Local semicircle law and complete delocalization for Wigner random matrices

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Abstract

We consider $N \times N$ Hermitian random matrices with i.i.d. entries. The matrix is normalized so that the average spacing between consecutive eigenvalues is of order $1/N$. Under suitable assumptions on the distribution of the single matrix element, we prove that, away from the spectral edges, the density of eigenvalues concentrates around the Wigner semicircle law on energy scales $\eta \gg N^{-1}(\log N)^8$. Up to the logarithmic factor, this is the smallest energy scale for which the semicircle law may be valid. We also prove that for all eigenvalues away from the spectral edges, the ℓ^∞-norm of the corresponding eigenvectors is of order $O(N^{-1/2})$, modulo logarithmic corrections. The upper bound $O(N^{-1/2})$ implies that every eigenvector is completely de-localized, i.e., the maximum size of the components of the eigenvector is of the same order as their average size.

In the Appendix, we include a lemma by J. Bourgain which removes one of our assumptions on the distribution of the matrix elements.

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1 Introduction

The Wigner semicircle law states that the empirical density of the eigenvalues of a random matrix is given by the universal semicircle distribution. This statement has been proved for many different ensembles, in particular for the case when the distributions of the entries of the matrix are independent, identically distributed (i.i.d.). To fix the scaling, we normalize the matrix so that the bulk of the spectrum lies in the energy interval $[-2, 2]$, i.e. the average spacing between consecutive eigenvalues is of order $1/N$. We now consider a window of size η in the bulk so that the typical number of eigenvalues is $N\eta$. In the usual statement of the semicircle law, η is a fixed number independent of N and it is taken to zero only after the limit $N \to \infty$. This can be viewed as the largest scale on which the semicircle law is valid. On the other extreme, for the smallest scale, one may take $\eta = k/N$ and take the limit $N \to \infty$ followed by $k \to \infty$. If the semicircle law is valid in this sense, we shall say that the *local semicircle law* holds. Below this smallest scale, the eigenvalue distribution is expected to be governed by the Dyson's statistics related to sine kernels. The Dyson statistics was proved for many ensembles (see [1, 4] for a review), including Wigner matrices with Gaussian convoluted distributions [7].

In this paper, we establish the local semicircle law up to logarithmic factors in the scales, i.e., for $\eta \sim N^{-1}(\log N)^8$. The result holds for any energy window in the bulk spectrum away from the spectral edges. In [5] we have proved the same statement for $\eta \gg N^{-2/3}$ (modulo logarithmic corrections). Prior to our work the best result was obtained in [2] for $\eta \gg N^{-1/2}$. See also [6] and [8] for related and earlier results. As a corollary, our result also proves that no gap between consecutive bulk eigenvalues can be bigger than $(\log N)^8/N$, to be compared with the expected average $1/N$ behavior given by Dyson's law.

It is widely believed that the eigenvalue distribution of the Wigner random matrix and the random Schrödinger operator in the extended (or delocalized) state regime are the same up to normalizations. Although this conjecture is far from the reach of the current method, a natural question arises as to whether the eigenvectors of random matrices are extended. More precisely, if $\mathbf{v} = (v_1, \cdots, v_N)$ is a normalized eigenvector, we say that $\mathbf{v}$ is *completely delocalized* if $\|\mathbf{v}\|_\infty = \max_j |v_j|$ is bounded from above by $CN^{-1/2}$, the average size of $|v_j|$. In this paper, we shall prove that all eigenvectors with eigenvalues away from the spectral edges are completely delocalized (modulo logarithmic corrections) in probability. Similar results, but with $CN^{-1/2}$ replaced by $CN^{-1/3}$ were proved in [5]. Notice that our new result, in particular, answers (up to logarithmic factors) the question posed by T. Spencer that $\|\mathbf{v}\|_4$ should be of order $N^{-1/4}$.

Denote the (ij)-th entry of an $N \times N$ matrix H by h_{ij}. We shall assume that the matrix is Hermitian, i.e., $h_{ij} = \overline{h_{ji}}$. These matrices form a *Hermitian Wigner ensemble* if

$$h_{ij} = N^{-1/2}[x_{ij} + \sqrt{-1}\, y_{ij}], \quad (i < j), \quad \text{and} \quad h_{ii} = N^{-1/2} x_{ii}, \tag{1.1}$$

where x_{ij}, y_{ij} $(i < j)$ and x_{ii} are independent real random variables with mean zero. We assume that x_{ij}, y_{ij} $(i < j)$ all have a common distribution ν with variance $1/2$ and with a strictly positive density function: $\mathrm{d}\nu(x) = (\text{const.})e^{-g(x)}\mathrm{d}x$. The diagonal elements, x_{ii}, also have a common distribution, $\mathrm{d}\widetilde{\nu}(x) = (\text{const.})e^{-\widetilde{g}(x)}\mathrm{d}x$, that may be different from $\mathrm{d}\nu$. Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability and the expectation value, respectively, w.r.t the joint distribution of all matrix elements.

We need to assume further conditions on the distributions of the matrix elements in addition to (1.1).

C1) The function g is twice differentiable and it satisfies

$$g''(x) \le M \tag{1.2}$$

with some finite M.

C2) There exists a $\delta > 0$ such that

$$\int e^{\delta x^2} \mathrm{d}\nu(x) < \infty, \tag{1.3}$$

and the same holds for $\widetilde{\nu}$.

C4) The measure ν satisfies the logarithmic Sobolev inequality, i.e. there exists a constant C such that for any density function $u > 0$ with $\int u \, \mathrm{d}\nu = 1$,

$$\int u \log u \, \mathrm{d}\nu \leq C \int |\nabla \sqrt{u}|^2 \mathrm{d}\nu \, . \tag{1.4}$$

Here we have followed the convention in [5] to use the label C4) and reserved C3) for an spectral gap condition in [5].

Notation. We will use the notation $|A|$ both for the Lebesgue measure of a set $A \subset \mathbb{R}$ and for the cardinality of a discrete set $A \subset \mathbf{Z}$. The usual Hermitian scalar product for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ will be denoted by $\mathbf{x} \cdot \mathbf{y}$ or by $(\mathbf{x}, \mathbf{y})$. We will use the convention that C denotes generic large constants and c denotes generic small positive constants whose values may change from line to line. Since we are interested in large matrices, we always assume that N is sufficiently large.

Let H be the $N \times N$ Wigner matrix with eigenvalues $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$. For any spectral parameter $z = E + i\eta \in \mathbb{C}$, $\eta > 0$, we denote the Green function by $G_z = (H - z)^{-1}$. Let $F(x) = F_N(x)$ be the empirical distribution function of the eigenvalues

$$F(x) = \frac{1}{N} \big| \{\alpha \, : \, \mu_\alpha \leq x\} \big| \, . \tag{1.5}$$

We define the Stieltjes transform of F as

$$m = m(z) = \frac{1}{N} \mathrm{Tr} \, G_z = \int_{\mathbb{R}} \frac{\mathrm{d}F(x)}{x - z} \, , \tag{1.6}$$

and we let

$$\rho = \rho_\eta(E) = \frac{\mathrm{Im} \, m(z)}{\pi} = \frac{1}{N\pi} \mathrm{Im} \, \mathrm{Tr} \, G_z = \frac{1}{N\pi} \sum_{\alpha=1}^{N} \frac{\eta}{(\mu_\alpha - E)^2 + \eta^2} \tag{1.7}$$

be the normalized density of states of H around energy E and regularized on scale η. The random variables m and ϱ also depend on N, when necessary, we will indicate this fact by writing m_N and ϱ_N.

For any $z = E + i\eta$ we let

$$m_{sc} = m_{sc}(z) = \int_{\mathbb{R}} \frac{\varrho_{sc}(x) \mathrm{d}x}{x - z}$$

be the Stieltjes transform of the Wigner semicircle distribution function whose density is given by

$$\varrho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}(|x| \leq 2) \, .$$

For $\kappa, \widetilde{\eta} > 0$ we define the set

$$S_{N,\kappa,\widetilde{\eta}} := \Big\{ z = E + i\eta \in \mathbb{C} \, : \, |E| \leq 2 - \kappa, \, \widetilde{\eta} \leq \eta \leq 1 \Big\}$$

and for $\widetilde{\eta} = N^{-1}(\log N)^8$ we write

$$S_{N,\kappa} := \Big\{ z = E + i\eta \in \mathbb{C} \ : \ |E| \le 2 - \kappa, \ \frac{(\log N)^8}{N} \le \eta \le 1 \Big\}.$$

The following two theorems are the main results of this paper.

Theorem 1.1 *Let H be an $N \times N$ Wigner matrix as described in* (1.1) *and assume the conditions* (1.2), (1.3) *and* (1.4). *Then for any $\kappa > 0$ and $\varepsilon > 0$, the Stieltjes transform $m_N(z)$ (see* (1.6)*) of the empirical eigenvalue distribution of the $N \times N$ Wigner matrix satisfies*

$$\mathbb{P}\Big\{ \sup_{z \in S_{N,\kappa}} |m_N(z) - m_{sc}(z)| \ge \varepsilon \Big\} \le e^{-c(\log N)^2} \tag{1.8}$$

where $c > 0$ depends on κ, ε. In particular, the density of states $\varrho_\eta(E)$ converges to the Wigner semicircle law in probability uniformly for all energies away from the spectral edges and for all energy windows at least $N^{-1}(\log N)^8$.

Furthermore, let $\eta^* = \eta^*(N)$ such that $(\log N)^8/N \ll \eta^* \ll 1$ as $N \to \infty$, then we have the convergence of the counting function as well:

$$\mathbb{P}\Big\{ \sup_{|E| \le 2-\kappa} \Big| \frac{\mathcal{N}_{\eta^*}(E)}{2N\eta^*} - \varrho_{sc}(E) \Big| \ge \varepsilon \Big\} \le e^{-c(\log N)^2} \tag{1.9}$$

for any $\varepsilon > 0$, where $\mathcal{N}_{\eta^*}(E) = |\{\alpha \ : \ |\mu_\alpha - E| \le \eta^*\}|$ denotes the number of eigenvalues in the interval $[E - \eta^*, E + \eta^*]$.

Theorem 1.2 *Let H be an $N \times N$ Wigner matrix as described in* (1.1) *and satisfying the conditions (1.2), (1.3) and (1.4). Fix $\kappa > 0$, and assume that C is large enough. Then there exists $c > 0$ such that*

$$\mathbb{P}\left\{ \exists \ \mathbf{v} \text{ with } H\mathbf{v} = \mu\mathbf{v}, \ \|\mathbf{v}\| = 1, \ \mu \in [-2 + \kappa, 2 - \kappa] \text{ and } \|\mathbf{v}\|_\infty \ge \frac{C(\log N)^{9/2}}{N^{1/2}} \right\} \le e^{-c(\log N)^2} \ .$$

Condition C1) is needed only because we use Lemma 2.3 of [5] in the following proof. J. Bourgain [3] has informed us that it can also be proved without this condition. We include his proof in the Appendix.

2 Proof of Theorem 1.1

The proof of (1.9) follows from (1.8) exactly as in Corollary 4.2 of [5], so we focus on proving (1.8). We first remove the supremum in (1.8).

For any two points $z, z' \in S_{N,\kappa,\eta}$ we have

$$|m_N(z) - m_N(z')| \le N^2 |z - z'|$$

since the gradient of $m_N(z)$ is bounded by $|\mathrm{Im}\ z|^{-2} \le N^2$ on $S_{N,\kappa}$. We can choose a set of at most $M = C\varepsilon^{-2}N^4$ points, $z_1, z_2, \ldots, z_M$, in $S_{N,\kappa,\eta}$ such that for any $z \in S_{N,\kappa,\eta}$ there exists a point z_j with $|z - z_j| \le \frac{1}{4}\varepsilon N^{-2}$. In particular, $|m_N(z) - m_N(z_j)| \le \varepsilon/4$ if N is large enough and $|m_{sc}(z) - m_{sc}(z_j)| \le \varepsilon/4$. Then

$$\mathbb{P}\Big\{ \sup_{z \in S_{N,\kappa}} |m_N(z) - m_{sc}(z)| \ge \varepsilon \Big\} \le \sum_{j=1}^{M} \mathbb{P}\Big\{ |m_N(z_j) - m_{sc}(z_j)| \ge \frac{\varepsilon}{2} \Big\} \le e^{-c(\log N)^2}$$

under the condition that $\eta \geq N^{-1}(\log N)^8$ since $\operatorname{Im} z_j \geq \eta$. So it is enough to prove

$$\mathbb{P}\Big\{|m_N(z) - m_{sc}(z)| \geq \varepsilon\Big\} \leq e^{-c(\log N)^2} \tag{2.1}$$

for each fixed $z \in S_{N,\kappa}$.

Let $B^{(k)}$ denote the $(N-1)\times(N-1)$ minor of H after removing the k-th row and k-th column. Note that $B^{(k)}$ is an $(N-1)\times(N-1)$ Hermitian Wigner matrix with a normalization factor off by $(1-\frac{1}{N})^{1/2}$. Let $\lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \ldots \leq \lambda_{N-1}^{(k)}$ denote its eigenvalues and $\mathbf{u}_1^{(k)}, \ldots, \mathbf{u}_{N-1}^{(k)}$ the corresponding normalized eigenvectors.

Let $\mathbf{a}^{(k)} = (h_{k1}, h_{k2}, \ldots h_{k(k-1)}, h_{k(k+1)}, \ldots h_{kN})^* \in \mathbb{C}^{N-1}$, i.e. the k-th column after removing the diagonal element h_{kk}. Computing the (k,k) diagonal element of the resolvent G_z, we have

$$G_z(k,k) = \frac{1}{h_{kk} - z - \mathbf{a}^{(k)} \cdot (B^{(k)} - z)^{-1}\mathbf{a}^{(k)}} = \Big[h_{kk} - z - \frac{1}{N}\sum_{\alpha=1}^{N-1} \frac{\xi_\alpha^{(k)}}{\lambda_\alpha^{(k)} - z}\Big]^{-1} \tag{2.2}$$

where we defined

$$\xi_\alpha^{(k)} := \big|\sqrt{N}\mathbf{a}^{(k)} \cdot \mathbf{u}_\alpha^{(k)}\big|^2.$$

Similarly to the definition of $m(z)$ in (1.6), we also define the Stieltjes transform of the density of states of $B^{(k)}$

$$m^{(k)} = m^{(k)}(z) = \frac{1}{N-1}\operatorname{Tr}\frac{1}{B^{(k)} - z} = \int_{\mathbb{R}} \frac{\mathrm{d}F^{(k)}(x)}{x - z}$$

with the empirical counting function

$$F^{(k)}(x) = \frac{1}{N-1}\big|\left\{\alpha \,:\, \lambda_\alpha^{(k)} \leq x\right\}\big|.$$

The spectral parameter z is fixed throughout the proof and we will omit from the argument of the Stieltjes transforms.

It follows from (2.2) that

$$m = m(z) = \frac{1}{N}\sum_{k=1}^{N} G_z(k,k) = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{h_{kk} - z - \mathbf{a}^{(k)} \cdot (B^{(k)} - z)^{-1}\mathbf{a}^{(k)}}\,. \tag{2.3}$$

Let $\mathbb{E}_k$ denote the expectation value w.r.t the random vector $\mathbf{a}^{(k)}$. Define the random variable

$$X_k := \mathbf{a}^{(k)} \cdot \frac{1}{B^{(k)} - z}\mathbf{a}^{(k)} - \mathbb{E}_k\, \mathbf{a}^{(k)} \cdot \frac{1}{B^{(k)} - z}\mathbf{a}^{(k)} = \frac{1}{N}\sum_{\alpha=1}^{N-1} \frac{\xi_\alpha^{(k)} - 1}{\lambda_\alpha^{(k)} - z} \tag{2.4}$$

where we used that $\mathbb{E}_k \xi_\alpha^{(k)} = \|\mathbf{u}_\alpha^{(k)}\|^2 = 1$. We note that

$$\mathbb{E}_k\, \mathbf{a}^{(k)} \cdot \frac{1}{B^{(k)} - z}\mathbf{a}^{(k)} = \frac{1}{N}\sum_{\alpha} \frac{1}{\lambda_\alpha^{(k)} - z} = \Big(1 - \frac{1}{N}\Big)m^{(k)}$$

With this notation it follows from (2.2) that

$$m = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{h_{kk} - z - \big(1 - \frac{1}{N}\big)m^{(k)} - X_k}\,. \tag{2.5}$$

We use that

$$\Big|m - \Big(1 - \frac{1}{N}\Big)m^{(k)}\Big| = \Big|\int \frac{\mathrm{d}F(x)}{x-z} - \Big(1-\frac{1}{N}\Big)\int \frac{\mathrm{d}F^{(k)}(x)}{x-z}\Big| = \frac{1}{N}\Big|\int \frac{NF(x) - (N-1)F^{(k)}(x)}{(x-z)^2}\mathrm{d}x\Big|.$$

We recall that the eigenvalues of H and $B^{(k)}$ are interlaced,

$$\mu_1 \le \lambda_1^{(k)} \le \mu_2 \le \lambda_2^{(k)} \le \ldots \le \lambda_{N-1}^{(k)} \le \mu_N, \tag{2.6}$$

(see e.g. Lemma 2.5 of [5]), therefore we have $\max_x |NF(x) - (N-1)F^{(k)}(x)| \le 1$. Thus

$$\Big|m - \Big(1 - \frac{1}{N}\Big)m^{(k)}\Big| \le \frac{1}{N}\int \frac{\mathrm{d}x}{|x-z|^2} \le \frac{C}{N\eta}\,.$$

We postpone the proof of the following lemma:

Lemma 2.1 *Suppose that $\mathbf{v}_\alpha$ and λ_α are eigenvectors and eigenvalues of an $N \times N$ random matrix with a law satisfying the assumption of Theorem 1.1. Let*

$$X = \frac{1}{N}\sum_\alpha \frac{\xi_\alpha - 1}{\lambda_\alpha - z}$$

with $z = E + i\eta$, $\xi_\alpha = |\mathbf{b}\cdot\mathbf{v}_\alpha|^2$, where the components of $\mathbf{b}$ are i.i.d. random variables satisfying (1.3) *and* (1.4)*. Then in the joint product probability space of $\mathbf{b}$ and the law of the random matrices we have*

$$\mathbb{P}[|X| \ge \varepsilon] \le e^{-c\varepsilon(\log N)^2}$$

with a sufficiently small c and $\varepsilon \le \varepsilon_0$ for any $\eta \ge (\log N)^8/N$.

For given $\varepsilon > 0$ we define the event

$$\Omega = \bigcup_{k=1}^N \{|X^{(k)}| \ge \varepsilon/3\} \cup \{|h_{kk}| \ge \varepsilon/3\}\,.$$

Since $h_{kk} = N^{-1/2}b_{kk}$ with b_{kk} satisfying (1.3), we have

$$\mathbb{P}\{|h_{kk}| \ge \varepsilon/3\} \le Ce^{-\delta\varepsilon^2 N/9}.$$

We now apply Lemma 2.1 for each $X^{(k)}$ and conclude that

$$\mathbb{P}(\Omega) \le e^{-c\varepsilon(\log N)^2}$$

with a sufficiently small $c > 0$. On the complement Ω^c we have from (2.5)

$$m = \frac{1}{N}\sum_{k=1}^N \frac{1}{-m - z + \delta_k}$$

where δ_k are random variables satisfying $|\delta_k| \le \varepsilon$. After expansion

$$\Big|m + \frac{1}{m+z}\Big| \le \frac{\varepsilon}{|m+z|^2}\,. \tag{2.7}$$

We note that the equation

$$M + \frac{1}{M+z} = 0 \tag{2.8}$$

has a unique solution for any $z \in S_{N,\kappa}$ with $\operatorname{Im} M > 0$, namely $M = m_{sc}(z)$, the Stieltjes transform of the semicircle law. Note that there exists $c(\kappa) > 0$ such that $\operatorname{Im} m_{sc}(E+i\eta) \geq c(\kappa)$ for any $|E| \leq 2-\kappa$, uniformly in η.

The equation (2.8) is stable in the following sense. For any small δ, let $M = M(z,\delta)$ be a solution to

$$M + \frac{1}{M+z} = \delta \tag{2.9}$$

with $\operatorname{Im} M > 0$. Explicitly, we have

$$M = \frac{-z + \sqrt{z^2 - 4 + 2z\delta + \delta^2}}{2} + \frac{\delta}{2},$$

where we have chosen the square root so that $\mathrm{I}mM > 0$ when $\delta = 0$ and $\mathrm{I}mz > 0$. On the compact set $z \in S_{N,\kappa}$, $|z^2 - 4|$ is bounded away from zero and thus

$$|M - m_{sc}| \leq C_\kappa \delta \tag{2.10}$$

for some constant C_κ depending only on κ.

Now we perform a continuity argument in η to prove that

$$|\, m(E+i\eta) - m_{sc}(E+i\eta)| \leq C^* \varepsilon \tag{2.11}$$

uniformly in $z \in S_{N,\kappa}$ with a sufficiently large constant C^*. Fix E with $|E| \leq 2-\kappa$. For $\eta = [\frac{1}{2}, 1]$, (2.11) follows from (2.7) with some small ε, since the right hand side of (2.7) is bounded by $C\varepsilon$. Suppose now that (2.11) has been proven for some $\eta \in [2N^{-1}(\log N)^8,\, 1]$ and we want to prove it for $\eta/2$. By integrating the inequality

$$\frac{\eta/2}{(x-E)^2 + (\eta/2)^2} \geq \frac{1}{2} \frac{\eta}{(x-E)^2 + \eta^2}$$

with respect to $\mathrm{d}F(x)$ we obtain that

$$\operatorname{Im} m\big(E + i\frac{\eta}{2}\big) \geq \frac{1}{2} \operatorname{Im} m(E+i\eta) \geq \frac{1}{2} c(\kappa) - C^*\varepsilon > \frac{c(\kappa)}{4}$$

for sufficiently small ε, where (2.11) and $\mathrm{I}m\, m_{sc}(E+i\eta) \geq c(\kappa)$ were used. Thus the right hand side of (2.7) for $z = E + i\frac{\eta}{2}$ is bounded by $C\varepsilon$, the constant depending only on κ. Applying the stability bound (2.10), we get (2.11) for η replaced with $\eta/2$. □

3 Proof of Lemma 2.1

Let $I_n = [n\eta, (n+1)\eta]$ and K_0 be a sufficiently large number. We have $[-K_0, K_0] \subset \cup_{n=-m}^{m} I_n$ with $m \leq CK_0/\eta$. Denote by Ω the event

$$\Omega := \Big\{ \max_n \mathcal{N}_{I_n} \geq N\eta(\log N)^2 \Big\} \cup \{ \max_\alpha |\lambda_\alpha| \geq K_0 \}$$

where $\mathcal{N}_{I_n} = |\{\alpha \,:\, \lambda_\alpha \in I_n\}|$ is the number of eigenvalues in the interval I_n. From Theorem 2.1 and Lemma 7.4 of [5], the probability of Ω is bounded by

$$\mathbb{P}(\Omega) \le e^{-c(\log N)^2}.$$

Therefore,

$$\begin{aligned} \mathbb{P}[|X| \ge \varepsilon] &\le e^{-c(\log N)^2} + \mathbb{E}\Big[\mathbf{1}_{\Omega^c}\mathbb{P}_{\mathbf{b}}[|X| \ge \varepsilon]\Big] \\ &\le e^{-c(\log N)^2} + e^{-\varepsilon T}\mathbb{E}\Big[\mathbf{1}_{\Omega^c}\cdot \mathbb{E}_{\mathbf{b}} e^{T|X|}\Big] \end{aligned} \tag{3.1}$$

for any $T > 0$, where $\mathbb{P}_{\mathbf{b}}$ and $\mathbb{E}_{\mathbf{b}}$ denote the probability and the expectation w.r.t. the variable $\mathbf{b}$.

On Ω^c we have

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big[e^{-\beta}\log \mathbb{E}_{\mathbf{b}} \exp\big(e^{\beta}\mathrm{Re}X\big)\Big] = e^{-\beta}\mathbb{E}_{\mathbf{b}} u\log u \le Ce^{-\beta}\mathbb{E}_{\mathbf{b}}|\nabla\sqrt{u}|^2$$

with

$$u = \frac{\exp\big(e^{\beta}\mathrm{Re}X\big)}{\mathbb{E}_{\mathbf{b}}\exp\big(e^{\beta}\mathrm{Re}X\big)}$$

and C being the constant in the logarithmic Sobolev inequality (1.4). Simple computation yields that

$$\begin{aligned} e^{-\beta}\mathbb{E}_{\mathbf{b}}|\nabla\sqrt{u}|^2 &\le e^{\beta}\mathbb{E}_{\mathbf{b}}\bigg[u\sum_k\Big(\Big|\frac{\partial X}{\partial\,\mathrm{Re}b_k}\Big|^2 + \Big|\frac{\partial X}{\partial\,\mathrm{Im}b_k}\Big|^2\Big)\bigg] \\ &= \frac{e^{\beta}}{N^2}\mathbb{E}_{\mathbf{b}}\bigg[u\sum_k\sum_{\alpha,\beta}\sum_{i,j}\Big(\frac{b_i\bar{b}_j\bar{\mathbf{v}}_\alpha(k)\mathbf{v}_\beta(k)\mathbf{v}_\alpha(j)\bar{\mathbf{v}}_\beta(i)}{(\lambda_\alpha - z)(\lambda_\beta - \bar{z})} + \frac{b_i\bar{b}_j\bar{\mathbf{v}}_\alpha(i)\mathbf{v}_\beta(j)\mathbf{v}_\alpha(k)\bar{\mathbf{v}}_\beta(k)}{(\lambda_\alpha - z)(\lambda_\beta - \bar{z})}\Big)\bigg] \\ &= \frac{2e^{\beta}}{N^2}\mathbb{E}_{\mathbf{b}}\bigg[u\sum_\alpha\sum_{i,j}\frac{b_i\bar{b}_j\bar{\mathbf{v}}_\alpha(i)\mathbf{v}_\alpha(j)}{|\lambda_\alpha - z|^2}\bigg] \\ &= \frac{e^{\beta}}{N^2}\mathbb{E}_{\mathbf{b}}\bigg[u\sum_\alpha\frac{\xi_\alpha}{|\lambda_\alpha - z|^2}\bigg] \le \frac{e^{\beta}}{N\eta}\mathbb{E}_{\mathbf{b}}\big[uY\big], \end{aligned}$$

where Y denotes

$$Y = \frac{1}{N}\sum_\alpha\frac{\xi_\alpha}{|\lambda_\alpha - z|}\,.$$

Recall the entropy inequality that for any $\gamma > 0$,

$$\mathbb{E}_{\mathbf{b}}\,[uY] \le \gamma^{-1}\mathbb{E}_{\mathbf{b}}\,u\log u + \gamma^{-1}\mathbb{E}_{\mathbf{b}}\,e^{\gamma Y}$$

Let $\gamma = 2Ce^{2\beta}/N\eta$ where $C \ge 1$ is the log-Sobolev constant from (1.4). We thus have

$$e^{-\beta}\mathbb{E}_{\mathbf{b}}\,|\nabla\sqrt{u}|^2 \le \frac{e^{\beta}}{N\eta\gamma}\Big[\mathbb{E}_{\mathbf{b}}\,u\log u + \mathbb{E}_{\mathbf{b}}e^{\gamma Y}\Big] \le \frac{e^{-\beta}}{2}\mathbb{E}_{\mathbf{b}}\,|\nabla\sqrt{u}|^2 + \frac{e^{-\beta}}{2}\mathbb{E}_{\mathbf{b}}\,e^{\gamma Y}.$$

Hence $\mathbb{E}_{\mathbf{b}}\,|\nabla\sqrt{u}|^2 \le \mathbb{E}_{\mathbf{b}}\,e^{\gamma Y}$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big[e^{-\beta}\log\mathbb{E}_{\mathbf{b}}\,\exp\big[e^{\beta}X\big]\Big] \le \frac{e^{-\beta}}{2}\mathbb{E}_{\mathbf{b}}\,e^{\gamma Y}.$$

We now choose β_0 such that

$$e^{\beta_0} = \frac{(N\eta)^{1/2}}{(\log N)^2}.$$

Integrating β from M to β_0 and using that

$$\gamma \leq \frac{2C}{(\log N)^4}$$

for all $\beta \leq \beta_0$, we have

$$e^{-\beta_0} \log \mathbb{E}_{\mathbf{b}} \exp\left[e^{\beta_0} \mathrm{Re} X\right] \leq e^{-M} \log \mathbb{E}_{\mathbf{b}} \exp\left[e^{M} \mathrm{Re} X\right] + e^{-\beta_0} \mathbb{E}_{\mathbf{b}} \exp\left[\frac{2C}{(\log N)^4} Y\right]. \tag{3.2}$$

The first term on the right hand side is expressed as

$$e^{-M} \log \mathbb{E}_{\mathbf{b}} \exp\left[e^{M} \mathrm{Re} X\right] = e^{-M} \log\left[1 + \mathbb{E}_{\mathbf{b}}\left[e^{M} \mathrm{Re} X\right] + \mathbb{E}_{\mathbf{b}}\Big(\exp\left[e^{M} \mathrm{Re} X\right] - 1 - e^{M} \mathrm{Re} X\Big)\right].$$

Since $\xi_\alpha = \left|\sum_j \bar{b}_j v_\alpha(j)\right|^2 \leq \sum_j |b_j|^2$, we have

$$|X| \leq \eta^{-1} + \frac{1}{N\eta} \sum_\alpha \xi_\alpha \leq \eta^{-1} \sum_j (|b_j|^2 + 1).$$

From the assumption (1.3), we have $\mathbb{E}_{\mathbf{b}}\, e^{\tau|X|} < \infty$ for τ small enough (depending on N and η). Therefore,

$$\lim_{M \to -\infty} \mathbb{E}_{\mathbf{b}}\Big(\exp\left[e^{M} \mathrm{Re} X\right] - 1 - e^{M} \mathrm{Re} X\Big) = 0.$$

Using $\mathbb{E}_{\mathbf{b}} X = 0$ and $|\log(1+c)| \leq 2|c|$ for any sufficiently small c, we have

$$e^{-M} \log \mathbb{E}_{\mathbf{b}} \exp\left[e^{M} \mathrm{Re} X\right] \leq 2e^{-M} \Big|\mathbb{E}_{\mathbf{b}}\Big(\exp\left[e^{M} \mathrm{Re} X\right] - 1 - e^{M} \mathrm{Re} X\Big)\Big|. \tag{3.3}$$

Since $|e^c - 1 - c| \leq |c|^2 e^{|c|}$ for any real c, we can bound the right hand side of (3.3) by

$$2e^{-M} \mathbb{E}_{\mathbf{b}} |e^{M} X|^2 e^{e^{M}|X|} = e^{M} \mathbb{E}_{\mathbf{b}} |X|^2 \exp\left[e^{M}|X|\right].$$

Since $|X|$ is exponentially integrable, the last term vanishes in the limit $M \to -\infty$. From (3.2) we have thus proved that

$$e^{-\beta_0} \log \mathbb{E}_{\mathbf{b}} \exp\left[e^{\beta_0} \mathrm{Re} X\right] \leq e^{-\beta_0} \mathbb{E}_{\mathbf{b}} \exp\left[\frac{2C}{(\log N)^4} Y\right]. \tag{3.4}$$

Denote by ν the constant $\nu = 2C/(\log N)^4$. By Hölder inequality, we can estimate

$$\mathbb{E}_{\mathbf{b}} e^{\nu Y} = \mathbb{E}_{\mathbf{b}} \prod_\alpha \exp\left[\frac{\nu}{N|\lambda_\alpha - z|} \xi_\alpha\right] \leq \prod_\alpha \left(\mathbb{E}_{\mathbf{b}} \exp\left[\frac{\nu c_\alpha}{N|\lambda_\alpha - z|} \xi_\alpha\right]\right)^{1/c_\alpha}, \tag{3.5}$$

where $\sum_\alpha \frac{1}{c_\alpha} = 1$. We shall choose

$$c_\alpha = \varrho \frac{N|\lambda_\alpha - z|}{\nu}$$

where ϱ is given by

$$\varrho = \frac{\nu}{N}\sum_{\alpha}\frac{1}{|\lambda_\alpha - z|} \le \frac{\nu \log N}{N\eta}\max_n \mathcal{N}_{I_n} \le \nu(\log N)^3 = \frac{2C}{\log N}\,.$$

Here we have used $\max_n \mathcal{N}_{I_n} \le N\eta(\log N)^2$ due to that we are in the set Ω^c. Notice that with this choice,

$$\frac{\nu c_\alpha}{N|\lambda_\alpha - z|} \le \frac{2C}{\log N}$$

is a small number. In the proof of Lemma 7.4 of [5] (see equation (7.13) of [5]) we showed that

$$\mathbb{E}_{\mathbf{b}}\, e^{\tau\xi_\alpha} < K$$

with a universal constant K if τ is sufficiently small depending on δ in (1.3). Thus from (3.4) and (3.5) we obtain

$$\log \mathbb{E}_{\mathbf{b}}\, \exp\left[e^{\beta_0}\mathrm{Re}X\right] \le \mathbb{E}_{\mathbf{b}}\, e^{\nu Y} \le \prod_{\alpha} K^{1/c_\alpha} = K\,.$$

The same inequality holds for $-\mathrm{Re}X$ as well as for the imaginary part of X. Thus we have proved that

$$\mathbb{E}_{\mathbf{b}} e^{e^{\beta_0}|X|} \le e^K.$$

Choosing an optimal T in (3.1), this proves that

$$\mathbb{P}[|X| \ge \varepsilon] \le e^{-c(\log N)^2} + e^{-\varepsilon e^{\beta_0}} \le e^{-c\varepsilon(\log N)^2}$$

as long as $N\eta \ge (\log N)^8$ and we thus conclude the Lemma. □

4 Delocalization of eigenvectors

Here we prove Theorem 1.2, the argument follows the same line as in [5] (Proposition 5.3). Let $\eta^* = N^{-1}(\log N)^9$ and partition the interval $[-2+\kappa, 2-\kappa]$ into $n_0 = O(1/\eta^*) \le O(N)$ intervals $I_1, I_2, \dots I_{n_0}$ of length η^*. As before, let $\mathcal{N}_I = |\{\beta \,:\, \mu_\beta \in I\}|$ denote the eigenvalues in I. By choosing (1.9) in Theorem 1.1, we have

$$\mathbb{P}\left\{\max_n \mathcal{N}_{I_n} \le \varepsilon N\eta^*\right\} \le e^{-c(\log N)^2}.$$

if ε is sufficiently small (depending on κ). Suppose that $\mu \in I_n$, and that $H\mathbf{v} = \mu\mathbf{v}$. Consider the decomposition

$$H = \begin{pmatrix} h & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix} \tag{4.1}$$

where $\mathbf{a} = (h_{12}, \dots h_{1N})^*$ and B is the $(N-1)\times(N-1)$ matrix obtained by removing the first row and first column from H. Let λ_α and $\mathbf{u}_\alpha$ (for $\alpha = 1, 2, \dots, N-1$) denote the eigenvalues and the normalized eigenvectors of B. From the eigenvalue equation $H\mathbf{v} = \mu\mathbf{v}$ and from (4.1) we find that

$$hv_1 + \mathbf{a}\cdot\mathbf{w} = \mu v_1, \quad \text{and} \quad \mathbf{a}v_1 + B\mathbf{w} = \mu\mathbf{w} \tag{4.2}$$

with $\mathbf{w} = (v_2, \dots, v_N)^t$. From these equations we obtain $\mathbf{w} = (\mu - B)^{-1}\mathbf{a}v_1$ and thus

$$\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = |v_1|^2 \mathbf{a} \cdot (\mu - B)^{-2}\mathbf{a}$$

Since $\|\mathbf{w}\|^2 = 1 - |v_1|^2$, we obtain

$$|v_1|^2 = \frac{1}{1 + \mathbf{a} \cdot (\mu - B)^{-2}\mathbf{a}} = \frac{1}{1 + \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{(\mu - \lambda_\alpha)^2}} \leq \frac{4N[\eta^*]^2}{\sum_{\lambda_\alpha \in I_n} \xi_\alpha}\,, \tag{4.3}$$

where in the second equality we set $\xi_\alpha = |\sqrt{N}\mathbf{a} \cdot \mathbf{u}_\alpha|^2$ and used the spectral representation of B. By the interlacing property of the eigenvalues of H and B, there exist at least $\mathcal{N}_{I_n} - 1$ eigenvalues λ_α in I_n. Therefore, using that the components of any eigenvector are identically distributed,

$$\begin{aligned}
&\mathbb{P}\Big(\exists\ \mathbf{v} \text{ with } H\mathbf{v} = \mu\mathbf{v},\ \|\mathbf{v}\| = 1,\ \mu \in [-2 + \kappa, 2 - \kappa] \text{ and } \|\mathbf{v}\|_\infty \geq \frac{C(\log N)^{9/2}}{N^{1/2}}\Big) \\
&\quad \leq N n_0 \sup_n \mathbb{P}\Big(\exists\ \mathbf{v} \text{ with } H\mathbf{v} = \mu\mathbf{v},\ \|\mathbf{v}\| = 1,\ \mu \in I_n \text{ and } |v_1|^2 \geq \frac{C(\log N)^9}{N}\Big) \\
&\quad \leq \text{const}\, N^2 \sup_n \mathbb{P}\left(\sum_{\lambda_\alpha \in I_n} \xi_\alpha \leq \frac{4N\eta^*}{C}\right) \\
&\quad \leq \text{const}\, N^2 \sup_n \mathbb{P}\left(\sum_{\lambda_\alpha \in I_n} \xi_\alpha \leq \frac{4N\eta^*}{C} \text{ and } \mathcal{N}_{I_n} \geq \varepsilon N\eta^*\right) + \text{const}\, N^2 \sup_n \mathbb{P}\left(\mathcal{N}_{I_n} \leq \varepsilon N\eta^*\right) \\
&\quad \leq \text{const}\, N^2 e^{-c(\log N)^9} + \text{const}\, N^2 e^{-c(\log N)^2} \leq e^{-c'(\log N)^2}\,.
\end{aligned} \tag{4.4}$$

using Corollary 2.4 of [5] and choosing C sufficiently large, depending on κ via ε. Here we used Corollary 2.4. of [5] that states that under condition (1.2) there exists a positive c such that for any δ small enough

$$\mathbb{P}\left(\sum_{\alpha \in A} \xi_\alpha \leq \delta m\right) \leq e^{-cm} \tag{4.5}$$

for all $A \subset \{1, \cdots, N-1\}$ with cardinality $|A| = m$. □

Appendix: Removal of the assumption C1)

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The following Lemma shows that the assumption C1) in Lemma 2.3 and its corollary in [5] can be removed.

Lemma 4.1 *Suppose that $z_1, \dots, z_N$ are bounded, complex valued i.i.d. random variables with $\mathbb{E}\, z_i = 0$ and $\mathbb{E}\,|z_i|^2 = a > 0$. Let $P : \mathbb{C}^N \to \mathbb{C}^N$ be a rank-m projection, and $\mathbf{z} = (z_1, \dots, z_N)$. Then, if δ is small enough, there exists $c > 0$ such that*

$$\mathbb{P}\left(|P\mathbf{z}|^2 \leq \delta m\right) \leq e^{-cm}\,.$$

Proof. It is enough to prove that

$$\mathbb{P}\left(\left||P\mathbf{z}|^2 - am\right| > \tau m\right) \le e^{-c\tau^2 m} \tag{4.6}$$

for all τ sufficiently small. Introduce the notation $\|X\|_q = \left[\mathbb{E}|X|^q\right]^{1/q}$. Since

$$\mathbb{P}\left(\left||P\mathbf{z}|^2 - am\right| > \tau m\right) \le \frac{\left\| |P\mathbf{z}|^2 - am\right\|_q^q}{(\tau m)^q}, \tag{4.7}$$

the bound (4.6) follows by showing that

$$\left\||P\mathbf{z}|^2 - am\right\|_q \le C\sqrt{q}\,\sqrt{m} \qquad \text{for all } q < m \tag{4.8}$$

(and then choosing $q = \alpha\tau^2 m$ with a small enough α). To prove (4.8), observe that (with the notation $\mathbf{e}_i = (0, \dots, 0, 1, 0 \dots, 0)$ for the standard basis of $\mathbb{C}^N$)

$$|P\mathbf{z}|^2 = \sum_{i=1}^N |z_i|^2\, |P\mathbf{e}_i|^2 + \sum_{i\neq j}^N \overline{z}_i\, z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j = am + \sum_{i=1}^N \left(|z_i|^2 - \mathbb{E}|z_i|^2\right)\, |P\mathbf{e}_i|^2 + \sum_{i\neq j} \overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \tag{4.9}$$

and thus

$$\left\||P\mathbf{z}|^2 - am\right\|_q \le \left\| \sum_{i=1}^N \left(|z_i|^2 - \mathbb{E}|z_i|^2\right)\, |P\mathbf{e}_i|^2 \right\|_q + \left\| \sum_{i\neq j}^N \overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q . \tag{4.10}$$

To bound the first term, we use that for arbitrary i.i.d. random variables $x_1, \dots, x_N$ with $\mathbb{E}\, x_j = 0$ and $\mathbb{E}\, e^{\delta|x_j|^2} < \infty$ for some $\delta > 0$, we have the bound

$$\|X\|_q \le C\sqrt{q}\,\|X\|_2 \tag{4.11}$$

for $X = \sum_{j=1}^N a_j x_j$, for arbitrary $a_j \in \mathbb{C}$. The bound (4.11) is an extension of Khintchine's inequality and it can be proven as follows using the representation

$$\|X\|_q^q = q \int_0^\infty \mathrm{d}y\; y^{q-1} P(|X| \ge y)\,. \tag{4.12}$$

Writing $a_j = |a_j| e^{i\theta_j}$, $\theta_j \in \mathbb{R}$, and decomposing $e^{i\theta_j} x_j$ into real and imaginary parts, it is clearly sufficient to prove (4.11) for the case when $a_j, x_j \in \mathbb{R}$ are real and x_j's are independent with $\mathbb{E}\, x_j = 0$ and $\mathbb{E}\, e^{\delta|x_j|^2} < \infty$. To bound the probability $\mathbb{P}(|X| \ge y)$ we observe that

$$\mathbb{P}(X \ge y) \le e^{-ty}\, \mathbb{E}\, e^{tX} = e^{-ty} \prod_{j=1}^N \mathbb{E}\, e^{ta_j x_j} \le e^{-ty}\, e^{Ct^2 \sum_{j=1}^N a_j^2}$$

because $\mathbb{E}\, e^{\tau x} \le e^{C\tau^2}$ from the moment assumptions on x_j with a sufficiently large C depending on δ. Repeating this argument for $-X$, we find

$$\mathbb{P}(|X| \ge y) \le 2\, e^{-ty}\, e^{Ct^2 \sum_{j=1}^N a_j^2} \le e^{-y^2/(2C\sum_{j=1}^N a_j^2)}$$

after optimizing in t. The estimate (4.11) follows then by plugging the last bound into (4.12) and computing the integral.

Applying (4.11) with $x_i = |z_i|^2 - \mathbb{E}\,|z_i|^2$ ($\mathbb{E}\,e^{\delta x_i^2} < \infty$ follows from the assumption $\|z_i\|_\infty < \infty$), the first term on the r.h.s. of (4.10) can be controlled by

$$\left\| \sum_{i=1}^N \left(|z_i|^2 - \mathbb{E}|z_i|^2 \right) |P\mathbf{e}_i|^2 \right\|_q \leq C\sqrt{q} \left(\sum_{i=1}^N |P\mathbf{e}_i|^4 \right)^{1/2} \leq C\sqrt{q} \left(\sum_{i=1}^N |P\mathbf{e}_i|^2 \right)^{1/2} = C\sqrt{q}\sqrt{m}\,. \tag{4.13}$$

As for the second term on the r.h.s. of (4.10), we define the functions $\xi_j(s), s \in [0,1], j = 1, \ldots, N$ by

$$\xi_j(s) = \begin{cases} 1 & \text{if } s \in \bigcup_{j=0}^{2^{k-1}-1} \left[\frac{2j}{2^k}, \frac{2j+1}{2^k}\right) \\ 0 & \text{otherwise} \end{cases} .$$

Since

$$\int_0^1 \mathrm{d}s\; \xi_i(s)(1-\xi_j(s)) = \frac{1}{4}$$

for all $i \neq j$, the second term on the r.h.s. of (4.10) can be estimated by

$$\left\| \sum_{i \neq j}^N \overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q \leq 4 \int_0^1 \mathrm{d}s\; \left\| \sum_{i \neq j}^N \xi_i(s)\,(1-\xi_j(s))\,\overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q . \tag{4.14}$$

For fixed $s \in [0,1]$, set

$$I(s) = \{1 \leq i \leq N : \xi_i(s) = 1\} \qquad \text{and} \qquad J(s) = \{1, \ldots, N\} \backslash I(s)\,.$$

Then

$$\left\| \sum_{i \neq j}^N \xi_i(s)\,(1-\xi_j(s))\,\overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q = \left\| \sum_{i \in I(s), j \in J(s)} \overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q = \left\| \sum_{j \in J(s)} z_j \left(\sum_{i \in I(s)} z_i P\mathbf{e}_i \right) \cdot \mathbf{e}_j \right\|_q .$$

Since by definition $I \cap J = \emptyset$, the variable $\{z_i\}_{i \in I}$ and the variable $\{z_j\}_{j \in J}$ are independent. Therefore, we can apply Khintchine's inequality (4.11) in the variables $\{z_j\}_{j \in J}$ (separating the real and imaginary parts) to conclude that

$$\begin{aligned} \left\| \sum_{i \neq j}^N \xi_i(s)\,(1-\xi_j(s))\,\overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q &\leq C\sqrt{q} \left\| \left(\sum_{j \in J(s)} \left| \left(\sum_{i \in I(s)} z_i P\mathbf{e}_i \right) \cdot \mathbf{e}_j \right|^2 \right)^{1/2} \right\|_q \\ &\leq C\sqrt{q} \left\| \sum_{i \in I(s)} z_i P\mathbf{e}_i \right\|_q \leq C\sqrt{q}\,\|P\mathbf{z}\|_q \end{aligned} \tag{4.15}$$

for every $s \in [0,1]$. It follows from (4.14) that

$$\left\| \sum_{i \neq j}^N \overline{z}_i z_j\, P\mathbf{e}_i \cdot P\mathbf{e}_j \right\|_q \leq C\,\sqrt{q}\,\|P\mathbf{z}\|_q\,.$$

Inserting the last equation and (4.13) into the r.h.s. of (4.10), it follows that

$$\left\| |P\mathbf{z}|^2 - am \right\|_q \leq C\sqrt{q}\left(\sqrt{m} + \|P\mathbf{z}\|_q\right) .$$

Since clearly

$$\|P\mathbf{z}\|_q \leq \left\| |P\mathbf{z}|^2 - am \right\|_q^{1/2} + \sqrt{am}$$

the bound (4.8) follows immediately. □

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