THE REVERSE POINCARÉ INEQUALITY

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Abstract. In this paper we show that, by the use of an equivalent norm, one can make unbounded linear operators exhibit the one of the most important properties of bounded operators without using the graph norm. We apply the results to obtain an interesting extension of the Poincaré inequality. As an application, we obtain a strong inequality for the nonlinear term in the Navier-Stokes equation.

Introduction

It is easy to show that any equivalent norm on a Banach space $\mathcal{B}$, can be identified with a transformation of $\mathcal{B}$ which preserves the topology. For example, any isometric isomorphism of $\mathcal{B}$ will preserve the topology with the same norm (see below). Recall that a norm on a Banach space $\mathcal{B}$ is said to be strict if $\|u+v\| = \|u\| + \|v\| \;\Rightarrow\; v = tu$ or $v = 0$ (see Holmes [H]). In many cases the space can be renormed to produce an equivalent strict norm. Clearly, such a replacement changes the geometric properties, but not the topology. It is implicitly assumed in analysis that there are no useful analytic advantages to a norm change. In order to see that this is not

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true and what such an apparently simple concept (renorming) can do to the analytic properties of the space, we begin with the following example:

Example 1. *Let* $\mathcal{H} = L^2[\mathbb{R}^3, d\mu]$*, where* $d\mu = (2\pi)^{-3/2} e^{-\frac{1}{2}|\mathbf{x}|^2} d\mathbf{x}$*, and consider the problem:*

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = \Delta u(t, \mathbf{x}) - \mathbf{x} \cdot \nabla \mathbf{u}(\mathbf{x}, t), \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

This is the well-known Ornstein-Uhlenbeck equation with solution $(T(t)u_0)(\mathbf{x}) = u(t, \mathbf{x})$*, where:*

$$(T(t)\mathbf{u}_0)(\mathbf{x}) = \frac{1}{\sqrt{[2\pi(1 - e^{-t})]^3}} \int_{\mathbb{R}^3} exp\left\{ -\frac{\left(e^{-t/2}\mathbf{x} - \mathbf{y}\right)^2}{2(1 - e^{-t})} \right\} \mathbf{u}_0(\mathbf{y}) d\mathbf{y}.$$

The operator $T(t)$ *is a (analytic) transition semigroup,* $T(t)\mathbf{1} = \mathbf{1}$*, with generator* $D^2 = \Delta - \mathbf{x} \cdot \nabla$*, so that* $\|T(t)u_0\|_2 = \|u_0\|_2$*. It follows that, for any* $\gamma \in (0, 1)$*,* $\|w\|_{2,\gamma} = \|T(\gamma)w\|_2$ *is an equivalent norm on* $\mathcal{H}$*. Moreover, since* $T(t)$ *is analytic and, if* w *is in the domain of* D^2*, there is a constant* $c > 0$ *such that* $\left\|D^2 w\right\|_{2,\gamma} = \left\|D^2 T(\gamma) w\right\|_2 \le c\gamma^{-1} \|w\|_2 = c\gamma^{-1} \|w\|_{2,\gamma}$*. We thus conclude that this equivalent norm on* $\mathcal{H}$ *makes* D^2 *bounded without using the graph norm. Note that* $\Delta = D^2 + \mathbf{x} \cdot \nabla$*. From here, it is easy to see that the equivalent norm also makes the operators* Δ *and* ∇ *bounded in the above sense. (For later reference, we set* $D = \nabla$*.)*

Purpose

The above example is very interesting and has interesting implications for analysis. (This will be discussed later.) The important question is: How general is the result? The main objective of this paper is to show that a variant of the above is possible on any separable Banach space.

Preliminaries

In what follows, $\mathcal{B}$, $\mathcal{H}$ is a separable Banach or Hilbert space, respectively, over the complex field. We let $L[\mathcal{B}], L[\mathcal{H}]$ denote the bounded linear operators on $\mathcal{B}$, $\mathcal{H}$ and assume that $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$. By a duality map, ϕ_u, defined on $\mathcal{B}$, we mean any linear functional $\phi_u \in \{f \in \mathcal{B}' | < u, f >= \|u\|_{\mathcal{B}}^2, u \in \mathcal{B}\}$, where $< . >$ is the natural pairing between a Banach space and its dual. Let $\mathbf{J} : \mathcal{H} \longrightarrow \mathcal{H}'$ be the standard conjugate isomorphism between a Hilbert space and its dual, so that $< u, \mathbf{J}(u) > = (u, u)_{\mathcal{H}} = \|u\|_{\mathcal{H}}^2$. We define the *Steadman duality map* of $\mathcal{B}$ associated with $\mathcal{H}$ by:

$$\phi_u^s = \frac{\|u\|_{\mathcal{B}}^2}{\|u\|_{\mathcal{H}}^2}\mathbf{J}(u).$$

It is easy to check that ϕ_u^s is a duality map for $\mathcal{B}$. A densely defined operator A is called dissipative if $\mathrm{Re}\,\langle A\phi, \phi_u^s\rangle \leq 0$ and, accretive if $\langle A\phi, \phi_u^s\rangle \geq 0$ for all $x \in D(A)$. (If A has no proper extensions it is called m-accretive.) It is called m-dissipative if in addition, A it is closed and $Ran(I - A) = \mathcal{B}$. It is well-known that an m-dissipative linear operator is the generator of a contraction semigroup (see Pazy [P]). We need the following results (see Lorenzi and Bertoldi [LB]).

Definition 2. *A closed densely defined linear operator $A : D(A) \subset \mathcal{B} \to \mathcal{B}$ is said to be m-sectorial if there exist three constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that the resolvent set of A, $\rho(A)$, contains the sector in the complex plane: $S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$ and,*

$$\|R(\lambda, A)\|_{L(\mathcal{B})} \leq \frac{M}{|\lambda - \omega|}.$$

Theorem 3. *If A satisfies the above conditions, then it is the generator of a strongly continuous analytic semigroup $T(t)$ such that:*

(1) *$T(t)u \in D(A^k)$ for any $t > 0$, any $u \in \mathcal{B}$ and any $k \in \mathbb{N}$. Furthermore, if $u \in D(A^k)$, then*

$$A^k T(t)u = T(t)A^k u, \ t \geq 0;$$

(2) *there exists a constant $M > 0$ such that*

$$\|T(t)\|_{L(\mathcal{B})} \leq Me^{\omega t}, \ t > 0,$$

where ω is the number in our definition;

(3) *for any $\delta > 0$ and any $k \in \mathbb{N}$, there exists a positive constant $c(k, \delta)$ such that*

$$\left\| t^k A^k T(t) \right\|_{L(\mathcal{B})} \leq c(k, \delta)e^{(\omega+\delta)t}, \ t > 0;$$

(4) *the function $t \mapsto T(t)$ is analytic in $(0, +\infty)$ with values in $L(\mathcal{B})$ and*

$$\frac{d^k}{dt^k}T(t) = A^k T(t), \quad t > 0, \ k \in \mathbb{N}.$$

The next theorem gives conditions for a closed densely defined linear operator to be m-sectorial on $\mathcal{H}$.

Theorem 4. *Let $-A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined selfadjoint positive operator. Then A is m-sectorial in $\mathcal{H}$ and satisfies the conditions of Theorem 3 with $\omega = 0$ and arbitrary $\theta < \pi$.*

A well-known result of von Neumann [VN] shows that on $\mathcal{H}$, any closed densely defined linear operator A has a well-defined adjoint A^*, with A^*A m-accretive. A well-known result in the classic book of Kato shows that $T = -[A^*A]^{1/2}$ is the generator of an analytic contraction semigroup on $\mathcal{H}$ and $D(T) = D(A)$. (Actually, $A = -WT$, where W is a partial isometry (see Kato [K]).)

Main Results

The following theorem is a variant of a result due to Gross and Kuelbs [G], [KB]. It shows that, given any closed densely defined linear operator A defined on $\mathcal{B}$, we can always find Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, and a positive trace class operator $\mathbf{T}_{12}$ defined on $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ (all as continuous dense embeddings), and $\mathbf{T}_{12}$ determines $\mathcal{H}_1$ when $\mathcal{B}$ and $\mathcal{H}_2$ are given. Furthermore, this can be done in such a way that $\mathcal{H}_1 \subset D(A)$.

Theorem 5. *Suppose $\mathcal{B}$ is a separable Banach space and A is a closed densely defined linear operator A on $\mathcal{B}$. Then there exist separable Hilbert* *spaces $\mathcal{H}_1, \mathcal{H}_2$ and a positive trace class operator $\mathbf{T}_{12}$ defined on $\mathcal{H}_2$ such that:*

1. *$\mathcal{H}_1 \subset D(A)$,*
2. *$\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ (all as continuous dense embeddings) and*
3. *$\mathcal{B}$ determines $\mathcal{H}_2$, while $\mathcal{B}$ and A determines $\mathbf{T}_{12}$ and $\mathcal{H}_1$.*

Proof. As $\mathcal{B}$ is separable, let $\{u_n\}$ be a dense set in $\mathcal{B}$ which is contained in $D(A)$, and let $\{f_n\}$ be any fixed set of corresponding duality mappings (i.e., $f_n \in \mathcal{B}'$ and $f_n(u_n) = \langle u_n, f_n\rangle = \|u_n\|_{\mathcal{B}}^2$). Let $\{t_n\}$ be a positive sequence of numbers such that $\sum_{n=1}^{\infty} t_n = 1$, and define $(u,v)_2$ by:

$$(u,v)_2 = \sum\nolimits_{n=1}^{\infty} t_n f_n(u)\bar{f}_n(v).$$

It is easy to see that $(u,v)_2$ is an inner product on $\mathcal{B}$. We let $\mathcal{H}_2$ be the Hilbert space generated by the completion of $\mathcal{B}$ with respect to this inner product. It is clear that $\mathcal{B}$ is dense in $\mathcal{H}_2$, and as

$$\|u\|_2^2 = \sum\nolimits_{n=1}^{\infty} t_n \left|f_n(u)\right|^2 \le \sup_n \left|f_n(u)\right|^2 = \|u\|_{\mathcal{B}}^2 \,,$$

we see that the embedding is continuous.

Now, let $\{\varphi_n\} \in D(A)$ be a complete orthonormal sequence for $\mathcal{H}_2$, and let $\{\lambda_n\}$ be a positive sequence such that $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $M = \sum_{n=1}^{\infty} \lambda_n^2 \left\|\phi_n\right\|_{\mathcal{B}}^2 < \infty$. Define the operator $\mathbf{T}_{12}$ on $\mathcal{B}$ by:

$$\mathbf{T}_{12}u = \sum\nolimits_{n=1}^{\infty} \lambda_n \left(u, \phi_n\right)_2 \phi_n.$$

Since

$$\mathcal{B} \subset \mathcal{H}_2 \Rightarrow \mathcal{H}'_2 \subset \mathcal{B}' \Rightarrow (\cdot, \phi_n)_2 \in \mathcal{B}', \ \forall n,$$

we have that $\mathbf{T}_{12}$ maps $\mathcal{B} \to \mathcal{B}$ and:

$$\|\mathbf{T}_{12}u\|_{\mathcal{B}}^2 \leq \left[\sum\nolimits_{n=1}^{\infty} \lambda_n^2 \|\phi_n\|_{\mathcal{B}}^2\right] \left[\sum\nolimits_{n=1}^{\infty} |(u, \phi_n)_2|^2\right] = M \|u\|_2^2 \leq M \|u\|_{\mathcal{B}}^2 .$$

Thus, $\mathbf{T}_{12}$ is a bounded operator on $\mathcal{B}$. Define $\mathcal{H}_1$ by:

$$\mathcal{H}_1 = \left\{u \in D(A) \,\middle|\, \sum\nolimits_{n=1}^{\infty} \lambda_n^{-1} |(u, \phi_n)_2|^2 < \infty\right\}, \quad (u, v)_1 = \sum\nolimits_{n=1}^{\infty} \lambda_n^{-1} (u, \phi_n)_2 (\phi_n, v)_2 .$$

With the above inner product, $\mathcal{H}_1$ is a Hilbert space and, since terms of the form $\{u_N = \sum_{k=1}^{N} \lambda_k^{-1} (u, \psi_k)_2 \phi_k \ : \ u, \psi_k \in D(A)\}$ are dense in $\mathcal{B}$, we see that $\mathcal{H}_1$ is dense in $\mathcal{B}$. It follows that $\mathcal{H}_1$ is also dense in $\mathcal{H}_2$. It is easy to see that $\mathbf{T}_{12}$ is a positive selfadjoint operator with respect to the $\mathcal{H}_2$ inner product so, by the theorem of Lax [L], $\mathbf{T}_{12}$ has a bounded extension to $\mathcal{H}_2$ and $\|\mathbf{T}_{12}\|_2 \leq \|\mathbf{T}_{12}\|_{\mathcal{B}}$. Finally, it is easy to see that, for $u, v \in \mathcal{H}_1$, $(u, v)_1 = (\mathbf{T}_{12}^{-1/2} u, \mathbf{T}_{12}^{-1/2} v)_2$ and $(u, v)_2 = (\mathbf{T}_{12}^{1/2} u, \mathbf{T}_{12}^{1/2} v)_1$. It follows that $\mathcal{H}_1$ is continuously embedded in $\mathcal{H}_2$, hence also in $\mathcal{B}$. It is clear that $\mathcal{H}_1 \subset D(A)$. $\square$

We remark that Theorem 5 (see (3)), suggests that both $\mathcal{H}_1$ and $\mathbf{T}_{12}$ depend on $D(A)$ and so are not unique. It is possible to give a proof which does not depend on the domain. However this requires the extension (generalization) of a substantial amount of operator theory from the Hilbert spaces to Banach spaces which does not provide any other benefits (for this paper). Thus, Theorem 5 represents a compromise.

Let us fix $A, \mathcal{H}_1, \mathcal{H}_2$ such that A is closed and densely defined on $\mathcal{B}$, with $\mathcal{H}_1 \subseteq \mathcal{B} \subseteq \mathcal{H}_2$ as continuous dense embeddings and, without loss of generality, assume that for $u \in \mathcal{H}_1$, $\|u\|_2 \leq \|u\|_{\mathcal{B}} \leq \|u\|_1$.

Theorem 6. *The operator A has a well-defined adjoint A^* defined on $\mathcal{B}$ such that:*

(1) *the operator $A^*A \geq 0$ (m-accretive),*

(2) $(A^*A)^* = A^*A$*, and*

(3) $I + A^*A$ *has a bounded inverse.*

Proof. **Part I**

We first prove that any closed densely defined linear operator C on $\mathcal{B}$ extends to a closed densely defined linear operator on $\mathcal{H}_2$, with $\rho(\bar{C}) = \rho(C)$ and $\sigma(\bar{C}) = \sigma(C)$. Let $\mathbf{J}_2 : \mathcal{H}_2 \longrightarrow \mathcal{H'}_2$ denote the standard conjuate isomorphism. Then, as $\mathcal{B}$ is strongly dense in $\mathcal{H}_2$, $\mathbf{J}_2[\mathcal{B}] \subset \mathcal{H'}_2 \subset \mathcal{B}'$ is (strongly) dense in $\mathcal{H'}_2$. If C is any closed densely defined linear operator on $\mathcal{B}$ with domain $D(C)$, then C' (the $\mathcal{B}$ adjoint of C) is closed on $\mathcal{B}'$. In addition, $C'|_{\mathcal{H'}_2}$ is closed and, for each $u \in D(C)$, $\mathbf{J}_2(u) \in \mathcal{H'}_2$ and $< Cv, \mathbf{J}_2(u) >$ is well defined $\forall v \in D(C)$. Hence $\mathbf{J}_2(u) \in D(C')$ for all $u \in D(C)$. Since $\mathbf{J}_2(\mathcal{B})$ is strongly dense in $\mathcal{H'}_2$, this implies that $\mathbf{J}_2(D(C)) \subset D(C')$ is strongly dense in $\mathcal{H'}_2$ so that $D(C')\,|_{\mathcal{H'}_2}$ is strongly dense in $\mathcal{H'}_2$. Thus, as $\mathcal{H}_2$ is reflexive, $[C'\,|_{\mathcal{H'}_2}]'$ is a closed densely defined operator on $\mathcal{H}_2$.

To prove the second assertion, note that, if $\hat{C}$ is the extension, and $\lambda I - \hat{C}$ has an inverse, then $\lambda I - C$ also has one, so $\rho(\hat{C}) \subset \rho(C)$ and $Ran(\lambda I - C) \subset Ran(\lambda I - \hat{C}) \subset \overline{Ran(\lambda I - C)}$ for any $\lambda \in \mathbb{C}$. For the other direction, assume that $\rho(C) \neq \emptyset$ so there is at least one $\lambda \in \rho(C)$. Then $(\lambda I - C)^{-1}$ is a continuous mapping from $Ran(\lambda I - C)$ onto $D(C)$ and $Ran(\lambda I - C)$ is dense in $\mathcal{B}$. Let $\varphi \in D(\hat{C})$, so that $(\varphi, C\varphi) \in \bar{G}(C)$, the closure of the graph of C (by definition). Thus, there exists a sequence $\{\varphi_n\} \subset D(C)$ such that $\|\varphi - \varphi_n\|_G = \|\varphi - \varphi_n\|_{\mathcal{B}} + \|C\varphi - C\varphi_n\|_{\mathcal{B}} \to 0$ as $n \to \infty$. It follows that $(\lambda I - \hat{C})\varphi = \lim_{n\to\infty}(\lambda I - C)\varphi_n$. However, by the boundedness of $(\lambda I - C)^{-1}$ on $R(\lambda I - C)$ we have that, for some $\delta > 0$,

$$\left\|(\lambda I - \hat{C})\varphi\right\|_{\mathcal{B}} = \lim_{n\to\infty} \|(\lambda I - C)\varphi_n\|_{\mathcal{B}} \geq \lim_{n\to\infty} \delta \|\varphi_n\|_{\mathcal{B}} = \delta \|\varphi\|_{\mathcal{B}}.$$

It follows that $\lambda I - \hat{C}$ has a bounded inverse and, since $D(C) \subset D(\hat{C})$ implies that $Ran(\lambda I - C) \subset Ran(\lambda I - \hat{C})$, we see that $Ran(\lambda I - \hat{C})$ is dense in $\mathcal{B}$ so that $\lambda \in \rho(\hat{C})$ and hence $\rho(C) \subset \rho(\hat{C})$. It follows that $\rho(C) = \rho(\hat{C})$ and necessarily, $\sigma(C) = \sigma(\hat{C})$.

Part II

If we let $\mathbf{J}_i : \mathcal{H}_i \to \mathcal{H}'_i\,(i = 1, 2)$, then $A_1 = A_{|\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_2$, and $A'_1 : \mathcal{H}'_2 \to \mathcal{H}'_1$. Since $\mathcal{H}_1$ is dense in $D(A)$, A'_1 is closed and densely defined.

It follows that $A'_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}'_1$ and $\mathbf{J}_1^{-1}A'_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}_1 \subset \mathcal{B}$ so that, if we define $A^* = [\mathbf{J}_1^{-1}A'_1\mathbf{J}_2]_{\mathcal{B}}$, then $A^* : \mathcal{B} \to \mathcal{B}$ (i.e., A^* is closed and densely defined). To prove (1), $\mathbf{J}'_i = \mathbf{J}_i$ and, if $u \in \mathcal{H}_1$, then

$\langle A^*Au, \mathbf{J}_2(u)\rangle = \langle Au, (A^*)'\mathbf{J}_2(u)\rangle$. Using the above definition of A^*, we get that $(A^*)'\mathbf{J}_2(u) = \{[\mathbf{J}_1^{-1}A'_1\mathbf{J}_2]_{\mathcal{B}}\}'\mathbf{J}_2(u) = [\mathbf{J}_2A_1\mathbf{J}_1^{-1}]\mathbf{J}_2(u) = \mathbf{J}_2(A_1u)$. Since, for $u \in \mathcal{H}_1$, $A_1u = Au$ and

$$\langle A^*Au, \phi_u^s\rangle = \frac{\|u\|_{\mathcal{B}}^2}{\|u\|_2^2}\langle Au, \mathbf{J}_2(A_1u)\rangle = \frac{\|u\|_{\mathcal{B}}^2}{\|u\|_2^2}\|Au\|_2^2 \geq 0,$$

we have that A^*A is accretive on a dense set. Thus, A^*A is accretive on $\mathcal{B}$. It is m-accretive because it has no proper extension. To prove (2), we have that for $u \in \mathcal{H}_1$,

$$\begin{aligned}
(A^*A)^*u &= (\{\mathbf{J}_1^{-1}[\{[\mathbf{J}_1^{-1}A'_1\mathbf{J}_2]|_{\mathcal{B}}A\}_1]'\mathbf{J}_2\}|_{\mathcal{B}})u \\
&= (\{\mathbf{J}_1^{-1}[\{A'_1[\mathbf{J}_2A_1\mathbf{J}_1^{-1}]|_{\mathcal{B}}\}]\mathbf{J}_2\}|_{\mathcal{B}})u \\
&= A^*Au.
\end{aligned}$$

It follows that the same result holds on $\mathcal{B}$. Finally, the proof that $I + A^*A$ is invertible follows the same lines as in von Neumann's theorem. □

If we put Parts I and II together, we see that A^* also has a closed densely defined extension to $\mathcal{H}_2$.

Definition 7. *Let S be bounded and A be a closed densely defined linear operator. Let $\mathcal{U}$, $\mathcal{V}$ subspaces of $\mathcal{B}$:*

(1) *A is said to be naturally self-adjoint if $A = A^*$ on $D(A)$.*

(2) *A is said to be normal if $AA^* = A^*A$ on $D(A)$.*

(3) *S is unitary if $SS^* = S^*S = I$.*

(4) *The subspace* $\mathcal{U}$ *is* $\perp$ *to* $\mathcal{V}$ *if* $\forall u \in \mathcal{U}$, $\langle v, \varphi_u^s \rangle = 0$ *and* $\forall v \in \mathcal{V}$, $\langle u, \varphi_v^s \rangle = 0$.

The last definition is transparent since, for example,

$$\langle v, \varphi_u^s \rangle = 0 \Leftrightarrow \langle v, J_2(u) \rangle = (v, u)_2 = 0 \;\; \forall v \in \mathcal{V}.$$

The following related definition is due to Palmer [PL],where the operator is called symmetric. This is essentially the same as a Hermitian operator as defined by Lumer [LU].

Definition 8. *A closed linear operator A, defined on* $\mathcal{B}$*, with dense domain is called self-conjugate if both* iA *and* $-iA$ *are dissipative.*

Theorem 9. *(Vidav-Palmer) A linear operator A, defined on* $\mathcal{B}$*, is self-conjugate if and only if* iA *and* $-iA$ *are generators of isometric semigroups.*

Theorem 10. *The operator A, defined on* $\mathcal{B}$*, is self-conjugate if and only if it is naturally self-adjoint.*

Proof. Let $\hat{A}$ and $\hat{A}^*$ be the closed densely defined extensions of A and A^* to $\mathcal{H}_2$. On $\mathcal{H}_2$, $\hat{A}$ is naturally self-adjoint if and only if $i\hat{A}$ generates a unitary group, if and only if it is self-conjugate. Thus, both definitions coincide on $\mathcal{H}_2$. It follows that the restrictions coincide on $\mathcal{B}$. $\square$

We now consider the general case.

Theorem 11. *If A is any closed densely defined linear operator on $\mathcal{B}$, there exists a positive m-accretive operator T and a partial isometry W such that $A = WT$, and $D(A) = D(T)$.*

Proof. Since both A and A^* have closed densely defined extensions $\hat{A}$ and $\hat{A}^*$ to $\mathcal{H}_2$, the operator $\hat{T} = [\hat{A}\hat{A}^*]^{1/2}$ is a well-defined m-accretive selfadjoint linear operator on $\mathcal{H}_2$, $\hat{A} = \hat{W}\hat{T}$ for some partial isometry $\hat{W}$ defined on $\mathcal{H}_2$ and $D(\hat{A}) = D(\hat{T})$. Our proof is complete when we notice that the restriction of $\hat{A}$ to $\mathcal{B}$ is A and $\hat{T}^2$ restricted to $\mathcal{B}$ is A^*A, so that the restriction of $\hat{W}$ to $\mathcal{B}$ is well-defined and must be a partial isometry. The equality of the domains is obvious. □

For any closed densely defined linear operator A on $\mathcal{B}$, let $T = [A^*A]^{1/2}$, $\bar{T} = [AA^*]^{1/2}$. Since both T and $\bar{T}$ are naturally self-adjoint, they generate unitary groups $U(t)$, $\bar{U}(t)$. From the above result, we know that $A = WT$, where W is a partial isometry, $D(A) = D(T)$ and $\|Au\|_{\mathcal{B}} = \|Tu\|_{\mathcal{B}}$, for each $u \in D(A)$. It is also easy to see that for $u \in D(A)$, $ATu = \bar{T}Au$. The following theorem is due to Lumer and Phillips [LP] (see remark 2 after Theorem 3.2).

Theorem 12. *If T is the generator of a unitary group $U(t)$, then there is a dense set $D_0 \subset D(A^\infty)$ such that, for all $v \in D_0$, $U(t)$ has the representation:*

$$U(\tau)v = \sum\nolimits_{n=0}^{\infty} \frac{\tau^n}{n!} T^n v.$$

Lemma 13. *$AU(\tau)v = \bar{U}(\tau)Av$ for all $v \in D(A)$.*

Proof. Since $ATv = \bar{T}Av$, it follows that $AT^n v = \bar{T}^n Av$ for all $v \in D_0$. Hence:

$$AU(\tau)v = \sum\nolimits_{n=0}^{\infty} \frac{\tau^n}{n!} AT^n v = \sum\nolimits_{n=0}^{\infty} \frac{\tau^n}{n!} \bar{T}^n Av = \bar{U}(\tau)Av.$$

It follows that the result extends to all $v \in D(A)$. □

Definition 14. *A linear operator A, defined on $\mathcal{B}$, is said to have an equivalent bound if $\mathcal{B}$ has an equivalent norm $\|\cdot\|_{\mathcal{B},1}$, for which $\|Au\|_{\mathcal{B},1} \leq c \|u\|_{\mathcal{B},1}$ for some constant c and each $u \in D(A)$.*

Theorem 15. *If A is any closed densely defined linear operator on $\mathcal{B}$, there exists an equivalent norm $\| \cdot \|_{\mathcal{B},1}$ for $\mathcal{B}$ such that A has an equivalent bound.*

Proof. Let A be given and set $T = [AA^*]^{1/2}$. Since T is a well-defined m-accretive naturally selfadjoint linear operator on $\mathcal{B}$, it follows that iT is m-sectorial and generates a strongly continuous unitary group $S(t) = exp\{tiT\}$ on $\mathcal{B}$. Let $\|u\|_{\mathcal{B},1} = \left\|S(\frac{1}{2})u\right\|_{\mathcal{B}}$. It is easy to see that for $u \in D(A)$:

$$\|Au\|_{\mathcal{B},1} = \left\|WTS(\tfrac{1}{2})u\right\|_{\mathcal{H}} = \left\|TS(\tfrac{1}{2})u\right\|_{\mathcal{B}} \leq 2c \|u\|_{\mathcal{B}} = 2c \|u\|_{\mathcal{B},1}.$$

Thus, A has an equivalent bound. □

1. Applications

In this section our references are Lorenzi and Bertoldi [LB], Pazy [P] and Sell and You [SY]. Here, we apply the main results to a few problems in probability theory and analysis. In the following, $L^p_\mu = L^p[\mathbb{R}^3, \mathfrak{B}(\mathbb{R}^3), \mu]$, where $\mathfrak{B}(\mathbb{R}^3)$ is the class of Borel sets and μ is a measure that is absolutely continuous with respect to Lebesgue measure. We assume that the operator (see [LB], pg. 156 and [P], pg. 209):

$$Au = \sum\nolimits_{i,j=1}^{3} D_i(q_{ij} D_j u) + \sum\nolimits_{i=1}^{3} b_i D_i u$$

is strongly elliptic (e.g., there exists a constant d such that $\sum\nolimits_{i,j=1}^{3} \xi_i(q_{ij}\xi_j) \geq d\sum\nolimits_{i=1}^{3} |\xi_i|^2$). The functions $q_{ij} = q_{ji},\ b_i,\ 1 \leq i, j \leq 3$ are assumed to be in $W^{1,p}_{loc}(\mathbb{R}^3)$.

1.1. **Markov processes.** There are a number of versions of the Poincaré inequality (see Evans [EV]). We consider the version that naturally appears in the theory of Markov processes. Let μ be a Borel probability measure associated with the transition semigroup $S(t)$ for a given Markov process with generator A. μ is called an invariant measure if:

$$\int_{\mathbb{R}^3} S(t)u(\mathbf{x})d\mu(\mathbf{x}) = \int_{\mathbb{R}^3} u(\mathbf{x})d\mu(\mathbf{x}), \quad t > 0,$$

for any $u(\mathbf{x}) \in \mathbb{C}^\infty_c$. If u is any function in L^p_μ and we set $\hat{u} = \int_{\mathbb{R}^3} u(\mathbf{x})d\mu(\mathbf{x})$, it is known that for $1 \leq p < \infty$:

$$\lim_{t\to\infty} \|S(t)u - \bar{u}\|_p = 0.$$

Since the generator of $S(t)$ is strongly elliptic and $u \in W^{1,p}_\mu$, in this case the Poincaré inequality satisfies:

$$\int_{\mathbb{R}^3} |u - \bar{u}|^p \, d\mu \leq C \int_{\mathbb{R}^3} |Du|^p \, d\mu, \tag{1}$$

where C is a positive constant. To simplify things, assume that $\bar{u} = 0$. In this case, we have $\|u\|_p^p \leq C \|Du\|_p^p$. First note that we can write the gradient operator D as $\mathbf{R}(-\Delta)^{1/2}u$, where $\mathbf{R}$ is the Riesz transform, $\|\mathbf{R}\| = 1$ (see Stein [S]). If we renorm our space using the semigroup of Example 1, we get that $\|u\|_{p,\gamma}^p \leq C \|Du\|_{p,\gamma}^p$. Using the analytic properties of $T(\gamma)$, we have that

$$\|Du\|_{p,\gamma}^p \leq c\gamma^{-1/2} \|u\|_{p,\gamma}^p \quad \text{for all } u \in W_\mu^{1,p}. \tag{2}$$

1.2. **Navier-Stokes Equations.** Let Ω be an open domain of class $\mathbb{C}^k$ contained in $\mathbb{R}^n$, $n \geq 2$, let $(\mathbb{L}^2[\Omega])^n$ be the complex Hilbert space of square integrable functions on Ω with values in $\mathbb{R}^n$, let $\mathbf{D}[\Omega]$ be $\{\mathbf{u} \in (\mathbb{C}_0^\infty[\Omega])^n \mid \nabla \cdot \mathbf{u} = 0\}$, let $\mathbb{H}[\Omega]$ be the completion of $\mathbf{D}[\Omega]$ with respect to the inner product of $(\mathbb{L}^2[\Omega])^n$, and let $\mathbb{V}[\Omega]$ be the completion of $\mathbf{D}[\Omega]$ with respect to the inner product of $\mathbb{H}^1[\Omega]$, the functions in $\mathbb{H}[\Omega]$ with weak derivatives in $(\mathbb{L}^2[\Omega])^n$. The classical Navier-Stokes initial-value problem (for $\Omega \subset \mathbb{R}^n$, and all $T > 0$) is

$$\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T) \times \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \text{ in } (0,T) \times \Omega, \\
&\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \text{ on } (0,T) \times \partial\Omega, \\
&\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.
\end{aligned} \tag{3}$$

The equations describe the time evolution of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ and the pressure p of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient ν in terms of a given initial velocity $\mathbf{u}_0(\mathbf{x})$ and given external body forces $\mathbf{f}(\mathbf{x},t)$. Applying the well-known Leray projection to equation (3), with $\mathbf{B}(\mathbf{u},\mathbf{u}) = \mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$ and $\mathbf{A} = \mathbb{P}(-\boldsymbol{\Delta})$, we can recast equation (3) in the standard form:

$$
\begin{aligned}
&\partial_t \mathbf{u} = -\nu\mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t) \text{ in } (0,T)\times\Omega,\\
&\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\partial\Omega,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega,
\end{aligned}
\tag{4}
$$

where we have used the fact that the orthogonal complement of $\mathbb{H}[\Omega]$ relative to $(\mathbb{L}^2[\Omega])^3$ is $\{\mathbf{v} : \mathbf{v} = \nabla q,\ q \in (H^1[\Omega])^3\}$ to eliminate the pressure term (see Sell and You [SY]). The difficulty in proving the existence of global-in-time strong solutions to this equation can be directly linked to the problem of getting good estimates for the nonlinear term $\mathbf{B}(\mathbf{u},\mathbf{u})$. For example, the following theorem is one of the major estimates used to study this equation (see equation 61.22 on page 366, in Sell and You [SY] and Constantin and Foias [CF]). (We assume that $u,\ v \in D(A)$.)

Theorem 16. *Let Ω be a bounded open set of class $\mathbb{C}^k$ in $\mathbb{R}^3$. Let $\alpha_i, 1 \le i \le 3$, satisfy $0 \le \alpha_1 \le k$, $0 \le \alpha_2 \le k-1$, $0 \le \alpha_3 \le k$, with $\alpha_1+\alpha_2+\alpha_3 \ge 3/2$ and*

$$(\alpha_1,\alpha_2,\alpha_3) \notin \{(3/2,0,0),(0,3/2,0),(0,0,3/2)\}.$$

Then there is a positive constant $c = c(\alpha_i, \Omega)$ *such that*

$$|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{\mathbb{H}}| \le c \left\| \mathbf{A}^{\alpha_1/2}\mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{(1+\alpha_2)/2}\mathbf{v} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{\alpha_3/2}\mathbf{w} \right\|_{\mathbb{H}}. \tag{5}$$

We plan to show that, by renorming $\mathbb{H}$, we can prove a very strong inequality for equation (5). First we need to investigate the Schrödinger version of the Stokes equation.

Stokes Equation. If we drop the nonlinear term, we get the well-known Stokes equation ($\mathbb{P}\mathbf{f}(t) = \mathbf{0}$):

$$\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} \text{ in } (0, T) \times \Omega,$$

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \text{ on } (0, T) \times \partial\Omega,$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.$$

The above problem has a unique solution, which is generated by an analytic contraction semigroup. If we replace the above by:

$$i\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} \text{ in } (0, T) \times \Omega,$$

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \text{ on } (0, T) \times \partial\Omega,$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.$$

We get the Schrödinger version. This equation also has a unique solution, which is generated by an analytic unitary group. $S(t) = exp\{it\mathbf{A}\}$. Let $\|\mathbf{u}\|_{\mathbb{H},1} = \left\|S(\frac{1}{3})\mathbf{u}\right\|_{\mathbb{H}}$, so that $\|\mathbf{u}\|_{\mathbb{H},1} = \|\mathbf{u}\|_{\mathbb{H}}$ for all $\mathbf{u} \in \mathbb{H}$. Since $\mathbf{A}$ is analytic, we have that $\|\mathbf{A}^z\mathbf{u}\|_{\mathbb{H},1} \le 3^z\bar{c}\,\|\mathbf{u}\|_{\mathbb{H},1}$. From Theorem 16, we have the following result:

Theorem 17. *Let $\mathbf{u} \in D(\mathbf{A})$, set $\mathbf{S} = S(\frac{1}{3})$ and renorm $\mathbb{H}$ so that $\|\mathbf{u}\|_{\mathbb{H},1} = \|\mathbf{S}\mathbf{u}\|_{\mathbb{H}}$. Then, with $k = 3$, if we let $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = 3$, there is a positive constant $c = c(\alpha_i, \Omega)$ such that*

$$\left|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{\mathbb{H},1}\right| \leq (9/4)c\bar{c}\, \|\mathbf{u}\|_{\mathbb{H},1} \|\mathbf{v}\|_{\mathbb{H},1} \|\mathbf{w}\|_{\mathbb{H},1}.$$

Proof. If we set $\mathbf{w}_1 = \mathbf{S}^2\mathbf{w}$, then

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w})_{\mathbb{H},1} = \langle \mathbf{S}\mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{S}\mathbf{w}\rangle_{\mathbb{H}} = \left\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{S}^2\mathbf{w}\right\rangle_{\mathbb{H}} = b(\mathbf{u}, \mathbf{v}, \mathbf{w}_1)_{\mathbb{H}}.$$

Using the selfadjoint property of $\mathbf{A}$, and integration by parts, we have

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w})_{\mathbb{H},1} = -b(\mathbf{u}, \mathbf{w_1}, \mathbf{v})_{\mathbb{H}}.$$

It now follows from Theorem 16 that:

$$\left|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{\mathbb{H},1}\right| \leq c \left\|\mathbf{A}^{\alpha_1/2}\mathbf{u}\right\|_{\mathbb{H}} \left\|\mathbf{A}^{(1+\alpha_2)/2}\mathbf{w}_1\right\|_{\mathbb{H}} \left\|\mathbf{A}^{\alpha_3/2}\mathbf{v}\right\|_{\mathbb{H}}.$$

If we set $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = 3$, we have

$$\begin{aligned}
\left|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{\mathbb{H},1}\right| &\leq c\, \|\mathbf{u}\|_{\mathbb{H}} \left\|\mathbf{A}^2\mathbf{w}_1\right\|_{\mathbb{H}} \|\mathbf{v}\|_{\mathbb{H}} \\
&\leq \tfrac{9}{4}c\bar{c}\, \|\mathbf{u}\|_{\mathbb{H}} \|\mathbf{w}\|_{\mathbb{H}} \|\mathbf{v}\|_{\mathbb{H}} \\
&= \tfrac{9}{4}c\bar{c}\, \|\mathbf{u}\|_{\mathbb{H},1} \|\mathbf{w}\|_{\mathbb{H},1} \|\mathbf{v}\|_{\mathbb{H},1}.
\end{aligned}$$

□

It should be noted that the choice of $1/3$ in the theorem is not important and not optimal. In practice, the choice is determined by the constants of the problem.

DISCUSSION. It is somewhat amazing that Theorem 15 is new for Hilbert spaces (its power is clear from Theorem 17). However, the possibility of such a result has been around since the discovery of the polar decomposition theorem for closed densely defined linear operators. We have found a number of other uses for this well-known result. For example, in [GBZS], we constructed a generalized Yosida approximator which provides a contractive approximator for all closed densely defined linear operators.

In the following, we use this result to show in what sense the Poincaré inequality can be generalized. Let A be a closed densely defined linear operator on $\mathcal{B}$ and let $T = -[A^*A]^{1/2}$, $\bar{T} = -[AA^*]^{1/2}$.

Theorem 18. *Let $S(t)$, $\bar{S}(t)$ be the contraction semigroups generated by $T, \bar{T}$. If for $u \in \mathcal{B}$, $\inf \|S(t)u\|_{\mathcal{B}} = r\,\|u\|_{\mathcal{B}}$, where $r < 1$ and the infimum is taken for $t \in [0,1]$. Then there exists a constant c such that for each $u \in D(A)$, $\|u\|_{\mathcal{B}} \leq c\,\|Au\|_{\mathcal{B}}$*

Proof. Since T is the generator of $S(t)$, we have for $u \in D(A)$:

$$|\|S(t)u\|_{\mathcal{B}} - \|u\|_{\mathcal{B}}| \leq \|S(t)u - u\|_{\mathcal{B}} = \left\| \int_0^t TS(\tau)u d\tau \right\|_{\mathcal{B}}$$

$$\leq \int_0^t \|TS(\tau)u\|_{\mathcal{B}}\, d\tau = \int_0^t \|AS(\tau)u\|_{\mathcal{B}}\, d\tau = \int_0^t \left\|\bar{S}(\tau)Au\right\|_{\mathcal{B}} d\tau \leq t\,\|Au\|_{\mathcal{B}}\,.$$

Hence, for $t \in [0,1]$,

$$\|u\|_{\mathcal{B}} - \|S(t)u\|_{\mathcal{B}} \leq \|u\|_{\mathcal{B}} - r\,\|u\|_{\mathcal{B}} \leq \|Au\|_{\mathcal{B}}\,.$$

If we set $c = \frac{1}{1-r}$, we are done. $\square$

The following result is known (see Pazy [P]):

Theorem 19. *Let A be the generator of an analytic contraction semigroup $S(t)$ on $\mathcal{B}$. If $\sigma(A)$ is the spectrum of A and,*

$$\sigma = \sup\{\mathrm{Re}(\lambda) : \ \lambda \in \sigma(A)\} < 0.$$

Then there exists a constant $0 < r < 1$ such that

$$\inf\{\|S(t)\| : \ t \in [0,1]\} = r.$$

In the above theorem, it is easy to show that $t = 1$ will give a lower bound.

CONCLUSION

In this paper, we have shown that, given an unbounded closed densely defined linear operator on a separable Banach space, it is always possible to renorm the space with an equivalent norm such that the operator has an equivalent bound. We have applied this result to show that one can obtain a certain general reverse of the Poincaré inequality. We have also shown how to get a much stronger inequality for the nonlinear term in the Navier-Stokes equation than is normally possible.

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