NONLOCAL ROBIN LAPLACIANS AND SOME REMARKS ON A PAPER BY FILONOV

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Dedicated with great pleasure to Sergio Albeverio on the occasion of his 70th birthday

Abstract. The aim of this paper is twofold: First, we characterize an essentially optimal class of boundary operators Θ which give rise to self-adjoint Laplacians $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ with (nonlocal and local) Robin-type boundary conditions on bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$. Second, we extend Friedlander's inequalities between Neumann and Dirichlet Laplacian eigenvalues to those between nonlocal Robin and Dirichlet Laplacian eigenvalues associated with bounded Lipschitz domains Ω, following an approach introduced by Filonov for this type of problems.

1. Introduction

In recent years, there has been a flurry of activity in connection with 2nd-order elliptic partial differential operators, particularly, Schrödinger–type operators on open domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, with nonempty boundary $\partial\Omega$, under various smoothness assumptions (resp., lack thereof) on Ω, and associated nonlocal Robin boundary conditions. We refer, for instance, to [2], [3], [4], [8], [10], [11], [12], [13], [15], [23], [24], [26], [27], [33], [35], [38], [47], and the literature cited therein.

If Ω is minimally smooth, that is, a Lipschitz domain, these Robin-type boundary conditions are formally of the type

$$\frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} + \Theta(u|_{\partial\Omega}) = 0 \tag{1.1}$$

in appropriate Sobolev spaces on the boundary $\partial\Omega$, where ν denotes the outward pointing normal unit vector to $\partial\Omega$, and Θ is an appropriate self-adjoint operator in $L^2(\partial\Omega; d^{n-1}\omega)$, with $d^{n-1}\omega$ the surface measure on $\partial\Omega$. The boundary condition in (1.1) is called *local* and then resembles the familiar classical Robin boundary condition for smooth domains Ω, if Θ equals the operator of multiplication M_θ by an appropriate function θ on the boundary $\partial\Omega$ (cf., e.g., [45]). Otherwise, the boundary condition (1.1) represents a *generalized* or *nonlocal* Robin boundary condition generated by the operator Θ. The case $\Theta = 0$ (resp., $\theta = 0$), of course, corresponds to the case of Neumann boundary conditions on $\partial\Omega$. The case of Dirichlet boundary conditions on $\partial\Omega$, that is, the condition $u|_{\partial\Omega} = 0$ (formally corresponding to $\Theta = \infty$, resp., $\theta = \infty$) will also play a major role in this paper.

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Schrödinger operators on bounded Lipschitz domains Ω with nonlocal Robin boundary conditions of the form (1.1), have been very recently discussed in great detail in [23] and [24], and our treatment of nonlocal Robin Laplacians in this paper naturally builds upon these two papers.

In addition to presenting a detailed approach to nonlocal Robin Laplacians on bounded Lipschitz domains, we also present an application to eigenvalue inequalities between the associated Robin and Dirichlet Laplacian eigenvalues, extending Friedlander's eigenvalue inequalities between Neumann and Dirichlet eigenvalues for bounded C^1-domains [19], employing its extension to very general bounded domains due to Filonov [18]. We briefly review the relevant history of these eigenvalue inequalities. We denote by

$$0 = \lambda_{N,\Omega,1} < \lambda_{N,\Omega,2} \leq \cdots \leq \lambda_{N,\Omega,j} \leq \lambda_{N,\Omega,j+1} \leq \cdots \tag{1.2}$$

the eigenvalues for the Neumann Laplacian $-\Delta_{N,\Omega}$ in $L^2(\Omega; d^n x)$, listed according to their multiplicity. Similarly,

$$0 < \lambda_{D,\Omega,1} < \lambda_{D,\Omega,2} \leq \cdots \leq \lambda_{D,\Omega,j} \leq \lambda_{D,\Omega,j+1} \leq \cdots \tag{1.3}$$

denote the eigenvalues for the Dirichlet Laplacian $-\Delta_{D,\Omega}$ in $L^2(\Omega; d^n x)$, again enumerated according to their multiplicity.

Then, for any open bounded domain $\Omega \subset \mathbb{R}^n$, the variational formulation of the Neumann and Dirichlet eigenvalue problem (in terms of Rayleigh quotients, cf. [9, Sect. VI.1]) immediately implies the inequalities

$$\lambda_{N,\Omega,j} \leq \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}. \tag{1.4}$$

Moreover, Pólya [44] proved in 1952 that

$$\lambda_{N,\Omega,2} < \lambda_{D,\Omega,1}, \tag{1.5}$$

answering a question of Kornhauser and Stakgold [34]. For a two-dimensional bounded convex domain $\Omega \subset \mathbb{R}^2$, with a piecewise C^2-boundary $\partial\Omega$, Payne [43] demonstrated in 1955 that

$$\lambda_{N,\Omega,j+2} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}. \tag{1.6}$$

For domains Ω with a C^2-boundary and $\partial\Omega$ having a nonnegative mean curvature, Aviles [6] showed in 1986 that

$$\lambda_{N,\Omega,j+1} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}. \tag{1.7}$$

This was reproved by Levine and Weinberger [37] in 1986 who also showed that

$$\lambda_{N,\Omega,j+n} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}, \tag{1.8}$$

for smooth bounded convex domains Ω, as well as

$$\lambda_{N,\Omega,j+n} \leq \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}, \tag{1.9}$$

for arbitrary bounded convex domains. In addition, they also proved inequalities of the type $\lambda_{N,\Omega,j+m} < \lambda_{D,\Omega,j}$, $j \in \mathbb{N}$, for all $1 \leq m \leq n$ under appropriate assumptions on $\partial\Omega$ in [37] (see also [36]). For additional eigenvalue inequalities we refer to Friedlander [20], [21].

In 1991, and most relevant to our paper, Friedlander [19] proved that actually

$$\lambda_{N,\Omega,j+1} \leq \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}, \tag{1.10}$$

for any bounded domain Ω with a C^1-boundary $\partial\Omega$. We also refer to Mazzeo [41] for an extension to certain smooth manifolds, and to Ashbaugh and Levine [5] and

Hsu and Wang [30] for the case of subdomains of the n-sphere S^n with a smooth boundary and nonnegative mean curvature. (For intriguing connections between these eigenvalue inequalities with the null variety of the Fourier transform of the characteristic function of the domain Ω, we also refer to [7].) Finally, inequality (1.10) was extended to any open domain Ω with finite volume, and with the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega; d^n x)$ compact, by Filonov [18] in 2004, who also proved strict inequality in (1.10), that is,

$$\lambda_{N,\Omega,j+1} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}. \tag{1.11}$$

We emphasize that Filonov's conditions on Ω are optimal as they are equivalent to $-\Delta_{N,\Omega}$, defined as the unique self-adjoint operator associated with the Neumann sesquilinear form in $L^2(\Omega; d^n x)$,

$$a_N(u,v) = \int_\Omega d^n x\, \overline{(\nabla u)(x)} \cdot (\nabla v)(x), \quad u,v \in H^1(\Omega), \tag{1.12}$$

having a purely discrete spectrum, that is,

$$\sigma_{\rm ess}(-\Delta_{N,\Omega}) = \emptyset \tag{1.13}$$

(cf. also our discussion in Lemmas 2.1, 2.2), where $\sigma_{\rm ess}(\cdot)$ abbreviates the essential spectrum. While Friedlander used techniques based on the Dirichlet-to-Neumann map and an appropriate trial function argument, Filonov found an elementary new proof directly based on eigenvalue counting functions (and the same trial functions). Friedlander's result (1.10) was recently reconsidered by Arendt and Mazzeo [2], which in turn motivated our present investigation into an extension of Filonov's result (1.11) to nonlocal Robin Laplacians $-\Delta_{\Theta,\Omega}$. In fact, if

$$\lambda_{\Theta,\Omega,1} \le \lambda_{\Theta,\Omega,2} \le \cdots \le \lambda_{\Theta,\Omega,j} \le \lambda_{\Theta,\Omega,j+1} \le \cdots, \tag{1.14}$$

denote the eigenvalues of the nonlocal Robin Laplacian $-\Delta_{\Theta,\Omega}$, counting multiplicity, we will prove that

$$\lambda_{\Theta,\Omega,j+1} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}, \tag{1.15}$$

assuming appropriate hypotheses on Θ, including, for instance,

$$\Theta \le 0 \tag{1.16}$$

in the sense that $\langle f, \Theta f\rangle_{1/2} \le 0$ for every $f \in H^{1/2}(\partial\Omega)$. Here, $\langle \cdot\,, \cdot \rangle_{1/2}$ denotes the duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega) = \big(H^{1/2}(\partial\Omega)\big)^*$. Most recently, Filonov's result was generalized to the Heisenberg Laplacian on certain three-dimensional domains by Hansson [29].

Condition (1.16) was anticipated by Filonov in the special case of local Robin Laplacians $-\Delta_{M_\theta,\Omega}$, where M_θ equals the operator of multiplication by an appropriate real-valued function θ on the boundary $\partial\Omega$. The case of local Robin Laplacians $-\Delta_{M_\theta,\Omega}$ associated with $C^{2,\alpha}$-domains $\Omega \subset \mathbb{R}^n$, $\alpha \in (0,1]$, was discussed by Levine [36] in 1988. Assuming $(n-1)h(\xi) \ge \theta(\xi)$, $\xi \in \partial\Omega$, $h(\cdot)$ the mean curvature on $\partial\Omega$, he established

$$\lambda_{M_\theta,\Omega,j+1} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}. \tag{1.17}$$

He also proved

$$\lambda_{M_\theta,\Omega,j+n} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}, \tag{1.18}$$

under the additional assumption of convexity of Ω. (In addition, he derived inequalities of the type $\lambda_{\Theta,\Omega,j+m} < \lambda_{D,\Omega,j}$, $j \in \mathbb{N}$, for all $1 \le m \le n$, under appropriate conditions on Ω.) Similarly, in the case of local Robin Laplacians $-\Delta_{M_\theta,\Omega}$ on smooth domains $\Omega \subset S^n$ and $(n-1)h(\xi) \geq \theta(\xi)$, $\xi \in \partial\Omega$, Ashbaugh and Levine [5] proved $\lambda_{M_\theta,\Omega,j+1} \leq \lambda_{D,\Omega,j}$, $j \in \mathbb{N}$, in 1997.

We conclude this introduction with a brief description of the content of each section: Section 2 succinctly reviews the basic facts on sesquilinear forms and their associated self-adjoint operators. Sobolev spaces on bounded Lipschitz domains and on their boundaries are presented in a nutshell in Section 3. Section 4 focuses on self-adjoint realizations of Laplacians with nonlocal Robin boundary conditions, and finally, Section 5 discusses the extension of Friedlander's eigenvalue inequalities between Neumann and Dirichlet eigenvalues to that of nonlocal Robin eigenvalues and Dirichlet eigenvalues for bounded Lipschitz domains, closely following a strategy of proof due to Filonov.

2. Sesquilinear Forms and Associated Operators

In this section we describe a few basic facts on sesquilinear forms and linear operators associated with them. Let $\mathcal{H}$ be a complex separable Hilbert space with scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ (antilinear in the first and linear in the second argument), $\mathcal{V}$ a reflexive Banach space continuously and densely embedded into $\mathcal{H}$. Then also $\mathcal{H}$ embeds continuously and densely into $\mathcal{V}^*$. That is,

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*. \tag{2.1}$$

Here the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{V}^*$ is accomplished via the identification

$$\mathcal{H} \ni v \mapsto (\,\cdot\,, v)_{\mathcal{H}} \in \mathcal{V}^*, \tag{2.2}$$

and we use the convention in this manuscript that if X denotes a Banach space, X^* denotes the *adjoint space* of continuous conjugate linear functionals on X, also known as the *conjugate dual* of X.

In particular, if the sesquilinear form

$${}_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*} \colon \mathcal{V} \times \mathcal{V}^* \to \mathbb{C} \tag{2.3}$$

denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, then

$${}_{\mathcal{V}}\langle u, v\rangle_{\mathcal{V}^*} = (u, v)_{\mathcal{H}}, \quad u \in \mathcal{V},\ v \in \mathcal{H} \hookrightarrow \mathcal{V}^*, \tag{2.4}$$

that is, the $\mathcal{V}, \mathcal{V}^*$ pairing ${}_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*}$ is compatible with the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ in $\mathcal{H}$.

Let $T \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*)$. Since $\mathcal{V}$ is reflexive, $(\mathcal{V}^*)^* = \mathcal{V}$, one has

$$T \colon \mathcal{V} \to \mathcal{V}^*, \quad T^* \colon \mathcal{V} \to \mathcal{V}^* \tag{2.5}$$

and

$${}_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{(\mathcal{V}^*)^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v, T^*u\rangle_{\mathcal{V}^*}}. \tag{2.6}$$

Self-adjointness of T is then defined by $T = T^*$, that is,

$${}_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle Tu, v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v, Tu\rangle_{\mathcal{V}^*}}, \quad u, v \in \mathcal{V}, \tag{2.7}$$

nonnegativity of T is defined by

$${}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq 0, \quad u \in \mathcal{V}, \tag{2.8}$$

and *boundedness from below of T by $c_T \in \mathbb{R}$* is defined by

$${}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c_T \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{2.9}$$

(By (2.4), this is equivalent to ${}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c_T\, {}_{\mathcal{V}}\langle u, u\rangle_{\mathcal{V}^*}$, $u \in \mathcal{V}$.)

Next, let the sesquilinear form $a(\,\cdot\,,\,\cdot\,)\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ (antilinear in the first and linear in the second argument) be $\mathcal{V}$-*bounded*, that is, there exists a $c_a>0$ such that

$$|a(u,v)| \leqslant c_a\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \quad u,v\in\mathcal{V}. \tag{2.10}$$

Then $\widetilde{A}$ defined by

$$\widetilde{A}\colon \begin{cases} \mathcal{V}\to\mathcal{V}^*, \\ v\mapsto \widetilde{A}v = a(\,\cdot\,,v), \end{cases} \tag{2.11}$$

satisfies

$$\widetilde{A}\in\mathcal{B}(\mathcal{V},\mathcal{V}^*) \ \text{ and } \ {}_{\mathcal{V}}\big\langle u,\widetilde{A}v\big\rangle_{\mathcal{V}^*} = a(u,v), \quad u,v\in\mathcal{V}. \tag{2.12}$$

Assuming further that $a(\,\cdot\,,\,\cdot\,)$ is *symmetric*, that is,

$$a(u,v)=\overline{a(v,u)}, \quad u,v\in\mathcal{V}, \tag{2.13}$$

and that a is $\mathcal{V}$-*coercive*, that is, there exists a constant $C_0>0$ such that

$$a(u,u)\geq C_0\|u\|_{\mathcal{V}}^2, \quad u\in\mathcal{V}, \tag{2.14}$$

respectively, then,

$$\widetilde{A}\colon \mathcal{V}\to\mathcal{V}^* \ \text{ is bounded, self-adjoint, and boundedly invertible.} \tag{2.15}$$

Moreover, denoting by A the part of $\widetilde{A}$ in $\mathcal{H}$ defined by

$$\operatorname{dom}(A)=\big\{u\in\mathcal{V}\,\big|\,\widetilde{A}u\in\mathcal{H}\big\}\subseteq\mathcal{H}, \quad A=\widetilde{A}\big|_{\operatorname{dom}(A)}\colon \operatorname{dom}(A)\to\mathcal{H}, \tag{2.16}$$

then A is a (possibly unbounded) self-adjoint operator in $\mathcal{H}$ satisfying

$$A\geq C_0 I_{\mathcal{H}}, \tag{2.17}$$

$$\operatorname{dom}\big(A^{1/2}\big)=\mathcal{V}. \tag{2.18}$$

In particular,

$$A^{-1}\in\mathcal{B}(\mathcal{H}). \tag{2.19}$$

The facts (2.1)–(2.19) are a consequence of the Lax–Milgram theorem and the second representation theorem for symmetric sesquilinear forms. Details can be found, for instance, in [14, §VI.3, §VII.1], [16, Ch. IV], and [39].

Next, consider a symmetric form $b(\,\cdot\,,\,\cdot\,)\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ and assume that b is *bounded from below by* $c_b\in\mathbb{R}$, that is,

$$b(u,u)\geq c_b\|u\|_{\mathcal{H}}^2, \quad u\in\mathcal{V}. \tag{2.20}$$

Introducing the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{V}_b}\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ (and the associated norm $\|\cdot\|_{\mathcal{V}_b}$) by

$$(u,v)_{\mathcal{V}_b}=b(u,v)+(1-c_b)(u,v)_{\mathcal{H}}, \quad u,v\in\mathcal{V}, \tag{2.21}$$

turns $\mathcal{V}$ into a pre-Hilbert space $(\mathcal{V};(\,\cdot\,,\,\cdot\,)_{\mathcal{V}_b})$, which we denote by $\mathcal{V}_b$. The form b is called *closed* in $\mathcal{H}$ if $\mathcal{V}_b$ is actually complete, and hence a Hilbert space. The form b is called *closable* in $\mathcal{H}$ if it has a closed extension. If b is closed in $\mathcal{H}$, then

$$|b(u,v)+(1-c_b)(u,v)_{\mathcal{H}}| \leqslant \|u\|_{\mathcal{V}_b}\|v\|_{\mathcal{V}_b}, \quad u,v\in\mathcal{V}, \tag{2.22}$$

and

$$|b(u,u)+(1-c_b)\|u\|_{\mathcal{H}}^2| = \|u\|_{\mathcal{V}_b}^2, \quad u\in\mathcal{V}, \tag{2.23}$$

show that the form $b(\,\cdot\,,\,\cdot\,)+(1-c_b)(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ is a symmetric, $\mathcal{V}$-bounded, and $\mathcal{V}$-coercive sesquilinear form. Hence, by (2.11) and (2.12), there exists a linear map

$$\widetilde{B}_{c_b}\colon \begin{cases} \mathcal{V}_b \to \mathcal{V}_b^*, \\ \qquad v \mapsto \widetilde{B}_{c_b} v = b(\,\cdot\,,v)+(1-c_b)(\,\cdot\,,v)_{\mathcal{H}}, \end{cases} \tag{2.24}$$

with

$$\widetilde{B}_{c_b} \in \mathcal{B}(\mathcal{V}_b,\mathcal{V}_b^*) \text{ and } {}_{\mathcal{V}_b}\big\langle u, \widetilde{B}_{c_b} v\big\rangle_{\mathcal{V}_b^*} = b(u,v)+(1-c_b)(u,v)_{\mathcal{H}}, \quad u,v\in\mathcal{V}. \tag{2.25}$$

Introducing the linear map

$$\widetilde{B} = \widetilde{B}_{c_b} + (c_b-1)\widetilde{I}\colon \mathcal{V}_b \to \mathcal{V}_b^*, \tag{2.26}$$

where $\widetilde{I}\colon \mathcal{V}_b \hookrightarrow \mathcal{V}_b^*$ denotes the continuous inclusion (embedding) map of $\mathcal{V}_b$ into $\mathcal{V}_b^*$, one obtains a self-adjoint operator B in $\mathcal{H}$ by restricting $\widetilde{B}$ to $\mathcal{H}$,

$$\operatorname{dom}(B) = \big\{u\in\mathcal{V}\,\big|\,\widetilde{B}u\in\mathcal{H}\big\} \subseteq \mathcal{H}, \quad B = \widetilde{B}\big|_{\operatorname{dom}(B)}\colon \operatorname{dom}(B)\to\mathcal{H}, \tag{2.27}$$

satisfying the following properties:

$$B \geq c_b I_{\mathcal{H}}, \tag{2.28}$$
$$\operatorname{dom}\big(|B|^{1/2}\big) = \operatorname{dom}\big((B-c_bI_{\mathcal{H}})^{1/2}\big) = \mathcal{V}, \tag{2.29}$$
$$\begin{aligned} b(u,v) &= \big(|B|^{1/2}u, U_B|B|^{1/2}v\big)_{\mathcal{H}} && (2.30)\\ &= \big((B-c_bI_{\mathcal{H}})^{1/2}u,(B-c_bI_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + c_b(u,v)_{\mathcal{H}} && (2.31)\\ &= {}_{\mathcal{V}_b}\big\langle u, \widetilde{B}v\big\rangle_{\mathcal{V}_b^*}, \quad u,v\in\mathcal{V}, && (2.32) \end{aligned}$$
$$b(u,v) = (u,Bv)_{\mathcal{H}}, \quad u\in\mathcal{V},\ v\in\operatorname{dom}(B), \tag{2.33}$$
$$\begin{aligned} \operatorname{dom}(B) = \{v\in\mathcal{V}\,|\,&\text{there exists an } f_v\in\mathcal{H} \text{ such that} \\ &b(w,v)=(w,f_v)_{\mathcal{H}} \text{ for all } w\in\mathcal{V}\}, \end{aligned} \tag{2.34}$$
$$\begin{aligned} &Bu = f_u, \quad u\in\operatorname{dom}(B), \\ &\operatorname{dom}(B) \text{ is dense in } \mathcal{H} \text{ and in } \mathcal{V}_b. \end{aligned} \tag{2.35}$$

Properties (2.34) and (2.35) uniquely determine B. Here U_B in (2.31) is the partial isometry in the polar decomposition of B, that is,

$$B = U_B|B|, \quad |B| = (B^*B)^{1/2} \geq 0. \tag{2.36}$$

The operator B is called the *operator associated with the form b.*

The norm in the Hilbert space $\mathcal{V}_b^*$ is given by

$$\|\ell\|_{\mathcal{V}_b^*} = \sup\{|{}_{\mathcal{V}_b}\langle u,\ell\rangle_{\mathcal{V}_b^*}|\,|\,\|u\|_{\mathcal{V}_b}\leqslant 1\}, \quad \ell\in\mathcal{V}_b^*, \tag{2.37}$$

with associated scalar product,

$$(\ell_1,\ell_2)_{\mathcal{V}_b^*} = {}_{\mathcal{V}_b}\big\langle \big(\widetilde{B}+(1-c_b)\widetilde{I}\big)^{-1}\ell_1,\ell_2\big\rangle_{\mathcal{V}_b^*}, \quad \ell_1,\ell_2\in\mathcal{V}_b^*. \tag{2.38}$$

Since

$$\big\|\big(\widetilde{B}+(1-c_b)\widetilde{I}\big)v\big\|_{\mathcal{V}_b^*} = \|v\|_{\mathcal{V}_b}, \quad v\in\mathcal{V}, \tag{2.39}$$

the Riesz representation theorem yields

$$\big(\widetilde{B}+(1-c_b)\widetilde{I}\big)\in\mathcal{B}(\mathcal{V}_b,\mathcal{V}_b^*) \text{ and } \big(\widetilde{B}+(1-c_b)\widetilde{I}\big)\colon\mathcal{V}_b\to\mathcal{V}_b^* \text{ is unitary.} \tag{2.40}$$

In addition,

$$\begin{aligned}{}_{\mathcal{V}_b}\big\langle u, \big(\widetilde{B} + (1-c_b)\widetilde{I}\,\big)v\big\rangle_{\mathcal{V}_b^*} &= \big(\big(B + (1-c_b)I_{\mathcal{H}}\big)^{1/2}u, \big(B + (1-c_b)I_{\mathcal{H}}\big)^{1/2}v\big)_{\mathcal{H}} \\ &= (u,v)_{\mathcal{V}_b}, \quad u,v \in \mathcal{V}_b.\end{aligned} \tag{2.41}$$

In particular,

$$\big\|(B + (1-c_b)I_{\mathcal{H}})^{1/2}u\big\|_{\mathcal{H}} = \|u\|_{\mathcal{V}_b}, \quad u \in \mathcal{V}_b, \tag{2.42}$$

and hence

$$(B+(1-c_b)I_{\mathcal{H}})^{1/2} \in \mathcal{B}(\mathcal{V}_b, \mathcal{H}) \text{ and } (B+(1-c_b)I_{\mathcal{H}})^{1/2} \colon \mathcal{V}_b \to \mathcal{H} \text{ is unitary.} \tag{2.43}$$

The facts (2.20)–(2.43) comprise the second representation theorem of sesquilinear forms (cf. [16, Sect. IV.2], [17, Sects. 1.2–1.5], and [32, Sect. VI.2.6]).

A special but important case of nonnegative closed forms is obtained as follows: Let $\mathcal{H}_j$, $j = 1,2$, be complex separable Hilbert spaces, and $T \colon \mathrm{dom}(T) \to \mathcal{H}_2$, $\mathrm{dom}(T) \subseteq \mathcal{H}_1$, a densely defined operator. Consider the nonnegative form $a_T \colon \mathrm{dom}(T) \times \mathrm{dom}(T) \to \mathbb{C}$ defined by

$$a_T(u,v) = (Tu, Tv)_{\mathcal{H}_2}, \quad u,v \in \mathrm{dom}(T). \tag{2.44}$$

Then the form a_T is closed (resp., closable) in $\mathcal{H}_1$ if and only if T is. If T is closed, the unique nonnegative self-adjoint operator associated with a_T in $\mathcal{H}_1$, whose existence is guaranteed by the second representation theorem for forms, then equals $T^*T \geq 0$. In particular, one obtains in addition to (2.44),

$$a_T(u,v) = (|T|u, |T|v)_{\mathcal{H}_1}, \quad u,v \in \mathrm{dom}(T) = \mathrm{dom}(|T|). \tag{2.45}$$

Moreover, since

$$\begin{aligned} b(u,v) + (1-c_b)(u,v)_{\mathcal{H}} &= \big((B + (1-c_b)I_{\mathcal{H}})^{1/2}u, (B + (1-c_b)I_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}}, \\ &\qquad u,v \in \mathrm{dom}(b) = \mathrm{dom}\big(|B|^{1/2}\big) = \mathcal{V},\end{aligned} \tag{2.46}$$

and $(B + (1-c_b)I_{\mathcal{H}})^{1/2}$ is self-adjoint (and hence closed) in $\mathcal{H}$, a symmetric, $\mathcal{V}$-bounded, and $\mathcal{V}$-coercive form is densely defined in $\mathcal{H} \times \mathcal{H}$ and closed in $\mathcal{H}$ (a fact we will be using in the proof of Theorem 4.5). We refer to [32, Sect. VI.2.4] and [48, Sect. 5.5] for details.

Next we recall that if a_j are sesquilinear forms defined on $\mathrm{dom}(a_j)$, $j = 1,2$, bounded from below and closed, then also

$$(a_1 + a_2) \colon \begin{cases} (\mathrm{dom}(a_1) \cap \mathrm{dom}(a_2)) \times (\mathrm{dom}(a_1) \cap \mathrm{dom}(a_2)) \to \mathbb{C}, \\ (u,v) \mapsto (a_1 + a_2)(u,v) = a_1(u,v) + a_2(u,v) \end{cases} \tag{2.47}$$

is bounded from below and closed (cf., e.g., [32, Sect. VI.1.6]).

Finally, we also recall the following perturbation theoretic fact: Suppose a is a sesquilinear form defined on $\mathcal{V} \times \mathcal{V}$, bounded from below and closed, and let b be a symmetric sesquilinear form bounded with respect to a with bound less than one, that is, $\mathrm{dom}(b) \supseteq \mathcal{V} \times \mathcal{V}$, and that there exist $0 \leqslant \alpha < 1$ and $\beta \geqslant 0$ such that

$$|b(u,u)| \leqslant \alpha |a(u,u)| + \beta \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{2.48}$$

Then

$$(a + b) \colon \begin{cases} \mathcal{V} \times \mathcal{V} \to \mathbb{C}, \\ (u,v) \mapsto (a+b)(u,v) = a(u,v) + b(u,v) \end{cases} \tag{2.49}$$

defines a sesquilinear form that is bounded from below and closed (cf., e.g., [32, Sect. VI.1.6]). In the special case where α can be chosen arbitrarily small, the form b is called infinitesimally form bounded with respect to a.

Finally we turn to a brief discussion of operators with purely discrete spectra. We denote by $\#S$ the cardinality of the set S.

Lemma 2.1. *Let $\mathcal{V}$, $\mathcal{H}$ be as in* (2.1), (2.2). *Assume that the inclusion $\iota_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathcal{H}$ is compact, and that the sesquilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is symmetric, $\mathcal{V}$-bounded, and suppose that there exists $\kappa > 0$ with the property that*

$$a_\kappa(u,v) := a(u,v) + \kappa\,(u,v)_{\mathcal{H}}, \quad u,v \in \mathcal{V}, \tag{2.50}$$

is $\mathcal{V}$-coercive. Then the operator A associated with $a(\cdot,\cdot)$ is self-adjoint and bounded from below. In addition, A has purely discrete spectrum

$$\sigma_{\rm ess}(A) = \emptyset, \tag{2.51}$$

and hence $\sigma(A) = \{\lambda_j(A)\}_{j\in\mathbb{N}}$, with $\lambda_j(A) \to \infty$ as $j \to \infty$,

$$-\kappa < \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_j(A) \leq \lambda_{j+1}(A) \leq \cdots . \tag{2.52}$$

Here, the eigenvalues $\lambda_j(A)$ of A are listed according to their multiplicity. Moreover, the following min-max principle holds:

$$\lambda_j(A) = \min_{\substack{L_j \textit{ subspace of } \mathcal{V} \\ \dim(L_j)=j}} \Big(\max_{0\neq u \in L_j} R_a[u] \Big), \quad j \in \mathbb{N}, \tag{2.53}$$

where $R_a[u]$ denotes the Rayleigh quotient

$$R_a[u] := \frac{a(u,u)}{\|u\|_{\mathcal{H}}^2}, \quad 0 \neq u \in \mathcal{V}. \tag{2.54}$$

As a consequence, if N_A is the eigenvalue counting function of A, that is,

$$N_A(\lambda) := \#\{j \in \mathbb{N} \,|\, \lambda_j(A) \leq \lambda\}, \quad \lambda \in \mathbb{R}, \tag{2.55}$$

then for each $\lambda \in \mathbb{R}$ one has

$$N_A(\lambda) = \max\{\dim(L) \in \mathbb{N}_0 \,|\, L \textit{ a subspace of } \mathcal{V} \textit{ with } a(u,u) \leq \lambda\|u\|_{\mathcal{H}}^2,\ u \in L\}. \tag{2.56}$$

Proof. Analogous claims for the operator B associated with the $\mathcal{V}$-coercive form $a_\kappa(\cdot,\cdot)$ are well-known (cf., e.g., [14, Sect. VI.3.2.5, Ch. VII]). Then the corresponding claims for A follow from these, after observing that $B = A + \kappa I_{\mathcal{H}}$, $R_{a_\kappa}[u] = R_a[u] + \kappa$, $\lambda_j(B) = \lambda_j(A) + \kappa$, and $N_B(\lambda) = N_A(\lambda - \kappa)$, $\lambda \in \mathbb{R}$. □

A closely related result is provided by the following elementary observations: Let $c \in \mathbb{R}$ and $B \geq cI_{\mathcal{H}}$ be a self-adjoint operator in $\mathcal{H}$, and introduce the sesquilinear form b in $\mathcal{H}$ associated with B via

$$\begin{aligned} &b(u,v) = \big((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + c(u,v)_{\mathcal{H}}, \\ &u,v \in \operatorname{dom}(b) = \operatorname{dom}\big(|B|^{1/2}\big). \end{aligned} \tag{2.57}$$

Given B and b, one introduces the Hilbert space $\mathcal{H}_b \subseteq \mathcal{H}$ by

$$\begin{aligned} &\mathcal{H}_b = \big(\operatorname{dom}\big(|B|^{1/2}\big), (\cdot,\cdot)_{\mathcal{H}_b}\big), \\ &(u,v)_{\mathcal{H}_b} = b(u,v) + (1-c)(u,v)_{\mathcal{H}} \\ &\qquad\quad\; = \big((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + (u,v)_{\mathcal{H}} \end{aligned} \tag{2.58}$$

$$= \big((B + (1 - c)I_{\mathcal{H}})^{1/2}u, (B + (1 - c)I_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}}.$$

Of course, $\mathcal{H}_b$ plays a role analogous to $\mathcal{V}_b$ in (2.21). As in (2.43) one then observes that

$$(B + (1 - c_b)I_{\mathcal{H}})^{1/2} \colon \mathcal{H}_b \to \mathcal{H} \text{ is unitary.} \tag{2.59}$$

Lemma 2.2. *Let $\mathcal{H}$, B, b, and $\mathcal{H}_b$ be as in* (2.57)–(2.59). *Then B has purely discrete spectrum, that is, $\sigma_{\rm ess}(B) = \emptyset$, if and only if $\mathcal{H}_b \hookrightarrow \mathcal{H}$ compactly.*

Proof. Denoting by $J_{\mathcal{H}_b} = I_{\mathcal{H}}|_{\mathcal{H}_b}$ the inclusion map from $\mathcal{H}_b$ into $\mathcal{H}$, one infers that

$$\mathcal{H} \xrightarrow{(B+(1-c)I_{\mathcal{H}})^{-1/2}} \mathcal{H}_b \overset{J_{\mathcal{H}_b}}{\hookrightarrow} \mathcal{H}. \tag{2.60}$$

Thus, one concludes that

$$\begin{aligned}
&\mathcal{H}_b \hookrightarrow \mathcal{H} \text{ compactly} \iff J_{\mathcal{H}_b} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{H}) \\
&\quad\iff \big[J_{\mathcal{H}_b}(B + (1 - c)I_{\mathcal{H}})^{-1/2}\big](B + (1 - c)I_{\mathcal{H}})^{1/2} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{H}) \\
&\quad\iff J_{\mathcal{H}_b}(B + (1 - c)I_{\mathcal{H}})^{-1/2} \in \mathcal{B}_\infty(\mathcal{H}) \iff (B + (1 - c)I_{\mathcal{H}})^{-1/2} \in \mathcal{B}_\infty(\mathcal{H}) \\
&\quad\iff (B - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_\infty(\mathcal{H}), \quad z \in \mathbb{C}\backslash\sigma(B) \\
&\quad\iff \sigma_{\rm ess}(B) = \emptyset,
\end{aligned} \tag{2.61}$$

since $(B + (1 - c_b)I_{\mathcal{H}})^{1/2} \colon \mathcal{H}_b \to \mathcal{H}$ is unitary by (2.59). □

3. Sobolev Spaces in Lipschitz Domains

The goal of this section is to introduce the relevant material pertaining to Sobolev spaces in subdomains of $\mathbb{R}^n$ and discuss various trace results.

We start by recalling some basic facts in connection with Sobolev spaces corresponding to open subsets $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. For an arbitrary $m \in \mathbb{N} \cup \{0\}$, we follow the customary way of defining L^2-Sobolev spaces of order $\pm m$ in Ω as

$$H^m(\Omega) := \big\{u \in L^2(\Omega; d^n x) \,\big|\, \partial^\alpha u \in L^2(\Omega; d^n x) \text{ for } 0 \leq |\alpha| \leq m\big\}, \tag{3.1}$$

$$H^{-m}(\Omega) := \bigg\{u \in \mathcal{D}'(\Omega) \,\bigg|\, u = \sum_{|\alpha| \leq m} \partial^\alpha u_\alpha, \text{ with } u_\alpha \in L^2(\Omega; d^n x),\, 0 \leq |\alpha| \leq m\bigg\}, \tag{3.2}$$

equipped with natural norms (cf., e.g., [1, Ch. 3], [40, Ch. 1]). Here $\mathcal{D}'(\Omega)$ denotes the usual set of distributions on $\Omega \subseteq \mathbb{R}^n$. Then one sets

$$H^m_0(\Omega) := \text{ the closure of } C^\infty_0(\Omega) \text{ in } H^m(\Omega), \quad m \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

As is well-known, all three spaces above are Banach, reflexive and, in addition,

$$\big(H^m_0(\Omega)\big)^* = H^{-m}(\Omega). \tag{3.4}$$

Again, see, for instance, [1, Ch. 3], [40, Sect. 1.1.14]. Throughout this paper, we agree to use the *adjoint* (rather than the dual) space X^* of a Banach space X.

One recalls that an open, nonempty, bounded set $\Omega \subset \mathbb{R}^n$ is called a *Lipschitz domain* if the following property holds: There exists an open covering $\{\mathcal{O}_j\}_{1 \leq j \leq N}$ of the boundary $\partial\Omega$ of Ω such that for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (considered in a new system of coordinates obtained from the original one via a rigid motion). The number $\max\{\nabla\varphi_j\|_{L^\infty(\mathbb{R}^{n-1}; d^{n-1}x')} \,|\, 1 \leq j \leq N\}$ is said to represent the *Lipschitz character* of Ω.

The classical theorem of Rademacher of almost everywhere differentiability of Lipschitz functions ensures that, for any Lipschitz domain Ω, the surface measure $d^{n-1}\omega$ is well-defined on $\partial\Omega$ and that there exists an outward pointing normal vector ν at almost every point of $\partial\Omega$.

In the remainder of this paper we shall make the following assumption:

Hypothesis 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded, nonempty Lipschitz domain.*

As regards L^2-based Sobolev spaces of fractional order $s \in \mathbb{R}$, in a bounded *Lipschitz domain* $\Omega \subset \mathbb{R}^n$ we set

$$H^s(\mathbb{R}^n) := \bigg\{U \in \mathcal{S}'(\mathbb{R}^n) \,\bigg|\, \|U\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} d^n\xi \, \big|\widehat{U}(\xi)\big|^2 \big(1 + |\xi|^{2s}\big) < \infty\bigg\}, \quad (3.5)$$

$$H^s(\Omega) := \big\{u \in \mathcal{D}'(\Omega) \,\big|\, u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\big\}. \quad (3.6)$$

Here $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$, and $\widehat{U}$ denotes the Fourier transform of $U \in \mathcal{S}'(\mathbb{R}^n)$. These definitions are consistent with (3.1)–(3.2). Moreover, so is

$$H_0^s(\Omega) := \big\{u \in H^s(\mathbb{R}^n) \,\big|\, \operatorname{supp}(u) \subseteq \overline{\Omega}\big\}, \quad s \in \mathbb{R}, \quad (3.7)$$

equipped with the natural norm induced by $H^s(\mathbb{R}^n)$, in relation to (3.3). One also has

$$\big(H_0^s(\Omega)\big)^* = H^{-s}(\Omega), \quad s \in \mathbb{R} \quad (3.8)$$

(cf., e.g., [31]). For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ it is known that

$$\big(H^s(\Omega)\big)^* = H^{-s}(\Omega), \quad -1/2 < s < 1/2. \quad (3.9)$$

See [46] for this and other related properties.

To discuss Sobolev spaces on the boundary of a Lipschitz domain, consider first the case when $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of a Lipschitz function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$. In this setting, we define the Sobolev space $H^s(\partial\Omega)$ for $0 \leq s \leq 1$, as the space of functions $f \in L^2(\partial\Omega; d^{n-1}\omega)$ with the property that $f(x', \varphi(x'))$, as a function of $x' \in \mathbb{R}^{n-1}$, belongs to $H^s(\mathbb{R}^{n-1})$. This definition is easily adapted to the case when Ω is a Lipschitz domain whose boundary is compact, by using a smooth partition of unity. Finally, for $-1 \leq s \leq 0$, we set

$$H^s(\partial\Omega) = \big(H^{-s}(\partial\Omega)\big)^*, \quad -1 \leqslant s \leqslant 0. \quad (3.10)$$

From the above characterization of $H^s(\partial\Omega)$ it follows that any property of Sobolev spaces (of order $s \in [-1, 1]$) defined in Euclidean domains, which are invariant under multiplication by smooth, compactly supported functions as well as compositions by bi-Lipschitz diffeomorphisms, readily extends to the setting of $H^s(\partial\Omega)$ (via localization and pull-back). For additional background information in this context we refer, for instance, to [16, Chs. V, VI], [28, Ch. 1].

Assuming Hypothesis 3.1, we introduce the boundary trace operator γ_D^0 (the Dirichlet trace) by

$$\gamma_D^0\colon C(\overline{\Omega}) \to C(\partial\Omega), \quad \gamma_D^0 u = u|_{\partial\Omega}. \quad (3.11)$$

Then there exists a bounded linear operator γ_D

$$\begin{aligned} &\gamma_D\colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \\ &\gamma_D\colon H^{3/2}(\Omega) \to H^{1-\varepsilon}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad \varepsilon \in (0,1) \end{aligned} \quad (3.12)$$

(cf., e.g., [42, Theorem 3.38]), whose action is compatible with that of γ_D^0. That is, the two Dirichlet trace operators coincide on the intersection of their domains. Moreover, we recall that

$$\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \text{ is onto for } 1/2 < s < 3/2. \tag{3.13}$$

Next, retaining Hypothesis 3.1, we introduce the operator γ_N (the strong Neumann trace) by

$$\gamma_N = \nu \cdot \gamma_D \nabla \colon H^{s+1}(\Omega) \to L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \tag{3.14}$$

where ν denotes the outward pointing normal unit vector to $\partial\Omega$. It follows from (3.12) that γ_N is also a bounded operator. We seek to extend the action of the Neumann trace operator (3.14) to other (related) settings. To set the stage, assume Hypothesis 3.1 and observe that the inclusion

$$\iota : H^{s_0}(\Omega) \hookrightarrow \big(H^r(\Omega)\big)^*, \quad s_0 > -1/2, \ r > 1/2, \tag{3.15}$$

is well-defined and bounded. We then introduce the weak Neumann trace operator

$$\widetilde{\gamma}_N \colon \big\{u \in H^{s+1/2}(\Omega) \,\big|\, \Delta u \in H^{s_0}(\Omega)\big\} \to H^{s-1}(\partial\Omega), \quad s \in (0,1), \ s_0 > -1/2, \tag{3.16}$$

as follows: Given $u \in H^{s+1/2}(\Omega)$ with $\Delta u \in H^{s_0}(\Omega)$ for some $s \in (0,1)$ and $s_0 > -1/2$, we set (with ι as in (3.15) for $r := 3/2 - s > 1/2$)

$$\langle \phi, \widetilde{\gamma}_N u\rangle_{1-s} = {}_{H^{1/2-s}(\Omega)}\langle \nabla\Phi, \nabla u\rangle_{(H^{1/2-s}(\Omega))^*} + {}_{H^{3/2-s}(\Omega)}\langle \Phi, \iota(\Delta u)\rangle_{(H^{3/2-s}(\Omega))^*}, \tag{3.17}$$

for all $\phi \in H^{1-s}(\partial\Omega)$ and $\Phi \in H^{3/2-s}(\Omega)$ such that $\gamma_D\Phi = \phi$. We note that the first pairing on the right-hand side of (3.17) is meaningful since

$$\big(H^{1/2-s}(\Omega)\big)^* = H^{s-1/2}(\Omega), \quad s \in (0,1), \tag{3.18}$$

and that the definition (3.17) is independent of the particular extension Φ of ϕ, and that $\widetilde{\gamma}_N$ is a bounded extension of the Neumann trace operator γ_N defined in (3.14).

4. Laplace Operators with Nonlocal Robin-Type Boundary Conditions

In this section we primarily focus on various properties of general Laplacians $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ including Dirichlet, $-\Delta_{D,\Omega}$, and Neumann, $-\Delta_{N,\Omega}$, Laplacians, nonlocal Robin-type Laplacians, and Laplacians corresponding to classical Robin boundary conditions associated with bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$.

For simplicity of notation we will denote the identity operators in $L^2(\Omega; d^n x)$ and $L^2(\partial\Omega; d^{n-1}\omega)$ by I_Ω and $I_{\partial\Omega}$, respectively. Also, in the sequel, the sesquilinear form

$$\langle \,\cdot\,, \,\cdot\, \rangle_s = {}_{H^s(\partial\Omega)}\langle \,\cdot\,, \,\cdot\, \rangle_{H^{-s}(\partial\Omega)} \colon H^s(\partial\Omega) \times H^{-s}(\partial\Omega) \to \mathbb{C}, \quad s \in [0,1], \tag{4.1}$$

(antilinear in the first, linear in the second factor), will denote the duality pairing between $H^s(\partial\Omega)$ and

$$H^{-s}(\partial\Omega) = \big(H^s(\partial\Omega)\big)^*, \quad s \in [0,1], \tag{4.2}$$

such that

$$\begin{aligned} &\langle f, g\rangle_s = \int_{\partial\Omega} d^{n-1}\omega(\xi)\, \overline{f(\xi)} g(\xi), \\ &f \in H^s(\partial\Omega), \ g \in L^2(\partial\Omega; d^{n-1}\omega) \hookrightarrow H^{-s}(\partial\Omega), \ s \in [0,1], \end{aligned} \tag{4.3}$$

where, as before, $d^{n-1}\omega$ stands for the surface measure on $\partial\Omega$.

Hypothesis 4.1. *Assume Hypothesis* 3.1*, suppose that $\delta > 0$ is a given number, and assume that $\Theta \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ is a self-adjoint operator which can be written as*

$$\Theta = \Theta_1 + \Theta_2 + \Theta_3, \tag{4.4}$$

where Θ_j, $j = 1, 2, 3$, have the following properties: There exists a closed sesquilinear form a_{Θ_0} in $L^2(\partial\Omega; d^{n-1}\omega)$, with domain $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, bounded from below by $c_{\Theta_0} \in \mathbb{R}$ (hence, a_{Θ_0} is symmetric) such that if $\Theta_0 \geqslant c_\Theta I_{\partial\Omega}$ denotes the self-adjoint operator in $L^2(\partial\Omega; d^{n-1}\omega)$ uniquely associated with a_{Θ_0} (cf. (2.27)*), then $\Theta_1 = \widetilde{\Theta}_0$, the extension of Θ_0 to an operator in $\mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ (as discussed in* (2.26) *and* (2.32)*). In addition,*

$$\Theta_2 \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big), \tag{4.5}$$

whereas $\Theta_3 \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ satisfies

$$\|\Theta_3\|_{\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} < \delta. \tag{4.6}$$

We recall the following useful result.

Lemma 4.2. *Assume Hypothesis* 3.1*. Then for every $\varepsilon > 0$ there exists a $\beta(\varepsilon) > 0$ (with $\beta(\varepsilon) \underset{\varepsilon\downarrow 0}{=} O(1/\varepsilon)$) such that*

$$\|\gamma_D u\|^2_{L^2(\partial\Omega; d^{n-1}\omega)} \leqslant \varepsilon \|\nabla u\|^2_{L^2(\Omega; d^n x)^n} + \beta(\varepsilon)\|u\|^2_{L^2(\Omega; d^n x)}, \quad u \in H^1(\Omega). \tag{4.7}$$

A proof from which it is possible to read off how the constant $\beta(\varepsilon)$ depends on the Lipschitz character of Ω appears in [24]. Below we discuss a general abstract scheme which yields results of this type, albeit with a less descriptive constant $\beta(\varepsilon)$. The lemma below is inspired by a result in [2].

Lemma 4.3. *Let $\mathcal{V}$ be a reflexive Banach space, $\mathcal{W}$ a Banach space, assume that $K \in \mathcal{B}_\infty(\mathcal{V}, \mathcal{V}^*)$, and that $T \in \mathcal{B}(\mathcal{V}, \mathcal{W})$ is one-to-one. Then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\big|{}_{\mathcal{V}}\langle u, Ku\rangle_{\mathcal{V}^*}\big| \leq \varepsilon\|u\|^2_{\mathcal{V}} + C_\varepsilon\|Tu\|^2_{\mathcal{W}}, \quad u \in \mathcal{V}. \tag{4.8}$$

Proof. Seeking a contradiction, assume that there exist $\varepsilon > 0$ along with a family of vectors $u_j \in \mathcal{V}$, $\|u_j\|_{\mathcal{V}} = 1$, $j \in \mathbb{N}$, for which

$$\big|{}_{\mathcal{V}}\langle u, Ku\rangle_{\mathcal{V}^*}\big| \geq \varepsilon + j\|Tu\|^2_{\mathcal{W}}, \quad j \in \mathbb{N}. \tag{4.9}$$

Furthermore, since $\mathcal{V}$ is reflexive, there is no loss of generality in assuming that there exists $u \in \mathcal{V}$ such that $u_j \to u$ as $j \to \infty$, in the weak-$*$ topology induced by $\mathcal{V}^*$ on $\mathcal{V}$. In particular, $Tu_j \to Tu$ as $j \to \infty$, in the weak-$*$ topology induced by $\mathcal{W}^*$ on $\mathcal{W}$. In addition, since K is compact, matters can also be arranged so that $Ku_j \to Ku$ as $j \to \infty$, in $\mathcal{V}^*$. This, in turn, yields that

$${}_{\mathcal{V}}\langle u_j, Ku_j\rangle_{\mathcal{V}^*} \to {}_{\mathcal{V}}\langle u, Ku\rangle_{\mathcal{V}^*} \text{ as } j \to \infty. \tag{4.10}$$

Together with

$$\|Tu_j\|^2_{\mathcal{W}} \leq j^{-1}\big|{}_{\mathcal{V}}\langle u_j, Ku_j\rangle_{\mathcal{V}^*}\big| + j^{-1}\varepsilon, \quad j \in \mathbb{N}, \tag{4.11}$$

this also shows that $Tu_j \to 0$ as $j \to \infty$, in $\mathcal{W}$. Hence, $Tu = 0$ in $\mathcal{W}$ which forces $u = 0$, since T is one-to-one. Given these facts, we note that, on the one hand, we have ${}_{\mathcal{V}}\langle u_j, Ku_j\rangle_{\mathcal{V}^*} \to 0$ as $j \to \infty$ by (4.10), while on the other hand

$\Big|{}_{\mathcal{V}}\langle u_j, Ku_j\rangle_{\mathcal{V}^*}\Big| \geq \varepsilon$ for every $j \in \mathbb{N}$ by (4.9). This contradiction concludes the proof. $\square$

Parenthetically, we note that Lemma 4.2 (with a less precise description of the constant $\beta(\varepsilon)$) follows from Lemma 4.3 by taking

$$\mathcal{V} := H^1(\Omega), \quad \mathcal{W} := L^2(\Omega, d^n x), \tag{4.12}$$

and, with $\gamma_D \in \mathcal{B}_\infty\big(H^1(\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big)$ denoting the Dirichlet trace,

$$K := \gamma_D^* \gamma_D \in \mathcal{B}_\infty\big(H^1(\Omega), \big(H^1(\Omega)\big)^*\big), \quad T := \iota : H^1(\Omega) \hookrightarrow L^2(\Omega, d^n x), \tag{4.13}$$

the inclusion operator.

Lemma 4.4. *Assume Hypothesis* 4.1*, where the number $\delta > 0$ is taken to be sufficiently small relative to the Lipschitz character of Ω. Consider the sesquilinear form $a_\Theta(\,\cdot\,,\,\cdot\,)$ defined on $H^1(\Omega) \times H^1(\Omega)$ by*

$$a_\Theta(u, v) := \int_\Omega d^n x\, \overline{(\nabla u)(x)} \cdot (\nabla v)(x) + \big\langle \gamma_D u, \Theta \gamma_D v \big\rangle_{1/2}, \quad u, v \in H^1(\Omega). \tag{4.14}$$

Then there exists $\kappa > 0$ with the property that the form

$$a_{\Theta,\kappa}(u, v) := a_\Theta(u, v) + \kappa\, (u, v)_{L^2(\Omega; d^n x)}, \quad u, v \in H^1(\Omega), \tag{4.15}$$

is $H^1(\Omega)$-coercive.

As a consequence, the form (4.14) *is symmetric, $H^1(\Omega)$-bounded, bounded from below, and closed in $L^2(\Omega; d^n x)$.*

Proof. We shall show that $\kappa > 0$ can be chosen large enough so that

$$\frac{1}{6}\|u\|^2_{H^1(\Omega)} \leq \frac{1}{3}\int_\Omega d^n x\, |(\nabla u)(x)|^2 + \frac{\kappa}{3}\int_\Omega d^n x\, |u(x)|^2 + \big\langle \gamma_D u, \Theta_j \gamma_D v \big\rangle_{1/2}, \quad u \in H^1(\Omega),\ j = 1, 2, 3, \tag{4.16}$$

where Θ_j, $j = 1, 2, 3$, are as introduced in Hypothesis 4.1. Summing up these three inequalities then proves that the form (4.15) is indeed $H^1(\Omega)$-coercive. To this end, we assume first $j = 1$ and recall that there exists $c_{\Theta_0} \in \mathbb{R}$ such that

$$\big\langle \gamma_D u, \Theta_1\, \gamma_D u \big\rangle_{1/2} \geq c_{\Theta_0} \|\gamma_D u\|^2_{L^2(\partial\Omega; d^{n-1}\omega)}, \quad u \in H^1(\Omega). \tag{4.17}$$

Thus, in this case, it suffices to show that

$$\begin{aligned} &\max\{-c_{\Theta_0}\,, 0\}\, \|\gamma_D u\|^2_{L^2(\partial\Omega; d^{n-1}\omega)} + \frac{1}{6}\|u\|^2_{H^1(\Omega)} \\ &\quad \leq \frac{1}{3}\int_\Omega d^n x\, |(\nabla u)(x)|^2 + \frac{\kappa}{3}\int_\Omega d^n x\, |u(x)|^2, \quad u \in H^1(\Omega), \end{aligned} \tag{4.18}$$

or, equivalently, that

$$\begin{aligned} &\max\{-c_{\Theta_0}\,, 0\}\, \|\gamma_D u\|^2_{L^2(\partial\Omega; d^{n-1}\omega)} \\ &\quad \leq \frac{1}{6}\int_\Omega d^n x\, |(\nabla u)(x)|^2 + \frac{2\kappa - 1}{6}\int_\Omega d^n x\, |u(x)|^2, \quad u \in H^1(\Omega). \end{aligned} \tag{4.19}$$

The fact that there exists $\kappa > 0$ for which (4.19) holds follows directly from Lemma 4.2.

Next, we observe that in the case where $j = 2, 3$, estimate (4.16) is implied by

$$\big|\langle\gamma_D u, \Theta_j\gamma_D u\rangle_{1/2}\big| \leq \frac{1}{6}\int_\Omega d^n x\,|(\nabla u)(x)|^2 + \frac{2\kappa-1}{6}\int_\Omega d^n x\,|u(x)|^2, \quad u \in H^1(\Omega), \tag{4.20}$$

or, equivalently, by

$$\big|\langle\gamma_D u, \Theta_j\gamma_D u\rangle_{1/2}\big| \leq \frac{1}{6}\|u\|^2_{H^1(\Omega)} + \frac{\kappa-1}{3}\|u\|^2_{L^2(\Omega;d^nx)}, \quad u \in H^1(\Omega). \tag{4.21}$$

When $j = 2$, in which case $\Theta_2 \in \mathcal{B}_\infty\big(H^1(\Omega), \big(H^1(\Omega)\big)^*\big)$, we invoke Lemma 4.3 with $\mathcal{V}$, $\mathcal{W}$ as in (4.12) and, with $\gamma_D \in \mathcal{B}\big(H^1(\Omega), H^{1/2}(\partial\Omega)\big)$ denoting the Dirichlet trace,

$$K := \gamma_D^*\Theta_2\gamma_D \in \mathcal{B}_\infty\big(H^1(\Omega), \big(H^1(\Omega)\big)^*\big), \quad T := \iota : H^1(\Omega) \hookrightarrow L^2(\Omega, d^nx), \tag{4.22}$$

the inclusion operator. Then, with $\varepsilon = 1/6$ and $\kappa := 3C_{1/6} + 1$, estimate (4.8) yields (4.21) for $j = 2$.

Finally, consider (4.21) in the case where $j = 3$ and note that by hypothesis,

$$\begin{aligned}\big|\langle\gamma_D u, \Theta_3\gamma_D u\rangle_{1/2}\big| &\leq \|\Theta_3\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\|\gamma_D u\|^2_{H^{1/2}(\partial\Omega)} \\ &\leq \delta\|\gamma_D\|^2_{\mathcal{B}(H^1(\Omega),H^{1/2}(\partial\Omega))}\|u\|^2_{H^1(\Omega)}, \quad u \in H^1(\Omega).\end{aligned} \tag{4.23}$$

Thus (4.21) also holds for $j = 3$ if

$$0 < \delta < \|\gamma_D\|^{-2}_{\mathcal{B}(H^1(\Omega),H^{1/2}(\partial\Omega))} \text{ and } \kappa > 1. \tag{4.24}$$

This completes the justification of (4.16), and hence finishes the proof. □

Next, we turn to a discussion of nonlocal Robin Laplacians in bounded Lipschitz subdomains of $\mathbb{R}^n$. Concretely, we describe a family of self-adjoint Laplace operators $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^nx)$ indexed by the boundary operator Θ. We will refer to $-\Delta_{\Theta,\Omega}$ as the nonlocal Robin Laplacian.

Theorem 4.5. *Assume Hypothesis* 4.1*, where the number $\delta > 0$ is taken to be sufficiently small relative to the Lipschitz character of Ω. Then the nonlocal Robin Laplacian, $-\Delta_{\Theta,\Omega}$, defined by*

$$\begin{aligned}&-\Delta_{\Theta,\Omega} = -\Delta, \\ &\mathrm{dom}(-\Delta_{\Theta,\Omega}) = \big\{u \in H^1(\Omega)\,\big|\,\Delta u \in L^2(\Omega; d^nx),\ \big(\widetilde{\gamma}_N + \Theta\gamma_D\big)u = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}\end{aligned} \tag{4.25}$$

is self-adjoint and bounded from below in $L^2(\Omega; d^nx)$. Moreover,

$$\mathrm{dom}\big(|-\Delta_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega), \tag{4.26}$$

and $-\Delta_{\Theta,\Omega}$, has purely discrete spectrum bounded from below, in particular,

$$\sigma_{\mathrm{ess}}(-\Delta_{\Theta,\Omega}) = \emptyset. \tag{4.27}$$

Proof. Denote by $a_{-\Delta_{\Theta,\Omega}}(\,\cdot\,,\,\cdot\,)$ the sesquilinear form introduced in (4.14). From Lemma 4.4, we know that $a_{-\Delta_{\Theta,\Omega}}$ is symmetric, $H^1(\Omega)$-bounded, bounded from below, as well as densely defined and closed in $L^2(\Omega; d^nx) \times L^2(\Omega; d^nx)$. Thus, if as in (2.34), we now introduce the operator $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^nx)$ by

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) = \bigg\{v \in H^1(\Omega)\,\bigg|\,\text{there exists a } w_v \in L^2(\Omega; d^nx) \text{ such that}$$

$$\int_\Omega d^nx\,\overline{\nabla w}\,\nabla v + \langle\gamma_D w, \Theta\gamma_D v\rangle_{1/2} = \int_\Omega d^nx\,\overline{w}w_v \text{ for all } w \in H^1(\Omega)\bigg\},$$

$$-\Delta_{\Theta,\Omega} u = w_u, \quad u \in \mathrm{dom}(-\Delta_{\Theta,\Omega}), \tag{4.28}$$

it follows from (2.20)–(2.43) (cf., in particular (2.27)) that $-\Delta_{\Theta,\Omega}$ is self-adjoint and bounded from below in $L^2(\Omega; d^n x)$ and that (4.26) holds. Next we recall that

$$H_0^1(\Omega) = \big\{u \in H^1(\Omega) \,\big|\, \gamma_D u = 0 \text{ on } \partial\Omega\big\}. \tag{4.29}$$

Taking $v \in C_0^\infty(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow H^1(\Omega)$, one concludes

$$\int_\Omega d^n x\, \overline{v} w_u = -\int_\Omega d^n x\, \overline{v}\, \Delta u \quad \text{for all } v \in C_0^\infty(\Omega), \text{ and hence } w_u = -\Delta u \text{ in } \mathcal{D}'(\Omega), \tag{4.30}$$

with $\mathcal{D}'(\Omega) = C_0^\infty(\Omega)'$ the space of distributions on Ω. Going further, suppose that $u \in \mathrm{dom}(-\Delta_{\Theta,\Omega})$ and $v \in H^1(\Omega)$. We recall that $\gamma_D \colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and compute

$$\begin{aligned}
\int_\Omega d^n x\, \overline{\nabla v}\, \nabla u &= -\int_\Omega d^n x\, \overline{v}\, \Delta u + \langle \gamma_D v, \widetilde{\gamma}_N u\rangle_{1/2} \\
&= \int_\Omega d^n x\, \overline{v} w_u + \big\langle \gamma_D v, \big(\widetilde{\gamma}_N + \Theta\gamma_D\big)u\big\rangle_{1/2} - \big\langle \gamma_D v, \Theta\gamma_D u\big\rangle_{1/2} \\
&= \int_\Omega d^n x\, \overline{\nabla v}\, \nabla u + \big\langle \gamma_D v, \big(\widetilde{\gamma}_N + \Theta\gamma_D\big)u\big\rangle_{1/2},
\end{aligned} \tag{4.31}$$

where we used the second line in (4.28). Hence,

$$\big\langle \gamma_D v, \big(\widetilde{\gamma}_N + \Theta\gamma_D\big)u\big\rangle_{1/2} = 0. \tag{4.32}$$

Since $v \in H^1(\Omega)$ is arbitrary, and the map $\gamma_D \colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is actually onto, one concludes that

$$\big(\widetilde{\gamma}_N + \Theta\gamma_D\big)u = 0 \text{ in } H^{-1/2}(\partial\Omega). \tag{4.33}$$

Thus,

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) \subseteq \big\{v \in H^1(\Omega) \,\big|\, \Delta v \in L^2(\Omega; d^n x),\ \big(\widetilde{\gamma}_N + \Theta\gamma_D\big)v = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}. \tag{4.34}$$

Next, assume that $u \in \big\{v \in H^1(\Omega) \,\big|\, \Delta v \in L^2(\Omega; d^n x),\ \big(\widetilde{\gamma}_N + \Theta\gamma_D\big)v = 0\big\}$, $w \in H^1(\Omega)$, and let $w_u = -\Delta u \in L^2(\Omega; d^n x)$. Then,

$$\begin{aligned}
\int_\Omega d^n x\, \overline{w} w_u &= -\int_\Omega d^n x\, \overline{w}\, \mathrm{div}(\nabla u) \\
&= \int_\Omega d^n x\, \overline{\nabla w}\, \nabla u - \langle \gamma_D w, \widetilde{\gamma}_N u\rangle_{1/2} \\
&= \int_\Omega d^n x\, \overline{\nabla w}\, \nabla u + \big\langle \gamma_D w, \Theta\gamma_D u\big\rangle_{1/2}.
\end{aligned} \tag{4.35}$$

Thus, applying (4.28), one concludes that $u \in \mathrm{dom}(-\Delta_{\Theta,\Omega})$ and hence

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) \supseteq \big\{v \in H^1(\Omega) \,\big|\, \Delta v \in L^2(\Omega; d^n x);\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)v = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}. \tag{4.36}$$

Finally, the last claim in the statement of Theorem 4.5 follows from the fact that $H^1(\Omega)$ embeds compactly into $L^2(\Omega; d^n x)$ (cf., e.g., [16, Theorem V.4.17]); see Lemma 2.1. □

In the special case $\Theta = 0$, that is, in the case of the Neumann Laplacian, we will also use the notation

$$-\Delta_{N,\Omega} := -\Delta_{0,\Omega}. \tag{4.37}$$

The case of the Dirichlet Laplacian $-\Delta_{D,\Omega}$ associated with Ω formally corresponds to $\Theta = \infty$ and so we isolate it in the next result (cf. also [22], [25]):

Theorem 4.6. *Assume Hypothesis* 3.1. *Then the Dirichlet Laplacian, $-\Delta_{D,\Omega}$, defined by*

$$\begin{aligned} & -\Delta_{D,\Omega} = -\Delta, \\ & \operatorname{dom}(-\Delta_{D,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \gamma_D u = 0 \text{ in } H^{1/2}(\partial\Omega)\big\} \\ & \qquad\qquad\qquad = \big\{u \in H_0^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}, \end{aligned} \tag{4.38}$$

is self-adjoint and strictly positive in $L^2(\Omega; d^n x)$. Moreover,

$$\operatorname{dom}\big((-\Delta_{D,\Omega})^{1/2}\big) = H_0^1(\Omega). \tag{4.39}$$

Since Ω is open and bounded, it is well-known that $-\Delta_{D,\Omega}$ has purely discrete spectrum contained in $(0,\infty)$, in particular,

$$\sigma_{\rm ess}(-\Delta_{D,\Omega}) = \emptyset. \tag{4.40}$$

This follows from (4.39) since $H_0^1(\Omega)$ embeds compactly into $L^2(\Omega; d^n x)$; the latter fact holds for arbitrary open, bounded sets $\Omega \subset \mathbb{R}^n$ (see, e.g., [16, Theorem V.4.18]).

5. Eigenvalue Inequalities

Assume Hypothesis 4.1 and denote by

$$\lambda_{\Theta,\Omega,1} \le \lambda_{\Theta,\Omega,2} \le \cdots \le \lambda_{\Theta,\Omega,j} \le \lambda_{\Theta,\Omega,j+1} \le \cdots \tag{5.1}$$

the eigenvalues for the Robin Laplacian $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$, listed according to their multiplicity. Similarly, we let

$$0 < \lambda_{D,\Omega,1} < \lambda_{D,\Omega,2} \le \cdots \le \lambda_{D,\Omega,j} \le \lambda_{D,\Omega,j+1} \le \cdots \tag{5.2}$$

be the eigenvalues for the Dirichlet Laplacian $-\Delta_{D,\Omega}$ in $L^2(\Omega; d^n x)$, again enumerated according to their multiplicity.

Theorem 5.1. *Assume Hypothesis* 4.1, *where the number $\delta > 0$ is taken to be sufficiently small relative to the Lipschitz character of Ω and, in addition, suppose that*

$$\big\langle \gamma_D(e^{ix\cdot\eta}), \Theta\gamma_D(e^{ix\cdot\eta})\big\rangle_{1/2} \le 0 \ \text{ for all } \eta \in \mathbb{R}^n. \tag{5.3}$$

Then

$$\lambda_{\Theta,\Omega,j+1} < \lambda_{D,\Omega,j}, \quad j \in \mathbb{N}. \tag{5.4}$$

Proof. One can follow Filonov [18] closely. The main reason we present Filonov's elegant argument is to ensure that this continues to hold in the case when a nonlocal Robin boundary condition is considered (in lieu of the Neumann boundary condition). Recalling the eigenvalue counting functions for the Dirichlet and Robin Laplacians, one sets for each $\lambda \in \mathbb{R}$,

$$N_D(\lambda) := \#\,\{\sigma(-\Delta_{D,\Omega}) \cap (-\infty,\lambda]\}, \quad N_\Theta(\lambda) := \#\,\{\sigma(-\Delta_{\Theta,\Omega}) \cap (-\infty,\lambda]\}. \tag{5.5}$$

Then Lemma 2.1 and Lemma 4.4 ensure that for each $\lambda \in \mathbb{R}$ one has

$$N_D(\lambda) = \max\left\{ \dim(L) \in \mathbb{N}_0 \,\middle|\, L \text{ a subspace of } H_0^1(\Omega) \text{ such that} \right.$$

$$\left.\int_{\Omega} d^n x\, |(\nabla u)(x)|^2 \leq \lambda \|u\|^2_{L^2(\Omega;d^n x)} \text{ for all } u \in L\right\}, \tag{5.6}$$

and

$$N_{\Theta}(\lambda) = \max \left\{ \dim(L) \in \mathbb{N}_0 \,\middle|\, L \text{ a subspace of } H^1_0(\Omega) \text{ with the property that} \right.$$
$$\left.\int_{\Omega} d^n x\, |(\nabla u)(x)|^2 + \langle \gamma_D u, \Theta \gamma_D u \rangle_{1/2} \leq \lambda \|u\|^2_{L^2(\Omega;d^n x)} \text{ for all } u \in L\right\}. \tag{5.7}$$

Next, observe that for any $\lambda \in \mathbb{C}$,

$$H^1_0(\Omega) \cap \ker(-\Delta_{\Theta,\Omega} - \lambda\, I_{\Omega}) = \{0\}. \tag{5.8}$$

Indeed, if $u \in H^1_0(\Omega) \cap \ker(-\Delta_{\Theta,\Omega} - \lambda\, I_{\Omega})$, then $u \in H^1(\Omega)$ satisfies $(-\Delta - \lambda)u = 0$ in Ω and $\gamma_D u = \widetilde{\gamma}_N u = 0$. It follows that the extension by zero of u to the entire $\mathbb{R}^n$ belongs to $H^1(\mathbb{R}^n)$, is compactly supported, and is annihilated by $-\Delta - \lambda$. Hence, this function vanishes identically, by unique continuation.

To continue, we fix $\lambda > 0$ and pick a subspace V of $H^1_0(\Omega)$ such that $\dim(V) = N_D(\lambda)$ and

$$\int_{\Omega} d^n x\, |(\nabla u)(x)|^2 \leq \lambda \int_{\Omega} d^n x\, |u(x)|^2, \quad u \in V. \tag{5.9}$$

Then the sum $W \dot{+} \ker(-\Delta_{\Theta,\Omega} - \lambda\, I_{\Omega})$ is direct, by (5.8). Since the functions $\{e^{ix\cdot\eta} \,|\, \eta \in \mathbb{R}^n,\, |\eta| = \sqrt{\lambda}\}$ are linearly independent, it follows that there exists a vector $\eta \in \mathbb{R}^n$ with $|\eta| = \sqrt{\lambda}$ and such that $e^{ix\cdot\eta}$ does not belong to the finite-dimensional space $V \dot{+} \ker(-\Delta_{\Theta,\Omega} - \lambda\, I_{\Omega})$. Assuming that this is the case, introduce

$$W := V \dot{+} \ker(-\Delta_{\Theta,\Omega} - \lambda\, I_{\Omega}) \dot{+} \{c e^{ix\cdot\eta} \,|\, c \in \mathbb{C}\}, \tag{5.10}$$

so that W is a finite-dimensional subspace of $H^1(\Omega)$. Let $w = u + v + c e^{ix\cdot\eta}$ be an arbitrary vector in V, where $u \in V$, $v \in \ker(-\Delta_{\Theta,\Omega} - \lambda\, I_{\Omega})$, and $c \in \mathbb{C}$. We then write

$$\begin{aligned}
&\int_{\Omega} d^n x |(\nabla w)(x)|^2 + \langle \gamma_D w, \Theta \gamma_D w \rangle_{1/2} \\
&\quad = \int_{\Omega} d^n x |\nabla(u + v + c e^{ix\cdot\eta})|^2 + \langle \gamma_D (v + c e^{ix\cdot\eta}), \Theta \gamma_D (v + c e^{ix\cdot\eta}) \rangle_{1/2} \\
&\quad = \int_{\Omega} d^n x \left(|\nabla u|^2 + |\nabla v|^2 + |c\eta|^2 \right) \\
&\qquad + 2\mathrm{Re} \left(\int_{\Omega} d^n x \left[\overline{\nabla v} \cdot \nabla(u + c e^{ix\cdot\eta}) + \overline{\nabla(c e^{ix\cdot\eta})} \cdot \nabla u \right] \right) \\
&\qquad + \langle \gamma_D (v + c e^{ix\cdot\eta}), \Theta \gamma_D (v + c e^{ix\cdot\eta}) \rangle_{1/2} \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned} \tag{5.11}$$

An integration by parts shows that

$$\begin{aligned}
\int_{\Omega} d^n x\, |\nabla v|^2 &= -\int_{\Omega} d^n x\, \overline{v} \Delta v + \langle \gamma_D v, \widetilde{\gamma}_N v \rangle_{1/2} \\
&= \lambda \int_{\Omega} d^n x\, |v|^2 - \langle \gamma_D v, \Theta \gamma_D v \rangle_{1/2}
\end{aligned} \tag{5.12}$$

where the last equality holds thanks to $-\Delta v = \lambda\, v$ and $\widetilde{\gamma}_N v = -\Theta\gamma_D v$. We now make use of this, (5.9), the fact that $|\eta|^2 = \lambda$, in order to estimate

$$I_1 \leq \lambda \int_\Omega d^n x \, \big[|u|^2 + |v|^2 + |c|^2\big] - \langle \gamma_D v, \Theta\gamma_D v\rangle_{1/2}. \tag{5.13}$$

Similarly,

$$\begin{aligned} I_2 &= -2\mathrm{Re}\bigg(\int_\Omega d^n x \, [\overline{\Delta v}(u + ce^{ix\cdot\eta}) + \overline{\Delta(ce^{ix\cdot\eta})}u]\bigg) + 2\mathrm{Re}\,\big(\langle\gamma_D(ce^{ix\cdot\eta}), \widetilde{\gamma}_N v\rangle_{1/2}\big) \\ &= 2\lambda\mathrm{Re}\bigg(\int_\Omega d^n x \, [\overline{v}(u + ce^{ix\cdot\eta}) + \overline{ce^{ix\cdot\eta}}u]\bigg) - 2\mathrm{Re}\,\big(\langle\gamma_D(ce^{ix\cdot\eta}), \Theta\gamma_D v\rangle_{1/2}\big). \end{aligned} \tag{5.14}$$

Thus, altogether,

$$\begin{aligned} &\int_\Omega d^n x |(\nabla w)(x)|^2 + \langle\gamma_D w, \Theta\gamma_D w\rangle_{1/2} \\ &\quad \leq \lambda \int_\Omega d^n x |w(x)|^2 + |c|^2 \langle\gamma_D(e^{ix\cdot\eta}), \Theta\gamma_D(e^{ix\cdot\eta})\rangle_{1/2}. \end{aligned} \tag{5.15}$$

Upon recalling (5.3), this yields

$$\int_\Omega d^n x |(\nabla w)(x)|^2 + \langle\gamma_D w, \Theta\gamma_D w\rangle_{1/2} \leq \lambda \int_\Omega d^n x |w(x)|^2, \quad w \in W. \tag{5.16}$$

Consequently,

$$\begin{aligned} N_\Theta(\lambda) \geq \dim(W) &= \dim(V) + \dim(\ker(-\Delta_{\Theta,\Omega} - \lambda\, I_\Omega)) + 1 \\ &= N_D(\lambda) + \dim(\ker(-\Delta_{\Theta,\Omega} - \lambda\, I_\Omega)) + 1. \end{aligned} \tag{5.17}$$

Specializing this to the case when $\lambda = \lambda_{D,\Omega,j}$ then yields

$$\begin{aligned} \#\,\{\sigma(-\Delta_{\Theta,\Omega}) \cap (-\infty, \lambda_{D,\Omega,j})\} &= N_\Theta(\lambda_{D,\Omega,j}) - \dim(\ker(-\Delta_{\Theta,\Omega} - \lambda_{D,\Omega,j}\, I_\Omega)) \\ &\geq N_D(\lambda_{D,\Omega,j}) + 1 \geq j + 1. \end{aligned} \tag{5.18}$$

Now, the fact that $\#\,\{\sigma(-\Delta_{\Theta,\Omega}) \cap (-\infty, \lambda_{D,\Omega,j})\} \geq j + 1$ is reinterpreted as (5.4). $\square$

The case treated in [18] is that of a local Robin boundary condition. That is, it was assumed that Θ is the operator of multiplication M_θ by a function θ defined on $\partial\Omega$ (which satisfies appropriate conditions). To better understand the way in which this scenario relates to the more general case treated here, we state and prove the following result:

Lemma 5.2. *Assume Hypothesis* 3.1 *and suppose that* $\Theta = M_\theta$*, the operator of multiplication with a measurable function* $\theta : \partial\Omega \to \mathbb{R}$*. Suppose that* $\theta \in L^p(\partial\Omega; d^{n-1}\omega)$*, where*

$$p = n - 1 \;\text{ if }\; n > 2, \;\text{ and }\; p \in (1, \infty] \;\text{ if }\; n = 2. \tag{5.19}$$

Then

$$\Theta \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big) \tag{5.20}$$

is a self-adjoint operator which satisfies

$$\|\Theta\|_{\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \leq C\|\theta\|_{L^p(\partial\Omega; d^{n-1}\omega)}, \tag{5.21}$$

where $C = C(\Omega, n, p) > 0$ *is a finite constant.*

Proof. Standard embedding results for Sobolev spaces (which continue to hold in the case when the ambient space is the boundary of a bounded Lipschitz domain) yield that

$$H^{1/2}(\partial\Omega) \hookrightarrow L^{q_0}(\partial\Omega; d^{n-1}\omega), \text{ where } q_0 := \begin{cases} \frac{2(n-1)}{n-2} \text{ if } n > 2, \\ \text{any number in } (1,\infty) \text{ if } n = 2. \end{cases} \tag{5.22}$$

Since the above embedding is continuous with dense range, via duality we also obtain that

$$L^{q_1}(\partial\Omega; d^{n-1}\omega) \hookrightarrow H^{-1/2}(\partial\Omega), \text{ where } q_1 := \begin{cases} \frac{2(n-1)}{n} \text{ if } n > 2, \\ \text{any number in } (1,\infty) \text{ if } n = 2. \end{cases} \tag{5.23}$$

Together, (5.22) and (5.23) yield that

$$\mathcal{B}\big(L^{q_0}(\partial\Omega; d^{n-1}\omega), L^{q_1}(\partial\Omega; d^{n-1}\omega)\big) \hookrightarrow \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big), \tag{5.24}$$

continuously. With p as in the statement of the lemma, Hölder's inequality yields that

$$M_\theta \in \mathcal{B}\big(L^{q_0}(\partial\Omega; d^{n-1}\omega), L^{q_1}(\partial\Omega; d^{n-1}\omega)\big) \tag{5.25}$$

and

$$\|M_\theta\|_{\mathcal{B}(L^{q_0}(\partial\Omega;d^{n-1}\omega), L^{q_1}(\partial\Omega;d^{n-1}\omega))} \leq C\|\theta\|_{L^p(\partial\Omega;d^{n-1}\omega)}, \tag{5.26}$$

for some finite constant $C = C(\partial\Omega, p, q_0, q_1) > 0$, granted that

$$\frac{1}{p} + \frac{1}{q_0} \leq \frac{1}{q_1}. \tag{5.27}$$

Inequality (5.27) then holds with equality when $n > 2$ and, given $p \in (1,\infty)$, q_0, q_1 can always be chosen as in (5.22) and (5.23) when $n = 2$ so that (5.27) continues to hold in this case as well. In summary, the above reasoning shows that $\Theta = M_\theta \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ and the estimate (5.21) holds. Let us also point out that Θ is a self-adjoint operator, since θ is real-valued.

It remains to establish (5.20), that is, to show that Θ is also a compact operator. To this end, fix $p_0 > p$ and let $\theta_j \in L^{p_0}(\partial\Omega; d^{n-1}\omega)$, $j \in \mathbb{N}$, be a sequence of functions with the property that $\theta_j \to \theta$ in $L^p(\partial\Omega; d^{n-1}\omega)$ as $j \to \infty$. Set $\Theta_j := M_{\theta_j}$, $j \in \mathbb{N}$. From what we proved above, it follows that

$$\Theta_j \to \Theta \text{ in } \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big) \text{ as } j \to \infty, \tag{5.28}$$

and there exists $r \in (1/2, 1)$ with the property that

$$\Theta_j \in \mathcal{B}\big(H^r(\partial\Omega), H^{-1/2}(\partial\Omega)\big), \quad j \in \mathbb{N}. \tag{5.29}$$

Since the embedding $H^r(\partial\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$ is compact, one concludes that

$$\Theta_j \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big), \quad j \in \mathbb{N}. \tag{5.30}$$

Thus, (5.20) follows from (5.30) and (5.28). □

We end by including a special case of Theorem 5.1 which is of independent interest. In particular, this links our conditions on Θ with Filonov's condition

$$\int_{\partial\Omega} d^{n-1}\omega\, \theta(\xi) \leq 0 \tag{5.31}$$

in the case where $\Theta = M_\theta$.

Corollary 5.3. *Assume Hypothesis* 4.1*, where the number* $\delta > 0$ *is taken to be sufficiently small relative to the Lipschitz character of* Ω *and, in addition, suppose that*

$$\Theta \leq 0 \tag{5.32}$$

in the sense that $\langle f, \Theta f\rangle_{1/2} \leq 0$ *for every* $f \in H^{1/2}(\partial\Omega)$*. Then* (5.4) *holds.*

In particular, assuming Hypothesis 3.1 *and* $\Theta = M_\theta$*, with* $\theta \in L^p(\partial\Omega; d^{n-1}\omega)$*, where* p *is as in* (5.19)*, is a function satisfying* (5.31)*, then* (5.4) *holds.*

Proof. The first part is directly implied by Theorem 5.1. The second part is a consequence of Lemma 5.2 and the conclusion in the first part of Corollary 5.3, since (5.32) reduces precisely to (5.31) for $\Theta = M_\theta$. □

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