# Asymptotic stability of ground states in 3D nonlinear Schrödinger equation including subcritical cases

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**Abstract**

We consider a class of nonlinear Schrödinger equation in three space dimensions with an attractive potential. The nonlinearity is local but rather general encompassing for the first time both subcritical and supercritical (in $L^2$) nonlinearities. We study the asymptotic stability of the nonlinear bound states, i.e. periodic in time localized in space solutions. Our result shows that all solutions with small initial data, converge to a nonlinear bound state. Therefore, the nonlinear bound states are asymptotically stable. The proof hinges on dispersive estimates that we obtain for the time dependent, Hamiltonian, linearized dynamics around a careful chosen one parameter family of bound states that "shadows" the nonlinear evolution of the system. Due to the generality of the methods we develop we expect them to extend to the case of perturbations of large bound states and to other nonlinear dispersive wave type equations.

## 1 Introduction

In this paper we study the long time behavior of solutions of the nonlinear Schrödinger equation (NLS) with potential in three space dimensions (3-d):

$$i\partial_t u(t,x) = [-\Delta_x + V(x)]u + g(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3 \tag{1.1}$$

$$u(0,x) = u_0(x) \tag{1.2}$$

where the local nonlinearity is constructed from the real valued, odd, $C^2$ function $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying

$$|g''(s)| \leq C(|s|^{\alpha_1} + |s|^{\alpha_2}), \quad s \in \mathbb{R}\ 0 < \alpha_1 \leqslant \alpha_2 < 3 \tag{1.3}$$

which is then extended to a complex function via the gauge symmetry:

$$g(e^{i\theta}s) = e^{i\theta}g(s) \tag{1.4}$$

The equation has important applications in statistical physics describing certain limiting behavior of Bose-Einstein condensates [7, 17, 9].

It is well known that this nonlinear equation admits periodic in time, localized in space solutions (bound states or solitary waves). They can be obtained via both variational techniques [1, 26, 21] and bifurcation methods [20, 21, 15], see also next section. Moreover the set of periodic solutions can be organized as a manifold (center manifold). Orbital stability of solitary waves, i.e. stability modulo the group of symmetries $u \mapsto e^{-i\theta}u$, was first proved in [21, 28], see also [11, 12, 22].

In this paper we show that solutions of (1.1)-(1.2) with small initial data asymptotically converge to the orbit of a certain bound state, see Theorem 3.1. Asymptotic stability studies of solitary waves were initiated in the work of A. Soffer and M. I. Weinstein [23, 24], see also [2, 3, 4, 6, 13]. Center manifold analysis was introduced in [20], see also [27].

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The main contribution of our result is to allow for subcritical and critical ($L^2$) nonlinearities, $0 < \alpha_1 \leqslant 1/3$ in (1.3). To accomplish this we develop an innovative technique in which linearization around a one parameter family of bound states is used to track the solution. Previously a fixed bound state has been used, see the papers cited in the previous paragraph. By continuously adapting the linearization to the actual evolution of the solution we are able to capture the correct effective potential induced by the nonlinearity $g$ into a time dependent linear operator. Once we have a good understanding of the semigroup of operators generated by the time dependent linearization, see Section 4, we obtain sharper estimates for the nonlinear dynamics via Duhamel formula and contraction principles for integral equations, see Section 3. They allow us to treat a large spectrum of nonlinearities including, for the first time, the subcritical ones.

The main challenge is to obtain good estimates for the semigroup of operators generated by the time dependent linearization that we use. This is accomplished in Section 4. The technique is perturbative, and similar to the one developed by the first author and A. Zarnescu for 2-D Schrödinger type operators in [15], see also [16]. The main difference is that in 3-D one needs to remove the non-integrable singularity in time at zero of the free Schrödinger propagator:

$$\|e^{i\Delta t}\|_{L^1\mapsto L^\infty} \sim |t|^{-3/2}.$$

We do this by generalizing a Fourier multiplier type estimate first introduced by Journé, Soffer, and Sogge in [14] and by proving certain smoothness properties of the effective potential induced by the nonlinearity, see the Appendix.

Since our methods rely on linearization around nonlinear bound states and estimates for integral operators we expect them to generalize to the case of large nonlinear ground states, see for example [6], or the presence of multiple families of bound states, see for example [25], where it should greatly reduce the restrictions on the nonlinearity. We are currently working on adapting the method to other spatial dimensions. The work in 2-D is almost complete, see [15, 16].

**Notations:** $H = -\Delta + V$;

$L^p = \{f : \mathbb{R}^2 \mapsto \mathbb{C} \mid f$ measurable and $\int_{\mathbb{R}^2} |f(x)|^p dx < \infty\}$, $\|f\|_p = \left(\int_{\mathbb{R}^2} |f(x)|^p dx\right)^{1/p}$ denotes the standard norm in these spaces;

$< x >= (1+|x|^2)^{1/2}$, and for $\sigma \in \mathbb{R}$, $L^2_\sigma$ denotes the $L^2$ space with weight $< x >^{2\sigma}$, i.e. the space of functions $f(x)$ such that $< x >^\sigma f(x)$ are square integrable endowed with the norm $\|f(x)\|_{L^2_\sigma} = \| < x >^\sigma f(x)\|_2$;

$\langle f, g\rangle = \int_{\mathbb{R}^2} \overline{f}(x) g(x) dx$ is the scalar product in $L^2$ where $\overline{z} =$ the complex conjugate of the complex number $z$;

$P_c$ is the projection on the continuous spectrum of $H$ in $L^2$;

$H^n$ denote the Sobolev spaces of measurable functions having all distributional partial derivatives up to order $n$ in $L^2$, $\|\cdot\|_{H^n}$ denotes the standard norm in this spaces.

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## 2 Preliminaries. The center manifold.

The center manifold is formed by the collection of periodic solutions for (1.1):

$$u_E(t,x) = e^{-iEt}\psi_E(x) \tag{2.1}$$

where $E \in \mathbb{R}$ and $0 \not\equiv \psi_E \in H^2(\mathbb{R}^3)$ satisfy the time independent equation:

$$[-\Delta + V]\psi_E + g(\psi_E) = E\psi_E \tag{2.2}$$

Clearly the function constantly equal to zero is a solution of (2.2) but (iii) in the following hypotheses on the potential $V$ allows for a bifurcation with a nontrivial, one parameter family of solutions:

**(H1)** Assume that

(i) There exists $C > 0$ and $\rho > 3$ such that:

1. $|V(x)| \leqslant C < x >^{-\rho}, \quad$ for all $x \in \mathbb{R}^3$;
2. $\nabla V \in L^p(\mathbb{R}^3)$ for some $2 \leqslant p \leqslant \infty$ and $|\nabla V(x)| \to 0$ as $|x| \to \infty$;
3. the Fourier transform of $V$ is in $L^1$.

(ii) 0 is a regular point[1] of the spectrum of the linear operator $H = -\Delta + V$ acting on $L^2$.

(iii) $H$ acting on $L^2$ has exactly one negative eigenvalue $E_0 < 0$ with corresponding normalized eigenvector $\psi_0$. It is well known that $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$, and can be chosen strictly positive.

Conditions (i)1. and (ii) guarantee the applicability of dispersive estimates of Murata [18] and Goldberg-Schlag [10] to the Schrödinger group $e^{-iHt}$. Condition (i)2. implies certain regularity of the nonlinear bound states while (i)3. allow us to use commutator type estimates, see Theorem 5.2. All these are needed to obtain estimates for the semigroup of operators generated by our time dependent linearization, see Theorems 4.1 and 4.2 in section 4. In particular (i)1. implies the local well posedness in $H^1$ of the initial value problem (1.1)-(1.2), see section 3.

By the standard bifurcation argument in Banach spaces [19] for (2.2) at $E = E_0$, condition (iii) guarantees existence of nontrivial solutions. Moreover, these solutions can be organized as a $C^1$ manifold (center manifold), see [15, section 2]. Since our main result requires, we are going to show in what follows that the center manifold is $C^2$. We note that for three and higher dimensions this has been sketched in [13], however they show smoothness by formal differentiation of certain equations without proof that at least one side has indeed derivatives.

As in [15] we decompose the solution of (2.2) in its projection onto the discrete and continuous part of the spectrum of $H$ :

$$\psi_E = a\psi_0 + h, \quad a = \langle \psi_0, \psi_E \rangle, \ h = P_c \psi_E.$$

Projecting now (2.2) onto $\psi_0$ and its orthogonal complement = Range $P_c$ we get:

$$0 = h + (H - E)^{-1} P_c g(a\psi_0 + h) \tag{2.3}$$

$$0 = E - E_0 - a^{-1} \langle \psi_0, g(a\psi_0 + h) \rangle \tag{2.4}$$

Although we are using milder hypothesis on $V$ the argument in the Appendix of [20] can be easily adapted to show that:

$$\mathcal{F}(E, a, h) = h + (H - E)^{-1} P_c g(a\psi_0 + h)$$

is a $C^2$ function from $(-\infty, 0) \times \mathbb{C} \times L^2_\sigma \cap H^2$ to $L^2_\sigma \cap H^2$ and $\mathcal{F}(E_0, 0, 0) = 0$, $D_h \mathcal{F}(E_0, 0, 0) = I$. Therefore the implicit function theorem applies to equation (2.3) and leads to the existence of $\delta_1 > 0$ and the $C^2$ function $\tilde{h}(E, a)$ from $(E_0 - \delta_1, E_0 + \delta_1) \times \{a \in \mathbb{C} \ : \ |a| < \delta_1\}$ to $L^2_\sigma \cap H^2$ such that (2.3) has a unique solution $h = \tilde{h}(E, a)$ for all $E \in (E_0 - \delta_1, E_0 + \delta_1)$ and $|a| < \delta_1$. Note that, by gauge invariance, if $(a, h)$ solves (2.3) then $(e^{i\theta} a, e^{i\theta} h)$, $\theta \in [0, 2\pi)$ is also a solution, hence by uniqueness we have:

$$\tilde{h}(E, a) = \frac{a}{|a|} \tilde{h}(E, |a|). \tag{2.5}$$

Because $\psi_0$ is real valued, we could apply the implicit function theorem to (2.3) under the restriction $a \in \mathbb{R}$ and $h$ in the subspace of real valued functions as it is actually done in [20]. By uniqueness of the solution we deduce that $\tilde{h}(E, |a|)$ is a real valued function.

Consider now the restriction of $\tilde{h}(E, a)$ to $a \in \mathbb{R}$, $|a| < \delta_1$. This is now a real valued $C^2$ function on $(E_0 - \delta_1, E_0 + \delta_1) \times (-\delta_1, \delta_1)$ which, by (2.5), is odd in the second variable. We now differentiate (2.3) with $h = \tilde{h}(E, a)$, to obtain the following estimates for the first and second derivatives of $\tilde{h}$ on $(E, a) \in$

[1] see [10, Definition 6] or $M_\mu = \{0\}$ in relation (3.1) in [18]

$(E_0-\delta_1, E_0+\delta_1)\times(-\delta_1,\delta_1)$ :

$$\begin{aligned}
\frac{\partial \tilde{h}}{\partial a}(E,a) &= -(D_h\mathcal{F})^{-1}(E,a,\tilde{h}(E,a))[(H-E)^{-1}P_c g'(a\psi_0+\tilde{h}(E,a))\psi_0] = \mathcal{O}(|a|^{1+\alpha_1})\\
\frac{\partial \tilde{h}}{\partial E}(E,a) &= (D_h\mathcal{F})^{-1}(E,a,\tilde{h}(E,a))[(H-E)^{-2}P_c g(a\psi_0+\tilde{h}(E,a))] = \mathcal{O}(|a|^{2+\alpha_1})\\
\frac{\partial^2 \tilde{h}}{\partial a^2}(E,a) &= -(D_h\mathcal{F})^{-1}\left[(H-E)^{-1}P_c g''(a\psi_0+\tilde{h}(E,a))\left(\psi_0+\frac{\partial \tilde{h}}{\partial a}\right)^2\right] = \mathcal{O}(|a|^{\alpha_1})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \tilde{h}}{\partial E\partial a}(E,a)=\mathcal{O}(|a|^{1+\alpha_1}) &= (D_h\mathcal{F})^{-1}\left[(H-E)^{-2}P_c g'(a\psi_0+\tilde{h})\left(\psi_0+\frac{\partial \tilde{h}}{\partial a}\right)\right]\\
&- (D_h\mathcal{F})^{-1}\left[(H-E)^{-1}P_c g''(a\psi_0+\tilde{h})\left(\psi_0+\frac{\partial \tilde{h}}{\partial a}\right)\frac{\partial \tilde{h}}{\partial E}\right]\\
\frac{\partial^2 \tilde{h}}{\partial E^2}(E,a)=\mathcal{O}(|a|^{2+\alpha_1}) &= +(D_h\mathcal{F})^{-1}\left[(H-E)^{-1}P_c g''(a\psi_0+\tilde{h})\left(\frac{\partial \tilde{h}}{\partial E}\right)^2\right]\\
&- (D_h\mathcal{F})^{-1}\left[2(H-E)^{-3}P_c g(a\psi_0+\tilde{h})-2(H-E)^{-2}P_c g'(a\psi_0+\tilde{h})\frac{\partial \tilde{h}}{\partial E}\right]
\end{aligned}$$

where we used $D_h\mathcal{F}(E,a,\tilde{h}(E,a))$ is invertible with bounded inverse and $D_h\mathcal{F}(E,0,0)=I$, $(H-E)^{-1}$ is bounded and analytic operator in $E\in(E_0-\delta_1,E_0+\delta_1)$, and $g'(s)=\mathcal{O}(s^{1+\alpha_1})$, $g''(s)=\mathcal{O}(s^{\alpha_1})$ as $s\to 0$.

Replacing now $h=\tilde{h}(E,a)$, $(E,a)\in(E_0-\delta_1,E_0+\delta_1)\times(-\delta_1,\delta_1)$ in (2.4) we get:

$$E-E_0 = a^{-1}\langle\psi_0, g(a\psi_0+\tilde{h}(E,a))\rangle. \tag{2.6}$$

To this we can apply again the implicit function theorem by observing that $G(E,a)=E-E_0-a^{-1}\langle\psi_0,g(a\psi_0+\tilde{h}(E,a))\rangle$ is a $C^1$ function from $(E_0-\delta_1,E_0+\delta_1)\times(-\delta_1,\delta_1)$ to $\mathbb{R}$ with the properties $G(E_0,0)=0$, $\partial_E G(E_0,0)=1$. We obtain the existence of $0<\delta\leqslant\delta_1$, and the $C^1$ even function $\tilde{E}:(-\delta,\delta)\mapsto(E_0-\delta,E_0+\delta)$ such that, for $|E-E_0|,|a|<\delta$, the unique solution of (2.4) with $h=\tilde{h}(E,a)$, is given by the $E=\tilde{E}(a)$. Note that $\tilde{E}$ is $C^2$ except at $a=0$ because $G$ is $C^2$ except at $a=0$, and:

$$\begin{aligned}
\frac{d\tilde{E}}{da}(a) &= -\frac{\partial_a G(E(a),a)}{\partial_E G(E(a),a)} = \mathcal{O}(|a|^{\alpha_1})\\
\frac{d^2\tilde{E}}{da^2}(a) &= \mathcal{O}(|a|^{\alpha_1-1}) \qquad \text{for } a\neq 0,\ \text{recall that } 0<\alpha_1\leqslant 1.
\end{aligned}$$

If we now define the odd function:

$$h(a)\equiv\tilde{h}(E(a),a), \qquad -\delta<a<\delta$$

we get a $C^2$ function because, for $a\neq 0$, based on the previous estimates on the derivatives of $\tilde{h}$ and $\tilde{E}$, we have

$$\frac{d^2h}{da^2}(a)=\frac{\partial \tilde{h}}{\partial E}\frac{d^2\tilde{E}}{da^2}+\frac{\partial^2\tilde{h}}{\partial E^2}\left(\frac{d\tilde{E}}{da}\right)^2+2\frac{\partial^2\tilde{h}}{\partial E\partial a}\frac{d\tilde{E}}{da}+\frac{\partial^2\tilde{h}}{\partial a^2}=\mathcal{O}(|a|^{\alpha_1}),$$

hence, by L'Hospital

$$\frac{d^2h}{da}(0)\stackrel{def}{=}\lim_{a\to 0}\frac{\frac{dh}{da}(a)-0}{a}=\lim_{a\to 0}\frac{d^2h}{da^2}(a)=0.$$

We now extend $h$ to complex values via the rotational symmetry (2.5):

$$h(a)=\frac{a}{|a|}\tilde{h}(E(|a|),|a|).$$

We have just proved:

**Proposition 2.1** *There exist $\delta > 0$, the $C^2$ function*

$$h : \{a \in \mathbb{R} \times \mathbb{R} \ : \ |a| < \delta\} \mapsto L^2_\sigma \cap H^2,$$

*and the $C^1$ function $E : (-\delta, \delta) \mapsto \mathbb{R}$ such that for $|E - E_0| < \delta$ and $|\langle \psi_0, \psi_E \rangle| < \delta$ the eigenvalue problem (2.2) has a unique solution up to multiplication with $e^{i\theta}$, $\theta \in [0, 2\pi)$, which can be represented as a center manifold:*

$$\psi_E = a\psi_0 + h(a), \ E = E(|a|), \quad \langle \psi_0, h(a) \rangle = 0, \quad h(e^{i\theta}a) = e^{i\theta}h(a), |a| < \delta. \tag{2.7}$$

*Moreover $E(|a|) = \mathcal{O}(|a|^{1+\alpha_1})$, $h(a) = \mathcal{O}(|a|^{2+\alpha_1})$, and for $a \in \mathbb{R}$, $|a| < \delta$, $h(a)$ is a real valued function with $\frac{d^2h}{da^2}(a) = \mathcal{O}(|a|^{\alpha_1})$*

Since $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$ the proposition implies that $\psi_E \in L^2_\sigma$. A regularity argument, see [23], gives a stronger result:

**Corollary 2.1** *For any $\sigma \in \mathbb{R}$, there exists a finite constant $C_\sigma$ such that:*

$$\| < x >^\sigma \psi_E \|_{H^2} \leqslant C_\sigma \|\psi_E\|_{H^2}.$$

**Remark 2.1** *By standard regularity methods, see for example [5, Theorem 8.1.1], one can show $\psi_E \in H^3$. Hence by Sobolev imbeddings both $\psi_E$ and $\nabla\psi_E$ are continuous and converge to zero as $|x| \to \infty$.*

**Remark 2.2** *By standard variational methods, see for example [21], one can show that the real valued solutions of (2.2) do not change sign. Then Harnack inequality for $H^2 \bigcap C(\mathbb{R}^3)$ solutions of (2.2) implies that these real solution cannot take the zero value. Hence $\psi_E$ given by (2.7) for $a \in \mathbb{R}$ is either strictly positive or strictly negative.*

In section 4 we also need some smoothness for the effective (linear) potential induced by the nonlinearity which modulo rotations of the complex plane is given by:

$$Dg|_{\psi_E}[u + iv] = g'(\psi_E)u + i\frac{g(\psi_E)}{\psi_E}v, \qquad \psi_E \geqslant 0$$

namely:

**(H2)** Assume that for the positive solution of (2.2) we have $\widehat{g'(\psi_E)}$, $\widehat{\frac{g(\psi_E)}{\psi_E}} \in L^1(\mathbb{R}^3)$ where $\hat{f}$ stands for the Fourier transform of the function $f$.

In concrete cases the hypothesis may be checked directly using the regularity of $\psi_E$, the solution of an uniform elliptic e-value problem. In general we can prove the following result:

**Proposition 2.2** *If the following holds*

**(H2')** *$g$ restricted to reals has third derivative except at zero and $|g'''(s)| < \frac{C}{s^{1-\alpha_1}} + Cs^{\alpha_2 - 1}$, $s > 0$, $0 < \alpha_1 \leqslant \alpha_2$;*

*then for the nonnegative solution of (2.2), $\psi_E$, we have $\widehat{g'(\psi_E)} \in L^1$ and $\widehat{\frac{g(\psi_E)}{\psi_E}} \in L^1$.*

We will give the proof in the Appendix.

We are going to decompose the solution of (1.1)-(1.2) into a projection onto the center manifold and a correction. For orbital stability the projection which minimizes the $H^1$ norm of the correction is used, see for example [28], while for asymptotic stability one wants to remove periodic in time components of the correction. Currently there are two different ways to accomplish this. First and most used one is to keep the correction orthogonal to the discrete spectrum of a fixed linear Schrödinger operator "close" to the dynamics, see [15, 20]. For example in [15] the linear Schrödinger operator is $-\Delta + V$ and the correction is always orthogonal on its sole eigenvector $\psi_0$, hence the decomposition becomes

$$u = a\psi_0 + h(a) + correction, \qquad \text{where } a = \langle \psi_0, u \rangle.$$

Second technique is to use the invariant subspaces of the actual linearized dynamics at the projection, see for example [13]. While more complicated the latter is the only one capable to render our main result. Since there are slight mistakes in the previous presentations of this decomposition we are going to describe it in what follows.

Consider the linearization of (1.1) at function on the center manifold $\psi_E = a\psi_0 + h(a),\ a = a_1 + ia_2 \in \mathbb{C},\ |a| < \delta$ :

$$\frac{\partial w}{\partial t} = -iL_{\psi_E}[w] - iEw \tag{2.8}$$

where

$$L_{\psi_E}[w] = (-\Delta + V - E)w + Dg_{\psi_E}[w] = (-\Delta + V - E)w + \lim_{\varepsilon\in\mathbb{R},\ \varepsilon\to 0} \frac{g(\psi_E + \varepsilon w) - g(\psi_E)}{\varepsilon} \tag{2.9}$$

**Properties of the linearized operator**:

1. $L_{\psi_E}$ is real linear and symmetric with respect to the real scalar product $\Re\langle\cdot,\cdot\rangle$, on $L^2(\mathbb{R}^3)$, with domain $H^2(\mathbb{R}^3)$.

2. Zero is an e-value for $-iL_{\psi_E}$ and its generalized eigenspace includes $\left\{\frac{\partial\psi_E}{\partial a_1}, \frac{\partial\psi_E}{\partial a_2}\right\}$

The real linearity of $L_{\psi_E}$ follows from (2.9). For symmetry consider first the case of a real valued $\psi_E = a\psi_0 + h(a),\ a \in (-\delta, \delta) \subset \mathbb{R}$. Then for $w = u + iv \in H^2(\mathbb{R}^3),\ u, v$ real valued we have

$$L_{\psi_E}[u + iv] = L_+[u] + iL_-[v]$$

with $L_+[u]$, $L_-[v]$ being real valued and symmetric:

$$\begin{aligned} L_+[u] &= (-\Delta + V - E)u + g'(\psi_E)u \\ L_-[v] &= (-\Delta + V - E)v + \frac{g(\psi_E)}{\psi_E}v. \end{aligned}$$

To determine the expression for $L_-$ we used the rotational symmetry (1.4):

$$g(e^{i\theta}\psi_E) = e^{i\theta}g(\psi_E)$$

and we differentiate it with respect to $\theta$ at $\theta = 0$ to get

$$Dg_{\psi_E}[i\psi_E] = ig(\psi_E). \tag{2.10}$$

Now,

$$\Re\langle L_{\psi_E}[u+iv], u_1+iv_1\rangle = \Re\langle L_+[u], u_1\rangle + \Re\langle L_-[v], v_1\rangle = \Re\langle u, L_+[u_1]\rangle + \Re\langle v, L_-[v_1]\rangle = \Re\langle u+iv, L_{\psi_E}[u_1+iv_1]\rangle$$

hence $L_{\psi_E}$ is symmetric for real valued $\psi_E$.

For a complex valued function on the center manifold $\psi_E = a\psi_0 + h(a),\ a \in \mathbb{C},\ |a| < \delta$ there exists $\theta \in [0, 2\pi)$ such that $a = |a|e^{i\theta}$ and

$$\psi_E = e^{i\theta}(|a|\psi_0 + h(|a|)) = e^{i\theta}\psi_E^{real}$$

where $\psi_E^{real}$ is real valued and on the center manifold. Using again the rotational symmetry of $g$ (1.4) we get:

$$L_{\psi_E}[w] = e^{i\theta}L_{\psi_E^{real}}[e^{-i\theta}w]. \tag{2.11}$$

Since $e^{i\theta}$ is a unitary linear operator on the real Hilbert space $L^2(\mathbb{R}^3)$ and, due to the argument above, $L_{\psi_E^{real}}$ is symmetric we get that $L_{\psi_E}$ is symmetric.

For the second property, we observe that substituting $w = i\psi_E$ in (2.9) and using (2.10), (2.2) we get

$$L_{\psi_E}[i\psi_E] = i[(-\Delta + V - E)\psi_E + g(\psi_E)] = 0.$$

Hence zero is an e-value for $-iL_{\psi_E}$ and $i\frac{\psi_E}{|a|}$ for $a \neq 0$ and $i\psi_0 = \lim_{a\to 0} i\frac{\psi_E}{a}$ for $a = 0$ are the corresponding eigenvectors. Moreover by differentiating (2.2) with respect to $a_1 = \Re a \in \mathbb{R}$ or $a_2 = \Im a \in \mathbb{R}$ we get

$$-iL_{\psi_E}\left[\frac{\partial \psi_E}{\partial a_j}\right] = -\frac{\partial E}{\partial a_j} i\psi_E, \quad j = 1, 2.$$

Since $\frac{\partial E}{\partial a_j} = E'(|a|)\frac{\partial |a|}{\partial a_j} \in \mathbb{R}$ we deduce that $\frac{\partial \psi_E}{\partial a_j}$, $j = 1, 2$ are in the generalized eigenspace of zero. Note that, by differentiating $h(e^{i\theta}a) = e^{i\theta}h(a)$ with respect to $\theta$ at $\theta = 0$ we get $Dh|_a[ia] = ih(a)$ and, via (2.7), $D\psi_E|_a[ia] = i\psi_E$. Since the differential can be written with the help of the gradient:

$$i\psi_E = D\psi_E|_a[ia] = \frac{\partial \psi_E}{\partial a_1}\Re[ia] + \frac{\partial \psi_E}{\partial a_2}\Im[ia],$$

we infer that

$$i\psi_E \in \operatorname{span}\left\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\right\} \quad \text{or equivalently } \psi_E \in \operatorname{span}\left\{i\frac{\partial \psi_E}{\partial a_1}, i\frac{\partial \psi_E}{\partial a_2}\right\}$$

where the span is taking over the reals[2].

One can now decompose $L^2(\mathbb{R}^3)$ into invariant subspaces with respect to $-iL_{\psi_E}$:

$$L^2(\mathbb{R}^3) = \operatorname{span}\left\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\right\} \oplus \mathcal{H}_a.$$

The standard choice is to use the projection along the dual basis:

$$\mathcal{H}_a = \{\phi_1, \phi_2\}^\perp$$

where the orthogonality is with respect to the real scalar product, and $\phi_1$, $\phi_2$ are in the generalized eigenspace of the adjoint of $-iL_{\psi_E}$ corresponding to the eigenvalue zero, and $\phi_1$ is orthogonal to $\frac{\partial \psi_E}{\partial a_2}$ but not to $\frac{\partial \psi_E}{\partial a_1}$ while $\phi_2$ is orthogonal to $\frac{\partial \psi_E}{\partial a_1}$ but not to $\frac{\partial \psi_E}{\partial a_2}$. Since $L_{\psi_E}$ is symmetric we have $(-iL_{\psi_E})^* = L_{\psi_E} i$ and a direct calculations shows that one can choose

$$\phi_1 = -i\frac{\partial \psi_E}{\partial a_2}, \qquad \phi_2 = i\frac{\partial \psi_E}{\partial a_1}$$

as long as $\Re\langle i\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\rangle \neq 0$. But

$$\Re\langle i\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\rangle = \Re\langle i\psi_0, i\psi_0\rangle = 1, \qquad \text{at } a = 0$$

and since $\psi_E$ is $C^2$ in $a_1$, $a_2$ we have:

**Remark 2.3** *By possible choosing $\delta > 0$ smaller than the one in Proposition 2.1 we get:*

$$\Re\langle i\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\rangle = \Re\langle i\psi_0, i\psi_0\rangle \geqslant \frac{1}{2}. \tag{2.12}$$

Consequently, for $|a| < \delta$,

$$\mathcal{H}_a = \left\{-i\frac{\partial \psi_E}{\partial a_2}, i\frac{\partial \psi_E}{\partial a_1}\right\}^\perp, \qquad \text{and } L^2(\mathbb{R}^3) = \operatorname{span}\left\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\right\} \oplus \mathcal{H}_a. \tag{2.13}$$

Our goal is to decompose the solution of (1.1) at each time into:

$$u = \psi_E + \eta = a\psi_0 + h(a) + \eta, \qquad \eta \in \mathcal{H}_a$$

which insures that $\eta$ is not in the non-decaying directions (tangent space of the central manifold) span $\left\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\right\}$ of the linearized equation (2.8) around $\psi_E$. The fact that this can be done in an unique manner is a consequence of the following lemma[3]:

[2]One can actually show that, for small $|a|$, zero is the only e-value of $-iL_{\psi_E}$ and the corresponding eigenspace is spanned by $\frac{\partial \psi_E}{\partial a_j}$, $j = 1, 2$. However this is not needed in our argument.

[3]This is an immediate consequence of the implicit function theorem but we find the proof in [13] to be incomplete.

**Lemma 2.1** *There exists $\delta_1 > 0$ such that any $\phi \in L^2(\mathbb{R}^3)$ satisfying $\|\phi\|_{L^2} \leqslant \delta_1$ can be uniquely decomposed:*

$$\phi = \psi_E + \eta = a\psi_0 + h(a) + \eta$$

*where $a = a_1 + ia_2 \in \mathbb{C},\ |a| < \delta,\ \eta \in \mathcal{H}_a$. Moreover the maps $\phi \mapsto a$ and $\phi \mapsto \eta$ are $C^1$ and there exist constant $C$ independent on $\phi$ such that*

$$|a| \leqslant 2\|\phi\|_{L^2}, \qquad \|\eta\|_{L^2} \leqslant C\|\phi\|_{L^2}.$$

**Proof:** Consider the map $F : \{a = (a_1, a_2) \in \mathbb{R}^2 \ : \ |a| < \delta\} \times L^2(\mathbb{R}^3) \mapsto \mathbb{R} \times \mathbb{R} :$

$$F(a_1, a_2, \phi) = \left( \Re\langle -i\frac{\partial \psi_E}{\partial a_2}, \phi - \psi_E\rangle, \Re\langle i\frac{\partial \psi_E}{\partial a_1}, \phi - \psi_E\rangle \right) \tag{2.14}$$

where $\psi_E = a\psi_0 + h(a),\ a = a_1 + ia_2$. Since $h(a)$ is $C^2$, $F$ is a $C^1$ map and:

$$\begin{aligned} F(0,0,0) &= 0 \\ \frac{\partial F}{\partial(a_1, a_2)}(0,0,0) &= \mathbb{I}_{\mathbb{R}^2} \end{aligned}$$

where for the calculation of the Jacobi matrix we used (2.12).

The implicit function theorem implies that there exist $\delta_2 \leqslant \delta$ and a $C^1$ map:

$$\tilde{F} = (\tilde{F}_1, \tilde{F}_2) : B(0, \delta_2) \subset L^2(\mathbb{R}^3) \mapsto \mathbb{R} \times \mathbb{R}$$

such that the only solutions of

$$F(a_1, a_2, \phi) = 0$$

in $|a| = |a_1 + ia_2| < \delta_2,\ \|\phi\|_{L^2} < \delta_2$ are given by

$$(a_1 = \tilde{F}_1(\phi), a_2 = \tilde{F}_2(\phi), \phi).$$

Now, for an arbitrary $\phi \in B(0, \delta_2) \subset L^2(\mathbb{R}^3)$, since

$$\phi = \psi_E + \eta = a\psi_0 + h(a) + \eta$$

with $a = a_1 + ia_2 \in \mathbb{C},\ |a| < \delta_2 \leqslant \delta,\ \eta \in \mathcal{H}_a$ is equivalent to $F(a_1, a_2, \phi) = 0$ we get that there is a unique choice:

$$a_1 = \tilde{F}_1(\phi), \quad a_2 = \tilde{F}_2(\phi), \quad \eta = \phi - a\psi_0 - h(a).$$

Moreover, by choosing $\delta_1 \leqslant \delta_2$ such that

$$\|D\tilde{F}_\phi\| \leqslant 2 \qquad \forall \phi \in L^2(\mathbb{R}^3),\ \|\phi\|_{L^2} \leqslant \delta_1$$

where the norm is the operator norm from $L^2(\mathbb{R}^3)$ into $\mathbb{R} \times \mathbb{R}$, we get, for all $\phi \in L^2(\mathbb{R}^3),\ \|\phi\|_{L^2} \leqslant \delta_1 :$

$$|a| = \sqrt{a_1^2 + a_2^2} \leqslant 2\|\phi\|_{L^2}$$

and

$$\|\eta\|_{L^2} \leqslant \|\phi\|_{L^2} + \|\psi_E\|_{L^2} \leqslant \|\phi\|_{L^2} + |a| + \|h(a)\|_{L^2} \leqslant C\|\phi\|_{L^2}$$

where $C \geqslant 3 + 2\sup_{a \in \mathbb{C}, |a| \leqslant \delta_2} \|Dh_a\|$. Note that the existence of $\delta_1$ is insured by the continuity of $D\tilde{F}$ and, from the implicit function theorem:

$$D\tilde{F}_0 = D_\phi F|_{\phi=0}$$

and the latter has norm one being the projection operator onto $\psi_0$.

This finishes the proof of Lemma 2.1. $\square$

**Remark 2.4** *Both the decomposition* (2.13) *and Lemma 2.1 can be extended without modifications to* $H^{-1}(\mathbb{R}^3)$ *the dual of* $H^1$ *because* $\frac{\partial\psi_E}{\partial a_j} \in H^1,\ j=1,2$. *In this case* $\langle u,\phi\rangle$ *denotes the evaluation of the functional* $\phi \in H^{-1}$ *at* $u \in H^1$.

We need one more technical result relating the spaces $\mathcal{H}_a$ and the space corresponding to the continuous spectrum of $-\Delta + V$ :

**Lemma 2.2** *There exists* $\delta > \delta_2 > 0$ *such that for any* $a \in \mathbb{C},\ |a| \leqslant \delta_2$ *the linear map* $P_c|_{\mathcal{H}_a} : \mathcal{H}_a \mapsto \mathcal{H}_0$ *is invertible, and its inverse* $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ *satisfies:*

$$\begin{aligned} \|R_a\zeta\|_{L^2_{-\sigma}} &\leqslant C_{-\sigma}\|\zeta\|_{L^2_{-\sigma}}, \qquad \sigma\in\mathbb{R} \text{ and for all } \zeta \in \mathcal{H}_0 \cap L^2_{-\sigma} && (2.15)\\ \|R_a\zeta\|_{L^p} &\leqslant C_p\|\zeta\|_{L^p}, \qquad 1 \leqslant p < \infty \text{ and for all } \zeta \in \mathcal{H}_0 \cap L^p && (2.16)\\ \overline{R_a\zeta} &= R_a\overline{\zeta} && (2.17)\end{aligned}$$

*where the constants* $C_{-\sigma},\ C_p > 0$ *are independent of* $a \in \mathbb{C},\ |a| \leqslant \delta_2$.

**Proof:** Since $\psi_0$ is orthogonal to $\mathcal{H}_0$, by continuity we can choose $\delta > \tilde{\delta}_2 > 0$ such that $\psi_0 \notin \mathcal{H}_a$ for $|a| < \tilde{\delta}_2$. Consequently $P_c|_{\mathcal{H}_a}$ is one to one, otherwise from $\phi \in \mathcal{H}_a,\ \phi \neq 0,\ P_c\phi = 0$ we get $\phi = z\psi_0$ for some $z \in \mathbb{C},\ z \neq 0$ which contradicts $\psi_0 \notin \mathcal{H}_a$.

Next, for $|a| < \tilde{\delta}_2$ we construct $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ such that:

$$P_cR_a\zeta = \zeta, \qquad \forall\zeta \in \mathcal{H}_0. \tag{2.18}$$

Since $P_c$ is the projection onto $\{\psi_0\}^\perp$, condition (2.18) is equivalent to

$$R_a\zeta = \zeta + z\psi_0 \tag{2.19}$$

for some $z \in \mathbb{C}$. To insure that the range of $R_a$ is in $\mathcal{H}_a$ we impose

$$\Re z\langle -i\frac{\partial\psi_E}{\partial a_2}, \psi_0\rangle = -\Re\langle -i\frac{\partial\psi_E}{\partial a_2}, \zeta\rangle, \qquad \Re z\langle i\frac{\partial\psi_E}{\partial a_1}, \psi_0\rangle = -\Re\langle i\frac{\partial\psi_E}{\partial a_1}, \zeta\rangle. \tag{2.20}$$

This linear system of two equations with two unknowns, $\Re z$ and $\Im z$, is uniquely solvable whenever $\psi_0 \notin \mathcal{H}_a$. Note that for $a = 0$ the system becomes: $z = \langle\psi_0, \zeta\rangle$.

In (2.19) we now choose $z$ to be the unique solution of (2.20) and obtain a well defined linear map $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ satisfying (2.18).

Consequently, $P_c|_{\mathcal{H}_a}$ is also onto, hence invertible and its inverse is $R_a$. Moreover, by the continuity of the coefficients of (2.20) with respect to $a$ we can choose $\delta_2 \leqslant \tilde{\delta}_2$ such that, for all $|a| \leqslant \delta_2$ :

$$|z| \leqslant 2\sqrt{(\Re\langle -i\frac{\partial\psi_E}{\partial a_2}, \zeta\rangle)^2 + (\Re\langle i\frac{\partial\psi_E}{\partial a_1}, \zeta\rangle)^2}. \tag{2.21}$$

Hence, via (2.19) and Hölder inequality we get:

$$\|R_a\zeta\|_Y \leqslant \|\zeta\|_Y + 2\|\psi_0\|_Y\|\zeta\|_Y\sqrt{\left\|\frac{\partial\psi_E}{\partial a_2}\right\|^2_{Y^*} + \left\|\frac{\partial\psi_E}{\partial a_1}\right\|^2_{Y^*}},$$

which, for the choice $Y = L^2_{-\sigma}(\mathbb{R}^3),\ Y^* = L^2_{-\sigma}(\mathbb{R}^3)$ respectively $Y = L^p(\mathbb{R}^3),\ Y^* = L^{p'}(\mathbb{R}^3),\ \frac{1}{p} + \frac{1}{p'} = 1$ give (2.15), respectively (2.16). The constants are independent of $a$ due to the continuous dependence of $\frac{\partial\psi_E}{\partial a_j},\ j = 1,2$ on $a \in \mathbb{C}$ in the compact $|a| \leqslant \delta_2$, and their exponential decay in time, see proposition 2.1 and corollary 2.1.

Now, $P_c$ commutes with complex conjugation because it is the orthogonal projection onto ${\psi_0}^\perp$ and $\psi_0$ is real valued. Then (2.17) follows from $R_a$ being the inverse of $P_c$.

The proof of Lemma 2.2 is now complete. □

We are now ready to prove our main result.

## 3 Main Result

**Theorem 3.1** *Assume that the nonlinear term in* (1.1) *satisfies* (1.3) *and* (1.4). *In addition assume that hypothesis (H1) and either (H2) or (H2') hold. Let* $p_1 = 3+\alpha_1,\ p_2 = 3+\alpha_2$. *Then there exists an* $\varepsilon_0$ *such that for all initial conditions* $u_0(x)$ *satisfying*

$$\max\{\|u_0\|_{L^{p_2'}}, \|u_0\|_{H^1}\} \le \varepsilon_0, \qquad \frac{1}{p_2'} + \frac{1}{p_2} = 1$$

*the initial value problem (1.1)-(1.2) is globally well-posed in* $H^1$ *and the solution decomposes into a radiative part and a part that asymptotically converges to a ground state.*

*More precisely, there exist a* $C^1$ *function* $a : \mathbb{R} \mapsto \mathbb{C}$ *such that, for all* $t \in \mathbb{R}$ *we have:*

$$u(t,x) = \underbrace{a(t)\psi_0(x) + h(a(t))}_{\psi_E(t)} + \eta(t,x) \tag{3.1}$$

*where* $\psi_E(t)$ *is on the central manifold (i.e it is a ground state) and* $\eta(t,x) \in \mathcal{H}_{a(t)}$, *see Proposition 2.1 and Lemma 2.1. Moreover there exists the ground states states* $\psi_{E_{\pm\infty}}$ *and the* $C^1$ *function* $\theta : \mathbb{R} \mapsto \mathbb{R}$ *such that* $\lim_{|t|\to\infty} \theta(t) = 0$ *and:*

$$\lim_{t\to\pm\infty} \|\psi_E(t) - e^{-it(E_\pm - \theta(t))}\psi_{E_{\pm\infty}}\|_{H^2 \bigcap L^2_\sigma} = 0,$$

*while* $\eta$ *satisfies the following decay estimates:*

$$\begin{aligned} \|\eta(t)\|_{L^2} &\le C_0(\alpha_1,\alpha_2)\varepsilon_0 \\ \|\eta(t)\|_{L^{p_1}} &\le C_1(\alpha_1,\alpha_2)\frac{\varepsilon_0}{(1+|t|)^{3(\frac{1}{2}-\frac{1}{p_1})}}, \quad p_1 = 3+\alpha_1 \end{aligned}$$

*and, for* $p_2 = 3+\alpha_2$ :

*(i) if* $\alpha_1 \ge \frac{1}{3}$ *or* $\frac{1}{3} > \alpha_1 > \frac{2\alpha_2}{3(3+\alpha_2)}$ *then*

$$\|\eta(t)\|_{L^{p_2}} \le C_2(\alpha_1,\alpha_2)\frac{\varepsilon_0}{(1+|t|)^{3(\frac{1}{2}-\frac{1}{p_2})}}$$

*(ii) if* $\alpha_1 = \frac{2\alpha_2}{3(3+\alpha_2)}$ *then*

$$\|\eta(t)\|_{L^{p_2}} \le C_2(\alpha_1,\alpha_2)\varepsilon_0\frac{\log(2+|t|)}{(1+|t|)^{3(\frac{1}{2}-\frac{1}{p_2})}}$$

*(iii) if* $\alpha_1 < \frac{2\alpha_2}{3(3+\alpha_2)}$ *then*

$$\|\eta(t)\|_{L^{p_2}} \le C_2(\alpha_1,\alpha_2)\frac{\varepsilon_0}{(1+|t|)^{\frac{1+3\alpha_1}{2}}}$$

*where the constants* $C_0,\ C_1$ *and* $C_2$ *are independent of* $\varepsilon_0$.

**Remark 3.1** *Note that the critical and supercritical cases* $\frac{1}{3} \le \alpha_1 < 3$ *are contained in* $(i)$. *Our results for these cases are stronger than the ones in [20, 23, 24] because we do not require the initial condition to be in* $L^2_\sigma,\ \sigma > 1$. *Compared to [13] we have sharper estimates for the asymptotic decay to the ground state but we require the initial data to be in* $L^{p_2'}$. *To the best of our knowledge the subcritical case* $\alpha_1 < 1/3$ *has not been treated previously.*

**Remark 3.2** *One can obtain estimates for the radiative part* $\eta$ *in* $L^p$, $2 \le p \le p_1 = 3+\alpha_1$, *or* $p_1 \le p \le p_2 = 3+\alpha_2$ *by Riesz-Thorin interpolation between* $L^2$ *and* $L^{p_1}$ *respectively between* $L^{p_1}$ *and* $L^{p_2}$.

**Proof of Theorem 3.1** It is well known that under hypothesis (H1)(i) the initial value problem (1)-(2) is locally well posed in the energy space $H^1$ and its $L^2$ norm is conserved, see for example [5, Cor. 4.3.3 at p. 92]. Global well posedness follows via energy estimates from $\|u_0\|_{H^1}$ small, see [5, Remark 6.1.3 at p. 165].

We choose $\varepsilon_0 \leqslant \delta_1$ given by Lemma 2.1. Then, for all times, $\|u(t)\|_{L^2} \leqslant \delta_1$ and we can decompose the solution into a solitary wave and a dispersive component as in (3.1):

$$u(t) = a(t)\psi_0 + h(a(t)) + \eta(t) = \psi_E(t) + \eta(t)$$

Moreover, by possible making $\varepsilon_0$ smaller we can insure that that $\|u(t)\|_{L^2} \leqslant \varepsilon_0$ implies $|a(t)| \leqslant \delta_2,\ t \in \mathbb{R}$ where $\delta_2$ is given by Lemma 2.2. In addition, since

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^3)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^3)),$$

and $u \mapsto a$ respectively $u \mapsto \eta$ are $C^1$, see Remark 2.4, we get that $a(t)$ is $C^1$ and $\eta \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$.

The solution is now described by the $C^1$ function $a : \mathbb{R} \in \mathbb{C}$ and $\eta(t) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. To obtain their equations we plug in (3.1) into (1.1). Then we get

$$\frac{\partial \eta}{\partial t} + D\psi_E|_a a' = -i(L_{\psi_E} + E)\eta - Ei\psi_E - iF_2(\psi_E, \eta) \tag{3.2}$$

where $L_{\psi_E}$ is defined by (2.9)

$$L_{\psi_E}\eta = (-\Delta + V - E)\eta - i\frac{d}{d\varepsilon} g(\psi_E + \varepsilon\eta)|_{\varepsilon=0}$$

and $F_2(\psi_E, \eta)$ denotes the nonlinear terms in $\eta$

$$F_2(\psi_E, \eta) = g(\psi_E + \eta) - g(\psi_E) - \underbrace{\frac{d}{d\varepsilon} g(\psi_E + \varepsilon\eta)|_{\varepsilon=0}}_{F_1(\psi_E,\eta)} \tag{3.3}$$

Then projecting (3.2) onto the invariant subspaces of $-iL_{\psi_E}$, $\mathcal{H}_a$, see (2.13) and the span$\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\}$, we obtain the equations for $\eta(t)$ and $a(t)$ :

$$\frac{\partial \eta}{\partial t} = -i(L_{\psi_E} + E)\eta - iF_2(\psi_E, \eta) - \tilde{F}_2(\psi_E, \eta) \tag{3.4}$$

$$D\psi_E|_a a' = -Ei\psi_E + \tilde{F}_2(\psi_E, \eta) \tag{3.5}$$

where

$$\tilde{F}_2(\psi_E, \eta) = \underbrace{\frac{\Re\langle -i\frac{\partial \psi_E}{\partial a_2}, -iF_2(\psi_E, \eta)\rangle}{\Re\langle -i\frac{\partial \psi_E}{\partial a_2}, \frac{\partial \psi_E}{\partial a_1}\rangle}}_{\beta_1(\psi_E,\eta)} \cdot \frac{\partial \psi_E}{\partial a_1} + \underbrace{\frac{\Re\langle i\frac{\partial \psi_E}{\partial a_1}, -iF_2(\psi_E, \eta)\rangle}{\Re\langle i\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\rangle}}_{\beta_2(\psi_E,\eta)} \cdot \frac{\partial \psi_E}{\partial a_2} \tag{3.6}$$

In order to obtain the estimates for $\eta(t)$, we analyze (3.4). The linear part of (3.4) is:

$$\begin{aligned} \frac{\partial \zeta}{\partial t} &= -i(L_{\psi_E} + E)\zeta = (-\Delta + V)\zeta - i\frac{d}{d\varepsilon} g(\psi_E(t) + \varepsilon\zeta)|_{\varepsilon=0} \\ \zeta(s) &= v \end{aligned} \tag{3.7}$$

Define $\Omega(t, s)v = \zeta(t)$. Then using Duhamel's principle (3.4) becomes

$$\eta(t) = \Omega(t, 0)\eta(0) - \int_0^t \Omega(t, s)[iF_2(\psi_E, \eta) + \tilde{F}_2(\psi_E, \eta)]ds \tag{3.8}$$

It is here where we differ from the approach [6, 20, 23, 24]. The right-hand side of our equation contains only nonlinear terms in $\eta$. However the challenge is to obtain good dispersive estimates for the propagator $\Omega(t, s)$ of the linearization (3.7), see Theorems 4.1 and 4.2.

In order to apply a contraction mapping argument for (3.8) we use the following Banach spaces. Let $p_1 = 3+\alpha_1$ and $p_2 = 3+\alpha_2$,

$$Y_i = \{u \in L^2 \cap L^{p_1} \cap L^{p_2} : \sup_t (1+|t|)^{3(\frac{1}{2}-\frac{1}{p_1})}\|u\|_{L^{p_1}} < \infty, \sup_t \frac{(1+|t|)^{n_i}}{[\log(2+|t|)]^{m_i}}\|u\|_{L^{p_2}} < \infty, \sup_t \|u\|_{L^2} < \infty\}$$

endowed with the norm

$$\|u\|_{Y_i} = \max\{\sup_t (1+|t|)^{3(\frac{1}{2}-\frac{1}{p_1})}\|u\|_{L^{p_1}}, \sup_t \frac{(1+|t|)^{n_i}}{[\log(2+|t|)]^{m_i}}\|u\|_{L^{p_2}}, \sup_t \|u\|_{L^2}\}$$

for $i = 1,2,3$, where $n_1 = n_2 = 3(\frac{1}{2}-\frac{1}{p_2})$, $n_3 = \frac{1+3\alpha_1}{2}$, $m_1 = m_3 = 0$ and $m_2 = 1$.

Consider the nonlinear operator in (3.8):

$$N(u) = \int_0^t \Omega(t,s)[iF_2(\psi_E, u) + \tilde{F}_2(\psi_E, u)]ds$$

**Lemma 3.1** *Consider the cases:*

$$1.\ \alpha_1 \geq \frac{1}{3} \text{ or } \frac{1}{3} > \alpha_1 > \frac{2\alpha_2}{3(3+\alpha_2)}; \quad 2.\ \alpha_1 = \frac{2\alpha_2}{3(3+\alpha_2)}; \quad 3.\ \alpha_1 < \frac{2\alpha_2}{3(3+\alpha_2)}.$$

*Then, for each case number i: $N : Y_i \to Y_i$ is well defined, and locally Lipschitz, i.e. there exists $\tilde{C}_i > 0$, such that*

$$\|Nu_1 - Nu_2\|_{Y_i} \leq \tilde{C}_i(\|u_1\|_{Y_i} + \|u_2\|_{Y_i} + \|u_1\|_{Y_i}^{1+\alpha_1} + \|u_2\|_{Y_i}^{1+\alpha_1} + \|u_1\|_{Y_i}^{1+\alpha_2} + \|u_2\|_{Y_i}^{1+\alpha_2})\|u_1 - u_2\|_{Y_i}.$$

Note that the Lemma gives the estimates for $\eta$ in the Theorem 3.1. Indeed, if we denote:

$$v = \Omega(t,0)\eta(0),$$

then

$$\|v\|_{Y_i} \leq C_0\|\eta(0)\|_{L^{p_2'}\cap H^1},$$

where $C_0 = \max\{C, C_p\}$, see theorem 4.1. We choose $\epsilon_0$ in the hypotheses of theorem 3.1, such that

$$C_0\epsilon_0 \leqslant \frac{1}{2}\Big(\sqrt{1+2/\tilde{C}_i} - 1\Big)$$

Then by continuity there exists $0 \leqslant Lip \leqslant 1$ such that:

$$\|v\|_{Y_i} \leq \frac{2-Lip}{4}\Big(\sqrt{1+2Lip/\tilde{C}_i} - 1\Big).$$

Let $R = L\|v\|_{Y_i}/(2-Lip)$ and $B(v,R)$ be the closed ball in $Y_i$ with center $v$ and radius $R$. A direct calculation shows that the right-hand side of (3.8):

$$Ku = v + Nu$$

leaves $B(v,R)$ invariant, i.e. $K : B(v,R) \mapsto B(v,R)$, and it is a contraction with Lipschitz constant $Lip$ on $B(v,R)$.

By the contraction mapping argument, (3.8) has a unique solution in $Y_i$. We now have two solutions of (3.4), one in $C(\mathbb{R}, H^1)$ from classical well posedness theory and one in $C(\mathbb{R}, L^2 \cap L^{p_1} \cap L^{p_2})$, $p_1 = 3+\alpha_1$, $p_2 = 3+\alpha_2$ from the above argument. Using uniqueness and the continuous embedding of $H^1$ in $L^2\cap L^{p_1}\cap L^{p_2}$, we infer that the solutions must coincide. Therefore, the time decaying estimates in the spaces $Y_{1-3}$ hold also for the $H^1$ solution.

**Proof of Lemma 3.1** Let $u_1, u_2$ be in one of the spaces $Y_i,\ i = 1,2,3$. Then at each $s \in \mathbb{R}$ we have:

$$\begin{aligned} F_2(\psi_E(s), u_1(s)) - F_2(\psi_E(s), u_2(s)) &= g(\psi_E + u_1) - g(\psi_E + u_2) - F_1(\psi_E, u_1) + F_1(\psi_E, u_2) \\ &= \int_0^1 \Big[\frac{d}{d\tau}g(\psi_E + u_2 + \tau(u_1 - u_2)) - \frac{d}{d\tau}g(\psi_E + \tau(u_1 - u_2))|_{\tau=0}\Big]d\tau \\ &= \int_0^1\int_0^1 \frac{d}{ds}\cdot\frac{d}{d\tau}g\big(\psi_E + s(u_2 + \tau(u_1 - u_2))\big)ds\ d\tau \end{aligned}$$

Using the hypothesis (1.3) we have $|g(u)| \le C(|u|^{2+\alpha_1} + |u|^{2+\alpha_2})$, then taking the derivatives with respect to $\tau$ and $s$ and estimating the integral we get:

$$|F_2(\psi_E, u_1) - F_2(\psi_E, u_2)| \le C\big[\underbrace{(|\psi_E|^{\alpha_1} + |\psi_E|^{\alpha_2})(|u_1| + |u_2|)|u_1 - u_2|}_{A_1} \tag{3.9}$$
$$+ \underbrace{(|u_1|^{1+\alpha_1} + |u_2|^{1+\alpha_1})|u_1 - u_2|}_{A_2} + \underbrace{(|u_1|^{1+\alpha_2} + |u_2|^{1+\alpha_2})|u_1 - u_2|}_{A_3}\big].$$

By (3.6) and Hölder inequality, for any $1 \leqslant q \leqslant \infty$ we have:

$$\begin{aligned} \|\tilde{F}_2(\psi_E, u_1) - \tilde{F}_2(\psi_E, u_2)\|_{L^q} \ &\le\ \tilde{C}\Big(\Big\|\frac{\partial\psi_E}{\partial a_2}\Big\|_{L^{p_2}}\Big\|\frac{\partial\psi_E}{\partial a_1}\Big\|_{L^q} + \Big\|\frac{\partial\psi_E}{\partial a_1}\Big\|_{L^{p_2}}\Big\|\frac{\partial\psi_E}{\partial a_2}\Big\|_{L^q}\Big)(\|A_1\|_{L^{p_2'}} + \|A_3\|_{L^{p_2'}}) \\ &+\ \tilde{C}\Big(\Big\|\frac{\partial\psi_E}{\partial a_2}\Big\|_{L^{p_1}}\Big\|\frac{\partial\psi_E}{\partial a_1}\Big\|_{L^q} + \Big\|\frac{\partial\psi_E}{\partial a_1}\Big\|_{L^{p_1}}\Big\|\frac{\partial\psi_E}{\partial a_2}\Big\|_{L^q}\Big)\|A_2\|_{L^{p_1'}} \\ &\le\ C(\|A_1\|_{L^{p_2'}} + \|A_2\|_{L^{p_1'}} + \|A_3\|_{L^{p_2'}}), \end{aligned} \tag{3.10}$$

where the uniform bounds on $\frac{\partial\psi_E}{\partial a_j} \in H^2(\mathbb{R}^3),\ j = 1, 2$, follow from their continuous dependence on scalar $a$, and $|a(t)| \leqslant \delta_2,\ t \in \mathbb{R}$.

Now let us consider the difference $Nu_1 - Nu_2$

$$(Nu_1 - Nu_2)(t) = \int_0^t \Omega(t,s)\big[iF_2(\psi_E(s), u_1(s)) - iF_2(\psi_E(s), u_2(s)) + \tilde{F}_2(\psi_E(s), u_1(s)) - \tilde{F}_2(\psi_E(s), u_2(s))]ds \tag{3.11}$$

- $L^{p_2}$ **Estimate :**

$$\|Nu_1 - Nu_2\|_{L^{p_2}} \le \int_0^t \|\Omega(t,s)\|_{L^{p_2'} \to L^{p_2}} C\Big(2\|A_1\|_{L^{p_2'}} + \|A_2\|_{L^{p_2'}} + \|A_2\|_{L^{p_1'}} + 2\|A_3\|_{L^{p_2'}}\Big)ds$$

To estimate the term containing $A_1$, observe that

$$\|(|\psi_E|^{\alpha_1} + |\psi_E|^{\alpha_2})(|u_1| + |u_2|)|u_1 - u_2|\|_{L^{p_2'}} \le \||\psi_E|^{\alpha_1} + |\psi_E|^{\alpha_2}\|_{L^\beta}(\|u_1\|_{L^{p_2}} + \|u_2\|_{L^{p_2}})\|u_1 - u_2\|_{L^{p_2}}$$

with $\frac{1}{\beta} + \frac{2}{p_2} = \frac{1}{p_2'}$. Using Theorem 4.2 (see also Remark 4.1), we have for each case number i: and $u_1,\ u_2 \in Y_i$ :

$$\begin{aligned} &\int_0^t \|\Omega(t,s)\|_{L^{p_2'} \to L^{p_2}} \|A_1\|_{L^{p_2'}} ds \\ &\quad\le \int_0^t \frac{C(p_2)}{|t-s|^{3(\frac{1}{2} - \frac{1}{p_2})}} \||\psi_E|^{\alpha_1} + |\psi_E|^{\alpha_2}\|_{L^\beta} \frac{[\log(2+|s|)]^{2m_i}}{(1+|s|)^{2n_i}} (\|u_1\|_{Y_i} + \|u_2\|_{Y_i})\|u_1 - u_2\|_{Y_i} ds \\ &\quad\le \frac{C(p_2)C_1C_2}{(1+|t|)^{3(\frac{1}{2} - \frac{1}{p_2})}} (\|u_1\|_{Y_i} + \|u_2\|_{Y_i})\|u_1 - u_2\|_{Y_i} \end{aligned}$$

where $C_2 = \sup_t \frac{(1+|t|)^{n_i}}{[\log(2+|t|)]^{m_i}} \int_0^t \frac{[\log(2+|s|)]^{2m_i} ds}{|t-s|^{3(\frac{1}{2} - \frac{1}{p_2})}(1+|s|)^{2n_i}} < \infty$ since $2n_i > 1$ and $C_1 = \sup_t \||\psi_E|^{\alpha_1} + |\psi_E|^{\alpha_2}\|_{L^\beta}$. The uniform bounds in $t \in \mathbb{R}$ for $\|\psi_E\|^{\alpha_j}_{L^{\alpha_j\beta}},\ j = 1, 2$ follow from the continuous dependence of $\psi_E = a(t)\psi_0 + h(a(t)) \in H^2(\mathbb{R}^3)$ on $a(t)$ and $|a(t)| \leqslant \delta_2,\ t \in \mathbb{R}$.

To estimate the terms containing $A_2$, observe that

$$\|(|u_1|^{1+\alpha_1} + |u_2|^{1+\alpha_1})|u_1 - u_2|\|_{L^{p_1'}} \le \big(\|u_1\|^{1+\alpha_1}_{L^{p_1}} + \|u_2\|^{1+\alpha_1}_{L^{p_1}}\big)\|u_1 - u_2\|_{L^{p_1}}$$

since $\frac{1}{p_1'} = \frac{2+\alpha_1}{p_1}$ and

$$\|(|u_1|^{1+\alpha_1}+|u_2|^{1+\alpha_1})|u_1-u_2|\|_{L^{p_2'}} \leq \left(\|u_1\|_{L^{p_1}}^{\theta(1+\alpha_1)}\|u_1\|_{L^2}^{(1-\theta)(1+\alpha_1)}+\|u_2\|_{L^{p_1}}^{\theta(1+\alpha_1)}\|u_2\|_{L^2}^{(1-\theta)(1+\alpha_1)}\right) \times\|u_1-u_2\|_{L^{p_1}}^{\theta}\|u_1-u_2\|_{L^2}^{1-\theta}$$

where $\frac{1}{p_2'} = (2+\alpha_1)(\frac{1-\theta}{2}+\frac{\theta}{p_1}),\ 0\leqslant\theta\leqslant 1$. Again using Theorem 4.2 (see also Remark 4.1), we have

$$\begin{aligned}
&\int_0^t \|\Omega(t,s)\|_{L^{p_2'}\to L^{p_2}}\|A_2\|_{L^{p_2'}}ds\\
&\leq \int_0^t \frac{C(p_2)}{|t-s|^{3(\frac{1}{2}-\frac{1}{p_2})}}\cdot\frac{(\|u_1\|_{Y_i}^{1+\alpha_1}+\|u_2\|_{Y_i}^{1+\alpha_1})\|u_1-u_2\|_{Y_i}}{(1+|s|)^{3(\frac{\alpha_1}{2}+\frac{1}{p_2})}}ds\\
&\leq \frac{C(p_2)C_3[\log(2+|t|)]^{m_i}}{(1+|t|)^{n_i}}(\|u_1\|_{Y_i}^{1+\alpha_1}+\|u_2\|_{Y_i}^{1+\alpha_1})\|u_1-u_2\|_{Y_i}
\end{aligned}$$

where the different decay rates $n_i$ depend on the case number in the hypotheses of this Lemma:

1. corresponds to $3(\frac{\alpha_1}{2}+\frac{1}{p_2})>1$, and $C_3=\sup_t(1+|t|)^{3(\frac{1}{2}-\frac{1}{p_2})}\int_0^t\frac{ds}{|t-s|^{3(\frac{1}{2}-\frac{1}{p_2})}(1+|s|)^{3(\frac{\alpha_1}{2}+\frac{1}{p_2})}}<\infty$;
2. corresponds to $3(\frac{\alpha_1}{2}+\frac{1}{p_2})=1$, and $C_3=\sup_t\frac{(1+|t|)^{3(\frac{1}{2}-\frac{1}{p_2})}}{\log(2+|t|)}\int_0^t\frac{ds}{|t-s|^{3(\frac{1}{2}-\frac{1}{p_2})}(1+|s|)}<\infty$;
3. corresponds to $3(\frac{\alpha_1}{2}+\frac{1}{p_2})<1$, and $C_3=\sup_t(1+|t|)^{\frac{1+3\alpha_1}{2}}\int_0^t\frac{ds}{|t-s|^{3(\frac{1}{2}-\frac{1}{p_2})}(1+|s|)^{3(\frac{\alpha_1}{2}+\frac{1}{p_2})}}<\infty$.

To estimate the term containing $A_3$, observe that

$$\|(|u_1|^{1+\alpha_2}+|u_2|^{1+\alpha_2})|u_1-u_2|\|_{L^{p_2'}} \leq \left(\|u_1\|_{L^{p_2}}^{1+\alpha_2}+\|u_2\|_{L^{p_2}}^{1+\alpha_2}\right)\|u_1-u_2\|_{L^{p_2}}$$

since $\frac{1}{p_2'}=\frac{2+\alpha_2}{p_2}$. Again using Theorem 4.2 (see also Remark 4.1), we have

$$\begin{aligned}
&\int_0^t \|\Omega(t,s)\|_{L^{p_2'}\to L^{p_2}}\|A_3\|_{L^{p_2'}}ds\\
&\leq \int_0^t \frac{C(p_2)}{|t-s|^{3(\frac{1}{2}-\frac{1}{p_2})}}\cdot\frac{[\log(2+|s|)]^{(2+\alpha_2)m_i}}{(1+|s|)^{(2+\alpha_2)n_i}}(\|u_1\|_{Y_i}^{1+\alpha_2}+\|u_2\|_{Y_i}^{1+\alpha_2})\|u_1-u_2\|_{Y_i}ds\\
&\leq \frac{C(p_2)C_4C_5[\log(2+|t|)]^{m_i}}{(1+|t|)^{n_i}}(\|u_1\|_{Y_i}^{1+\alpha_2}+\|u_2\|_{Y_i}^{1+\alpha_2})\|u_1-u_2\|_{Y_i}
\end{aligned}$$

where $C_5=\sup_t\frac{(1+|t|)^{n_i}}{[\log(2+|t|)]^{m_i}}\int_0^t\frac{[\log(2+|s|)]^{(2+\alpha_2)m_i}ds}{|t-s|^{3(\frac{1}{2}-\frac{1}{p_2})}(1+|s|)^{(2+\alpha_2)n_i}}<\infty$ since $(2+\alpha_2)n_i>1$.

- $L^{p_1}$ **Estimate :** From (3.11) we have

$$\begin{aligned}
\|Nu_1-Nu_2\|_{L^{p_1}}(t) \leqslant\ & \|\int_0^t\Omega(t,s)[iF_2(\psi_E(s),u_1(s))-iF_2(\psi_E(s),u_2(s))]ds\|_{L^{p_1}}\\
&+\ C\int_0^t\|\Omega(t,s)\|_{L^{p_1'}\mapsto L^{p_1}}\|\tilde F_2(\psi_E(s),u_1(s))-\tilde F_2(\psi_E(s),u_2(s))\|_{L^{p_1'}}ds
\end{aligned}$$

For the second integral we use (3.10) with $q=p_1'$ and the previous estimates on $A_1,\ A_2,\ A_3$ to obtain the required bound. For the first integral moving the norm inside the integration and applying $L^{p_1'}\mapsto L^{p_1}$ estimates for $\Omega(t,s)$ and (3.9) for the nonlinear term would require the control of $A_3$ in $L^{p_1'}$. The latter, unfortunately, can no longer be interpolated between $L^2$ and $L^{p_2}$. To avoid this difficulty we separate and treat differently the part of the nonlinearity having an $A_3$ like behavior by decomposing $\mathbb{R}^3$ in two disjoints measurable sets related to the inequality (3.9):

$$V_1(s)=\{x\in\mathbb{R}^3\mid|F_2(\psi_E(s,x),u_2(s,x))-F_2(\psi_E(s,x),u_1(s,x))|\leqslant CA_3(s,x)\},\qquad V_2(s)=\mathbb{R}^3\setminus V_1(s)$$

On $V_2(s)$, using polar representation of complex numbers, we further split the nonlinear term into:

$$iF_2(\psi_E(s,x),u_1(s,x)) - iF_2(\psi_E(s,x),u_2(s,x)) = e^{i\theta(s,x)}CA_3(s,x) \\ + \underbrace{e^{i\theta(s,x)}[|iF_2(\psi_E(s,x),u_1(s,x)) - iF_2(\psi_E(s,x),u_2(s,x))| - CA_3(s,x)]}_{G(s,x)}$$

where, due to inequality (3.9), $|G(s,x)| \leqslant C(A_1(s,x)+A_2(s,x))$ on $V_2(s)$. Then we have:

$$\int_0^t \Omega(t,s)[iF_2(\psi_E(s),u_1(s)) - iF_2(\psi_E(s),u_2(s))]ds = \int_0^t \Omega(t,s)(1-\chi(s))G(s)ds \\ + \underbrace{\int_0^t \Omega(t,s)[\chi(s)(iF_2(\psi_E(s),u_1(s)) - iF_2(\psi_E(s),u_2(s))) + (1-\chi(s))e^{i\theta(s)}CA_3(s)]ds}_{I(t)},$$

where $\chi(s)$ is the characteristic function of $V_1(s)$. Now

$$\|\int_0^t \Omega(t,s)(1-\chi(s))G(s)ds\|_{L^{p_1}} \leqslant \int_0^t \|\Omega(t,s)\|_{L^{p_1'}\mapsto L^{p_1}} C(\|A_1(s)\|_{L^{p_1'}} + \|A_2(s)\|_{L^{p_1'}})ds$$

and estimates as in the previous step for $A_1$ and $A_2$ give the required decay. For $I(t)$ we use interpolation:

$$\|I(t)\|_{L^{p_1}} \le \|I(t)\|_{L^2}^{1-\theta}\|I(t)\|_{L^{p_2}}^{\theta} \le \|I(t)\|_{L^2}^{1-\theta}\left(\int_0^t \|\Omega(t,s)\|_{L^{p_2'}\mapsto L^{p_2}}\|A_3\|_{L^{p_2'}}ds\right)^{\theta}$$

where $\frac{1}{p_1} = \frac{1-\theta}{2} + \frac{\theta}{p_2}$. We know from previous step that the above integral decays as $(1+|t|)^{-3(\frac{1}{2}-\frac{1}{p_2})}$ and below we will show its $L^2$ norm will be bounded. Therefore

$$\sup_t (1+|t|)^{3(\frac{1}{2}-\frac{1}{p_1})}\|I(t)\|_{L^{p_1}} < \infty$$

and the $L^{p_1}$ estimates are complete.

- $L^2$ **Estimate :** To estimate $L^2$ norm we cannot use $L^2 \to L^2$ estimate for $\Omega(t,s)$ because that would force us to control $L^{2(\alpha_2+2)}$ which cannot pe interpolated between $L^2$ and $L^{p_2}$, $p_2 = \alpha_2 + 3$. We avoid this by using the decomposition:

$$T(t,s)v = [P_c\Omega(t,s) - e^{-iH(t-s)}P_c]v \quad \text{i.e.} \quad \Omega(t,s) = R_{a(t)}T(t,s) + R_{a(t)}e^{-iH(t-s)}P_c$$

For $T(t,s)$ we will use $L^{p'} \to L^2$ estimates, see Theorem 4.1, while for $e^{-iH(t-s)}P_c$ we will use Stricharz estimates $L_t^\infty L_x^2$. We will also use a decomposition of the nonlinear term similar to the one for $L^{p_1}$ estimates that will allow us to estimate in a different manner this time the terms behaving like $A_2$, see (3.9). All in all we have:

$$\|Nu_1 - Nu_2\|_{L^2} \le \int_0^t \|\Omega(t,s)\|_{L^2\to L^2}\|\tilde{F}_2(\psi_E,u_1) - \tilde{F}_2(\psi_E,u_2)\|_{L^2}ds \\ + \|R_{a(t)}\|_{L^2\mapsto L^2}\int_0^t \|T(t,s)\|_{L^{p_2'}\to L^2}C(\|A_1\|_{L^{p_2'}} + \|A_3\|_{L^{p_2'}})ds \\ + \|R_{a(t)}\|_{L^2\mapsto L^2}\int_0^t \|T(t,s)\|_{L^{p_1'}\to L^2}C\|A_2\|_{L^{p_1'}}ds \\ + \|R_{a(t)}\|_{L^2\mapsto L^2}\|\int_0^t e^{-iH(t-s)}P_c(A_1(s)+A_3(s)ds\|_{L^2} + \|R_{a(t)}\|_{L^2\mapsto L^2}\|\int_0^t e^{-iH(t-s)}P_cA_2ds\|_{L^2}$$

For the first integral we use Theorem 4.2 part (i), (3.10) with $q=2$ and the estimates we have already obtained for $A_1$, $A_2$ and $A_3$. We deduce that this integral is uniformly bounded in $t \in \mathbb{R}$. Similarly we get uniform boundedness of the second and third integral by using Theorem 4.1 part (iv).

For the fourth integral we use Stricharz estimate:

$$\sup_{t\in\mathbb{R}}\|\int_0^t e^{-iH(t-s)}P_cA_1ds\|_{L^2}\le C_s\left[\left(\int_{\mathbb{R}}\|A_1(s)\|_{L^{p_2'}}^{\gamma_2'}ds\right)^{\frac{1}{\gamma_2'}}+\left(\int_{\mathbb{R}}\|A_3(s)\|_{L^{p_2'}}^{\gamma_2'}ds\right)^{\frac{1}{\gamma_2'}}\right]$$

where $\frac{1}{\gamma_2'}+\frac{1}{\gamma_2}=1$, and $\frac{2}{\gamma_2}=3\left(\frac{1}{2}-\frac{1}{p_2}\right)$. Using again the estimates we obtained before for $A_1$ and $A_3$. we get:

$$\begin{aligned}\|A_1\|_{L_s^{\gamma_2'}L^{p_2'}}&\le C_{11}\Big[\int_{\mathbb{R}}\frac{(\log(2+|s|))^{2m_i\gamma_2'}}{(1+|s|)^{2n_i\gamma_2'}}ds\Big]^{\frac{1}{\gamma_2'}}(\|u_1\|_{Y_i}+\|u_2\|_{Y_i})\|u_1-u_2\|_{Y_i}\\&\le C_{11}C_8(\|u_1\|_{Y_i}+\|u_2\|_{Y_i})\|u_1-u_2\|_{Y_i}\end{aligned}$$

where $C_8=\int_{\mathbb{R}}\frac{(\log(2+|s|))^{2m_i\gamma_1'}}{(1+|s|)^{2n_i\gamma_1'}}ds<\infty$ since $2n_i\gamma'>1$ and:

$$\begin{aligned}\|A_3\|_{L_s^{\gamma_2'}L^{p_2'}}&\le C_{12}\Big[\int_{\mathbb{R}}\frac{(\log(2+|s|))^{(2+\alpha_2)m_i\gamma_3'}}{(1+|s|)^{(2+\alpha_2)n_i\gamma_3'}}ds\Big]^{\frac{1}{\gamma_3'}}(\|u_1\|_{Y_i}^{1+\alpha_2}+\|u_2\|_{Y_i}^{1+\alpha_2})\|u_1-u_2\|_{Y_i}\\&\le C_9(\|u_1\|_{Y_i}^{1+\alpha_2}+\|u_2\|_{Y_i}^{1+\alpha_2})\|u_1-u_2\|_{Y_i}\end{aligned}$$

where $C_9=\int_{\mathbb{R}}\frac{(\log(2+|s|))^{(2+\alpha_2)m_i\gamma_3'}}{(1+|s|)^{(2+\alpha_2)n_i\gamma_2'}}ds<\infty$ since $(2+\alpha_2)n_1\gamma_3'>1$.

Similarly, for the fifth integral:

$$\sup_{t\in\mathbb{R}}\|\int_0^t e^{-iH(t-s)}P_cA_2ds\|_{L^2}\le C_s\Big(\int_{\mathbb{R}}\|A_2\|_{L^{p_1'}}^{\gamma_1'}ds\Big)^{\frac{1}{\gamma_1'}}$$

where $\frac{1}{\gamma_1'}+\frac{1}{\gamma_1}=1$, and $\frac{2}{\gamma_1}=3\left(\frac{1}{2}-\frac{1}{p_1}\right)$. Furthermore we have

$$\begin{aligned}\|A_3\|_{L_s^{\gamma_1'}L^{p_1'}}&\le C_{13}\Big[\int_{\mathbb{R}}\frac{ds}{(1+|s|)^{3(2+\alpha_1)\gamma_2'(\frac{1}{2}-\frac{1}{p_1})}}\Big]^{\frac{1}{\gamma_2'}}(\|u_1\|_{Y_i}^{1+\alpha_1}+\|u_2\|_{Y_i}^{1+\alpha_1})\|u_1-u_2\|_{Y_i}\\&\le C_{13}C_{10}(\|u_1\|_{Y_i}^{1+\alpha_1}+\|u_2\|_{Y_i}^{1+\alpha_1})\|u_1-u_2\|_{Y_i}\end{aligned}$$

where $C_{10}=\int_{\mathbb{R}}\frac{ds}{(1+|s|)^{3(2+\alpha_1)\gamma_2'(\frac{1}{2}-\frac{1}{p_1})}}ds<\infty$ since $3(2+\alpha_1)\gamma_2'(\frac{1}{2}-\frac{1}{p_1})>1$ .

The $L^2$ estimates are now complete and the proof of Lemma 3.1 is finished. □

We now finish the proof of Theorem 3.1 by analyzing the dynamics on the center manifold and showing it converges to a ground state. Using the fact that

$$i\psi_E=D\psi_E|_a[ia]=\frac{\partial\psi_E}{\partial a_1}\Re[ia]+\frac{\partial\psi_E}{\partial a_2}\Im[ia]$$

equation (3.5) becomes

$$D\psi_E|_a(a'+iEa)=\frac{\partial\psi_E}{\partial a_1}\Re[a'+iEa]+\frac{\partial\psi_E}{\partial a_2}\Im[a'+iEa]=\tilde{F}_2(\psi_E,\eta)=\beta_1(\psi_E,\eta)\frac{\partial\psi_E}{\partial a_1}+\beta_2(\psi_E,\eta)\frac{\partial\psi_E}{\partial a_2}$$

Hence

$$|a'+iEa|=\sqrt{\beta_1^2+\beta_2^2}=b(t)$$

and

$$\Big|[a(t)e^{i\int_0^t E(s)ds}]'\Big|=b(t)$$

Since $b(t)=\sqrt{\beta_1^2+\beta_2^2}$, and

$$\beta_1\le\left\|\frac{\partial\psi_E}{\partial a_2}\right\|_{L^{p_2}}(\|A_1\|_{L^{p_2'}}+\|A_3\|_{L^{p_2'}})+\left\|\frac{\partial\psi_E}{\partial a_2}\right\|_{L^{p_1}}\|A_2\|_{L^{p_1'}}\le C(\|\eta\|_{L^{p_2}}^2+\|\eta\|_{L^{p_2}}^{2+\alpha_2}+\|\eta\|_{L^{p_1}}^{2+\alpha_1})$$

$$\beta_2 \leq \left\|\frac{\partial \psi_E}{\partial a_1}\right\|_{L^{p_2}} (\|A_1\|_{L^{p_2'}} + \|A_3\|_{L^{p_2'}}) + \left\|\frac{\partial \psi_E}{\partial a_1}\right\|_{L^{p_1}} \|A_2\|_{L^{p_1'}} \leq C(\|\eta\|^2_{L^{p_2}} + \|\eta\|^{2+\alpha_2}_{L^{p_2}} + \|\eta\|^{2+\alpha_1}_{L^{p_1}})$$

we get $0 \leqslant b(t) \leqslant C(1+|t|)^{1+\delta}$ for some $\delta > 0$, in each of the cases $(i)$, $(ii)$ and $(iii)$ in the Theorem 3.1. Then, for any $\varepsilon > 0$ we have

$$\left|a(t)e^{i\int_0^t E(s)ds} - a(t')e^{i\int_0^{t'} E(s)ds}\right| \leqslant \int_{t'}^t b(s)ds < \varepsilon \tag{3.12}$$

for $t$, $t'$ sufficiently large respectively sufficiently small. Therefore $a(t)e^{i\int_0^t E(s)ds}$ has a limit when $t \to \pm\infty$. This means

$$e^{i\int_0^t E(s)ds}\psi_E = a(t)e^{i\int_0^t E(s)ds}\psi_0 + e^{i\int_0^t E(s)ds}h(a(t)) = a(t)e^{i\int_0^t E(s)ds}\psi_0 + h(a(t)e^{i\int_0^t E(s)ds}) \to \psi_{E_{\pm\infty}}$$

Above we used $h(e^{i\theta}a) = e^{i\theta}h(a)$, see Proposition 2.1. In addition $|a(t)| \to a_\pm$ as $t \to \pm$ at a rate $|t|^{-\delta}$. Since $E(s) = E(|a(s)|$ is $C^1$ in $|a|$ on $|a| \leqslant \delta_2$, we deduce $|E(\pm s) - E_\pm| \leqslant C(1+s)^{-\delta}$ for $s \geqslant 0$ and some constant $C > 0$. If we denote

$$\theta(\pm t) = \frac{1}{\pm t}\int_0^{\pm t} E(s) - E_\pm ds, \qquad t \geqslant 0$$

then $\lim_{|t|\to\infty} \theta(t) = 0$ and

$$\lim_{t\to\pm} e^{it(E_\pm - \theta(t))}\psi_E(t) = \psi_{E_\pm}.$$

This finishes the proof of Theorem 3.1. $\square$

# 4 Linear Estimates

Consider the linear Schrödinger equation with a potential in three space dimensions:

$$\begin{aligned} i\frac{\partial u}{\partial t} &= (-\Delta + V(x))u \\ u(0) &= u_0 \end{aligned}$$

It is known that if V satisfies hypothesis (H1) (i) and (ii) then the radiative part of the solution, i.e. its projection onto the continuous spectrum of $H = -\Delta + V$, satisfies the estimates:

$$\|e^{-iHt}P_c u_0\|_{L^2_{-\sigma}} \leq C_M \frac{1}{|t|^{\frac{3}{2}}}\|u_0\|_{L^2_\sigma} \tag{4.1}$$

for $\sigma > 1$ and some constant $C_M > 0$ independent of $u_0$ and $t \in \mathbb{R}$, and

$$\|e^{-iHt}P_c u_0\|_{L^p} \leq C_p \frac{1}{|t|^{3(\frac{1}{2}-\frac{1}{p})}}\|u_0\|_{L^{p'}} \tag{4.2}$$

for some constant $C_p > 0$ depending only on $2 \leq p$. The case $p = \infty$ in (4.2) is proved by Goldberg and Schlag in [10]. The conservation of the $L^2$ norm gives the $p = 2$ case:

$$\|e^{-iHt}P_c u_0\|_{L^2} = \|u_0\|_{L^2}$$

The general result (4.2) follows from Riesz-Thorin interpolation.

We would like to extend these estimates to the linearized dynamics around the center manifold. We consider the linear equation, with initial data at time $s$,

$$\begin{aligned} i\frac{d\zeta}{dt} &= H\zeta + F_1(\psi_E, \zeta) \\ \zeta(s) &= v \end{aligned}$$

where $F_1(\psi_E,\zeta) = \frac{d}{d\varepsilon} g(\psi_E + \varepsilon\zeta)|_{\varepsilon=0} = \frac{\partial}{\partial u} g(u)|_{u=\psi_E}\zeta + \frac{\partial}{\partial \bar{u}} g(u)|_{u=\psi_E}\overline{\zeta}$. For the sake of simpler notation, we will use $F_1(\zeta)$.

By Duhamel's principle we have:

$$\zeta(t) = e^{-iH(t-s)}v(s) - i\int_s^t e^{-iH(t-\tau)}F_1(\zeta)d\tau \tag{4.3}$$

In the next theorems we will extend estimates of type (4.1)-(4.2) to the operators $\Omega(t,s)$ and $T(t,s)$ considering the fact that $\psi_E(t)$ is small. Recall that

$$T(t,s) = P_c\Omega(t,s) - e^{-iH(t-s)}P_c \quad \text{i.e.} \quad \Omega(t,s) = R_{a(t)}T(t,s) + R_{a(t)}e^{-iH(t-s)}P_c$$

**Theorem 4.1** *There exists $\varepsilon_1 > 0$ such that for $\|\langle x\rangle^\sigma \psi_E\|_{H^2} < \varepsilon_1$ there exist constants $C$, $C_p > 0$ with the property that for any $t$, $s \in \mathbb{R}$ the followings hold:*

$$(i) \quad \|\Omega(t,s)\|_{L^2_\sigma \to L^2_{-\sigma}} \le \frac{C}{(1+|t-s|)^{\frac{3}{2}}}$$

$$(ii) \quad \|T(t,s)\|_{L^1\to L^2_{-\sigma}} \le \begin{cases} \frac{C}{|t-s|^{\frac{1}{2}}} & \text{for } s\le t\le s+1 \\ \frac{C}{(1+|t-s|)^{\frac{3}{2}}} & \text{for } t > s+1 \end{cases}$$

$$(iii) \quad T(t,s) \in L^2_t(\mathbb{R}, L^2 \to L^2_{-\sigma}) \cap L^\infty_t(\mathbb{R}, L^2 \to L^2_{-\sigma})$$

$$(iv) \quad \|\Omega(t,s)\|_{L^{p'}\to L^2_{-\sigma}} \le \frac{C}{|t-s|^{3(\frac{1}{2}-\frac{1}{p})}} \quad \text{for all } 2\le p\le L^\infty$$

$$\|T(t,s)\|_{L^{p'}\to L^2_{-\sigma}} \le \begin{cases} \frac{C}{|t-s|^{(\frac{1}{2}-\frac{1}{p})}} & \text{for } s\le t\le s+1 \\ \frac{C}{(1+|t-s|)^{3(\frac{1}{2}-\frac{1}{p})}} & \text{for } t > s+1 \end{cases}$$

**Proof of Theorem 4.1** Fix $s \in \mathbb{R}$.

($i$) By definition, we have $\Omega(t,s)v = \zeta(t)$ where $\zeta(t)$ satisfies equation (4.3). We project (4.3) onto continuous spectrum of $H = -\Delta + V$ :

$$\xi(t) = e^{-iH(t-s)}P_cv - i\int_s^t e^{-iH(t-\tau)}P_cF_1(R_a\xi)d\tau \tag{4.4}$$

where $\xi = P_c\zeta$. We are going to prove the estimate for $P_c\Omega(t,s)$ by showing that the nonlinear equation (4.4) can be solved via contraction principle argument in an appropriate functional space. To this extent let us consider the functional space

$$X_1 := \{u \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^3)) | \sup_{t>s}(1+(t-s))^{\frac{3}{2}}\|u(t)\|_{L^2_{-\sigma}} < \infty\}$$

endowed with the norm

$$\|u\|_{X_1} := \sup_{t>s}\{(1+(t-s))^{\frac{3}{2}}\|u(t)\|_{L^2_{-\sigma}}\} < \infty$$

Note that the inhomogeneous term in (4.4) $\xi_0 = e^{-iH(t-s)}P_cv$ satisfies $\xi_0 \in X_1$ and

$$\|\xi_0\|_{X_1} \le C_M\|v\|_{L^2_\sigma} \tag{4.5}$$

because of (4.1). We collect the $\xi$ dependent part of the right hand side of (4.4) in a linear operator $L(s) : X_1 \to X_1$

$$[L(s)\xi](t) = -i\int_s^t e^{-iH(t-\tau)}P_c[F_1(R_a\xi)]d\tau \tag{4.6}$$

We will show that $L$ is a well defined bounded operator from $X_1$ to $X_1$ whose operator norm can be made less or equal to 1/2 by choosing $\varepsilon_1$ sufficiently small. Consequently $Id - L$ is invertible and the solution of the equation (4.4) can be written as $\xi = (Id - L)^{-1}\xi_0$. In particular

$$\|\xi\|_{X_1} \le (1 - \|L\|)^{-1}\|\xi_0\|_{X_1} \le 2\|\xi_0\|_{X_1}$$

which in combination with the definition of $\Omega$, the definition of the norm $X_1$ and the estimate (4.5), finishes the proof of $(i)$.

It remains to prove that $L$ is a well defined bounded operator from $X_1$ to $X_1$ whose operator norm can be made less than $1/2$ by choosing $\varepsilon_1$ sufficiently small.

$$\|L(s)\xi(t)\|_{L^2_{-\sigma}} \leq \int_s^t \|e^{-iH(t-\tau)}P_c\|_{L^2_\sigma \to L^2_{-\sigma}} \|F_1(R_a\xi)\|_{L^2_\sigma} d\tau \tag{4.7}$$

On the other hand

$$\|F_1(R_a\xi)\|_{L^2_\sigma} \leq \|\langle x\rangle^{2\sigma}(|\psi_E|^{1+\alpha_1} + |\psi_E|^{1+\alpha_2})\|_{L^\infty}\|R_a\xi\|_{L^2_{-\sigma}} \leq (\varepsilon_1^{1+\alpha_1} + \varepsilon_1^{1+\alpha_2})\|\xi\|_{L^2_{-\sigma}}$$

and using the last three relations, as well as the estimate (4.1) and the fact that $\xi \in X_1$ we obtain that

$$\begin{aligned}\|L(s)\|_{X_1 \to X_1} &\leq (\varepsilon_1^{1+\alpha_1} + \varepsilon_1^{1+\alpha_2}) \sup_{t>0}(1+|t-s|)^{\frac{3}{2}} \int_s^t \frac{1}{(1+|t-\tau|)^{\frac{3}{2}}} \cdot \frac{1}{(1+|\tau-s|)^{\frac{3}{2}}}\ d\tau \\ &\leq (\varepsilon_1^{1+\alpha_1} + \varepsilon_1^{1+\alpha_2}) \sup_{t>0}(1+|t-s|)^{\frac{3}{2}} \frac{1}{(1+|\frac{t-s}{2}|)^{\frac{3}{2}}} \leq C(\varepsilon_1^{1+\alpha_1} + \varepsilon_1^{1+\alpha_2})\end{aligned}$$

Now choosing $\varepsilon_1$ small enough we get

$$\|L\|_{X_1 \to X_1} < \frac{1}{2}$$

Therefore

$$\|P_c\Omega(t,s)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \frac{\tilde{C}}{(1+|t-s|)^{\frac{3}{2}}}$$

and

$$\|\Omega(t,s)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \|R_{a(t)}\|_{L^2_{-\sigma} \to L^2_{-\sigma}} \|P_c\Omega(t,s)\|_{L^2_\sigma \to L^2_{-\sigma}} \leqslant \frac{C}{(1+|t-s|)^{\frac{3}{2}}}$$

by Lemma 2.2.

$(ii)$ Recall that

$$P_c\Omega(t,s)v = T(t,s)v + e^{-iH(t-s)}P_cv \tag{4.8}$$

Denote:

$$T(t,s)v = W(t) \tag{4.9}$$

then, by plugging in (4.4), $W(t)$ satisfies the following equation:

$$W(t) = \underbrace{-i\int_s^t e^{-iH(t-\tau)}P_c[F_1(R_ae^{-iH(\tau-s)}P_cv)]d\tau}_{f(t)} + [L(s)W](t) \tag{4.10}$$

By definition of $T(t,s)$ (4.9) it is sufficient to prove that the solution of (4.10) satisfies

$$\|W(t)\|_{L^2_{-\sigma}} \leq \begin{cases} \frac{C\|v\|_{L^1}}{|t-s|^{\frac{1}{2}}} & \text{for } s \leq t \leq s+1 \\ \frac{C\|v\|_{L^1}}{(1+|t-s|)^{\frac{3}{2}}} & \text{for } t > s+1 \end{cases}$$

Let us also observe that it suffices to prove this estimate only for the forcing terms $f(t)$ because then we will be able to do the contraction principle in the functional space in which $f(t)$ will be, and thus obtain the same decay for $W$ as for $f(t)$.

This time we will consider the functional space

$$X_2 = \{u \in C(\mathbb{R}, L^2_{-\sigma}\mathbb{R}^3)| \sup_{|t-s|>1}(1+|t-s|)^{\frac{3}{2}}\|u\|_{L^2_{-\sigma}} < \infty, \sup_{|t-s|\leq 1}|t-s|^{\frac{1}{2}}\|u\|_{L^2_{-\sigma}} < \infty\}$$

endowed with the norm

$$\|u\|_{X_2} = \begin{cases} \sup_t |t-s|^{\frac{1}{2}}\|u\|_{L^2_{-\sigma}} & \text{for } |t-s| \le 1 \\ \sup_t (1+|t-s|)^{\frac{3}{2}}\|u\|_{L^2_{-\sigma}} & \text{for } |t-s| > 1 \end{cases}$$

Now we will estimate $f(t)$. First we will investigate the short time behavior of this term. If $s \le t \le s+1$. Recall that $F_1(u) = \frac{d}{d\tau}g(\psi_E + \tau u) = \frac{\partial}{\partial u}g(u)|_{u=\psi_E}u + \frac{\partial}{\partial \bar{u}}g(u)|_{u=\psi_E}\overline{u} = g_u u + g_{\bar{u}}\overline{u}$.

$$\|f(t)\|_{L^2_{-\sigma}} \le \|\langle x\rangle^{-\sigma}\int_s^t e^{-iH(t-\tau)}P_cF_1(R_a e^{-iH(\tau-s)}P_c v)d\tau\|_{L^2}$$

For the term $g_u R_a e^{-iH(\tau-s)}P_c v$ we have

$$\begin{aligned} &\|\langle x\rangle^{-\sigma}\int_s^t \|e^{-iH(t-\tau)}P_c g_u R_a e^{-iH(\tau-s)}P_c v d\tau\|_{L^2} \\ \le &\|\langle x\rangle^{-\sigma}\|_{L^2}\int_s^t \|e^{-iH(t-s)}e^{iH(\tau-s)}P_c g_u R_a e^{-iH(\tau-s)}P_c v\|_{L^\infty}d\tau \\ \le &\int_s^t \frac{C}{|t-s|^{\frac{3}{2}}}\sup\|\widehat{g_u}\|_{L^1}\|v\|_{L^1}d\tau \le C\frac{\|v\|_{L^1}}{|t-s|^{\frac{1}{2}}}\sup\|\widehat{g_u}\|_{L^1} < \infty \end{aligned}$$

and for the term $g_{\bar{u}}\overline{R}_a e^{iH(\tau-s)}P_c v$ we have

$$\begin{aligned} &\|\langle x\rangle^{-\sigma}\int_s^t \|e^{-iH(t-\tau)}P_c g_{\bar{u}}\overline{R}_a e^{iH(\tau-s)}P_c v d\tau\|_{L^2} \\ &\le \|\langle x\rangle^{-\sigma}\|_{L^2}\int_s^{s+\frac{t-s}{4}} \|e^{-iH(t+s-2\tau)}e^{-iH(\tau-s)}P_c g_{\bar{u}}\overline{R}_a e^{iH(\tau-s)}P_c v\|_{L^\infty}d\tau \\ &\quad + \int_{s+\frac{t-s}{4}}^t \|e^{-iH(t-\tau)}P_c\|_{L^2_\sigma\to L^2_{-\sigma}}\|\langle x\rangle^\sigma g_{\bar{u}}\overline{R}_a e^{-iH(\tau-s)}P_c v\|_{L^2}d\tau \\ &\le \int_s^{s+\frac{t-s}{4}} \frac{C}{|t+s-2\tau|^{\frac{3}{2}}}\sup\|\widehat{g_{\bar{u}}}\|_{L^1}\|v\|_{L^1}d\tau + \int_{s+\frac{t-s}{4}}^t \frac{C}{(1+|t-\tau)^{3/2}}\|\langle x\rangle^\sigma g_{\bar{u}}\|_{L^2}\|e^{-iH(\tau-s)}P_c v\|_{L^\infty}d\tau \\ &\le C\frac{\|v\|_{L^1}}{|t-s|^{\frac{1}{2}}}\sup(\|\widehat{g_{\bar{u}}}\|_{L^1} + \|\langle x\rangle^\sigma g_{\bar{u}}\|_{L^2}) < \infty \end{aligned}$$

There we used J-S-S type estimate; see Appendix for $t = \tau - s$; $|\tau - s| \le 1$. For the long time behavior of $f(t)$, we will split this integral into three parts to be estimated differently. For $t > s+1$,

$$f(t) = \underbrace{\int_s^{s+\frac{1}{2}}\cdots}_{I_1} + \underbrace{\int_{s+\frac{1}{2}}^{\frac{t+s}{2}}\cdots}_{I_2} + \underbrace{\int_{\frac{t+s}{2}}^t\cdots}_{I_3}$$

Then for $t > s+1$,

$$
\begin{aligned}
\|I_1\|_{L^2_{-\sigma}} &\leq \|\langle x\rangle^{-\sigma}\int_s^{s+\frac12} e^{-iH(t-s)}P_cF_1(R_ae^{-iH(\tau-s)}P_cv)d\tau\|_{L^2}\\
&\leq \|\langle x\rangle^{-\sigma}\|_{L^2}\int_s^{s+\frac12}\|e^{-iH(t-s)}e^{iH(\tau-s)}P_cg_uR_ae^{-iH(\tau-s)}P_cv\|_{L^\infty}d\tau\\
&\quad+\|\langle x\rangle^{-\sigma}\|_{L^2}\int_s^{s+\frac12}\|e^{-iH(t+s-2\tau)}e^{-iH(\tau-s)}P_cg_{\bar u}\overline{R}_ae^{iH(\tau-s)}P_cv\|_{L^\infty}d\tau\\
&\leq \|\langle x\rangle^{-\sigma}\|_{L^2}\int_s^{s+\frac12}\frac{C}{|t-s|^{\frac32}}\|e^{iH(\tau-s)}P_cg_uR_ae^{-iH(\tau-s)}P_cv\|_{L^1}d\tau\\
&\quad+\|\langle x\rangle^{-\sigma}\|_{L^2}\int_s^{s+\frac12}\frac{C}{|t+s-2\tau|^{\frac32}}\|e^{-iH(\tau-s)}P_cg_{\bar u}\overline{R}_ae^{iH(\tau-s)}P_cv\|_{L^1}d\tau\\
&\leq C\|\langle x\rangle^{-\sigma}\|_{L^2}\Big(\frac{1}{|t-s|^{\frac32}}+\frac{1}{|t-s-1|^{\frac32}}\Big)\int_s^{s+\frac12}\sup(\|\widehat{g_u}\|_{L^1}+\|\widehat{g_{\bar u}}\|_{L^1})\|v\|_{L^1}d\tau\\
&\leq C\|\langle x\rangle^{-\sigma}\|_{L^2}\sup(\|\widehat{g_u}\|_{L^1}+\|\widehat{g_{\bar u}}\|_{L^1})\frac{1}{(1+|t-s|)^{\frac32}}\|v\|_{L^1}
\end{aligned}
$$

For the second integral we have

$$
\begin{aligned}
\|I_2\|_{L^2_{-\sigma}} &\leq \int_{s+\frac12}^{\frac{t+s}{2}}\|e^{-iH(t-\tau)}P_c\|_{L^2_\sigma\to L^2_{-\sigma}}\|F_1(R_ae^{-iH(\tau-s)}P_cv)\|_{L^2_\sigma}d\tau\\
&\leq \int_{s+\frac12}^{\frac{t+s}{2}}\frac{C}{(1+|t-\tau|)^{\frac32}}\|\langle x\rangle^\sigma|\psi_E|^{1+\alpha}\|_{L^2}\|e^{-iH(\tau-s)}P_cv\|_{L^\infty}d\tau\\
&\leq \frac{C\|v\|_{L^1}}{(1+|\frac{t-s}{2}|)^{\frac32}}\int_{s+\frac12}^{\frac{t+s}{2}}\frac{d\tau}{|\tau-s|^{\frac32}}\leq\frac{C\|v\|_{L^1}}{(1+|t-s|)^{\frac32}}
\end{aligned}
$$

$I_3$ is estimated similiar to $I_2$.

$(iii)$ From (4.10) we have

$$
\langle x\rangle^{-\sigma}W(t)=\int_s^t\langle x\rangle^{-\sigma}e^{-iH(t-\tau)}P_c[F_1(R_ae^{-iH(\tau-s)}P_cv)]d\tau+\int_s^t\langle x\rangle^{-\sigma}e^{-iH(t-\tau)}P_c[F_1(R_aW(\tau))]d\tau
$$

Then

$$
\begin{aligned}
\|\langle x\rangle^{-\sigma}W(t)\|_{L^2_tL^2_x} &\leq \Big\|\int_s^t\frac{C}{(1+|t-\tau|)^{3/2}}\|\langle x\rangle^\sigma F_1(R_ae^{-iH(\tau-s)}P_cv)\|_{L^2}d\tau\Big\|_{L^2_t}\\
&\quad+\Big\|\int_s^t\frac{C}{(1+|t-\tau|)^{3/2}}(\|\langle x\rangle^{2\sigma}g_u\|_{L^\infty}+\|\langle x\rangle^{2\sigma}g_{\bar u}\|_{L^\infty})\|\langle x\rangle^{-\sigma}W(\tau)\|_{L^2_x}d\tau\Big\|_{L^2_t}\\
&\leq C\|K\|_{L^1}\|v\|_{L^2}+\varepsilon_1C\|K\|_{L^1}\|\langle x\rangle^{-\sigma}W\|_{L^2_xL^2_t}
\end{aligned}
$$

Where $K(t)=(1+|t|)^{-3/2}$. For the term $\langle x\rangle^\sigma F_1(R_ae^{-iHt}P_cv)=\langle x\rangle^\sigma(g_uR_ae^{-iHt}P_cv+g_{\bar u}\overline{R}_ae^{iHt}P_cv)$ we used $\|\langle x\rangle^{2\sigma}g_u\|_{L^\infty}$ and $\|\langle x\rangle^{2\sigma}g_{\bar u}\|_{L^\infty}$ is uniformly bounded in $t$ since $|g_u|=|g_{\bar u}|\leq C(|\psi_E|^{1+\alpha_1}+|\psi_E|^{1+\alpha_2})$ and the Kato smoothing estimate $\|\langle x\rangle^{-\sigma}e^{-iHt}P_cv\|_{L^2_t(\mathbb{R},L^2_x)}\leq C\|v\|_{L^2_x}$. Choosing $\varepsilon_1$ small enough we get $\|\langle x\rangle^{-\sigma}W\|_{L^2_xL^2_t}<\infty$. In other words $T(t,s)\in L^2_t(\mathbb{R},L^2\to L^2_{-\sigma})$. And similarly

$$
\begin{aligned}
\|\langle x\rangle^{-\sigma}W(t)\|_{L^2_x} &\leq \int_s^t\frac{C}{(1+|t-\tau|)^{3/2}}\|\langle x\rangle^\sigma F_1(R_ae^{-iH(\tau-s)}P_cv)\|_{L^2}d\tau\\
&\quad+\int_s^t\frac{C}{(1+|t-\tau|)^{3/2}}(\|\langle x\rangle^{2\sigma}g_u\|_{L^\infty}+\|\langle x\rangle^{2\sigma}g_{\bar u}\|_{L^\infty})\|\langle x\rangle^{-\sigma}W(\tau)\|_{L^2_x}d\tau\\
&\leq C\|v\|_{L^2}+\varepsilon_1C\|\langle x\rangle^{-\sigma}W\|_{L^2_x}
\end{aligned}
$$

This finishes the proof of $(iii)$, $T(t,s) \in L^2_t(\mathbb{R}, L^2 \to L^2_{-\sigma}) \cap L^\infty_t(\mathbb{R}, L^2 \to L^2_{-\sigma})$.

$(iv)$ By Riesz-Thorin interpolation between $(ii)$ and $(iii)$ (the $L^\infty_t$ part) we get the desired estimates. $\square$

The next step is to obtain estimates for $\Omega(t,s)$ and $T(t,s)$ in unweighted $L^p$ spaces.

**Theorem 4.2** *Assume that $\|\langle x\rangle^\sigma \psi_E\|_{H^2} < \varepsilon_1$ (where $\varepsilon_1$ is the one used in Theorem 4.1). Then there exist constants $C_2$, $C_2'$ and $C_\infty$ for all $t$, $s \in \mathbb{R}$ the following estimates hold:*

$$\begin{aligned}
&(i) \quad \|\Omega(t,s)\|_{L^2\to L^2} \le C_2, \quad \|T(t,s)\|_{L^2\to L^2} \le C_2\\
&(ii) \quad \|\Omega(t,s)\|_{L^1\to L^\infty} \le \frac{C_\infty}{|t-s|^{\frac{3}{2}}}, \quad \|T(t,s)\|_{L^1\to L^\infty} \le \begin{cases} C_\infty |t-s|^{\frac{1}{2}} & \text{for } |t-s| \le 1\\ \frac{C_\infty}{|t-s|^{\frac{3}{2}}} & \text{for } |t-s| > 1\end{cases}\\
&(iii) \quad \|T(t,s)\|_{L^{p'}\to L^2} \le C_2', \quad \text{for} \quad p = 6
\end{aligned}$$

**Remark 4.1** *By Riesz-Thorin interpolation from $(i)$ and $(ii)$, and from $(i)$ and $(iii)$ we get*

$$\begin{aligned}
\|\Omega(t,s)\|_{L^{p'}\to L^p} &\le \frac{C_p}{|t-s|^{3(\frac{1}{2}-\frac{1}{p})}}, \quad \text{for all} \quad 2 \le p \le \infty\\
\|T(t,s)\|_{L^{p'}\to L^p} &\le \begin{cases} C_p |t-s|^{(\frac{1}{2}-\frac{1}{p})} & \text{for } |t-s| \le 1\\ \frac{C}{|t-s|^{3(\frac{1}{2}-\frac{1}{p})}} & \text{for } |t-s| > 1\end{cases}, \quad \text{for all} \quad 2 \le p \le \infty\\
\|T(t,s)\|_{L^{p'}\to L^2} &\le C_p, \quad \text{for all} \quad 2 \le p \le 6
\end{aligned}$$

**Proof of Theorem 4.2** Because of the estimate (4.2) and relation $P_c\Omega = T + e^{-iH(t-s)}P_c$, It suffices to prove the theorem for $T(t,s)$.

$(i)$ To estimate the $L^2$ norm we will use duality argument to make use of cancelations.

$$\begin{aligned}
&\|f(t)\|^2_{L^2} = \langle f(t), f(t)\rangle\\
&= \int_s^t \int_s^t \langle e^{-iH(t-\tau)} P_c F_1(R_a e^{-iH(\tau-s)} P_c v), e^{-iH(t-\tau')} P_c F_1(R_a e^{-iH(\tau'-s)} P_c v)\rangle d\tau' d\tau\\
&= \int_s^t \int_s^t \langle F_1(R_a e^{-iH(\tau-s)} P_c v), e^{-iH(\tau-\tau')} P_c F_1(R_a e^{-iH(\tau'-s)} P_c v)\rangle d\tau' d\tau\\
&= \int_s^t \int_s^t \langle \langle x\rangle^\sigma F_1(R_a e^{-iH(\tau-s)} P_c v), \langle x\rangle^{-\sigma} e^{-iH(\tau-\tau')} P_c F_1(R_a e^{-iH(\tau'-s)} P_c v)\rangle d\tau' d\tau\\
&\le \int_s^t \int_s^t \|F_1(R_a e^{-iH(\tau-s)} P_c v)\|_{L^2_\sigma} \|e^{-iH(\tau-\tau')} P_c F_1(R_a e^{-iH(\tau'-s)} P_c v)\|_{L^2_{-\sigma}} d\tau' d\tau\\
&\le \int_s^t \|\langle x\rangle^\sigma F_1(R_a e^{-iH(\tau-s)} P_c v)\|_{L^2} \int_s^t \frac{C}{(1+|\tau-\tau'|)^{3/2}} \|\langle x\rangle^\sigma F_1(R_a e^{-iH(\tau'-s)} P_c v)\|_{L^2} d\tau' d\tau\\
&\le C\|\langle x\rangle^\sigma F_1(R_a e^{-iH(\tau-s)} P_c v)\|_{L^2_\tau L^2_x} \left\| \int_s^t \frac{C}{(1+|\tau-\tau'|)^{3/2}} \|\langle x\rangle^\sigma F_1(R_a e^{-iH(\tau'-s)} P_c v)\|_{L^2_x} d\tau \right\|_{L^2_\tau}\\
&\le C\|K\|_{L^1} \|\langle x\rangle^\sigma F_1(R_a e^{-iHt} P_c v)\|^2_{L^2_t L^2_x} \le C\|v\|^2_{L^2} < \infty
\end{aligned}$$

At the last line, $K(t) = (1+|t|)^{-3/2}$ and we used convolution estimate. For the term $\langle x\rangle^\sigma F_1(R_a e^{-iHt} P_c v) = \langle x\rangle^\sigma (g_u R_a e^{-iHt} P_c v + g_{\bar{u}} \overline{R}_a e^{iHt} P_c v)$ we used the Kato smoothing estimate $\|\langle x\rangle^{-\sigma} e^{-iHt} P_c v\|_{L^2_t(\mathbb{R}, L^2_x)} \le$

$C\|v\|_{L^2_x}$. We will estimate $L^2$ norm of $L$ similiar to $f$.

$$\begin{aligned}
&\|L(s)W\|_{L^2}^2 = \langle L(s)W, L(s)W\rangle \\
&= \langle \int_s^t e^{-iH(t-\tau)}P_cF_1(W(\tau))d\tau, \int_s^t e^{-iH(t-\tau')}P_cF_1(W(\tau'))d\tau'\rangle \\
&= \int_s^t \int_s^t \langle F_1(W(\tau)), e^{-iH(\tau-\tau')}P_cF_1(W(\tau'))\rangle] d\tau' d\tau \\
&\leq \int_s^t (\|\langle x\rangle^\sigma g_u\|_{L^\infty} + \|\langle x\rangle^\sigma g_{\bar{u}}\|_{L^\infty})\|\langle x\rangle^{-\sigma}W\|_{L^2} \\
&\qquad\qquad \times \int_s^t CK(\tau-\tau')(\|\langle x\rangle^\sigma g_u\|_{L^\infty} + \|\langle x\rangle^\sigma g_{\bar{u}}\|_{L^\infty})\|\langle x\rangle^{-\sigma}W\|_{L^2} d\tau' d\tau \\
&\leq C\|\langle x\rangle^{-\sigma}W\|_{L^2_xL^2_\tau}\Big\|\int_s^t CK(\tau-\tau')\|\langle x\rangle^{-\sigma}W\|_{L^2_x}d\tau'\Big\|_{L^2_\tau} \\
&\leq C\|K\|_{L^1}\|\langle x\rangle^{-\sigma}W\|_{L^2_\tau L^2_x} < \infty
\end{aligned}$$

By Theorem 4.1 $(iii)$, $\|\langle x\rangle^{-\sigma}W\|_{L^2_\tau L^2_x} < \infty$.
Therefore we conclude $\|T(s,t)\|_{L^2\to L^2} \leq C$ and $\|\Omega(s,t)\|_{L^2\to L^2} \leq C$

$(ii)$ Let us first investigate the short time behavior of the forcing term $f(t)$. We will assume $s \leq t \leq s+1$,

$$\begin{aligned}
\|f(t)\|_{L^\infty} &= \|\int_s^t e^{-iH(t-\tau)}P_cF_1(R_ae^{-iH(\tau-s)}P_cv)d\tau\|_{L^\infty} \\
&\leq \int_s^t \|e^{-iH(t-s)}P_c\|_{L^1\to L^\infty}\|e^{iH(\tau-s)}P_cg_uR_ae^{-iH(\tau-s)}P_cv\|_{L^1}d\tau \\
&+ \int_s^{s+\frac{t-s}{4}} \|e^{-iH(t+s-2\tau)}P_c\|_{L^1\to L^\infty}\|e^{-iH(\tau-s)}P_cg_{\bar{u}}\overline{R}_ae^{iH(\tau-s)}P_c\bar{v}\|_{L^1}d\tau \\
&+ \int_{s+\frac{t-s}{4}}^{t-\frac{t-s}{4}} \|e^{-iH(t-\tau)}P_c\|_{L^1\to L^infty}\|g_{\bar{u}}\overline{R}_ae^{iH(\tau-s)}P_c\bar{v}\|_{L^1}d\tau \\
&+ \int_{t-\frac{t-s}{4}}^t \|e^{-iH(t+s-2\tau)}P_c\|_{L^1\to L^\infty}\|e^{-iH(\tau-s)}P_cg_{\bar{u}}\overline{R}_ae^{iH(\tau-s)}P_c\bar{v}\|_{L^1}d\tau \\
&\leq \int_s^t \frac{C}{|t-s|^{\frac{3}{2}}}\sup\|\widehat{g_u}\|_{L^1}\|v\|_{L^1}d\tau + \int_s^{s+\frac{t-s}{4}} \frac{C}{|t+s-2\tau|^{\frac{3}{2}}}\sup\|\widehat{g_{\bar{u}}}\|_{L^1}\|v\|_{L^1}d\tau \\
&+ \int_{s+\frac{t-s}{4}}^{t-\frac{t-s}{4}} \frac{C}{|t-\tau|^{\frac{3}{2}}}\|g_{\bar{u}}\|_{L^1}\|v\|_{L^1}d\tau + \int_{t-\frac{t-s}{4}}^t \frac{C}{|t+s-2\tau|^{\frac{3}{2}}}\sup\|\widehat{g_{\bar{u}}}\|_{L^1}\|v\|_{L^1}d\tau \\
&\leq C\frac{\|v\|_{L^1}}{|t-s|^{\frac{1}{2}}}(\|g_{\bar{u}}\|_{L^1} + \sup(\|\widehat{g_u}\|_{L^1} + \|\widehat{g_{\bar{u}}}\|_{L^1}))
\end{aligned}$$

Now let us investigate the long time bevaviour of the forcing term $f(t)$. We will assume $t > s+1$ and seperate $f(t)$ into four parts as follows,

$$f(t) = \underbrace{\int_s^{s+\frac{1}{4}}\cdots}_{I_1} + \underbrace{\int_{s+\frac{1}{4}}^{t-\frac{1}{4}}\cdots}_{I_2} + \underbrace{\int_{t-\frac{1}{4}}^t\cdots}_{I_3}$$

We will start with $I_2$ for which we are away from the singularities around $\tau = s$ and $\tau = t$. Then for $I_1$ and

$I_3$ we will use J-S-S type estimate to remove the singularities.

$$\begin{aligned}
\|I_2\|_{L^\infty} &\le \int_{s+\frac{1}{4}}^{t-\frac{1}{4}} \|e^{-iH(t-\tau)}P_c(g_u R_a e^{-iH(\tau-s)}P_c v + g_{\bar u}\overline{R}_a e^{iH(\tau-s)}P_c\bar v)\|_{L^\infty} d\tau \\
&\le \int_{s+\frac{1}{4}}^{t-\frac{1}{4}} \frac{C}{|t-\tau|^{\frac{3}{2}}}(\|g_u R_a e^{-iH(\tau-s)}P_c v\|_{L^1} + \|g_{\bar u}\overline{R}_a e^{iH(\tau-s)}P_c\bar v\|_{L^1}) d\tau \\
&\le \int_{s+\frac{1}{4}}^{t-\frac{1}{4}} \frac{C}{|t-\tau|^{\frac{3}{2}}}(\|g_u\|_{L^1} + \|g_{\bar u}\|_{L^1})\|e^{-iH(\tau-s)}P_c v\|_{L^\infty} d\tau \\
&\le \int_{s+\frac{1}{4}}^{t-\frac{1}{4}} \frac{C}{|t-\tau|^{\frac{3}{2}}}(\|g_u\|_{L^1} + \|g_{\bar u}\|_{L^1})\frac{\|v\|_{L^1}}{|\tau-s|^{\frac{3}{2}}} d\tau \\
&\le C\frac{\|v\|_{L^1}}{|t-s|^{\frac{3}{2}}}(\|g_u\|_{L^1} + \|g_{\bar u}\|_{L^1})
\end{aligned}$$

$$\begin{aligned}
\|I_1\|_{L^\infty} &\le \int_{s}^{s+\frac{1}{4}} \|e^{-iH(t-s)}P_c e^{iH(\tau-s)}P_c g_u R_a e^{-iH(\tau-s)}P_c v\|_{L^\infty} d\tau \\
&\quad + \int_{s}^{s+\frac{1}{4}} \|e^{-iH(t+s-2\tau)}e^{-iH(\tau-s)}P_c g_{\bar u}\overline{R}_a e^{iH(\tau-s)}P_c\bar v\|_{L^\infty} d\tau \\
&\le \int_{s}^{s+\frac{1}{4}} \frac{C}{|t-s|^{\frac{3}{2}}}\|e^{iH(\tau-s)}P_c g_u R_a e^{-iH(\tau-s)}P_c v\|_{L^1} d\tau \\
&\quad + \int_{s}^{s+\frac{1}{4}} \frac{C}{|t+s-2\tau|^{\frac{3}{2}}}\|e^{-iH(\tau-s)}P_c g_{\bar u}\overline{R}_a e^{iH(\tau-s)}P_c\bar v\|_{L^1} d\tau \\
&\le \int_{s}^{s+\frac{1}{4}} C\Big(\frac{1}{|t-s|^{\frac{3}{2}}} + \frac{1}{|t-s-1|^{\frac{3}{2}}}\Big)\sup(\|\widehat{g_u}\|_{L^1} + \|\widehat{g_{\bar u}}\|_{L^1})\|v\|_{L^1} d\tau \\
&\le C\frac{\|v\|_{L^1}}{|t-s|^{\frac{3}{2}}}
\end{aligned}$$

$$\begin{aligned}
\|I_3\|_{L^\infty} &\le \int_{t-\frac{1}{4}}^{t} \|e^{-iH(t-\tau)}P_c g_u R_a e^{iH(t-\tau)}P_c e^{-iH(t-s)}P_c v\|_{L^\infty} d\tau \\
&\quad + \int_{t-\frac{1}{4}}^{t} \|e^{-iH(t-\tau)}P_c g_{\bar u}\overline{R}_a e^{iH(t-\tau)}P_c e^{iH(t+s-2\tau)}P_c\bar v\|_{L^\infty} d\tau \\
&\le C\frac{\|v\|_{L^1}}{|t-s|^{\frac{3}{2}}}
\end{aligned}$$

Now it remains to show that $L(s)W$ is bounded in $L^\infty$. Again to remove the singularities we will split the integral in different parts. Let us consider $s \le t \le s+1$,

$$\begin{aligned}
L(s)W &= \int_s^t e^{-iH(t-\tau)}P_c F_1(R_a W(\tau)) d\tau \\
&= \int_s^t e^{-iH(t-\tau)}P_c g_u R_a \Big[\int_s^\tau e^{-iH(\tau-\tau')}P_c[F_1(R_a e^{-iH(\tau'-s)}P_c v) + F_1(R_a W(\tau'))]d\tau'\Big] d\tau \\
&\quad + \int_s^t e^{-iH(t-\tau)}P_c g_{\bar u}\overline{R}_a \Big[\int_s^\tau e^{iH(\tau-\tau')}P_c[\overline{F_1(R_a e^{-iH(\tau'-s)}P_c v)} + \overline{F_1(R_a W(\tau'))}]d\tau'\Big] d\tau
\end{aligned}$$

All the terms will be either of the following forms

$$L_1 = \int_s^t e^{-iH(t-\tau)}P_c g_u R_a \int_s^\tau e^{-iH(\tau-\tau')}P_c X(\tau') d\tau' d\tau$$

$$L_2 = \int_s^t e^{-iH(t-\tau)} P_c g_{\bar{u}} \overline{R}_a \int_s^\tau e^{iH(\tau-\tau')} P_c \overline{X(\tau')} d\tau' d\tau$$

where $X(\tau') = g_u R_a e^{-iH(\tau'-s)} P_c v, g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}, g_u R_a W(\tau'), g_{\bar{u}} \overline{R_a W(\tau')}$

In what follows we will add $e^{iH(t-\tau)}$ and $e^{-iH(t-\tau)}$ terms after $g_u R_a$ and $g_{\bar{u}} \overline{R}_a$ then we will estimate the terms in a similiar way as we estimated $I_1$ and $I_3$.

$$L_1 = \int_s^t e^{-iH(t-\tau)} P_c g_u R_a e^{iH(t-\tau)} \int_s^\tau e^{-iH(t-\tau')} P_c X(\tau') d\tau' d\tau \tag{4.11}$$

$$L_2 = \int_s^t e^{-iH(t-\tau)} P_c g_{\bar{u}} \overline{R}_a e^{iH(t-\tau)} \int_s^\tau e^{-iH(t-2\tau+\tau')} P_c \overline{X(\tau')} d\tau' d\tau \tag{4.12}$$

- For $X(\tau') = g_u R_a e^{-iH(\tau'-s)} P_c v$ we have

$$\begin{aligned}
\|L_1\|_{L^\infty} &\le \int_s^t \|e^{-iH(t-\tau)} P_c g_u R_a e^{iH(t-\tau)}\|_{L^\infty \to L^\infty} \int_s^\tau \|e^{-iH(t-s)} P_c\|_{L^1 \to L^\infty} \|e^{iH(\tau'-s)} P_c g_u R_a e^{-iH(\tau'-s)} P_c v\|_{L^1} d\tau' d\tau \\
&\le \int_s^t \|\widehat{g_u}\|_{L^1} \int_s^\tau \frac{C}{|t-s|^{\frac{3}{2}}} \|\widehat{g_u}\|_{L^1} \|v\|_{L^1} d\tau' d\tau \le C\sqrt{t-s}\|v\|_{L^1} \le C\|v\|_{L^1} \quad \text{for } s \le t \le s+1
\end{aligned}$$

$$\begin{aligned}
\|L_2\|_{L^\infty} &\le \int_s^{t-\frac{t-s}{4}} \|e^{-iH(t-\tau)} P_c\|_{L^1 \to L^\infty} \|g_{\bar{u}} \overline{R}_a \int_s^\tau e^{-iH(\tau-s)} P_c e^{-iH(\tau'-s)} P_c g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}\|_{L^1} d\tau' d\tau \\
&+ \int_{t-\frac{t-s}{4}}^t \|e^{-iH(t-\tau)} P_c g_{\bar{u}} \overline{R}_a e^{iH(t-\tau)}\|_{L^\infty \to L^\infty} \\
&\qquad \times \int_s^\tau \|e^{-iH(t+s-2\tau)} P_c\|_{L^1 \to L^\infty} \|e^{-iH(\tau'-s)} P_c g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}\|_{L^1} d\tau' d\tau \\
&\le \int_s^{t-\frac{t-s}{4}} \frac{C}{|t-\tau|^{\frac{3}{2}}} \|g_{\bar{u}}\|_{L^1} \int_s^\tau \frac{C}{|\tau-s|^{\frac{3}{2}}} \|\widehat{g_{\bar{u}}}\|_{L^1} \|v\|_{L^1} d\tau' d\tau \\
&+ \int_{t-\frac{t-s}{4}}^t \|\widehat{g_{\bar{u}}}\|_{L^1} \int_s^\tau \frac{C}{|t+s-2\tau|^{\frac{3}{2}}} \|\widehat{g_{\bar{u}}}\|_{L^1} \|v\|_{L^1} d\tau' d\tau \le C\sqrt{|t-s|}\|v\|_{L^1}
\end{aligned}$$

- For $X(\tau') = g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}$ we have

$$\begin{aligned}
\|L_1\|_{L^\infty} \le &\int_s^t \|e^{-iH(t-\tau)} P_c g_u R_a e^{iH(t-\tau)}\|_{L^\infty \to L^\infty} \\
&\qquad \times \Big[ \int_s^{s+\frac{t-s}{4}} \|e^{-iH(t+s-2\tau')} P_c\|_{L^1 \to L^\infty} \|e^{-iH(\tau'-s)} P_c g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}\|_{L^1} d\tau' \\
&\qquad + \int_{s+\frac{t-s}{4}}^{t-\frac{t-s}{4}} \|e^{-iH(t-\tau')}\|_{L^1 \to L^\infty} \|g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}\|_{L^1} d\tau' \\
&\qquad + \int_{t-\frac{t-s}{4}}^\tau \|e^{-iH(t+s-2\tau')} P_c\|_{L^1 \to L^\infty} \|e^{-iH(\tau'-s)} P_c g_{\bar{u}} \overline{R}_a e^{iH(\tau'-s)} P_c \bar{v}\|_{L^1} d\tau' \Big] d\tau \\
\le &\int_s^t \|\widehat{g_u}\|_{L^1} \Big[ \int_s^{s+\frac{t-s}{4}} \frac{C}{|t+s-2\tau'|^{\frac{3}{2}}} \|\widehat{g_{\bar{u}}}\|_{L^1} d\tau' + \int_{s+\frac{t-s}{4}}^{t-\frac{t-s}{4}} \frac{C}{|t-\tau'|^{\frac{3}{2}}} \|g_{\bar{u}}\|_{L^1} \frac{C}{|\tau-s|^{\frac{3}{2}}} d\tau' \\
&\qquad + \int_{t-\frac{t-s}{4}}^\tau \frac{C}{|t+s-2\tau'|^{\frac{3}{2}}} \|\widehat{g_{\bar{u}}}\|_{L^1} d\tau' \Big] \|v\|_{L^1} d\tau \\
\le &C\sqrt{|t-s|}\|v\|_{L^1}
\end{aligned}$$

$$
\begin{aligned}
\|L_2\|_{L^\infty} \le& \int_s^t \|e^{-iH(t-\tau)}P_c g_{\bar u}\overline{R}_a e^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\Big[\int_s^{s+\frac{t-s}{4}} \|e^{-iH(t-2\tau+2\tau'-s)}e^{iH(\tau'-s)}P_c g_u R_a e^{-iH(\tau'-s)}P_c v\|_{L^\infty}d\tau' \\
&+\int_{s+\frac{t-s}{4}}^{\tau} \|e^{-iH(t-2\tau+\tau')}P_c\|_{L^1\to L^\infty}\|g_u R_a e^{-iH(\tau'-s)}P_c v\|_{L^1}d\tau'\Big]d\tau \\
\le& \int_s^t \|\widehat{g_{\bar u}}\|_{L^1}\Big[\int_s^{s+\frac{t-s}{4}} \frac{C}{|t-2\tau+2\tau'-s|^{\frac32}}\|\widehat{g_{\bar u}}\|_{L^1}\|v\|_{L^1}d\tau' + \int_{s+\frac{t-s}{4}}^{\tau} \frac{C}{|t-2\tau+\tau'|^{\frac32}}\|g_{\bar u}\|_{L^1}\frac{\|v\|_{L^1}}{|\tau'-s|^{\frac32}}d\tau'\Big]d\tau \\
\le& C\sqrt{|t-s|}\|v\|_{L^1}
\end{aligned}
$$

- For $X(\tau') = g_u R_a W(\tau')$ and $g_{\bar u}\overline{R_a W(\tau')}$ we will change the order of integration,

$$
\begin{aligned}
\|L_1\|_{L^\infty} \le& \int_s^t\int_{\tau'}^t \|e^{-iH(t-\tau)}P_c g_u R_a e^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\|e^{-iH(t-\tau')}P_c\|_{L^1\to L^\infty}\|g_u R_a W(\tau')\|_{L^1}d\tau d\tau' \\
\le& \int_s^t\int_{\tau'}^t \|\widehat{g_u}\|_{L^1}\frac{C}{|t-\tau'|^{\frac32}}\|\langle x\rangle^\sigma g_u\|_{L^2}\|W\|_{L^2_{-\sigma}}d\tau d\tau' \\
\le& \int_s^t \|\widehat{g_u}\|_{L^1}\frac{C}{|t-\tau'|^{\frac12}}\|\langle x\rangle^\sigma g_u\|_{L^2}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac12}}d\tau' \\
\le& C\sqrt{|t-s|}\|v\|_{L^1}
\end{aligned}
$$

$$
\begin{aligned}
\|L_2\|_{L^\infty} \le& \int_s^t\int_{\tau'}^{t-\frac{t-\tau'}{4}} \|e^{-iH(t-\tau)}P_c\|_{L^1\to L^\infty}\|g_u R_a e^{-iH(\tau-\tau')}P_c g_u R_a W(\tau')\|_{L^1}d\tau d\tau' \\
&+\int_s^t\int_{t-\frac{t-\tau'}{4}}^t \|e^{-iH(t-\tau)}P_c g_u R_a e^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\|e^{-iH(t+\tau'-2\tau)}P_c g_u R_a W(\tau')\|_{L^\infty}d\tau d\tau' \\
\le& \int_s^t\int_{\tau'}^{t-\frac{t-\tau'}{4}} \frac{C}{|t-\tau|^{\frac32}}\|g_{\bar u}\|_{L^2}\|e^{-iH(\tau-\tau')}P_c\|_{L^2\to L^2}\|g_u R_a W(\tau')\|_{L^2}d\tau d\tau' \\
&+\int_s^t\int_{\tau'}^{t-\frac{t-\tau'}{4}} \|\widehat{g_u}\|_{L^1}\frac{C}{|t+\tau'-2\tau|^{\frac32}}\|g_u R_a W(\tau')\|_{L^1}d\tau d\tau' \\
\le& \int_s^t \frac{C}{|t-\tau'|^{\frac12}}\|g_{\bar u}\|_{L^2}\|\langle x\rangle^\sigma g_u\|_{L^\infty}\|W(\tau')\|_{L^2_{-\sigma}}d\tau' \\
&+\int_s^t \|\widehat{g_u}\|_{L^1}\frac{C}{|t-\tau'|^{\frac12}}\|\langle x\rangle^\sigma g_u\|_{L^2}\|W(\tau')\|_{L^2_{-\sigma}}d\tau' \\
\le& C\sqrt{|t-s|}\|v\|_{L^1}
\end{aligned}
$$

Similarly we will investigate the long time behavior of the operator $L(s)$ for $t > s+1$.

$$
L(s)W(t) = \underbrace{\int_s^{t-\frac14}\cdots}_{L_3} + \underbrace{\int_{t-\frac14}^t\cdots}_{L_4}
$$

$$\begin{aligned}\|L_3\|_{L^\infty} &\le \int_s^{t-\frac14} \|e^{-iH(t-\tau)}P_cF_1(R_aW(\tau))\|_{L^\infty}d\tau\\ &\le \int_s^{t-\frac14} \|e^{-iH(t-\tau)}P_c\|_{L^1\to L^\infty}\|F_1(R_aW(\tau))\|_{L^1}d\tau\\ &\le \int_s^{t-\frac14} \frac{C}{|t-s|^{\frac32}}(\|\langle x\rangle^\sigma g_u\|_{L^2}+\|\langle x\rangle^\sigma g_{\bar u}\|_{L^2})\|W\|_{L^2_{-\sigma}}d\tau\\ &\le \frac{C}{|t-s|^{\frac32}}\int_s^{t-\frac14}\frac{\|v\|_{L^1}}{(1+|\tau-s|)^{\frac32}}d\tau \le \frac{C}{|t-s|^{\frac32}}\|v\|_{L^1}\end{aligned}$$

In $L_4$ we will plug in (4.10) once more:

$$\begin{aligned}L_4 &= \int_{t-\frac14}^t e^{-iH(t-\tau)}P_cF_1(R_aW(\tau))d\tau\\ &= \int_{t-\frac14}^t e^{-iH(t-\tau)}P_cg_uR_a\Big[\int_s^\tau e^{-iH(\tau-\tau')}P_c[F_1(R_ae^{-iH(\tau'-s)}P_cv)+F_1(R_aW(\tau'))]d\tau'\Big]d\tau\\ &\quad+\int_{t-\frac14}^t \|e^{-iH(t-\tau)}P_cg_{\bar u}\overline{R}_a\Big[\int_s^\tau e^{iH(\tau-\tau')}P_c[\overline{F_1(R_ae^{-iH(\tau'-s)}P_cv)}+\overline{F_1(R_aW(\tau'))}]d\tau'\Big]d\tau\end{aligned}$$

Again we will add $e^{iH(t-\tau)}$ and $e^{-iH(t-\tau)}$ terms after $g_uR_a$ and $g_{\bar u}\overline{R}_a$. Then all the terms will be similar to $L_1$, $L_2$, (4.11) – (4.12) respectively. After seperating the the inside integrals into pieces, we will estimate short time step integrals exactly the same way we did short time behavior by using JSS estimate, and the other integrals will be estimated using the usual norms.

- For $X(\tau') = g_uR_ae^{-iH(\tau'-s)}P_cv$ we have

$$\begin{aligned}\|L_1\|_{L^\infty} &\le \int_{t-\frac14}^t \|e^{-iH(t-\tau)}P_cg_uR_ae^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\\ &\qquad\Big[\int_s^{s+\frac14}\|e^{-iH(t-s)}P_c\|_{L^1\to L^\infty}\|e^{iH(\tau'-s)}P_cg_uR_ae^{-iH(\tau'-s)}P_cv\|_{L^1}d\tau'\\ &\qquad+\int_{s+\frac14}^{t-\frac14}\|e^{-iH(t-\tau')}P_c\|_{L^1\to L^\infty}\|g_uR_ae^{-iH(\tau'-s)}P_cv\|_{L^\infty}d\tau'\\ &\qquad+\int_{t-\frac14}^{\tau}\|e^{-iH(t-s)}P_c\|_{L^1\to L^\infty}\|e^{iH(\tau'-s)}P_cg_uR_ae^{-iH(\tau'-s)}P_cv\|_{L^1}d\tau'\Big]d\tau\\ &\le \frac{C\|v\|_{L^1}}{|t-s|^{\frac32}}\end{aligned}$$

$$\begin{aligned}\|L_2\|_{L^\infty} &\le \int_{t-\frac14}^t \|e^{-iH(t-\tau)}P_cg_{\bar u}\overline{R}_ae^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\\ &\qquad\times\Big[\int_s^{s+\frac14}\|e^{-iH(t+s-2\tau)}P_c\|_{L^1\to L^\infty}\|e^{-iH(\tau'-s)}P_cg_{\bar u}\overline{R}_ae^{iH(\tau'-s)}P_c\bar v\|_{L^1}d\tau'\\ &\qquad+\int_{s+\frac14}^{t-\frac14}\|e^{-iH(t-2\tau+\tau')}P_c\|_{L^1\to L^\infty}\|g_{\bar u}\overline{R}_ae^{iH(\tau'-s)}P_c\bar v\|_{L^1}d\tau'\\ &\qquad+\int_{t-\frac14}^{\tau}\|e^{-iH(t+s-2\tau)}P_c\|_{L^1\to L^\infty}\|e^{-iH(\tau'-s)}P_cg_{\bar u}\overline{R}_ae^{iH(\tau'-s)}P_c\bar v\|_{L^1}d\tau'\Big]d\tau\\ &\le \frac{C\|v\|_{L^1}}{|t-s|^{\frac32}}\end{aligned}$$

- For $X(\tau')=g_{\bar u}\overline{R}_a e^{iH(\tau'-s)}P_c\bar v$ we have

$$\begin{aligned}
\|L_1\|_{L^\infty}\le&\int_{t-\frac14}^{t}\|e^{-iH(t-\tau)}P_cg_uR_ae^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\\
&\times\Big[\int_s^{s+\frac14}\|e^{-iH(t+s-2\tau')}P_c\|_{L^1\to L^\infty}\|e^{-iH(\tau'-s)}P_cg_{\bar u}\overline{R}_ae^{iH(\tau'-s)}P_c\bar v\|_{L^1}d\tau'\\
&\quad+\int_{s+\frac14}^{t-\frac14}\|e^{-iH(t-\tau')}\|_{L^1\to L^\infty}\|g_{\bar u}\overline{R}_ae^{iH(\tau'-s)}P_c\bar v\|_{L^1}d\tau'\\
&\quad+\int_{t-\frac14}^{\tau}\|e^{-iH(t+s-2\tau')}P_c\|_{L^1\to L^\infty}\|e^{-iH(\tau'-s)}P_cg_{\bar u}\overline{R}_ae^{iH(\tau'-s)}P_c\bar v\|_{L^1}d\tau'\Big]d\tau\\
\le&\int_{t-\frac14}^{t}\|\widehat{g_u}\|_{L^1}\Big[\int_s^{s+\frac14}\frac{C}{|t+s-2\tau'|^{\frac32}}\|\widehat{g_{\bar u}}\|_{L^1}d\tau'+\int_{s+\frac14}^{t-\frac14}\frac{C}{|t-\tau'|^{\frac32}}\|g_{\bar u}\|_{L^1}\frac{C}{|\tau-s|^{\frac32}}d\tau'\\
&\quad+\int_{t-\frac14}^{\tau}\frac{C}{|t+s-2\tau'|^{\frac32}}\|\widehat{g_{\bar u}}\|_{L^1}d\tau'\Big]\|v\|_{L^1}d\tau\\
\le&C\frac{\|v\|_{L^1}}{|t-s|^{\frac32}}
\end{aligned}$$

$$\begin{aligned}
\|L_2\|_{L^\infty}\le&\int_s^t\|e^{-iH(t-\tau)}P_cg_{\bar u}\overline{R}_ae^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\Big[\int_s^{s+\frac14}\|e^{-iH(t-2\tau+2\tau'-s)}e^{iH(\tau'-s)}P_cg_uR_ae^{-iH(\tau'-s)}P_cv\|_{L^\infty}d\tau'\\
&+\int_{s+\frac14}^{\tau}\|e^{-iH(t-2\tau+\tau')}P_c\|_{L^1\to L^\infty}\|g_uR_ae^{-iH(\tau'-s)}P_cv\|_{L^1}d\tau'\Big]d\tau\\
\le&\int_s^t\|\widehat{g_{\bar u}}\|_{L^1}\Big[\int_s^{s+\frac14}\frac{C}{|t-2\tau+2\tau'-s|^{\frac32}}\|\widehat{g_{\bar u}}\|_{L^1}\|v\|_{L^1}d\tau'+\int_{s+\frac14}^{\tau}\frac{C}{|t-2\tau+\tau'|^{\frac32}}\|g_{\bar u}\|_{L^1}\frac{\|v\|_{L^1}}{|\tau'-s|^{\frac32}}d\tau'\Big]d\tau\\
\le&\frac{C\|v\|_{L^1}}{|t-s|^{\frac32}}
\end{aligned}$$

- $L_1$, $L_2$ terms corresponding to $X(\tau')=g_uR_aW(\tau')$ and $g_{\bar u}\overline{R_aW(\tau')}$

$$\begin{aligned}
\|L_1\|_{L^\infty}\le&\int_{t-\frac14}^{t}\|e^{-iH(t-\tau)}P_cg_uR_ae^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\int_s^\tau\|e^{-iH(t-\tau')}P_c\|_{L^1\to L^\infty}\|g_uR_aW(\tau')\|_{L^1}d\tau'd\tau\\
\le&\int_{t-\frac14}^{t}\|\widehat{g_u}\|_{L^1}\int_s^\tau\frac{C}{|t-\tau'|^{\frac32}}\|\langle x\rangle^\sigma g_u\|_{L^2}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac32}}d\tau'd\tau\\
\le&\int_{t-\frac14}^{t}\left[\int_s^{\frac{t+s}{2}}\frac{C}{|t-\tau'|^{\frac32}}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac32}}d\tau+\int_{\frac{t+s}{2}}^{\tau}\frac{C}{|t-\tau'|^{\frac32}}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac32}}d\tau\right]d\tau'\\
\le&\frac{C\|v\|_{L^1}}{|t-s|^{\frac32}}
\end{aligned}$$

$$\begin{aligned}
\|L_2\|_{L^\infty}\le&\int_{t-\frac14}^{t}\|e^{-iH(t-\tau)}P_cg_{\bar u}\overline{R_a}e^{iH(t-\tau)}\|_{L^\infty\to L^\infty}\int_s^\tau\|e^{-iH(t+\tau'-2\tau)}P_c\|_{L^1\to L^\infty}\|g_uR_aW(\tau')\|_{L^1}d\tau'd\tau\\
\le&\int_{t-\frac14}^{t}\|\widehat{g_{\bar u}}\|_{L^1}\int_s^\tau\frac{C}{|t+\tau'-2\tau|^{\frac32}}\|\langle x\rangle^\sigma g_u\|_{L^2}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac32}}d\tau'd\tau\\
\le&\int_{t-\frac14}^{t}\left[\int_s^{\frac{\tau+s}{2}}\frac{C}{|t+\tau'-2\tau|^{\frac32}}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac32}}d\tau+\int_{\frac{\tau+s}{2}}^{\tau}\frac{C}{|t-\tau'|^{\frac32}}\frac{C\|v\|_{L^1}}{(1+|\tau'-s|)^{\frac32}}d\tau\right]d\tau'\\
\le&\frac{C\|v\|_{L^1}}{|t-s|^{\frac32}}
\end{aligned}$$

Now combining all the above estimates we get

$$\|W(t)\|_{L^\infty} \leq \begin{cases} C|t-s|^{\frac{1}{2}} & \text{for } |t-s| \leq 1 \\ \frac{C}{|t-s|^{\frac{3}{2}}} & \text{for } |t-s| > 1 \end{cases}$$

This finishes the proof of $(ii)$.

$(iii)$ We split $f$ given by (4.10):

$$f = \underbrace{\int_s^{s+1} \cdots}_{I_1} + \underbrace{\int_{s+1}^{t} \cdots}_{I_2}$$

For $I_1$ integral, it suffices to show that $\|g_u(\tau)R_a e^{-iH(\tau-s)}P_c v\|_{L^2} \in L^1_\tau[s,s+1]$. Since $g_u(\tau)$ has bounded derivatives we have $\|g_u(\tau)-g_u(s)\|_{L^3} \leq C|\tau-s|$, then by Hölder inequality in space,

$$\|(g_u(\tau)-g_u(s))R_a e^{-iH(\tau-s)}P_c v\|_{L^2} \leq \|g_u(\tau)-g_u(s)\|_{L^3}\|e^{-iH(\tau-s)}P_c v\|_{L^6} \in L^1_\tau$$

Now it suffices to show $\|g_u(s)R_a e^{-iH(\tau-s)}P_c v\|_{L^2} \in L^1_\tau$. For any $\tilde{v} \in L^2$ we have

$$\|g_u(s)R_a e^{-iH(\tau-s)}P_c v\|_{L^2} = \langle \tilde{v}, g_u(s)R_a e^{-iH(\tau-s)}P_c v\rangle = \langle e^{iH(\tau-s)}P_c g_u(s)R_a\tilde{v}, v\rangle \leq \underbrace{\|e^{iH(\tau-s)}P_c R_a g_u(s)\tilde{v}\|_{L^6}}_{\in L^2_\tau}\|v\|_{L^{6/5}}$$

Since $L^2_\tau[s,s+1] \hookrightarrow L^1_\tau[s,s+1]$, $\|e^{iH(\tau-s)}P_c R_a g_u(s)\tilde{v}\|_{L^6} \in L^1_\tau$.

$$\begin{aligned}
\|I_2\|_{L^2} &\leq C\Big(\int_{s+1}^t \|g_u R_a e^{-iH(\tau-s)}P_c v\|^{\gamma'}_{L^{\rho'}}d\tau\Big)^{\frac{1}{\gamma'}} + C\Big(\int_{s+1}^t \|g_{\bar{u}}\overline{R}_a e^{iH(\tau-s)}P_c\bar{v}\|^{\gamma'}_{L^{\rho'}}d\tau\Big)^{\frac{1}{\gamma'}} \\
&\leq C\Big(\int_{s+1}^t \|\langle x\rangle^\sigma g_u\|^{\gamma'}_{L^{\frac{3\gamma}{2}}}\|e^{-iH(\tau-s)}P_c v\|^{\gamma'}_{L^2_{-\sigma}}d\tau\Big)^{\frac{1}{\gamma'}} + C\Big(\int_{s+1}^t \|\langle x\rangle^\sigma g_{\bar{u}}\|^{\gamma'}_{L^{\frac{3\gamma}{2}}}\|e^{iH(\tau-s)}P_c\bar{v}\|^{\gamma'}_{L^2_{-\sigma}}d\tau\Big)^{\frac{1}{\gamma'}} \\
&\leq C\Big(\int_{s+1}^t \frac{d\tau}{|\tau-s|^{3(\frac{1}{2}-\frac{1}{p})\gamma'}}\Big)^{\frac{1}{\gamma'}} < \infty
\end{aligned}$$

At the first inequality we used Strichartz estimate with $(\gamma,\rho)$ with $\gamma > 2$ and the last inequality holds since $3(\frac{1}{2}-\frac{1}{p})\gamma' > 1$ for $p=6$ and $\gamma > 2$. Similarly we will estimate $L(s)W$.

$$\begin{aligned}
\|L(s)W(t)\|_{L^2} &\leq C\Big(\int_s^t \|g_u R_a W + g_{\bar{u}}\overline{R_a W}\|^{\gamma'}_{L^{\rho'}}d\tau\Big)^{\frac{1}{\gamma'}} \\
&\leq C\Big(\int_s^t \|\langle x\rangle^\sigma(g_u+g_{\bar{u}})\|^{\gamma'}_{L^{\frac{3\gamma}{2}}}\|W\|^{\gamma'}_{L^2_{-\sigma}}d\tau\Big)^{\frac{1}{\gamma'}} \\
&\leq C\Big(\int_s^t \frac{1}{(1+|\tau-s|)^{3(\frac{1}{2}-\frac{1}{p})\gamma'}}\Big)^{\frac{1}{\gamma'}} < \infty
\end{aligned}$$

Hence $T(t,s): L^{p'} \to L^2$ is bounded for $p=6$. This finishes the proof of part $(iii)$ and the theorem. $\square$

# 5 Appendix

## 5.1 J-S-S type estimates

In [14] the authors obtain the following estimate[4]:

**Theorem 5.1** *If $W:\mathbb{R}^n \mapsto \mathbb{C}$ has Fourier transform $\widehat{W} \in L^1(\mathbb{R}^n)$ then for any $t \in \mathbb{R}$ and any $1 \leqslant p \leqslant \infty$ we have:*

$$\|e^{-i\Delta t}We^{i\Delta t}\|_{L^p\mapsto L^p} \leqslant \|\widehat{W}\|_{L^1}$$

[4]Their theorem is stated differently but the proof can be easily adapted to obtain the advertised estimate

In what follows we are going to generalize the estimate to the semigroup of operators generated by $-\Delta+V$ :

**Theorem 5.2** *Assume $V:\mathbb{R}^n\mapsto\mathbb{R}$ and $W:\mathbb{R}^n\mapsto\mathbb{C}$ have Fourier transforms in $L^1(\mathbb{R}^n)$. Then for any $T>0$ there exist a constant $C_T$ independent of $W$ such that for any $-T\leqslant t\leqslant T$ and any $1\leqslant p\leqslant\infty$ we have:*

$$\|e^{-i(-\Delta+V)t}We^{i(-\Delta+V)t}\|_{L^p\mapsto L^p}\leqslant C_T\|\widehat{W}\|_{L^1}.$$

*One can choose $C_T=\exp(2\|\widehat{V}\|_{L^1}T)$.*

The proof relies on existence of finite time wave operators:

**Lemma 5.1** *If $V:\mathbb{R}^n\mapsto\mathbb{R}$ has Fourier transform in $L^1(\mathbb{R}^n)$ then for any $T>0$ there exist a constant $C_T$ such that for any $-T\leqslant t\leqslant T$ and any $1\leqslant p\leqslant\infty$ we have:*

$$\|e^{-i(-\Delta+V)t}e^{-i\Delta t}\|_{L^p\mapsto L^p}\leqslant C_T\|\widehat{W}\|_{L^1},\quad \|e^{i\Delta t}e^{i(-\Delta+V)t}\|_{L^p\mapsto L^p}\leqslant C_T\|\widehat{W}\|_{L^1}.$$

*One can choose $C_T=\exp(\|\widehat{V}\|_{L^1}T)$.*

**Proof of Lemma:** Let

$$H=-\Delta+V$$

then $H$ is a self adjoint operator on $L^2$ with domain $H^2$, (note that $V\in L^\infty$,) hence it generates a group of isometric operators:

$$e^{-iHt}:L^2\mapsto L^2,\qquad t\in\mathbb{R}.$$

Consequently:

$$Q(t)=e^{-iHt}e^{-i\Delta t}:L^2\mapsto L^2,\qquad t\in\mathbb{R},\tag{5.1}$$

is also a family of isometric operators. Their infinitesimal generators are:

$$\frac{dQ}{dt}=-ie^{-iHt}Ve^{-i\Delta t}=-i\underbrace{e^{-iHt}e^{-i\Delta t}}_{Q(t)}\ \underbrace{e^{i\Delta t}Ve^{-i\Delta t}}_{Q_0(t)}$$

Hence

$$Q(t)=\mathbb{I}_d-i\int_0^t Q(s)Q_0(s)ds\tag{5.2}$$

where

$$Q_0(t)=e^{i\Delta t}Ve^{-i\Delta t}:L^p\mapsto L^p,\qquad 1\leqslant p\leqslant\infty$$

is bounded uniformly by $\|\widehat{V}\|_{L^1}$, see Theorem 5.1.

The contraction principle shows that for any $T>0$ and any $1\leqslant p\leqslant\infty$ the linear equation (5.2) has a unique solution in the Banach space $C([-T,T],B(L^p,L^p))$. Since on $L^2\bigcap L^p$ the solution is given by (5.1) and $L^2\bigcap L^p$ is dense in $L^p$ we obtain that for any $-T\leqslant t\leqslant T$ and any $1\leqslant p\leqslant\infty$, $e^{-iHt}e^{-i\Delta t}$ has a unique extension to a bounded operator on $L^p$. Applying the $L^p$ norm in (5.2) we get:

$$\|Q(t)\|_{L^p}\leqslant 1+\int_0^t\|Q(s)\|_{L^p}\|\widehat{V}\|_{L^1}ds$$

and by Gronwall inequality:

$$\|Q(t)\|_{L^p}\leqslant e^{\|\widehat{V}\|_{L^1}|t|}\leqslant e^{\|\widehat{V}\|_{L^1}T},\quad \text{for}-T\leqslant t\leqslant T.$$

A similar argument can be made for $Q^*(t)=e^{i\Delta t}e^{iHt}$.

The Lemma is now completely proven.

**Proof of Theorem 5.2:** For $H$, $Q$ and $Q^*$ as in the proof of the previous Lemma we have:

$$e^{-iHt}WE^{iHt}=\underbrace{e^{-iHt}e^{-i\Delta t}}_{Q(t)}\ \underbrace{e^{i\Delta t}We^{-i\Delta t}}_{L^p\mapsto L^p\ \text{bounded}}\ \underbrace{e^{i\Delta t}e^{iHt}}_{Q^*(t)}.$$

Hence using Theorem 5.1 and Lemma 5.1 we get for any $1\leqslant p\leqslant\infty$ :

$$\|e^{-iHt}WE^{iHt}\|_{L^p\mapsto L^p}\leqslant e^{2\|\widehat{V}\|_{L^1}T}\|\widehat{W}\|_{L^1},\qquad \text{for }-T\leqslant t\leqslant T.$$

The theorem is now completely proven.

**Remark 5.1** *To obtain the linear estimates in Section 4 we used Theorem 5.2 in the form:*

$$\|e^{iHt}WR_ae^{-iHt}\|_{L^p\mapsto L^p}\leqslant C\|\widehat{W}\|_{L^1},\qquad \text{for } 0\leqslant t\leqslant 1$$

*where $W$ is the effective potential induced by the nonlinearity, see next subsection, while $R_a$ is the linear operator defined in Lemma 2.2.*

To see why the above estimate holds consider $f\in L^p\bigcap L^2\bigcap \mathcal{H}_0$. Then by (2.19) we have for a certain $z=z(f)\in\mathbb{C}$ :

$$e^{iHt}WR_ae^{-iHt}f=e^{iHt}We^{-iHt}f+ze^{iHt}W\psi_0.$$

Theorem 5.2 applies directly to the first term on the right hand side, while for the second term we use, see (2.21):

$$|z|\leqslant 2\|f\|_{L^p}\sqrt{\left\|\frac{\partial\psi_E}{\partial a_2}\right\|_{L^{p'}}+\left\|\frac{\partial\psi_E}{\partial a_1}\right\|_{L^{p'}}},\qquad \frac{1}{p}+\frac{1}{p'}=1$$

and the fact that $\psi_0$ is an e-vector of $H$ with e-value $E_0<0$ hence

$$\|e^{iHt}W\psi_0\|_{L^p}=\|e^{iHt}We^{-iHt}e^{iE_0t}\psi_0\|_{L^p}\leqslant C\|\widehat{W}\|_{L^1}\|\psi_0\|_{L^p},$$

where again we used Theorem 5.2.

## 5.2 Smoothness of the effective potential

In this section we will prove Proposition 2.2 i.e. $\widehat{g'(\psi_E)}$ and $(\widehat{\frac{g(\psi_E)}{\psi_E}})$

From by Corollary 2.1, we have $\psi_E\in H^2$ which implies $\psi_E\in L^p$ for $2\le p\le\infty$. Also from (1.3), by integrating, we get $|g'(s)|\le C(|s|^{1+\alpha_1}+|s|^{1+\alpha_2})$. Hence $|g'(\psi_E)|\le C(|\psi_E|^{1+\alpha_1}+|\psi_E|^{1+\alpha_2})\in L^2$ and $|g''(\psi_E)|\le C(|\psi_E|^{\alpha_1}+|\psi_E|^{\alpha_2})\in L^\infty$. Now we have

$$\begin{aligned}\|\widehat{g'(\psi_E)}\|_{L^1}&=\|\frac{1}{1+|\xi|^2}(1+|\xi|^2)\widehat{g'(\psi_E)}\|_{L^1}\\&\le\|\frac{1}{1+|\xi|^2}\|_{L^2}\|(1+|\xi|^2)\widehat{g'(\psi_E)}\|_{L^2}\\&\le C(\|\widehat{g'(\psi_E)}\|_{L^2}+\|\widehat{\Delta g'(\psi_E)}\|_{L^2})\\&\le C(\|\underbrace{g'(\psi_E)}_{\in L^2}\|_{L^2}+\|\Delta g'(\psi_E)\|_{L^2})\end{aligned}$$

So it suffices to show that $\Delta g'(\psi_E)\in L^2$. Similarly it is enough to show that $\Delta(\frac{g(\psi_E)}{\psi_E})\in L^2$.

$$\Delta g'(\psi_E)=g'''(\psi_E)|\nabla\psi_E|^2+\underbrace{g''(\psi_E)}_{\in L^\infty}\underbrace{\Delta\psi_E}_{\in L^2}\tag{5.3}$$

and

$$\Delta(\frac{g(\psi_E)}{\psi_E})=(\frac{g''(\psi_E)}{\psi_E}-2\frac{g'(\psi_E)}{\psi_E^2}+2\frac{g(\psi_E)}{\psi_E^3})|\nabla\psi_E|^2+\underbrace{(\frac{g'(\psi_E)}{\psi_E}-\frac{g(\psi_E)}{\psi_E^2})}_{\in L^\infty}\underbrace{\Delta\psi_E}_{\in L^2}\tag{5.4}$$

We will use the following comparison theorem proved in [8, Theorem 2.1] to get the upper bound for the $\nabla\psi_E$ and lower bound for $\psi_E$:

**Theorem 5.3** *Let $\varphi\ge 0$ be continuous on $\overline{\mathbb{R}^3\setminus K}$ and $A\ge B\ge 0$ for some closed set $K$. Suppose that on $\overline{\mathbb{R}^3\setminus K}$, in the distributional sense,*

$$\Delta|\psi|\ge A|\psi|;\qquad \Delta\varphi\le B\varphi$$

*and that $|\psi|\le\varphi$ on $\partial K$ and $\psi$, $\varphi\to 0$ as $x\to\infty$. Then $|\psi|\le\varphi$ on all of $\mathbb{R}^3\setminus K$.*

Note that $\frac{\partial \psi_E}{\partial x_1}$ and $\psi_E$ are continuous and $\frac{\partial \psi_E}{\partial x_1}$, $\psi_E \to 0$ as $|x| \to \infty$. Hence $\left|\frac{g(\psi_E)}{\psi_E}\right| \leq C(|\psi_E|^{1+\alpha_1} + |\psi_E|^{1+\alpha_2}) \to 0$ as $x \to \infty$.

First we need the standard upper bound for $\psi_E \geq 0$. For any $A < -E$, there exists $C_A$ depending on $A$ such that $\psi_E \leq C_A e^{-\sqrt{A}|x|}$. Indeed if $R$ is sufficiently large, on $\overline{\mathbb{R}^3 \setminus B(0,R)}$ we have

$$\Delta \psi_E = [-E + V(x) + \frac{g(\psi_E)}{\psi_E}]\psi_E \geq A\psi_E, \quad \text{and} \quad \Delta\varphi = A\varphi - \frac{2\sqrt{A}}{|x|}\varphi \leq A\varphi$$

and on $\partial B(0,R)$ we have $\psi_E \leq C_A e^{-\sqrt{A}|x|}$ for $C_A$ big enough. Then by Theorem 5.3 we have $\psi_E \leq C_A e^{-\sqrt{A}|x|}$ on $\mathbb{R}^3 \setminus B(0,R)$.
To get the lower bound for $\psi_E$ we will choose $\varphi = \psi_E$ and $\psi = Ce^{-\sqrt{A_2}|x|}$ in Theorem 5.3. On $\overline{\mathbb{R}^3 \setminus B(0,R)}$, fix $\varepsilon > 0$, $A_2 \geq -E + 2\varepsilon$ and choose $R$ large enough such that $\frac{2\sqrt{A_2}}{|x|} \leq \varepsilon$ for $|x| \geq R$. Then from (2.2) we have

$$\Delta\psi_E = [-E + V(x)]\psi_E + g(\psi_E) \leq [-E + V + \frac{g(\psi_E)}{\psi_E}]\psi_E \leq (-E + \varepsilon)\psi_E$$

and for $A_2 \geq -E + 2\varepsilon$ we have

$$\Delta\psi = A_2\psi - \frac{2\sqrt{A_2}}{|x|}\psi \geq (-E + \varepsilon)\psi$$

Choose $C$ such that $Ce^{-\sqrt{A_2}|x|} \leq \psi_E$ on $\partial B(0,R)$. Then by theorem 5.3, we have $Ce^{-\sqrt{A_2}|x|} \leq \psi_E$ for $|x| > R$.
We will show that for $\psi = \frac{\partial \psi_E}{\partial x_1}$ and $\varphi = Ce^{-\sqrt{A_1}|x|}$ where $A_1 < -E$ hypothesis of the theorem 5.3 is satisfied.
Differentiating the eigenvalue equation (2.2) with respect to $x_1$ we get

$$\Delta\frac{\partial\psi_E}{\partial x_1} = [-E + V(x)]\frac{\partial\psi_E}{\partial x_1} + g'(\psi_E)\frac{\partial\psi_E}{\partial x_1} + \frac{\partial V}{\partial x_1}\psi_E$$

Let

$$f^{\pm} = \max\{0, \pm f\} \quad \text{and} \quad S_{\leq} = \{x \in \mathbb{R}^3 | \left|\frac{\partial\psi_E}{\partial x_1}\right| \leq \psi_E\} \quad \text{and} \quad S_{\geq} = \{x \in \mathbb{R}^3 | \left|\frac{\partial\psi_E}{\partial x_1}\right| \geq \psi_E\}$$

Fix $A_1 < -E$, choose $R$ large enough such that $-E + V(x) + g'(\psi_E) - \left|\frac{\partial V}{\partial x_1}\right| \geq A_1$ on $|x| \geq R$. Let $S = S_{\leq} \cup B(0,R)$, then on $\overline{\mathbb{R} \setminus S}$ we have

$$\Delta\left|\frac{\partial\psi_E}{\partial x_1}\right| \geq A_1\left|\frac{\partial\psi_E}{\partial x_1}\right|$$

Now, by continuity of $\frac{\partial\psi_E}{\partial x_1}$ there exists $C_1$ such that $\left|\frac{\partial\psi_E}{\partial x_1}\right| e^{\sqrt{A_1}|x|} \leq C_1$ on $|x| = R$. Since both on $\frac{\partial\psi_E}{\partial x_1}$ and $\psi_E$ are continuous we have $\left|\frac{\partial\psi_E}{\partial x_1}\right| = \psi_E \leq C_2 e^{\sqrt{A_1}|x|}$ on $\partial S_{\leq}$. So on $\partial S$, we have $\left|\frac{\partial\psi_E}{\partial x_1}\right| \leq \max\{C_1, C_2\} e^{\sqrt{A_1}|x|}$
Therefore by theorem 5.3, we have $|\nabla\psi_E| \leq Ce^{-\sqrt{A_1}|x|}$

Now we can prove Proposition 2.2

**Proof of Proposition 2.2** By **(H2')** we have $|g'''(s)| < \frac{C}{s^{1-\alpha_1}} + Cs^{\alpha_2 - 1}$, $s > 0$, $0 < \alpha_1 \leqslant \alpha_2$; then

$$|g'''(\psi_E)|\nabla\psi_E|^2| \leq \frac{C}{\psi_E^{1-\alpha_1}}|\nabla\psi_E|^2$$

and

$$|(\frac{g''(\psi_E)}{\psi_E} - 2\frac{g'(\psi_E)}{\psi_E^2} + 2\frac{g(\psi_E)}{\psi_E^3})|\nabla\psi_E|^2| \leq \frac{C}{\psi_E^{1-\alpha_1}}|\nabla\psi_E|^2$$

Using the estimates for $|\nabla\psi_E|$ and $\psi_E$ and choosing $2\sqrt{A_1} > \sqrt{A_2}$, we get that $\Delta g'(\psi_E), \Delta(\frac{g(\psi_E)}{\psi_E}) \in L^2$. Hence we get the desired estimates for $\widehat{g'(\psi_E)}$ and $\widehat{\frac{g(\psi_E)}{\psi_E}}$.

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