51}$$

$P_\chi(d\chi)$ is the Gaussian measure with propagator $g_\chi(\mathbf{x})$, which is obtained from $g(\mathbf{x})$ by replacing $T(\mathbf{k})$ with $T^\chi(\mathbf{k})$, $\mathcal{Q}(\psi,\chi)$ is the sum of the quadratic terms $Q(\psi,\chi)$ and $Q(\psi',\chi')$, represented in terms of the new variables; $B(A)$ and $\mathcal{V}(\psi,\chi)$ are defined in (39) and (40).

If $J > 0$ and $J_4$ is any real number, $u$ is a strictly increasing function of $\tanh(\beta J)$ and has range $(0,1)$, as one can check by using the definition of $s$, see (28). On the other hand, $\det T(\mathbf{k}) = 0$ only if $\mathbf{k} = 0$ and $\mu(\mathbf{k}) = 0$; hence, $g(\mathbf{x})$ has a singularity at $u = u_c = \sqrt{2} - 1$, which is an allowed value; moreover, if $\beta|J_4| \ll 1$ (as we shall suppose in the following), $u = \tanh(\beta J) + O(\beta J_4)$. Since we expect that the interaction will move this singularity, it is convenient to modify the interaction by adding a finite *counterterm* $i\nu\frac{1}{L^2}\sum_{\omega,\mathbf{k}} \omega\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},-\omega}$, which is compensated by replacing, in the matrix $T(\mathbf{k})$, $\mu(\mathbf{k})$ with

$$\mu_1(\mathbf{k}) = (\cos k_0 + \cos k_1 - 2) + 2(1 - \frac{u^*}{u}) \quad , \quad u^* = \sqrt{2} - 1 - \nu \;. \tag{52}$$

Let us call $T_1(\mathbf{k})$ the new matrix and $P_1(d\psi)$ the corresponding measure; we get

$$\bar{Z}(\bar{A}) = \frac{1}{\mathcal{N}_1} \int P_1(d\psi) P_\chi(d\chi)\; e^{\mathcal{Q}(\psi,\chi)+\mathcal{V}^{(1)}(\psi,\chi)+B(\bar{A})} \;, \tag{53}$$

where

$$\mathcal{V}^{(1)}(\psi,\chi) = i\nu\frac{1}{L^2} \sum_{\omega,\mathbf{k}} \omega\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},-\omega} + \mathcal{V}(\psi,\chi) \;, \tag{54}$$

and $\nu$ has to be determined so that the interacting propagator has an infrared singularity at $u = u^*$; the critical temperature is uniquely determined by the value of $u^*$.

Let us now remark that $\det T^\chi(\mathbf{k})$ is strictly positive for any $\mathbf{k}$, as one can easily see by using the fact that $u \in (0,1)$. On the other hand, it is easy to see that

$$\mathcal{Q}(\psi,\chi) = -\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}}\sum_{\omega,\omega'}[\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\chi}^-_{\mathbf{k},\omega'} + \widehat{\chi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},\omega'}]Q_{\omega,\omega'}(\mathbf{k}) , \tag{55}$$

where $Q(\mathbf{k})$ is a matrix which vanishes at $\mathbf{k} = 0$. Hence, if we define

$$\widetilde{\psi}^+ = \psi^+ Q T_\chi^{-1} \quad , \quad \widetilde{\psi}^- = T_\chi^{-1} Q \psi^- , \tag{56}$$

the change of variables $\chi^+ \to \chi^+ + \widetilde{\psi}^+$, $\chi^- \to \chi^- + \widetilde{\psi}^-$, allows us to rewrite (53) in the form

$$\bar{Z}(\bar{A}) = \frac{1}{\mathcal{N}}\int P_{Z_1,\mu_1}(d\psi)P_\chi(d\chi)\, e^{\mathcal{V}^{(1)}(\psi,\chi-\widetilde{\psi})+\widetilde{B}(\bar{A})} , \tag{57}$$

where $\widetilde{B}(\bar{A})$ is the functional obtained from $B(\bar{A})$ by replacing $\chi$ with $\chi-\widetilde{\psi}$ and $P_{Z_1,\mu_1}(d\psi)$ is the Gaussian measure with propagator

$$g(\mathbf{x}) = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}\mathbf{x}}(T^{(1)})^{-1}(\mathbf{k}) , \tag{58}$$

where $T^{(1)}(\mathbf{k}) = T(\mathbf{k}) - Q(\mathbf{k})T_\chi^{-1}Q(\mathbf{k})$. In order to agree with the conventions about fermion models we used in our previous papers, we make also the trivial change of variables

$$\widehat{\psi}^+_{\mathbf{k},\omega} \to -i\omega\widehat{\psi}^+_{\widetilde{\mathbf{k}},\omega} \quad , \quad \widehat{\psi}^-_{\mathbf{k},\omega} \to \widehat{\psi}^-_{\widetilde{\mathbf{k}},\omega} \quad , \quad \mathbf{k} = (k_0, k_1) \quad , \quad \widetilde{\mathbf{k}} = (k_1, k_0) . \tag{59}$$

Hence, by an explicit calculation of $Q(\mathbf{k})$ and using the identity $u^*/u = 1 - \mu_1(0)/2$, one can see that $T^{(1)}(\mathbf{k})$ is the matrix

$$C_1(\mathbf{k})\begin{pmatrix} Z_1(-i\sin k_0 + \sin k_1) + \mu_{+,+}(\mathbf{k}) & -\mu_1 - \mu_{+,-}(\mathbf{k}) \\ -\mu_1 - \mu_{+,-}(\mathbf{k}) & Z_1(-i\sin k_0 - \sin k_1) + \mu_{-,-}(\mathbf{k}) \end{pmatrix} \tag{60}$$

with $C_1(\mathbf{k}) = 1$, $\mu_1 = 2u^*\mu_1(0)/(2-\mu_1(0))$ and $Z_1 = u^*$; moreover $\mu_{+,+}(\mathbf{k}) = -\mu_{-,-}(\mathbf{k})^*$ is an odd function of $\mathbf{k}$ of the form $\mu_{+,+}(\mathbf{k}) = 2u^*\mu_1(0)(-i\sin k_0 + \sin k_1)/(4-2\mu_1(0)) + O(|\mathbf{k}|^3)$, while $\mu_{+,-}(\mathbf{k})$ is a real even function, of order $|\mathbf{k}|^2$, which vanishes only at $\mathbf{k} = 0$. Finally, $\det T^{(1)}(\mathbf{k}) \ge C(2 - \cos k_0 - \cos k_1)$, so that $P_{Z_1,\mu_1}(d\psi)$ has the same type of infrared singularity as $P_1(d\psi)$.

The fact that $\det T_\chi(\mathbf{k})$ is strictly positive implies that $g_\chi(\mathbf{x})$ is an exponential decaying function; hence, we can safely perform the integration over the field $\chi$ in (57). The result can be written in the following form (see Lemma 1 of [18])

$$\bar{Z}(\bar{A}) \equiv e^{\mathcal{S}(\bar{A})} = \int P_{Z_1,\mu_1}(d\psi)e^{L^2\mathcal{N}^{(1)}+\bar{\mathcal{V}}^{(1)}(\psi)+B^{(1)}(\bar{A})} , \tag{61}$$

where $\mathcal{N}^{(1)}$ is a constant and the *effective potential* $\bar{\mathcal{V}}^{(1)}(\psi)$ can be represented as

$$\bar{\mathcal{V}}^{(1)} = \sum_{n\ge1}\sum_{\underline{\alpha},\underline{\omega},\underline{\varepsilon}}\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline{\omega},\underline{\alpha},\underline{\varepsilon},2n}(\mathbf{x}_1,..,\mathbf{x}_{2n})\partial^{\alpha_1}\psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1}...\partial^{\alpha_{2n}}\psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}} . \tag{62}$$

The kernels $W_{\underline{\omega},\underline{\alpha},\underline{\varepsilon},2n}$ in the previous expansions are analytic functions of $\lambda$ and $\nu$ near the origin; if we suppose that $\nu = O(\lambda)$, their Fourier transforms satisfy, for any $n \ge 1$, the bounds, see [18]

$$|\widehat{W}_{\underline{\alpha},\underline{\omega},\underline{\varepsilon},2n}(\mathbf{k}_1,...\mathbf{k}_{2n-1})| \le L^2 C^n |\lambda|^n . \tag{63}$$

A similar representation can be written for the functional of the external field $B^{(1)}(\bar{A})$.

As explained in detail in [18], the symmetries of the two models we are considering imply that, in the r.h.s. of (62), there are no local terms quadratic in the field, which are relevant or marginal, except those which are already present in the free measure and are all marginal. It follows that the integration in (61) can be done by iteratively integrating the fields with decreasing momentum scale and by moving to the free measure all the marginal terms quadratic in the field. We introduce a scaling parameter $\gamma = 2$, a decomposition of the unity $1 = f_1 + \sum_{h=-\infty}^{0} f_h(\mathbf{k})$, with $f_h(\mathbf{k})$ a function with support $\{\gamma^{h-1}\pi/4 \leq |\mathbf{k}| \leq \gamma^{h+1}\pi/4\}$, and the corresponding decomposition of the field $\psi = \sum_{j=-\infty}^{1} \psi^{(j)}$. If the fields $\psi^{(1)}, .., \psi^{(h+1)}$ are integrated, we get

$$e^{\mathcal{S}(\bar{A})} = e^{S^{(h)}(\bar{A})} \int P_{Z_h,\mu_h}(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\bar{A})} , \tag{64}$$

where $\psi^{(\leq h)} = \sum_{j=-\infty}^{h} \psi^{(j)}$ and $P_{Z_h,\mu_h}(d\psi)$ is the Gaussian measure with the propagator obtained from (58) by replacing in (60) $C_1(\mathbf{k})$ with $C_h(\mathbf{k}) = [\sum_{k=-\infty}^{h} f_h(\mathbf{k})]^{-1}$, $\mu_1$ with $\mu_h$, $Z_1$ with $Z_h$ and the functions $\mu_{\sigma,\sigma'}(\mathbf{k})$ with similar functions $\mu^{(h)}_{\sigma,\sigma'}(\mathbf{k})$ (which turn out to be negligible for $h \to -\infty$, as a consequence of the following analysis). The *effective interaction* $\mathcal{V}^{(h)}(\psi)$ can be written as

$$\mathcal{V}^{(h)}(\psi) = \gamma^h \nu_h F_\nu^{(h)} + \lambda_h F_\lambda^{(h)} + R^{(h)}(\psi) \equiv \mathcal{L}\mathcal{V}^{(h)}(\psi) + R^{(h)}(\psi) , \tag{65}$$

where $\nu_h$ and $\lambda_h$ are suitable real numbers,

$$\begin{aligned} F_\nu^{(h)} &= \frac{1}{L^2} \sum_\omega \sum_{\mathbf{k}} \hat{\psi}^{(\leq h)+}_{\mathbf{k},\omega} \hat{\psi}^{(\leq h)-}_{\mathbf{k},-\omega} , \\ F_\lambda^{(\leq h)} &= \frac{1}{L^8} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_4} \hat{\psi}^{(\leq h)+}_{\mathbf{k}_1,+} \hat{\psi}^{(\leq h)+}_{\mathbf{k}_3,-} \hat{\psi}^{(\leq h)-}_{\mathbf{k}_2,+} \hat{\psi}^{(\leq h)-}_{\mathbf{k}_4,-} \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) , \end{aligned} \tag{66}$$

and $R^{(h)}(\psi)$ is expressed by a sum over monomials similar to (62), with $2n+\alpha_1+..+\alpha_{2n} > 4$ ; the kernels are bounded if $\sup_{k\geq h}(|\lambda_k| + |\nu_k|)$ is small enough. According to power counting, $F_\nu$ is relevant, $F_\lambda$ is marginal while all terms in $R^h$ are irrelevant. Moreover

$$\begin{aligned} &\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \bar{A}) = \sum_{\varepsilon,\mathbf{x}} Z_h^{(\varepsilon)} \bar{A}^\varepsilon_{\mathbf{x}} O^{(\leq h)\varepsilon}_{\mathbf{x}} + R_1^{(h)}(\psi^{(\leq h)}, \bar{A}) \equiv \\ &\mathcal{L}\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \bar{A}) + R_1^{(h)}(\psi^{(\leq h)}, \bar{A}) , \end{aligned} \tag{67}$$

where

$$\begin{aligned} O^{(\leq h)+}_{\mathbf{x}} &= \psi^{(\leq h)+}_{\mathbf{x},+} \psi^{(\leq h)-}_{\mathbf{x},-} + \psi^{(\leq h)+}_{\mathbf{x},-} \psi^{(\leq h)-}_{\mathbf{x},+} , \\ O^{(\leq h)-}_{\mathbf{x}} &= i[\psi^{(\leq h)+}_{\mathbf{x},+} \psi^{(\leq h)+}_{\mathbf{x},-} + \psi^{(\leq h)-}_{\mathbf{x},+} \psi^{(\leq h)-}_{\mathbf{x},-}] , \end{aligned} \tag{68}$$

and $R_1^{(h)}(\psi^{(\leq h)}, \bar{A})$ is a sum of irrelevant terms. Note that many other possible local marginal or relevant terms could be generated in the RG integration, which are however absent due to the symmetry of the problem, as proved in [18], App.F (see also [11], §A2.2). The above integration procedure is done till the scale $h^*$ defined as the maximal $j$ such that $\gamma^j \leq |\mu_j|$, and the integration of the fields $\psi^{(\leq h^*)}$ can be done in a single step. Roughly speaking, $h^*$ defines the momentum scale of the mass.

The propagator of the field $\psi^{(\leq h)}$ can be written, for $h \leq 0$, as

$$g^{(\leq h)}(\mathbf{x}, \mathbf{y}) = g_T^{(\leq h)}(\mathbf{x}, \mathbf{y}) + r^{(\leq h)}(\mathbf{x}, \mathbf{y}) , \tag{69}$$

where

$$g_T^{(\leq h)}(\mathbf{x},\mathbf{y}) = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{Z_h}T_h^{-1}(\mathbf{k}) \ , \tag{70}$$

$$T_h(\mathbf{k}) = C_h(\mathbf{k})\begin{pmatrix} -ik_0+k_1 & -\mu_h \\ \mu_h & -ik_0-k_1 \end{pmatrix} \ , \tag{71}$$

and, for any positive integer $M$,

$$|r^{(\leq h)}(\mathbf{x},\mathbf{y})| \leq C_M \frac{\gamma^{2h}}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|^M)} \ . \tag{72}$$

The propagator $g_T^{(h)}(\mathbf{x},\mathbf{y})$ verifies a similar bound with $\gamma^h$ replacing $\gamma^{2h}$. A similar decomposition can be done for $g^{(h)}(\mathbf{x},\mathbf{y})$.

The effective couplings $\lambda_j$ (which, by construction, are the same in the massless $\mu=0$ or in the massive $\mu\neq 0$ case, see [11]), satisfy a recursive equation of the form

$$\lambda_{j-1} = \lambda_j + \beta_\lambda^{(j)}(\lambda_j,...,\lambda_0) + \bar\beta_\lambda^{(j)}(\lambda_j,\nu_j;...;\lambda_0,\nu_0) \tag{73}$$

where $\beta_\lambda^{(j)}$, $\bar\beta_\lambda^{(j)}$ are $\mu$-independent and expressed by a *convergent* expansion in $\lambda_j,\nu_j..,\lambda_0,\nu_0$; moreover $\bar\beta_\lambda^{(j)}$ vanishing if at least one of the $\nu_k$ is zero. From the decomposition (69), the smaller bound on propagators $r$ and because of a special feature of the propagator $g_T$, the following property, called *vanishing of the Beta function*, was proved in Theorem 2 of [9] for suitable positive constants $C$ and $\vartheta<1$:

$$|\beta_\lambda^{(j)}(\lambda_j,...,\lambda_j)| \leq C|\lambda_j|^2\gamma^{\vartheta j} \ . \tag{74}$$

Moreover, it is possible to prove that, for a suitable choice of $\nu_1 = O(\lambda)$, $\nu_j = O(\gamma^{\vartheta j}\bar\lambda_j)$, if $\bar\lambda_j = \sup_{k\geq j}|\lambda_k|$, and this implies, by the *short memory* property ( see for instance A4.6 of [11]), $\bar\beta_\lambda^{(j)} = O(\gamma^{\vartheta j}\bar\lambda_j^2)$ so that the sequence $\lambda_j$ converges, as $j\to-\infty$, to a smooth function $\lambda_{-\infty}(\lambda) = \lambda + O(\lambda^2)$, such that

$$|\lambda_j - \lambda_{-\infty}| \leq C\lambda^2\gamma^{\vartheta j} \ . \tag{75}$$

Moreover

$$\frac{Z_{j-1}}{Z_j} = 1 + \beta_z^{(j)}(\lambda_j,...,\lambda_0) + \bar\beta_z^{(j)}(\lambda_j,\nu_j;..,\lambda_0,\nu_0) \ , \tag{76}$$

with $\bar\beta_z^{(j)}$ vanishing if at least one of the $\nu_k$ is zero so that, by $\nu_j = O(\gamma^{\vartheta j}\bar\lambda_j)$ and the short memory property, $\bar\beta_z^{(j)} = O(\lambda_j\gamma^{\vartheta j})$. Finally

$$\beta_z(\lambda_j,...,\lambda_0) = \beta_z(\lambda_{-\infty},...,\lambda_{-\infty}) + O(\lambda\gamma^{\vartheta h}) \ , \tag{77}$$

where the last identity follows from (75) and the *short memory* property. An important point is that the function $\beta_z(\lambda_{-\infty},...,\lambda_{-\infty})$ is model independent. Similar equations hold for $Z_h^{(\pm)}, \mu_h$, with leading terms again model independent.

By an explicit computation and (77) there exist $\eta_+(\lambda_{-\infty}) = c_1\lambda_{-\infty}+O(\lambda_{-\infty}^2)$, $\eta_-(\lambda_{-\infty}) = -c_1\lambda_{-\infty}+O(\lambda_{-\infty}^2)$, $\eta_\mu(\lambda_{-\infty}) = c_1\lambda_{-\infty}+O(\lambda_{-\infty}^2)$ and $\eta_z(\lambda_{-\infty}) = c_2\lambda_{-\infty}^2+O(\lambda_{-\infty}^3)$, with $c_1$ and $c_2$ strictly positive, such that, for any $j\leq 0$,

$$\begin{aligned}
&|\log_\gamma(Z_{j-1}/Z_j) - \eta_z(\lambda_{-\infty})| \leq C\lambda^2\gamma^{\vartheta j} \ , \\
&|\log_\gamma(\mu_{j-1}/\mu_j) - \eta_\mu(\lambda_{-\infty})| \leq C|\lambda|\gamma^{\vartheta j}, \\
&|\log_\gamma(Z_{j-1}^{(\pm)}/Z_j^{(\pm)}) - \eta_\pm(\lambda_{-\infty})| \leq C\lambda^2\gamma^{\vartheta j} \ .
\end{aligned} \tag{78}$$

The critical indices are functions of $\lambda_{-\infty}$ only, as it is clear from (77); moreover from (6.28) ad (5.4) of [18],

$$x_\pm = 1 - \eta_\pm + \eta_z \quad , \quad \eta_\mu = \eta_+ - \eta_z = 1 - x_+ \; . \tag{79}$$

When the limit $\mu \to 0$ is taken (after the limit $L \to \infty$, so that all the $Z_{\gamma,\gamma'}$ have the same limit), the multiscale integration procedure implies the power law decay of the correlations given by (8).

If $\mu \neq 0$ (that is, if the temperature is not the critical one), the correlations decay faster than any power with rate proportional to $\mu_{h^*}$, where, if $[x]$ denotes the largest integer $\leq x$, $h^*$ is given by

$$h^* = \left[\frac{\log_\gamma |\mu|}{1+\eta_\mu}\right] \; , \tag{80}$$

so that

$$\alpha = \frac{1}{2 - x_+} \; . \tag{81}$$

# 3 Equivalence with an effective QFT

## 3.1 The effective QFT

We introduce a QFT model, which has a large distance behavior of the same type as that of the formal scaling limit of the spin models with Hamiltonian (1.1). As a general fact, the relations between the critical indices and the coupling depend on the regularization procedure used to define the QFT model; the kind of regularization that we are going to use allows us to get expressions for the critical indices, simple enough to prove the extended scaling relations.

The QFT model is defined as the limit $N \to \infty$, followed by the limit $-l \to \infty$, to be called *the removed cutoff limit*, of a model with an infrared $\gamma^l$ and an ultraviolet $\gamma^N$ momentum cut-off, $-l, N \geq 0$. This model is expressed in terms of the following Grassmann integral

$$\begin{aligned} e^{\mathcal{W}_N(A,J,\varphi)} = \int P(d\psi^{[l,N]}) \exp\Bigg\{ \mathcal{V}^{(N)}(\psi^{[l,N]}) + \sum_\varepsilon \int d\mathbf{x} A^\varepsilon_{\mathbf{x}} O_{\varepsilon,\mathbf{x}} + \\ + \sum_\omega \int d\mathbf{x}\, [J_{\mathbf{x},\omega} \psi^{[l,N]+}_{\mathbf{x},\omega} \psi^{[l,N]-}_{\mathbf{x},\omega} + \psi^{+[l,N]}_{\mathbf{x},\omega} \varphi^-_{\mathbf{x},\omega} + \varphi^+_{\mathbf{x},\omega} \psi^{[l,N]-}_{\mathbf{x},\omega}] \Bigg\} \; , \end{aligned} \tag{82}$$

where $\mathbf{x} \in \widetilde{\Lambda}$, a square subset of $\mathbb{R}^2$, $O^+_{\mathbf{x}}$ and $O^-_{\mathbf{x}}$ are defined in (68) and $P(d\psi^{[l,N]})$ is a Gaussian measure with propagator $g^{[l,N]}_T(\mathbf{x},\mathbf{y})$ given by (70) with $\mu_h = \mu, Z_h = 1$ and $C_h^{-1}(\mathbf{k})$ replaced by $C_{l,N}^{-1}(\mathbf{k}) = \sum_{k=l}^N f_k(\mathbf{k})$. The interaction is

$$\mathcal{V}^{(N)}(\psi) = \frac{\lambda_\infty}{2} \sum_\omega \int d\mathbf{x} \int d\mathbf{y} v_K(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},\omega} \psi^+_{\mathbf{y},-\omega} \psi^-_{\mathbf{x},\omega} \psi^-_{\mathbf{y},-\omega} \; , \tag{83}$$

where $K < N$ and $v_K(\mathbf{x}-\mathbf{y})$ is given by

$$v_K(\mathbf{x}-\mathbf{y}) = \frac{1}{L^2} \sum_{\mathbf{p}} \chi_0(\gamma^{-K}\mathbf{p}) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \; , \tag{84}$$

$\chi_0(\mathbf{p})$ being a smooth function with support in $\{|\mathbf{p}| \leq 2\}$ and equal to 1 for $\{|\mathbf{p}| \leq 1\}$. The correlation functions are found by making suitable derivatives with respect to the external fields $A_{\mathbf{x}}$, $J_{\mathbf{x}}$, $\varphi_{\mathbf{x}}$ and setting them equal to zero.

Note that $\lim_{K\to\infty} v_K(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$, so that the model becomes the Thirring model in the limit $K \to \infty$ (taken after the limit $N \to \infty$), if one also introduces an ultraviolet renormalization of the field, $\lambda_\infty$ and $\mu$. However, in the following we shall take $K$ fixed, for example $K = 0$, so that no ultraviolet regularization is needed.

We shall study the functional $\mathcal{W}_N(A, J, \varphi)$ by performing a multiscale integration of (82); we have to distinguish two different regimes: the first regime, called *ultraviolet*, contains the scales $h \in [K+1, N]$, while the second one contains the scales $h \leq K$, and is called *infrared*.

## 3.2 The ultraviolet integration

We shall briefly describe how to control the integration of the ultraviolet scales, without encountering any divergence We shall assume that the reader is familiar with the tree expansion, as described, for example, in [7], and we only sketch the proofs, omitting many details. Moreover, for simplicity, we shall only consider the case $A = \varphi = 0$ and $\mu = 0$, but the result is valid for the full problem; for more details in a similar case, see [19, 20].

If the fields $\psi^{(N)}, \psi^{(N-1)}, ..., \psi^{(h+1)}$ are integrated, we get an expression like (64) in which the fermionic integration is $P(d\psi^{[l,h]})$ with propagator $g_T^{[l,h]}$, and $V^{(h)}$ is sum of integrated monomials in $m$ $\psi^+_{\mathbf{x}_i,\omega_i}$ variables, $i = 1, \ldots, m$, $m$ $\psi^-_{\mathbf{y}_i,\omega_i}$ variables and $n$ $J_{\mathbf{z}_j,\omega'_j}$ external fields, $j = 1, \ldots, n$, multiplied by suitable kernels $W^{(n;2m)(h)}_{\underline{\omega}';\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})$. These kernels are represented as power expansions in $\lambda$ and $\nu$, with coefficients which are finite sums of products of delta functions (of the difference between couples of space variables) times smooth functions of the variables which remains after the constraints implied by the the delta functions are taken into account. With an abuse of notation, we shall denote by $\int d\underline{\mathbf{z}}d\underline{\mathbf{x}}d\underline{\mathbf{y}} \left| W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})\right|$ the expansion which is obtained by summing, for each coefficient, the $L^1$ norm of these smooth functions. We introduce the following norm

$$\|W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}\| \stackrel{def}{=} \frac{1}{|\widetilde{\Lambda}|}\int d\underline{\mathbf{z}}d\underline{\mathbf{x}}d\underline{\mathbf{y}} \left| W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})\right| . \tag{85}$$

**Theorem 3.1** *If $\lambda_\infty$ is small enough, there exist two constants $C_1 > 1$ and $C_2$, such that, if $K \leq h \leq N$, the relevant or marginal contributions to the effective potential satisfy the bounds:*

$$\|W^{(0;2)(h)}_{\omega}\| \leq C_1|\lambda_\infty|\gamma^h\gamma^{-2(h-K)} , \tag{86}$$

$$\|W^{(1;2)(h)}_{\omega';\omega} - \delta_2\delta_{\omega,\omega'}\| \leq C_2|\lambda_\infty|\gamma^{-(h-K)} , \tag{87}$$

$$\|W^{(0;4)(h)}_{\omega,\omega'} - \lambda_\infty v\delta_4\delta_{\omega,-\omega'}\| \leq C_2|\lambda_\infty|^2\gamma^{-(h-K)} , \tag{88}$$

*where $\delta_2(\mathbf{z};\mathbf{x},\mathbf{y}) \equiv \delta(\mathbf{z}-\mathbf{x})\delta(\mathbf{z}-\mathbf{y})$ and $v\delta_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2) \equiv \delta(\mathbf{x}_1-\mathbf{y}_1)v_K(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{y}_2)$.*

**Proof.** The proof is by induction: we assume that the bounds (86)-(88) hold for $h$ : $k+1 \leq h \leq N$ (for $h = N$ they are true with $C_1 = C_2 = 0$) and we prove them for $h = k$.

The starting point is the following remark. Suppose that we build the tree expansion, by defining the *localization operation* so that it acts as the identity on the relevant or marginal terms, that is $W^{(0;2)(h)}_{\omega}$, $W^{(1;2)(h)}_{\omega';\omega}$ and $W^{(0;4)(h)}_{\omega,\omega'}$, while it annihilates, as always, all the other contributions to the effective potential. Then, it is easy to see that the inductive assumption implies the following "dimensional" bound, for $\lambda_\infty$ small enough:

$$\|W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}\| \leq C^{n+d_{n,m}}|C_1\lambda_\infty|^{d_{n,m}}\gamma^{k(2-n-m)} , \tag{89}$$

where $d_{n,m} = \max\{m-1, 0\}$, if $n > 0$, and $d_{n,m} = \max\{m-1, 1\}$, if $n = 0$, and $C$ is a suitable constant larger, at least, of $\gamma$. In fact, the localization procedure and the bounds (86)-(88) imply that all the tree vertices have positive dimension and there are three types of endpoints, associated to $W^{(0;2)(h)}_{\omega}$, $W^{(1;2)(h)}_{\omega';\omega}$, $W^{(0;4)(h)}_{\omega,\omega'}$, which contribute (up to dimensional factors and for $\lambda_\infty$ small enough) a factor $C_1|\lambda_\infty|$, $1 + C_2|\lambda_\infty| \le C$ and $|\lambda_\infty|[1 + C_2|\lambda_\infty|] \le C_1|\lambda_\infty|$, respectively. Note that the condition $C > \gamma$ comes from the bound of the trivial tree (that with only one endpoint) contributing to the tree expansion of $W^{(0;2)(k)}_{\omega}$.

We need to improve the bound (89) when $2 - n - m \ge 0$. We can write, by using the properties of the fermionic truncated expectations and the fact that, by the oddness of the free propagator, $W^{(1;0)}_{\omega}(\mathbf{k}) = 0$,

$$\begin{aligned} &W^{(0;2)(k)}_{\omega}(\mathbf{x}, \mathbf{y}) = \\ &= \lambda_\infty \int d\mathbf{w} d\mathbf{w}'\; v_K(\mathbf{x} - \mathbf{w}) g^{[k+1,N]}_{\omega}(\mathbf{x} - \mathbf{w}') W^{(1;2)(k)}_{-\omega;\omega}(\mathbf{w}; \mathbf{w}', \mathbf{y}) \;, \end{aligned} \tag{90}$$

which can be bounded, by using (89), as

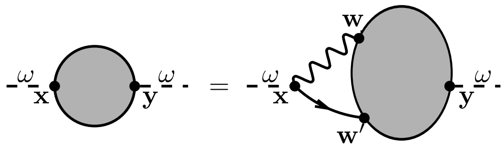


Figure 2: : Graphical representation of (90)

$$\begin{aligned} &\|W^{(0;2)(k)}_{\omega}\| \le |\lambda_\infty| \|v_K\|_{L^\infty} \|W^{(1;2)(k)}_{-\omega;\omega}\| \sum_{j=k+1}^{N} \|g^{(j)}_{\omega}\|_{L^1} \le \\ &\le \frac{c_1}{1-\gamma^{-1}} \gamma^{2K} C|\lambda_\infty|\gamma^{-k} \le C_1|\lambda_\infty|\gamma^{k}\gamma^{-2(k-K)} \;, \end{aligned} \tag{91}$$

where, for example, $C_1 = \max\{2, \frac{c_1}{1-\gamma^{-1}} C\}$; hence (86) is proved. Note that the condition $C_1 \ge 2$ is introduced only because $C_1$ is the same constant appearing in (89).

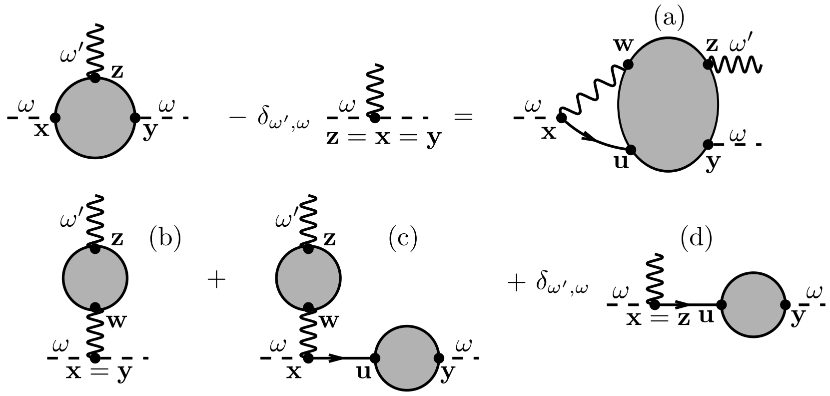


Figure 3: : Graphical representation of $W^{(1;2)(k)}_{\omega';\omega}(\mathbf{z}; \mathbf{x}, \mathbf{y})$

Let us now consider $W^{(1;2)(k)}_{\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y})$ and note that it can be decomposed as the sum of the five terms in Fig.3, The term denoted by $(a)$ in Fig.3 can be bounded as

$$\|W^{(1;2)(k)}_{(a);\omega';\omega}\| \leq |\lambda_\infty| \|v_K\|_{L^\infty} \|W^{(2;2)(k)}_{\omega',-\omega;\omega}\| \sum_{j=k+1}^{N} \|g^{(j)}_\omega\|_{L^1} \leq CC_1|\lambda_\infty|\gamma^{-2(k-K)} \; . \tag{92}$$

The bounds for the graphs $(c)$ and $(d)$ are an easy consequence of the the bound for $W^{(0;2)(k)}_{\omega}$.

In order to obtain an improved bound also for the graph $(b)$ of Fig. 3, we need to further expand $W^{(2;0)(k)}_{\omega,\omega'}$ as done in Fig 4, if we suppose that the arrows in the fermion lines of graph $(b2)$ can be reversed.

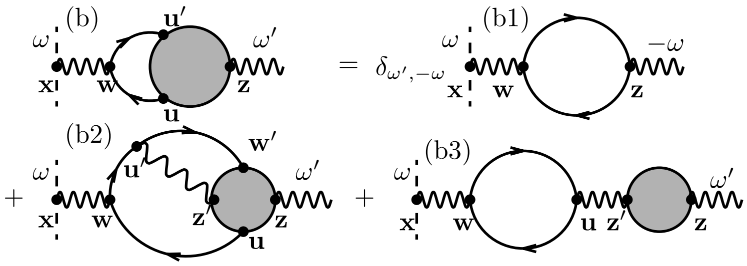


Figure 4: : Graphical representation of graph (b) in Fig.3

The bound for the graph $(b2)$ can be done by using the previous arguments. We can write

$$\begin{aligned} &W^{(1;2)(k)}_{(b2)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \lambda_\infty^2 \delta(\mathbf{x}-\mathbf{y}) \int d\mathbf{w} d\mathbf{u}' d\mathbf{z}' \; v_K(\mathbf{x}-\mathbf{w}) v_K(\mathbf{u}'-\mathbf{z}') \cdot \\ &\cdot \int d\mathbf{u} d\mathbf{w}' \; g^{[k+1,N]}_\omega(\mathbf{w}-\mathbf{u}) g^{[k+1,N]}_\omega(\mathbf{u}'-\mathbf{w}) g^{[k+1,N]}_\omega(\mathbf{w}'-\mathbf{u}') \cdot \\ &\cdot W^{(2;2)(k)}_{\omega',\omega;-\omega}(\mathbf{z},\mathbf{z}';\mathbf{w}',\mathbf{u}) \; . \end{aligned} \tag{93}$$

In order to get the right bound, it is convenient to decompose the three propagators $g_\omega$ into scales and then bound by the $L^\infty$ norm the propagator of lowest scale, while the two others are used to control the integration over the inner space variables through their $L^1$ norm. Hence we get:

$$\|W^{(1;2)(k)}_{(b2)\omega';\omega}\| \leq |\lambda_\infty|^2 \|v_K\|_{L^\infty} \|v_K\|_{L^1} \|W^{(2;2)(k)}_{\omega',-\omega;\omega}\| \cdot \tag{94}$$

$$\cdot 3! \sum_{k+1\leq i'\leq j\leq i\leq N} \|g^{(j)}_\omega\|_{L^1} \|g^{(i)}_\omega\|_{L^1} \|g^{(i')}_\omega\|_{L^\infty} \leq C_3|\lambda_\infty|^2\gamma^{-2(k-K)} \; . \tag{95}$$

for some constant $C_3$.

The bound of $(b1)$ and $(b3)$ requires a new argument, based on a cancelation following from the particular form of the free propagator. Let us consider, for instance, $(b1)$:

$$\begin{aligned} &W^{(1;2)(k)}_{(b1)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \\ &= \lambda_\infty \delta_{\omega',-\omega} \delta(\mathbf{x}-\mathbf{y}) \int d\mathbf{w} \; v_K(\mathbf{x}-\mathbf{w}) \left[ g^{[k+1,N]}_{-\omega}(\mathbf{w}-\mathbf{z}) \right]^2 \; . \end{aligned} \tag{96}$$

On the other hand, since the cutoff function $C_{k,N}(\mathbf{k})$ is symmetric in the exchange between $k_0$ and $k_1$, it is easy to see that $g^{[k,N]}_\omega(x_0,x_1) = -i\omega g^{[k,N]}_\omega(x_1,-x_0)$; hence

$$\int d\mathbf{u} \; \left[ g^{[k+1,N]}_{-\omega}(\mathbf{u}) \right]^2 = 0 \; . \tag{97}$$

It follows, by using (97) and the identity

$$v_K(\mathbf{x}-\mathbf{w}) = v_K(\mathbf{x}-\mathbf{z}) + \sum_{j=0,1} (z_j - w_j) \int_0^1 d\tau \, \big(\partial_j v_K\big)\big(\mathbf{x}-\mathbf{z}+\tau(\mathbf{z}-\mathbf{w})\big) \;, \tag{98}$$

that we can write

$$\begin{aligned} & W^{(1;2)(k)}_{(b1)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \lambda_\infty \delta_{\omega',-\omega}\delta(\mathbf{x}-\mathbf{y}) \cdot \\ & \cdot \sum_{j=0,1} \int_0^1 d\tau \int d\mathbf{w} \, \big(\partial_j v_K\big)\big(\mathbf{x}-\mathbf{z}+\tau(\mathbf{z}-\mathbf{w})\big)(z_j - w_j) \left[g^{[k+1,N]}_{-\omega}(\mathbf{w}-\mathbf{z})\right]^2 \;. \end{aligned} \tag{99}$$

Hence,

$$\|W^{(1;2)(k)}_{(b1)\omega';\omega}\| \le 4|\lambda_\infty| \sum_{i=k}^N \sum_{j=k}^i \|g^{(j)}_{-\omega}\|_{L^\infty} \int d\mathbf{x} \, \big|(\partial_j v_K)(\mathbf{x})\big| \cdot \tag{100}$$

$$\cdot \int d\mathbf{w} \, |w_j| |g^{(i)}_{-\omega}(\mathbf{w})| \le C_4 |\lambda_\infty| \gamma^{-(k-K)} \;. \tag{101}$$

By summing all the bounds, we see that there is a constant $C_2$ such that

$$\|W^{(1;2)(k)}_{\omega';\omega} - \delta_{\omega,\omega'}\delta_2\| \le C_2 |\lambda_\infty| \gamma^{-(k-K)} \;, \tag{102}$$

which proves (87). The bound (88) for $W^{(0;4)(k)}$ follows from similar arguments.

### 3.3 Equivalence of the spin and the QFT models

As a consequence of the integration of the ultraviolet scales discussed in the previous section, we can write the removed cutoffs limit of (82), with $\varphi = J = 0$ and with the choice $K = 0$, as

$$\lim_{l\to-\infty} \lim_{N\to\infty} \int P_{\mu_0,Z_0}(d\psi^{(\le 0)}) e^{\mathcal{V}^{(0)}(\psi^{(\le 0)}) + \mathcal{B}^{(0)}(\psi^{(\le 0)},A)} \;, \tag{103}$$

where the propagator of the integration measure in (103) coincides with $g^{(\le 0)}_T(\mathbf{x},\mathbf{y})$, defined in (70), $\mathcal{L}\mathcal{V}^{(0)} = \lambda_0 F^{(0)}_\lambda$ and $\mathcal{L}\mathcal{B}^{(0)}$ is defined as in (67); from the analysis of the previous section it follows that $\lambda_0$ is a smooth function of $\lambda_\infty$, such that $\lambda_0 = \lambda_\infty + O(\lambda_\infty^2)$.

The multiscale integration for the negative scales can be done exactly as described in §2.4, with the only difference that, by the oddness of the free propagator, $\nu_j = 0$ and

$$\lambda_{j-1} = \lambda_j + \widehat\beta^{(j)}_\lambda(\lambda_j, \ldots\lambda_0) \;, \tag{104}$$

where, by (69) and the short memory property,

$$\widehat\beta^{(j)}_\lambda(\lambda_j, \ldots\lambda_0) = \beta^{(j)}_\lambda(\lambda_j, \ldots\lambda_0) + O(\bar\lambda_j^2 \gamma^{\vartheta j}) \;, \tag{105}$$

$\beta^{(j)}_\lambda(\lambda_j, \ldots\lambda_j)$ being the function appearing in the bound (74), so that we can prove in the usual way that $\lambda_{-\infty} = \lambda_0 + O(\lambda_0^2)$; since $\lambda_0 = \lambda_\infty + O(\lambda_\infty^2)$, we have

$$\lambda_{-\infty} = h(\lambda_\infty) = \lambda_\infty + O(\lambda_\infty^2) \;, \tag{106}$$

for some analytic function $h(\lambda_\infty)$, invertible for $\lambda_\infty$ small enough. Moreover

$$\frac{Z^\pm_{j-1}}{Z^\pm_j} = 1 + \widehat\beta^{(j)}_\pm(\lambda_j, \ldots, \lambda_0) \;, \tag{107}$$

with
$$\widehat{\beta}_{\pm}^{(j)}(\lambda_j,...,\lambda_0)=\beta_{\pm}^{(j)}(\lambda_j,...\lambda_0)+O(\bar{\lambda}_j^2\gamma^{\vartheta j})\;, \tag{108}$$
$\beta_{\pm}^{(j)}$ being the functions appearing in the analogous equations for the model of §2.4. This implies that
$$\eta_{\pm}=\log_{\gamma}[1+\beta_{\pm}^{(-\infty)}(\lambda_{-\infty},...\lambda_{-\infty})]\;, \tag{109}$$
that is *the critical indices in the AT or 8V or in the model (82) are the same as functions of* $\lambda_{-\infty}$.

Of course $\lambda_{-\infty}$ is a rather complex function of all the details of the models. However, if we call $\lambda'_j(\lambda)$ the effective couplings of the lattice model of the previous sections, the invertibility of $h(\lambda_\infty)$ implies that we can choose $\lambda_\infty$ so that
$$h(\lambda_\infty)=\lambda'_{-\infty}(\lambda)\;. \tag{110}$$
With this choice of $\lambda_\infty(\lambda)$, the critical indices are the same, as they depend only on $\lambda_{-\infty}$; the rest of this chapter is devoted to the proof that the critical indices have, as functions of $\lambda_\infty$, simple expressions, which imply the scaling relations in the main theorem.

**Remark** (109) and (110) play a central role in our analysis; they say that the critical indices of the spin lattice models (1.1) are equal to the ones of the QFT model (82), provided that its coupling is chosen properly; such a model is defined in the continuum but the non locality of the interaction has the effect that no ultraviolet divergences are generated. On the other hand, the model (82) verifies extra symmetries, involving Ward Identities and closed equation, which allow us to derive simple expressions for the indices in terms of $\lambda_\infty$, as we will see in the following sections.

## 3.4 Ward Identities

We consider the case $\mu=0$ and we call $D_\omega(\mathbf{k})=-ik_0+\omega k$. We shorten the notation of $\mathcal{W}_N(0,J,\eta)$ into $\mathcal{W}_N(J,\eta)$. By the change of variables $\psi^{\pm}_{\mathbf{x},\omega}\to e^{\pm i\alpha_{\mathbf{x},\omega}}\psi^{\pm}_{\mathbf{x},\omega}$ we obtain the identity
$$\begin{aligned}&D_\omega(\mathbf{p})\frac{\partial\mathcal{W}_N}{\partial\widehat{J}_{\mathbf{p},\omega}}(0,\eta)-\nu\,\widehat{v}_K(\mathbf{p})D_{-\omega}(\mathbf{p})\frac{\partial\mathcal{W}_N}{\partial\widehat{J}_{\mathbf{p},-\omega}}(0,\eta)=\\&=\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}\frac{\partial\mathcal{W}_N}{\partial\widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta)-\frac{\partial\mathcal{W}_N}{\partial\widehat{\eta}^-_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta)\widehat{\eta}^-_{\mathbf{k},\omega}\right]+\frac{\partial\mathcal{W}_{\mathcal{A}}}{\partial\widehat{\alpha}_{\mathbf{p},\omega}}(0,0,\eta)\;,\end{aligned} \tag{111}$$
where $\nu$ is a constant to be chosen later,
$$\begin{aligned}e^{\mathcal{W}_{\mathcal{A}}(J,\alpha,\eta)}&=\int P(d\psi^{[l,N]})e^{\mathcal{V}^{(N)}(\psi^{[l,N]})+\sum_\omega\int d\mathbf{x}\;J_{\mathbf{x},\omega}\psi^{[l,N]+}_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}}\\&\cdot e^{\sum_\omega\int d\mathbf{x}[\psi^{[l,N]+}_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega}+\eta^+_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}]}e^{[\mathcal{A}_0-\nu\mathcal{A}_-](\alpha,\psi^{[l,N]})}\;,\end{aligned}$$
$$\mathcal{A}_0(\alpha,\psi)\stackrel{def}{=}\sum_{\omega=\pm}\int\frac{d\mathbf{q}\,d\mathbf{p}}{(2\pi)^4}\;C_\omega(\mathbf{q},\mathbf{p})\widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q},\omega}\widehat{\psi}^-_{\mathbf{p},\omega}\;, \tag{112}$$
$$\mathcal{A}_-(\alpha,\psi)\stackrel{def}{=}\sum_{\omega=\pm}\int\frac{d\mathbf{q}\,d\mathbf{p}}{(2\pi)^4}\;D_{-\omega}(\mathbf{p}-\mathbf{q})\widehat{v}_K(\mathbf{p}-\mathbf{q})\widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q},-\omega}\widehat{\psi}^-_{\mathbf{p},-\omega}\;, \tag{113}$$
$$C_\omega(\mathbf{q},\mathbf{p})=[\chi^{-1}_{l,N}(\mathbf{p})-1]D_\omega(\mathbf{p})-[\chi^{-1}_{l,N}(\mathbf{q})-1]D_\omega(\mathbf{q})\;, \tag{114}$$

and $\chi_{l,N}(\mathbf{k}) = \sum_{k=l}^{N} f_k(\mathbf{k})$.

**Remark** - As explained in §2.2 of [8], (111) is obtained by introducing a cut-off function $\chi^{\varepsilon}_{l,N}(\mathbf{k})$ never vanishing for all values of $\mathbf{k} \neq 0$ and equivalent to $\chi_{l,N}(\mathbf{k})$ as far as the scaling properties of the theory are concerned; $\varepsilon$ is a small parameter and $\lim_{\varepsilon\to 0^+} \chi^{\varepsilon}_{l,N}(\mathbf{k}) = \chi_{l,N}(\mathbf{k})$. This further regularization (to be removed before taking the removed cutoffs limit) ensures that the identity $[(\chi^{\varepsilon}_{l,N})^{-1}(\mathbf{k}) - 1]\chi^{\varepsilon}_{l,N}(\mathbf{k}) = 1 - \chi^{\varepsilon}_{l,N}(\mathbf{k})$ is satisfied for all $\mathbf{k} \neq 0$. When this further regularization is removed, all the quantities we shall study have a well defined expression.

The two equations obtained from (111) by putting $\omega = \pm 1$ can be solved w.r.t. $\partial e^{\mathcal{W}_N}/\partial \widehat{J}_{\mathbf{p},\omega}$ and, if we define

$$\begin{aligned} a(\mathbf{p}) &= \frac{1}{1-\nu\,\widehat{v}_K(\mathbf{p})} \quad , \quad \bar{a}(\mathbf{p}) = \frac{1}{1+\nu\,\widehat{v}_K(\mathbf{p})} \ , \\ A_\varepsilon(\mathbf{p}) &= \frac{a(\mathbf{p}) + \varepsilon \bar{a}(\mathbf{p})}{2} \ , \end{aligned} \tag{115}$$

we obtain the identity

$$\begin{aligned} &\frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{J}_{\mathbf{p},\omega}}(0,\eta) - \sum_{\omega'} \frac{A_{\omega\omega'}(\mathbf{p})}{D_\omega(\mathbf{p})} \frac{\partial e^{\mathcal{W}_\mathcal{A}}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'}}(0,0,\eta) = \\ &= \sum_{\omega'} \frac{A_{\omega\omega'}(\mathbf{p})}{D_\omega(\mathbf{p})} \int \frac{d\mathbf{k}}{(2\pi)^2} \left[ \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega'} \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega'}}(0,\eta) - \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^-_{\mathbf{k}+\mathbf{p},\omega'}}(0,\eta) \widehat{\eta}^-_{\mathbf{k},\omega'} \right] . \end{aligned} \tag{116}$$

Given a correlation function with $m$ external fields of momenta $\mathbf{k}_1, \ldots, \mathbf{k}_m$, we shall say that its *external momenta are non exceptional*, if, for any subset $I$ of $\{1, \ldots, m\}$, $\sum_{i\in I} \sigma_i \mathbf{k}_i \neq 0$, where $\sigma_i = +1$ for the incoming momenta and $\sigma_i = -1$ for the outcoming momenta. Note that our definitions are such that $\eta^+$ is an incoming field, while $\eta^-$, $J$ and $\alpha$ are outcoming.

An important role in this paper will have the following lemma, which was already proved in [19].

**Lemma 3.2** *If $\lambda_\infty$ is small enough, there exists a choice of $\nu$, independent of $l$ and $N$, such that*

$$\nu = \frac{\lambda_\infty}{4\pi} \tag{117}$$

*and, in the limit of removed cut-offs,*

$$\sum_{\omega'} \frac{A_{\omega\omega'}(\mathbf{p})}{D_\omega(\mathbf{p})} \frac{\partial e^{\mathcal{W}_\mathcal{A}}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'}}(0,0,\eta) = 0 \ , \tag{118}$$

*in the sense that the correlation functions generated by deriving w.r.t. $\eta$ the l.h.s. of (118) vanish in the limit of removed cutoffs, if the external momenta are non exceptional.*

**Proof.** We sketch here the proof, as it will be useful in the following, referring for more details to [19] (see also [10] and [5, 9]). The starting point is the remark that $\mathcal{W}_\mathcal{A}(\alpha, 0, \eta)$ is very similar to $\mathcal{W}_N(J, \eta)$, see (82), the difference being that $\int J_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$ is replaced by $\mathcal{A}_0 - \nu\mathcal{A}_-$. A crucial role in the analysis is played by the function $C_\omega(\mathbf{p}, \mathbf{q})$ appearing in the definition of $\mathcal{A}_0$; this function is very singular, but it appears in the various equations relating the correlation functions only through the regular function

$$\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) \stackrel{def}{=} \widetilde{\chi}_N(\mathbf{p})C_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{g}^{(i)}_\omega(\mathbf{q}+\mathbf{p})\widehat{g}^{(j)}_\omega(\mathbf{q}) \;, \tag{119}$$

where $\widetilde{\chi}_N(\mathbf{p})$ is a smooth function, with support in the set $\{|\mathbf{p}|\le 3\gamma^{N+1}\}$ and equal to 1 in the set $\{|\mathbf{p}|\le 2\gamma^{N+1}\}$; we can add freely this factor in the definition, since $\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})$ will only be used for values of $\mathbf{p}$ such that $\widetilde{\chi}_N(\mathbf{p})=1$, thanks to the support properties of the propagator. It is easy to see that $\widehat{U}^{(i,j)}_\omega$ vanishes if neither $j$ nor $i$ equals $N$ or $l$; this has the effect that at least one of the fields in $\mathcal{A}_0$ has to be integrated at the $N$ or $l$ scale.

As a matter of fact, the terms in which at least one field is integrated at scale $l$ are much easier to analyze, see below. In order to study the others, it is convenient to introduce the function $\widehat{S}^{(i,j)}_{\bar\omega,\omega}$ defined by the equation

$$\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) = \sum_{\bar\omega} D_{\bar\omega}(\mathbf{p})\widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \;. \tag{120}$$

One can show that, if we define

$$S^{(i,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \int \frac{d\mathbf{p}\, d\mathbf{q}}{(2\pi)^4}\, e^{-i\mathbf{p}(\mathbf{x}-\mathbf{z})}e^{i\mathbf{q}(\mathbf{y}-\mathbf{z})}\widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{p},\mathbf{q}) \;, \tag{121}$$

then, given any positive integer $M$, there exists a constant $C_M$ such that, if $j>l$,

$$|S^{(N,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y})| \le C_M \frac{\gamma^N}{1+[\gamma^N|\mathbf{x}-\mathbf{z}|]^M}\frac{\gamma^j}{1+[\gamma^j|\mathbf{y}-\mathbf{z}|]^M} \;, \tag{122}$$

a bound which is used to control the renormalization of the marginal terms containing a vertex of type $\mathcal{A}_0$. We choose $\nu$ as given by

$$\nu = \lambda_\infty \sum_{i,j=l+1}^{N} \int \frac{d\mathbf{q}}{(2\pi)^2}\; \widehat{S}^{(i,j)}_{-\omega,\omega}(\mathbf{q},\mathbf{q}) \;; \tag{123}$$

by an explicit calculation one can see that, for any $l<0$ and $N>0$, $\nu$ satisfies (117). We remark that, to get this result, it is important to exclude from the sum in the r.h.s. of (123) the couples $(i,j)$ with one of the indices equal to $l$; without this restriction, $\nu$ would be equal to 0, for any $N>0$.

The fact that the external momenta are non exceptional is important to avoid the infrared singularities of the correlation functions. This condition on the momenta is taken into account by using the fact that, in the tree expansion of the correlation functions, there are important constraints on the scale indices of the trees. This allows us to safely bound the Fourier transforms of the correlation functions by the sum over the $L^1$ norms in the coordinate space of the contributions associated to the different trees; see [5], §3.1, for an example of this strategy. Moreover, the tree structure of the expansion allows us to express the $L^1$ norm of the correlation functions in terms of the $L^1$ norm of the effective potential on the different scales; hence, in the following, in order to study the effect on the Fourier transform of the correlations of the ultraviolet region, we shall study the $L^1$ norm of the kernels in the coordinate space.

We will proceed as in the analysis of $\mathcal{W}_N(J,\eta)$, integrating first the ultraviolet scales $N, N-1,\dots,h+1$, $h\ge K$, following a procedure very similar to the one described in §3.2, the main difference being that there appear in the effective potential new monomials in the external field $\alpha$ and in $\psi$.

We consider first the terms contributing to $\mathcal{W}_{\mathcal{A}}(\alpha,0,\eta)$ in which at least one of the two fields in $\mathcal{A}_0$ or $\mathcal{A}_-$ is contracted at scale $N$. The marginal terms such that only one of these two fields is contracted are proportional to $W^{(0;2)(k)}$, so that one can use (86) to bound them. Hence, we shall consider in detail only the terms such that both fields of $\mathcal{A}_0$ or $\mathcal{A}_1$ are contracted and we shall call $\widehat{K}^{(n;2m)(k)}_{\Delta;\omega;\underline{\omega}'}$ the corresponding kernels of the monomials with $2m$ $\psi$-fields and $n$ $\alpha$-fields. In the case $n=1$, we decompose them as follows:

$$\widehat{K}^{(1;2m)(k)}_{\Delta;\omega;\underline{\omega}'}(\mathbf{p};\underline{\mathbf{k}}) = \sum_\sigma D_{\sigma\omega}(\mathbf{p})\widehat{W}^{(1;2m)(k)}_{\Delta;\sigma,\omega;\underline{\omega}'}(\mathbf{p};\underline{\mathbf{k}})\;, \tag{124}$$

where $\mathbf{p}$ is the momentum flowing along the external $\alpha$-field. As in §3.2, we have to improve the dimensional bound of $W^{(1;2)(k)}_{\Delta;\sigma,\omega;\omega'}$. We can write the following identity, which is represented the first line of Fig.5 in the case $\sigma=-1$:

$$\begin{aligned} W^{(1;2)(k)}_{\Delta;\sigma,\omega;\omega'}(\mathbf{z};\mathbf{x},\mathbf{y}) &= \sum_{i,j=k}^{N}\int d\mathbf{u}d\mathbf{w}\; S^{(i,j)}_{\sigma\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})W^{(0;4)(k)}_{\omega,\omega'}(\mathbf{u},\mathbf{w},\mathbf{x},\mathbf{y}) - \\ &-\nu\,\delta_{-1,\sigma}\int d\mathbf{w}\; v_K(\mathbf{z}-\mathbf{w})W^{(1;2)(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y})\;. \end{aligned} \tag{125}$$

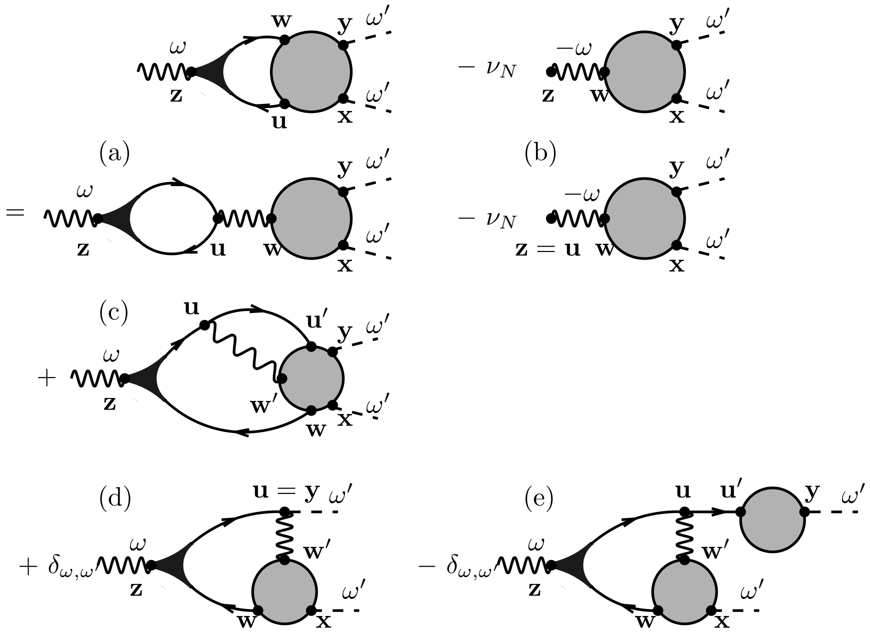


Figure 5: : Graphical representation of $W^{(1;2)(k)}_{\Delta;-1,\omega;\omega'}$

We can further decompose $W^{(1;2)(k)}_{\Delta;-1,\omega;\omega'}$ as in the last there lines of Fig.5. The term (c) can be written as

$$\begin{aligned} &\lambda_\infty\sum_{i,j=k}^{N}\int d\mathbf{u}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'\; S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})g^{[k,N]}_{\omega}(\mathbf{u}-\mathbf{u}')v_K(\mathbf{u}-\mathbf{w}')\,\cdot \\ &\cdot W^{(1;4)(k)}_{-\omega;\omega,\omega'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y})\;. \end{aligned} \tag{126}$$

Hence, if we put $b_j(\mathbf{x}) \stackrel{def}{=} \gamma^j/(1+[\gamma^j|\mathbf{x}|]^3)$, we recall that $S^{(i,j)}_{-\omega,\omega}$ is different from 0 only if either $i$ or $j$ is equal to $N$, and we use the bound (122), we see that the norm of (c) is bounded by

$$C_3|\lambda_\infty|\|v_K\|_{L^\infty} \sum_{i,j,m=k}^{N\,*} \int d\mathbf{x}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'\ |W^{(1;4)(k)}_{-\omega;\omega,\omega'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y})| \cdot$$
$$\cdot \int d\mathbf{z}d\mathbf{u}\ b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|g^{(m)}_\omega(\mathbf{u}-\mathbf{u}')| \ , \tag{127}$$

where $*$ reminds that $\max\{i,j\}=N$. Since the $L^1$ and the $L^\infty$ norm of $b_j$ satisfy a bound similar to analogous bounds of $g^{(j)}_\omega$, we can proceed as in the previous section to bound $\int d\mathbf{z}d\mathbf{u}\ b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|g^{(m)}_\omega(\mathbf{u}-\mathbf{u}')|$, by taking the $L^\infty$ norm for the factor with the smaller index and the $L^1$ norm for the other two. By also using (89), we get the bound

$$C_\vartheta|\lambda_\infty|^2\gamma^{-2(k-K)}\gamma^{-\vartheta(N-k)} \ , \tag{128}$$

for any $0<\vartheta<1$ ($C_\vartheta$ is divergent for $\vartheta\to1$). With respect to analogous bound in §3.2 ((b2) in Fig.4), there is an improvement of a factor $\gamma^{-\vartheta(N-k)}$. The term (d) can be bounded by

$$C|\lambda_\infty|\|v_K\|_{L^\infty} \sum_{i,j=k}^{N\ *} \|b_i\|_{L^1}\ \|b_j\|_{L^1} \le C|\lambda_\infty|\gamma^{-(k-K)}\gamma^{-(N-k)} \ ;$$

for the term (e) we get the bound $C|\lambda_\infty|^2\gamma^{-3(k-K)}\gamma^{-(N-k)}$. By putting together all the previous bounds, we get

$$\|(c)+(d)+(e)\| \le C_\vartheta|\lambda_\infty|\gamma^{-(k-K)}\gamma^{-\vartheta(N-k)} \ . \tag{129}$$

We consider now the terms (a) and (b), whose sum can be written as

$$\int d\mathbf{u}\ \left[\lambda_\infty \sum_{i,j=k}^{N} S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) - \nu\delta(\mathbf{z}-\mathbf{u})\right] \cdot$$
$$\cdot \int d\mathbf{w}\ v_K(\mathbf{u}-\mathbf{w})W^{(1;2)(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) \ . \tag{130}$$

By using the identity (98), (130) can be written also as

$$\left[\lambda_\infty \sum_{i,j=k}^{N} \int d\mathbf{u}\ S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) - \nu\right] \int d\mathbf{w}\ v_K(\mathbf{z}-\mathbf{w})W^{(1;2),(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) +$$
$$+\lambda_\infty \sum_{p=0,1}\sum_{i,j=k}^{N} \int d\mathbf{u}\ S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})(u_p-z_p) \cdot \tag{131}$$
$$\cdot \int_0^1 d\tau \int d\mathbf{w}\ (\partial_p v_K)(\mathbf{z}-\mathbf{w}+\tau(\mathbf{u}-\mathbf{z}))W^{(1;2),(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) \ .$$

The latter term is again irrelevant and vanishing for $N-k\to+\infty$; in fact, its norm can be bounded by

$$2|\lambda_\infty|\|W^{(1;2),(k)}_{-\omega;\omega'}\|\ \|\partial v_K\|_{L^1} \sum_{i,j=k}^{N\ *} \int d\mathbf{z}\ b_i(\mathbf{z}-\mathbf{u})b_j(\mathbf{z}-\mathbf{u})|\mathbf{u}-\mathbf{z}_p| \le$$
$$\le C|\lambda_\infty|\gamma^{-(k-K)}\gamma^{-(N-k)} \ . \tag{132}$$

Contrary to what happened for the graph (b1) of Fig4, the contribution of the graph (a) to the first term in the r.h. side of (131) is not zero (that is, *the fermionic bubble is not vanishing*); however, in this case its value is compensated by the graph (b), thanks to the explicit choice we made for $\nu$. Indeed we have

$$\lambda_\infty \sum_{i,j=k}^{N} \int d\mathbf{u}\, S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) - \nu = -2\lambda_\infty \sum_{j=l+1}^{k-1} \int d\mathbf{u}\, S^{(N,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) \;, \tag{133}$$

that easily implies that the first term in the r.h. side of (131) is bounded by $C|\lambda_\infty|\gamma^{-(N-k)}$.

Let us finally consider $W^{(1;2)(k)}_{\Delta;+1,\omega;\omega'}$, for which we can use a graph expansion similar to that of Fig.5, the only differences being that $\nu$ is replaced by 0 and the indices $-\omega$ are replaced by $\omega$. Hence a bound can be obtained with the same arguments used above, with only one important difference: the contribution that in the previous analysis was compensated by the graph (b) now is zero by symmetry reasons. Indeed, if we call $\mathbf{k}^*$ the vector $\mathbf{k}$ rotated by $\pi/2$, it is easy to see that $\widehat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{k}^*,\mathbf{p}^*) = -\omega\bar\omega \widehat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{k},\mathbf{p})$, which implies that

$$\sum_{i,j=k}^{N} \int d\mathbf{u}\, S^{(i,j)}_{\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) = \sum_{i,j=k}^{N} \int \frac{d\mathbf{k}}{(2\pi)^2}\, \widehat S^{(i,j)}_{\omega,\omega}(\mathbf{k},-\mathbf{k}) = 0 \;. \tag{134}$$

We have then proved that

$$\|W^{(1;2)(k)}_{\Delta;\sigma,\omega;\underline{\omega}'}\| \le C|\lambda_\infty|\gamma^{-\vartheta(N-k)} \;, \tag{135}$$

which implies, by dimensional bounds and the short memory property, that, for $K \le k \le N$,

$$\|W^{(1;2m)(k)}_{\Delta;\sigma,\omega;\underline{\omega}'}\| \le (C|\lambda_\infty|)^m \gamma^{(1-m)k}\gamma^{-\vartheta(N-k)} \;. \tag{136}$$

It remains to analyze the terms contributing to $\mathcal{W}_{\mathcal{A}}(\alpha,0,\eta)$ in which no one of the two fields in $\mathcal{A}_0$ is contracted at scale $N$. If $i \ge l$ we can use the bound

$$\left| \frac{\widehat U^{(i,l)}_{\omega'}(\mathbf{q}+\mathbf{p},\mathbf{q})}{D_\omega(\mathbf{p})} \right| \le C\gamma^{-(i-l)} \frac{\gamma^{-l-i}}{Z_{i-1}} \quad , \quad \text{if } |\mathbf{p}| \ge 2\gamma^{l+1} \;, \tag{137}$$

and the factor $\gamma^{-(i-l)}$ in the r.h.s. of this bound is an improvement w.r.t. the dimensional bound and makes indeed irrelevant the marginal terms containing a vertex of type $\mathcal{A}_0$, if one of the $\psi$ fields is contracted on scale $l$ and $\mathbf{p}$ has a fixed value different from 0, as we are supposing.

The contributions to the correlation functions generated by the l.h.s. of (118), such that one of the $\psi$-fields in $\mathcal{A}_0$ is contracted at scale $l$ (hence it is an external field at scale $k$), vanish in the limit $l \to -\infty$, if the momentum $\mathbf{p}$ of the $\alpha$ field is fixed at a value different from 0, as we are supposing. This follows from the bound (137), since the value of $i$ is essentially fixed at a value of order $\log_\gamma |\mathbf{p}|$ and the extra factor $\gamma^{-(i-l)}$ vanishes for $l \to -\infty$. The correlations generated by the terms containing $W^{(1;2m)(k)}_{\Delta}$ are vanishing in the limit of removed cut-offs, thanks to the extra factor $\gamma^{-\vartheta(N-k)}$ in (136), with respect to the dimensional one, and the short memory property.

## 3.5 Closed equations

The Schwinger-Dyson equations for $\mu = 0$ are generated by the identity, see [6],

$$D_\omega(\mathbf{k}) \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat\eta^+_{\mathbf{k},\omega}}(0,\eta) = \chi_{l,N}(\mathbf{k}) \Big[ \widehat\eta^-_{\mathbf{k},\omega} e^{\mathcal{W}_N(0,\eta)} -$$

$$-\lambda_\infty \int \frac{d\mathbf{p}}{(2\pi)^2}\ \widehat{v}_K(\mathbf{p}) \frac{\partial^2 e^{\mathcal{W}_N}}{\partial \widehat{J}_{\mathbf{p},-\omega}\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta)\Bigg]\ . \tag{138}$$

By using (116) we easily get:

$$\begin{aligned}
&D_\omega(\mathbf{k})\frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta) = \chi_{l,N}(\mathbf{k})\Bigg\{\widehat{\eta}^-_{\mathbf{k},\omega} e^{\mathcal{W}_N(0,\eta)} - \\
&-\lambda_\infty \sum_{\omega'}\int \frac{d\mathbf{p}}{(2\pi)^2}\, v_K(\mathbf{p})\frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\ \cdot \\
&\cdot \int \frac{d\mathbf{q}}{(2\pi)^2}\ \Bigg[\widehat{\eta}^+_{\mathbf{q}+\mathbf{p},\omega'} \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{q},\omega'}\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta) - \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}\partial \widehat{\eta}^-_{\mathbf{q}+\mathbf{p},\omega'}}(0,\eta)\widehat{\eta}^-_{\mathbf{q},\omega'}\Bigg] - \\
&-\lambda_\infty \sum_{\omega'}\int \frac{d\mathbf{p}}{(2\pi)^2}\ v_K(\mathbf{p})\frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\frac{\partial^2 e^{\mathcal{W}_{\mathcal{A}}}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'}\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,0,\eta)\Bigg\}\ .
\end{aligned} \tag{139}$$

We now want to prove that the last term in the r.h.s. of (139) is negligible in the limit of removed cutoffs, if $\mathbf{k}$ is fixed at a value far from the cutoffs.

**Theorem 3.3** *In the limit of removed cutoffs, the correlation functions generated by deriving w.r.t. $\eta$ the functional*

$$\sum_{\omega'}\int \frac{d\mathbf{p}}{(2\pi)^2}\ v_K(\mathbf{p})\frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\frac{\partial^2 e^{\mathcal{W}_{\mathcal{A}}(0,0,\eta)}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'}\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}} \tag{140}$$

*vanish, if the external momenta are non exceptional.*

**Proof.** It is convenient to write (140) as $\sum_{\varepsilon=\pm}\frac{\partial \mathcal{W}_{T,\varepsilon}}{\partial \widehat{\beta}_{\mathbf{k},\omega}}(0,\eta)$, where

$$\begin{aligned}
e^{\mathcal{W}_{T,\varepsilon}(\beta,\eta)} &= \int P(d\psi^{[l,N]}) e^{\mathcal{V}^{(N)}(\psi^{[l,N]})+\sum_\omega \int d\mathbf{x}[\psi^{[l,N]+}_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega}+\eta^+_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}]}\ . \\
&\cdot\quad e^{\left[T_1^{(\varepsilon)}-\nu T_-^{(\varepsilon)}\right](\psi^{l,N},\beta)}
\end{aligned} \tag{141}$$

and

$$\begin{aligned}
T_1^{(\varepsilon)}(\psi,\beta) &= \sum_\omega \int \frac{d\mathbf{k}\ d\mathbf{p}\ d\mathbf{q}}{(2\pi)^4}\ \widehat{v}^{(\varepsilon)}_K(\mathbf{p})\frac{C_{-\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})}{D_{-\omega}(\mathbf{p})}\ \cdot \\
&\cdot\quad \widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},-\varepsilon\omega}\widehat{\psi}^-_{\mathbf{q},-\varepsilon\omega}\ ,
\end{aligned} \tag{142}$$

$$T_-^{(\varepsilon)}(\psi,\beta) = \sum_\omega \int \frac{d\mathbf{k}\ d\mathbf{p}\ d\mathbf{q}}{(2\pi)^4}\ \widehat{u}^{(\varepsilon)}_K(\mathbf{p})\widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\varepsilon\omega}\widehat{\psi}^-_{\mathbf{q},\varepsilon\omega}\ , \tag{143}$$

where

$$\widehat{v}^{(\varepsilon)}_K(\mathbf{p}) \stackrel{def}{=} v_K(\mathbf{p})\widehat{A}_\varepsilon(\mathbf{p})\quad ,\quad \widehat{u}^{(\varepsilon)}_K(\mathbf{p}) = \widehat{v}^{(\varepsilon)}_K(\mathbf{p})\widehat{v}_K(\mathbf{p})\frac{D_{\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\ . \tag{144}$$

Note that $v^{(\pm)}_K(\mathbf{x})$ and $u^{(-)}_K(\mathbf{x})$ are smooth functions of fast decay, hence they are equivalent to $v_K(\mathbf{x})$ in the bounds. This is not true for $u^{(+)}_K(\mathbf{x})$, whose Fourier transform is bounded but discontinuous in $\mathbf{p}=0$. However, in the following we shall only need to know that $\|u^{(+)}_K\|_{L^\infty} \le C\gamma^{2K}$ and that $|\widehat{u}^{(+)}_K(\mathbf{p})| \le |\widehat{v}^{(+)}_K(\mathbf{p})\widehat{v}_K(\mathbf{p})|$, which are easy to prove.

As in §3.4, we now perform a multiscale integration for the ultraviolet scales $N, N-1,\ldots,k+1$, $k\ge K$, very similar to the one described in §3.2, the main difference being that that there appear in the effective potential new monomials in the external field $\beta$ and in $\psi$. As explained in the previous section, in order to control the Fourier transform at non exceptional momenta of the correlation functions, it is in general sufficient to control, in the ultraviolet region, the $L^1$ norm in coordinate space of the kernels appearing in the effective potential. This is in general true also in the proof of this theorem, except for a bound, where one has to be more careful, see below.

The contributions to the correlation functions such that one of the $\psi$-fields in $T_1^{(\varepsilon)}(\psi,\beta)$ with momentum $\mathbf{q}+\mathbf{p}$ or $\mathbf{q}$, see (142), is contracted at scale $l$ (hence it is an external field at scale $k$), vanish in the limit $l\to-\infty$, if the momentum $\mathbf{k}$ of $\beta$ is fixed at a value different from 0, as we are supposing. In fact, in this case either $|\mathbf{p}|$ or $|\mathbf{k}+\mathbf{p}|$ is greater than $|\mathbf{k}|/2$; hence, by using (137) or the short memory property, these contributions satisfy a bound containing the extra factor $\gamma^l|\mathbf{k}/2|$, which vanishes for $l\to-\infty$. We consider then just the terms contributing to $\mathcal{W}_{T,\varepsilon}(\beta,\eta)$, in which at least one of the two $\psi$-fields in $T_1^{(\varepsilon)}(\psi,\beta)$ with momentum $\mathbf{q}+\mathbf{p}$ or $\mathbf{q}$ is contracted at scale $N$. We shall call $W^{(1;2m-1)}_{T,\varepsilon;\omega;\underline{\omega}'}$ the corresponding kernels of the monomials with $2m-1$ $\psi$-fields and 1 $\alpha$-field. We claim that

$$\|W^{(1;2m-1)(k)}_{T,\varepsilon;\omega;\underline{\omega}'}\|\le C\gamma^{(2-m)k}\gamma^{-\vartheta(N-k)}\;. \tag{145}$$

By the usual arguments, this is a consequence of the improved bounds:

$$\|W^{(1;1)(k)}_{T,\varepsilon;\omega,\omega}\| \;\le\; C|\lambda_\infty|\gamma^k\gamma^{-\vartheta(N-k)}\gamma^{-2(k-K)}\;, \tag{146}$$

$$\|W^{(1;3)(k)}_{T,\varepsilon;\omega,\underline{\omega}'}\| \;\le\; C|\lambda_\infty|\gamma^{-\vartheta(N-k)}\;. \tag{147}$$

We prove first the bound (146). We can write

$$W^{(1;1)(k)}_{T,\varepsilon;\omega,\omega}=W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}+W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega} \tag{148}$$

where

a) $W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}$ is the sum over the terms such that the field $\beta$ belongs only to a $T_1^{(\varepsilon)}$-vertex, whose $\psi$-field $\widehat\psi^+_{\mathbf{q}+\mathbf{p},-\varepsilon\omega}$ either is contracted with $\widehat\psi^-_{\mathbf{k}+\mathbf{p},\omega}$ (this can happen only for $\varepsilon=-1$) or is connected to it through a kernel $\widehat W^{(0;2)(k)}_\omega(\mathbf{q}+\mathbf{p})$.

b) $W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega}$ is the sum over the remaining terms.

Let us consider the first term. Given $\mathbf{k}$, for $N$ large enough, $\chi^{-1}_{l,N}(\mathbf{k})-1=0$; hence we can write:

$$\widehat W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}(\mathbf{k})=\delta_{\varepsilon,-1}\int\frac{d\mathbf{p}}{(2\pi)^2}\,\frac{\widehat v_K^{(-1)}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}[\chi_{-\infty,N}(\mathbf{p}+\mathbf{k})-1]\cdot \tag{149}$$
$$\cdot\left[1+\widehat g_\omega^{[k+1,N]}(\mathbf{p}+\mathbf{k})\widehat W^{(0;2)(k)}_\omega(\mathbf{p}+\mathbf{k})\right]\left[1+\widehat g_\omega^{[k+1,N]}(\mathbf{k})\widehat W^{(0;2)(k)}_\omega(\mathbf{k})\right]\;.$$

Moreover, since $\widehat v_K^{(-1)}(\mathbf{p})=0$ for $|\mathbf{p}|\ge 2\gamma^K$, then $\chi_{-\infty,N}(\mathbf{p}+\mathbf{k})-1=0$, if $\widehat v_K^{(-1)}(\mathbf{p})\ne0$ and $N$ is large enough. It follows that, given a fixed $\mathbf{k}$, for $N$ large enough,

$$\widehat W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}(\mathbf{k})=0\;. \tag{150}$$

Let us now consider $W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega}(\mathbf{x}-\mathbf{y})$, which can be decomposed as in Fig. 6. By using (125), it can be written as

$$\sum_\sigma\int d\mathbf{z}\;u_K^{(\varepsilon)}(\mathbf{x}-\mathbf{z})g_\omega^{[k,N]}(\mathbf{x}-\mathbf{w})W^{(1;2)(k)}_{\Delta;\sigma,-\varepsilon\omega;\omega}(\mathbf{z};\mathbf{y},\mathbf{w})\;, \tag{151}$$

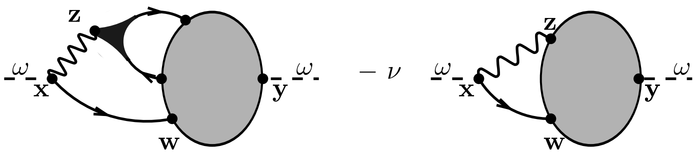


Figure 6: : Graphical representation of $W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega}$

hence its norm, by using (135), can be bounded by

$$\|u^{(\varepsilon)}_K\|_{L^\infty}\sum_{j=k}^{N}|g^{(j)}_\omega|_{L^1}\|W^{(1;2)(k)}_{\Delta;\sigma,-\varepsilon\omega;\omega}\|\le C|\lambda_\infty|\gamma^k\gamma^{-2(k-K)}\gamma^{-\vartheta(N-k)}\ . \tag{152}$$

In order to prove the bound (147), we write

$$W^{(1;3)(k)}_{T,\varepsilon;\omega;\underline{\omega}'}=W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}+W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\underline{\omega}'}\ , \tag{153}$$

where $W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}$ contains the terms in which the field $\widehat\psi_{\mathbf{k}+\mathbf{p},\omega}$ of $T_1$ and $T_-$ is not contracted or is linked to a kernel $\widehat W^{(0;2)(k)}_\omega$, while the other terms are collected in $W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\underline{\omega}'}$. Let us consider first $W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}$; its Fourier transform, if we call $\mathbf{k}^+$ and $\mathbf{k}^-$ the momenta of the two fields connected to the line $u^{(\varepsilon)}_K$, can be written as (note that $\underline{\omega}'$ is of the form $(\omega,\omega',\omega')$):

$$\begin{aligned}\widehat W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}(\mathbf{k};\mathbf{k}^+,\mathbf{k}^-)&=\Big[1+\widehat g^{[k+1,N]}_\omega(\mathbf{k}+\mathbf{k}^+-\mathbf{k}^-)\widehat W^{(0;2)(k)}_\omega(\mathbf{k}+\mathbf{k}^+-\mathbf{k}^-)\Big]\cdot\\ &\cdot\widehat u^{(\varepsilon)}_K(\mathbf{k}^+-\mathbf{k}^-)\sum_\sigma\widehat W^{(1;2)(k)}_{\Delta;\sigma,-\varepsilon\omega,\omega'}(\mathbf{k}^-+\mathbf{k}^+-\mathbf{k}^-,\mathbf{k}^-)\ .\end{aligned} \tag{154}$$

Then, if $\varepsilon=-1$, since $\|v^{(-1)}_K\|_{L^1}\le C$, by using the bounds (135) and (86), we find

$$\|W^{(1;3)(k)}_{(a)T,-1;\omega;\underline{\omega}'}\|\le C|\lambda_\infty|\gamma^{-\vartheta(N-k)} \tag{155}$$

This bound is not true in the case $\varepsilon=+1$, where it is necessary to take carefully into account that we are indeed bounding the Fourier transform of the correlation functions generated by (140), at fixed (non exceptional) external momenta.

The terms contributing to these correlations and containing $W^{(1;3)(k)}_{(a)T,+1;\omega;\underline{\omega}'}$ as a cluster can be of two different types. There are terms such that the line corresponding to $\widehat\psi_{\mathbf{k}+\mathbf{p},\omega}$ is connected to the rest of the graph only through the vertex of the field $\beta$. In this case, we have to bound an expression of the type

$$\widehat u^{(+1)}_K(\mathbf{k}-\mathbf{q})\widehat G_1(\underline{\mathbf{k}}')\widehat G_2(\underline{\mathbf{k}}'')\ , \tag{156}$$

where $\underline{\mathbf{k}}'$ and $\underline{\mathbf{k}}''$ are a set of independent external momenta, $\mathbf{q}=-\sum_i\sigma_i\mathbf{k}'_i$, $\mathbf{q}-\mathbf{k}=\sum_i\sigma_i\mathbf{k}''_i$ and $\widehat G_2(\underline{\mathbf{k}}'')$ contains the cluster associate to $\sum_\sigma\widehat W^{(1;2)(k)}_{\Delta;\sigma,-\omega,\omega'}$; this expression is bounded by $C\|G_1\|\,|\Gamma_2\|$, the same result that we should get in the case $\varepsilon=-1$, by bounding the full expression with the $\|\cdot\|$ norm. Hence, the final bound is the same we would obtain by using (155) for $\varepsilon=+1$.

We still have to consider the terms such that the line corresponding to $\widehat\psi_{\mathbf{k}+\mathbf{p},\omega}$ is connected to the rest of the graph even if we erase the vertex of the field $\beta$. Now we have to bound an expression of the type

$$\int \frac{d\mathbf{p}}{(2\pi)^2}\ \widehat{u}_K^{(+1)}(\mathbf{p})\widehat{g}^{(j)}(\mathbf{p}+\mathbf{k})\widehat{G}(\mathbf{p},\underline{\mathbf{k}}')\ , \tag{157}$$

where $\sum_i \sigma_i \mathbf{k}_i = \mathbf{k}$ and $\widehat{G}(\mathbf{p},\underline{\mathbf{k}}')$ contains the cluster associate to $\sum_\sigma \widehat{W}^{(1;2)(k)}_{\Delta;\sigma,-\omega,\omega'}$; this expression can be bounded by $C\|\widehat{g}\|_{L^1}\|G\|$, the same result that we should get in the case $\varepsilon=-1$, by bounding the full expression with the $\|\cdot\|$ norm.

Let us finally consider $W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\underline{\omega}'}$, which can be represented as in Fig.7.

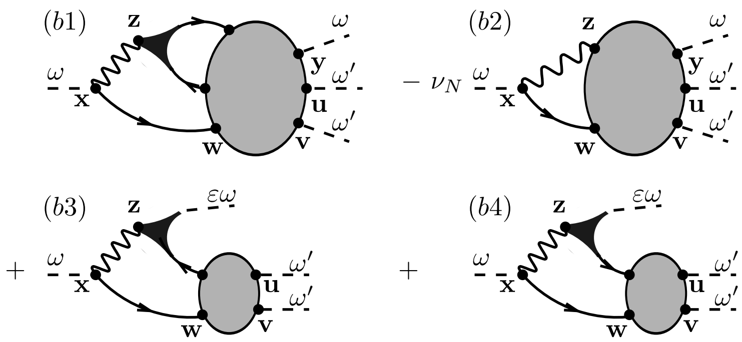


Figure 7: : Graphical representation of $W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\omega'}$

We can write

$$\begin{aligned} &W^{(1;3)(k)}_{(b)T,\varepsilon;\omega,\underline{\omega}'}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) = \\ &= \int d\mathbf{z}d\mathbf{w}\ u_K^{(\varepsilon)}(\mathbf{x}-\mathbf{z})g_\omega^{[k,N]}(\mathbf{x}-\mathbf{w})W^{(1;4)(k)}_{\Delta,\varepsilon;\omega,\omega'}(\mathbf{z};\mathbf{w},\mathbf{y},\mathbf{u},\mathbf{v})\ , \end{aligned} \tag{158}$$

so that, by the bounds (135), $\|W^{(1;4)(k)}_{\Delta,\varepsilon;\omega,\omega'}\| \le C|\lambda_\infty|\gamma^{-k}\gamma^{-\vartheta(N-k)}$ and $\|u_K^{(\varepsilon)}\|_{L^\infty} \le C\gamma^{2K}$, we get:

$$\|W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\omega'}\| \le C|\lambda_\infty|\gamma^{-2(k-K)}\gamma^{-\vartheta(N-k)}\ . \tag{159}$$

Again, with respect to the analogous bound in §3.2,we have an extra factor $\gamma^{-\vartheta(N-k)}$ and this implies, proceeding for instance as in §4.1 of [5], the proof of the Theorem.

## 3.6 Solution of the closed equations and proof of $x_+x_-=1$

We want to solve the closed equations for the correlation functions

$$\langle \psi^-_{\mathbf{x},\omega}\psi^+_{\mathbf{y},\omega}\rangle \stackrel{def}{=} S_\omega(\mathbf{x}-\mathbf{y})\ , \tag{160}$$

$$\langle \psi^-_{\mathbf{x},\omega}\psi^-_{\mathbf{y},-\omega}\psi^+_{\mathbf{u},-\omega}\psi^+_{\mathbf{v},\omega}\rangle \stackrel{def}{=} G_\omega(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v})\ , \tag{161}$$

in the limit of removed cutoffs. By taking in (139) one derivative w.r.t. $\widehat{\eta}^-_{k,\omega}$ and then putting $\eta\equiv 0$, we find

$$D_\omega(\mathbf{k})\widehat{S}_\omega(\mathbf{k}) = 1 + \lambda_\infty \int \frac{d\mathbf{p}}{(2\pi)^2}\ \widehat{F}_{K,-}(\mathbf{p})\widehat{S}_\omega(\mathbf{k}+\mathbf{p})\ , \tag{162}$$

where

$$\widehat{F}_{K,\varepsilon}(\mathbf{p}) \stackrel{def}{=} \frac{v_K(\mathbf{p})A_\varepsilon(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\ . \tag{163}$$

In the space coordinates, equation (162) becomes

$$(\partial_\omega S_\omega)(\mathbf{x}) - \lambda_\infty F_{K,-}(\mathbf{x}) S_\omega(\mathbf{x}) = \delta(\mathbf{x}) , \tag{164}$$

where $\partial_\omega = \partial_{x_0} + i\omega\partial_{x_1}$ and $F_{K,-}(\mathbf{x}) = \int d\mathbf{p}/(2\pi)^2 e^{-i\mathbf{p}\mathbf{x}} \widehat{F}_{K,-}(-\mathbf{p})$. Hence, if we define

$$\Delta_\varepsilon(\mathbf{x}|\mathbf{z}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{e^{-i\mathbf{k}\mathbf{x}} - e^{-i\mathbf{k}\mathbf{z}}}{D_\omega(\mathbf{k})} \widehat{F}_{K,\varepsilon}(-\mathbf{k}) , \tag{165}$$

its solution is:

$$S_\omega(\mathbf{x}) = e^{\lambda_\infty \Delta_-(\mathbf{x}|0)} g_\omega(\mathbf{x}) . \tag{166}$$

Note that, for large $|\mathbf{x}|$, thanks to (115),

$$\Delta_\varepsilon(\mathbf{x}|0) \sim -\frac{A_\varepsilon(0)}{2\pi} \ln|\mathbf{x}| = -\frac{a(0) + \varepsilon\bar{a}(0)}{4\pi} \ln|\mathbf{x}| , \tag{167}$$

which implies, in particular, that the critical index $\eta_z$, defined in (78) is equal to $[a(0) - \bar{a}(0)]/(4\pi)$.

Let us now consider the 4-point correlation (161). If we take in (139) three derivatives w.r.t. $\widehat{\eta}^+_{\mathbf{q},-\omega}$, $\widehat{\eta}^-_{\mathbf{k}+\mathbf{q}-\mathbf{s},\omega}$ and $\widehat{\eta}^-_{\mathbf{s},-\omega}$, we find:

$$\begin{aligned} D_\omega(\mathbf{k})\widehat{G}_\omega(\mathbf{k},\mathbf{q},\mathbf{s}) &= \delta(\mathbf{q}-\mathbf{s})\widehat{S}_{-\omega}(\mathbf{q}) + \lambda_\infty \int \frac{d\mathbf{p}}{(2\pi)^2}\, \widehat{F}_{K,-}(\mathbf{p})\widehat{G}_\omega(\mathbf{k}+\mathbf{p},\mathbf{q},\mathbf{s}) + \\ &+\lambda_\infty \int \frac{d\mathbf{p}}{(2\pi)^2}\, \widehat{F}_{K,+}(\mathbf{p})\Big[\widehat{G}_\omega(\mathbf{k}+\mathbf{p},\mathbf{q}-\mathbf{p},\mathbf{s}) - \widehat{G}_\omega(\mathbf{k}+\mathbf{p},\mathbf{q},\mathbf{s}+\mathbf{p})\Big] , \end{aligned} \tag{168}$$

which, in the space coordinates, becomes:

$$\begin{aligned} &(\partial^{\mathbf{x}}_\omega G_\omega)(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) = \delta(\mathbf{x}-\mathbf{v})S_{-\omega}(\mathbf{y}-\mathbf{u}) + \\ &+\lambda_\infty\Big[F_{K,+}(\mathbf{x}-\mathbf{y}) - F_{K,+}(\mathbf{x}-\mathbf{u}) + F_{K,-}(\mathbf{x}-\mathbf{v})\Big]G_\omega(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) . \end{aligned} \tag{169}$$

By using (166), we find that the solution of this equation is given by

$$\begin{aligned} G_\omega(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) &= e^{-\lambda_\infty\Big[\Delta_+(\mathbf{x}-\mathbf{y}|\mathbf{v}-\mathbf{y}) - \Delta_+(\mathbf{x}-\mathbf{u},\mathbf{v}-\mathbf{u})\Big]} \cdot \\ &\cdot\ S_\omega(\mathbf{x}-\mathbf{v})S_{-\omega}(\mathbf{y}-\mathbf{u}) . \end{aligned} \tag{170}$$

If we put in this equation $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = \mathbf{v}$, we find, using also (161) and (167), that

$$\begin{aligned} \langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},\omega}\rangle &= \langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},\omega}\rangle_0 e^{-2\lambda_\infty[\Delta_+(\mathbf{x}-\mathbf{y},0) - \Delta_-(\mathbf{x}-\mathbf{y},0)]} \\ \underset{|\mathbf{x}-\mathbf{y}|\to\infty}{\sim} &\quad \frac{C}{|\mathbf{x}-\mathbf{y}|^{2[1-\bar{a}(0)(\lambda_\infty/2\pi)]}} . \end{aligned} \tag{171}$$

If we put instead $\mathbf{x} = \mathbf{y}$ and $\mathbf{u} = \mathbf{v}$, we get

$$\begin{aligned} \langle \psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{u},-\omega}\psi^-_{\mathbf{u},\omega}\rangle &= \langle \psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{u},-\omega}\psi^-_{\mathbf{u},\omega}\rangle_0 e^{2\lambda_\infty[\Delta_+(\mathbf{x}-\mathbf{u},0) + \Delta_-(\mathbf{x}-\mathbf{u},0)]} \\ \underset{|\mathbf{x}-\mathbf{u}|\to\infty}{\sim} &\quad \frac{C}{|\mathbf{x}-\mathbf{u}|^{2[1+a(0)(\lambda_\infty/2\pi)]}} . \end{aligned} \tag{172}$$

By using (171), (172), the first line of (115), (117) and the definition (8) of $x_\pm$, we finally get the first identity in (10).

# 4 Appendix: the anisotropic Ashkin-Teller model

In this appendix, in order to derive (12), we briefly recall the analysis of the anisotropic Ashkin-Teller model in [13]. The integration procedure is similar to that described in §2, the main difference being that the quadratic part (42) of the interaction now contains also terms of the form $\psi^{\varepsilon(\leq h)}_{\mathbf{x},\omega}\psi^{\varepsilon(\leq h)}_{\mathbf{x},-\omega}$. It follows, see (12) (where different definitions of the fermion fields were used) for details, that we have to substitute the Grassmann integration $P_{Z_h,\mu_h}(d\psi^{(\leq h)})$ in (64) with a new measure $P_{Z_h,\mu_h,\sigma_h}(d\psi^{(\leq h)})$, where $\mu_h$ and $\sigma_h$ are the constants multiplying, respectively, the quadratic *mass terms*

$$2\sum_{\omega=\pm}\psi^{(\leq h)+}_{\mathbf{x},\omega}\psi^{(\leq h)-}_{\mathbf{x},-\omega} \qquad \text{and} \qquad -2i\sum_{\varepsilon=\pm}\psi^{(\leq h)\varepsilon}_{\mathbf{x},+}\psi^{(\leq h)\varepsilon}_{\mathbf{x},-}\ . \tag{173}$$

One can see that

$$\begin{aligned}&|\log_\gamma(\mu_{j-1}/\mu_j)-\eta_\mu(\lambda_{-\infty})|\leq C\lambda^2\gamma^{\vartheta j}\ ,\\ &|\log_\gamma(\sigma_{j-1}/\sigma_j)-\eta_\sigma(\lambda_{-\infty})|\leq C\lambda^2\gamma^{\vartheta j}\ .\end{aligned} \tag{174}$$

Hence, since the two mass terms are clearly proportional, respectively, to the operators $O^+$ and $O^-$, we find that

$$\eta_\mu=\eta_+-\eta_z\ ,\qquad \eta_\sigma=\eta_--\eta_z\ . \tag{175}$$

It turns out that the difference of the critical temperatures scales as $|v|^{x_T}$ where $x_T$, see (5.26) of [13] (where the indices are defined with a different sign and the definitions of $\mu_h$ and $\sigma_h$ are exchanged), is given by

$$x_T=\frac{1+\eta_\mu}{1+\eta_\sigma}\ , \tag{176}$$

which implies (12), since $\eta_\mu=1-x_+$ and $\eta_\sigma=1-x_-$.

**Acknowledgments** P.F. is indebited with David Brydges for stimulating his interest in the topic with the request of a review seminar on the papers [24] and [18].

# References

[1] Ashkin J., Teller E.: *Statistics of Two-Dimensional Lattices with Four Components.* Phys. Rev. **64**, 178 - 184, (1943).

[2] Baxter R.J.: *Eight-Vertex Model in Lattice Statistics.* Phys. Rev. Lett. **26**, 832–833, (1971).

[3] Baxter R.J.: *Exactly solved models in statistical mechanics.* Academic Press, Inc. London, (1989).

[4] Barber M., Baxter R.J.: *On the spontaneous order of the eight-vertex model.* J. Phys. C **6**, 2913–2921, (1973).

[5] Benfatto G., Falco P., Mastropietro V.: *Functional Integral Construction of the Massive Thirring model: Verification of Axioms and Massless Limit.* Comm. Math. Phys. **273**, 67–118, (2007).

[6] Benfatto G., Falco P., Mastropietro V.: *Massless Sine-Gordon and Massive Thirring Models: proof of the Coleman's equivalence.* Comm. Math. Phys., to appear (2008).

[7] Benfatto G., Mastropietro V.: *Rev. Math. Phys.* **13**, 1323–1435, (2001).

[8] Benfatto G., Mastropietro V.: *On the Density-Density Critical Indices in Interacting Fermi Systems.* Comm. Math. Phys. **231**, 97–134, (2002).

[9] Benfatto G., Mastropietro V.: *Ward Identities and Chiral Anomaly in the Luttinger Liquid.* Comm. Math. Phys. **258**, 609–655, (2005).

[10] Falco P., Mastropietro V.: *Renormalization Group and Asymtotic Spin–Charge Separation for Chiral Luttinger Liquid.* J.Stat.Phys. **131**, 79–116, (2008).

[11] Giuliani A., Mastropietro V.: Anomalous Critical Exponents in the Anisotropic Ashkin-Teller Mode *Phys. Rev. Lett.* **93**, 190603–07, (2004).

[12] Giuliani A., Mastropietro V.: *Anomalous Universality in the Anisotropic AshkinâTeller Model.* Comm. Math. Phys. **256**, 681–725, (2005).

[13] Kadanoff L.P.: *Connections between the Critical Behavior of the Planar Model and That of the Eight-Vertex Model.* Phys. Rev. Lett. **39**, 903–905, (1977).

[14] Kadanoff L.P., Brown A.C.: *Correlation functions on the critical lines of the Baxter and Ashkin-Teller models. Ann. Phys.* **121**, 318–345, (1979).

[15] Kadanoff L.P., Wegner F.J.: *Some Critical Properties of the Eight-Vertex Model. Phys. Rev. B* **4**, 3989–3993, (1971).

[16] Luther A., Peschel I.: *Calculations of critical exponents in two dimension from quantum field theory in one dimension.* Phys. Rev. B **12**, 3908–3917, (1975).

[17] Mastropietro V.: *Non-Universality in Ising Models with Four Spin Interaction.* J. Stat. Phys. **111**, 201–259, (2003).

[18] Mastropietro V.: *Ising Models with Four Spin Interaction at Criticality.* Comm. Math. Phys. **244** 595–64 (2004).

[19] Mastropietro V.: *Nonperturbative Adler-Bardeen theorem. J. Math. Phys* **48**, 022302, (2007).

[20] Mastropietro V.: *Non-perturbative aspects of chiral anomalies.* J. Phys. A **40**, 10349–10365, (2007).

[21] Mastropietro V.: *Non-perturbative Renormalization.* World Scientific, (2008).

[22] den Nijs M.P.M.: *Derivation of extended scaling relations between critical exponents in two dimensional models from the one dimensional Luttinger model.* Phys. Rev. B **23**, 6111–6125, (1981).

[23] Pruisken A.M.M. Brown A.C.: *Universality for the critical lines of the eight vertex, Ashkin-Teller and Gaussian models.* Phys. Rev. B **23**, 1459–1468, (1981).

[24] Pinson H., Spencer T.: *Unpublished.*

[25] Samuel S. *The use of anticommuting variable integrals in statistical mechanics. I. The computation of partition functions.* J. Math. Phys. **21**, 2806, (1980).

[26] Smirnov S.: *Towards conformal invariance of 2D lattice models.* Proceedings Madrid ICM, Europ. Math. Soc, 2006 - arXiv:0708.0032

[27] Spencer T. *A mathematical approach to universality in two dimensions.* Physica A **279**, 250–259, (2000).

[28] Zamolodchikov A.B., Zamolodchikov Al. B.: *Conformal field theory and 2D critical phenomena, part 1.* Soviet Scientific Reviews A **10**, 269, (1989).