GENERALIZED ROBIN BOUNDARY CONDITIONS, ROBIN-TO-DIRICHLET MAPS, AND KREIN-TYPE RESOLVENT FORMULAS FOR SCHRÖDINGER OPERATORS ON BOUNDED LIPSCHITZ DOMAINS

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Dedicated with great pleasure to Vladimir Maz'ya on the occasion of his 70th birthday.

Abstract. We study generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains in $\mathbb{R}^n$, $n \geqslant 2$. We also discuss the case of bounded $C^{1,r}$-domains, $(1/2) < r < 1$.

1. Introduction

This paper is a continuation of the earlier papers [40] and [43], where we studied general, not necessarily self-adjoint, Schrödinger operators on $C^{1,r}$-domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, with compact boundaries $\partial\Omega$, $(1/2) < r < 1$ (including unbounded domains, i.e., exterior domains) with Dirichlet and Neumann boundary conditions on $\partial\Omega$. Our results also applied to convex domains Ω and to domains satisfying a uniform exterior ball condition. In addition, a careful discussion of locally singular potentials V with close to optimal local behavior of V was provided in [40] and [43].

In this paper we push the envelope in a different direction: Rather than discussing potentials with close to optimal local behavior, we will assume that $V \in L^\infty(\Omega; d^n x)$ and hence essentially replace it by zero nearly everywhere in this paper. On the other hand, instead of treating Dirichlet and Neumann boundary conditions at $\partial\Omega$, we now consider generalized Robin and again Dirichlet boundary conditions, but under minimal smoothness conditions on the domain Ω, that is, we now consider Lipschitz domains Ω. Additionally, to reduce some technicalities, we will assume that Ω is bounded throughout this paper. Occasionally we also discuss the case of bounded $C^{1,r}$-domains, $(1/2) < r < 1$. The principal new result in this paper is a derivation of Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains Ω in connection with the case of Dirichlet and generalized Robin boundary conditions on $\partial\Omega$.

In Section 2 we provide a detailed discussion of self-adjoint Laplacians with generalized Robin (and Dirichlet) boundary conditions on $\partial\Omega$. In Section 3 we then treat generalized Robin and Dirichlet boundary value problems and introduce associated Robin-to-Dirichlet and Dirichlet-to-Robin maps. Section 4 contains the principal new results of this paper; it is devoted to Krein-type resolvent formulas connecting Dirichlet and generalized Robin Laplacians with the help of the Robin-to-Dirichlet map. Appendix A collects useful material on Sobolev spaces and trace maps for $C^{1,r}$ and Lipschitz domains. Appendix B summarizes pertinent facts on sesquilinear forms and their associated linear operators. Estimates on the fundamental solution of the Helmholtz equation in

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$\mathbb{R}^n$, $n \geqslant 2$, are recalled in Appendix C. Finally, certain results on Calderón–Zygmund theory on Lipschitz surfaces of fundamental relevance to the material in the main body of this paper are presented in Appendix D.

While we formulate and prove all results in this paper for self-adjoint generalized Robin Laplacians and Dirichlet Laplacians, we emphasize that all results in this paper immediately extend to closed Schrödinger operators $H_{\Theta,\Omega} = -\Delta_{\Theta,\Omega} + V$, $\operatorname{dom}\big(H_{\Theta,\Omega}\big) = \operatorname{dom}\big(-\Delta_{\Theta,\Omega}\big)$ in $L^2(\Omega; d^n x)$ for (not necessarily real-valued) potentials V satisfying $V \in L^\infty(\Omega; d^n x)$, by consistently replacing $-\Delta$ by $-\Delta + V$, etc. More generally, all results extend directly to Kato–Rellich bounded potentials V relative to $-\Delta_{\Theta,\Omega}$ with bound less than one.

Next, we briefly list most of the notational conventions used throughout this paper. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\operatorname{dom}(T)$ and $\operatorname{ran}(T)$ denoting the domain and range of T. The spectrum (resp., essential spectrum) of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\,\cdot\,)$ (resp., $\sigma_{\rm ess}(\,\cdot\,)$). The Banach spaces of bounded and compact linear operators in $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ will be used for bounded and compact operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$. Throughout this manuscript, if X denotes a Banach space, X^* denotes the *adjoint space* of continuous conjugate linear functionals on X (rather than the usual dual space of continuous linear functionals on X). This avoids the well-known awkward distinction between adjoint operators in Banach and Hilbert spaces (cf., e.g., the pertinent discussion in [35, p. 3–4]).

Finally, a notational comment: For obvious reasons in connection with quantum mechanical applications, we will, with a slight abuse of notation, dub $-\Delta$ (rather than Δ) as the "Laplacian" in this paper.

2. Laplace Operators with Generalized Robin Boundary Conditions

In this section we primarily focus on various properties of general Laplacians $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ including Dirichlet, $-\Delta_{D,\Omega}$, and Neumann, $-\Delta_{N,\Omega}$, Laplacians, generalized Robin-type Laplacians, and Laplacians corresponding to classical Robin boundary conditions associated with open sets $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, introduced in Hypothesis 2.1 below.

We start with introducing our assumptions on the set Ω and the boundary operator Θ which subsequently will be employed in defining the boundary condition on $\partial\Omega$:

Hypothesis 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded, nonempty Lipschitz domain.*

We refer to Appendix A for more details on Lipschitz domains.

For simplicity of notation we will denote the identity operators in $L^2(\Omega; d^n x)$ and $L^2(\partial\Omega; d^{n-1}\omega)$ by I_Ω and $I_{\partial\Omega}$, respectively. Also, we refer to Appendix A for our notation in connection with Sobolev spaces.

Hypothesis 2.2. *Assume Hypothesis* 2.1 *and suppose that a_Θ is a closed sesquilinear form in $L^2(\partial\Omega; d^{n-1}\omega)$ with domain $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, bounded from below by $c_\Theta \in \mathbb{R}$ (hence, in particular, a_Θ is symmetric). Denote by $\Theta \geqslant c_\Theta I_{\partial\Omega}$ the self-adjoint operator in $L^2(\partial\Omega; d^{n-1}\omega)$ uniquely associated with a_Θ (cf.* (B.27)*) and by $\widetilde{\Theta} \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ the extension of Θ as discussed in* (B.26) *and* (B.32)*.*

Thus one has

$$\big\langle f, \widetilde{\Theta}\, g \big\rangle_{1/2} = \overline{\big\langle g, \widetilde{\Theta}\, f \big\rangle_{1/2}}, \quad f, g \in H^{1/2}(\partial\Omega). \tag{2.1}$$

$$\big\langle f, \widetilde{\Theta}\, f \big\rangle_{1/2} \geq c_\Theta \|f\|^2_{L^2(\partial\Omega; d^{n-1}\omega)}, \quad f \in H^{1/2}(\partial\Omega). \tag{2.2}$$

Here the sesquilinear form

$$\langle \,\cdot\,, \,\cdot\, \rangle_s = {}_{H^s(\partial\Omega)}\langle \,\cdot\,, \,\cdot\, \rangle_{H^{-s}(\partial\Omega)} \colon H^s(\partial\Omega) \times H^{-s}(\partial\Omega) \to \mathbb{C}, \quad s \in [0,1], \tag{2.3}$$

(antilinear in the first, linear in the second factor), denotes the duality pairing between $H^s(\partial\Omega)$ and

$$H^{-s}(\partial\Omega) = \big(H^s(\partial\Omega)\big)^*, \quad s \in [0,1], \tag{2.4}$$

such that

$$\langle f, g\rangle_s = \int_{\partial\Omega} d^{n-1}\omega(\xi)\, \overline{f(\xi)} g(\xi), \quad f \in H^s(\partial\Omega),\ g \in L^2(\partial\Omega; d^{n-1}\omega) \hookrightarrow H^{-s}(\partial\Omega),\ s \in [0,1], \tag{2.5}$$

and $d^{n-1}\omega$ denotes the surface measure on $\partial\Omega$.

Hypothesis 2.1 on Ω is used throughout this paper. Similarly, Hypothesis 2.2 is assumed whenever the boundary operator $\widetilde{\Theta}$ is involved. (Later in this section, and the next, we will occasionally strengthen our hypotheses.)

We introduce the boundary trace operator γ_D^0 (the Dirichlet trace) by

$$\gamma_D^0 \colon C(\overline{\Omega}) \to C(\partial\Omega), \quad \gamma_D^0 u = u|_{\partial\Omega}. \tag{2.6}$$

Then there exists a bounded, linear operator γ_D (cf., e.g., [65, Theorem 3.38]),

$$\begin{aligned} &\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \\ &\gamma_D \colon H^{3/2}(\Omega) \to H^{1-\varepsilon}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad \varepsilon \in (0,1), \end{aligned} \tag{2.7}$$

whose action is compatible with that of γ_D^0. That is, the two Dirichlet trace operators coincide on the intersection of their domains. Moreover, we recall that

$$\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \text{ is onto for } 1/2 < s < 3/2. \tag{2.8}$$

While, in the class of bounded Lipschitz subdomains in $\mathbb{R}^n$, the end-point cases $s = 1/2$ and $s = 3/2$ of $\gamma_D \in \mathcal{B}\big(H^s(\Omega), H^{s-(1/2)}(\partial\Omega)\big)$ fail, we nonetheless have

$$\gamma_D \in \mathcal{B}\big(H^{(3/2)+\varepsilon}(\Omega), H^1(\partial\Omega)\big), \quad \varepsilon > 0. \tag{2.9}$$

See Lemma A.4 for a proof. Below we augment this with the following result:

Lemma 2.3. *Assume Hypothesis* 2.1*. Then for each $s > -3/2$, the restriction to boundary operator* (2.6) *extends to a linear operator*

$$\gamma_D : \big\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\big\} \to L^2(\partial\Omega; d^{n-1}\omega), \tag{2.10}$$

is compatible with (2.7)*, and is bounded when $\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{1/2}(\Omega)} + \|\Delta u\|_{H^s(\Omega)}$.*

Furthermore, for each $s > -3/2$, the restriction to boundary operator (2.6) *also extends to a linear operator*

$$\gamma_D : \big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in H^{1+s}(\Omega)\big\} \to H^1(\partial\Omega), \tag{2.11}$$

which, once again, is compatible with (2.7)*, and is bounded when $\big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in H^{1-s}(\Omega)\big\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{H^{1+s}(\Omega)}$.*

Proof. For each $s \in \mathbb{R}$ set $H^s_\Delta(\Omega) = \{u \in H^s(\Omega) \,|\, \Delta u = 0 \text{ in } \Omega\}$ and observe that this is a closed subspace of $H^s(\Omega)$. In particular, $H^s_\Delta(\Omega)$ is a Banach space when equipped with the norm inherited from $H^s(\Omega)$. Next we recall the nontangential maximal operator M defined in (D.9). According to [36], or Corollary 5.7 in [47], one has

$$H^{1/2}_\Delta(\Omega) = \big\{u \text{ harmonic in } \Omega \,|\ M(u) \in L^2(\partial\Omega; d^{n-1}\omega)\big\} \tag{2.12}$$

and $u \mapsto \|M(u)\|_{L^2(\partial\Omega;d^{n-1}\omega)}$ is an equivalent norm on $H^{1/2}_{\Delta}(\Omega)$. To continue, fix some $\kappa > 0$ and set $d(y) = \operatorname{dist}(y, \partial\Omega)$ for $y \in \Omega$. According to [26], the nontangential trace operator

$$(\gamma_{\rm n.t.} u)(x) = \lim_{\substack{\Omega \ni y \to x \\ |x-y| < (1+\kappa)\, d(y)}} u(y) \tag{2.13}$$

is then well-defined when considered as a mapping

$$\gamma_{\rm n.t.} : \big\{ u \text{ harmonic in } \Omega \,\big|\, M(u) \in L^2(\partial\Omega; d^{n-1}\omega) \big\} \to L^2(\partial\Omega; d^{n-1}\omega). \tag{2.14}$$

Furthermore, the operator (2.13), (2.14) is bounded.

Granted these results, for a fixed $s > -3/2$ we may then attempt to define

$$\widetilde{\gamma}_D : H^{1/2}_{\Delta}(\Omega) + H^{s+2}(\Omega) \to L^2(\partial\Omega; d^{n-1}\omega) \tag{2.15}$$

by setting

$$\widetilde{\gamma}_D(u+v) = \gamma_{\rm n.t.} u + \gamma_D v, \quad u \in H^{1/2}_{\Delta}(\Omega),\ v \in H^{s+2}(\Omega). \tag{2.16}$$

A moment's reflection shows that, in order to establish that the mapping (2.15), (2.16) is well-defined, it suffices to prove that

$$\gamma_{\rm n.t.} u = \gamma_D u \text{ in } L^2(\partial\Omega; d^{n-1}\omega) \text{ whenever } u \in H^{(1/2)+\varepsilon}(\Omega),\ \varepsilon > 0. \tag{2.17}$$

in the case when Ω is a bounded Lipschitz domain which is star-like with respect to the origin in $\mathbb{R}^n$ (cf. (A.6)).

Assuming that this is the case, pick $u \in H^{(1/2)+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, and for each $t \in (0,1)$ set $u_t(x) = u(tx)$, $x \in \Omega$. We claim that

$$u_t \to u \ \text{ in } \ H^{(1/2)+\varepsilon}(\Omega) \ \text{ as } \ t \to 1. \tag{2.18}$$

To justify this, it suffices to prove that this is the case when $u \in C^\infty(\overline{\Omega})$ as the result in its full generality then follows from a standard density argument. However, for every $u \in C^\infty(\overline{\Omega})$ one trivially has $u_t \to u$ as $t \to 1$ in $H^1(\Omega)$, hence in $H^{(1/2)+\varepsilon}(\Omega)$. Having disposed of (2.18), we may then conclude that $\gamma_D u_t \to \gamma_D u$ in $L^2(\partial\Omega; d^{n-1}\omega)$ as $t \to 1$. Since for each $t \in (0,1)$ we have $u_t \in C(\overline{\Omega})$, it follows that $\gamma_D u_t = \gamma_D^0 u_t = u_t|_{\partial\Omega}$. Thus, altogether,

$$u_t|_{\partial\Omega} \to \gamma_D u \ \text{ in } \ L^2(\partial\Omega; d^{n-1}\omega) \ \text{ as } \ t \to 1. \tag{2.19}$$

On the other hand, for almost every $x \in \partial\Omega$, and every $t \in (0,1)$, we have that $y = tx$ belongs to Ω, converges to x as $t \to 1$, and $|x - y| \leq (1+\kappa) \operatorname{dist}(y, \partial\Omega)$ for some sufficiently large $\kappa = \kappa(\Omega) > 0$ (independent of x and t). This implies that

$$u_t(x) \to (\gamma_{\rm n.t.} u)(x) \ \text{ pointwise, for a.e. } x \in \partial\Omega, \text{ as } t \to 1. \tag{2.20}$$

Combining (2.19), (2.20) we therefore conclude that the functions $\gamma_{\rm n.t.} u$, $\gamma_D u \in L^2(\partial\Omega; d^{n-1}\omega)$ coincide pointwise a.e. on $\partial\Omega$. This proves (2.17) and finishes the justification of the fact that the mapping (2.15), (2.16) is well-defined.

Granted (2.15), (2.16) is well-defined, it is implicit in its own definition that the mapping (2.15), (2.16) is also bounded when we equip $H^{1/2}_{\Delta}(\Omega) + H^{s+2}(\Omega)$ with the canonical norm

$$w \mapsto \inf_{\substack{w = u+v \\ u \in H^{1/2}_{\Delta}(\Omega),\, v \in H^{s+2}(\Omega)}} \|u\|_{H^{1/2}_{\Delta}(\Omega)} + \|v\|_{H^{s+2}(\Omega)}. \tag{2.21}$$

The same type of argument as above (i.e., restricting attention to pieces of Ω which are star-like Lipschitz domains, and using dilations with respect to the respective center of star-likeness) shows the following: If $w \in C(\overline{\Omega})$ can be decomposed as $u + v$ with $u \in H^{1/2}_{\Delta}(\Omega)$ and $v \in H^{s+2}(\Omega)$ for some $s > -3/2$, then $w|_{\partial\Omega} = \gamma_{\rm n.t.} u + \gamma_D v$. In other words, the action of the trace operator $\widetilde{\gamma}_D$ in (2.15), (2.16) is compatible with that of (2.6). This completes the study of the nature and properties of $\widetilde{\gamma}_D$ in (2.15), (2.16).

Consider next the claim made about (2.10). As far as this is concerned, it suffices to prove that

$$\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\} \hookrightarrow H^{1/2}_\Delta(\Omega) + H^{s+2}(\Omega), \quad s > -3/2, \tag{2.22}$$

continuously. To see this, pick $u \in H^{1/2}(\Omega)$ such that $\Delta u \in H^s(\Omega)$ and extend (cf. [81]) Δu to a compactly supported distribution $w \in H^{-s}(\mathbb{R}^n)$. Next, set

$$v(x) = \int_{\mathbb{R}^n} d^n y\, E_n(x-y)w(y), \quad x \in \Omega, \tag{2.23}$$

where

$$E_n(x) = \begin{cases} \frac{1}{2\pi}\ln(|x|), & n = 2, \\ \frac{1}{(2-n)\omega_{n-1}}|x|^{2-n}, & n \geq 3, \end{cases} \tag{2.24}$$

is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ (cf. (C.1) for $z = 0$). Here $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\,\cdot\,)$ the Gamma function, cf. [1, Sect. 6.1]) represents the area of the unit sphere S^{n-1} in $\mathbb{R}^n$. Then $v \in H^{s+2}(\Omega)$ and $\Delta v = \Delta u$ in Ω. As a consequence, the function $w = u - v$ is harmonic and belongs to $H^{1/2}(\Omega)$, that is, $u = w + v$ with $w \in H^{1/2}_\Delta(\Omega)$, $v \in H^{s+2}(\Omega)$. Furthermore, the estimate

$$\|w\|_{H^{1/2}(\Omega)} + \|v\|_{H^{s+2}(\Omega)} \leq C\big(\|u\|_{H^{1/2}(\Omega)} + \|\Delta u\|_{H^s(\Omega)}\big) \tag{2.25}$$

for some $C = C(\Omega, s) > 0$ is implicit in the above construction. Thus, the inclusion (2.22) is well-defined and continuous, so that the claim about the operator (2.10) follows from this and the fact that $\widetilde{\gamma}_D$ in (2.15), (2.16) is well-defined and bounded.

We now digress momentarily for the purpose of developing an integration by parts formula which will play a significant role shortly. First, if Ω is a bounded star-like Lipschitz domain in $\mathbb{R}^n$ and G is a vector field with components in $H^{1/2}_\Delta(\Omega) + H^{s+2}_\Delta(\Omega)$, $s > -3/2$, such that $\operatorname{div}(G) \in L^1(\Omega)$, then

$$\int_\Omega dx^n \operatorname{div}(G) = \int_{\partial\Omega} d^{n-1}\omega\, \nu \cdot \widetilde{\gamma}_D G. \tag{2.26}$$

Indeed, if as before $G_t(x) = G(tx)$, $x \in \Omega$, $t \in (0,1)$, then

$$\operatorname{div}(G_t) = t(\operatorname{div}(G))_t \text{ in the sense of distributions in } \Omega. \tag{2.27}$$

Writing (2.26) for G_t in place of G, with $0 < t < 1$, and then passing to the limit $t \to \infty$ yields the desired result. As a corollary of (2.26) and (2.22), we also have that (2.26) holds if Ω is a bounded star-like Lipschitz domain in $\mathbb{R}^n$ and G is a vector field with components in $\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\}$, $s > -3/2$, such that $\operatorname{div}(G) \in L^1(\Omega)$. Since the latter space is a module over $C_0^\infty(\mathbb{R}^n)$ and any Lipschitz domain is locally star-like, a simple argument based on a smooth partition of unity shows that the star-likeness condition on Ω can be eliminated. More precisely,

$$\left.\begin{array}{l} \text{Hypothesis 2.1,} \\ G \in \big\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\big\}^n, \; s > -3/2, \\ \operatorname{div}(G) \in L^1(\Omega; d^n x) \end{array}\right\} \Longrightarrow (2.26) \text{ holds.} \tag{2.28}$$

Moving on, consider the operator (2.11). To get started, we fix $s > -3/2$ and assume that the function $u \in H^{3/2}(\Omega)$ is such that $\Delta u \in H^{1+s}(\Omega)$. Then

$$\gamma_D u \in H^{1-\varepsilon}(\partial\Omega) \text{ for every } \varepsilon > 0. \tag{2.29}$$

To continue, we recall the discussion (results and notation) in the paragraph containing (A.11)–(A.16) in Appendix A. For every $j,k\in\{1,...,n\}$, we now claim that

$$\frac{\partial(\gamma_D u)}{\partial\tau_{j,k}}=\nu_j\gamma_D(\partial_k u)-\nu_k\gamma_D(\partial_j u). \tag{2.30}$$

Since the functions $\partial_j u,\partial_k u$ belong to the space $\{w\in H^{1/2}(\Omega)\,|\,\Delta w\in H^s(\Omega)\}$, we may then conclude from (2.30) and (2.10) that

$$\frac{\partial(\gamma_D u)}{\partial\tau_{j,k}}\in L^2(\partial\Omega;d^{n-1}\omega), \tag{2.31}$$

and, in addition,

$$\Big\|\frac{\partial(\gamma_D u)}{\partial\tau_{j,k}}\Big\|_{L^2(\partial\Omega;d^{n-1}\omega)}\leq C\big(\|u\|_{H^{3/2}(\Omega)}+\|\Delta u\|_{H^{1+s}(\Omega)}\big), \tag{2.32}$$

for every $j,k\in\{1,...,n\}$. In concert with (2.32) and (2.29), the characterization in (A.16) then entails that $\gamma_D u\in H^1(\partial\Omega)$ and $\|\gamma_D u\|_{H^1(\partial\Omega)}\leq C\big(\|u\|_{H^{3/2}(\Omega)}+\|\Delta u\|_{H^{1+s}(\Omega)}\big)$. In summary, the proof of the claims made about (2.11) is finished, modulo establishing (2.30).

To deal with (2.30), let $\psi\in C_0^\infty(\mathbb{R}^n)$ and fix $j,k\in\{1,...,n\}$. Consider next the vector fields

$$\begin{aligned}F_{j,k}&=\big(0,...,0,u\partial_k\psi,0...,0,-u\partial_j\psi,0...,0\big),\\ G_{j,k}&=\big(0,...,0,\psi\partial_k u,0...,0,-\psi\partial_j u,0...,0\big),\end{aligned} \tag{2.33}$$

with the nonzero components on the j-th and k-th slots. Then $F_{j,k}$, $G_{j,k}$ have components in the space $\{u\in H^{1/2}(\Omega)\,|\,\Delta u\in H^s(\Omega)\}$ with $s>-3/2$ and satisfy

$$\mathrm{div}(F_{j,k})=-\mathrm{div}(G_{j,k})=\partial_j u\partial_k\psi-\partial_k u\partial_j\psi\in L^2(\Omega;d^nx), \tag{2.34}$$

in the sense of distributions. Also,

$$\begin{aligned}\nu\cdot\widetilde\gamma_D(F_{j,k})&=(\gamma_D u)\big(\nu_k\partial_j\psi-\nu_j\partial_k\psi\big),\\ \nu\cdot\widetilde\gamma_D(G_{j,k})&=\psi\big(\nu_k\gamma_D(\partial_j u)-\nu_j\gamma_D(\partial_k u)\big).\end{aligned} \tag{2.35}$$

Hence, using (2.28), we obtain

$$\begin{aligned}\int_{\partial\Omega}d^{n-1}\omega\,(\gamma_D u)\big(\nu_k\partial_j\psi-\nu_j\partial_k\psi\big)&=\int_{\partial\Omega}d^{n-1}\omega\,\nu\cdot\widetilde\gamma_D(F_{j,k})\\ &=\int_\Omega d^nx\,\mathrm{div}(F_{j,k})=-\int_\Omega d^nx\,\mathrm{div}(G_{j,k})\\ &=-\int_{\partial\Omega}d^{n-1}\omega\,\psi\big(\nu_k\gamma_D(\partial_j u)-\nu_j\gamma_D(\partial_k u)\big).\end{aligned} \tag{2.36}$$

This justifies (2.30) and finishes the proof of Lemma 2.3. □

Next, we introduce the operator γ_N (the Neumann trace) by

$$\gamma_N=\nu\cdot\gamma_D\nabla\colon H^{s+1}(\Omega)\to L^2(\partial\Omega;d^{n-1}\omega),\quad 1/2<s<3/2, \tag{2.37}$$

where ν denotes the outward pointing normal unit vector to $\partial\Omega$. It follows from (2.7) that γ_N is also a bounded operator. In addition, we denote by

$$\widetilde\gamma_N\colon\big\{u\in H^1(\Omega)\,\big|\,\Delta u\in\big(H^1(\Omega)\big)^*\big\}\to H^{-1/2}(\partial\Omega) \tag{2.38}$$

a weak Neumann trace operator defined by

$$\langle\phi,\widetilde\gamma_N u\rangle_{1/2}=\int_\Omega d^nx\,\overline{\nabla\Phi(x)}\cdot\nabla u(x)+{}_{H^1(\Omega)}\langle\Phi,\Delta u\rangle_{(H^1(\Omega))^*},\quad u\in H^1(\Omega),\ \Delta u\in\big(H^1(\Omega)\big)^*, \tag{2.39}$$

for all $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D\Phi = \phi$. We note that this definition is independent of the particular extension Φ of ϕ, and that $\widetilde{\gamma}_N$ is a bounded extension of the Neumann trace operator γ_N defined in (2.37). As was the case of the Dirichlet trace, the (weak) Neumann trace operator (2.38), (2.39) is onto (cf. the discussion in [42, Sect. 4]). For additional details we refer to equations (A.28)–(A.30). Here we only wish to discuss the end-point case $s = 1/2$ of (2.37).

Lemma 2.4. *Assume Hypothesis* 2.1. *Then the Neumann trace operator* (2.37) *also extends to*

$$\widetilde{\gamma}_N : \big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to L^2(\partial\Omega; d^{n-1}\omega) \tag{2.40}$$

in a bounded fashion when the space $\big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\big\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$. *This extension is compatible with* (2.38).

Proof. Fix $\psi \in C^\infty(\overline{\Omega})$. Applying (2.28) to the vector field $G = \overline{\psi}\nabla u$ yields

$$\int_{\partial\Omega} d^{n-1}\omega\, \overline{\psi}\, \nu \cdot \gamma_D(\nabla u) = \int_\Omega dx^n\, \overline{\nabla\psi} \cdot \nabla u + \int_\Omega dx^n\, \overline{\psi}\, \Delta u. \tag{2.41}$$

Consider now $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D\Phi = \phi$. Since $C^\infty(\overline{\Omega} \hookrightarrow H^1(\Omega)$ is dense, it is possible to select a sequence $\psi_j \in C^\infty(\overline{\Omega})$, $j \in \mathbb{N}$, such that $\psi_j \to \Phi$ in $H^1(\Omega)$ as $j \to \infty$. This entails $\nabla\psi_j \to \nabla\Phi$ in $L^2(\Omega; d^n x)$ and $\psi_j|_{\partial\Omega} \to \phi$ in $H^{1/2}(\partial\Omega)$ as $j \to \infty$. Writing (2.41) for ψ_j in place of ψ and passing to the limit $j \to \infty$ then yields

$$\int_{\partial\Omega} d^{n-1}\omega\, \overline{\phi}\, \nu \cdot \gamma_D(\nabla u) = \int_\Omega dx^n\, \overline{\nabla\Phi} \cdot \nabla u + \int_\Omega dx^n\, \overline{\Phi}\, \Delta u. \tag{2.42}$$

This shows that the Neumann trace of u in the sense of (2.38), (2.39) is actually $\nu \cdot \gamma_D(\nabla u)$. In addition,

$$\begin{aligned}
\|\widetilde{\gamma}_N u\|_{L^2(\partial\Omega;d^{n-1}\omega)} &= \|\nu \cdot \gamma_D(\nabla u)\|_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&\leq \|\gamma_D(\nabla u)\|_{L^2(\partial\Omega;d^{n-1}\omega)^n} \\
&\leq C\big(\|\nabla u\|_{H^{1/2}(\Omega)^n} + \|\Delta(\nabla u)\|_{H^{-1}(\Omega)^n}\big) \\
&= C\big(\|\nabla u\|_{H^{1/2}(\Omega)^n} + \|\nabla(\Delta u)\|_{H^{-1}(\Omega)^n}\big) \\
&\leq C\big(\|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}\big),
\end{aligned} \tag{2.43}$$

where we have used the boundedness of the Dirichlet trace operator in (2.10) with $s = -1$. This shows that, in the context of (2.40), the Neumann trace operator

$$\widetilde{\gamma}_N u = \nu \cdot \gamma_D(\nabla u) \tag{2.44}$$

has the desired properties. □

Before starting our discussion of generalized Robin Laplacians, we formulate the following useful result:

Lemma 2.5. *For all* $\varepsilon > 0$, *there exists a* $\beta(\varepsilon) > 0$ ($\beta(\varepsilon) \underset{\varepsilon\downarrow 0}{=} O(1/\varepsilon)$) *such that*

$$\|\gamma_D u\|^2_{L^2(\partial\Omega;d^{n-1}\omega)} \leqslant \varepsilon\|\nabla u\|^2_{L^2(\Omega;d^n x)^n} + \beta(\varepsilon)\|u\|^2_{L^2(\Omega;d^n x)} \ \text{ for all } \ u \in H^1(\Omega). \tag{2.45}$$

Proof. Since Ω is a bounded Lipschitz domain, there exists an $h \in C_0^\infty(\mathbb{R}^n)^n$ with real-valued components and $\kappa > 0$ such that (cf., [45, Lemma 1.5.1.9, p. 40])

$$(\nu \cdot h)_{\mathbb{C}^n} \geqslant \kappa \ \text{ a.e. on } \partial\Omega. \tag{2.46}$$

Thus,

$$\|\gamma_D u\|^2_{L^2(\partial\Omega;d^{n-1}\omega)} \leqslant \frac{1}{\kappa}\int_{\partial\Omega} d^{n-1}\omega\, (\nu \cdot h)_{\mathbb{C}^n} |\gamma_D u|^2$$

$$
\begin{aligned}
&= \frac{1}{\kappa} \int_\Omega d^n x \, \mathrm{div}(|u|^2 h), \\
&= \frac{1}{\kappa} \bigg(\int_\Omega d^n x \, \big(\nabla |u|^2, h\big)_{\mathbb{C}^n} + |u|^2 \mathrm{div}(h) \bigg), \quad u \in H^1(\Omega), \qquad (2.47)
\end{aligned}
$$

using the divergence theorem in the second step. Since for arbitrary $\varepsilon > 0$,

$$
|2u\nabla u| \leqslant \varepsilon |\nabla u|^2 + (1/\varepsilon)|u|^2, \quad u \in H^1(\Omega), \tag{2.48}
$$

and $h \in C_0^\infty(\mathbb{R}^n)^n$, one arrives at (2.45). □

Next we describe a family of self-adjoint Laplace operators $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ indexed by the boundary operator Θ. We will refer to $-\Delta_{\Theta,\Omega}$ as the generalized Robin Laplacian.

Theorem 2.6. *Assume Hypothesis* 2.2. *Then the generalized Robin Laplacian,* $-\Delta_{\Theta,\Omega}$*, defined by*

$$
-\Delta_{\Theta,\Omega} = -\Delta, \quad \mathrm{dom}(-\Delta_{\Theta,\Omega}) = \big\{ u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x); \ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big) u = 0 \text{ in } H^{-1/2}(\partial\Omega) \big\}, \tag{2.49}
$$

is self-adjoint and bounded from below in $L^2(\Omega; d^n x)$*. Moreover,*

$$
\mathrm{dom}\big(| - \Delta_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega). \tag{2.50}
$$

Proof. We introduce the sesquilinear form $a_{-\Delta_{\Theta,\Omega}}(\,\cdot\,,\,\cdot\,)$ on the domain $H^1(\Omega) \times H^1(\Omega)$ by

$$
a_{-\Delta_{\Theta,\Omega}}(u,v) = a_{-\Delta_{0,\Omega}}(u,v) + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v \big\rangle_{1/2}, \quad u,v \in H^1(\Omega), \tag{2.51}
$$

where $a_{-\Delta_{0,\Omega}}(\,\cdot\,,\,\cdot\,)$ on $H^1(\Omega) \times H^1(\Omega)$ denotes the Neumann Laplacian form

$$
a_{-\Delta_{0,\Omega}}(u,v) = \int_\Omega d^n x \, \overline{(\nabla u)(x)} \, (\nabla v)(x), \quad u,v \in H^1(\Omega). \tag{2.52}
$$

One verifies that $a_{-\Delta_{\Theta,\Omega}}(\,\cdot\,,\,\cdot\,)$ is well-defined on $H^1(\Omega) \times H^1(\Omega)$ since $\gamma_D \in \mathcal{B}\big(H^1(\Omega), H^{1/2}(\partial\Omega)\big)$, $\widetilde{\Theta} \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$, and $(\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2} \in \mathcal{B}\big(H^{1/2}(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big)$ (cf. (B.43)). This also implies that

$$
(\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D \in \mathcal{B}\big(H^1(\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{2.53}
$$

Employing (2.1) and (2.2), $a_{-\Delta_{\Theta,\Omega}}$ is symmetric and bounded from below by c_Θ. Next, we intend to show that $a_{-\Delta_{\Theta,\Omega}}$ is a closed form in $L^2(\Omega; d^n x) \times L^2(\Omega; d^n x)$. For this purpose we rewrite $\big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v \big\rangle_{1/2}$ as

$$
\begin{aligned}
&\big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v \big\rangle_{1/2} \\
&\quad = \big((\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D u, (\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D v\big)_{L^2(\partial\Omega; d^{n-1}\omega)} \\
&\qquad - (1-c_\Theta)(\gamma_D u, \gamma_D v)_{L^2(\partial\Omega; d^{n-1}\omega)}, \quad u,v \in H^1(\Omega), \qquad (2.54)
\end{aligned}
$$

(cf. (B.31), (B.32)), and notice that the last form on the right-hand side of (2.54) is nonclosable in $L^2(\Omega; d^n x)$ since γ_D is nonclosable as an operator defined on a dense subspace from $L^2(\Omega; d^n x)$ into $L^2(\partial\Omega; d^{n-1}\omega)$ (cf. the discussion in connection with (B.44)).

To deal with this noncloseability issue, we now split off the last form on the right-hand side of (2.54) and hence introduce

$$
\begin{aligned}
b_{-\Delta_{\Theta,\Omega}}(u,v) &= (\nabla u, \nabla v)_{L^2(\Omega; d^n x)^n} \\
&\quad + \big((\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D u, (\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D v\big)_{L^2(\partial\Omega; d^{n-1}\omega)} \\
&\quad + d_b(u,v)_{L^2(\Omega; d^n x)}, \quad u,v \in H^1(\Omega), \qquad (2.55)
\end{aligned}
$$

for $d_b > 0$. Then due to the nonnegativity of the second form on the right-hand side in (2.55), $b_{-\Delta_{\Theta,\Omega}}$ is $H^1(\Omega)$-coercive, that is, for some $c_1 > 0$,

$$b_{-\Delta_{\Theta,\Omega}}(u,u) \geqslant c_1 \|u\|^2_{H^1(\Omega)}, \tag{2.56}$$

where $\|u\|^2_{H^1(\Omega)} = \|\nabla u\|^2_{L^2(\Omega;d^nx)^n} + \|u\|^2_{L^2(\Omega;d^nx)}$. Next, we note that by (2.53),

$$\begin{aligned} &\Big|\big((\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D u, (\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D v\big)_{L^2(\partial\Omega;d^{n-1}\omega)}\Big| \\ &\quad \leqslant \big\|(\Theta + (1-c_\Theta)I_{\partial\Omega})^{1/2}\gamma_D\big\|^2_{\mathcal{B}(H^1(\Omega),L^2(\partial\Omega;d^{n-1}\omega))}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad u,v \in H^1(\Omega). \end{aligned} \tag{2.57}$$

Since trivially, $\|\nabla u\|^2_{L^2(\Omega;d^nx)} + d_b\|u\|^2_{L^2(\Omega;d^nx)} \leqslant c\|u\|^2_{H^1(\Omega)}$ for some $c > 0$, one infers that $b_{-\Delta_{\Theta,\Omega}}$ is also $H^1(\Omega)$-bounded, that is, for some $c_2 > 0$,

$$b_{-\Delta_{\Theta,\Omega}}(u,u) \leqslant c_2\|u\|^2_{H^1(\Omega)}. \tag{2.58}$$

Thus, the symmetric form $b_{-\Delta_{\Theta,\Omega}}$ is $H^1(\Omega)$-bounded and $H^1(\Omega)$-coercive and hence densely defined and closed in $L^2(\Omega;d^nx) \times L^2(\Omega;d^nx)$ by the discussion following (B.46).

To deal with the nonclosable form $(\gamma_D u, \gamma_D v)_{L^2(\partial\Omega;d^{n-1}\omega)}$, $u,v \in H^1(\Omega)$, it suffices to note that by Lemma 2.5 this form is infinitesimally bounded with respect to the Neumann Laplacian form $a_{-\Delta_{0,\Omega}}$ on $H^1(\Omega)\times H^1(\Omega)$, and since the form $\big((\Theta+1-c_\Theta)^{1/2}\gamma_D u, (\Theta+1-c_\Theta)^{1/2}\gamma_D v\big)_{L^2(\partial\Omega;d^{n-1}\omega)}$, $u,v \in H^1(\Omega)$, is nonnegative, the form $(\gamma_D u, \gamma_D v)_{L^2(\partial\Omega;d^{n-1}\omega)}$ is also infinitesimally bounded with respect to the form $b_{-\Delta_{\Theta,\Omega}}$. By the discussion in connection with (B.48), (B.49), the form $a_{-\Delta_{\Theta,\Omega}}$ (possibly shifted by an irrelevant real constant) defined on $H^1(\Omega)\times H^1(\Omega)$, is thus densely defined in $L^2(\Omega;d^nx)\times L^2(\Omega;d^nx)$, bounded from below, and closed.

According to (B.34) we thus introduce the operator $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega;d^nx)$ by

$$\begin{aligned} &\mathrm{dom}(-\Delta_{\Theta,\Omega}) = \bigg\{v \in H^1(\Omega)\,\bigg|\, \text{there exists an } w_v \in L^2(\Omega;d^nx) \text{ such that} \\ &\qquad \int_\Omega d^nx\,\overline{\nabla w}\,\nabla v + \big\langle\gamma_D w, \widetilde\Theta\gamma_D v\big\rangle_{1/2} = \int_\Omega d^nx\,\overline{w}w_v \text{ for all } w \in H^1(\Omega)\bigg\}, \\ &-\Delta_{\Theta,\Omega}u = w_u, \quad u \in \mathrm{dom}(-\Delta_{\Theta,\Omega}). \end{aligned} \tag{2.59}$$

By the formalism displayed in (B.20)–(B.43) (cf., in particular (B.27)), $-\Delta_{\Theta,\Omega}$ is self-adjoint in $L^2(\Omega;d^nx)$ and (2.50) holds. Recall that

$$H^1_0(\Omega) = \{u \in H^1(\Omega)\,|\,\gamma_D u = 0 \text{ on } \partial\Omega\}. \tag{2.60}$$

Taking $v \in C_0^\infty(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow H^1(\Omega)$, one concludes

$$\int_\Omega d^nx\,\overline{v}w_u = -\int_\Omega d^nx\,\overline{v}\,\Delta u \quad \text{for all } v \in C_0^\infty(\Omega), \text{ and hence } w_u = -\Delta u \text{ in } \mathcal{D}'(\Omega), \tag{2.61}$$

with $\mathcal{D}'(\Omega) = C_0^\infty(\Omega)'$ the space of distributions in Ω.

Next, we suppose that $u \in \mathrm{dom}(-\Delta_{\Theta,\Omega})$ and $v \in H^1(\Omega)$. We recall that $\gamma_D\colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and compute

$$\begin{aligned} \int_\Omega d^nx\,\overline{\nabla v}\,\nabla u &= -\int_\Omega d^nx\,\overline{v}\,\Delta u + \langle\gamma_D v, \widetilde\gamma_N u\rangle_{1/2} \\ &= \int_\Omega d^nx\,\overline{v}w_u + \big\langle\gamma_D v, \big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)u\big\rangle_{1/2} - \big\langle\gamma_D v, \widetilde\Theta\gamma_D u\big\rangle_{1/2} \\ &= \int_\Omega d^nx\,\overline{\nabla v}\,\nabla u + \big\langle\gamma_D v, \big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)u\big\rangle_{1/2}, \end{aligned} \tag{2.62}$$

where we used the second line in (2.59). Hence,

$$\big\langle \gamma_D v, \big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)u\big\rangle_{1/2} = 0. \tag{2.63}$$

Since $v \in H^1(\Omega)$ is arbitrary, and the map $\gamma_D\colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is actually onto, one concludes that

$$\big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)u = 0 \ \text{ in } \ H^{-1/2}(\partial\Omega). \tag{2.64}$$

Thus,

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) \subseteq \big\{v \in H^1(\Omega)\,\big|\,\Delta v \in L^2(\Omega; d^n x);\ \big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)v = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}. \tag{2.65}$$

Finally, assume that $u \in \big\{v \in H^1(\Omega)\,\big|\,\Delta v \in L^2(\Omega; d^n x);\ \big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)v = 0\big\}$, $w \in H^1(\Omega)$, and let $w_u = -\Delta u \in L^2(\Omega; d^n x)$. Then,

$$\begin{aligned}
\int_\Omega d^n x\, \overline{w} w_u &= -\int_\Omega d^n x\, \overline{w}\,\mathrm{div}(\nabla u) \\
&= \int_\Omega d^n x\, \overline{\nabla w}\,\nabla u - \langle \gamma_D w, \widetilde\gamma_N u\rangle_{1/2} \\
&= \int_\Omega d^n x\, \overline{\nabla w}\,\nabla u + \big\langle \gamma_D w, \widetilde\Theta\gamma_D u\big\rangle_{1/2}.
\end{aligned} \tag{2.66}$$

Thus, applying (2.59), one concludes that $u \in \mathrm{dom}(-\Delta_{\Theta,\Omega})$ and hence

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) \supseteq \big\{v \in H^1(\Omega)\,\big|\,\Delta v \in L^2(\Omega; d^n x);\ \big(\widetilde\gamma_N + \widetilde\Theta\gamma_D\big)v = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}, \tag{2.67}$$

finishing the proof of Theorem 2.6. □

Corollary 2.7. *Assume Hypothesis* 2.2. *Then the generalized Robin Laplacian, $-\Delta_{\Theta,\Omega}$, has purely discrete spectrum bounded from below, in particular,*

$$\sigma_{\mathrm{ess}}(-\Delta_{\Theta,\Omega}) = \emptyset. \tag{2.68}$$

Proof. Since $\mathrm{dom}\big(|-\Delta_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega)$, by (2.50), and $H^1(\Omega)$ embeds compactly into $L^2(\Omega; d^n x)$ (cf., e.g., [35, Theorem V.4.17]), one infers that $(-\Delta_{\Theta,\Omega} + I_\Omega)^{-1/2} \in \mathcal{B}_\infty(L^2(\Omega; d^n x))$. Consequently, one obtains

$$(-\Delta_{\Theta,\Omega} + I_\Omega)^{-1} \in \mathcal{B}_\infty(L^2(\Omega; d^n x)), \tag{2.69}$$

which is equivalent to (2.68). □

The important special case where Θ corresponds to the operator of multiplication by a real-valued, essentially bounded function θ leads to Robin boundary conditions we discuss next:

Corollary 2.8. *In addition to Hypothesis* 2.1, *assume that Θ is the operator of multiplication in $L^2(\partial\Omega; d^{n-1}\omega)$ by the real-valued function θ satisfying $\theta \in L^\infty(\partial\Omega; d^{n-1}\omega)$. Then Θ satisfies the conditions in Hypothesis* 2.2 *resulting in the self-adjoint and bounded from below Laplacian $-\Delta_{\theta,\Omega}$ in $L^2(\Omega; d^n x)$ with Robin boundary conditions on $\partial\Omega$ in* (2.49) *given by*

$$(\widetilde\gamma_N + \theta\gamma_D)u = 0 \ \text{ in } \ H^{-1/2}(\partial\Omega). \tag{2.70}$$

Proof. By Lemma 2.5, the sesquilinear form

$$\langle \gamma_D u, \theta\gamma_D v\rangle_{1/2}, \quad u, v \in H^1(\Omega), \tag{2.71}$$

is infinitesimally form bounded with respect to the Neumann Laplacian form $a_{-\Delta_{0,\Omega}}$. By (B.48) and (B.49) this in turn proves that the form $a_{-\Delta_{\Theta,\Omega}}$ in (2.51) is closed and one can now follow the proof of Theorem 2.6 from (2.59) on, step by step. □

Remark 2.9. (i) In the case of a smooth boundary $\partial\Omega$, the boundary conditions in (2.70) are also called "classical" boundary conditions (cf., e.g., [85]); in the more general case of bounded Lipschitz domains we also refer to [6] and [96, Ch. 4] in this context. Next, we point out that, in [58], the authors have dealt with the case of Laplace operators in bounded Lipschitz domains, equipped with local boundary conditions of Robin-type, with boundary data in $L^p(\partial\Omega; d^{n-1}\omega)$, and produced nontangential maximal function estimates. For the case $p = 2$, when our setting agrees with that of [58], some of our results in this section and the following are a refinement of those in [58]. Maximal L^p-regularity and analytic contraction semigroups of Dirichlet and Neumann Laplacians on bounded Lipschitz domains were studied in [100]. Holomorphic C_0-semigroups of the Laplacian with Robin boundary conditions on bounded Lipschitz domains have been discussed in [97]. Moreover, Robin boundary conditions for elliptic boundary value problems on arbitrary open domains were first studied by Maz'ya [63], [64, Sect. 4.11.6], and subsequently in [27] (see also [28] which treats the case of the Laplacian). In addition, Robin-type boundary conditions involving measures on the boundary for very general domains Ω were intensively discussed in terms of quadratic forms and capacity methods in the literature, and we refer, for instance, to [6], [7], [17], [96], and the references therein.
(ii) In the special case $\theta = 0$ (resp., $\widetilde{\Theta} = 0$), that is, in the case of the Neumann Laplacian, we will also use the notation

$$-\Delta_{N,\Omega} = -\Delta_{0,\Omega}. \tag{2.72}$$

The case of the Dirichlet Laplacian $-\Delta_{D,\Omega}$ associated with Ω formally corresponds to $\Theta = \infty$ and so we isolate it in the next result:

Theorem 2.10. *Assume Hypothesis* 2.1. *Then the Dirichlet Laplacian, $-\Delta_{D,\Omega}$, defined by*

$$\begin{aligned} -\Delta_{D,\Omega} = -\Delta, \quad \operatorname{dom}(-\Delta_{D,\Omega}) &= \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \gamma_D u = 0 \text{ in } H^{1/2}(\partial\Omega)\big\} \\ &= \big\{u \in H_0^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}, \end{aligned} \tag{2.73}$$

is self-adjoint and strictly positive in $L^2(\Omega; d^n x)$. Moreover,

$$\operatorname{dom}\big((-\Delta_{D,\Omega})^{1/2}\big) = H_0^1(\Omega). \tag{2.74}$$

Proof. We introduce the sesquilinear form $a_{D,\Omega}(\,\cdot\,,\,\cdot\,)$ on the domain $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$a_{D,\Omega}(u,v) = \int_\Omega d^n x\, \overline{(\nabla u)(x)}\, (\nabla v)(x), \quad u, v \in H_0^1(\Omega). \tag{2.75}$$

Clearly, $a_{D,\Omega}$ is symmetric, nonnegative, and well-defined on $H_0^1(\Omega) \times H_0^1(\Omega)$. Since Ω is bounded, that is, $|\Omega| < \infty$, $H_0^1(\Omega)$-coercivity of $a_{D,\Omega}$ then immediately follows from Poincaré's inequality for $H_0^1(\Omega)$-functions (cf., e.g., [99, Theorem I.7.6]).

Next we introduce the operator $-\Delta_{D,\Omega}$ in $L^2(\Omega; d^n x)$ by

$$\begin{aligned} \operatorname{dom}(-\Delta_{D,\Omega}) = \bigg\{ v \in H_0^1(\Omega) \,\bigg|\, &\text{ there exists an } w_v \in L^2(\Omega; d^n x) \text{ such that} \\ &\int_\Omega d^n x\, \overline{\nabla w}\, \nabla v = \int_\Omega d^n x\, \overline{w} w_v \text{ for all } w \in H_0^1(\Omega) \bigg\}, \\ -\Delta_{D,\Omega} u = w_u, \quad u \in \operatorname{dom}(-\Delta_{D,\Omega}).& \end{aligned} \tag{2.76}$$

By the formalism displayed in (B.1)–(B.19), $-\Delta_{D,\Omega}$ is self-adjoint in $L^2(\Omega; d^n x)$ and (2.74) holds. Taking $v \in C_0^\infty(\Omega) \hookrightarrow H_0^1(\Omega)$, one concludes

$$\int_\Omega d^n x\, \overline{v} w_u = -\int_\Omega d^n x\, \overline{v}\, \Delta u \ \text{ in } \mathcal{D}'(\Omega) \ \text{ and hence } w_u = -\Delta u \ \text{ in } \mathcal{D}'(\Omega). \tag{2.77}$$

Since $v \in H_0^1(\Omega)$ if and only if $v \in H^1(\Omega)$ and $\gamma_D v = 0$ in $H^{1/2}(\partial\Omega)$ (cf., e.g., [45, Corollary 1.5.1.6]), and $v \in \mathrm{dom}(-\Delta_{D,\Omega})$ implies $\Delta v \in L^2(\Omega; d^n x)$, one computes for $u \in \mathrm{dom}(-\Delta_{D,\Omega})$ and $v \in H_0^1(\Omega)$ that

$$\int_\Omega d^n x\, \overline{\nabla v}\, \nabla u = -\int_\Omega d^n x\, \overline{v} \Delta u = \int_\Omega d^n x\, \overline{v} w_u. \tag{2.78}$$

Thus, $w_u = -\Delta u \in L^2(\Omega; d^n x)$ and hence

$$\mathrm{dom}(-\Delta_{D,\Omega}) \subseteq \big\{v \in H_0^1(\Omega) \,\big|\, \Delta v \in L^2(\Omega; d^n x)\big\}. \tag{2.79}$$

Finally, assume that $u \in \big\{v \in H_0^1(\Omega) \,\big|\, \Delta v \in L^2(\Omega; d^n x)\big\}$, $w \in H_0^1(\Omega)$, and let $w_u = -\Delta u \in L^2(\Omega; d^n x)$. Then,

$$\int_\Omega d^n x\, \overline{w} w_u = -\int_\Omega d^n x\, \overline{w}\, \mathrm{div}(\nabla u) = \int_\Omega d^n x\, \overline{\nabla w}\, \nabla u, \tag{2.80}$$

since $\gamma_D w = 0$ in $L^2(\partial\Omega; d^{n-1}\omega)$. Thus, applying (2.76), one concludes that $u \in \mathrm{dom}(-\Delta_{D,\Omega})$ and hence

$$\mathrm{dom}(-\Delta_{D,\Omega}) \supseteq \big\{v \in H_0^1(\Omega) \,\big|\, \Delta v \in L^2(\Omega; d^n x)\big\}, \tag{2.81}$$

finishing the proof of Theorem 2.10. □

Since Ω is open and bounded, it is well-known that $-\Delta_{D,\Omega}$ has purely discrete spectrum contained in $(0,\infty)$, in particular,

$$\sigma_{\mathrm{ess}}(-\Delta_{D,\Omega}) = \emptyset \tag{2.82}$$

(this follows from (2.74) since $H_0^1(\Omega)$ embeds compactly into $L^2(\Omega; d^n x)$; the latter fact holds for arbitrary open, bounded sets $\Omega \subset \mathbb{R}^n$, cf., e.g., [35, Theorem V.4.18]).

While the principal objective of this paper was to prove the results in this section and the subsequent for minimally smooth domains Ω, it is of interest to study similar problems when Hypothesis 2.1 is further strengthen to:

Hypothesis 2.11. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{1,r}$ for some $1/2 < r < 1$.*

We refer to Appendix A for some details on $C^{1,r}$-domains.

Correspondingly, the natural strengthening of Hypothesis 2.2 reads:

Hypothesis 2.12. *In addition to Hypothesis 2.2 and 2.11 assume that*

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^{3/2}(\partial\Omega), H^{1/2}(\partial\Omega)\big). \tag{2.83}$$

We note that a sufficient condition for (2.83) to hold is

$$\widetilde{\Theta} \in \mathcal{B}\big(H^{3/2-\varepsilon}(\partial\Omega), H^{1/2}(\partial\Omega)\big) \text{ for some } \varepsilon > 0. \tag{2.84}$$

Notational comment. To avoid introducing an additional sub- or superscript into our notation of $-\Delta_{\Theta,\Omega}$ and $-\Delta_{D,\Omega}$, we will use the same symbol for these operators irrespective of whether the pair of Hypothesis 2.1 and 2.2 or the pair of Hypothesis 2.11 and 2.12 is involved. Our results will be carefully stated so that it is always evident which set of hypotheses is used.

Next, we discuss certain regularity results for fractional powers of the resolvents of the Dirichlet and Robin Laplacians, first in Lipschitz then in smoother domains.

Lemma 2.13. *Assume Hypothesis 2.1 in connection with $-\Delta_{D,\Omega}$ and Hypothesis 2.2 in connection with $-\Delta_{\Theta,\Omega}$. Then the following boundedness properties hold for all $q \in [0,1]$ and $z \in \mathbb{C}\backslash[0,\infty)$,*

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-q/2},\ (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-q/2} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^q(\Omega)\big). \tag{2.85}$$

The fractional powers in (2.85) (and in subsequent analogous cases) are defined via the functional calculus implied by the spectral theorem for self-adjoint operators. As discussed in [40, Lemma A.2] in the closely related situation of Lemma 2.14, the key ingredients in proving Lemma 2.13 are the inclusions

$$\operatorname{dom}(-\Delta_{D,\Omega}) \subset H^1(\Omega), \quad \operatorname{dom}(-\Delta_{\Theta,\Omega}) \subset H^1(\Omega) \tag{2.86}$$

and real interpolation methods. The above results should be compared with its analogue for smoother domains. Specifically, we have:

Lemma 2.14. *Assume Hypothesis* 2.11 *in connection with* $-\Delta_{D,\Omega}$ *and Hypothesis* 2.12 *in connection with* $-\Delta_{\Theta,\Omega}$*. Then the following boundedness properties hold for all* $q \in [0,1]$ *and* $z \in \mathbb{C}\backslash[0,\infty)$*,*

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-q},\ (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-q} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^{2q}(\Omega)\big). \tag{2.87}$$

As explained in [40, Lemma A.2], the key ingredients in proving Lemma 2.14 are the inclusions

$$\operatorname{dom}(-\Delta_{D,\Omega}) \subset H^2(\Omega), \quad \operatorname{dom}(-\Delta_{\Theta,\Omega}) \subset H^2(\Omega) \tag{2.88}$$

and real interpolation methods.

Moving on, we now consider mapping properties of powers of the resolvents of generalized Robin Laplacians multiplied (to the left) by the Dirichlet boundary trace operator:

Lemma 2.15. *Assume Hypothesis* 2.1 *and let* $\varepsilon > 0$*,* $z \in \mathbb{C}\backslash[0,\infty)$*. Then,*

$$\gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-(1+\varepsilon)/4} \in \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{2.89}$$

As in [40, Lemma 6.9], Lemma 2.15 follows from Lemma 2.13 and from (2.7) and (2.37). Once again, we wish to contrast this with the corresponding result for smoother domains, recorded below.

Lemma 2.16. *Assume Hypothesis* 2.11 *in connection with* $-\Delta_{D,\Omega}$ *and Hypothesis* 2.12 *in connection with* $-\Delta_{\Theta,\Omega}$*, and let* $\varepsilon > 0$*,* $z \in \mathbb{C}\backslash[0,\infty)$*. Then,*

$$\gamma_N(-\Delta_{D,\Omega} - zI_\Omega)^{-(3+\varepsilon)/4}, \gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-(1+\varepsilon)/4} \in \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{2.90}$$

As in [40, Lemma 6.9], Lemma 2.16 follows from Lemma 2.14 and from (2.7) and (2.37). In contrast to Lemma 2.16 under the stronger Hypothesis 2.12, we cannot obtain an analog of (2.90) for $-\Delta_{D,\Omega}$ under the weaker Hypothesis 2.1.

The analog of Theorem 2.6 for smoother domains reads as follows:

Theorem 2.17. *Assume Hypothesis* 2.12*. Then the generalized Robin Laplacian,* $-\Delta_{\Theta,\Omega}$*, defined by*

$$-\Delta_{\Theta,\Omega} = -\Delta, \quad \operatorname{dom}(-\Delta_{\Theta,\Omega}) = \big\{u \in H^2(\Omega) \,\big|\, \big(\gamma_N + \widetilde{\Theta}\gamma_D\big)u = 0 \text{ in } H^{1/2}(\partial\Omega)\big\}, \tag{2.91}$$

is self-adjoint and bounded from below in $L^2(\Omega; d^n x)$*. Moreover,*

$$\operatorname{dom}\big(|-\Delta_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega). \tag{2.92}$$

Proof. We adapt the proof of [40, Lemma A.1], dealing with the special case of Neumann boundary conditions (i.e., in the case $\widetilde{\Theta} = 0$), to the present situation. For convenience of the reader we produce a complete proof below.

By Theorem 2.6, the operator $T_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$, defined by

$$T_{\Theta,\Omega} = -\Delta, \quad \operatorname{dom}(T_{\Theta,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}, \tag{2.93}$$

is self-adjoint and bounded from below, and

$$\operatorname{dom}\big(|T_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega) \tag{2.94}$$

holds. Thus, we need to prove that $\operatorname{dom}(T_{\Theta,\Omega}) \subseteq H^2(\Omega)$.

Consider $u \in \operatorname{dom}(T_{\Theta,\Omega})$ and set $f = -\Delta u + u \in L^2(\Omega; d^n x)$. Viewing f as an element in $\big(H^1(\Omega)\big)^*$, the classical Lax–Milgram Lemma implies that u is the unique solution of the boundary-value problem

$$\begin{cases} (-\Delta + I_\Omega)u = f \in L^2(\Omega) \hookrightarrow \big(H^1(\Omega)\big)^*, \\ u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = 0. \end{cases} \tag{2.95}$$

One convenient way to actually show that

$$u \in H^2(\Omega), \tag{2.96}$$

is to use layer potentials. Specifically, let $E_n(z;x)$ be the fundamental solution of the Helmholtz differential expression $(-\Delta - z)$ in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, that is,

$$E_n(z;x) = \begin{cases} (i/4)\big(2\pi|x|/z^{1/2}\big)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x|\big), & n \geq 2,\ z \in \mathbb{C}\backslash\{0\}, \\ \frac{-1}{2\pi}\ln(|x|), & n = 2,\ z = 0, \\ \frac{1}{(n-2)\omega_{n-1}}|x|^{2-n}, & n \geq 3,\ z = 0, \end{cases} \tag{2.97}$$
$$\operatorname{Im}\big(z^{1/2}\big) \geq 0,\ x \in \mathbb{R}^n\backslash\{0\}.$$

Here $H^{(1)}_\nu(\,\cdot\,)$ denotes the Hankel function of the first kind with index $\nu \geq 0$ (cf. [1, Sect. 9.1]).

We also define the associated single layer potential

$$(\mathcal{S}_z g)(x) = \int_{\partial\Omega} d^{n-1}\omega(y)\, E_n(z; x-y) g(y), \quad x \in \Omega,\ z \in \mathbb{C}, \tag{2.98}$$

where g is an arbitrary measurable function on $\partial\Omega$. As is well-known (the interested reader may consult, e.g., [68], [95] for jump relations in the context of Lipschitz domains), if

$$(K^\#_z g)(x) = \int_{\partial\Omega} d^{n-1}\omega(y)\, \partial_{\nu_x} E_n(z; x-y) g(y), \quad x \in \partial\Omega,\ z \in \mathbb{C}, \tag{2.99}$$

stands for the so-called adjoint double layer on $\partial\Omega$, the following jump formula holds

$$\widetilde{\gamma}_N \mathcal{S}_z g = \big(-\tfrac{1}{2} I_{\partial\Omega} + K^\#_z\big) g. \tag{2.100}$$

It should be noted that

$$K^\#_z \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}, \tag{2.101}$$

whenever Ω is a bounded Lipschitz domain. See Lemma D.3.

Now, if we denote by w the convolution of $f \in L^2(\Omega; d^n x)$ with $E_n(-1;\,\cdot\,)$ in Ω, then $w \in H^2(\Omega)$ and the solution u of (2.95) is given by

$$u = w + \mathcal{S}_{-1} g \tag{2.102}$$

for a suitably chosen function g on $\partial\Omega$. Concretely, we shall then require that

$$\big(\gamma_N + \widetilde{\Theta}\gamma_D\big)\mathcal{S}_{-1} g = -\big(\gamma_N + \widetilde{\Theta}\gamma_D\big) w, \tag{2.103}$$

or equivalently,

$$\big(-\tfrac{1}{2} I_{\partial\Omega} + K^\#_{-1}\big) g + \widetilde{\Theta}\gamma_D \mathcal{S}_{-1} g = -\big(\gamma_N + \widetilde{\Theta}\gamma_D\big) w \in H^{1/2}(\partial\Omega). \tag{2.104}$$

By hypothesis, $\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^{3/2}(\Omega), H^{1/2}(\partial\Omega)\big)$ and hence

$$\widetilde{\Theta}\gamma_D \mathcal{S}_{-1} \in \mathcal{B}_\infty\big(H^{1/2}(\Omega), H^{1/2}(\partial\Omega)\big) \tag{2.105}$$

as soon as one proves that $\mathcal{S}_{-1}$ satisfies

$$\mathcal{S}_{-1} \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^2(\Omega)\big). \tag{2.106}$$

To prove this, as a preliminary step we note (cf. [69]) that

$$\mathcal{S}_{-1}\colon H^{-s}(\partial\Omega) \to H^{-s+3/2}(\Omega) \tag{2.107}$$

is well-defined and bounded for each $s \in [0,1]$, even when Ω is only a bounded Lipschitz domain. For a fixed, arbitrary $j \in \{1,...,n\}$, consider next the operator $\partial_{x_j}\mathcal{S}_{-1}$ whose integral kernel is $\partial_{x_j}E_n(-1;x-y) = -\partial_{y_j}E_n(-1;x-y)$. We write

$$\partial_{y_j} = \sum_{k=1}^n \nu_k(y)\nu_k(y)\partial_{y_j} = \sum_{k=1}^n \nu_k(y)\frac{\partial}{\partial\tau_{k,j}(y)} + \nu_j(y)\nu(y)\cdot\nabla_y \tag{2.108}$$

where the tangential derivative operators $\partial/\partial\tau_{k,j} = \nu_k\partial_j - \nu_j\partial_k,\ j,k = 1,\dots,n$, satisfy (A.17). Using the boundary integration by parts formula (A.24) it follows that

$$\partial_j\mathcal{S}_{-1}h = \mathcal{D}_{-1}(\nu_j h) + \sum_{k=1}^n \mathcal{S}_{-1}\bigg(\frac{\partial(\nu_k h)}{\partial\tau_{k,j}}\bigg), \quad h \in H^{1/2}(\partial\Omega), \tag{2.109}$$

where, for $z \in \mathbb{C}$,

$$\mathcal{D}_z h(x) = \int_{\partial\Omega} d^{n-1}\omega(y)\,\nu(y)\cdot\nabla_y[E_n(z;x-y)]h(y), \quad x \in \Omega, \tag{2.110}$$

is the so-called (acoustic) double layer potential operator. Its mappings properties on the scale of Sobolev spaces have been analyzed in [69] and we note here that

$$\mathcal{D}_{-1}\colon H^s(\partial\Omega) \to H^{s+1/2}(\Omega), \quad 0 \le s \le 1, \tag{2.111}$$

requires only that Ω is Lipschitz. Assuming that multiplication by (the components of) ν preserves the space $H^{1/2}(\partial\Omega)$ (which is the case if, e.g., Ω is of class $C^{1,r}$ for some $(1/2) < r < 1$; cf. Lemma A.5), the desired conclusion about the operator (2.106) follows from (2.107), (2.109) and (2.111).

Going further, if one proves that

$$-\tfrac{1}{2}I_{\partial\Omega} + K^{\#}_{-1} \text{ is boundedly invertible on } H^{1/2}(\partial\Omega), \tag{2.112}$$

the operator $-\frac{1}{2}I_{\partial\Omega} + K^{\#}_{-1} + \widetilde{\Theta}\gamma_D\mathcal{S}_{-1}$ will be a Fredholm operator in $H^{1/2}(\partial\Omega)$ with index zero, implying the validity of (2.96). That (2.112) holds is essentially well-known. See, for instance, [92, Proposition 4.5] which requires that Ω is of class $C^{1,r}$ for some $(1/2) < r < 1$. This concludes the proof of the fact that $\operatorname{dom}(T_{\Theta,\Omega}) \subseteq H^2(\Omega)$. ☐

Again we isolate the Neumann Laplacian $-\Delta_{N,\Omega}$, that is, the special case $\widetilde{\Theta} = 0$ in (2.91), under Hypothesis 2.11,

$$-\Delta_{N,\Omega} = -\Delta, \quad \operatorname{dom}(-\Delta_{N,\Omega}) = \big\{u \in H^2(\Omega)\,\big|\,\widetilde{\gamma}_N u = 0 \text{ in } H^{1/2}(\partial\Omega)\big\}. \tag{2.113}$$

Similarly, one can now treat the case of the Dirichlet Laplacian. This has originally been done under more general conditions on Ω (assuming the boundary of Ω to be compact rather than Ω bounded) in [40, Lemmas A.1]. For completeness we repeat the short argument below:

Theorem 2.18. *Assume Hypothesis* 2.11. *Then the Dirichlet Laplacian, $-\Delta_{D,\Omega}$, defined by*

$$-\Delta_{D,\Omega} = -\Delta, \quad \operatorname{dom}(-\Delta_{D,\Omega}) = \big\{u \in H^2(\Omega)\,\big|\,\gamma_D u = 0 \textit{ in } H^{3/2}(\partial\Omega)\big\}, \tag{2.114}$$

is self-adjoint and strictly positive in $L^2(\Omega;d^nx)$. Moreover,

$$\operatorname{dom}\big((-\Delta_{D,\Omega})^{1/2}\big) = H^1_0(\Omega). \tag{2.115}$$

Proof. For convenience of the reader we reproduce the short proof of [40, Lemma A.1] in the special case of Dirichlet boundary conditions, given the proof of Theorem 2.17.

By Theorem 2.10, the operator $T_{D,\Omega}$ in $L^2(\Omega; d^n x)$, defined by

$$T_{D,\Omega} = -\Delta, \quad \mathrm{dom}(T_{D,\Omega}) = \big\{u \in H^1_0(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \gamma_D u = 0 \text{ in } L^2(\partial\Omega; d^{n-1}\omega)\big\}, \tag{2.116}$$

is self-adjoint and strictly positive, and

$$\mathrm{dom}\big((T_{D,\Omega})^{1/2}\big) = H^1_0(\Omega) \tag{2.117}$$

holds. Thus, we need to prove that $\mathrm{dom}(T_{D,\Omega}) \subseteq H^2(\Omega)$. To achieve this, we follow the proof of Theorem 2.17, starting with the same representation (2.102). This time, the requirement on g is that $\gamma_D \mathcal{S} g = h = \gamma_D w \in H^{3/2}(\partial\Omega)$. Thus, it suffices to know that

$$\gamma_D \mathcal{S} \colon H^{1/2}(\partial\Omega) \to H^{3/2}(\partial\Omega) \tag{2.118}$$

is an isomorphism. When $\partial\Omega$ is of class C^∞, it has been proved in [91, Proposition 7.9] that $\gamma_D \mathcal{S} \colon H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ is an isomorphism for each $s \in \mathbb{R}$ and, if Ω is of class $C^{1,r}$ with $(1/2) < r < 1$, the validity range of this result is limited to $-1-r < s < r$, which covers (2.118). The latter fact follows from an inspection of Taylor's original proof of [91, Proposition 7.9]. Here we just note that the only significant difference is that if $\partial\Omega$ is of class $C^{1,r}$ (instead of class C^∞), then S is a pseudodifferential operator whose symbol exhibits a limited amount of regularity in the space-variable. Such classes of operators have been studied in, e.g., [68], [90, Chs. 1, 2]. □

Remark 2.19. We emphasize that all results in this section extend to closed Schrödinger operators

$$H_{\Theta,\Omega} = -\Delta_{\Theta,\Omega} + V, \quad \mathrm{dom}\big(H_{\Theta,\Omega}\big) = \mathrm{dom}\big(-\Delta_{\Theta,\Omega}\big) \tag{2.119}$$

for (not necessarily real-valued) potentials V satisfying $V \in L^\infty(\Omega; d^n x)$, consistently replacing $-\Delta$ by $-\Delta + V$, etc. More generally, all results extend to Kato–Rellich bounded potentials V relative to $-\Delta_{\Theta,\Omega}$ with bound less than one. Extensions to potentials permitting stronger local singularities, and an extensions to (not necessarily bounded) Lipschitz domains with compact boundary, will be pursued elsewhere.

3. Generalized Robin and Dirichlet Boundary Value Problems and Robin-to-Dirichlet and Dirichlet-to-Robin Maps

This section is devoted to generalized Robin and Dirichlet boundary value problems associated with the Helmholtz differential expression $-\Delta - z$ in connection with the open set Ω. In addition, we provide a detailed discussion of Robin-to-Dirichlet maps, $M^{(0)}_{\Theta,D,\Omega}$, in $L^2(\partial\Omega; d^{n-1}\omega)$.

In this section we strengthen Hypothesis 2.2 by adding assumption (3.1) below:

Hypothesis 3.1. *In addition to Hypothesis* 2.2 *suppose that*

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^1(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{3.1}$$

We note that (3.1) is satisfied whenever there exists some $\varepsilon > 0$ such that

$$\widetilde{\Theta} \in \mathcal{B}\big(H^{1-\varepsilon}(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{3.2}$$

We recall the definition of the weak Neumann trace operator $\widetilde{\gamma}_N$ in (2.38), (2.39) and start with the Helmholtz Robin boundary value problems:

Theorem 3.2. *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then for every* $g \in L^2(\partial\Omega; d^{n-1}\omega)$, *the following generalized Robin boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^{3/2}(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

has a unique solution $u = u_\Theta$. This solution u_Θ satisfies

$$\begin{aligned} &\gamma_D u_\Theta \in H^1(\partial\Omega), \quad \tilde{\gamma}_N u_\Theta \in L^2(\partial\Omega; d^{n-1}\omega), \\ &\|\gamma_D u_\Theta\|_{H^1(\partial\Omega)} + \|\widetilde{\gamma}_N u_\Theta\|_{L^2(\partial\Omega;d^{n-1}\omega)}, \leq C\|g\|_{L^2(\partial\Omega;d^{n-1}\omega)} \end{aligned} \tag{3.4}$$

and

$$\|u_\Theta\|_{H^{3/2}(\Omega)} \leq C\|g\|_{L^2(\partial\Omega;d^{n-1}\omega)}, \tag{3.5}$$

for some constant constant $C = C(\Theta, \Omega, z) > 0$. Finally,

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^{3/2}(\Omega)\big), \tag{3.6}$$

and the solution u_Θ is given by the formula

$$u_\Theta = \big(\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big)^* g. \tag{3.7}$$

Proof. It is clear from Lemma 2.3 and Lemma 2.4 that the boundary value problem (3.3) has a meaningful formulation and that any solution satisfies the first line in (3.4). Uniqueness for (3.3) is an immediate consequence of the fact that $z \in \mathbb{C} \setminus \sigma(-\Delta_{\Theta,\Omega})$. As for existence, as in the proof of Theorem 2.17, we look for a candidate expressed as

$$u(x) = (\mathcal{S}_z h)(x), \quad x \in \Omega \tag{3.8}$$

for some $h \in L^2(\partial\Omega; d^{n-1}\omega)$. This ensures that $u \in H^{3/2}(\Omega)$ and $(-\Delta - z)u = 0$ in Ω. Above, the single layer potential $\mathcal{S}_z$ has been defined in (2.98). The boundary condition $\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g$ on $\partial\Omega$ is then equivalent to

$$\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)(\mathcal{S}_z h) = g, \tag{3.9}$$

respectively, to

$$\big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big)h + \widetilde{\Theta}\gamma_D\mathcal{S}_z h = g. \tag{3.10}$$

Here $K_z^{\#}$ has been defined in (2.99).

To obtain unique solvability of (3.10) for $h \in L^2(\partial\Omega; d^{n-1}\omega)$, given $g \in L^2(\partial\Omega; d^{n-1}\omega)$, at least when $z \in \mathbb{C}\backslash D$, where $D \subset \mathbb{C}$ is a discrete set, we proceed in a series of steps. The first step is to observe that the operator in question is Fredholm with index zero for every $z \in \mathbb{C}$. To see this, we decompose

$$\big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) = \big(-\tfrac{1}{2}I_{\partial\Omega} + K_0^{\#}\big) + \big(K_z^{\#} - K_0^{\#}\big), \tag{3.11}$$

and recall that $\big(K_z^{\#} - K_0^{\#}\big) \in \mathcal{B}_\infty(L^2(\partial\Omega; d^{n-1}\omega))$ (cf. Lemma D.3) and that $-\frac{1}{2}I_{\partial\Omega} + K_0^{\#}$ is a Fredholm operator in $L^2(\partial\Omega; d^{n-1}\omega)$ with Fredholm index equal to zero as proven by Verchota [95]. In addition, we note that $\widetilde{\Theta}\gamma_D\mathcal{S}_z \in \mathcal{B}_\infty(L^2(\partial\Omega; d^{n-1}\omega))$ which follows from Hypothesis 3.1 and the fact that the following operators are bounded

$$\begin{aligned} &\mathcal{S}_z : L^2(\partial\Omega; d^{n-1}\omega) \to \{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\}, \\ &\gamma_D : \{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\} \to L^2(\partial\Omega; d^{n-1}\omega), \end{aligned} \tag{3.12}$$

(where the space $\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$). See Lemma 2.3 and Theorem D.6. Thus, $\big(-\frac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_z$ is a Fredholm operator in $L^2(\partial\Omega; d^{n-1}\omega)$ with Fredholm index equal to zero, for every $z \in \mathbb{C}$. In particular, it is invertible if and only if it is injective.

In the second step, we study the injectivity of $\big(-\frac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_z$ on $L^2(\partial\Omega; d^{n-1}\omega)$. For this purpose we now suppose that

$$\big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big)k + \widetilde{\Theta}\gamma_D\mathcal{S}_z k = 0 \text{ for some } k \in L^2(\partial\Omega; d^{n-1}\omega). \tag{3.13}$$

Introducing $w=\mathcal{S}_z k$ in Ω one then infers that w satisfies

$$\begin{cases}(-\Delta-z)w=0 \text{ in } \Omega, \quad w\in H^{3/2}(\Omega),\\ \big(\widetilde\gamma_N+\widetilde\Theta\gamma_D\big)w=0 \text{ on } \partial\Omega.\end{cases} \tag{3.14}$$

Thus one obtains,

$$\begin{aligned} 0 &\leq \int_\Omega d^n x\,|\nabla w|^2 = \sum_{j=1}^n \int_\Omega d^n x\,\overline{\partial_j w}\partial_j w = -\int_\Omega d^n x\,\overline{\Delta w} w + \int_{\partial\Omega} d^{n-1}\omega\,\overline{\partial_j w}\,\nu_j\gamma_D w \\ &= \overline{z}\int_\Omega d^n x\,|w|^2 + \langle\gamma_D w,\widetilde\gamma_N w\rangle_{1/2} = \overline{z}\int_\Omega d^n x\,|w|^2 - \big\langle\gamma_D w,\widetilde\Theta\gamma_D w\big\rangle_{1/2}. \end{aligned} \tag{3.15}$$

At this point we will first consider the case when $z\in\mathbb{C}\backslash\mathbb{R}$ (so that, in particular, $\operatorname{Im} z\neq 0$). In this scenario, recalling (2.1) and taking the imaginary parts of the two most extreme sides of (3.15) imply that $\int_\Omega d^n x\,|w|^2=0$ and, hence, $w=0$ in Ω.

To continue, let $\widetilde\gamma_N^{ext}$ and γ_D^{ext} denote, respectively, the Neumann and Dirichlet traces for the exterior domain $\mathbb{R}^n\backslash\overline{\Omega}$. Also, parallel to (2.98), set

$$(\mathcal{S}_{ext,z} g)(x)=\int_{\partial\Omega} d^{n-1}\omega(y)\,E_n(z;x-y)g(y), \quad x\in\mathbb{R}^n\backslash\overline{\Omega},\ z\in\mathbb{C}, \tag{3.16}$$

where g is an arbitrary measurable function on $\partial\Omega$. Then, due to the weak singularity in the integral kernel of $\mathcal{S}_z$,

$$\gamma_D\mathcal{S}_z g=\gamma_D^{ext}\mathcal{S}_{ext,z} g \quad \text{for every } \ g\in L^2(\partial\Omega;d^{n-1}\omega), \tag{3.17}$$

whereas the counterpart of (2.100) becomes

$$\widetilde\gamma_N^{ext}\mathcal{S}_{ext,z} g=\big(\tfrac12 I_{\partial\Omega}+K_z^{\#}\big)g \quad \text{for every } \ g\in L^2(\partial\Omega;d^{n-1}\omega). \tag{3.18}$$

Compared with (2.100), the change in sign is due to the fact that the outward unit normal to $\mathbb{R}^n\backslash\overline{\Omega}$ is $-\nu$.

Moving on, if we set $w^{ext}(x)=(\mathcal{S}_{ext,z}k)(x)$ for $x\in\mathbb{R}^n\backslash\overline{\Omega}$, then from what we have proved so far

$$0=\gamma_D w=\gamma_D\mathcal{S}_z k=\gamma_D^{ext}w^{ext} \quad \text{in } \ L^2(\partial\Omega;d^{n-1}\omega). \tag{3.19}$$

Fix now a sufficiently large $R>0$ such that $\overline{\Omega}\subset B(0;R)$ and write the analogue of (3.15) for the restriction of w^{ext} to $B(0;R)\backslash\overline{\Omega}$:

$$\begin{aligned} \int_{B(0;R)\backslash\overline{\Omega}} d^n x\,|\nabla w^{ext}|^2 &= \overline{z}\int_{B(0;R)\backslash\overline{\Omega}} d^n x\,|w^{ext}|^2 - \langle\gamma_D^{ext}w^{ext},\widetilde\gamma_N^{ext}w^{ext}\rangle_{1/2} \\ &\quad - \int_{|\xi|=R} d^{n-1}\omega(\xi)\,w^{ext}(\xi)\frac{\xi}{|\xi|}\cdot\nabla w^{ext}(\xi). \end{aligned} \tag{3.20}$$

In view of (3.19), the above identity reduces to

$$\int_{B(0;R)\backslash\overline{\Omega}} d^n x\,|\nabla w^{ext}|^2 = \overline{z}\int_{B(0;R)\backslash\overline{\Omega}} d^n x\,|w^{ext}|^2 - \int_{|\xi|=R} d^{n-1}\omega(\xi)\,w^{ext}(\xi)\frac{\xi}{|\xi|}\cdot\nabla w^{ext}(\xi). \tag{3.21}$$

Recall that we are assuming $z\in\mathbb{C}\backslash\mathbb{R}$. Given that, by (C.17) (and the comment following right after it), the integral kernel of $\mathcal{S}_{ext,z}k$ has exponential decay at infinity, it follows that w^{ext} decays exponentially at infinity. Thus, after passing to limit $R\to\infty$, we arrive at

$$\int_{\mathbb{R}^n\backslash\overline{\Omega}} d^n x\,|\nabla w^{ext}|^2 = \overline{z}\int_{\mathbb{R}^n\backslash\overline{\Omega}} d^n x\,|w^{ext}|^2. \tag{3.22}$$

Consequently, taking the imaginary parts of both sides we arrive at the conclusion that $w^{ext} = 0$ in $\mathbb{R}^n\backslash\overline{\Omega}$. With this in hand, we may then invoke (2.100), (3.18) to deduce that

$$k = \widetilde{\gamma}_N^{ext}\mathcal{S}_z k - \widetilde{\gamma}_N\mathcal{S}_z k = \widetilde{\gamma}_N^{ext} w^{ext} - \widetilde{\gamma}_N w = 0, \tag{3.23}$$

given that w, w^{ext} vanish in Ω and $\mathbb{R}^n\backslash\overline{\Omega}$, respectively. Hence, $k = 0$ in $L^2(\partial\Omega; d^{n-1}\omega)$. This proves that the operator $\big(-\frac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_z$ is injective, hence, invertible on $L^2(\partial\Omega; d^{n-1}\omega)$ whenever $z \in \mathbb{C}\backslash\mathbb{R}$.

In the third step, the goal is to extend this result to other values of the parameter z. To this end, fix some $z_0 \in \mathbb{C}\backslash\mathbb{R}$, and for $z \in \mathbb{C}$, consider

$$\mathcal{A}_z = [\big(-\tfrac{1}{2}I_{\partial\Omega} + K_{z_0}^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_{z_0}]^{-1}[\big(K_z^{\#} - K_{z_0}^{\#}\big) + \widetilde{\Theta}\gamma_D(\mathcal{S}_z - \mathcal{S}_{z_0})]. \tag{3.24}$$

Observe that the operator-valued mapping $z \mapsto \mathcal{A}_z \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$ is analytic and, thanks to Lemma D.3, $\mathcal{A}_z \in \mathcal{B}_\infty\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$. The analytic Fredholm theorem then yields invertibility of $I + \mathcal{A}_z$ except for z in a discrete set $D \subset \mathbb{C}$. Thus,

$$\big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_z = [\big(-\tfrac{1}{2}I_{\partial\Omega} + K_{z_0}^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_{z_0}][I + \mathcal{A}_z] \tag{3.25}$$

is invertible for $z \in \mathbb{C}\backslash D$ where D is a discrete set which, by the invertibility result proved in the previous paragraph, is contained in $\mathbb{R}$.

The above argument proves unique solvability of (3.3) for $z \in \mathbb{C}\backslash D$, where D is a discrete subset of $\mathbb{R}$. The representation (3.8) and the fact that $\gamma_D\mathcal{S}_z \colon L^2(\partial\Omega; d^{n-1}\omega) \to H^1(\partial\Omega)$ boundedly then yields $\gamma_D u_\Theta \in H^1(\partial\Omega)$. Moreover, by (2.100) and (3.8),

$$\widetilde{\gamma}_N u_\Theta = \widetilde{\gamma}_N\mathcal{S}_z h = \big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big)h \in L^2(\partial\Omega; d^{n-1}\omega) \tag{3.26}$$

since by (2.101), $K_z^{\#} \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$. This proves (3.4) when $z \in \mathbb{C}\backslash D$. For $z \in \mathbb{C}\backslash D$, the natural estimate (3.5) is a consequence of the integral representation formula (3.8) and (D.26).

Next, fix a complex number $z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{\Theta,\Omega}))$ along with two functions, $v \in L^2(\Omega; d^n x)$ and $g \in L^2(\partial\Omega; d^{n-1}\omega)$. Also, let u_Θ solve (3.3). One computes

$$\begin{aligned}
\big(u_\Theta, v\big)_{L^2(\Omega;d^nx)} &= \big(u_\Theta, (-\Delta - \overline{z})(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\Omega;d^nx)} \\
&= \big((-\Delta - z)u_\Theta, (-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\Omega;d^nx)} \\
&\quad + \big(\widetilde{\gamma}_N u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&\quad - \big\langle \gamma_D u_\Theta, \widetilde{\gamma}_N\big(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega\big)^{-1}v\big\rangle_{1/2} \\
&= \big(\widetilde{\gamma}_N u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&\quad + \big\langle \gamma_D u_\Theta, \widetilde{\Theta}\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big\rangle_{1/2} \\
&= \big(\widetilde{\gamma}_N u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&\quad + \overline{\big\langle \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v, \widetilde{\Theta}\gamma_D u_\Theta\big\rangle_{1/2}} \\
&= \big(\widetilde{\gamma}_N u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&\quad + \big\langle \widetilde{\Theta}\gamma_D u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big\rangle_{1/2} \\
&= \big\langle \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big\rangle_{1/2} \\
&= \big(\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_\Theta, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&= \big(g, \gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&= \big(\big(\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big)^*g, v\big)_{L^2(\Omega;d^nx)}.
\end{aligned} \tag{3.27}$$

Since $v \in L^2(\Omega; d^n x)$ was arbitrary, this yields

$$u_\Theta = \big(\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z} I_\Omega)^{-1}\big)^* g \ \text{ in } \ L^2(\Omega; d^n x), \quad \text{for } \ z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{\Theta,\Omega})), \tag{3.28}$$

which proves (3.7) for $z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{\Theta,\Omega}))$. From this and (3.5), the membership (3.7) also follows when $z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{\Theta,\Omega}))$.

The extension to the more general case when $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$ is then done by resorting to analytic continuation with respect to z. More specifically, fix $z_0 \in \mathbb{C}\backslash\sigma(\Delta_{\Theta,\Omega})$. Then there exists $r > 0$ such that

$$\big(\overline{B(z_0, r)} \setminus \{z_0\}\big) \cap D = \emptyset, \quad \overline{B(z_0, r)} \cap \sigma(-\Delta_{\Theta,\Omega}) = \emptyset, \tag{3.29}$$

since D is discrete and $\sigma(-\Delta_{\Theta,\Omega})$ is closed. We may then write

$$[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z_0} I_\Omega)^{-1}]^* = -\frac{1}{2\pi i} \int_{|z-z_0|=r} dz\, [\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z} I_\Omega)^{-1}]^* \tag{3.30}$$

as operators in $\mathcal{B}(H^{-1/2}(\partial\Omega), L^2(\Omega; d^n x))$. However, granted (3.29), what we have shown so far yields that $[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z} I_\Omega)^{-1}]^* \in \mathcal{B}(L^2(\partial\Omega; d^{n-1}\omega), H^{3/2}(\Omega))$ whenever $|z - z_0| = r$, with a bound

$$\|[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z} I_\Omega)^{-1}]^*\|_{\mathcal{B}(L^2(\partial\Omega; d^{n-1}\omega), H^{3/2}(\Omega))} \leq C = C(\Omega, \Theta, z_0, r) \tag{3.31}$$

independent of the complex parameter $z \in \partial B(z_0, r)$. This estimate and Cauchy's representation formula (3.30) then imply that $[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z_0} I_\Omega)^{-1}]^* \in \mathcal{B}(L^2(\partial\Omega; d^{n-1}\omega), H^{3/2}(\Omega))$. This further entails that $u = [\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z_0} I_\Omega)^{-1}]^* g$ solves (3.3), written with z_0 in place of z, and satisfies (3.5). Finally, the memberships in (3.4) (along with naturally accompanying estimates) follow from Lemma 2.3 and Lemma 2.4. This shows that (3.6), along with the well-posedness of (3.3) and all the desired properties of the solution, hold whenever $z \in \mathbb{C}\backslash\sigma(\Delta_{\Theta,\Omega})$. $\square$

Next, we turn to the Dirichlet case originally treated in [43, Theorem 3.1] but under stronger regularity conditions on Ω. In order to facilitate the subsequent considerations, we isolate a useful technical result in the lemma below.

Lemma 3.3. *Assume Hypothesis* 2.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Omega,\Omega})$. *Then*

$$(-\Delta_{D,\Omega} - z I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\big\} \tag{3.32}$$

is a well-defined bounded operator, where the space $\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega; d^n x)}$.

Proof. Consider $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Omega,\Omega})$, $f \in L^2(\Omega; d^n x)$ and set $w = (-\Delta_{D,\Omega} - z I_\Omega)^{-1} f$. It follows that u is the unique solution of the problem

$$(-\Delta - z)w = f \ \text{ in } \ \Omega, \quad w \in H^1_0(\Omega). \tag{3.33}$$

The strategy is to devise an alternative representation for w from which it is clear that w has the claimed regularity in Ω. To this end, let $\widetilde{f}$ denote the extension of f by zero to $\mathbb{R}^n$ and denote by E the operator of convolution by $E_n(z; \cdot\,)$. Since the latter is smoothing of order 2, it follows that $v = \big(E\widetilde{f}\big)\big|_\Omega \in H^2(\Omega)$ and $(-\Delta - z)v = f$ in Ω. In particular, $g = -\gamma_D v \in H^1(\partial\Omega)$.

We now claim that the problem

$$(-\Delta - z)u = 0 \ \text{ in } \ \Omega, \quad u \in H^{3/2}(\Omega), \quad \gamma_D u = g \text{ on } \partial\Omega, \tag{3.34}$$

has a solution (satisfying natural estimates). To see this, we look for a solution in the form (3.8) for some $h \in L^2(\partial\Omega; d^{n-1}\omega)$. This guarantees that $u \in H^{3/2}(\Omega)$ by Theorem D.6, and $(-\Delta - z)u = 0$ in

Ω. Ensuring that the boundary condition holds comes down to solving $\gamma_D \mathcal{S}_z h = g$. In this regard, we recall that

$$\gamma_D \mathcal{S}_0 \colon L^2(\partial\Omega; d^{n-1}\omega) \to H^1(\partial\Omega) \text{ is invertible} \tag{3.35}$$

(cf. [95]). With this in hand, by relying on Theorem D.6 and arguing as in the proof of Theorem 3.2, one can show that there exists a discrete set $D \subset \mathbb{C}$ such that

$$\gamma_D \mathcal{S}_z \colon L^2(\partial\Omega; d^{n-1}\omega) \to H^1(\partial\Omega) \text{ is invertible for } z \in \mathbb{C}\backslash D. \tag{3.36}$$

Thus, a solution of (3.34) is given by

$$u = \mathcal{S}_z\big((\gamma_D \mathcal{S}_z)^{-1}(\gamma_D v)\big) \text{ if } z \in \mathbb{C}\backslash D. \tag{3.37}$$

Moreover, by Theorem D.6, this satisfies

$$\|u\|_{H^{3/2}(\Omega)} \leq C(\Omega, z)\|g\|_{H^1(\partial\Omega)} \leq C(\Omega, z)\|f\|_{L^2(\Omega; d^n x)}, \quad z \in \mathbb{C}\backslash D. \tag{3.38}$$

Consequently, if $z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{D,\Omega}))$, then $u+v$ solves (3.33). Hence, by uniqueness, $w = u+v$ in this case. This shows that $w = (-\Delta_{D,\Omega} - zI_\Omega)^{-1} f$ belongs to $H^{3/2}(\Omega)$ and satisfies $\Delta w \in L^2(\Omega; d^n x)$ with

$$\|w\|_{H^{3/2}(\Omega)} + \|\Delta w\|_{L^2(\Omega; d^n x)} \leq C(\Omega, z)\|f\|_{L^2(\Omega; d^n x)}, \quad z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{D,\Omega})). \tag{3.39}$$

In summary, the above argument shows that the operator (3.32) is well-defined and bounded whenever $z \in \mathbb{C}\backslash(D \cup \sigma(-\Delta_{D,\Omega}))$. The extension to $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$ is then achieved via analytic continuation (as in the last part of the proof of Theorem 3.2). $\square$

Having established Lemma 3.3, we can now readily prove the following result.

Lemma 3.4. *Assume Hypothesis* 2.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. *Then*

$$\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\omega)\big), \tag{3.40}$$

and

$$\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big]^* \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), L^2(\Omega; d^n x)\big). \tag{3.41}$$

Proof. Obviously, it suffices to only prove (3.40). However, this is an immediate consequence of Lemma 3.3 and Lemma 2.4. $\square$

We note that Lemma 3.4 corrects an inaccuracy in the proof of [43, Theorem 3.1] in the following sense: The proof of (3.20) and (3.21) in [43] relies on [43, Lemma 2.4], which in turn requires the stronger assumptions [43, Hypothesis 2.1] on Ω than merely the Lipschitz assumption on Ω. However, the current Lemmas 2.15 and 3.4 (and the subsequent Theorem 3.5) show that (3.20) and (3.21) in [43], as well as the results stated in [43, Theorem 3.1], are actually all correct.

After this preamble, we are ready to state the result about the well-posedness of the Dirichlet problem, alluded to above.

Theorem 3.5. *Assume Hypothesis* 2.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. *Then for every* $f \in H^1(\partial\Omega)$, *the following Dirichlet boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^{3/2}(\Omega), \\ \gamma_D u = f \text{ on } \partial\Omega, \end{cases} \tag{3.42}$$

has a unique solution $u = u_D$. *This solution* u_D *satisfies*

$$\widetilde{\gamma}_N u_D \in L^2(\partial\Omega; d^{n-1}\omega) \quad \text{and} \quad \|\widetilde{\gamma}_N u_D\|_{L^2(\partial\Omega; d^{n-1}\omega)} \leq C_D\|f\|_{H^1(\partial\Omega)}, \tag{3.43}$$

for some constant $C_D = C_D(\Omega, z) > 0$. *Moreover,*

$$\|u_D\|_{H^{3/2}(\Omega)} \leq C_D\|f\|_{H^1(\partial\Omega)}. \tag{3.44}$$

Finally,

$$\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(H^1(\partial\Omega), H^{3/2}(\Omega)\big), \tag{3.45}$$

and the solution u_D *is given by the formula*

$$u_D = -\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^* f. \tag{3.46}$$

Proof. Uniqueness for (3.42) is a direct consequence of the fact that $z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega})$. Existence, at least when $z\in\mathbb{C}\setminus D$ for a discrete set $D\subset\mathbb{C}$, is implicit in the proof of Lemma 3.3 (cf. (3.34)). Note that a solution thus constructed obeys (3.44) and satisfies (3.43) (cf. Lemmas 2.3-2.4).

Next, we turn to the proof of (3.46). Assume that $z\in\mathbb{C}\setminus(D\cup\sigma(-\Delta_{D,\Omega}))$ and denote by u_D the unique solution of (3.42). Also, recall (3.40)–(3.41). Based on these and Green's formula, one computes

$$\begin{aligned}
(u_D, v)_{L^2(\Omega;d^nx)} &= (u_D, (-\Delta-\overline{z})(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}v)_{L^2(\Omega;d^nx)} \\
&= ((-\Delta-z)u_D, (-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}v)_{L^2(\Omega;d^nx)} \\
&\quad + (\widetilde{\gamma}_N u_D, \gamma_D(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}v)_{L^2(\partial\Omega;d^{n-1}\omega)} \\
&\quad - \langle \gamma_D u_D, \widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}v\rangle_{1/2} \\
&= -\langle f, \widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}v\rangle_{1/2} \\
&= -\big(\big(\widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big)^* f, v\big)_{L^2(\Omega;d^nx)}
\end{aligned} \tag{3.47}$$

for any $v\in L^2(\Omega;d^nx)$. This proves (3.46) with the operators involved understood in the sense of (3.41). Given (3.44), one obtains (3.45) granted that $z\in\mathbb{C}\setminus(D\cup\sigma(-\Delta_{D,\Omega}))$.

Finally, extending the above results to the more general case in which $z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega})$ is done using analytic continuation, as in the last part of the proof of Theorem 3.2. □

Assuming Hypothesis 3.1, we now introduce the Dirichlet-to-Robin map $M_{D,\Theta,\Omega}^{(0)}(z)$ associated with $(-\Delta-z)$ on Ω, as follows,

$$M_{D,\Theta,\Omega}^{(0)}(z)\colon \begin{cases} H^1(\partial\Omega)\to L^2(\partial\Omega;d^{n-1}\omega), \\ \qquad f\mapsto -\big(\widetilde{\gamma}_N+\widetilde{\Theta}\gamma_D\big)u_D, \end{cases} \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{3.48}$$

where u_D is the unique solution of

$$(-\Delta-z)u=0 \text{ in } \Omega, \quad u\in H^{3/2}(\Omega), \quad \gamma_D u=f \text{ on } \partial\Omega. \tag{3.49}$$

Continuing to assume Hypothesis 3.1, we next introduce the Robin-to-Dirichlet map $M_{\Theta,D,\Omega}^{(0)}(z)$ associated with $(-\Delta-z)$ on Ω, as follows,

$$M_{\Theta,D,\Omega}^{(0)}(z)\colon \begin{cases} L^2(\partial\Omega;d^{n-1}\omega)\to H^1(\partial\Omega), \\ \qquad g\mapsto \gamma_D u_\Theta, \end{cases} \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}), \tag{3.50}$$

where u_Θ is the unique solution of

$$(-\Delta-z)u=0 \text{ in } \Omega, \quad u\in H^{3/2}(\Omega), \quad \big(\widetilde{\gamma}_N+\widetilde{\Theta}\gamma_D\big)u=g \text{ on } \partial\Omega. \tag{3.51}$$

We note that Robin-to-Dirichlet maps have also been studied in [10].

We conclude with the following theorem, one of the main results of this paper:

Theorem 3.6. *Assume Hypothesis* 3.1. *Then*

$$M_{D,\Theta,\Omega}^{(0)}(z)\in\mathcal{B}\big(H^1(\partial\Omega), L^2(\partial\Omega;d^{n-1}\omega)\big), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{3.52}$$

and

$$M_{D,\Theta,\Omega}^{(0)}(z) = \big(\widetilde{\gamma}_N+\widetilde{\Theta}\gamma_D\big)\big[\big(\widetilde{\gamma}_N+\widetilde{\Theta}\gamma_D\big)(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*, \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{3.53}$$

Moreover,

$$M_{\Theta,D,\Omega}^{(0)}(z) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^1(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}), \tag{3.54}$$

and, in fact,

$$M_{\Theta,D,\Omega}^{(0)}(z) \in \mathcal{B}_\infty\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{3.55}$$

In addition,

$$M_{\Theta,D,\Omega}^{(0)}(z) = \gamma_D\big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{3.56}$$

Finally, let $z \in \mathbb{C}\backslash(\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{\Theta,\Omega}))$. *Then*

$$M_{\Theta,D,\Omega}^{(0)}(z) = -M_{D,\Theta,\Omega}^{(0)}(z)^{-1}. \tag{3.57}$$

Proof. The membership in (3.52) is a consequence of Theorem 3.5. In this context we note that by the first line in (2.73), $\gamma_D(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1} = 0$, and hence

$$u_D = -\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* f = -\big[\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* f \tag{3.58}$$

by (3.46). Moreover, applying $-\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)$ to u_D in (3.46) implies formula (3.53). Likewise, (3.54) follows from Theorem 3.2. In addition, since $H^1(\partial\Omega)$ embeds compactly into $L^2(\partial\Omega; d^{n-1}\omega)$ (cf. (A.10) and [67, Proposition 2.4]), $M_{\Theta,D,\Omega}^{(0)}(z)$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$, are compact operators in $L^2(\partial\Omega; d^{n-1}\omega)$, justifying (3.55). Applying γ_D to u_Θ in (3.7) implies formula (3.56).

There remains to justify (3.57). To this end, let $g \in L^2(\partial\Omega; d^{n-1}\omega)$ be arbitrary. Then

$$M_{D,\Theta,\Omega}^{(0)}(z) M_{\Theta,D,\Omega}^{(0)}(z) g = M_{D,\Theta,\Omega}^{(0)}(z)\gamma_D u_\Theta = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big) u_D, \quad f = \gamma_D u_\Theta \in H^1(\partial\Omega). \tag{3.59}$$

Here u_Θ is the unique solution of $(-\Delta - z)u = 0$ with $u \in H^{3/2}(\Omega)$ and $\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g$, and u_D is the unique solution of $(-\Delta - z)u = 0$ with $u \in H^{3/2}(\Omega)$ and $\gamma_D u = f \in H^1(\partial\Omega)$. Since $(u_D - u_\Theta) \in H^{3/2}(\Omega)$ and $\gamma_D u_D = f = \gamma_D u_\Theta$, one concludes

$$\gamma_D(u_D - u_\Theta) = 0 \text{ and } (-\Delta - z)(u_D - u_\Theta) = 0. \tag{3.60}$$

Uniqueness of the Dirichlet problem proved in Theorem 3.5 then yields $u_D = u_\Theta$ which further entails that $-\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_D = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_\Theta = -g$. Thus,

$$M_{D,\Theta,\Omega}^{(0)}(z) M_{\Theta,D,\Omega}^{(0)}(z) g = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big) u_D = -g, \tag{3.61}$$

implying $M_{D,\Theta,\Omega}^{(0)}(z) M_{\Theta,D,\Omega}^{(0)}(z) = -I_{\partial\Omega}$. Conversely, let $f \in H^1(\partial\Omega)$. Then

$$M_{\Theta,D,\Omega}^{(0)}(z) M_{D,\Theta,\Omega}^{(0)}(z) f = M_{\Theta,D,\Omega}^{(0)}(z)\big(-\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_D\big) = \gamma_D u_\Theta, \tag{3.62}$$

and we set

$$g = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_D \in L^2(\partial\Omega; d^{n-1}\omega). \tag{3.63}$$

Here $u_D, u_\Theta \in H^{3/2}(\Omega)$ are such that $(-\Delta - z)u_\Theta = (-\Delta - z)u_D = 0$ in Ω and $\gamma_D u_D = f$, $\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_\Theta = g$. Thus $\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)(u_\Theta + u_D) = 0$, $(-\Delta - z)(u_\Theta + u_D) = 0$ and $(u_D + u_\Theta) \in H^{3/2}(\Omega)$. Uniqueness of the generalized Robin problem proved in Theorem 3.2 then yields $u_\Theta = -u_D$ and hence $\gamma_D u_\Theta = -\gamma_D u_D = -f$. Thus,

$$M_{\Theta,D,\Omega}^{(0)}(z) M_{D,\Theta,\Omega}^{(0)}(z) f = \gamma_D u_\Theta = -f, \tag{3.64}$$

implying $M_{\Theta,D,\Omega}^{(0)}(z) M_{D,\Theta,\Omega}^{(0)}(z) \subseteq -I_{\partial\Omega}$. The desired conclusion now follows. $\square$

Remark 3.7. In the above considerations, the special case $\Theta = 0$ represents the frequently studied Neumann-to-Dirichlet and Dirichlet-to-Neumann maps $M_{N,D,\Omega}^{(0)}(z)$ and $M_{D,N,\Omega}^{(0)}(z)$, respectively. That is, $M_{N,D,\Omega}^{(0)}(z) = M_{0,D,\Omega}^{(0)}(z)$ and $M_{D,N,\Omega}^{(0)}(z) = M_{D,0,\Omega}^{(0)}(z)$. Thus, as a corollary of Theorem 3.6 we have

$$M_{N,D,\Omega}^{(0)}(z) = -M_{D,N,\Omega}^{(0)}(z)^{-1}, \tag{3.65}$$

whenever Hypothesis 2.1 holds and $z \in \mathbb{C}\backslash(\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{N,\Omega}))$.

Remark 3.8. We emphasize again that all results in this section immediately extend to Schrödinger operators $H_{\Theta,\Omega} = -\Delta_{\Theta,\Omega} + V$, $\operatorname{dom}(H_{\Theta,\Omega}) = \operatorname{dom}(-\Delta_{\Theta,\Omega})$ in $L^2(\Omega; d^n x)$ for (not necessarily real-valued) potentials V satisfying $V \in L^\infty(\Omega; d^n x)$, or more generally, for potentials V which are Kato–Rellich bounded with respect to $-\Delta_{\Theta,\Omega}$ with bound less than one. Denoting the corresponding M-operators by $M_{D,N,\Omega}(z)$ and $M_{\Theta,D,\Omega}(z)$, respectively, we note, in particular, that (3.48)–(3.57) extend replacing $-\Delta$ by $-\Delta + V$ and restricting $z \in \mathbb{C}$ appropriately.

Our discussion of Weyl–Titchmarsh operators follows the earlier papers [40] and [43]. For related literature on Weyl–Titchmarsh operators, relevant in the context of boundary value spaces (boundary triples, etc.), we refer, for instance, to [3], [5], [12], [13], [18]–[21], [30]– [33], [39], [41], [44, Ch. 3], [61], [62], [66], [74], [75], [78], [79], [82], [83], [94].

4. Some Variants of Krein's Resolvent Formula

In this section we present our principal new results, variants of Krein's formula for the difference of resolvents of generalized Robin Laplacians and Dirichlet Laplacians on bounded Lipschitz domains.

We start by weakening Hypothesis 3.1 by using assumption (4.1) below:

Hypothesis 4.1. *In addition to Hypothesis* 2.2 *suppose that*

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big). \tag{4.1}$$

We note that condition (4.1) is satisfied if there exists some $\varepsilon > 0$ such that

$$\widetilde{\Theta} \in \mathcal{B}\big(H^{1/2-\varepsilon}(\partial\Omega), H^{-1/2}(\partial\Omega)\big). \tag{4.2}$$

Before proceeding with the main topic of this section, we will comment to the effect that Hypothesis 3.1 is indeed stronger than Hypothesis 4.1, as the latter follows from the former via duality and interpolation, implying

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^s(\partial\Omega), H^{s-1}(\partial\Omega)\big), \quad 0 \le s \le 1. \tag{4.3}$$

To see this, one first employs the fact that

$$(H^{s_0}(\partial\Omega), H^{s_1}(\partial\Omega))_{\theta,2} = H^s(\partial\Omega) \tag{4.4}$$

for $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$, $0 \le s_0, s_1 \le 1$, and $s_0 \ne s_1$, where $(\cdot,\cdot)_{\theta,q}$ denotes the *real interpolation* method.

Second, one uses the fact that if $T : X_j \to Y_j$, $j = 0, 1$, is a linear and bounded operator between two pairs of compatible Banach spaces, which is compact for $j = 0$, then $T \in \mathcal{B}_\infty((X_0, X_1)_{\theta,p}, (Y_0, Y_1)_{\theta,p})$ for every $\theta \in (0, 1)$. This is a result due to Cwikel [25]:

Theorem 4.2 ([25])**.** *Let X_j, Y_j, $j = 0, 1$, be two compatible Banach space couples and suppose that the linear operator $T : X_j \to Y_j$ is bounded for $j = 0$ and compact for $j = 1$. Then $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is compact for all $\theta \in (0, 1)$ and $p \in [1, \infty]$.*

(Interestingly, the corresponding result for the complex method of interpolation remains open.)

In our next two results below (Theorems 4.3–4.5) we discuss the solvability of the Dirichlet and Robin boundary value problems with solution in the energy space $H^1(\Omega)$.

Theorem 4.3. *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then for every* $g \in H^{-1/2}(\partial\Omega)$*, the following generalized Robin boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \text{ on } \partial\Omega, \end{cases} \tag{4.5}$$

has a unique solution $u = u_\Theta$. *Moreover, there exists a constant* $C = C(\Theta, \Omega, z) > 0$ *such that*

$$\|u_\Theta\|_{H^1(\Omega)} \leq C\|g\|_{H^{-1/2}(\partial\Omega)}. \tag{4.6}$$

In particular,

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(H^{-1/2}(\partial\Omega), H^1(\Omega)\big), \tag{4.7}$$

and the solution u_Θ *of* (4.5) *is once again given by formula* (3.7).

Proof. The argument follows a pattern similar to that in the proof of Theorem 2.17. In a first stage, we look for a solution for (4.5) in the form

$$u(x) = (\mathcal{S}_z h)(x), \quad x \in \Omega, \tag{4.8}$$

for some $h \in H^{-1/2}(\partial\Omega)$. Here the single layer potential $\mathcal{S}_z$ has been defined in (2.98), and the fundamental Helmholtz solution E_n is given by (2.97) (cf. also (C.1)). Any such choice of u guarantees that u belongs to $H^1(\Omega)$ and satisfies $(-\Delta - z)u = 0$ in Ω. See (D.27). To ensure that the boundary condition in (4.5) is verified, one then takes

$$h = \big[\big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_z\big]^{-1}g. \tag{4.9}$$

That the operator

$$\big(-\tfrac{1}{2}I_{\partial\Omega} + K_z^{\#}\big) + \widetilde{\Theta}\gamma_D\mathcal{S}_z : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \tag{4.10}$$

is invertible for all but a discreet set of real values of the parameter z, can be established based on Hypothesis 4.1 by reasoning as before. The key result in this context is that the operator $-\frac{1}{2}I_{\partial\Omega} + K_z^{\#} \in \mathcal{B}(H^{-1/2}(\partial\Omega))$ is Fredholm, with Fredholm index zero for every $z \in \mathbb{C}$. □

Remark 4.4. As a byproduct of Theorem 4.3 (with $\Theta = 0$) we obtain that the weak Neumann trace $\widetilde{\gamma}_N$ in (2.38), (2.39) is onto.

In the following we denote by $\widetilde{I}_\Omega$ the continuous inclusion (embedding) map of $H^1(\Omega)$ into $\big(H^1(\Omega)\big)^*$. By a slight abuse of notation, we also denote the continuous inclusion map of $H_0^1(\Omega)$ into $\big(H_0^1(\Omega)\big)^*$ by the same symbol $\widetilde{I}_\Omega$.

Theorem 4.5. *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then for every* $w \in (H^1(\Omega))^*$*, the following generalized inhomogeneous Robin problem,*

$$\begin{cases} (-\Delta - z)u = w \text{ in } \Omega, \quad u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.11}$$

has a unique solution $u = u_{\Theta,w}$. *Moreover, there exists a constant* $C = C(\Theta, \Omega, z) > 0$ *such that*

$$\|u_{\Theta,w}\|_{H^1(\Omega)} \leq C\|w\|_{(H^1(\partial\Omega))^*}. \tag{4.12}$$

In particular, the operator $(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$*, originally defined as a bounded operator on* $L^2(\Omega; d^n x)$*,*

$$(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \tag{4.13}$$

can be extended to a mapping

$$\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \in \mathcal{B}\big(\big(H^1(\Omega)\big)^*; H^1(\Omega)\big). \tag{4.14}$$

Here $-\widetilde{\Delta}_{\Theta,\Omega} \in \mathcal{B}\big(H^1(\Omega), \big(H^1(\Omega)\big)^*\big)$ *is the extension of* $-\Delta_{\Theta,\Omega}$ *in accordance with* (B.26).

Proof. It is well-known (see, e.g., [47]) that the adjoint space $\big(H^1(\Omega)\big)^*$ can be naturally identified with $\{v \in H^{-1}(\mathbb{R}^n) \,|\, \operatorname{supp} v \subseteq \overline{\Omega}\}$. Hence, if $w \in \big(H^1(\Omega)\big)^*$, taking the convolution of w with $E_n(z;\,\cdot\,)$ in (C.1) and then restricting back to Ω yields a function $u_0 \in H^1(\Omega)$ for which $(-\Delta - z)u_0 = w$ in Ω. A solution of (4.11) is then given by $u = u_0 + u_1$, where u_1 satisfies

$$\begin{cases} (-\Delta - z)u_1 = 0 \text{ in } \Omega, \quad u_1 \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_1 = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_0 \in H^{-1/2}(\partial\Omega) \text{ on } \partial\Omega. \end{cases} \tag{4.15}$$

That the latter boundary problem is solvable is ensured by Theorem 4.3. Note that the solution thus constructed obeys (4.12). Uniqueness for (4.11) follows from the corresponding uniqueness statement in Theorem 4.3. $\square$

Remark 4.6. Similar (yet simpler) considerations also show that the operator $(-\Delta_{D,\Omega} - zI_\Omega)^{-1}$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, originally defined as bounded operator on $L^2(\Omega; d^nx)$,

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^nx)\big), \tag{4.16}$$

extends to a mapping

$$\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \in \mathcal{B}\big(H^{-1}(\Omega); H^1_0(\Omega)\big). \tag{4.17}$$

Here $-\widetilde{\Delta}_{D,\Omega} \in \mathcal{B}\big(H^1_0(\Omega), H^{-1}(\Omega)\big)$ is the extension of $-\Delta_{D,\Omega}$ in accordance with (B.26). Indeed, the Lax–Milgram lemma applies and yields that

$$\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big) \colon H^1_0(\Omega) \to \big(H^1_0(\Omega)\big)^* = H^{-1}(\Omega) \tag{4.18}$$

is, in fact, an isomorphism whenever $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$.

Corollary 4.7. *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then the operator* $M^{(0)}_{\Theta,D,\Omega}(z) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$ *in* (3.50), (3.51) *extends* (*in a compatible manner*) *to*

$$\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z) \in \mathcal{B}\big(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{4.19}$$

In addition, $\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)$ *permits the representation*

$$\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z) = \gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{4.20}$$

The same applies to the adjoint $M^{(0)}_{\Theta,D,\Omega}(z)^* \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$ *of* $M^{(0)}_{\Theta,D,\Omega}(z)$, *resulting in the bounded extension* $\big(\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)\big)^* \in \mathcal{B}\big(H^{-1/2}(\partial\Omega),\, H^{1/2}(\partial\Omega)\big)$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$.

Proof. The claim (4.19) is a direct consequence of Theorem 4.3, while the claim (4.20) follows from the fact that

$$\gamma_D^* \colon \big(H^{1/2}(\partial\Omega)\big)^* = H^{-1/2}(\partial\Omega) \to \big(H^1(\Omega)\big)^* \tag{4.21}$$

(cf. (A.32), (4.14) and (3.56)). The rest follows from dualizing these claims. $\square$

Lemma 4.8. *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$. *Then the following resolvent relation holds on* $\big(H^1(\Omega)\big)^*$, *regarded as a subspace of* $H^{-1}(\Omega)$*:*

$$\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} = \big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} + \big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^*\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1}. \tag{4.22}$$

Proof. To set the stage, we recall (2.38)–(2.39) and (4.21). Together with (4.14) and (4.17), these ensure that the composition of operators appearing on the right-hand side of (4.22) is well-defined. Next, let $\phi_1, \psi_1 \in L^2(\Omega; d^nx)$ be arbitrary and define

$$\begin{aligned} &\phi = (-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1 \in \operatorname{dom}(\Delta_{\Theta,\Omega}) \subset H^1(\Omega), \\ &\psi = (-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1 \in \operatorname{dom}(\Delta_{D,\Omega}) \subset \big(H^1_0(\Omega) \cap \operatorname{dom}(\widetilde{\gamma}_N)\big) \subset H^1(\Omega). \end{aligned} \tag{4.23}$$

As a consequence of our earlier results, both sides of (4.22) are bounded operators from $(H^1(\Omega))^*$ into $H^1(\Omega)$. By density, it therefore suffices to show that the following identity holds:

$$\begin{aligned}(\phi_1, (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega;d^nx)} - (\phi_1, (-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega;d^nx)} \\ = (\phi_1, (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\gamma_D^*\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega;d^nx)}.\end{aligned} \tag{4.24}$$

We note that according to (4.23) one has,

$$(\phi_1, (-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega;d^nx)} = ((-\Delta_{D,\Omega} - \overline{z}I_\Omega)\phi, \psi)_{L^2(\Omega;d^nx)}, \tag{4.25}$$

$$\begin{aligned}(\phi_1, (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega;d^nx)} &= \big(\big((-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\big)^*\phi_1, \psi_1\big)_{L^2(\Omega;d^nx)} \\ &= ((-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1, \psi_1)_{L^2(\Omega;d^nx)} \\ &= (\phi, (-\Delta_{D,\Omega} - zI_\Omega)\psi)_{L^2(\Omega;d^nx)},\end{aligned} \tag{4.26}$$

and, further,

$$\begin{aligned}&(\phi_1, (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\gamma_D^*\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega;d^nx)} \\ &\quad = {}_{H^1(\Omega)}\langle(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1, \gamma_D^*\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1\rangle_{(H^1(\Omega))^*} \\ &\quad = \langle\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1, \widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\psi_1\rangle_{1/2} = \langle\gamma_D\phi, \widetilde{\gamma}_N\psi\rangle_{1/2}.\end{aligned} \tag{4.27}$$

Thus, matters have been reduced to proving that

$$((-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)\phi, \psi)_{L^2(\Omega;d^nx)} - (\phi, (-\Delta_{D,\Omega} - zI_\Omega)\psi)_{L^2(\Omega;d^nx)} = \langle\gamma_D\phi, \widetilde{\gamma}_N\psi\rangle_{1/2}. \tag{4.28}$$

Using (A.31) for the left-hand side of (4.28) one obtains

$$\begin{aligned}&((-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)\phi, \psi)_{L^2(\Omega;d^nx)} - (\phi, (-\Delta_{D,\Omega} - zI_\Omega)\psi)_{L^2(\Omega;d^nx)} \\ &\quad = -(\Delta\phi, \psi)_{L^2(\Omega;d^nx)} + (\phi, \Delta\psi)_{L^2(\Omega;d^nx)} \\ &\quad = (\nabla\phi, \nabla\psi)_{L^2(\Omega;d^nx)^n} - \langle\widetilde{\gamma}_N\phi, \gamma_D\psi\rangle_{1/2} - (\nabla\phi, \nabla\psi)_{L^2(\Omega;d^nx)^n} + \langle\gamma_D\phi, \widetilde{\gamma}_N\psi\rangle_{1/2} \\ &\quad = -\langle\widetilde{\gamma}_N\phi, \gamma_D\psi\rangle_{1/2} + \langle\gamma_D\phi, \widetilde{\gamma}_N\psi\rangle_{1/2}.\end{aligned} \tag{4.29}$$

Observing that $\gamma_D\psi = 0$ since $\psi \in \mathrm{dom}(\Delta_{D,\Omega})$, one concludes (4.28). ☐

The following regularity result for the Robin resolvent will also play an important role shortly.

Lemma 4.9. *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then*

$$(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} : L^2(\Omega; d^nx) \to \big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^nx)\big\} \tag{4.30}$$

is a well-defined bounded operator, where the space $\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^nx)\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^nx)}$.

Proof. Consider $f \in L^2(\Omega; d^nx)$ and set $u = (-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}f$. It follows that u is the unique solution of the problem

$$(-\Delta - z)u = f \text{ in } \Omega, \quad u \in H^1(\Omega), \quad \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = 0 \text{ on } \partial\Omega. \tag{4.31}$$

The strategy is to devise an alternative representation for u from which it is clear that u has the claimed regularity in Ω. To this end, let $\widetilde{f}$ denote the extension of f by zero to $\mathbb{R}^n$ and denote by E the operator of convolution by $E_n(z;\,\cdot\,)$. Since the latter is smoothing of order 2, it follows that $w = (E\widetilde{f})|_\Omega \in H^2(\Omega)$ and $(-\Delta - z)w = f$ in Ω. Also, let v be the unique solution of the problem

$$(-\Delta - z)v = 0 \text{ in } \Omega, \quad v \in H^{3/2}(\Omega), \quad \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)v = -\big(\gamma_N + \widetilde{\Theta}\gamma_D\big)w \text{ on } \partial\Omega. \tag{4.32}$$

That (4.32) is solvable is a consequence of Theorem 4.3. Then $v + w$ also solves (4.31) so that, by uniqueness, $u = v + w$. This shows that u has the desired regularity properties and, hence, the operator (4.30) is well-defined and bounded. ☐

Under the Hypothesis 4.1, (4.14) and (2.7) give

$$\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-zI_\Omega\big)^{-1}\in\mathcal{B}\big((H^1(\Omega))^*,H^{1/2}(\partial\Omega)\big). \tag{4.33}$$

Hence, by duality,

$$\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-zI_\Omega\big)^{-1}\big]^*\in\mathcal{B}\big(H^{-1/2}(\partial\Omega),H^1(\Omega)\big). \tag{4.34}$$

We wish to complement this with the following result.

Corollary 4.10. *Assume Hypothesis* 3.1 *and suppose that* $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then*

$$\gamma_D(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^1(\partial\Omega)\big). \tag{4.35}$$

In particular,

$$\big[\gamma_D(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}\big]^*\in\mathcal{B}\big(H^{-1}(\partial\Omega),L^2(\Omega;d^nx)\big)\hookrightarrow\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\Omega),L^2(\Omega;d^nx)\big). \tag{4.36}$$

In addition, the operator (4.36) *is compatible with* (4.34) *in the sense that*

$$\big[\gamma_D(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}\big]^*f=\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-zI_\Omega\big)^{-1}\big]^*f\ \text{ in } L^2(\Omega;d^nx),\quad f\in H^{-1/2}(\partial\Omega). \tag{4.37}$$

As a consequence,

$$\big[\gamma_D(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}\big]^*f=\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-zI_\Omega\big)^{-1}\big]^*f\ \text{ in } L^2(\Omega;d^nx),\quad f\in L^2(\partial\Omega;d^{n-1}\omega). \tag{4.38}$$

Proof. The first part of the statement is an immediate consequence of Lemma 4.9 and Lemma 2.3. As for (4.37), pick $f\in H^{-1/2}(\partial\Omega)\hookrightarrow H^{-1}(\partial\Omega)$ and $u\in L^2(\Omega;d^nx)\hookrightarrow(H^1(\Omega))^*$ arbitrary. We may then write

$$\begin{aligned}\langle[\gamma_D(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}]^*f\,,\,u\rangle_{L^2(\Omega;d^nx)}&=\langle f\,,\,\gamma_D(-\Delta_{\Theta,\Omega}-\overline{z}I_\Omega)^{-1}u\rangle_1\\&=\langle f\,,\,\gamma_D(-\Delta_{\Theta,\Omega}-\overline{z}I_\Omega)^{-1}u\rangle_{1/2}\\&=\big\langle f\,,\,\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-\overline{z}I_\Omega\big)^{-1}u\big\rangle_{1/2}\\&={}_{H^1(\Omega)}\big\langle\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-zI_\Omega\big)^{-1}\big]^*f\,,\,u\big\rangle_{(H^1(\Omega))^*}\\&=\big\langle\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega}-zI_\Omega\big)^{-1}\big]^*f\,,\,u\big\rangle_{L^2(\Omega;d^nx)},\end{aligned} \tag{4.39}$$

since (4.13) and (4.14) are compatible. This gives (4.37). Since $L^2(\partial\Omega;d^{n-1}\omega)\hookrightarrow H^{-1/2}(\partial\Omega)$, (4.38) also follows. □

We will need a similar compatibility result for the composition between the Neumann trace and resolvents of the Dirichlet Laplacian.

Lemma 4.11. *Assume Hypothesis* 2.1. *Then*

$$\big(-\widetilde{\Delta}_{D,\Omega}-zI_\Omega\big)^{-1}:(H^1(\Omega))^*\to\big\{u\in H^1(\Omega)\,|\,\Delta u\in(H^1(\Omega))^*\big\},\quad z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega}), \tag{4.40}$$

is a well-defined, linear and bounded operator when the space $\{u\in H^1(\Omega)\,|\,\Delta u\in(H^1(\Omega))^*\}$ *is equipped with the natural graph norm* $u\mapsto\|u\|_{H^1(\Omega)}+\|\Delta u\|_{(H^1(\Omega))^*}$. *Consequently,*

$$\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega}-zI_\Omega\big)^{-1}\in\mathcal{B}\big((H^1(\Omega))^*,H^{-1/2}(\partial\Omega)\big),\quad z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega}), \tag{4.41}$$

and, hence,

$$\big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega}-zI_\Omega\big)^{-1}\big]^*\in\mathcal{B}\big(H^{1/2}(\partial\Omega),H^1(\Omega)\big),\quad z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega}). \tag{4.42}$$

Furthermore, the operators (4.41), (4.42) *are compatible with* (3.40) *and* (3.41), *respectively, in the sense that for each* $z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega})$,

$$\widetilde{\gamma}_N(-\Delta_{D,\Omega}-zI_\Omega)^{-1}f=\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{\Omega,\Omega}-zI_\Omega\big)^{-1}f\ \text{ in } H^{-1/2}(\partial\Omega),\quad f\in L^2(\Omega;d^nx), \tag{4.43}$$

and

$$\big[\widetilde\gamma_N(-\Delta_{D,\Omega}-zI_\Omega)^{-1}\big]^*f=\big[\widetilde\gamma_N\big(-\widetilde\Delta_{\Omega,\Omega}-zI_\Omega\big)^{-1}\big]^*f \text{ in } L^2(\Omega;d^nx), \quad f\in H^{1/2}(\partial\Omega). \tag{4.44}$$

Proof. Let $z\in\mathbb{C}\setminus\sigma(-\Delta_{D,\Omega})$. If $f\in(H^1(\Omega))^*\hookrightarrow H^{-1}(\Omega)$ and $u=\big(-\widetilde\Delta_{D,\Omega}-zI_\Omega\big)^{-1}f$ then $u\in H^1_0(\Omega)$ satisfies $(-\Delta-z)u=f$. Hence, $\Delta u=-zu-f\in(H^1(\Omega))^*$ which shows that the operator (4.40) is well-defined and bounded. Then (4.41) is a consequence of this and (2.38), whereas (4.42) follows from (4.41) and duality.

Going further, (4.43) is implied by Lemma 3.3, the compatibility statement in Lemma 2.4, and Remark 4.6. There remains to justify (4.44). To this end, if $f\in H^{1/2}(\partial\Omega)\hookrightarrow L^2(\partial\Omega;d^{n-1}\omega)$ and $u\in L^2(\Omega;d^nx)\hookrightarrow(H^1(\Omega))^*$ are arbitrary, we may write

$$\begin{aligned}
\langle[\widetilde\gamma_N(-\Delta_{D,\Omega}-zI_\Omega)^{-1}]^*f\,,\,u\rangle_{L^2(\Omega;d^nx)}&=\langle f\,,\,\widetilde\gamma_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}u\rangle_0\\
&=\langle f\,,\,\widetilde\gamma_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}u\rangle_{1/2}\\
&=\big\langle f\,,\,\widetilde\gamma_N\big(-\widetilde\Delta_{D,\Omega}-\overline{z}I_\Omega\big)^{-1}u\big\rangle_{1/2}\\
&={}_{H^1(\Omega)}\big\langle\big[\widetilde\gamma_N\big(-\widetilde\Delta_{D,\Omega}-zI_\Omega\big)^{-1}\big]^*f\,,\,u\big\rangle_{(H^1(\Omega))^*}\\
&=\big\langle\big[\widetilde\gamma_N\big(-\widetilde\Delta_{D,\Omega}-zI_\Omega\big)^{-1}\big]^*f\,,\,u\big\rangle_{L^2(\Omega;d^nx)},
\end{aligned}\tag{4.45}$$

where the second equality is based on (4.43). This justifies (4.44) and finishes the proof of the lemma. □

The stage is now set for proving the L^2-version of Lemma 4.8.

Lemma 4.12. *Assume Hypothesis* 3.1 *and suppose that* $z\in\mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega})\cup\sigma(-\Delta_{D,\Omega}))$. *Then the following resolvent relation holds on* $L^2(\Omega;d^nx)$*,*

$$\begin{aligned}
(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}&=(-\Delta_{D,\Omega}-zI_\Omega)^{-1}+\big[\gamma_D(-\Delta_{\Theta,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\big[\widetilde\gamma_N(-\Delta_{D,\Omega}-zI_\Omega)^{-1}\big]\\
&=(-\Delta_{D,\Omega}-zI_\Omega)^{-1}+\big[\widetilde\gamma_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\big[\gamma_D(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}\big].
\end{aligned}\tag{4.46}$$

Proof. Consider the first equality in (4.46). To begin with, we note that the following operators are well-defined, linear and bounded:

$$(-\Delta_{D,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx)\big),\quad(-\Delta_{\Theta,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx)\big),\tag{4.47}$$

$$\widetilde\gamma_N(-\Delta_{D,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\partial\Omega;d^{n-1}\omega)\big),\tag{4.48}$$

$$\big[\gamma_D(-\Delta_{\Theta,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\omega),L^2(\Omega;d^nx))\big).\tag{4.49}$$

Indeed, (4.47) follows from the fact that $z\in\mathbb{C}\backslash\big(\sigma(-\Delta_{D,\Omega})\cup\sigma(-\Delta_{\Theta,\Omega})\big)$, (4.48) is covered by (3.40), and (4.49) is taken care of by (4.36). Together, these memberships show that both sides of (4.46) are bounded operators on $L^2(\Omega;d^nx)$. Having established this, the first equality in (4.46) follows from Lemma 4.8, granted the compatibility results from Corollary 4.10 and Lemma 4.11. Then the second equality in (4.46) is a consequence of what we have proved so far and duality. □

We note that the special case $\Theta=0$ in Lemma 4.12 was discussed by Nakamura [71] (in connection with cubic boxes Ω) and subsequently in [40, Lemma A.3] (in the case of a Lipschitz domain with a compact boundary).

Lemma 4.13. *Assume Hypothesis* 4.1 *and suppose that* $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then*

$$\big[\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)\big]^*=\widetilde{M}^{(0)}_{\Theta,D,\Omega}(\overline{z})\tag{4.50}$$

as operators in $\mathcal{B}\big(H^{-1/2}(\partial\Omega); H^{1/2}(\partial\Omega)\big)$. *In particular, assuming Hypothesis* 3.1*, then*

$$\big[M^{(0)}_{\Theta,D,\Omega}(z)\big]^* = M^{(0)}_{\Theta,D,\Omega}(\overline{z}). \tag{4.51}$$

Proof. Let $f,g \in H^{-1/2}(\partial\Omega)$. Then using the definition of $\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)$ one infers

$$\big\langle \widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)f, g\big\rangle_{1/2} = \big\langle \gamma_D u, \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)v\big\rangle_{1/2} \tag{4.52}$$

where u,v solve the Robin boundary value problems

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = f \text{ on } \partial\Omega, \end{cases} \tag{4.53}$$

and

$$\begin{cases} (-\Delta - \overline{z})v = 0 \text{ in } \Omega, \quad v \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)v = g \text{ on } \partial\Omega, \end{cases} \tag{4.54}$$

respectively. That this is possible is ensured by Theorem 4.3. Using (A.31) we may then write

$$\begin{aligned}
\big\langle \gamma_D u, \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)v\big\rangle_{1/2} &= \langle \gamma_D u, \widetilde{\gamma}_N v\rangle_{1/2} + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2} \\
&= (\nabla u, \nabla v)_{L^2(\Omega;d^nx)^n} + {}_{H^1(\Omega)}\langle u, \Delta v\rangle_{(H^1(\Omega))^*} + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2} \\
&= (\nabla u, \nabla v)_{L^2(\Omega;d^nx)^n} - \overline{z}\,(u,v)_{L^2(\Omega;d^nx)^n} + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2} \\
&= \overline{(\nabla v, \nabla u)_{L^2(\Omega;d^nx)^n} - z\,(v,u)_{L^2(\Omega;d^nx)^n}} + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2} \\
&= \overline{(\nabla v, \nabla u)_{L^2(\Omega;d^nx)^n} + {}_{H^1(\Omega)}\langle v, \Delta u\rangle_{(H^1(\Omega))^*}} + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2} \\
&= \overline{\langle \gamma_D v, \widetilde{\gamma}_N u\rangle_{1/2}} + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2} \\
&= \overline{\langle \gamma_D v, \widetilde{\gamma}_N u\rangle_{1/2}} + \big\langle \widetilde{\Theta}\gamma_D u, \gamma_D v\big\rangle_{1/2} \\
&= \overline{\langle \gamma_D v, \widetilde{\gamma}_N u\rangle_{1/2} + \big\langle \gamma_D v, \widetilde{\Theta}\gamma_D u\big\rangle_{1/2}} \\
&= \overline{\big\langle \gamma_D v, \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u\big\rangle_{1/2}} \\
&= \overline{\big\langle \widetilde{M}^{(0)}_{\Theta,D,\Omega}(\overline{z})g, f\big\rangle_{1/2}}. \\
&= \big\langle f, \widetilde{M}^{(0)}_{\Theta,D,\Omega}(\overline{z})g\big\rangle_{1/2}.
\end{aligned} \tag{4.55}$$

Now (4.50) follows from (4.52) and (4.55). Finally, (4.51) follows from (4.50) by restriction of the latter to $L^2(\partial\Omega; d^{n-1}\omega)$. ☐

Next we briefly turn to the Herglotz property of the Robin-to-Dirichlet map. We recall that an operator-valued function $M(z) \in \mathcal{B}(\mathcal{H})$, $z \in \mathbb{C}_+$ (where $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0)$, for some separable complex Hilbert space $\mathcal{H}$, is called an *operator-valued Herglotz function* if $M(\,\cdot\,)$ is analytic on $\mathbb{C}_+$ and

$$\mathrm{Im}(M(z)) \geqslant 0, \quad z \in \mathbb{C}_+. \tag{4.56}$$

Here, as usual, $\mathrm{Im}(M) = (M - M^*)/(2i)$.

Lemma 4.14. *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}_+$*. Then for every* $g \in H^{-1/2}(\partial\Omega)$*,* $g \neq 0$*,*

$$\frac{1}{2i}\big\langle\, g, \big[\widetilde{M}_{\Theta,D}(z) - \widetilde{M}_{\Theta,D}(z)^*\big]g\big\rangle_{1/2} = \mathrm{Im}(z)\|u_\Theta\|^2_{L^2(\Omega;d^nx)} > 0, \tag{4.57}$$

where u_Θ *satisfies*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \text{ on } \partial\Omega. \end{cases} \tag{4.58}$$

In particular, assuming Hypothesis 3.1*, then*

$$\operatorname{Im}\big(M^{(0)}_{\Theta,D,\Omega}(z)\big) \geqslant 0, \quad z \in \mathbb{C}_+, \tag{4.59}$$

and hence $M^{(0)}_{\Theta,D,\Omega}(\,\cdot\,)$ *is an operator-valued Herglotz function on* $L^2(\partial\Omega; d^{n-1}\omega)$*.*

Proof. Let u_Θ be given by the solution of (4.58). Then $M^{(0)}_{\Theta,D,\Omega}g = \gamma_D u_\Theta$ by (3.50), and using self-adjointness of $\widetilde{\Theta}$ (in the sense of (B.7)) and the Green's formula (A.31), one computes,

$$\begin{aligned} \frac{1}{2i}\big\langle g, \big[\widetilde{M}_{\Theta,D}(z) - \widetilde{M}_{\Theta,D}(z)^*\big]g\big\rangle_{1/2} &= \frac{1}{2i}\big[\big\langle g, \widetilde{M}_{\Theta,D}(z)g\big\rangle_{1/2} - \big\langle \widetilde{M}_{\Theta,D}(z)g, g\big\rangle_{1/2}\big] \\ &= \frac{1}{2i}\big[\big\langle\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_\Theta, \gamma_D u_\Theta\big\rangle_{1/2} - \big\langle\gamma_D u_\Theta, \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u_\Theta\big\rangle_{1/2}\big] \\ &= \frac{1}{2i}\big[\langle\widetilde{\gamma}_N u_\Theta, \gamma_D u_\Theta\rangle_{1/2} - \langle\gamma_D u_\Theta, \widetilde{\gamma}_N u_\Theta\rangle_{1/2}\big] \\ &\quad + \frac{1}{2i}\big[\big\langle\widetilde{\Theta}\gamma_D u_\Theta, \gamma_D u_\Theta\big\rangle_{1/2} - \big\langle\gamma_D u_\Theta, \widetilde{\Theta}\gamma_D u_\Theta\big\rangle_{1/2}\big] \\ &= \operatorname{Im}(\langle\widetilde{\gamma}_N u_\Theta, \gamma_D u_\Theta\rangle_{1/2}) \\ &= \operatorname{Im}[(\nabla u_\Theta, \nabla u_\Theta)_{L^2(\Omega;d^nx)} + {}_{H^1(\Omega)}\langle\Delta u_\Theta, u_\Theta\rangle_{(H^1(\Omega))^*}] \\ &= \operatorname{Im}(-\overline{z}\,{}_{H^1(\Omega)}\langle u_\Theta, u_\Theta\rangle_{(H^1(\Omega))^*}) \\ &= \operatorname{Im}(z)\,{}_{H^1(\Omega)}\langle u_\Theta, u_\Theta\rangle_{(H^1(\Omega))^*} \\ &= \operatorname{Im}(z)\|u_\Theta\|^2_{L^2(\Omega;d^nx)} > 0 \end{aligned} \tag{4.60}$$

since $g \neq 0$ implies $u_\Theta \neq 0$. This proves (4.57). Restriction of (4.57) to $g \in L^2(\partial\Omega; d^{n-1}\omega)$ then yields (4.59). ∎

Returning to the principal goal of this section, we now prove the following variant of a Krein-type resolvent formula relating $\widetilde{\Delta}_{\Theta,\Omega}$ and $\widetilde{\Delta}_{D,\Omega}$:

Theorem 4.15. *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$*. Then the following Krein formula holds on* $\big(H^1(\Omega)\big)^*$*, regarded as a subspace of* $H^{-1}(\Omega)$*:*

$$\begin{aligned} \big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} &= \big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \\ &\quad + \big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1}\big]^*\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)\big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\big]. \end{aligned} \tag{4.61}$$

Proof. Applying γ_D from the left to both sides of (4.22) yields

$$\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} = \gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^*\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1}, \tag{4.62}$$

since $\gamma_D\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} = 0$. Thus, by (4.20),

$$\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} = \widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1}, \tag{4.63}$$

as operators in $\mathcal{B}\big(\big(H^1(\Omega)\big)^*, H^{1/2}(\partial\Omega)\big)$. Taking adjoints in (4.63) (written with $\overline{z}$ in place of z) then leads to

$$\begin{aligned} \big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^* &= \big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1}\big]^* \\ &= \big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1}\big]^*\big[\widetilde{M}^{(0)}_{\Theta,D,\Omega}(\overline{z})\big]^* \end{aligned}$$

$$= \big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1}\big]^* \widetilde{M}^{(0)}_{\Theta,D,\Omega}(z), \tag{4.64}$$

by Lemma 4.13. Replacing this back in (4.22) then readily yields (4.61). □

The $L^2(\Omega; d^n x)$-analog of Theorem 4.15 then reads as follows:

Theorem 4.16. *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$. *Then the following Krein formula holds on* $L^2(\Omega; d^n x)$*:*

$$\begin{aligned}(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} \\ &\quad + \big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* M^{(0)}_{\Theta,D,\Omega}(z)\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big].\end{aligned} \tag{4.65}$$

Proof. This follows from Theorem 4.15 and the compatibility results established in Lemma 4.11. □

An attractive feature of the Krein-type formula (4.65) lies in the fact that $M^{(0)}_{\Theta,D,\Omega}(z)$ encodes spectral information about $\Delta_{\Theta,\Omega}$. This will be pursued in future work.

Assuming Hypothesis 2.1, the special case $\Theta = 0$ then connects the Neumann and Dirichlet resolvents,

$$\begin{aligned}\big(-\widetilde{\Delta}_{N,\Omega} - z\widetilde{I}_\Omega\big)^{-1} &= \big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \\ &\quad + \big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1}\big]^* \widetilde{M}^{(0)}_{N,D,\Omega}(z)\big[\widetilde{\gamma}_N\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\big], \\ &\qquad z \in \mathbb{C}\backslash(\sigma(-\Delta_{N,\Omega}) \cup \sigma(-\Delta_{D,\Omega})),\end{aligned} \tag{4.66}$$

on $\big(H^1(\Omega)\big)^*$, regarded as a subspace of $H^{-1}(\Omega)$, and similarly,

$$\begin{aligned}(-\Delta_{N,\Omega} - zI_\Omega)^{-1} &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} \\ &\quad + \big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* M^{(0)}_{N,D,\Omega}(z)\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big], \\ &\qquad z \in \mathbb{C}\backslash(\sigma(-\Delta_{N,\Omega}) \cup \sigma(-\Delta_{D,\Omega})),\end{aligned} \tag{4.67}$$

on $L^2(\Omega; d^n x)$. Here $\widetilde{M}^{(0)}_{N,D,\Omega}(z)$ and $M^{(0)}_{N,D,\Omega}(z)$ denote the corresponding Neumann-to-Dirichlet operators.

Remark 4.17. In the case when Hypothesis 2.11 is enforced, it can be shown that

$$M^{(0)}_{N,D,\Omega}(z) \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{3/2}(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash(\sigma(-\Delta_{N,\Omega}) \cup \sigma(-\Delta_{D,\Omega})), \tag{4.68}$$

and

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^2(\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{4.69}$$

Note that, by duality, the latter membership also entails

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big((H^2(\Omega))^*, L^2(\Omega; d^n x)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{4.70}$$

and, given (2.37),

$$\gamma_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^{1/2}(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{4.71}$$

Since, in the current scenario, we also have

$$\gamma_N^* \in \mathcal{B}\big(H^{-1/2}(\partial\Omega), (H^2(\Omega))^*\big), \tag{4.72}$$

it follows that (4.67) takes the form

$$\begin{aligned}(-\Delta_{N,\Omega} - zI_\Omega)^{-1} &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} \\ &\quad + (-\Delta_{D,\Omega} - zI_\Omega)^{-1}\gamma_N^* M^{(0)}_{N,D,\Omega}(z)\gamma_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}, \\ &\qquad z \in \mathbb{C}\backslash(\sigma(-\Delta_{N,\Omega}) \cup \sigma(-\Delta_{D,\Omega})),\end{aligned} \tag{4.73}$$

on $L^2(\Omega; d^n x)$, where the composition of the various operators involved is well-defined by the above discussion. Formula (4.73) should be viewed as a variant of (4.65) in which the Neumann trace operator can be decoupled from the two resolvents of $-\Delta_{D,\Omega}$ in the second term on the right-hand side of (4.65).

Due to the fundamental importance of Krein-type resolvent formulas (and more generally, Robin-to-Dirichlet maps) in connection with the spectral and inverse spectral theory of ordinary and partial differential operators, abstract versions, connected to boundary value spaces (boundary triples) and self-adjoint extensions of closed symmetric operators with equal (possibly infinite) deficiency spaces, have received enormous attention in the literature. In particular, we note that Robin-to-Dirichlet maps in the context of ordinary differential operators reduce to the celebrated (possibly, matrix-valued) Weyl–Titchmarsh function, the basic object of spectral analysis in this context. Since it is impossible to cover the literature in this paper, we refer, for instance, to [2, Sect. 84], [4], [8], [9], [12], [14], [15], [21], [22], [38], [41], [48], [50]–[57], [60], [61], [66], [72]–[79], [84], [87]–[89], and the references cited therein. We add, however, that the case of infinite deficiency indices in the context of partial differential operators (in our concrete case, related to the deficiency indices of the operator closure of $-\Delta \restriction_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$), is much less studied and the results obtained in this section, especially, under the assumption of Lipschitz (i.e., minimally smooth) domains, to the best of our knowledge, are new.

Finally, we emphasize once more that Remark 3.8 also applies to the content of this section (assuming that V is real-valued in connection with Lemmas 4.13 and 4.14).

Appendix A. Properties of Sobolev Spaces and Boundary Traces for $C^{1,r}$ and Lipschitz Domains

The purpose of this appendix is to recall some basic facts in connection with Sobolev spaces corresponding to Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and on domains satisfying Hypothesis 2.11.

In this manuscript we use the following notation for the standard Sobolev Hilbert spaces ($s \in \mathbb{R}$),

$$H^s(\mathbb{R}^n) = \bigg\{U \in \mathcal{S}(\mathbb{R}^n)' \,\big|\, \|U\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} d^n\xi \, \big|\widehat{U}(\xi)\big|^2 \big(1 + |\xi|^{2s}\big) < \infty \bigg\}, \tag{A.1}$$

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) \,|\, u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\}, \tag{A.2}$$

$$H_0^s(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in the norm of } H^s(\Omega). \tag{A.3}$$

Here $\mathcal{D}'(\Omega)$ denotes the usual set of distributions on $\Omega \subseteq \mathbb{R}^n$, Ω open and nonempty, $\mathcal{S}(\mathbb{R}^n)'$ is the space of tempered distributions on $\mathbb{R}^n$, and $\widehat{U}$ denotes the Fourier transform of $U \in \mathcal{S}(\mathbb{R}^n)'$. It is then immediate that

$$H^{s_1}(\Omega) \hookrightarrow H^{s_0}(\Omega) \text{ for } -\infty < s_0 \leq s_1 < +\infty, \tag{A.4}$$

continuously and densely.

Next, we recall the definition of a $C^{1,r}$-domain $\Omega \subseteq \mathbb{R}^n$, Ω open and nonempty, for convenience of the reader: Let $\mathcal{N}$ be a space of real-valued functions in $\mathbb{R}^{n-1}$. One calls a bounded domain $\Omega \subset \mathbb{R}^n$ of class $\mathcal{N}$ if there exists a finite open covering $\{\mathcal{O}_j\}_{1 \leq j \leq N}$ of the boundary $\partial\Omega$ of Ω with the property that, for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a function $\varphi_j \in \mathcal{N}$ (considered in a new system of coordinates obtained from the original one via a rigid motion). Two special cases are going to play a particularly important role in the sequel. First, if $\mathcal{N}$ is $\mathrm{Lip}\,(\mathbb{R}^{n-1})$, the space of real-valued functions satisfying a (global) Lipschitz condition in $\mathbb{R}^{n-1}$, we shall refer to Ω as being a Lipschitz domain; cf. [86, p. 189], where such domains are called "minimally smooth". Second, corresponding to the case when $\mathcal{N}$ is the subspace of $\mathrm{Lip}\,(\mathbb{R}^{n-1})$ consisting of functions whose first-order derivatives satisfy a (global) Hölder condition of order $r \in (0,1)$, we shall say that Ω is of class $C^{1,r}$. The classical theorem of Rademacher of almost everywhere differentiability of Lipschitz functions ensures that, for any Lipschitz domain Ω, the surface measure $d^{n-1}\omega$ is well-defined on $\partial\Omega$ and that there exists an outward pointing normal vector ν at almost every point of $\partial\Omega$.

Call a bounded, open set $\Omega \subset \mathbb{R}^n$ a star-like Lipschitz domain with respect to a point x^* (called center of star-likeness) if Ω is Lipschitz domain and

$$x^* + t(x - x^*) \in \Omega \quad \text{for every } x \in \Omega \text{ and } t \in [0.1]. \tag{A.5}$$

The above geometrical characterization of Lipschitz domains can be used to show that, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ then there exists a finite family of open sets Ω_j, $1 \leq j \leq N$, such that

$$\Omega = \bigcup_{j=1}^{N} \Omega_j, \qquad \Omega_j \text{ star-like Lipschitz domain, } 1 \leq j \leq N. \tag{A.6}$$

For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ it is known that

$$\big(H^s(\Omega)\big)^* = H^{-s}(\Omega), \quad -1/2 < s < 1/2. \tag{A.7}$$

See [93] for this and other related properties. We also refer to our convention of using the *adjoint* (rather than the dual) space X^* of a Banach space X as described near the end of the introduction.

Next, assume that $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of a function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ of class $C^{1,r}$. Then for $0 \leq s < 1 + r$, the Sobolev space $H^s(\partial\Omega)$ consists of functions $f \in L^2(\partial\Omega; d^{n-1}\omega)$ such that $f(x', \varphi(x'))$, as a function of $x' \in \mathbb{R}^{n-1}$, belongs to $H^s(\mathbb{R}^{n-1})$. This definition is easily adapted to the case when Ω is a domain of class $C^{1,r}$ whose boundary is compact, by using a smooth partition of unity. Finally, for $-1-r < s < 0$, we set $H^s(\partial\Omega) = \big(H^{-s}(\partial\Omega)\big)^*$. The same construction concerning $H^s(\partial\Omega)$ applies in the case when $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain (i.e., $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is only Lipschitz) provided $0 \leqslant s \leqslant 1$. In this scenario we set

$$H^s(\partial\Omega) = \big(H^{-s}(\partial\Omega)\big)^*, \quad -1 \leqslant s \leqslant 0. \tag{A.8}$$

It is useful to observe that this entails

$$\|f\|_{H^{-s}(\partial\Omega)} \approx \|\sqrt{1 + |\nabla\varphi(\,\cdot\,)|^2} f(\,\cdot\,, \varphi(\,\cdot\,))\|_{H^{-s}(\mathbb{R}^{n-1})}, \quad 0 \leq s \leq 1. \tag{A.9}$$

To define $H^s(\partial\Omega)$, $0 \leq s \leqslant 1$, when Ω is a Lipschitz domain with compact boundary, we use a smooth partition of unity to reduce matters to the graph case. More precisely, if $0 \leq s \leq 1$ then $f \in H^s(\partial\Omega)$ if and only if the assignment $\mathbb{R}^{n-1} \ni x' \mapsto (\psi f)(x', \varphi(x'))$ is in $H^s(\mathbb{R}^{n-1})$ whenever $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with the property that if Σ is an appropriate rotation and translation of $\{(x', \varphi(x')) \in \mathbb{R}^n \,|\, x' \in \mathbb{R}^{n-1}\}$, then $(\operatorname{supp}(\psi) \cap \partial\Omega) \subset \Sigma$ (this appears to be folklore, but a proof will appear in [67, Proposition 2.4]). Then Sobolev spaces with a negative amount of smoothness are defined as in (A.8) above.

From the above characterization of $H^s(\partial\Omega)$ it follows that any property of Sobolev spaces (of order $s \in [-1,1]$) defined in Euclidean domains, which are invariant under multiplication by smooth, compactly supported functions as well as composition by bi-Lipschitz diffeomorphisms, readily extends to the setting of $H^s(\partial\Omega)$ (via localization and pull-back). As a concrete example, for each Lipschitz domain Ω with compact boundary, one has

$$H^s(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega) \text{ compactly if } 0 < s \leq 1. \tag{A.10}$$

For additional background information in this context we refer, for instance, to [10], [11], [35, Chs. V, VI], [45, Ch. 1], [65, Ch. 3], [99, Sect. I.4.2].

For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with compact boundary, an equivalent definition of the Sobolev space $H^1(\partial\Omega)$ is the collection of functions in $L^2(\partial\Omega; d^{n-1}\omega)$ with the property that the (pointwise, Euclidean) norm of their tangential gradient belongs to $L^2(\partial\Omega; d^{n-1}\omega)$. To make this precise, consider the first-order tangential derivative operators $\partial/\partial\tau_{j,k}$, $1 \leq j,k \leq n$, acting on a function ψ of class C^1 in a neighborhood of $\partial\Omega$ by

$$\partial\psi/\partial\tau_{j,k} = \nu_j(\partial_k\psi)\Big|_{\partial\Omega} - \nu_k(\partial_j\psi)\Big|_{\partial\Omega}. \tag{A.11}$$

For every $f \in L^1(\partial\Omega)$ define the functional $\partial f/\partial\tau_{j,k}$ by setting

$$\partial f/\partial\tau_{j,k} : C^1(\mathbb{R}^n) \ni \psi \mapsto \int_{\partial\Omega} d^{n-1}\omega\, f\,(\partial\psi/\partial\tau_{k,j}) \tag{A.12}$$

When $f \in L^1(\partial\Omega; d^{n-1}\omega)$ has $\partial f/\partial\tau_{j,k} \in L^1(\partial\Omega; d^{n-1}\omega)$, the following integration by parts formula holds:

$$\int_{\partial\Omega} d^{n-1}\omega\, f\,(\partial\psi/\partial\tau_{k,j}) = \int_{\partial\Omega} d^{n-1}\omega\,(\partial f/\partial\tau_{j,k}\,\psi, \quad \psi \in C^1(\mathbb{R}^n). \tag{A.13}$$

We then have the Sobolev-type description of $H^1(\partial\Omega)$:

$$H^1(\partial\Omega) = \big\{f \in L^2(\partial\Omega; d^{n-1}\omega)\,\big|\,\partial f/\partial\tau_{j,k} \in L^2(\partial\Omega; d^{n-1}\omega),\ j,k = 1,\dots,n\big\} \tag{A.14}$$

with

$$\|f\|_{H^1(\partial\Omega)} \approx \|f\|_{L^2(\partial\Omega;d^{n-1}\omega)} + \sum_{j,k=1}^{n} \|\partial f/\partial\tau_{j,k}\|_{L^2(\partial\Omega;d^{n-1}\omega)}, \tag{A.15}$$

($\approx$ denoting equivalent norms), or equivalently,

$$H^1(\partial\Omega) = \bigg\{f \in L^2(\partial\Omega; d^{n-1}\omega)\,\bigg|\ \text{there exists a constant } c > 0 \text{ such that for every } v \in C_0^\infty(\mathbb{R}^n),$$
$$\bigg|\int_{\partial\Omega} d^{n-1}\omega f\,\partial v/\partial\tau_{j,k}\bigg| \leq c\,\|v\|_{L^2(\partial\Omega;d^{n-1}\omega)}\,,\ j,k = 1,\dots,n\bigg\}. \tag{A.16}$$

Let us also point out here that if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain then for any $j,k \in \{1,...,n\}$ the operator

$$\partial/\partial\tau_{j,k} : H^s(\partial\Omega) \to H^{s-1}(\partial\Omega), \quad 0 \leq s \leq 1, \tag{A.17}$$

is well-defined, linear and bounded. This is proved by interpolating the case $s = 1$ and its dual version. In fact, the following more general result (extending (A.14)) is true.

Lemma A.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then for every $s \in [0,1]$,*

$$H^s(\partial\Omega) = \{f \in L^2(\partial\Omega; d^{n-1}\omega)\,|\,\partial f/\partial\tau_{j,k} \in H^{s-1}(\partial\Omega),\, 1 \leq j,k \leq n\} \tag{A.18}$$

and

$$\|f\|_{H^s(\partial\Omega)} \approx \|f\|_{L^2(\partial\Omega;d^{n-1}\omega)} + \sum_{j,k=1}^{n} \|\partial f/\partial\tau_{j,k}\|_{H^{s-1}(\partial\Omega)}. \tag{A.19}$$

Proof. The left-to-right inclusion in (A.18) along with the right-pointing inequality in (A.19) are consequences of the boundedness of (A.17). As for the opposite directions, we note that using a smooth partition of unity and making a rigid transformation of the space, matters can be localized near a boundary point where $\partial\Omega$ coincides with the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. Then for each sufficiently nice function $f : \partial\Omega \to \mathbb{R}$ the Chain Rule yields

$$\Big(\frac{\partial f}{\partial\tau_{j,n}}\Big)(x,\varphi(x)) = \frac{1}{\sqrt{1+|\nabla\varphi(x)|^2}}\frac{\partial}{\partial x_j}\big[f(x,\varphi(x))\big], \quad 1 \leq j \leq n-1. \tag{A.20}$$

On account of this and (A.9), we then deduce (upon noticing that $\partial/\partial\tau_{n,n} = 0$) that

$$\sum_{j=1}^{n} \|\partial f/\partial\tau_{j,n}\|_{H^{s-1}(\partial\Omega)} \approx \sum_{j=1}^{n-1} \|\partial_j[f(\,\cdot\,,\varphi(\,\cdot\,))]\|_{H^{s-1}(\mathbb{R}^{n-1})}. \tag{A.21}$$

Furthermore, we also have

$$\|f\|_{L^2(\partial\Omega;d^{n-1}\omega)} \approx \|f(\,\cdot\,,\varphi(\,\cdot\,))\|_{L^2(\mathbb{R}^{n-1};d^{n-1}x)}. \tag{A.22}$$

Next, we recall the general Euclidean lifting result

$$H^s(\mathbb{R}^{n-1}) = \{f \in H^{s-1}(\mathbb{R}^{n-1}) \,|\, \partial_j f \in H^{s-1}(\mathbb{R}^{n-1}),\, 1 \leq j \leq n-1\}, \quad s \in \mathbb{R}, \tag{A.23}$$

which can be found in [80, Section 2.1.4]. Now, the right-to-left inclusion in (A.18), as well as the left-pointing inequality in (A.19), follow based on (A.21), (A.22) and the estimate which naturally accompanies (A.23). □

Lemma A.2. *Assume Hypothesis* 2.1. *Then for every $s \in [0,1]$ and $j,k \in \{1,...,n\}$*

$$\langle \partial f/\partial\tau_{j,k}\,,\, g\rangle_{1-s} = \langle f\,,\, \partial g/\partial\tau_{k,j}\rangle_{1-s} \tag{A.24}$$

for every $f \in H^s(\partial\Omega)$ and $g \in H^{1-s}(\partial\Omega)$.

Proof. Since for every $s \in [0,1]$

$$C^\infty(\mathbb{R}^n)|_{\partial\Omega} \hookrightarrow H^s(\partial\Omega) \text{ densely}, \tag{A.25}$$

it suffices to prove (A.24) in the case when $f = u|_{\partial\Omega}$ and $g = v|_{\partial\Omega}$ for $u, v \in C^\infty(\mathbb{R}^n)$. In this scenario, we need to establish that

$$\int_{\partial\Omega} d^{n-1}\omega\,(\partial u/\partial\tau_{j,k})v = \int_{\partial\Omega} d^{n-1}\omega\, u(\partial v/\partial\tau_{k,j}), \quad 1 \leq j,k \leq n. \tag{A.26}$$

To this end, we rely on Green's formula (valid for Lipschitz domains) to write

$$\begin{aligned}\int_{\partial\Omega} d^{n-1}\omega\,(\partial u/\partial\tau_{j,k})v &= \int_{\partial\Omega} d^{n-1}\omega\,(\nu_j\partial_k u - \nu_k\partial_j u)v \\ &= \int_\Omega d^n x\,[\partial_j(v\partial_k u) - \partial_k(v\partial_j u)] \\ &= \int_\Omega d^n x\,[(\partial_j v)(\partial_k u) - (\partial_k v)(\partial_j u)]. \end{aligned}\tag{A.27}$$

One observes that the right-most integrand above is an antisymmetric expression in the indices j,k. Consequently, so is the left-most integral in (A.27). This, however, is equivalent to (A.26). □

Moving on, we next consider the following bounded linear map

$$\begin{cases} \big\{w \in L^2(\Omega; d^n x)^n \,\big|\, \mathrm{div}(w) \in \big(H^1(\Omega)\big)^*\big\} \to H^{-1/2}(\partial\Omega) = \big(H^{1/2}(\partial\Omega)\big)^* \\ \qquad\qquad\qquad\qquad w \mapsto \nu\cdot w \end{cases} \tag{A.28}$$

by setting

$${}_{H^{1/2}(\partial\Omega)}\langle \phi, \nu\cdot w\rangle_{(H^{1/2}(\partial\Omega))^*} = \int_\Omega d^n x\, \overline{\nabla\Phi(x)}\cdot w(x) + {}_{H^1(\Omega)}\langle \Phi, \mathrm{div}(w)\rangle_{(H^1(\Omega))^*} \tag{A.29}$$

whenever $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ is such that $\gamma_D\Phi = \phi$. Here ${}_{H^1(\Omega)}\langle \Phi, \mathrm{div}(w)\rangle_{(H^1(\Omega))^*}$ in (A.29) is the natural pairing between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$ (which, in turn, is compatible with the (bilinear) distributional pairing). It should be remarked that the above definition is independent of the particular extension $\Phi \in H^1(\Omega)$ of ϕ.

Going further, one can introduce a (weak) Neumann trace operator $\widetilde{\gamma}_N$ as follows:

$$\widetilde{\gamma}_N \colon \begin{cases} \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in \big(H^1(\Omega)\big)^*\big\} \to H^{-1/2}(\partial\Omega) \\ \hfill u \mapsto \widetilde{\gamma}_N u = \nu \cdot \nabla u, \end{cases} \tag{A.30}$$

with the dot product understood in the sense of (A.28). We emphasize that the weak Neumann trace operator $\widetilde{\gamma}_N$ in (A.30) is a bounded extension of the operator γ_N introduced in (2.37). Indeed, to see that $\mathrm{dom}(\gamma_N) \subseteq \mathrm{dom}(\widetilde{\gamma}_N)$, we note that if $u \in H^{s+1}(\Omega)$ for some $1/2 < s < 3/2$, then $\Delta u \in H^{-1+s}(\Omega) = \big(H^{1-s}(\Omega)\big)^* \hookrightarrow \big(H^1(\Omega)\big)^*$, by (A.7) and (A.4). With this in hand, it is then easy to show that $\widetilde{\gamma}_N$ in (A.30) and γ_N in (2.37) agree (on the smaller domain), as claimed. In addition, one can show that the (weak) Neumann trace operator (A.30) is onto (cf., e.g., the discussion in [42, Sect. 4]).

We note that (A.30) and (A.29) yield the following Green's formula

$$\langle \gamma_D \Phi, \widetilde{\gamma}_N u\rangle_{1/2} = (\nabla\Phi, \nabla u)_{L^2(\Omega;d^nx)^n} + {}_{H^1(\Omega)}\langle \Phi, \Delta u\rangle_{(H^1(\Omega))^*}, \tag{A.31}$$

valid for any $u \in H^1(\Omega)$ with $\Delta u \in \big(H^1(\Omega)\big)^*$, and any $\Phi \in H^1(\Omega)$. The pairing on the left-hand side of (A.31) is between functionals in $\big(H^{1/2}(\partial\Omega)\big)^*$ and elements in $H^{1/2}(\partial\Omega)$, whereas the last pairing on the right-hand side is between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$. For further use, we also note that the adjoint of (2.7) maps boundedly as follows

$$\gamma_D^* \colon \big(H^{s-1/2}(\partial\Omega)\big)^* \to \big(H^s(\Omega)\big)^*, \quad 1/2 < s < 3/2. \tag{A.32}$$

Remark A.3. While it is tempting to view γ_D as an unbounded but densely defined operator on $L^2(\Omega; d^nx)$ whose domain contains the space $C_0^\infty(\Omega)$, one should note that in this case its adjoint γ_D^* is not densely defined: Indeed (cf. [40, Remark A.4]), $\mathrm{dom}(\gamma_D^*) = \{0\}$ and hence γ_D is not a closable linear operator in $L^2(\Omega; d^nx)$.

Next we recall the following result from [43] (and reproduce its proof for subsequent use in the proofs of Lemmas A.5 and D.3).

Lemma A.4 (cf. [43], Lemma A.6)**.** *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open Lipschitz domain with a compact, nonempty boundary $\partial\Omega$. Then the Dirichlet trace operator γ_D (originally considered as in* (2.7)*) satisfies* (2.9)*.*

Proof. Let $u \in H^{(3/2)+\varepsilon}(\Omega)$, $v \in C_0^\infty(\mathbb{R}^n)$, and $u_\ell \in C^\infty(\overline{\Omega}) \hookrightarrow H^{(3/2)+\varepsilon}(\Omega)$, $\ell \in \mathbb{N}$, be a sequence of functions approximating u in $H^{(3/2)+\varepsilon}(\Omega)$. It follows from (2.7) and (A.4) that $\gamma_D u, \gamma_D(\nabla u) \in L^2(\partial\Omega; d^{n-1}\omega)$. Utilizing (A.13), one computes for all $j,k = 1,\dots,n$,

$$\begin{aligned} \bigg| \int_{\partial\Omega} d^{n-1}\omega\, \gamma_D u \frac{\partial v}{\partial \tau_{j,k}} \bigg| &= \bigg| \lim_{\ell\to\infty} \int_{\partial\Omega} d^{n-1}\omega\, u_\ell \frac{\partial v}{\partial \tau_{j,k}} \bigg| = \bigg| \lim_{\ell\to\infty} \int_{\partial\Omega} d^{n-1}\omega\, v \frac{\partial u_\ell}{\partial \tau_{j,k}} \bigg| \\ &\leq c \bigg| \lim_{\ell\to\infty} \int_{\partial\Omega} d^{n-1}\omega\, v\, \gamma_D(\nabla u_\ell) \bigg| \leq c\, \|\gamma_D(\nabla u)\|_{L^2(\partial\Omega;d^{n-1}\omega)}\, \|v\|_{L^2(\partial\Omega;d^{n-1}\omega)}\,. \end{aligned} \tag{A.33}$$

Thus, it follows from (A.16) and (A.33) that $\gamma_D u \in H^1(\partial\Omega)$. □

Next, we prove the following fact:

Lemma A.5. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then for each $r \in (1/2, 1)$, the space $C^r(\partial\Omega)$ is a module over $H^{1/2}(\partial\Omega)$. More precisely, if M_f denotes the operator of multiplication by f, then there exists $C = C(\Omega, r) > 0$ such that*

$$M_f \in \mathcal{B}\big(H^{1/2}(\partial\Omega)\big) \text{ and } \|M_f\|_{\mathcal{B}\left(H^{1/2}(\partial\Omega)\right)} \leq C\|f\|_{C^r(\partial\Omega)} \text{ for every } f \in C^r(\partial\Omega). \tag{A.34}$$

As a consequence, if Ω is actually a bounded $C^{1,r}$ domain with $r \in (1/2,1)$, then the Neumann and Dirichlet trace operators γ_N, γ_D satisfy

$$\gamma_N \in \mathcal{B}\big(H^2(\Omega), H^{1/2}(\partial\Omega)\big) \tag{A.35}$$

and

$$\gamma_D \in \mathcal{B}\big(H^2(\Omega), H^{3/2}(\partial\Omega)\big). \tag{A.36}$$

Proof. The first part of the lemma is a direct consequence of general results about pointwise multiplication of functions in Triebel–Lizorkin spaces (a scale which contains both Hölder and Sobolev spaces); see [80, Theorem 2 on p. 177]. Then (A.35) follows from this, (2.7), the fact that $\gamma_N = \nu\cdot\gamma_D$, and $\nu \in C^r(\partial\Omega)$. Next, one observes that for each $u \in H^2(\Omega)$ one has $\gamma_D u \in H^1(\partial\Omega)$ by Lemma A.4. In addition,

$$\frac{\partial}{\partial\tau_{j,k}}(\gamma_D u) = \nu_j\gamma_D(\partial_k u) - \nu_k\gamma_D(\partial_j u) \in H^{1/2}(\partial\Omega), \tag{A.37}$$

with a naturally accompanying estimate, by (2.7) and the fact that, as observed in the first part of the current proof, multiplication by ν_j (for $1 \leq j \leq n$) preserves $H^{1/2}(\partial\Omega)$. Consequently, (A.36) follows from this and (A.38), (A.39) below. □

Our next result should be compared with (A.14) and Lemma A.1.

Lemma A.6. *If $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,r}$ domain with $r \in (1/2,1)$ then*

$$H^{3/2}(\partial\Omega) = \{f \in H^1(\partial\Omega)\,|\,\partial f/\partial\tau_{j,k} \in H^{1/2}(\partial\Omega)\, 1 \leq j,k \leq n\} \tag{A.38}$$

and

$$\|f\|_{H^{3/2}(\partial\Omega)} \approx \|f\|_{H^1(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial f/\partial\tau_{j,k}\|_{H^{1/2}(\partial\Omega)}. \tag{A.39}$$

Proof. To justify (A.38) and (A.39) we use a smooth cut-off function to localize the problem near a boundary point where $\partial\Omega$ coincides with the graph of a $C^{1,r}$ function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. In this setting, the desired conclusions follow from (A.20), (A.34), and (A.23) used with $s = 3/2$. □

Appendix B. Sesquilinear Forms and Associated Operators

In this appendix we describe a few basic facts on sesquilinear forms and linear operators associated with them.

Let $\mathcal{H}$ be a complex separable Hilbert space with scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ (antilinear in the first and linear in the second argument), $\mathcal{V}$ a reflexive Banach space continuously and densely embedded into $\mathcal{H}$. Then also $\mathcal{H}$ embeds continuously and densely into $\mathcal{V}^*$.

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*. \tag{B.1}$$

Here the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{V}^*$ is accomplished via the identification

$$\mathcal{H} \ni u \mapsto (\,\cdot\,, u)_{\mathcal{H}} \in \mathcal{V}^*, \tag{B.2}$$

and we recall the convention in this manuscript (cf. the discussion at the end of the introduction) that if X denotes a Banach space, X^* denotes the *adjoint space* of continuous conjugate linear functionals on X.

In particular, if the sesquilinear form

$${}_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*} \colon \mathcal{V} \times \mathcal{V}^* \to \mathbb{C} \tag{B.3}$$

denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, then

$${}_{\mathcal{V}}\langle u, v\rangle_{\mathcal{V}^*} = (u,v)_{\mathcal{H}}, \quad u \in \mathcal{V},\ v \in \mathcal{H} \hookrightarrow \mathcal{V}^*, \tag{B.4}$$

that is, the $\mathcal{V}, \mathcal{V}^*$ pairing ${}_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*}$ is compatible with the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ in $\mathcal{H}$.

Let $T \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*)$. Since $\mathcal{V}$ is reflexive, $(\mathcal{V}^*)^* = \mathcal{V}$, one has

$$T\colon \mathcal{V} \to \mathcal{V}^*, \quad T^*\colon \mathcal{V} \to \mathcal{V}^* \tag{B.5}$$

and

$$ {}_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{(\mathcal{V}^*)^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v, T^*u\rangle_{\mathcal{V}^*}}. \tag{B.6}$$

Self-adjointness of T is then defined by $T = T^*$, that is,

$$ {}_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle Tu, v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v, Tu\rangle_{\mathcal{V}^*}}, \quad u, v \in \mathcal{V}, \tag{B.7}$$

nonnegativity of T is defined by

$$ {}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq 0, \quad u \in \mathcal{V}, \tag{B.8}$$

and *boundedness from below of T by $c_T \in \mathbb{R}$* is defined by

$$ {}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c_T \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{B.9}$$

(By (B.4), this is equivalent to ${}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c_T \, {}_{\mathcal{V}}\langle u, u\rangle_{\mathcal{V}^*}$, $u \in \mathcal{V}$.)

Next, let the sesquilinear form $a(\,\cdot\,,\,\cdot\,)\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ (antilinear in the first and linear in the second argument) be *$\mathcal{V}$-bounded*, that is, there exists a $c_a > 0$ such that

$$|a(u, v)| \leqslant c_a \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \quad u, v \in \mathcal{V}. \tag{B.10}$$

Then $\widetilde{A}$ defined by

$$\widetilde{A}\colon \begin{cases} \mathcal{V} \to \mathcal{V}^*, \\ v \mapsto \widetilde{A}v = a(\,\cdot\,, v), \end{cases} \tag{B.11}$$

satisfies

$$\widetilde{A} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*) \text{ and } {}_{\mathcal{V}}\big\langle u, \widetilde{A}v\big\rangle_{\mathcal{V}^*} = a(u, v), \quad u, v \in \mathcal{V}. \tag{B.12}$$

Assuming further that $a(\,\cdot\,,\,\cdot\,)$ is *symmetric*, that is,

$$a(u, v) = \overline{a(v, u)}, \quad u, v \in \mathcal{V}, \tag{B.13}$$

and that a is *$\mathcal{V}$-coercive*, that is, there exists a constant $C_0 > 0$ such that

$$a(u, u) \geq C_0 \|u\|_{\mathcal{V}}^2, \quad u \in \mathcal{V}, \tag{B.14}$$

respectively, then,

$$\widetilde{A}\colon \mathcal{V} \to \mathcal{V}^* \text{ is bounded, self-adjoint, and boundedly invertible.} \tag{B.15}$$

Moreover, denoting by A the part of $\widetilde{A}$ in $\mathcal{H}$ defined by

$$\mathrm{dom}(A) = \big\{u \in \mathcal{V} \,|\, \widetilde{A}u \in \mathcal{H}\big\} \subseteq \mathcal{H}, \quad A = \widetilde{A}\big|_{\mathrm{dom}(A)}\colon \mathrm{dom}(A) \to \mathcal{H}, \tag{B.16}$$

then A is a (possibly unbounded) self-adjoint operator in $\mathcal{H}$ satisfying

$$A \geq C_0 I_{\mathcal{H}}, \tag{B.17}$$

$$\mathrm{dom}\big(A^{1/2}\big) = \mathcal{V}. \tag{B.18}$$

In particular,

$$A^{-1} \in \mathcal{B}(\mathcal{H}). \tag{B.19}$$

The facts (B.1)–(B.19) are a consequence of the Lax–Milgram theorem and the second representation theorem for symmetric sesquilinear forms. Details can be found, for instance, in [29, §VI.3, §VII.1], [35, Ch. IV], and [59].

Next, consider a symmetric form $b(\,\cdot\,,\,\cdot\,)\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ and assume that b is *bounded from below by $c_b \in \mathbb{R}$*, that is,

$$b(u, u) \geq c_b \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{B.20}$$

Introducing the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{V}(b)}\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ (with associated norm $\|\cdot\|_{\mathcal{V}(b)}$) by

$$(u,v)_{\mathcal{V}(b)} = b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}, \quad u,v\in\mathcal{V}, \tag{B.21}$$

turns $\mathcal{V}$ into a pre-Hilbert space $(\mathcal{V};(\,\cdot\,,\,\cdot\,)_{\mathcal{V}(b)})$, which we denote by $\mathcal{V}(b)$. The form b is called *closed* if $\mathcal{V}(b)$ is actually complete, and hence a Hilbert space. The form b is called *closable* if it has a closed extension. If b is closed, then

$$|b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}| \leqslant \|u\|_{\mathcal{V}(b)}\|v\|_{\mathcal{V}(b)}, \quad u,v\in\mathcal{V}, \tag{B.22}$$

and

$$|b(u,u) + (1-c_b)\|u\|_{\mathcal{H}}^2| = \|u\|_{\mathcal{V}(b)}^2, \quad u\in\mathcal{V}, \tag{B.23}$$

show that the form $b(\,\cdot\,,\,\cdot\,)+(1-c_b)(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ is a symmetric, $\mathcal{V}$-bounded, and $\mathcal{V}$-coercive sesquilinear form. Hence, by (B.11) and (B.12), there exists a linear map

$$\widetilde{B}_{c_b}\colon \begin{cases} \mathcal{V}(b)\to\mathcal{V}(b)^*, \\ \qquad v\mapsto \widetilde{B}_{c_b}v = b(\,\cdot\,,v) + (1-c_b)(\,\cdot\,,v)_{\mathcal{H}}, \end{cases} \tag{B.24}$$

with

$$\widetilde{B}_{c_b}\in\mathcal{B}(\mathcal{V}(b),\mathcal{V}(b)^*) \text{ and } {}_{\mathcal{V}(b)}\langle u,\widetilde{B}_{c_b}v\rangle_{\mathcal{V}(b)^*} = b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}, \quad u,v\in\mathcal{V}. \tag{B.25}$$

Introducing the linear map

$$\widetilde{B} = \widetilde{B}_{c_b} + (c_b-1)\widetilde{I}\colon \mathcal{V}(b)\to\mathcal{V}(b)^*, \tag{B.26}$$

where $\widetilde{I}\colon \mathcal{V}(b)\hookrightarrow\mathcal{V}(b)^*$ denotes the continuous inclusion (embedding) map of $\mathcal{V}(b)$ into $\mathcal{V}(b)^*$, one obtains a self-adjoint operator B in $\mathcal{H}$ by restricting $\widetilde{B}$ to $\mathcal{H}$,

$$\operatorname{dom}(B) = \big\{u\in\mathcal{V}\,\big|\,\widetilde{B}u\in\mathcal{H}\big\}\subseteq\mathcal{H}, \quad B = \widetilde{B}\big|_{\operatorname{dom}(B)}\colon \operatorname{dom}(B)\to\mathcal{H}, \tag{B.27}$$

satisfying the following properties:

$$B \ge c_b I_{\mathcal{H}}, \tag{B.28}$$

$$\operatorname{dom}\big(|B|^{1/2}\big) = \operatorname{dom}\big((B-c_bI_{\mathcal{H}})^{1/2}\big) = \mathcal{V}, \tag{B.29}$$

$$\begin{aligned} b(u,v) &= \big(|B|^{1/2}u, U_B|B|^{1/2}v\big)_{\mathcal{H}} && \text{(B.30)}\\ &= \big((B-c_bI_{\mathcal{H}})^{1/2}u, (B-c_bI_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + c_b(u,v)_{\mathcal{H}} && \text{(B.31)}\\ &= {}_{\mathcal{V}(b)}\langle u,\widetilde{B}v\rangle_{\mathcal{V}(b)^*}, \quad u,v\in\mathcal{V}, && \text{(B.32)} \end{aligned}$$

$$b(u,v) = (u,Bv)_{\mathcal{H}}, \quad u\in\mathcal{V},\ v\in\operatorname{dom}(B), \tag{B.33}$$

$$\begin{aligned} \operatorname{dom}(B) = \{v\in\mathcal{V}\,|\,&\text{there exists an } f_v\in\mathcal{H} \text{ such that}\\ &b(w,v) = (w,f_v)_{\mathcal{H}} \text{ for all } w\in\mathcal{V}\}, \end{aligned} \tag{B.34}$$

$$Bu = f_u, \quad u\in\operatorname{dom}(B),$$

$$\operatorname{dom}(B) \text{ is dense in } \mathcal{H} \text{ and in } \mathcal{V}(b). \tag{B.35}$$

Properties (B.34) and (B.35) uniquely determine B. Here U_B in (B.31) is the partial isometry in the polar decomposition of B, that is,

$$B = U_B|B|, \quad |B| = (B^*B)^{1/2}. \tag{B.36}$$

The operator B is called the *operator associated with the form b*.

The norm in the Hilbert space $\mathcal{V}(b)^*$ is given by

$$\|\ell\|_{\mathcal{V}(b)^*} = \sup\{|{}_{\mathcal{V}(b)}\langle u,\ell\rangle_{\mathcal{V}(b)^*}|\,|\,\|u\|_{\mathcal{V}(b)}\leqslant 1\}, \quad \ell\in\mathcal{V}(b)^*, \tag{B.37}$$

with associated scalar product,

$$(\ell_1, \ell_2)_{\mathcal{V}(b)^*} = {}_{\mathcal{V}(b)}\big\langle \big(\widetilde{B} + (1-c_b)\widetilde{I}\big)^{-1}\ell_1, \ell_2\big\rangle_{\mathcal{V}(b)^*}, \quad \ell_1, \ell_2 \in \mathcal{V}(b)^*. \tag{B.38}$$

Since

$$\big\|\big(\widetilde{B} + (1-c_b)\widetilde{I}\big)v\big\|_{\mathcal{V}(b)^*} = \|v\|_{\mathcal{V}(b)}, \quad v \in \mathcal{V}, \tag{B.39}$$

the Riesz representation theorem yields

$$\big(\widetilde{B} + (1-c_b)\widetilde{I}\big) \in \mathcal{B}(\mathcal{V}(b), \mathcal{V}(b)^*) \text{ and } \big(\widetilde{B} + (1-c_b)\widetilde{I}\big) \colon \mathcal{V}(b) \to \mathcal{V}(b)^* \text{ is unitary.} \tag{B.40}$$

In addition,

$$\begin{aligned} {}_{\mathcal{V}(b)}\big\langle u, \big(\widetilde{B} + (1-c_b)\widetilde{I}\big)v\big\rangle_{\mathcal{V}(b)^*} &= \big(\big(B + (1-c_b)I_{\mathcal{H}}\big)^{1/2}u, \big(B + (1-c_b)I_{\mathcal{H}}\big)^{1/2}v\big)_{\mathcal{H}} \\ &= (u,v)_{\mathcal{V}(b)}, \quad u, v \in \mathcal{V}(b). \end{aligned} \tag{B.41}$$

In particular,

$$\big\|(B + (1-c_b)I_{\mathcal{H}})^{1/2}u\big\|_{\mathcal{H}} = \|u\|_{\mathcal{V}(b)}, \quad u \in \mathcal{V}(b), \tag{B.42}$$

and hence

$$(B + (1-c_b)I_{\mathcal{H}})^{1/2} \in \mathcal{B}(\mathcal{V}(b), \mathcal{H}) \text{ and } (B + (1-c_b)I_{\mathcal{H}})^{1/2} \colon \mathcal{V}(b) \to \mathcal{H} \text{ is unitary.} \tag{B.43}$$

The facts (B.20)–(B.43) comprise the second representation theorem of sesquilinear forms (cf. [35, Sect. IV.2], [37, Sects. 1.2–1.5], and [49, Sect. VI.2.6]).

A special but important case of nonnegative closed forms is obtained as follows: Let $\mathcal{H}_j$, $j = 1, 2$, be complex separable Hilbert spaces, and $T \colon \mathrm{dom}(T) \to \mathcal{H}_2$, $\mathrm{dom}(T) \subseteq \mathcal{H}_1$, a densely defined operator. Consider the nonnegative form $a_T \colon \mathrm{dom}(T) \times \mathrm{dom}(T) \to \mathbb{C}$ defined by

$$a_T(u,v) = (Tu, Tv)_{\mathcal{H}_2}, \quad u, v \in \mathrm{dom}(T). \tag{B.44}$$

Then the form a_T is closed (resp., closable) if and only if T is. If T is closed, the unique nonnegative self-adjoint operator associated with a_T in $\mathcal{H}_1$, whose existence is guaranteed by the second representation theorem for forms, then equals T^*T. In particular, one obtains

$$a_T(u,v) = (|T|u, |T|v)_{\mathcal{H}_1}, \quad u, v \in \mathrm{dom}(T) = \mathrm{dom}(|T|). \tag{B.45}$$

In addition, since

$$\begin{aligned} b(u,v) + (1-c_b)(u,v)_{\mathcal{H}} &= \big((B + (1-c_b)I_{\mathcal{H}})^{1/2}u, (B + (1-c_b)I_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}}, \\ &\qquad u, v \in \mathrm{dom}(b) = \mathrm{dom}\big(|B|^{1/2}\big) = \mathcal{V}, \end{aligned} \tag{B.46}$$

and $(B + (1-c_b)I_{\mathcal{H}})^{1/2}$ is self-adjoint (and hence closed) in $\mathcal{H}$, a symmetric, $\mathcal{V}$-bounded, and $\mathcal{V}$-coercive form is densely defined in $\mathcal{H} \times \mathcal{H}$ and closed (a fact we used in the proof of Theorem 2.6). We refer to [49, Sect. VI.2.4] and [98, Sect. 5.5] for details.

Next we recall that if a_j are sesquilinear forms defined on $\mathrm{dom}(a_j) \times \mathrm{dom}(a_j)$, $j = 1, 2$, bounded from below and closed, then also

$$(a_1 + a_2) \colon \begin{cases} (\mathrm{dom}(a_1) \cap \mathrm{dom}(a_2)) \times (\mathrm{dom}(a_1) \cap \mathrm{dom}(a_2)) \to \mathbb{C}, \\ (u,v) \mapsto (a_1 + a_2)(u,v) = a_1(u,v) + a_2(u,v) \end{cases} \tag{B.47}$$

is bounded from below and closed (cf. [49, Sect. VI.1.6]).

Finally, we also recall the following perturbation theoretic fact: Suppose a is a sesquilinear form defined on $\mathcal{V} \times \mathcal{V}$, bounded from below and closed, and let b be a symmetric sesquilinear form bounded with respect to a with bound less than one, that is, $\mathrm{dom}(b) \supseteq \mathcal{V} \times \mathcal{V}$, and that there exist $0 \leqslant \alpha < 1$ and $\beta \geqslant 0$ such that

$$|b(u,u)| \leqslant \alpha |a(u,u)| + \beta \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{B.48}$$

Then

$$(a+b)\colon \begin{cases} \mathcal{V}\times\mathcal{V}\to\mathbb{C}, \\ (u,v)\mapsto (a+b)(u,v)=a(u,v)+b(u,v) \end{cases} \tag{B.49}$$

defines a sesquilinear form that is bounded from below and closed (cf. [49, Sect. VI.1.6]). In the special case where α can be chosen arbitrarily small, the form b is called infinitesimally form bounded with respect to a.

Appendix C. Estimates for the Fundamental Solution of the Helmholtz Equation

The principal aim of this appendix is to recall and prove some estimates for the fundamental solution (i.e., the Green's function) of the Helmholtz equation and its x-derivatives up to the second order.

Let $E_n(z;x)$ be the fundamental solution of the Helmholtz equation $(-\Delta - z)\psi(z;\,\cdot\,)=0$ in $\mathbb{R}^n$, $n\in\mathbb{N}$, $n\geqslant 2$, already introduced in (2.97), and reproduced for convenience below:

$$E_n(z;x) = \begin{cases} \frac{i}{4}\Big(\frac{2\pi|x|}{z^{1/2}}\Big)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x|\big), & n\geqslant 2,\ z\in\mathbb{C}\backslash\{0\}, \\ \frac{-1}{2\pi}\ln(|x|), & n=2,\ z=0, \\ \frac{1}{(n-2)\omega_{n-1}}|x|^{2-n}, & n\geqslant 3,\ z=0, \end{cases} \tag{C.1}$$
$$\operatorname{Im}\big(z^{1/2}\big)\geq 0,\ x\in\mathbb{R}^n\backslash\{0\},$$

where $H^{(1)}_{\nu}(\,\cdot\,)$ denotes the Hankel function of the first kind with index $\nu\geq 0$ (cf. [1, Sect. 9.1]) and $\omega_{n-1}=2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\,\cdot\,)$ the Gamma function, cf. [1, Sect. 6.1]) represents the area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

As $z\to 0$, $E_n(z,x)$, $x\in\mathbb{R}^n\backslash\{0\}$ is continuous for $n\geq 3$,

$$E_n(z,x) \underset{z\to 0}{=} E_n(0,x) = \frac{1}{(n-2)\omega_{n-1}}|x|^{2-n}, \quad x\in\mathbb{R}^n\backslash\{0\},\ n\geqslant 3, \tag{C.2}$$

but discontinuous for $n=2$ as

$$E_2(z,x) \underset{z\to 0}{=} \frac{-1}{2\pi}\ln\big(z^{1/2}|x|/2\big)\big[1+O\big(z|x|^2\big)\big] + \frac{1}{2\pi}\psi(1) + O\big(z|x|^2\big), \quad x\in\mathbb{R}^2\backslash\{0\},\ n=2. \tag{C.3}$$

Here $\psi(w)=\Gamma'(w)/\Gamma(w)$ denotes the digamma function (cf. [1, Sect. 6.3]). Thus, we simply *define* $E_2(0;x)=\frac{-1}{2\pi}\ln(|x|)$, $x\in\mathbb{R}^2\backslash\{0\}$ as in (C.1).

To estimate E_n we recall that (cf. [1, Sect. 9.1])

$$H^{(1)}_{(n-2)/2}(\,\cdot\,) = J_{(n-2)/2}(\,\cdot\,) + iY_{(n-2)/2}(\,\cdot\,), \tag{C.4}$$

with J_ν and Y_ν the regular and irregular Bessel functions, respectively.

We start considering small values of $|x|$ and for this purpose recall the following absolutely convergent expansions (cf. [1, Sect. 9.1]):

$$J_\nu(\zeta) = \Big(\frac{\zeta}{2}\Big)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\zeta^{2k}}{4^k k!\Gamma(\nu+k+1)}, \quad \zeta\in\mathbb{C}\backslash(-\infty,0],\ \nu\in\mathbb{R}\backslash(-\mathbb{N}), \tag{C.5}$$

$$J_{-m}(\zeta) = (-1)^m J_m(\zeta), \quad \zeta\in\mathbb{C},\ m\in\mathbb{N}_0, \tag{C.6}$$

$$Y_\nu(\zeta) = \frac{J_\nu(\zeta)\cos(\nu\pi) - J_{-\nu}(\zeta)}{\sin(\nu\pi)}, \quad \zeta\in\mathbb{C}\backslash(-\infty,0],\ \nu\in(0,\infty)\backslash\mathbb{N}, \tag{C.7}$$

$$Y_m(\zeta) = -\frac{\zeta^{-m}}{2^m\pi}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}\frac{\zeta^{2k}}{4^k} + \frac{2}{\pi}J_m(\zeta)\ln(\zeta/2)$$

$$- \frac{\zeta^m}{2^m\pi}\sum_{k=0}^{\infty}[\psi(k+1)+\psi(m+k=1)]\frac{(-1)^k\zeta^{2k}}{4^k k!(m+k)!}, \quad \zeta\in\mathbb{C}\backslash(-\infty,0],\ m\in\mathbb{N}_0. \tag{C.8}$$

We note that all functions in (C.5), (C.7), and (C.8) are analytic in $\mathbb{C}\backslash(-\infty,0]$ and that $J_m(\,\cdot\,)$ is entire for $m\in\mathbb{Z}$. In addition, all functions in (C.5)–(C.8) have continuous nontangential limits as $\zeta\to\eta<0$, with generally different values on either side of the cut $(-\infty,0]$ due to the presence of the functions ζ^ν and $\ln(\zeta)$. (We chose $\nu\in\mathbb{R}$ and subsequently usually $\nu\geqslant 0$ for simplicity only; complex values of ν are discussed in [1, Ch. 9].)

Due to the presence of the logarithmic term for even dimensions we next distinguish even and odd space dimensions n:

(*i*) $n=2m+2$, $m\in\mathbb{N}_0$, and $z\in\mathbb{C}\backslash\{0\}$ fixed:

$$\begin{aligned}
E_{2m+2}(z;x) &= \frac{i}{4}\left(\frac{2\pi|x|}{z^{1/2}}\right)^{-m} H^{(1)}_m\big(z^{1/2}|x|\big)\\
&= \frac{i}{4}\left(\frac{2\pi|x|}{z^{1/2}}\right)^{-m}\big[J_m\big(z^{1/2}|x|\big)+iY_m\big(z^{1/2}|x|\big)\big]\\
&= \frac{i}{4}\left(\frac{2\pi|x|}{z^{1/2}}\right)^{-m}\bigg\{O\big(|x|^m\big)+\frac{2i}{\pi}\ln\left(\frac{z^{1/2}|x|}{2}\right)O\big(|x|^m\big)\\
&\quad -\frac{i}{\pi}\left(\frac{z^{1/2}|x|}{2}\right)^{-m}(1-\delta_{m,0})\left[(m-1)!+(1-\delta_{m,1})(m-2)!\left(\frac{z|x|^2}{4}\right)+O\big(|x|^4\big)\right]\bigg\}.
\end{aligned} \tag{C.9}$$

(*ii*) $n=2m+1$, $m\in\mathbb{N}$, and $z\in\mathbb{C}\backslash\{0\}$ fixed:

$$\begin{aligned}
E_{2m+1}(z;x) &= \frac{i}{4}\left(\frac{2\pi|x|}{z^{1/2}}\right)^{(1/2)-m} H^{(1)}_{m-(1/2)}\big(z^{1/2}|x|\big)\\
&= \frac{i}{4}z^{m/2}\pi^{-m}2^{1-m}|x|^{1-m}h^{(1)}_{m-1}\big(z^{1/2}|x|\big)\\
&= \frac{i}{2z^{1/2}}(2\pi i|x|)^{-m}e^{iz^{1/2}|x|}\sum_{k=0}^{m-1}\frac{(m+k-1)!}{k!(m-k-1)!}\big(-2iz^{1/2}|x|\big)^{-k}\\
&\underset{|x|\to 0}{=}\begin{cases}(4\pi|x|)^{-1}\big[1+iz^{1/2}|x|+O\big(|x|^2\big)\big], & m=1,\\ [(2m-1)\omega_{2m}]^{-1}|x|^{1-2m}\big[1+O\big(|x|^2\big)\big], & m\geq 2,\end{cases}
\end{aligned} \tag{C.10}$$

with $h^{(1)}_\ell(\,\cdot\,)$ defined in [1, Sect. 10.1].

Given these expansions we can now summarize the behavior of $E_n(z;x)$ and its derivatives up to the second order as $|x|\to 0$:

Lemma C.1. *Fix $z\in\mathbb{C}\backslash\{0\}$. Then the fundamental solution $E_n(z;\,\cdot\,)$ of the Helmholtz equation $(-\Delta-z)\psi(z;\,\cdot\,)=0$ and its derivatives up to the second order satisfy the following estimates for $0<|x|<R$, with $R>0$ fixed:*

$$|E_n(z;x)-E_n(0;x)|\leq\begin{cases}C, & n=2,3,\\ C[|\ln(|x|)|+1], & n=4,\\ C\big[|x|^{4-n}+1\big], & n\geqslant 5,\end{cases} \tag{C.11}$$

$$|\partial_j E_n(z;x)-\partial_j E_n(0;x)|\leq\begin{cases}C, & n=2,3,\\ C\big[|x|^{3-n}+1\big], & n\geqslant 4,\end{cases} \tag{C.12}$$

$$|\partial_j\partial_k E_n(z;x) - \partial_j\partial_k E_n(0;x)| \leq \begin{cases} C[|\ln(|x|)| + 1], & n = 2, \\ C\big[|x|^{2-n} + 1\big], & n \geqslant 3. \end{cases} \tag{C.13}$$

Here $C = C(R, n, z)$ represent various different constants in (C.11)–(C.13) *and $\partial_j = \partial/\partial x_j$, $1 \leqslant j \leqslant n$.*

Proof. The estimates in (C.11) follow from combining (C.1), (C.9), and (C.10). The estimates in (C.12) follow from the fact that

$$\partial_j E_n(z;x) = -2\pi x_j E_{n+2}(z;x), \quad z \in \mathbb{C}\backslash\{0\},\ x \in \mathbb{R}^n\backslash\{0\},\ 1 \leqslant j \leqslant n,\ n \geqslant 2, \tag{C.14}$$

which permits one to reduce them essentially to (C.11) with n replaced by $n+2$. The recursion relation (C.14) is a consequence of the well-known identity (cf. [1, Sect. 9.1])

$$\frac{d}{d\zeta}\big(\zeta^{-\nu}\mathcal{C}_\nu(\zeta)\big) = -\zeta^{-\nu}\mathcal{C}_{\nu+1}(\zeta), \quad \zeta \in \mathbb{C}\backslash\{0\},\ \nu \in \mathbb{R}, \tag{C.15}$$

where $\mathcal{C}_\nu(\,\cdot\,)$ denotes any linear combination of Bessel functions of order ν with ζ and ν independent coefficients. Iterating (C.14) yields

$$\begin{aligned} &\partial_j\partial_k E_n(z;x) = 4\pi^2 x_j x_k E_{n+4}(z;x) - 2\pi\delta_{j,k}E_{n+2}(z;x), \\ &z \in \mathbb{C}\backslash\{0\},\ x \in \mathbb{R}^n\backslash\{0\},\ 1 \leqslant j,k \leqslant n,\ n \geqslant 2. \end{aligned} \tag{C.16}$$

Combining (C.11) and (C.16) then yields (C.13). $\square$

Finally, we mention for completeness that for large values of $|x|$, (C.1) implies the following simple asymptotic behavior (cf. [1, Sect. 9.1]):

$$E_n(z;x) \underset{|x|\to\infty}{=} \frac{i}{z^{1/2}}\left(\frac{2\pi|x|}{z^{1/2}}\right)^{(1-n)/2} e^{i\left(z^{1/2}|x| - \pi((n-1)/4)\right)}\big[1 + O\big(|x|^{-1}\big)\big], \quad z \in \mathbb{C}\backslash\{0\},\ \mathrm{Im}\big(z^{1/2}\big) \geqslant 0. \tag{C.17}$$

In particular, as long as $z \in \mathbb{C}\backslash[0,\infty)$ (and hence $\mathrm{Im}\big(z^{1/2}\big) > 0$), $E_n(z;x)$ decays exponentially with respect to x as $|x| \to \infty$.

Appendix D. Calderón–Zygmund Theory on Lipschitz Surfaces

This appendix records various useful consequences of the Calderón–Zygmund theory on Lipschitz surfaces.

Our first result, Lemma D.1 below, is modeled upon a more general result in [46]. For the sake of completeness we include the full argument.

Lemma D.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with compact boundary and let $k(\,\cdot\,,\,\cdot\,)$ be a real-valued, measurable function on $\partial\Omega \times \partial\Omega$ satisfying*

$$|k(x,y)| \leq \frac{\psi(|x-y|)}{|x-y|^{n-1}}, \quad x, y \in \partial\Omega, \tag{D.1}$$

where ψ is monotone increasing and satisfies

$$\int_0^1 \frac{\psi(t)}{t}\, dt < \infty. \tag{D.2}$$

Consider

$$(Kf)(x) = \int_{\partial\Omega} d^{n-1}\omega(y)\, k(x,y) f(y), \quad x \in \partial\Omega. \tag{D.3}$$

Then

$$K \in \mathcal{B}_\infty\big(L^p(\partial\Omega; d^{n-1}\omega)\big) \tag{D.4}$$

for each $p \in (1,\infty)$.

Proof. For a fixed, arbitrary $\varepsilon > 0$, decompose $k(x,y) = k_\varepsilon(x,y) + k_b(x,y)$, where

$$k_\varepsilon(x,y) = \begin{cases} k(x,y), & |x-y| \leq \varepsilon, \\ 0, & |x-y| > \varepsilon. \end{cases} \tag{D.5}$$

Then $K = K_\varepsilon + K_b$, where K_ε, K_b are integral operators on $\partial\Omega$ with integral kernels $k_\varepsilon(x,y)$ and $k_b(x,y)$, respectively. Setting

$$S_j(x) = \big\{y \in \partial\Omega \,|\, 2^{-j-1} \leq |x-y| < 2^{-j}\big\}, \quad x \in \partial\Omega,\ j \in \mathbb{N}, \tag{D.6}$$

for each $x \in \partial\Omega$ we may then compute (with the logarithm taken in base 2)

$$\begin{aligned} \int_{\partial\Omega} d^{n-1}\omega(y)\,|k_\varepsilon(x,y)| &\leq C \sum_{j \geq \log 1/\varepsilon} \int_{S_j(x)} d^{n-1}\omega(y)\, \frac{\psi(|x-y|)}{|x-y|^{n-1}} \\ &\leq C \sum_{j \geq \log 1/\varepsilon} \psi(e^{-j}) \leq C \int_0^\varepsilon \frac{\psi(t)}{t}\,dt. \end{aligned} \tag{D.7}$$

Of course, there is a similar estimate for $\int_{\partial\Omega} d^{n-1}\omega(x)\,|k_\varepsilon(x,y)|$, uniformly for $y \in \partial\Omega$. Schur's lemma then yields

$$\|K_\varepsilon\|_{\mathcal{B}(L^p(\partial\Omega;d^{n-1}\omega))} \leq C \int_0^\varepsilon \frac{\psi(t)}{t}\,dt \to 0 \text{ as } \varepsilon \to 0. \tag{D.8}$$

Thus, it suffices to show that K_b is compact on each $L^p(\partial\Omega;d^{n-1}\omega)$ space, for $p \in (1,\infty)$, under the hypothesis that $k_b(x,y)$ is bounded. First note that K_b is compact on $L^2(\partial\Omega, d^{n-1}\omega)$, since it is Hilbert-Schmidt, due to the fact that $\omega(\partial\Omega) < \infty$. The compactness of K_b on $L^p(\partial\Omega, d^{n-1}\omega)$ for each $p \in (1,\infty)$ then follows from an interpolation theorem of Krasnoselski (see, e.g., [16, Theorem 2.9, p. 203]). □

We now record a basic result from the theory of singular integral operators of Calderón–Zygmund-type on Lipschitz domains. To state it, we recall that $\hat{f}$ denotes the Fourier transform of appropriate functions $f\colon \mathbb{R}^n \to \mathbb{C}$. Moreover, given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, set $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^n \backslash \overline{\Omega}$, we define the nontangential approach regions $\Gamma^\pm_\kappa(x)$, $x \in \partial\Omega$, by $\Gamma^\pm_\kappa(x) = \{y \in \Omega_\pm \,|\, |x-y| < (1+\kappa)\,\mathrm{dist}\,(y,\partial\Omega)\}$, where $\kappa > 0$ is a fixed parameter. Next, at every boundary point the nontangential maximal function of a mapping u (defined in either Ω_+ or Ω_-) is given by

$$M(u)(x) = \sup\,\{|u(y)| \,|\, y \in \Gamma^\pm_\kappa(x)\} \tag{D.9}$$

(with the choice of sign depending on whether u is defined in Ω_+, or Ω_-) and, for u defined in $\Omega_\pm$, we set

$$\gamma_{\text{n.t.}} u(x) = \lim_{\substack{y \to x \\ y \in \Gamma^\pm_\kappa(x)}} u(y) \text{ for a.e. } x \in \partial\Omega. \tag{D.10}$$

For future reference, let us record here a useful estimate proved in [34], valid for any Lipschitz domain $\Omega \subset \mathbb{R}^n$ which is either bounded or has an unbounded boundary. In this setting, for any $p \in (0,\infty)$ and any function u defined in Ω,

$$\|u\|_{L^{np/(n-1)}(\Omega;d^n x)} \leq C(\Omega,n,p)\|Mu\|_{L^p(\partial\Omega;d^{n-1}\omega)}. \tag{D.11}$$

Theorem D.2. *There exists a positive integer $N = N(n)$ with the following significance. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with compact boundary, and assume that*

$$k \in C^N(\mathbb{R}^n\backslash\{0\}) \text{ with } k(-x) = -k(x) \text{ and } k(\lambda x) = \lambda^{-(n-1)}k(x),\ \lambda > 0,\ x \in \mathbb{R}^n\backslash\{0\}. \tag{D.12}$$

Define the singular integral operator

$$(\mathcal{T}f)(x) = \int_{\partial\Omega} d^{n-1}\omega(y)\, k(x-y)f(y), \quad x \in \mathbb{R}^n \backslash \partial\Omega. \tag{D.13}$$

Then for each $p \in (1,\infty)$ there exists a finite constant $C = C(p,n,\partial\Omega) > 0$ such that

$$\|M(\mathcal{T}f)\|_{L^p(\partial\Omega;d^{n-1}\omega)} \leq C\|k|_{S^{n-1}}\|_{C^N}\|f\|_{L^p(\partial\Omega;d^{n-1}\omega)}. \tag{D.14}$$

Furthermore, for each $p \in (1,\infty)$, $f \in L^p(\partial\Omega; d^{n-1}\omega)$, the limit

$$(Tf)(x) = \text{p.v.}\int_{\partial\Omega} d^{n-1}\omega(y)\, k(x-y)f(y) = \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} d^{n-1}\omega(y)\, k(x-y)f(y) \tag{D.15}$$

exists for a.e. $x \in \partial\Omega$, and the jump-formula

$$\gamma_{\text{n.t.}}(\mathcal{T}f)(x) = \lim_{\substack{z\to x\\ z\in\Gamma^{\pm}_{\kappa}(x)}} (\mathcal{T}f)(z) = \pm\tfrac{1}{2i}\hat{k}(\nu(x))f(x) + (Tf)(x) \tag{D.16}$$

is valid at a.e. $x \in \partial\Omega$, where ν denotes the unit normal pointing outwardly relative to Ω.

Finally,

$$\|\mathcal{T}f\|_{H^{1/2}(\Omega)} \leq C\|f\|_{L^2(\partial\Omega;d^{n-1}\omega)}. \tag{D.17}$$

See the discussion in [23], [24], [68].

Lemma D.3. *Whenever Ω is a Lipschitz domain with compact boundary,*

$$K^{\#}_z \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}, \tag{D.18}$$

and

$$\big(K^{\#}_{z_1} - K^{\#}_{z_2}\big) \in \mathcal{B}_\infty\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z_1, z_2 \in \mathbb{C}, \tag{D.19}$$

$$\gamma_D \mathcal{S}_z \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^1(\partial\Omega)\big), \quad z \in \mathbb{C}. \tag{D.20}$$

Proof. Recall the fundamental solution $E_n(z;\,\cdot\,)$ for the Helmholtz equation $(-\Delta - z)\psi(z;\,\cdot\,) = 0$ in $\mathbb{R}^n$ introduced in (2.97). Then the integral kernel of the operator $K^{\#}_z - K^{\#}_0$ is given by

$$k(x,y) = \nu(x)\cdot\big(\nabla E_n(z; x-y) - \nabla E_n(0; x-y)\big), \quad x, y \in \partial\Omega. \tag{D.21}$$

By (C.12) we therefore have $|k(x,y)| \leq C|x-y|^{2-n}$, hence (D.1) holds with $\psi(t) = t$. Note that (D.2) is satisfied for this choice of ψ, so (D.19) is a consequence of Lemma D.1. In addition, (D.18) follows from (D.19) and Theorem D.2, according to which $K^{\#}_0 \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$. Finally, the reasoning for (D.20) is similar (here (A.15) is useful). □

Lemma D.4. *Let $0 < \alpha < (n-1)$ and $1 < p < q < \infty$ be related by*

$$\frac{1}{q} = \frac{1}{p} - \Big(\alpha + \frac{1}{p}\Big)\frac{1}{n}. \tag{D.22}$$

Then the the operator J_α defined by

$$J_\alpha f(x) = \int_{\mathbb{R}^{n-1}} d^{n-1}y\, \frac{1}{|x-y|^{n-1-\alpha}} f(y), \quad x \in \overline{\mathbb{R}^n_+},\ f \in L^p(\mathbb{R}^{n-1}; d^{n-1}x), \tag{D.23}$$

is bounded from $L^p(\mathbb{R}^{n-1}; d^{n-1}x)$ to $L^q(\mathbb{R}^n_+; d^n x)$, that is, for some constant $C_{\alpha,p,q} > 0$,

$$\|J_\alpha f\|_{L^q(\mathbb{R}^n_+;d^n x)} \leq C_{\alpha,p,q}\|f\|_{L^p(\mathbb{R}^{n-1};d^{n-1}x)}, \quad f \in L^p(\mathbb{R}^{n-1}; d^{n-1}x). \tag{D.24}$$

Proof. A direct proof appears in [70]. An alternative argument is to observe that $M(J_\alpha f)(x) \leq CJ_\alpha(|f|)(x)$, uniformly for $x \in \partial\mathbb{R}^n_+$, and then to invoke the general estimate (D.11) in concert with the classical Hardy-Littlewood-Sobolev fractional integration theorem (cf., e.g., [86], Theorem 1 on p. 119). □

Next, we record a lifting result for Sobolev spaces in Lipschitz domains in [47].

Theorem D.5. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with compact boundary. Then, for every $\alpha > 0$, the following equivalence of norms holds:*

$$\|u\|_{H^{\alpha+1}(\Omega)} \approx \|u\|_{L^2(\Omega)} + \|\nabla u\|_{H^\alpha(\Omega)}. \tag{D.25}$$

Theorem D.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then for every $z \in \mathbb{C}$,*

$$\mathcal{S}_z \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^{3/2}(\Omega)\big), \tag{D.26}$$

and

$$\mathcal{S}_z \in \mathcal{B}\big(H^{-1}(\partial\Omega), H^{1/2}(\Omega)\big). \tag{D.27}$$

In particular,

$$\mathcal{S}_z \in \mathcal{B}\big(H^{s-1}(\partial\Omega), H^{s+(1/2)}(\Omega)\big), \quad 0 \leqslant s \leqslant 1. \tag{D.28}$$

Proof. Given $f \in L^2(\partial\Omega; d^{n-1}\omega)$, write $\mathcal{S}_z f = \mathcal{S}_0 f + (\mathcal{S}_z - \mathcal{S}_0)f$. From (D.17) and Lemma D.5 we know that $\|\mathcal{S}_0 f\|_{H^{3/2}(\Omega)} \leq C\|f\|_{L^2(\partial\Omega; d^{n-1}\omega)}$, for some constant $C > 0$ independent of f. Using (C.13) and Lemma D.4 (with $\alpha = 1$) one concludes that

$$\nabla^2(\mathcal{S}_z - \mathcal{S}_0) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), L^2(\Omega; d^n x)\big), \tag{D.29}$$

and (D.26) follows from this. The proof of (D.27), is analogous and has as starting point the fact that $\mathcal{S}_0 f \in \mathcal{B}\big(H^{-1}(\partial\Omega), H^{1/2}(\Omega)\big)$, itself a consequence of (D.17) and the following description of $H^{-1}(\partial\Omega)$:

$$H^{-1}(\partial\Omega) = \bigg\{ g + \sum_{1 \leq j,k \leq n} (\partial f_{j,k}/\partial\tau_{j,k}) \,\bigg|\, g, f_{j,k} \in L^2(\partial\Omega; d^{n-1}\omega) \bigg\}. \tag{D.30}$$

Then (D.29) ensures that

$$(\mathcal{S}_z - \mathcal{S}_0) \in \mathcal{B}\big(H^{-1}(\partial\Omega), H^1(\Omega)\big), \tag{D.31}$$

and (D.27) follows. □

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Dover, New York, 1993.

[3] S. Albeverio, J. F. Brasche, M. M. Malamud, and H. Neidhardt, *Inverse spectral theory for symmetric operators with several gaps: scalar-type Weyl functions*, J. Funct. Anal. **228**, 144–188 (2005).

[4] S. Albeverio, M. Dudkin, A. Konstantinov, and V. Koshmanenko, *On the point spectrum of $\mathcal{H}_{-2}$-singular perturbations*, Math. Nachr. **280**, 20–27 (2007).

[5] W. O. Amrein and D. B. Pearson, *M operators: a generalization of Weyl–Titchmarsh theory*, J. Comp. Appl. Math. **171**, 1–26 (2004).

[6] W. Arendt and M. Warma, *Dirichlet and Neumann boundary conditions: What is in between?*, J. Evolution Eq. **3**, 119–135 (2003).

[7] W. Arendt and M. Warma, *The Laplacian with Robin boundary conditions on arbitrary domains*, Potential Anal. **19**, 341–363 (2003).

[8] Yu. M. Arlinskii and E. R. Tsekanovskii, *Some remarks on singular perturbations of self-adjoint operators*, Meth. Funct. Anal. Top. **9**, No. 4, 287–308 (2003).

[9] Yu. Arlinskii and E. Tsekanovskii, *The von Neumann problem for nonnegative symmetric operators*, Int. Eq. Operator Theory, **51**, 319–356 (2005).

[10] G. Auchmuty, *Steklov eigenproblems and the representation of solutions of elliptic boundary value problems*, Num. Funct. Anal. Optimization **25**, 321–348 (2004).

[11] G. Auchmuty, *Spectral characterization of the trace spaces $H^s(\partial\Omega)$*, SIAM J. Math. Anal. **38**, 894–905 (2006).

[12] J. Behrndt and M. Langer, *Boundary value problems for partial differential operators on bounded domains*, J. Funct. Anal. **243**, 536–565 (2007).

[13] J. Behrndt, M. M. Malamud, and H. Neidhardt, *Scattering matrices and Weyl functions*, preprint, 2006.
[14] S. Belyi, G. Menon, and E. Tsekanovskii, *On Krein's formula in the case of non-densely defined symmetric operators*, J. Math. Anal. Appl. **264**, 598–616 (2001).
[15] S. Belyi and E. Tsekanovskii, *On Krein's formula in indefinite metric spaces*, Lin. Algebra Appl. **389**, 305–322 (2004).
[16] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, Vol. 129, Academic Press, Inc., Boston, MA, 1988.
[17] M. Biegert and M. Warma, *Removable singularities for a Sobolev space*, J. Math. Anal. Appl. **313**, 49–63 (2006).
[18] J. F. Brasche, M. M. Malamud, and H. Neidhardt, *Weyl functions and singular continuous spectra of self-adjoint extensions*, in *Stochastic Processes, Physics and Geometry: New Interplays. II. A Volume in Honor of Sergio Albeverio*, F. Gesztesy, H. Holden, J. Jost, S. Paycha, M. Röckner, and S. Scarlatti (eds.), Canadian Mathematical Society Conference Proceedings, Vol. 29, Amer. Math. Soc., Providence, RI, 2000, pp. 75–84.
[19] J. F. Brasche, M. M. Malamud, and H. Neidhardt, *Weyl function and spectral properties of self-adjoint extensions*, Integral Eqs. Operator Theory **43**, 264–289 (2002).
[20] B. M. Brown and M. Marletta, *Spectral inclusion and spectral exactness for PDE's on exterior domains*, IMA J. Numer. Anal. **24**, 21–43 (2004).
[21] B. M. Brown, M. Marletta, S. Naboko, and I. Wood, *Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices*, J. London Math. Soc., to appear.
[22] J. Brüning, V. Geyler, and K. Pankrashkin, *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, preprint, 2007.
[23] R.R. Coifman, G. David and Y. Meyer, *La solution des conjecture de Calderón*, Adv. in Math. **48**, 144–148 (1983).
[24] R.R. Coifman, A. McIntosh and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. of Math. (2) **116**, 361–387 (1982).
[25] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. **65**, 333–343 (1992).
[26] B.E.J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. **65**, 275–288 (1977).
[27] E. N. Dancer and D. Daners, *Domain perturbation for elliptic equations subject to Robin boundary conditions*, J. Diff. Eq. **138**, 86–132 (1997).
[28] D. Daners, *Robin boundary value problems on arbitrary domains*, Trans. Amer. Math. Soc. **352**, 4207–4236 (2000).
[29] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Volume 2, Functional and Variational Methods*, Springer, Berlin, 2000.
[30] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Meth. Funct. Anal. Top. **6**, No. 3, 24–55 (2000).
[31] V. Derkach, S. Hassi, M. Malamud, and H. de Snoo, *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc. **358**, 5351–5400 (2006).
[32] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).
[33] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73**, 141–242 (1995).
[34] M. Dindoš and M. Mitrea, *Semilinear Poisson problems in Sobolev–Besov spaces on Lipschitz domains*, Publ. Mat. **46**, 353–403 (2002).
[35] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.
[36] E. Fabes, *Layer potential methods for boundary value problems on Lipschitz domains*, Potential theory—surveys and problems (Prague, 1987), pp. 55–80, Lecture Notes in Math., Vol. **1344**, Springer, Berlin, 1988.
[37] W. G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. **433**, Springer, Berlin, 1975.
[38] F. Gesztesy, K. A. Makarov, and E. Tsekanovskii, *An addendum to Krein's formula*, J. Math. Anal. Appl. **222**, 594–606 (1998).
[39] F. Gesztesy, N. J. Kalton, K. A. Makarov, and E. Tsekanovskii, *Some Applications of Operator-Valued Herglotz Functions*, in *Operator Theory, System Theory and Related Topics. The Moshe Livšic Anniversary Volume*, D. Alpay and V. Vinnikov (eds.), Operator Theory: Advances and Applications, Vol. 123, Birkhäuser, Basel, 2001, pp. 271–321.
[40] F. Gesztesy, Y. Latushkin, M. Mitrea, and M. Zinchenko, *Nonselfadjoint operators, infinite determinants, and some applications*, Russ. J. Math. Phys. **12**, 443–471 (2005).
[41] F. Gesztesy and M. M. Malamud, *Boundary value problems for elliptic operators from the extension theory point of view and Weyl–Titchmarsh functions*, in preparation.
[42] F. Gesztesy and M. Mitrea, *Robin-to-Robin maps and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains*, preprint, 2008.

[43] F. Gesztesy, M. Mitrea, and M. Zinchenko, *Variations on a Theme of Jost and Pais*, J. Funct. Anal. **253**, 399–448 (2007).
[44] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer, Dordrecht, 1991.
[45] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.
[46] S. Hofmann, M. Mitrea and M. Taylor, *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains*, preprint, 2007.
[47] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130**, 161–219 (1995).
[48] W. Karwowski and S. Kondej, *The Laplace operator, null set perturbations and boundary conditions*, in *Operator Methods in Ordinary and Partial Differential Equations*, S. Albeverio, N. Elander, W. N. Everitt, and P. Kurasov (eds.), Operator Theory: Advances and Applications, Vol. 132, Birkhäuser, Basel, 2002, pp. 233–244.
[49] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.
[50] V. Koshmanenko, *Singular operators as a parameter of self-adjoint extensions*, in *Operator Theory and Related Topics*, V. M. Adamyan, I. Gohberg, M. Gorbachuk, V. Gorbachuk, M. A. Kaashoek, H. Langer, and G. Popov (eds.), Operator Theory: Advances and Applications, Vol. 118, Birkhäuser, Basel, 2000, pp. 205–223.
[51] M. G. Krein, *On Hermitian operators with deficiency indices one*, Dokl. Akad. Nauk SSSR **43**, 339–342 (1944). (Russian).
[52] M. G. Krein, *Resolvents of a hermitian operator with defect index* (m,m), Dokl. Akad. Nauk SSSR **52**, 657–660 (1946). (Russian).
[53] M. G. Krein and I. E. Ovcharenko, *Q-functions and sc-resolvents of nondensely defined hermitian contractions*, Sib. Math. J. **18**, 728–746 (1977).
[54] M. G. Krein and I. E. Ovčarenko, *Inverse problems for Q-functions and resolvent matrices of positive hermitian operators*, Sov. Math. Dokl. **19**, 1131–1134 (1978).
[55] M. G. Krein, S. N. Saakjan, *Some new results in the theory of resolvents of hermitian operators*, Sov. Math. Dokl. **7**, 1086–1089 (1966).
[56] P. Kurasov and S. T. Kuroda, *Krein's resolvent formula and perturbation theory*, J. Operator Theory **51**, 321–334 (2004).
[57] H. Langer, B. Textorius, *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math. **72**, 135–165 (1977).
[58] L. Lanzani and Z. Shen, *On the Robin boundary condition for Laplace's equation in Lipschitz domain*, Comm. Partial Diff. Eqs. **29**, 91–109 (2004).
[59] J. L. Lions, *Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs*, J. Math. Soc. Japan **14**, 233–241 (1962).
[60] M. M.Malamud, *On a formula of the generalized resolvents of a nondensely defined hermitian operator*, Ukrain. Math. J. **44**, 1522–1547 (1992).
[61] M. M. Malamud and V. I. Mogilevskii, *Krein type formula for canonical resolvents of dual pairs of linear relations*, Methods Funct. Anal. Topology, **8**, No. 4, 72–100 (2002).
[62] M. Marletta, *Eigenvalue problems on exterior domains and Dirichlet to Neumann maps*, J. Comp. Appl. Math. **171**, 367–391 (2004).
[63] V. G. Maz'ja, *Einbettungssätze für Sobolewsche Räume*, Teubner Texte zur Mathematik, Teil 1, 1979; Teil 2, 1980, Teubner, Leipzig.
[64] V. G. Maz'ja, *Sobolev Spaces*, Springer, Berlin, 1985.
[65] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.
[66] A. B. Mikhailova, B. S. Pavlov, and L. V. Prokhorov, *Intermediate Hamiltonian via Glazman's splitting and analytic perturbation for meromorphic matrix-functions*, Math. Nachr. **280**, 1376–1416 (2007).
[67] I. Mitrea and M. Mitrea, *Multiple Layer Potentials for Higher Order Elliptic Boundary Value Problems*, preprint, 2008.
[68] D. Mitrea, M. Mitrea, and M. Taylor, *Layer Potentials, the Hodge Laplacian, and Global Boundary Problems in Nonsmooth Riemannian Manifolds*, Mem. Amer. Math. Soc. **150**, No. 713, (2001).
[69] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*, J. Funct. Anal. **176**, 1–79 (2000).
[70] M. Mitrea and M. Taylor, *Sobolev and Besov space estimates for solutions to second order PDE on Lipschitz domains in manifolds with Dini or Hölder continuous metric tensors*, Comm. Part. Diff. Eq. **30**, 1–37 (2005).
[71] S. Nakamura, *A remark on the Dirichlet–Neumann decoupling and the integrated density of states*, J. Funct. Anal. **179**, 136–152 (2001).

[72] G. Nenciu, *Applications of the Krein resolvent formula to the theory of self-adjoint extensions of positive symmetric operators*, J. Operator Theory **10**, 209–218 (1983).
[73] K. Pankrashkin, *Resolvents of self-adjoint extensions with mixed boundary conditions*, Rep. Math. Phys. **58**, 207–221 (2006).
[74] B. Pavlov, *The theory of extensions and explicitly-soluble models*, Russ. Math. Surv. **42:6**, 127–168 (1987).
[75] B. Pavlov, *S-matrix and Dirichlet-to-Neumann operators*, Ch. 6.1.6 in *Scattering: Scattering and Inverse Scattering in Pure and Applied Science*, Vol. 2, R. Pike and P. Sabatier (eds.), Academic Press, San Diego, 2002, pp. 1678–1688.
[76] A. Posilicano, *A Krein-like formula for singular perturbations of self-adjoint operators and applications*, J. Funct. Anal. **183**, 109–147 (2001).
[77] A. Posilicano, *Self-adjoint extensions by additive perturbations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (5) Vol. II, 1–20 (2003).
[78] A. Posilicano, *Boundary triples and Weyl functions for singular perturbations of self-adjoint operators*, Meth. Funct. Anal. Topology **10**, No. 2, 57–63 (2004).
[79] A. Posilicano, *Self-adjoint extensions of restrictions*, Operators and Matrices, to appear; arXiv:math-ph/0703078.
[80] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear partial Differential Operators*, de Gruyter, Berlin, New York, 1996.
[81] V. S. Rychkov, *On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc. (2) **60**, 237–257 (1999).
[82] V. Ryzhov, *A general boundary value problem and its Weyl function*, Opuscula Math. **27**, 305–331(2007).
[83] V. Ryzhov, *Weyl–Titchmarsh function of an abstract boundary value problem, operator colligations, and linear systems with boundary control*, Complex Anal. Operator Theory, to appear.
[84] Sh. N. Saakjan, *On the theory of the resolvents of a symmetric operator with infinite deficiency indices*, Dokl. Akad. Nauk Arm. SSR **44**, 193–198 (1965). (Russian.)
[85] B. Simon, *Classical boundary conditions as a tool in quantum physics*, Adv. Math. **30**, 268–281 (1978).
[86] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970.
[87] A. V. Straus, *Generalized resolvents of symmetric operators*, Dokl. Akad. Nauk SSSR **71**, 241–244 (1950). (Russian.)
[88] A. V. Straus, *On the generalized resolvents of a symmetric operator*, Izv. Akad. Nauk SSSR Ser. Math. **18**, 51–86 (1954). (Russian.)
[89] A. V. Straus, *Extensions and generalized resolvents of a non-densely defined symmetric operator*, Math. USSR Izv. **4**, 179–208 (1970).
[90] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Progress in Mathematics Vol. 100, Birkhäuser, Boston, MA, 1991.
[91] M. E. Taylor, *Partial Differential Equations, Vol. II*, Springer, New York, 1996.
[92] M. E. Taylor, *Tools for Partial Differential Equations*, American Mathematical Society, Providence, RI, 2000.
[93] H. Triebel, *Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Rev. Mat. Complut. **15**, 475–524 (2002).
[94] E. R. Tsekanovskii and Yu. L. Shmul'yan, *The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions*, Russ. Math. Surv. **32:5**, 73–131 (1977).
[95] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59**, 572–611 (1984).
[96] M. Warma, *The Laplacian with general Robin boundary conditions*, Ph.D. Thesis, University of Ulm, 2002.
[97] M. Warma, *The Robin and Wentzell-Robin Laplacians on Lipschitz domains*, Semigroup Forum **73**, 10–30 (2006).
[98] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.
[99] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.
[100] I. Wood, *Maximal L^p-regularity for the Laplacian on Lipschitz domains*, Math. Z. **255**, 855–875 (2007).

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