Reduction of systems of first-order differential equations via Λ-symmetries

Giampaolo Cicogna*

Dipartimento di Fisica “E.Fermi” dell’Università di Pisa
and Istituto Nazionale di Fisica Nucleare, Sez. di Pisa

Largo B. Pontecorvo 3, Ed. B-C, I-56127, Pisa, Italy

Abstract

The notion of λ-symmetries, originally introduced by C. Muriel and J.L. Romero, is extended to the case of systems of first-order ODE’s (and of dynamical systems in particular). It is shown that the existence of a symmetry of this type produces a reduction of the differential equations, restricting the presence of the variables involved in the problem. The results are compared with the case of standard (i.e. exact) Lie-point symmetries and are also illustrated by some examples.

PACS: 02.20.Sv; 02.30.Hq

Keywords: λ-symmetries; reduction procedures; dynamical systems

1 Introduction

It is a well known property that if an ordinary differential equation (ODE) admits a Lie point-symmetry, then the order of the equation can be lowered by one (see e.g. [1]). The notion of λ-symmetry has been introduced in 2001 by Muriel and Romero [2, 3] with the main purpose of obtaining this reduction even in the absence of standard Lie symmetries. The idea consists in introducing a suitable modification of the prolongation rules of the vector field in such a way that the lowering procedure still works, even if λ-symmetries are not symmetries in the proper sense, as they do not map in general solutions into solutions.

λ-Symmetries are related to symmetries of integral exponential type [1, 2, 3], to hidden and to some classes of potential symmetries (see [3, 4, 5] and references therein). The meaning of λ-prolongation has been clarified (together with a possible generalization of the procedure) by means of classical theory

*Email: cicogna@df.unipi.it

of characteristics of vector fields [6]. λ-Symmetries have been extended to partial differential equations [7, 8] (and called in that context μ-symmetries), and also interpreted in terms of a deformed Lie derivative in a more geometrical approach [9]. A nontrivial relationship with nonlocal symmetries has been recently pointed out [10]; an interpretation in terms of appropriately defined changes of reference frames has been also proposed [11]. For the implications of λ- and μ-symmetries in Noether-type conservation rules see [12, 13].

In the case of first-order ODE's, Lie symmetries cannot lower the order of the equations, but they provide a sort of "reduction" of the complexity of the system, or – more precisely – a reduction of the number of the involved variables (see [1], Theorem 2.66). In this paper, we will restrict precisely to the case of *systems* of *first-order* ODE's (with usual regularity and nondegeneracy assumptions: see e.g. [1, 14]), and of dynamical systems (DS) in particular, where the application of λ-symmetries requires some attention and where they exhibit some relevant peculiarities. We shall prove that some forms of reduction are allowed also in these cases.

An application of λ-symmetries to systems of ODE's has been already considered in a particular case [15]; in the present paper we want to examine more general situations.

2 Systems of ODE's

Let us recall first of all that in the case of a *single* dependent variable $u = u(t)$ (we shall always denote by $t \in \mathbf{R}$ the independent variable, according to its natural interpretation as the time variable in the case of DS), the first-order λ-prolongation $X_\lambda^{(1)}$ of a vector field X

$$X \; = \; \tau(t,u)\frac{\partial}{\partial t} + \varphi(t,u)\frac{\partial}{\partial u} \tag{1}$$

is defined as

$$X_\lambda^{(1)} \; = \; X^{(1)} + \lambda\, Q \frac{\partial}{\partial \dot{u}} \tag{2}$$

where $X^{(1)}$ is the *standard* prolongation [1, 14], $\dot{u} = \mathrm{d}u/\mathrm{d}t$, $\lambda \; = \; \lambda(t,u,\dot{u})$ is an arbitrary C^∞ function, and $Q \; = \; \varphi - \tau\,\dot{u}$.

Considering systems of equations, and then $q > 1$ dependent variables $u_a = u_a(t)$, the natural extension of definition (2) is

$$X_\Lambda^{(1)} \; = \; X^{(1)} + (\Lambda Q)_a \frac{\partial}{\partial \dot{u}_a} \tag{3}$$

where the sum over $a = 1, \ldots, q$ is understood, with

$$X \; = \; \tau \frac{\partial}{\partial t} + \varphi_a \frac{\partial}{\partial u_a}\,, \qquad\qquad Q_a \; = \; \varphi_a - \tau\,\dot{u}_a \tag{4}$$

and where now Λ is a $q \times q$ matrix of C^∞ functions depending on $t, u_a, \dot{u}_a$. The case $\Lambda = \lambda I$ is the one considered, in the context of DS and also for systems of ODE's of any order, by Muriel and Romero [15].

Given a system of q first-order ODE's (we shall assume for simplicity that the number of the equations is the same as the number of dependent variables $u_a(t)$)

$$F_a(t, u_b, \dot{u}_b) \;=\; 0 \qquad\qquad a, b = 1, \ldots, q \tag{5}$$

we shall say that this system is Λ-*symmetric* under a vector field X if there is a matrix Λ such that

$$X_\Lambda^{(1)}\, F_a \big|_{F_a=0} \;=\; 0\ . \tag{6}$$

It is clear from (3) that Λ is not uniquely defined: indeed, for any matrix R such that $RQ = Q$ then also $\Lambda' = \Lambda R$ satisfies the above condition. This arbitrariness in the definition of Λ, far from being disturbing, may be useful in practice, as it allows the choice of the more convenient matrix Λ in view of the given problem.

We shall say that the system (5) is Λ-*invariant* under X if there is a matrix Λ such that

$$X_\Lambda^{(1)}\, F_a \;=\; 0\ . \tag{7}$$

It is not too restrictive to assume that the system of ODE's we are going to consider can be put into a Λ-invariant form. Indeed, extending to Λ-symmetries a well known result [1, 6], it can be shown that if a system is Λ-symmetric, then there exists a $q \times q$ matrix $A = A(t, u, \dot{u})$ such that

$$X_\Lambda^{(1)}\, F_a \;=\; A_{ab} F_b\ .$$

It is now enough to prove, applying standard arguments (cf. e.g. [16, 17, 18]), the existence of some some $q \times q$ invertible matrix S, possibly depending on $t, u, \dot{u}$, such that

$$X_\Lambda^{(1)}\, S + S\, A \;=\; 0$$

and the (locally) equivalent system $G_a \equiv S_{ab} F_b \;=\; 0$ turns out to be Λ-invariant. We will consider in the following, unless otherwise stated, only Λ-invariant systems.

The matrix Λ plays the role of an additional "unknown" in the determining equations which are deduced from the Λ-invariance condition

$$\tau \frac{\partial F_a}{\partial t} + \varphi_b \frac{\partial F_a}{\partial u_b} + \left(\varphi_b^{(1)} + (\Lambda Q)_b \right) \frac{\partial F_a}{\partial \dot{u}_b} \;=\; 0 \tag{8}$$

where $\varphi_b^{(1)}$ is the coefficient of the standard first-order prolongation, and which clearly strongly depend on the explicit form of the functions F_a. For instance, in the case where $F_a = \dot{u}_a - f_a(t, u)$, i.e. the case of dynamical systems (see Section 3), these equations are

$$\{\varphi\,,\, f\}_a \;=\; \frac{\partial}{\partial t}(\varphi_a - \tau\, f_a) - \frac{\partial \tau}{\partial u_b} f_a f_b + \Lambda_{ab} Q_b\ ,$$

where $\{\varphi\,,\, f\}_a = \varphi_b \dfrac{\partial f_a}{\partial u_b} - f_b \dfrac{\partial \varphi_a}{\partial u_b}$, which clearly become

$$\{\varphi\,,\, f\}_a \;=\; (\Lambda\, \varphi)_a \tag{9}$$

in the case of autonomous systems and time-independent vector fields X with $\tau \equiv 0$ (see [19]).

Let us now introduce "symmetry-adapted" coordinates w_a (sometimes also called canonical coordinates) characterized by the property of being invariant under the action of the vector field X:

$$X\, w_a \;=\; \tau \frac{\partial w_a}{\partial t} + \varphi_b \frac{\partial w_a}{\partial u_b} \;=\; 0 \;; \tag{10}$$

they are obtained through the associated characteristic equations

$$\frac{\mathrm{d}t}{\tau} \;=\; \frac{\mathrm{d}u_a}{\varphi_a} \;.$$

In this way we introduce exactly q new variables w_a; one at least of these, say w_q, will depend explicitly on t, and we will choose this as the new independent variable and call it η. In particular, if $\tau \equiv 0$, we can choose $\eta = t$. As $(q+1)$-th variable, which will be called z, we will take the coordinate "along the action of X", i.e. such that $X\,z = 1$. Summarizing, the new set of variables is

$$\eta,\ w_\alpha(\eta),\ z(\eta), \qquad\qquad \alpha = 1, \ldots, q-1 \tag{11}$$

(among these, w_α and $w_q \equiv \eta$ are invariant under X) and clearly do *not* depend on Λ.

We have now to write the given vector field X and its first Λ-prolongation (3) in terms of these coordinates. We get first $X \;=\; X^{(1)} \;=\; \partial/\partial z$, and we then find that eq. (3) takes the form

$$X^{(1)}_\Lambda \;=\; \frac{\partial}{\partial z} + M_\alpha \frac{\partial}{\partial w'_\alpha} + M_q \frac{\partial}{\partial z'} \tag{12}$$

where $w'_\alpha = \mathrm{d}w_\alpha / \mathrm{d}\eta$, $z' = \mathrm{d}z/\mathrm{d}\eta$ and (here and in the following the sum will be always understood over the repeated indices $\alpha = 1, \ldots, q-1$ and $a = 1, \ldots, q$; D_t is the total derivative)

$$M_\alpha \;=\; (D_t\eta)^{-2}\Big(D_t\eta \frac{\partial w_\alpha}{\partial u_a} - D_t w_\alpha \frac{\partial \eta}{\partial u_a}\Big)(\Lambda Q)_a \tag{13}$$

$$M_q \;=\; (D_t\eta)^{-2}\Big(D_t\eta \frac{\partial z}{\partial u_a} - D_t z \frac{\partial \eta}{\partial u_a}\Big)\;(\Lambda Q)_a \;. \tag{14}$$

In particular, if $\tau \equiv 0$, and then with $\eta = t$, we have more simply

$$\frac{\partial w'_\alpha}{\partial \dot u_a} = \frac{\partial w_\alpha}{\partial u_a} \qquad , \qquad \frac{\partial z'}{\partial \dot u_a} = \frac{\partial z}{\partial u_a}$$

and

$$M_\alpha \;=\; \frac{\partial w_\alpha}{\partial u_a}(\Lambda\, Q)_a \qquad , \qquad M_q \;=\; \frac{\partial z}{\partial u_a}(\Lambda\, Q)_a \;. \tag{15}$$

The above expressions (13,14) can be obtained either by direct calculation expressing by the chain rule the operators $\partial/\partial\dot{u}_a$ in terms of $\partial/\partial w'_\alpha$, $\partial/\partial z'$, or – more elegantly – starting from the algebraic relation

$$[X_\Lambda^{(1)},\, D_t] \;=\; -D_t(\tau)D_t + (\Lambda Q)_a \frac{\partial}{\partial u_a} \tag{16}$$

which can be easily proved and generalizes to Λ-symmetries other similar known identities [2, 3]. From this, one directly gets indeed

$$X_\Lambda^{(1)}(w'_\alpha) \;=\; X_\Lambda^{(1)}\Big(\frac{D_t w_\alpha}{D_t\eta}\Big) \;=\; \frac{X_\Lambda^{(1)}(D_t w_\alpha)(D_t\eta) - (D_t w_\alpha)X_\Lambda^{(1)}(D_t\eta)}{(D_t\eta)^2} \;=$$

$$= (D_t\eta)^{-2}\Big(D_t\eta\frac{\partial w_\alpha}{\partial u_a} - D_t w_\alpha \frac{\partial\eta}{\partial u_a}\Big)(\Lambda Q)_a \;=\; M_\alpha \tag{17}$$

thanks to $X_\Lambda^{(1)} w_\alpha = X w_\alpha = 0$, $X_\Lambda^{(1)}\eta = X\eta = 0$; similarly for $X_\Lambda^{(1)}(z')$.

It can be interesting to point out that eq. (17) puts in clear evidence the difference with respect to exact symmetries: indeed, starting from the q (X-invariant) variables w_α, η one obtains $q-1$ first-order differential quantities w'_α which are invariant under $X^{(1)}$, but in general *not* under $X_\Lambda^{(1)}$.

In turn, the given system of differential equations will take the form (we will use the $\widetilde{\cdot}$ to denote the expressions in the new variables)

$$\widetilde{F}_a(\eta, w_\alpha, w'_\alpha, z, z') \;=\; 0 \tag{18}$$

and the condition of its Λ-invariance under X now becomes

$$\frac{\partial\widetilde{F}_a}{\partial z} + M_\alpha \frac{\partial\widetilde{F}_a}{\partial w'_\alpha} + M_q \frac{\partial\widetilde{F}_a}{\partial z'} \;=\; 0\ . \tag{19}$$

This allows us to state the following first form of reduction:

Theorem 1. *If the system (5) is Λ-invariant under a vector field X, then, once written in the symmetry-adapted coordinates $\eta, w_\alpha, w'_\alpha, z, z'$, it turns out to depend on only $2q$ quantities (instead of $2q+1$): i.e. on the q variables w_α, η and on other q first-order differential Λ-invariant quantities $\zeta_a = \zeta_a(\eta, z, w_\alpha, w'_\alpha, z')$ which are obtained from the characteristic equations*

$$\mathrm{d}z \;=\; \frac{\mathrm{d}w'_\alpha}{M_\alpha} \;=\; \frac{\mathrm{d}z'}{M_q} \tag{20}$$

coming from condition (19).

Examples 1 and 2 will illustrate this result.

3 The case of Dynamical Systems

Let us now consider the particularly important case of the dynamical systems, i.e. the systems of first-order ODE's which are written "in explicit form":

$$\dot{u}_a \;=\; f_a(t,u) \; . \tag{21}$$

Clearly, once symmetry-adapted coordinates are introduced, the system becomes "automatically" a function of the $2q$ quantities w_α, η, ζ_a, as granted by Theorem 1.

But it can be preferable or more convenient (e.g. in view of the physical interpretation in terms of "evolution" problem, or also if the explicit expression of the ζ_a is not known[1]) to adopt a different point of view, i.e. to preserve the form of the system as an explicit DS, i.e. to rewrite it as follows

$$w'_\alpha \;=\; \widetilde{f}_\alpha(\eta, w, z) \tag{22}$$

$$z' \;=\; \widetilde{f}_q(\eta, w, z) \; . \tag{23}$$

and to look for the dependence on z of the r.h.s. This point of view will be elucidated by Examples 3 and 4.

Recalling the expression (12) of the first Λ-prolongation of X, we then easily deduce in this case:

Theorem 2. *If a DS is Λ-invariant under X, the dependence on z of the r.h.s. of eq.s (22,23) is given by*

$$\frac{\partial \widetilde{f}_\alpha}{\partial z} \;=\; M_\alpha \qquad ; \qquad \frac{\partial \widetilde{f}_q}{\partial z} \;=\; M_q \; .$$

Then, if for some $\overline{\alpha}$ one has $M_{\overline{\alpha}} = 0$, the corresponding $\widetilde{f_{\overline{\alpha}}}$ does not depend on z. If $M_\alpha = 0$ for all $\alpha = 1, \ldots, q-1$, then only $\widetilde{f}_q$ depends on z and the system splits into a system for the $q-1$ variables $w_\alpha = w_\alpha(\eta)$ and the last equation (23) which is an ODE for the variables z and η.

It is useful to compare the situation covered by Theorems 1 and 2 with the case of exact symmetry: the difference is that in the case of exact symmetry all terms of eq. (18) are independent of z; the same is true for all the terms at the r.h.s. of (22,23): then, in this case, the last equation for z and η turns out to be a quadrature, as is well known [1].

Clearly, if $\Lambda = 0$ i.e. if X is an exact symmetry, then $M_\alpha = M_q = 0$. Conversely, it can be shown that if $M_\alpha = M_q = 0$ then the symmetry X is exact. This is particularly clear in the case $\tau \equiv 0$ (and then $\eta = t$): indeed, in this case the conditions $M_\alpha = M_q = 0$ can be written (see (15)) $\mathbf{J}_{ab}(\Lambda\, Q)_b = 0$ where $\mathbf{J}$ is the (invertible !) Jacobian matrix of the transformation from u_a to

[1] If one is interested to know "a priori" the expressions of the q differential Λ-invariant quantities $\zeta_a = \zeta_a(\eta, z, w_\alpha, w'_\alpha, z')$, one has to express M_α, M_q in terms of w_α, η in order to solve (20).

w_α, z. Then $\Lambda\, Q = 0$, which is the same as $\Lambda = 0$ (recall that Λ is not uniquely defined).

Notice in particular that the term $(\Lambda\, Q)_a$ appearing in the expressions (13,14), when written in the new coordinates, becomes

$$(\widetilde{\Lambda}\, \widetilde{Q})_a \;=\; \widetilde{\Lambda}_{aq}$$

indeed $\widetilde{Q} \equiv (0, 0, \ldots, 1)$. This shows that only the last column of $\widetilde{\Lambda}$ is relevant.

Finally, let us recall the following result:

Theorem 3. (Muriel-Romero [15]) *If $\Lambda = \lambda\, I$, then $M_\alpha = 0$ for all $\alpha = 1, \ldots, q-1$, and the conclusion of the last part of Theorem 2 holds.*

Indeed, from $X\, w_\alpha = X\, \eta = 0$, $X\, z = 1$ and the definition of Q, one easily deduces

$$Q_a \frac{\partial w_\alpha}{\partial u_a} \;=\; -\tau\, D_t w_\alpha \qquad ; \qquad Q_a \frac{\partial \eta}{\partial u_a} \;=\; -\tau\, D_t \eta$$

and

$$Q_a \frac{\partial z}{\partial u_a} \;=\; -\tau\, D_t z + 1$$

hence, in the case $(\Lambda\, Q)_a = \lambda Q_a$ considered in [15], one gets

$$M_\alpha \;=\; 0 \qquad ; \qquad M_q \;=\; (D_t \eta)^{-2} \lambda \ .$$

Notice also that (16) becomes in this case

$$[X^{(1)}_\lambda,\, D_t] \;=\; -D_t(\tau) D_t + \lambda Q_a \frac{\partial}{\partial u_a} \;=\; -D_t(\tau) D_t + \lambda X - \lambda \tau D_t \ .$$

4 Examples

Example 1. This is a very simple example, which can provide a clear illustration of Theorem 1. Consider any system $F_a(t, u_1, u_2, \dot{u}_1, \dot{u}_2) = 0$ $(a = 1, 2)$ of two first-order ODE's for the variables $u_1 = u_1(t)\,, u_2 = u_2(t)$ and consider the vector field

$$X \;=\; \frac{\partial}{\partial u_2} \ .$$

It is easily seen that if one chooses

$$\Lambda \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

then, with our notation, $w_1 = u_1$, $\eta = t$ and $z = u_2$; eq. (13,14) give $M_1 = M_2 = 1$ and therefore from (20) $\zeta_1 = \dot{w}_1 - z = \dot{u}_1 - u_2$, $\zeta_2 = \dot{z} - z = \dot{u}_2 - u_2$. Then, Λ-invariance under X gives that the F_a depend only on the quantities t, $\dot{u}_1 - u_2$, $\dot{u}_2 - u_2$, in agreement with Theorem 1. Extension to more than 2 variables u_a is immediate.

Example 2. Consider a system of ODE's for the two variables $u_1 = u_1(t)$, $u_2 = u_2(t)$ of the form

$$h(s_1, s_2)\,(\dot u_1 - u_1 u_2) + a(t)(u_1^2 + u_2^2)u_1 + b^2(t)(u_1^2 + u_2^2)u_2 \;=\; 0$$
$$h(s_1, s_2)\,(\dot u_2 + u_1^2) + a(t)(u_1^2 + u_2^2)u_2 - b^2(t)(u_1^2 + u_2^2)u_1 \;=\; 0$$

where h is a function of $s_1 = u_1\dot u_1 + u_2\dot u_2$, $s_2 = u_1\dot u_2 - \dot u_1 u_2 + u_1^3 + u_1 u_2^2$ and where $a(t)$, $b(t)$ are arbitrary functions of t; it is clearly not symmetric under the rotation operator

$$X \;=\; u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}$$

(unless $h \equiv 0$), however it turns out to be Λ-symmetric (but not Λ-invariant) under rotations if $\Lambda = \lambda I$ with $\lambda = u_2$. Indeed, e.g., one has $X_\Lambda^{(1)}\, h = 0$, $X_\Lambda^{(1)}\,(\dot u_1 - u_1 u_2) = (\dot u_2 + u_1^2)$, etc. Introducing symmetry-adapted coordinates, which are $w_1 = r = (u_1^2 + u_2^2)^{1/2}$, $\eta = t$, $z = \theta$, with obvious notations, the system becomes

$$\dot r\, h(r\dot r, r^2(\dot\theta + r\cos\theta)) + a(t)r^3 \;=\; 0$$
$$(\dot\theta + r\cos\theta)\, h(r\dot r, r^2(\dot\theta + r\cos\theta)) - b^2(t)r^2 \;=\; 0$$

which turns to be Λ-invariant under $X = \partial/\partial\theta$ with $\lambda = r\sin\theta$. As expected, thanks to Theorem 1, this system contains only the four quantities r, t and $\zeta_1 = \dot r, \zeta_2 = \dot\theta + r\cos\theta$.

If, e.g., $h = s_2$, the system can be also put in the explicit form of a DS:

$$\dot r = \pm(a(t)/b(t))r \qquad \dot\theta \;=\; \pm b(t) - r\cos\theta$$

and – as a consequence – according to Theorems 2 and 3, one (and only one) of the above equations does not contain z (here: θ). Then the system can be easily solved.

Example 3. Consider any DS for $u_a = u_a(t)$, $a = 1, 2, 3$, of the form

$$\dot u_1 \;=\; h_1(t, w_1, w_2) + (a - 3b)u_2 u_3 + b\,u_3^3 + h_2(t, w_1, w_2)\,u_3 + h_3(t, w_1, w_2)\,u_3^2$$
$$\dot u_2 \;=\; h_2(t, w_1, w_2) + 2u_3 h_3(t, w_1, w_2) + a u_3^2$$
$$\dot u_3 \;=\; c u_3 + 2h_3(t, w_1, w_2)$$

where a, b, c are constants and h_a are functions of t, $w_1 = 2u_2 - u_3^2$, $w_2 = 3u_1 - 3u_2 u_3 + u_3^3$. Systems of this form are Λ-invariant under the vector field

$$X \;=\; u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3} \qquad \text{with} \qquad \Lambda = \begin{pmatrix} 0 & 0 & (a-3b)u_2 \\ 0 & 0 & (2a-c)u_3 \\ 0 & 0 & c \end{pmatrix}.$$

The X-invariant quantities are just w_1, w_2, together with $\eta = t$. The coefficients M_α, M_q (see eq.s (13,14)), with $z = u_3$, are

$$M_1 \;=\; 4(a - c)z \qquad , \qquad M_3 = c \qquad ,$$

$$M_2 \,=\, 3(a-3b-c)u_2 - 6(a-c)u_3^2 = 3(a-3b-c)(w_1+z^2)/2 - 6(a-c)z^2 \ .$$

The characteristic equations (20) can then be easily solved to obtain the three first-order differential Λ-invariant quantities $\zeta_1 = \dot{w}_1 - 2(a-c)z^2$, $\zeta_2 = \dot{w}_2 - (3/2)(a-3b-c)w_1 z - (3/2)(a+b-c)z^3$, $\zeta_3 = \dot{z} - c\,z$. Direct calculation shows that this system becomes

$$\zeta_a - g_a(t,w_1,w_2) \,=\, 0$$

where $g_1 = 2h_1$, $g_2 = 3h_1 - 3h_3 w_1$, $g_3 = 2h_3$, and then contains only the quantities t, w_1, w_2, ζ_a, in agreement with Theorem 1. If instead one prefers to write the system as an explicit DS, then it is

$$\begin{aligned} \dot{w}_1 \,&=\, 2(a-c)z^2 + g_1(t,w_1,w_2) \\ \dot{w}_2 \,&=\, \frac{3}{2}(a-3b-c)w_1 z + \frac{3}{2}(-a-b+c)z^3 + g_2(t,w_1,w_2) \\ \dot{z} \,&=\, cz + g_3(t,w_1,w_2) \ . \end{aligned}$$

Now, if $a=c$, the first equation does not contain z; if $a=c$ and $b=0$, only the third equation contains z, in agreement with Theorem 2. If $a=c=0$, then only the second equation contains z and "plays the role" of the $q-$th equation in our notation. The case $a=b=c=0$ is of course the case of exact symmetry $\Lambda = 0$.

Example 4. This is an example with non-autonomous DS and vector field X with $\tau \neq 0$ and therefore $\eta \neq t$. Consider the DS for $u_a = u_a(t)$, $a = 1,2,3$,

$$\begin{aligned} &\dot{u}_1 = t + h_1(s,w_1,w_2)\,\exp(-\lambda_1 t) \\ &\dot{u}_2 = 1 + h_2(s,w_1,w_2)\,\exp(-\lambda_2 t) \\ &\dot{u}_3 = u_2 + h_2\exp(-\lambda_2 t)\frac{1-\exp(u_2\big(\lambda_2-\lambda_3)\big)}{\lambda_3-\lambda_2} + h_3\exp(-\lambda_3 u_2) \end{aligned}$$

where h_a are nonvanishing functions of $s = u_2 - t$, $w_1 = u_1 - t^2/2$, $w_2 = u_3 - u_2^2/2$. This system is Λ-invariant under

$$X \,=\, \frac{\partial}{\partial t} + t\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_3}$$

with $\Lambda = \mathrm{diagonal}(\lambda_1,\lambda_2,\lambda_3)$. We can choose as invariants under X just w_1, w_2 and $\eta = u_2 - t$, with $z = u_2$. It is now more useful to rewrite the system in these coordinates preserving its form of explicit DS, we get then

$$\begin{aligned} w_1' \,&=\, (h_1/h_2)\exp\left((\lambda_2-\lambda_1)(z-\eta)\right) \\ w_2' \,&=\, \Big(\frac{1-\exp(\lambda_2-\lambda_3)z}{\lambda_3-\lambda_2} - z\Big) + (h_3/h_2)\exp\big(z(\lambda_2-\lambda_3)-\lambda_2\eta)\big) \\ z' \,&=\, (1/h_2)\exp\big(\lambda_2(z-\eta)\big) \ . \end{aligned}$$

We see that if $\lambda_1 = \lambda_2$, or $\lambda_2 = \lambda_3$ (notice that the case $\lambda_2 = \lambda_3$ is well defined), one of the above equations does not contain z in agreement with Theorem 2; whereas if $\lambda_1 = \lambda_2 = \lambda_3$ only one equation contains z, as stated by Theorem 3; the same happens also if $\lambda_1 = \lambda_2 = 0$, according to Theorem 2.

5 Concluding remarks

An interesting property which relates invariance with λ-symmetries is the following. Consider the case of an autonomous DS $\dot{u}_a = f_a(u)$ which is Λ-invariant under a vector field X of the form $X = \varphi_a(u)\partial/\partial u_a$. If $w = w(u)$ is any invariant under X, i.e. $\varphi_a \partial w/\partial u_a$, then its *Lie derivative along* f_a, i.e.

$$D_t^{(f)}\, w \equiv f_a \frac{\partial w}{\partial u_a}$$

satisfies

$$X\left(D_t^{(f)}\, w\right) \;=\; \left(\varphi_a \frac{\partial}{\partial u_a}\right)\left(f_b \frac{\partial}{\partial u_b}\right) w \;=\; \Lambda_{ab}\varphi_b \frac{\partial w}{\partial u_a}$$

hence $X\left(D_t^{(f)}\, w\right) = 0$ if $\Lambda = \lambda\, I$, having used the commutation rule (9) and the invariance property of w. On the other hand, eq. (17) gives directly, for X of the above form,

$$X_\Lambda^{(1)}\,(D_t\, w) \;=\; (\Lambda\varphi)_a \frac{\partial w}{\partial u_a}\ .$$

The strong difference is that the latter result is purely algebraic, being a consequence of the relation (16), and expresses a property of the vector field X which holds *independently* of the presence of any DS (i.e., of any choice of the functions f_a). The former result, instead, states that the time evolution under the dynamics described by the DS $\dot{u}_a = f_a$ of a quantity $w(u)$ which is invariant under a vector field X *preserves* this invariance *even if X is not* a (standard) symmetry of the DS; it is enough to require that X is a λ-symmetry of the DS.

It can be noticed that the present statement, concerning Lie derivatives, can be suitably extended to the case of several vector fields X for the given DS (see [20], Prop. 2.1).

Several other aspects of λ-symmetries (and of all their generalizations as well) could be further investigated. Apart from their geometrical interpretation (see the papers quoted in the Introduction), their action on changes of coordinates should be better understood, as well as their general role in finding solutions of differential equations which do not admit standard symmetries: see e.g. [2, 3, 5, 6] and the references therein; compare also, for instance, with [21], for what concerns the problem of finding integrating factors for ODE's and its relationship with symmetry properties.

It can be observed, finally, that any ODE $\Delta(t, u, \dot{u}, \ddot{u}, \ldots) = 0$ of arbitrary order > 1 can be transformed into a system of first-order ODE's, and therefore our results could be applied also to this case. This is true in principle: the only nearly obvious remark is that one has to consider no longer vector fields of the form $X = \tau(\partial/\partial t) + \varphi(\partial/\partial u)$ involving only the two variables t and u, but also extended vector fields $X = \tau(\partial/\partial t) + \varphi_{(0)}(\partial/\partial u) + \varphi_{(1)}(\partial/\partial \dot{u}) + \varphi_{(2)}(\partial/\partial \ddot{u}) + \ldots$. It is "conceptually" different to look for vector fields of the former or of the latter form; on the other hand, the "concrete effect" of the existence of a symmetry is different in the two contexts (i.e., lowering the order in the case of the ODE's, and respectively reducing the presence of the involved variables in the case of first-order systems, as shown). This holds in particular for λ- and Λ-symmetries, where also the prolongation rules of the vector fields are markedly different in the two cases. To emphasize this different role of Λ-symmetries in the context of first-order systems, it should be perhaps more appropriate to call them ρ-symmetries (where ρ stands for "reducing", in contrast with λ, which could stand for "lowering").

Acknowledgments

It is a pleasure to thank Giuseppe Gaeta and Diego Catalano Ferraioli for detailed discussions and useful comments. Thanks are also due to one of the referees for his/her accurate and appropriate comments and suggestions, which helped me to improve the presentation of these results.

References

[1] P.J. Olver, Application of Lie groups to Differential Equations, Springer, Berlin 1986

[2] C. Muriel, J.L. Romero, IMA J. Appl. Math. 66 (2001) 111, and J. Appl. Math. 66 (2001) 477

[3] C. Muriel, J.L. Romero, Theor. Math. Phys. 133 (2002) 1565; and J. Lie theory 13 (2003) 167

[4] B. Abraham-Shrauner, A. Guo, in: Modern Group Analysis, (N.H. Ibragimov et al. ed.s) Dordrecht, Kluwer, 1993, p.1

[5] M.L. Gandarias, E. Medina, C. Muriel, J. Nonlin. Math. Phys. 9 (2002) 47

[6] E. Pucci, G. Saccomandi, J. Phys. A 35 (2002) 6145

[7] G. Gaeta, P. Morando, J. Phys. A 37 (2004) 6955

[8] G. Cicogna, G. Gaeta, P. Morando, J. Phys. A 37 (2004) 9467

[9] P. Morando, J. Phys. A 40 (2007) 11547, and in: Symmetry and Perturbation Theory (SPT2007) (G. Gaeta et al. eds.), World Scientific, Singapore 2007, p. 140

[10] D. Catalano Ferraioli, J. Phys. A 40 (2007) 5479

[11] G. Gaeta, Int. J. Geom. Meth. Mod. Phys. 4 (2007) 807

[12] C. Muriel, J.L. Romero, P.J. Olver, J. Diff. Eqs. 222 (2006) 164

[13] G. Cicogna, G. Gaeta, J. Phys. A 40 (2007) 11899

[14] G.W. Bluman, S.C. Anco, Symmetry and Integration Methods for Differential Equations, Springer, New York, 2002

[15] C. Muriel, J.L. Romero, in: Proceedings of the I Colloquium on Lie Theory and Applications (I. Bajo and E. Sanmartin eds.), Publicacións Universidade de Vigo, Vigo 2002, p. 143

[16] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York 1955

[17] J.F. Cariñena, M.A. Del Olmo, P. Winternitz, Lett. Math. Phys. 29 (1993) 151

[18] G. Cicogna, M. Laino, Rev. Math. Phys. 18 (2006) 1

[19] G. Cicogna, G. Gaeta, Symmetry and Perturbation Theory in Nonlinear Dynamics, Lecture Notes in Physics, LNP monographs Vol. 57, Springer, Berlin 1999

[20] K.P. Hadeler, S. Walcher, J. Nonlin. Sci. 16 (2006) 583

[21] E.S. Cheb–Terrab, A.D. Roche, J. Symb. Comput. 27 (1999) 501