A Model of Heat Conduction

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Abstract

We define a deterministic "scattering" model for heat conduction which is *continuous* in space, and which has a Boltzmann type flavor, obtained by a closure based on memory loss between collisions. We prove that this model has, for stochastic driving forces at the boundary, close to Maxwellians, a unique non-equilibrium steady state.

1 Introduction

In this paper, we consider the problem of heat conduction in a model which is a *continuous* approximation of a discrete model of a chain of cells, each of which contains a (very simple) scatterer in its interior. Particles move between the cells, interacting with the scatterers, but not among themselves, similar to the model put forward in [3].

After a quite detailed description of this scatterer-model, we will finally arrive in Eq. (3.5) at a formulation with a continuous space variable x varying in $[0, 1]$. This approximation will be obtained by taking formally the limit of $N \to \infty$ cells, but taking each cell of length $1/N$. (The reader who is interested only in the formulation of the x-continuous equation can directly skip to Eq. (3.5).) We then proceed to show our main result, namely the existence of solutions to this Boltzmann-like equation for initial conditions close to equilibrium. Therefore, this will show that, although the model is deterministic (except for the boundary conditions), with no internal dissipation, it has a (unique) non-equilibrium steady state when driven weakly out of equilibrium. The only approximations of the model are the limit of $N \to \infty$, and a closure relation which models a loss of memory between collisions.

1.1 One cell

To define the model, we begin by describing the scattering process in one cell. We begin with the description of one "cell". The cell is 1-dimensional, of length $2L$, and with particles entering on either side. These particles have all mass m, velocity

v and momentum $p = mv$. These particles do not interact among themselves. Note that $v \in \mathbb{R}$ and more precisely, $v > 0$ if the particle enters from the left, while $v < 0$ if it enters on the right side of the cell. In the center of the cell, we imagine a "scatterer" which is a point-like particle which can exchange energy and momentum with the particles, but does not change its own position. (This scatterer is to be thought of as a 1-dimensional variant of the rotating disks used in [3].) The scatterer has mass M and its "velocity" will be denoted by V. The collision rules are those of an elastic collision, where $\tilde{v}$ and $\tilde{V}$ denote quantities after the collision while v, V are those before the collision. In equations,

$$\begin{aligned} \tilde{v} &= -\varrho v + (1+\varrho)V \ , \\ \tilde{V} &= (1-\varrho)v + \varrho V \ , \end{aligned}$$

with

$$\varrho = \frac{M-m}{M+m} \ , \qquad \mu \equiv \frac{m}{M} = \frac{1-\varrho}{1+\varrho} \ . \tag{1.1}$$

Note that $\varrho \in (-1, 1)$, since we assume m and M to be finite and non-zero. If $\tilde{v} > 0$, we say that the particle leaves the cell to the right; if $\tilde{v} < 0$, we say it leaves to the left.

For simplicity, we will assume $\varrho > 0$, that is, $M > m$. For the momenta, we get the analogous rules

$$\begin{aligned} \tilde{p} &= -\varrho p + (1-\varrho)P \ , \\ \tilde{P} &= (1+\varrho)p + \varrho P \ . \end{aligned}$$

Note that the matrix

$$S = \begin{pmatrix} -\varrho & 1-\varrho \\ 1+\varrho & \varrho \end{pmatrix} \tag{1.2}$$

has determinant equal to -1 and furthermore $S^2 = 1$.

We next formulate scattering in terms of probability densities (for momenta) for just one cell. We denote by $g(t, P)$ the probability density that at time t the scatterer has momentum P $(= MV)$ and we will establish the equation for the time evolution of this function. To begin with, we assume that particles enter only from the left of the cell, with momentum distribution (in a neighboring cell or a bath) $p \mapsto f_{\rm L}^{+}(t, p)$, where $p = mv$. Thus, there are, on average, $p f_{\rm L}^{+}(t, p){\rm d}p/m$ particles entering the cell (per unit of time) from the left with momentum in $[p, p + {\rm d}p]$. Note that $f_{\rm L}^{+}$ has support on $p \geq 0$ only, (indicated by the exponent "$+$"); it is the distribution of particles *going to enter the cell* from the left. Also note that the distribution of the momenta after collision, *i.e.*, before *leaving* the cell, is in general not the same as $f_{\rm L}^{+}$.

Denote by $\mathscr{P}(t)$ the stochastic process describing the momentum of the scatterer. We have for any interval (measurable set) of momenta A, for the probabilities **P**:

$$\mathbf{P}(\mathscr{P}(t + {\rm d}t) \in A) = \mathbf{P}(\mathscr{P}(t) \in A \ ; \ \text{no collision in } [t, t + {\rm d}t])$$

$$+\mathbf{P}(\mathscr{P}(t+\mathrm{d}t) \in A\ ;\ \text{collisions occurred in } [t, t+\mathrm{d}t])\ .$$

We assume for simplicity that with probability one, only one collision can occur in an interval $[t, t+\mathrm{d}t]$. If there is a collision in $[t, t+\mathrm{d}t]$ with a particle of velocity $v = p/m > 0$, this particle must have left the boundary at time $t - \frac{m}{p}L$ with momentum p. Therefore,

$$\begin{aligned}&\mathbf{P}(\mathscr{P}(t) \in A\ ;\ \text{a collision occurred in } [t, t+\mathrm{d}t])\\ &= \mathrm{d}t \int_A \mathrm{d}\tilde{P} \int_{\mathbb{R}^+} \mathrm{d}p\, \delta(\tilde{P} - \varrho P - (1+\varrho)p)\, g(t,P)\, \tfrac{p}{m} f_{\mathrm{L}}^+(t - \tfrac{m}{p}L, p)\ .\end{aligned}$$

This equation neglects memory effects coming from the fact that a particle may have hit the scatterer, bounce out of the cell and reenter to hit again the scatterer. Similarly,

$$\begin{aligned}&\mathbf{P}(\mathscr{P}(t) \in A\ ;\ \text{no collision occurred in } [t, t+\mathrm{d}t])\\ &= \left(1 - \mathrm{d}t \int_{\mathbb{R}^+} \mathrm{d}p\, \tfrac{p}{m} f_{\mathrm{L}}^+(t - \tfrac{m}{p}L, p)\right) \int_A \mathrm{d}P\, g(t,P)\ .\end{aligned}$$

We immediately deduce the evolution equation,

$$\begin{aligned}\partial_t g(t,P) = &- g(t,P) \int_{\mathbb{R}^+} \mathrm{d}p\, \tfrac{p}{m} f_{\mathrm{L}}^+(t - \tfrac{m}{p}L, p)\\ &+ \frac{1}{\varrho} \int_{\mathbb{R}^+} \mathrm{d}p\, g(t, \tfrac{P-(1+\varrho)p}{\varrho})\, \tfrac{p}{m}\, f_{\mathrm{L}}^+(t - \tfrac{m}{p}L, p)\ .\end{aligned} \tag{1.3}$$

Note that this equation preserves the integral of g over P, *i.e.*, it preserves probability.

This identity generalizes immediately to the inclusion of injection from the right, with distribution f_{R}^- having support in $p < 0$. One gets

$$\begin{aligned}\partial_t g(t,P) = &-g(t,P) \int_{\mathbb{R}} \mathrm{d}p\, \tfrac{|p|}{m} \left(f_{\mathrm{L}}^+(t - \tfrac{m}{p}L, p) + f_{\mathrm{R}}^-(t + \tfrac{m}{p}L, p)\right)\\ &+ \frac{1}{\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g(t, \tfrac{P-(1+\varrho)p}{\varrho})\, \tfrac{|p|}{m} \left(f_{\mathrm{L}}^+(t - \tfrac{m}{p}L, p) + f_{\mathrm{R}}^-(t + \tfrac{m}{p}L, p)\right)\ .\end{aligned} \tag{1.4}$$

In the stationary case, this leads to

$$g(P) = \frac{1}{\varrho\lambda} \int_{\mathbb{R}} \mathrm{d}p\, g(\tfrac{P-(1+\varrho)p}{\varrho})\, \tfrac{|p|}{m} \left(f_{\mathrm{L}}^+(p) + f_{\mathrm{R}}^-(p)\right)\ , \tag{1.5}$$

where

$$\lambda = \int_{\mathbb{R}} \mathrm{d}p\, \tfrac{|p|}{m} \left(f_{\mathrm{L}}^+(p) + f_{\mathrm{R}}^-(p)\right) \tag{1.6}$$

is the particle flux (see Sect. 3.1 below).

It is important to note that the solution g of Eq. (1.5) only depends on the *sum*: $f = f_{\mathrm{L}}^+ + f_{\mathrm{R}}^-$, and thus, we can define a map

$$f \mapsto g_f\ ,$$

where g_f is the (unique) solution of Eq. (1.5). It will be discussed in detail in Sect. 5.

We can also compute the distribution of the momenta of the particles after collision. We have

$$\mathbf{P}(\tilde{p} \in A\ ;\ \text{a collision occurred in } [t, t+\mathrm{d}t])$$

$$= \mathrm{d}t \int_A \mathrm{d}\tilde{p} \int_{\mathbb{R}} \mathrm{d}P\, \delta(\tilde{p} + \varrho p - (1-\varrho)P)\, g(t,P)\, \tfrac{|p|}{m} f(t - \tfrac{m}{|p|}L, p)\ .$$

This particle reaches the left or right boundary of the cell (according to the sign of $\tilde{p}$) after a time $mL/|\tilde{p}|$ (assuming the scatterer is located in the center of the cell). Therefore, we have for the ejection distributions f_{L}^- (on the left) and f_{R}^+ (on the right):

$$\tfrac{|\tilde{p}|}{m} f_{\mathrm{L}}^-(t,\tilde{p}) = \frac{\theta(-\tilde{p})}{1-\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g(t - \tfrac{m}{|\tilde{p}|}L, \tfrac{\tilde{p}+\varrho p}{1-\varrho}) \tfrac{|p|}{m}\, f(t - \tfrac{m}{|\tilde{p}|}L - \tfrac{m}{|p|}L, p)\ ,$$

and

$$\tfrac{|\tilde{p}|}{m} f_{\mathrm{R}}^+(t,\tilde{p}) = \frac{\theta(+\tilde{p})}{1-\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g(t - \tfrac{m}{|\tilde{p}|}L, \tfrac{\tilde{p}+\varrho p}{1-\varrho}) \tfrac{|p|}{m}\, f(t - \tfrac{m}{|\tilde{p}|}L - \tfrac{m}{|p|}L, p)\ ,$$

where θ is the Heaviside function. In the stationary case we get

$$\tfrac{|\tilde{p}|}{m} f_{\mathrm{L}}^-(\tilde{p}) = \frac{\theta(-\tilde{p})}{1-\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g(\tfrac{\tilde{p}+\varrho p}{1-\varrho})\, \tfrac{|p|}{m} f(p)\ , \tag{1.7}$$

and

$$\tfrac{|\tilde{p}|}{m} f_{\mathrm{R}}^+(\tilde{p}) = \frac{\theta(+\tilde{p})}{1-\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g(\tfrac{\tilde{p}+\varrho p}{1-\varrho})\, \tfrac{|p|}{m} f(p)\ . \tag{1.8}$$

Since $g = g_f$ is determined by the incoming distribution $f_{\mathrm{in}} = f_{\mathrm{L}}^+ + f_{\mathrm{R}}^-$ (and is unique if we normalize the integral of g to 1)

$$\int_{\mathbb{R}} \mathrm{d}P\, g(t,P) = 1\ , \tag{1.9}$$

we see that the outgoing distribution $f_{\mathrm{out}} = f_{\mathrm{L}}^- + f_{\mathrm{R}}^+$ is entirely determined by the incoming distribution. Note also that the flux is preserved:

$$\int \mathrm{d}p\, \tfrac{|p|}{m} f_{\mathrm{in}}(p)\ =\ \int \mathrm{d}p\, \tfrac{|p|}{m} f_{\mathrm{out}}(p)\ .$$

1.2 Stationary solutions for one cell

Here, we will look for stationary states of the evolution equation (1.3), which have also the property that the ejected distributions are equal to the injected ones. It is almost obvious that Maxwellian fixed points can be found, but for completeness, we write down the formulas. The reader should note that the distributions f_{in} and f_{out} have singularities at $p = 0$. This reflects the well-known fact that slow particles need very more time to leave the cell than fast ones. However $F(p) \equiv \frac{p}{m} f(p)$ is a very nice function, and it is this function which appears in all the calculations of the fluxes, and stationary profiles. In this section we do the calculations with the quantity f. Starting from Sect. 3, we will use F.

We impose the two incoming distributions

$$f_{\rm L}^{+}(p) = \sigma\theta(+p)\tfrac{m}{|p|}e^{-\beta p^2/(2m)} \ ,$$

and

$$f_{\rm R}^{-}(p) = \sigma\theta(-p)\tfrac{m}{|p|}e^{-\beta p^2/(2m)} \ ,$$

where σ is an arbitrary positive constant (related to λ in (1.6)) and θ is the Heaviside function. It is easy to verify, using Gaussian integration and the identity $M = M\varrho^2 + m(1+\varrho)^2$, that the solution of equation (1.5) is given by

$$g(P) = \sqrt{\frac{\beta}{2\pi M}}e^{-\beta P^2/(2M)} = \sqrt{\frac{\beta}{2\pi M}}e^{-\beta P^2(1-\varrho)/((1+\varrho)2m)} \ .$$

Moreover, using the same identity several times one gets from Eqs.(1.7) and (1.8) for the exiting distributions

$$f_{\rm L}^{-}(p) = \sigma\theta(-p)\tfrac{m}{|p|}e^{-\beta p^2/(2m)} \ ,$$

and

$$f_{\rm R}^{+}(p) = \sigma\theta(+p)\tfrac{m}{|p|}e^{-\beta p^2/(2m)} \ .$$

Therefore, we see that the Maxwellian fixed points (divided by $|p|$) preserve both the distribution g of the scatterer, as well as the distributions of the particles.

In fact, there are also non-Maxwellian fixed points of the form

$$f_{\rm L}^{+}(p) = \sigma\theta(+p)\tfrac{m}{|p|}e^{-\beta(p-ma)^2/(2m)} \ ,$$

and

$$f_{\rm R}^{-}(p) = \sigma\theta(-p)\tfrac{m}{|p|}e^{-\beta(p-ma)^2/(2m)} \ .$$

It is easy to verify that the solution of equation (1.5) is now given by

$$g(P) = \sqrt{\frac{\beta}{2\pi M}}e^{-\beta(P-Ma)^2/(2M)} \ .$$

Moreover,

$$f_{\rm L}^{-}(p) = \sigma\theta(-p)\tfrac{m}{|p|}e^{-\beta(p-ma)^2/(2m)} \ ,$$

and

$$f_{\rm R}^{+}(p) = \sigma\theta(+p)\tfrac{m}{|p|}e^{-\beta(p-ma)^2/(2m)} \ .$$

The verification that this is a solution for any $a \in \mathbb{R}$ is again by Gaussian integration. Note that if $a \neq 0$ there is in fact a flux through the cell.

2 N cells

The model generalizes immediately to the case of N cells which are arranged in a row, by requiring that the exit distributions of any given cell are equal to the entry distributions of the neighboring cells: The cells are numbered from 1 to N and we have the collections of functions $f^+_{\mathrm{L},i}$, $f^-_{\mathrm{L},i}$, $f^+_{\mathrm{R},i}$, and $f^-_{\mathrm{R},i}$ for the particle fluxes and g_i for the scatterers, $i = 1, \dots, N$. The equality of entrance and exit distributions is given by the identities $f^+_{\mathrm{L},i+1} = f^+_{\mathrm{R},i}$, and $f^-_{\mathrm{R},i} = f^-_{\mathrm{L},i+1}$ for $1 \le i < N$. The system is completely determined by the two functions $f^+_{\mathrm{L},1}$ and $f^-_{\mathrm{R},N}$. The equations (1.5) generalize to

$$g_i(P) = \frac{1}{\varrho\lambda} \int_{\mathbb{R}} \mathrm{d}p\, g_i\big(\tfrac{P-(1+\varrho)p}{\varrho}\big) \tfrac{|p|}{m} \left(f^+_{\mathrm{L},i}(p) + f^-_{\mathrm{R},i}(p) \right) , \tag{2.1}$$

and similarly (1.7) and (1.8) lead to

$$\begin{aligned} \tfrac{|\tilde p|}{m} f^-_{\mathrm{L},i}(\tilde p) &= \frac{\theta(-\tilde p)}{1-\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g_i\big(\tfrac{\tilde p + \varrho p}{1-\varrho}\big)\, \tfrac{|p|}{m} f_i(p) , \\ \tfrac{|\tilde p|}{m} f^+_{\mathrm{R},i}(\tilde p) &= \frac{\theta(+\tilde p)}{1-\varrho} \int_{\mathbb{R}} \mathrm{d}p\, g_i\big(\tfrac{\tilde p + \varrho p}{1-\varrho}\big)\, \tfrac{|p|}{m} f_i(p) , \end{aligned} \tag{2.2}$$

where $f_i = f^+_{\mathrm{L},i} + f^-_{\mathrm{R},i}$. Clearly, the Gaussians of the previous section are still solutions to the full equations for N contiguous cells.

Here we have closed the model by assuming independence between the particles leaving and entering from the left (and from the right). In concrete systems this is not true since a particle can leave a cell to the left and re-bounce back into the original cell after just one collision with the scatterer in the neighboring cell, and, in such a situation there is too much memory to allow for full independence. It is possible to imagine several experimental arrangements for which independence is a very good approximation, see also [3, 2] for discussions of such issues. One of them could be to imagine long channels between the scatterers where time decorrelation would produce independence. Note that "chaotic" channels may be more complicated since they can modify the distribution of left (right) traveling particles between two cells.

3 Continuous space

We are now ready to formulate the model in its final form. The cells are now replaced by a continuum, with a variable $x \in [0, 1]$ and the relations we have derived so far will be generalized to this continuum formulation. So we have moving particles, of mass m and described by a time-dependent density $f(t, p, x)$.

The scatterers have mass M and their momentum distribution is called $g(t, P, x)$. It is best to think that the continuous variable $x \in [0, 1]$ replaces the discrete index $i \in \{0, \dots, N\}$. There is then an implicit rescaling of the form $x \approx i/N$. Recall that the scatterers are *fixed* in space (although they have momentum) but that the particles will move in the domain $[0, 1]$.

We first impose, for all $x \in [0, 1]$, the normalization

$$\int_{\mathbb{R}} \mathrm{d}P\, g(t, P, x) = 1 \ , \tag{3.1}$$

which is the generalization of Eq.(1.9). The particles again do not interact with each other, but only with the scatterers and, expressed in momenta, the matrix maps (p, P) to $(\tilde{p}, \tilde{P})$:

$$\begin{pmatrix} \tilde{p} \\ \tilde{P} \end{pmatrix} = \begin{pmatrix} -\varrho & (1-\varrho) \\ (1+\varrho) & \varrho \end{pmatrix} \begin{pmatrix} p \\ P \end{pmatrix} \equiv S \begin{pmatrix} p \\ P \end{pmatrix} \ , \tag{3.2}$$

as in Eq.(1.2).

We now rewrite the problem in the form of a Boltzmann equation, which takes into account this matrix, as well as the particle transport. One obtains, with $(\tilde{p}, \tilde{P})$ related to (p, P) as above:

$$\begin{aligned} &\partial_t f(t,p,x) + \frac{p}{m}\partial_x f(t,p,x) \\ &\qquad = \int \mathrm{d}P \left(\frac{|\tilde{p}|}{m} f(t,\tilde{p},x)\, g(t,\tilde{P},x) - \frac{|p|}{m} f(t,p,x)\, g(t,P,x) \right) , \\ &\partial_t g(t,P,x) = \int \mathrm{d}p \left(\frac{|\tilde{p}|}{m} f(t,\tilde{p},x)\, g(t,\tilde{P},x) - \frac{|p|}{m} f(t,p,x)\, g(t,P,x) \right) . \end{aligned} \tag{3.3}$$

The time independent version of the equation will be derived below from the model with a chain of cells. It is useful to introduce the function

$$F(t,p,x) = \frac{|p|}{m} f(t,p,x) \ ,$$

and then Eq.(3.3) takes the form

$$\begin{aligned} &m\partial_t F(t,p,x) + p\partial_x F(t,p,x) \\ &\qquad = |p| \int \mathrm{d}P \left(F(t,\tilde{p},x)\, g(t,\tilde{P},x) - F(t,p,x)\, g(t,P,x) \right) , \\ &\partial_t g(t,P,x) = \int \mathrm{d}p \left(F(t,\tilde{p},x)\, g(t,\tilde{P},x) - F(t,p,x)\, g(t,P,x) \right) . \end{aligned} \tag{3.4}$$

Remark. One can also imitate a scattering cross section by introducing a factor $\gamma \in [0, 1]$ in Eq.(3.4) (in the integrals) but this can be scaled away by a change of time and space scales. See also Sect. 8.3.

We come now to the **main equations** whose solutions will be discussed in detail in the remainder of the paper. The equation (3.4) takes, for the stationary solution, the form

$$\mathrm{sign}(p)\partial_x F(p,x) = \int \mathrm{d}P \left(F(\tilde{p},x)\, g(\tilde{P},x) - F(p,x)\, g(P,x) \right) , \tag{3.5a}$$

$$0 = \int \mathrm{d}p \, \Big(F(\tilde{p}, x)\, g(\tilde{P}, x) - F(p, x)\, g(P, x) \Big) \ . \tag{3.5b}$$

We will show that this equation has non-equilibrium solutions.

Remark. Note that the model we have obtained here is *not* momentum-translation invariant, because of the term sign(p), except when the r.h.s. of the equation is 0.

Derivation of (3.5). The derivation of (3.5) from (2.1–2.2) is based on the following formal limit: We replace the index i by the continuous variable $x = i/N$ and set $\varepsilon = 1/(2N)$. We consider that $f^{\pm}_{\mathrm{L},i}$ is at $(i - \frac{1}{2})/N = x - \varepsilon$, while $f^{\pm}_{\mathrm{R},i}$ is at $x + \varepsilon$. We have the correspondences, with $\theta_{\pm}(p) \equiv \theta(\pm p)$:

$$\begin{aligned}
\theta_+(p)F(p, x - \varepsilon) &= \tfrac{|p|}{m} f^+_{\mathrm{L},i} \ , \qquad & \theta_-(p)F(p, x - \varepsilon) &= \tfrac{|p|}{m} f^-_{\mathrm{L},i} \ , \\
\theta_+(p)F(p, x + \varepsilon) &= \tfrac{|p|}{m} f^+_{\mathrm{R},i} \ , \qquad & \theta_-(p)F(p, x + \varepsilon) &= \tfrac{|p|}{m} f^-_{\mathrm{R},i} \ , \\
g(P, x) &= g_i(P).
\end{aligned}$$

To simplify momentarily the notation, let

$$\begin{aligned}
F_-(p, x)\ \theta_+(p) &\equiv \theta_+(p)F(p, x - \varepsilon) \ , \quad & F_-(p, x)\ \theta_-(p) &\equiv \theta_-(p)F(p, x - \varepsilon) \ , \\
F_+(p, x)\ \theta_+(p) &\equiv \theta_+(p)F(p, x + \varepsilon) \ , \quad & F_+(p, x)\ \theta_-(p) &\equiv \theta_-(p)F(p, x + \varepsilon) \ .
\end{aligned}$$

With these conventions, (2.1) becomes (setting $\lambda = 1$):

$$g(P, x) \ = \ \frac{1}{\varrho} \int_{\mathbb{R}} \mathrm{d}q \, g(\tfrac{P-(1+\varrho)q}{\varrho}, x)\big(F_-(q, x)\theta_+(q) + F_+(q, x)\theta_-(q)\big) \ , \tag{3.6}$$

which is equivalent to (3.5b). Similarly, Eq.(2.2) leads to

$$\begin{aligned}
F_-(p, x)\theta_-(p) &= \frac{\theta(-p)}{1 - \varrho} \int_{\mathbb{R}} \mathrm{d}q \, g(\tfrac{p+\varrho q}{1-\varrho}, x) \big(F_-(q, x)\theta_+(q) + F_+(q, x)\theta_-(q)\big) \ , \\
F_+(p, x)\theta_+(p) &= \frac{\theta(+p)}{1 - \varrho} \int_{\mathbb{R}} \mathrm{d}q \, g(\tfrac{p+\varrho q}{1-\varrho}, x) \big(F_-(q, x)\theta_+(q) + F_+(q, x)\theta_-(q)\big) \ .
\end{aligned} \tag{3.7}$$

Subtracting the first equation from the second in (3.7) leads to

$$\begin{aligned}
&F_+(p, x)\theta_+(p) - F_-(p, x)\theta_-(p) \\
&\quad = \frac{\mathrm{sign}(p)}{1 - \varrho} \int_{\mathbb{R}} \mathrm{d}q \, g(\tfrac{p+\varrho q}{1-\varrho}, x) \big(F_-(q, x)\theta_+(q) + F_+(q, x)\theta_-(q)\big) \ .
\end{aligned} \tag{3.8}$$

On the other hand, since $\mathrm{d}p\, \mathrm{d}P = \mathrm{d}\tilde{p}\, \mathrm{d}\tilde{P}$ we can, by (3.5b), impose the condition

$$\int \mathrm{d}P \, g(P, x) = 1 \ ,$$

for all x. Then we have the trivial identity

$$F_-(p, x)\theta_+(p) - F_+(p, x)\theta_-(p) = \int_{\mathbb{R}} \mathrm{d}P \, g(P, x)\big(F_-(p, x)\theta_+(p) - F_+(p, x)\theta_-(p)\big) \ . \tag{3.9}$$

Subtracting (3.9) from (3.8), we get for $p > 0$,

$$F_+(p,x) - F_-(p,x) = \frac{1}{1-\varrho}\int \mathrm{d}q\, g(\tfrac{p+\varrho q}{1-\varrho},x)(F_-(q,x)\theta_+(q) + F_+(q,x)\theta_-(q)) - \int \mathrm{d}q\, g(q,x)F_-(p,x) \tag{3.10}$$

while for $p < 0$,

$$F_+(p,x) - F_-(p,x) = -\frac{1}{1-\varrho}\int \mathrm{d}q\, g(\tfrac{p+\varrho q}{1-\varrho},x)(F_-(q,x)\theta_+(q) + F_+(q,x)\theta_-(q)) + \int \mathrm{d}q\, g(q,x)F_+(p,x) \tag{3.11}$$

Note now that

$$\begin{aligned} F_-(q,x)\theta_+(q) + F_+(q,x)\theta_-(q) &= F(q,x-\varepsilon)\theta_+(q) + F(q,x+\varepsilon)\theta_-(q) \\ &= F(q,x) + \theta_+(q)(F(q,x-\varepsilon) - F(q,x)) \\ &\quad + \theta_-(q)(F(q,x+\varepsilon) - F(q,x)) . \end{aligned}$$

Therefore, replacing the $F_\pm$ in the r.h.s. in (3.10) and (3.11) by $F(p,x)$ is a higher order correction in ε, and we finally get

$$F(p,x+\varepsilon) - F(p,x-\varepsilon) = \frac{\text{sign}(p)}{1-\varrho}\int_{\mathbb{R}} \mathrm{d}q \left(g(\tfrac{p+\varrho q}{1-\varrho},x)F(q,x) - g(q,x)F(p,x) \right) .$$

A further change of integration variables leads to (3.5a), while (3.6) leads to (3.5b). (We have not taken into account the scaling by $\varepsilon = 1/(2N)$ which is needed to get the derivative; we will come back to this question in the discussion in Sect. 8.3.) This ends the derivation of (3.5).

The derivative term in Eq.(3.5) reflects the gradients which have to appear when the system is out of equilibrium. However, if the system is at equilibrium, the equivalence between Eq.(3.5) and Eqs.(1.5)–(1.8) immediately tells us that stationary solutions in the form of Gaussians (for F, not for f) exist:

$$F(p) = \gamma\sqrt{\frac{\beta}{2\pi m}}e^{-\beta p^2/(2m)} , \qquad g(P) = \sqrt{\frac{\beta}{2\pi M}}e^{-\beta P^2/(2M)} . \tag{3.12}$$

Furthermore, we have again translated versions of this fixed point,

$$F(p) = \gamma\sqrt{\frac{\beta}{2\pi m}}e^{-\beta(p-ma)^2/(2m)} , \qquad g(P) = \sqrt{\frac{\beta}{2\pi M}}e^{-\beta(P-Ma)^2/(2M)} , \tag{3.13}$$

because in this case, the r.h.s. of Eq.(3.5) is zero.

3.1 Flux

We can define various fluxes of the particles (recall that $F(p,x) = |p| f(p,x)/m$):

$$\begin{aligned}\Phi_{\mathrm{P}} &= \text{particle flux} = \int \mathrm{d}p \, \mathrm{sign}(p) \, F(p,x) \; , \\ \Phi_{\mathrm{M}} &= \text{momentum activity} = \int \mathrm{d}p \, |p| \, F(p,x) \; , \\ \Phi_{\mathrm{E}} &= \text{energy flux} = \int \mathrm{d}p \, \frac{\mathrm{sign}(p) p^2}{2m} \, F(p,x) \; .\end{aligned} \tag{3.14}$$

Note that for the stationary Maxwellians of (3.13) these fluxes are equal to

$$\begin{aligned}\Phi_{\mathrm{P}} &= \sqrt{\frac{2m\pi}{\beta}} \; \mathrm{erf}(a\sqrt{\beta m/2}) \; , \\ \Phi_{\mathrm{M}} &= \frac{2m}{\beta} e^{-\beta m a^2/2} + a\sqrt{\frac{2m^3\pi}{\beta}} \mathrm{erf}(a\sqrt{\beta m/2}) \; , \\ \Phi_{\mathrm{E}} &= \frac{2am^2}{\beta} e^{-\beta m a^2/2} + (1+\beta m a^2)\sqrt{\frac{2m^3\pi}{\beta^3}} \mathrm{erf}(a\sqrt{\beta m/2}) \; .\end{aligned}$$

Also note that for $a = 0$ the quantity Φ_{M} does not vanish. This is because it measures the total outgoing flux, not the directed outgoing flux (which is of course 0 when $a = 0$).

Lemma 3.1. *For every stationary solution of (3.5) the 3 fluxes of (3.14) are independent of* $x \in [0,1]$.

Proof. From (3.5a) we deduce that

$$\partial_x \int \mathrm{d}p \, \mathrm{sign}(p) \, F(p,x) = \int \mathrm{d}p \, \mathrm{d}P \, \Big(F(\tilde p, x) \, g(\tilde P, x) - F(p,x) \, g(P,x) \Big) \; ,$$

which vanishes since $\mathrm{d}p \, \mathrm{d}P = \mathrm{d}\tilde p \, \mathrm{d}\tilde P$. Similarly, multiplying (3.5a) by p and integrating over p, we get

$$\partial_x \int \mathrm{d}p \, |p| \, F(p,x) = \int \mathrm{d}p \, \mathrm{d}P \;\; p \, \Big(F(\tilde p, x) \, g(\tilde P, x) - F(p,x) \, g(P,x) \Big) \; .$$

Multiplying (3.5b) by P and integrating over P, we get

$$0 = \int \mathrm{d}p \, \mathrm{d}P \;\; P \, \Big(F(\tilde p, x) \, g(\tilde P, x) - F(p,x) \, g(P,x) \Big) \; .$$

Adding these two equations, we see that

$$\partial_x \int \mathrm{d}p \, |p| \, F(p,x) = \int \mathrm{d}p \, \mathrm{d}P \;\; (p+P) \, \Big(F(\tilde p, x) \, g(\tilde P, x) - F(p,x) \, g(P,x) \Big) \; .$$

But this vanishes, since $P + p = \tilde{P} + \tilde{p}$ by momentum conservation, and using again $\mathrm{d}p\,\mathrm{d}P = \mathrm{d}\tilde{p}\,\mathrm{d}\tilde{P}$. In a similar way, we first have

$$\begin{aligned}&\partial_x \int \mathrm{d}p\, \frac{|p|p}{2m}\, F(p,x)\\ &= \int \mathrm{d}p\, \mathrm{d}P\; \frac{p^2}{2m} \Big(F(\tilde{p},x)\, g(\tilde{P},x) - F(p,x)\, g(P,x) \Big)\end{aligned}$$

Finally, multiplying this time (3.5b) by P^2/M, integrating over P, adding to the above equation and using energy conservation, we get

$$\begin{aligned}&\partial_x \int \mathrm{d}p\, \frac{|p|p}{2m}\, F(p,x)\\ &= \int \mathrm{d}p\, \mathrm{d}P\; \left(\frac{p^2}{2m} + \frac{P^2}{2M} \right) \Big(F(\tilde{p},x)\, g(\tilde{P},x) - F(p,x)\, g(P,x) \Big)\\ &= 0\ .\end{aligned}$$

Thus, all three fluxes are independent of x, as asserted. □

4 Formulating the heat-conduction problem

Based on the stationarity equation (3.5), we now formulate the problem of heat conduction in mathematical terms. We imagine that the input of the problem is given by prescribing the *incoming* fluxes on both sides of the system. The system, in its stationary state, should then adapt all the other quantities, F and g, to this given input, which describes really the forcing of the system. In particular, the distribution of the outgoing fluxes will be entirely determined by the incoming fluxes.

We now formulate this question in mathematical terms: The incoming fluxes are described by two functions $\mathscr{F}_0(p)$ (defined for $p \geq 0$) and $\mathscr{F}_1(p)$ (defined for $p \leq 0$). These are the incoming distributions on the left end (index 0) and the right end (index 1) of the system.

In terms of these 2 functions, the problem of existence of a stationary state can be formulated as (recall that the rescaled system has length one):

Is there a solution (F, g) of the equations (3.5) with the boundary conditions

$$F(p,0) = \mathscr{F}_0(p)\ ,\ \ \forall p \geq 0 \qquad \textit{and} \qquad F(p,1) = \mathscr{F}_1(p)\ ,\ \ \forall p \leq 0\ . \tag{4.1}$$

Assume for a moment that, instead of the boundary conditions (4.1) we were given just $F(p,0)$, but now *for all* $p \in \mathbb{R}$, not only for $p > 0$. Assume furthermore, that $g(p,x)$ is determined by (3.5b). In that case, the relation (3.5) can be written as a dynamical system in the variable x:

$$\partial_x F(\cdot,x) = \mathscr{X}(F(\cdot,x))\ . \tag{4.2}$$

Thus, if $F(\cdot,0)$ is given, then, in principle, $F(\cdot,1)$ is determined (uniquely) by Eq.(4.2), provided such a solution exists. We denote this map by $\mathscr{Y}_0$:

$$\mathscr{Y}_0 : F(\cdot,0) \mapsto F(\cdot,1) \ .$$

What is of interest for our problem is the restriction of the image of $\mathscr{Y}$ to functions of negative p only, since that corresponds to the incoming particles from the right side, and so we define

$$\Bigl(\mathscr{Y}(F(\cdot,0))\Bigr)(p) \equiv \theta_-(p) \ \cdot \ \Bigl(\mathscr{Y}_0(F(\cdot,0))\Bigr)(p) = \theta_-(p)\cdot F(p,1) \ .$$

Using this map $\mathscr{Y}$, we will show that when $F(\cdot,0)$ varies in a small neighborhood the map $\mathscr{Y}$ is invertible on its image. By taking inverses the problem of heat conduction for our model will be solved for small temperature and flux difference.

Of course, this needs a careful study of the function space on which $\mathscr{Y}$ is supposed to act. This will be done below.

To formulate the problem more precisely, we change notation, and let

$$\begin{aligned}
F_0^+(p) &= \theta_+(p)F(p,0) \ ,\\
F_0^-(p) &= \theta_-(p)F(p,0) \ ,\\
F_1^+(p) &= \theta_+(p)F(p,1) \ ,\\
F_1^-(p) &= \theta_-(p)F(p,1) \ .
\end{aligned}$$

We assume now that F_0^+ is *fixed* once and for all and omit it from the notation. Then, we see that $\mathscr{Y}$ can be interpreted as a map which maps the function F_0^- to F_1^-, and we call this map Φ.

We will show below that for F_0^- in a small neighborhood $\mathcal{D}$ the map Φ is 1-1 onto its image $\Phi(\mathcal{D})$ and can therefore be inverted. For any $\hat{F}$ in $\Phi(\mathcal{D})$, we can take $F_0^- = \Phi^{-1}(\hat{F})$, and we will have solved the problem of existence of heat flux.

5 The g equation

We start here with the study of existence of g for given F. Since (3.5b) does not couple different x, we can fix x. The equation (3.5b) is then equivalent to

$$g = \mathcal{A}_F(g) \ ,$$

where the operator $\mathcal{A}_F$ acting on the function h is defined by

$$\mathcal{A}_F(h)(\tilde{P},x) = \frac{\int \mathrm{d}p\, F(\tilde{p}(p,\tilde{P}),x)h(P(p,\tilde{P}),x)}{\int \mathrm{d}p\, F(p,x)} \ ,$$

(provided the denominator does not vanish). Note that for fixed $\tilde{P}$ and p, we can solve the collision system (3.2) to find the corresponding P and $\tilde{p}$, namely

$$P = \frac{1}{\varrho}\,\tilde{P} - \frac{1+\varrho}{\varrho}\,p \qquad \text{and} \qquad \tilde{p} = \frac{1-\varrho}{\varrho}\,\tilde{P} - \frac{1}{\varrho}\,p \ .$$

The action of the operator $\mathcal{A}_F$ can then be rewritten as

$$\mathcal{A}_F(h)(\tilde{P},x) = \frac{\varrho^{-1}\int \mathrm{d}p\, F(p,x)h(\tilde{P}/\varrho - (1+\varrho)p/\varrho, x)}{\int \mathrm{d}s\, F(s,x)} \ . \tag{5.1}$$

In order to study this operator notice that it does not depend explicitly on x. It is convenient to study instead a family of operators indexed by functions φ of the momentum only. We define (assuming the integral of φ does not vanish)

$$(\mathscr{L}_\varphi \psi)(\tilde{P}) = \frac{\int \mathrm{d}p\, \varphi(p)\psi(\tilde{P}/\varrho - (1+\varrho)p/\varrho)}{\varrho \int \mathrm{d}p\, \varphi(p)} \ .$$

A final change of variables will be useful when we study $\mathscr{L}_\varphi$:

$$(\mathscr{L}_\varphi \psi)(p) = \frac{\int \mathrm{d}q\, \varphi(\frac{p-\varrho q}{1+\varrho})\psi(q)}{(1+\varrho)\int \mathrm{d}q\, \varphi(q)} \ . \tag{5.2}$$

6 The mathematical setup and the main result

Having formulated the problem of existence of the stationary solution in general, we now fix the mathematical framework in which we can prove this existence. This framework, while quite general, depends nevertheless on a certain number of technical assumptions which we formulate now.

We fix once and for all the ratio $\mu = m/M$ of the masses, and assume, for definiteness, that $\mu \in (0,1)$. It seems that this condition is not really necessary, and probably the condition $m \neq M$ (and the masses non-zero) should work as well, but we have not pursued this.

We next describe a condition on the incoming distribution, called F in the earlier sections. The basic idea, inspired from the equilibrium calculations, is that $F(p,x)$ should be close to

$$F_{\text{reference}}(p) = \exp(-\beta p^2/(2m)) \equiv \exp(-\alpha p^2) \ ,$$

while the derived quantity $g(p,x)$ should be close to

$$g_{\text{reference}}(p) = \exp(-\beta p^2/(2M)) \equiv \exp(-\mu\alpha p^2) \ .$$

Upon rescaling p, we may assume henceforth that $\alpha = 1$.

The operators of the earlier sections will now be described in spaces with weights

$$\mathcal{W}_\nu(p) = \exp(-\nu p^2) \ ,$$

where we will choose $\nu = 1$ for the F and $\nu = \mu$ for the g.

We recall that the operator $\mathscr{L}_F$ in "flat" space is

$$(\mathscr{L}_F g)(p) \ = \ \frac{\int \mathrm{d}q\, F(\frac{p-\varrho q}{1+\varrho})g(q)}{(1+\varrho)\int \mathrm{d}q F(q)} \ .$$

We then define the integral kernel in the space with weights $\exp(-p^2)$ for F and $\exp(-\mu p^2)$ for g, and write

$$F(p) \;=\; e^{-p^2} v(p)\;, \qquad g(p) \;=\; e^{-\mu p^2} u(p)\;.$$

Here, $\mu = m/M = (1-\varrho)/(1+\varrho)$, as before. Expressed with u and v the operator $\mathscr{L}_F$ takes the form

$$(K_v u)(p) = \frac{1}{(1+\varrho)\int \mathrm{d}q e^{-q^2} v(q)} \cdot (L_v u)(p)\;, \tag{6.1}$$

where

$$(L_v u)(p) \;=\; \int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q) u(q)\;,$$

and

$$K(p,q) = \mathcal{W}_1(\tfrac{p-\varrho q}{1+\varrho}) \cdot \mathcal{W}_\mu(q)/\mathcal{W}_\mu(p)\;.$$

A simple calculation shows that

$$K(p,q) \;=\; e^{-(\varrho p-q)^2/(1+\varrho)^2}\;. \tag{6.2}$$

Our task will be to understand under which conditions the linear operator $\mathscr{L}_F$ has an eigenvalue 1. This will be done by showing that K_v is quasi-compact. It is here that we were not able to give reasonable bounds on $K(p,q)$ in the case of different exponentials for $p > 0$ and $p < 0$, which represents different temperatures for ingoing and outgoing particles.

6.1 Function spaces

We now define spaces which are adapted to the simultaneous requirement of functions being close to a Gaussian near $|p| = \infty$ and u and v having limits, *and* K_v being quasi-compact. We define a space $\mathscr{G}_1$ of functions u with norm

$$\|u\|_{\mathscr{G}_1} \;=\; \int e^{-\mu p^2} |u(p)| \mathrm{d}p\;.$$

Similarly, $\mathscr{F}_1$ is the space of functions v with norm

$$\|v\|_{\mathscr{F}_1} \;=\; \int e^{-p^2} |v(p)| \mathrm{d}p\;.$$

Thus, the only difference is the absence of the factor $\mu = (1-\varrho)/(1+\varrho)$ in the exponent.

We also define a smaller space $\mathscr{G}_2$, contained in $\mathscr{G}_1$, with the norm

$$\|u\|_{\mathscr{G}_2} = \int |\mathrm{d}u(p)| + \int e^{-\mu p^2} |u(p)| \mathrm{d}p\;,$$

and the analogous space $\mathscr{F}_2$ contained in $\mathscr{F}_1$ with the norm

$$\|v\|_{\mathscr{F}_2} = \int |\mathrm{d}v(p)| + \int e^{-p^2} |v(p)| \mathrm{d}p\;,$$

Remark. To simplify notation we write $\int |\mathrm{d}u(p)|$ instead of the variation norm. However, the "integration by parts" formula would hold with the "correct" definition of variation as well.

Lemma 6.1. *One has the inclusion $\mathscr{G}_2 \subset \mathrm{L}^\infty$, and more precisely*

$$\|u\|_{\mathrm{L}^\infty} \le \int |\mathrm{d}u| + e^\mu \int e^{-\mu p^2} |u(p)| \mathrm{d}p \le e^\mu \|u\|_{\mathscr{G}_2} .$$

Furthermore, if $u \in \mathscr{G}_2$, then $\lim_{p\to\pm\infty} u(p)$ exists. The maps $u \mapsto \lim_{p\to\pm\infty} u(p)$ and $u \mapsto \int \mathrm{d}p \, \exp(-\mu p^2) \cdot u(p)$ are continuous functions from $\mathscr{G}_2$ to $\mathbb{R}$. The unit ball of $\mathscr{G}_2$ is compact in $\mathscr{G}_1$.

Analogous statements hold for the spaces $\mathscr{F}_2$ (defined without the factor μ).

Proof. The first statement is easy, but it will be convenient to have the explicit estimates. We have

$$u(y) - u(x) = \int_{[x,y]} \mathrm{d}u ,$$

and therefore

$$|u(x)| \le \int |\mathrm{d}u| + \int_{-1/2}^{1/2} |u(y)| \mathrm{d}y \le \int |\mathrm{d}u| + e^\mu \int e^{-\mu p^2} |u(p)| \mathrm{d}p .$$

The second statement follows at once since the functions in $\mathscr{G}_2$ are of bounded variation.

For the last assertions, it follows from the inclusion in L^∞ that the unit ball of $\mathscr{G}_2$ is equi-integrable at infinity in $\mathrm{L}^1(e^{-\mu p^2}\mathrm{d}p)$. Moreover, a set of uniformly bounded functions of uniformly bounded variation is compact in any $\mathrm{L}^1(K, \mathrm{d}p)$ for any compact subset K of $\mathbb{R}$ (see [1], Helly's selection principle). □

6.2 A cone in $\mathscr{F}_2$

We will work in the space $\mathscr{F}_2$ but we will need a cone (of positive functions, with adequate decay) in this space, in order to prove quasi-compactness of K_v.

We define a cone $\mathcal{C}_{\mathscr{F}}$ in $\mathscr{F}_2$ by the condition

$$\mathcal{C}_{\mathscr{F}} = \left\{ v \in \mathscr{F}_2 , \quad v \ge 0 \text{ and } Z \cdot \lim_{p\to\pm\infty} v(p) < 1 \right\} , \tag{6.3}$$

where

$$Z = Z(v) = \frac{\sqrt{\pi}}{\int \mathrm{d}p \; e^{-p^2} v(p)} . \tag{6.4}$$

Lemma 6.2. *The cone $\mathcal{C}_{\mathscr{F}}$ has non empty interior (in $\mathscr{F}_2$) and is convex.*

Proof. By Lemma 6.1 the maps $v \mapsto \lim_{p\to\pm\infty} v(p)$ and $v \mapsto \int \exp(-p^2) v(p) \mathrm{d}p$ are continuous in $\mathscr{F}_2$ and hence the assertion follows. □

Remark. Note that a function in the interior of the cone is necessarily bounded away from zero, since at infinity it must have a non-zero limit and in any compact set, if it is never zero, it is bounded away from zero.

Remark. Note that the function $v \equiv 1$ (the Gaussian) is *not* in the cone $\mathcal{C}_{\mathscr{F}}$. In fact, we require that $\lim_{p\to\pm\infty} F(p) e^{p^2} \cdot \int e^{-p'^2} \mathrm{d}p' / \int F(p') \mathrm{d}p' < 1$.

6.3 The main result

On the set $\mathcal{C}_{\mathscr{F}}$, we consider now the spatial evolution equations (3.5) in the variables v and u_v (which is the solution of $K_v u = u$ with K_v defined in (6.1)):

$$\begin{aligned}
&\partial_x v(p,x) \\
&\quad = \text{sign}(p) \int \mathrm{d}P \Big((\mathcal{W}_1 \cdot v)(\tilde{p},x)\ (\mathcal{W}_\mu \cdot u_{v(\cdot,x)})(\tilde{P},x) \\
&\qquad\qquad - (\mathcal{W}_1 \cdot v)(p,x)\ (\mathcal{W}_\mu \cdot u_{v(\cdot,x)})(P,x) \Big) \\
&\quad = \text{sign}(p) \int \mathrm{d}P\, (\mathcal{W}_1 \cdot v)(-\varrho p + (1-\varrho)P, x)\, (\mathcal{W}_\mu \cdot u_{v(\cdot,x)})((1+\varrho)p + \varrho P, x) \\
&\qquad - \text{sign}(p)\ v(p,x) \int \mathrm{d}P\, e^{-\mu P^2} u_{v(\cdot,x)}(P,x)\ ,
\end{aligned} \tag{6.5}$$

with initial condition $v(\cdot, x = 0) \in \mathcal{C}_{\mathscr{F}}$. We will give a more explicit variant in (7.9).

Any solution of this equation is a function of p and x, and it is easy to verify that it satisfies the equation (3.5a). Together with the definition of u_v we have a complete solution of the nonlinear system (3.5). Here we assume of course that the r.h.s. of the above equation is well defined as a function, so that we can multiply by sign(p).

Theorem 6.3. *For any $v_0 \in \mathcal{C}_{\mathscr{F}}$, there are a number $x_{v_0} > 0$ and a neighborhood $\mathscr{V}_{v_0}$ of v_0 in $\mathcal{C}_{\mathscr{F}}$ such that the solution of (6.5) exists for any initial condition $v_0 = v(p,0) \in \mathscr{V}_{v_0}$ and for any x in the interval $[0, x_{v_0}]$. The function $v_0 \mapsto x_{v_0}$ is continuous from $\mathcal{C}_{\mathscr{F}}$ to $\mathbb{R}^+$ compactified at infinity. We denote by Φ_x the semi-flow integrating (6.5). For any $x \in [0, x_{v_0}]$, the map $\Phi_x : v \mapsto \Phi_x(v)$ is a local diffeomorphism, i.e., a diffeomorphism on $\mathscr{V}_{v_0}$.*

Note that this implies in particular that the probability densities for $g(\cdot, x)$ and $F(\cdot, x)$ remain positive for all $x \in [0, x_{v_0}]$, which is of course crucial from the physics point of view.

We will prove this in Sect. 7 (and in the appendix).

7 Bound on the operator K_v and proof of Theorem 6.3

These bounds are the crux of the matter. They actually show, that, under the conditions on $\mathscr{F}_2$ and the set $\mathcal{C}_{\mathscr{F}}$, the operator K_v is quasi-compact. In terms of the physical problem, this means that the scatterer is not heating up if the incoming fluxes are in $\mathscr{F}_2$.

The object of study is, for $v \in \mathcal{C}_{\mathscr{F}}$, the operator

$$(K_v u)(p) = \frac{1}{(1+\varrho)\int e^{-q^2} v(q)\mathrm{d}q} \int v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q) u(q) \mathrm{d}q\ .$$

and we are asking for a solution u of the equation $K_v(u) = u$.

Lemma 7.1. *If $v \geq 0$, K_v is a positive (nonnegative) operator, and*

$$\int \mathrm{d}p\, e^{-\mu p^2}(K_v u)(p) = \int \mathrm{d}p\, e^{-\mu p^2} u(p)$$

and

$$\|K_v\|_{\mathscr{G}_1} = 1 \ .$$

Proof. Easy, compute and take absolute values. Alternately, consider that the probability is conserved in the original space. ☐

Our main technical result is

Proposition 7.2. *For $v \in \mathcal{C}_{\mathscr{F}}$, there exist a ζ, $0 \leq \zeta < 1$ and an $R > 0$ (both depend on v continuously) such that for any $u \in \mathscr{G}_2$ one has the bound*

$$\int |\mathrm{d}K_v(u)| \leq \zeta \int |\mathrm{d}u| + R\|u\|_{\mathscr{G}_1} \ .$$

Proof. Since v will be fixed throughout the study of K_v, it will be useful to introduce the abbreviation $Q = Q_v$ for the normalizing factor

$$Q = \frac{1}{(1+\varrho)\int e^{-q^2} v(q)\mathrm{d}q} \ . \tag{7.1}$$

We will use a family of smooth cut-off functions χ_L ($L > 1$) which are equal to 1 on $[-L+\frac{1}{2}, L-\frac{1}{2}]$ and which vanish on $|q| > L+\frac{1}{2}$. Let Θ be a C^∞ function satisfying $0 \leq \Theta \leq 1$, with $\Theta(q) = 0$ for $q \leq -\frac{1}{2}$ and $\Theta(q) = 1$ for $q \geq \frac{1}{2}$. We define χ_L by

$$\chi_L(q) = \begin{cases} \Theta(q+L) & \text{if } q \leq -L+\frac{1}{2} \ , \\ 1 & \text{if } -L+\frac{1}{2} \leq q \leq L-\frac{1}{2} \ , \\ \Theta(L-q) & \text{if } q \geq L-\frac{1}{2} \ . \end{cases}$$

The functions χ_L are C^∞, satisfy $0 \leq \chi_L \leq 1$ and $\|\chi_L'\|_{\mathrm{L}^\infty}$ is independent of L. Let L_1 and L_2 be two positive numbers to be chosen large enough later on (depending on v). We will use the partition of unity

$$1 = \chi_L + \chi_L^\perp \ .$$

Using this decomposition of unity with $L = L_1$ and $L = L_2$, we write $K_v = K_v^{(1)} + K_v^{(2)} + K_v^{(3)}$ with

$$(K_v^{(1)}u)(p) = Q\int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho})K(p,q)u(q)\cdot \chi_{L_1}(q) \ ,$$

$$(K_v^{(2)}u)(p) = Q\int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho})K(p,q)u(q)\cdot \chi_{L_1}^\perp(q)\, \chi_{L_2}(p-\varrho q) \ ,$$

$$(K_v^{(3)}u)(p) = Q\int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho})K(p,q)u(q)\cdot \chi_{L_1}^\perp(q)\, \chi_{L_2}^\perp(p-\varrho q) \ .$$

We will now estimate the variation of the three operators separately.

For the variation of the first term, we find

$$\begin{aligned}\mathrm{d}(K_v^{(1)}u)(p) = &+ Q\int \mathrm{d}q\,\mathrm{d}v(\tfrac{p-\varrho q}{1+\varrho})\tfrac{1}{1+\varrho}\cdot K(p,q)u(q)\chi_{L_1}(q)\\ &+ Q\,\mathrm{d}p\int \mathrm{d}q\,v(\tfrac{p-\varrho q}{1+\varrho})\partial_p K(p,q)\cdot u(q)\chi_{L_1}(q)\ .\end{aligned}$$

Using the explicit form of $K(p,q)$ (see Eq.(6.2)), and some easy bounds which we defer to the Appendix, we get the bound

$$\begin{aligned}\int|\mathrm{d}(K_v^{(1)}u)(p)| &\le \mathcal{O}(1)Q\left(\|v\|_{\mathrm{L}^\infty}+\int|\mathrm{d}v|\right)\int_{-L_1-\frac{1}{2}}^{L_1+\frac{1}{2}}|u(q)|\mathrm{d}q\\ &\le \mathcal{O}(1)Q\left(\|v\|_{\mathrm{L}^\infty}+\int|\mathrm{d}v|\right)e^{\mu(L_1+\frac{1}{2})^2}\int e^{-\mu q^2}|u(q)|\mathrm{d}q\\ &\le \ \mathrm{const.}\|u\|_{\mathscr{G}_1}\cdot\|v\|_{\mathscr{F}_2}\ .\end{aligned}\tag{7.2}$$

The variation of $K_v^{(2)}$ leads to three terms:

$$\begin{aligned}&\mathrm{d}(K_v^{(2)}u)(p)\\ &\quad= Q\,\mathrm{d}p\int\mathrm{d}q\,v(\tfrac{p-\varrho q}{1+\varrho})K(p,q)u(q)\cdot\chi_{L_1}^{\perp}(q)\,\chi'_{L_2}(p-\varrho q)\\ &\quad+Q\int\mathrm{d}q\,\mathrm{d}v(\tfrac{p-\varrho q}{1+\varrho})\tfrac{1}{1+\varrho}\cdot K(p,q)u(q)\cdot\chi_{L_1}^{\perp}(q)\,\chi_{L_2}(p-\varrho q)\\ &\quad+Q\,\mathrm{d}p\int\mathrm{d}q\,v(\tfrac{p-\varrho q}{1+\varrho})\ \partial_p K(p,q)\ \cdot u(q)\cdot\chi_{L_1}^{\perp}(q)\,\chi_{L_2}(p-\varrho q)\\ &\quad:=\mathrm{d}J_{21}+\mathrm{d}J_{22}+\mathrm{d}J_{23}\ .\end{aligned}$$

In these terms, the variables p and q are in the domain

$$\mathcal{D}=\{(p,q)\in\mathbb{R}^2\ :\ |p-\varrho q|<L_2+\tfrac{1}{2}\text{ and }|q|>L_1-\tfrac{1}{2}\}\ ,$$

and for $L_1=3\varrho L_2/(1-\varrho^2)$ and L_2 sufficiently large we have from Lemma A.3:

$$K(p,q)<\exp\left(-C_1(\varrho p-q)^2-C_2L_2^2\right)\ .\tag{7.3}$$

Therefore, we get for $\mathrm{d}J_{21}$:

$$\int|\mathrm{d}J_{21}|\le\mathrm{const.}e^{-C_2L_2^2}\|u\|_\infty\|v\|_\infty\int_{(p,q)\in\mathcal{D}}\mathrm{d}p\,\mathrm{d}q e^{-C_1(\varrho p-q)^2}\ .\tag{7.4}$$

The integral exists and is uniformly bounded in L_2 (since $|\varrho p-q|\to\infty$ when $|q|\to\infty$).

The term $\mathrm{d}J_{22}$ is handled in a similar way and leads to the bound

$$\int|\mathrm{d}J_{22}|\le\mathrm{const.}e^{-C_2L_2^2}\|u\|_\infty\int|\mathrm{d}v|\ .\tag{7.5}$$

The following identity is useful:

$$\partial_p K(p,q) = -\varrho \partial_q K(p,q)\ . \tag{7.6}$$

For the term $\mathrm{d}J_{23}$ we observe that from (7.6) one gets, upon integrating by parts, with the notation

$$\begin{aligned}\mathrm{d}J_{23} =& \varrho Q\,\mathrm{d}p \int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q)\cdot \partial_q \left(\chi^{\perp}_{L_1}(q)\chi_{L_2}(p-\varrho q)\right)\\ &+ \varrho Q\,\mathrm{d}p \int \mathrm{d}v(\tfrac{p-\varrho q}{1+\varrho})\tfrac{-\varrho}{1+\varrho} K(p,q)u(q)\cdot \chi^{\perp}_{L_1}(q)\chi_{L_2}(p-\varrho q)\\ &+ \varrho Q\,\mathrm{d}p \int v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q)\mathrm{d}u(q)\cdot \chi^{\perp}_{L_1}(q)\chi_{L_2}(p-\varrho q)\\ :=& \mathrm{d}J_{231} + \mathrm{d}J_{232} + \mathrm{d}J_{233}\ .\end{aligned}$$

All these terms are localized in the domain $\mathcal{D}$. In $\mathrm{d}J_{231}$ there appears a derivative

$$\begin{aligned}X =& \partial_q \left(\chi^{\perp}_{L_1}(q)\chi_{L_2}(p-\varrho q)\right)\\ =& -\chi'_{L_1}(q)\chi_{L_2}(p-\varrho q)\\ &- \varrho\chi^{\perp}_{L_1}(q)\chi'_{L_2}(p-\varrho q)\\ :=& X_1 + X_2\ .\end{aligned}$$

The terms involving X_1 and X_2 can be bounded as $\mathrm{d}J_{21}$ and $\mathrm{d}J_{22}$ by observing that $\mathrm{supp}\chi'_{L_1} \subset \{|q| < L_1 + \frac{1}{2}\}$, and similarly for X_2.

The terms $\mathrm{d}J_{232}$ and $\mathrm{d}J_{233}$ are bounded similarly.

Together, these lead to a bound

$$\int |\mathrm{d}K^{(2)}_v(u)| \le \mathrm{const.} e^{-C_2 L_2^2} \|v\|_{\mathscr{F}_2} \|u\|_{\mathscr{G}_2}\ . \tag{7.7}$$

Remark. Note that in this term, the norm $\|u\|_{\mathscr{G}_2}$ appears with a *small* coefficient, while in (7.2) it was $\|u\|_{\mathscr{G}_1}$ (with a large coefficient).

Finally, we estimate the total variation of $K^{(3)}_v(u)$ and here, the nature of the set $\mathcal{C}_{\mathscr{F}}$ will be important. We have

$$\begin{aligned}&\mathrm{d}(K^{(3)}_v u)(p)\\ &= Q \int \mathrm{d}q\, \mathrm{d}v(\tfrac{p-\varrho q}{1+\varrho})\tfrac{1}{1+\varrho} K(p,q)u(q)\cdot \chi^{\perp}_{L_1}(q)\,\chi^{\perp}_{L_2}(p-\varrho q)\\ &- Q\,\mathrm{d}p \int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q)u(q)\cdot \chi^{\perp}_{L_1}(q)\,\chi'_{L_2}(p-\varrho q)\\ &+ Q\,\mathrm{d}p \int \mathrm{d}q\, v(\tfrac{p-\varrho q}{1+\varrho}) \partial_p K(p,q)\cdot u(q)\cdot \chi^{\perp}_{L_1}(q)\,\chi^{\perp}_{L_2}(p-\varrho q)\\ &:= \mathrm{d}J_{31} + \mathrm{d}J_{32} + \mathrm{d}J_{33}\ .\end{aligned}$$

The critical term is $\mathrm{d}J_{33}$, but we first deal with the two others which are treated similar to earlier cases.

For the first term we have by Lemma A.4 which tells us that K is exponentially bounded on $\mathcal{D}'$:

$$\int |\mathrm{d}J_{31}| \leq \text{const.} \|u\|_{\mathrm{L}^\infty} \int_{|s|>(L_2-\frac{1}{2})/(1+\varrho)} |\mathrm{d}v(s)| \; .$$

where $\mathcal{D}'$ is the domain

$$\mathcal{D}' = \{(p,q) \; : \; |q| > L_1 \text{ and } |p-\varrho q| > L_2\} \; .$$

For the second term, we have, again by Lemma A.4 below,

$$\int |\mathrm{d}J_{32}| \leq \text{const.} \|u\|_{\mathrm{L}^\infty} \, \|v\|_{\mathrm{L}^\infty}$$

The last term is more delicate, and uses the property $Z \cdot \lim_{p\to\pm\infty} v(p) < 1$ of the definition of the cone $vcone$, Eq.(6.3). integrate by parts as before using (7.6) and get

$$\begin{aligned}
\mathrm{d}J_{33} =& \varrho Q \,\mathrm{d}p \int \mathrm{d}v(\tfrac{p-\varrho q}{1+\varrho}) \tfrac{-\varrho}{1+\varrho} K(p,q) u(q) \cdot \chi^{\perp}_{L_1}(q) \, \chi^{\perp}_{L_2}(p-\varrho q) \\
& \varrho Q \,\mathrm{d}p \int \mathrm{d}q \, v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q) u(q) \cdot \partial_q(\chi_{L_1}(q) \chi^{\perp}_{L_2}(p-\varrho q)) \\
& \varrho Q \mathrm{d}p \int v(\tfrac{p-\varrho q}{1+\varrho}) K(p,q) \mathrm{d}u(q) \cdot \chi^{\perp}_{L_1}(q) \, \chi^{\perp}_{L_2}(p-\varrho q) \\
:=& \, \mathrm{d}J_{331} + \mathrm{d}J_{332} + \mathrm{d}J_{333} \; .
\end{aligned}$$

The term $\mathrm{d}J_{331}$ is bounded like $\mathrm{d}J_{31}$.

In a similar way $\mathrm{d}J_{332}$ and $\mathrm{d}J_{32}$ are bounded by the same methods.

The term $\mathrm{d}J_{333}$ makes use of the limit condition in $\mathcal{C}_{\mathscr{F}}$. Consider the integral of $|\mathrm{d}J_{333}|$. This leads to a bound and setting $L_2' = (L_2 - \frac{1}{2})/(1+\varrho)$:

$$\begin{aligned}
\int |\mathrm{d}J_{333}(p)| \leq & \, \varrho \, Q \sup_{|s|>L_2'} |v(s)| \cdot \int |\mathrm{d}u(q)| \\
& \cdot \int \mathrm{d}p \, K(p,q) \cdot \chi^{\perp}_{L_1}(q) \, \chi^{\perp}_{L_2}(p-\varrho q) \\
\leq & \, \varrho \, Q \left(\sup_{|q|>L_1-\frac{1}{2}} \int \mathrm{d}p \, K(p,q) \right) \cdot \sup_{|s|>L_2'} |v(s)| \int |\mathrm{d}u| \qquad (7.8) \\
= & \frac{\sqrt{\pi}}{\int \mathrm{d}q e^{-q^2} v(q)} \cdot \sup_{|s|>L_2'} |v(s)| \int |\mathrm{d}u| \\
= & \, Z \cdot \sup_{|s|>L_2'} |v(s)| \int |\mathrm{d}u| \; ,
\end{aligned}$$

where Z was defined in Eq.(6.4). Collecting all the estimates, we get

$$\int |\mathrm{d}K_v(u)| \leq C \int e^{-\mu q^2} |u(q)| \mathrm{d}q + \zeta(L_2) \int |\mathrm{d}u|$$

where

$$\zeta(L_2) = \mathcal{O}(1) e^{-C_2 L_2^2} \|v\|_{\mathscr{F}_2} + \mathcal{O}(1) \int_{|s|>L_2'} |\mathrm{d}v(s)| + Z \cdot \sup_{|s|>L_2'} |v(s)| \ .$$

Since v belongs to $\mathcal{C}_{\mathscr{F}}$, it follows that

$$\lim_{L_2 \to \infty} \zeta(L_2) < 1 \ ,$$

and the Lemma follows by taking L_2 large enough. □

Proposition 7.3. *For any $v \in \mathcal{C}_{\mathscr{F}}$, the equation $K_v(u) = u$ has a solution in $\mathscr{G}_2$. This solution can be chosen positive, it is then unique if we impose $\|u\|_{\mathscr{G}_1} = 1$. We call it u_v. The map $v \mapsto u_v$ is differentiable.*

Proof. We apply the theorem of Ionescu-Tulcea and Marinescu [4] to prove the existence of u. Since for $v > 0$, the operator K_v is positivity improving, it follows by a well known argument, see *e.g.*, [6] that the peripheral spectrum consists only of the simple eigenvalue one and the eigenvector can be chosen positive. If normalized, it is then unique. Since the operator K_v depends linearly and continuously on v (in $\mathscr{F}_2$), the last result follows by analytic perturbation theory (see [5]). □

We next consider the equation (6.5) for v:

$$\partial_x v(p) = \mathrm{sign}(p) \left(\frac{1}{\varrho} \int e^{(1-\varrho^2)(p-q/(1+\varrho))^2/\varrho^2} v((1-\varrho)q - p)/\varrho) \, u_v(q) \, \mathrm{d}q \right.$$
$$\left. - v(p) \int e^{-\mu q^2} u_v(q) \, \mathrm{d}q \right) \ . \qquad (7.9)$$

Proposition 7.4. *The r.h.s. of the equation for v is a C^1 vector field on $\mathscr{F}_2$.*

Proof. This follows easily from the fact that the map $v \mapsto u_v$ is C^1. □

Theorem 7.5. *Let $v_0 \in \mathcal{C}_{\mathscr{F}}$, and assume that v_0 is bounded below away from zero and has nonzero limits at $\pm\infty$. Then there is a number $s = s(v) > 0$ such that the solution of equation (7.9) with initial condition v_0 exists in $\mathscr{F}_2$ and is nonnegative (moreover, it belongs to $\mathcal{C}_{\mathscr{F}}$).*

Proof. Follows at once from the previous proposition and the fact that v_0 is in the interior of $\mathcal{C}_{\mathscr{F}}$. □

The proof of Theorem 6.3 is now completed by observing that the map $\Phi : v_0 \mapsto \Phi(v_0)$ is indeed a local diffeomorphism, since it is given as the solution of an evolution equation.

8 Remarks and Discussion

8.1 The behavior of the solution at $p = \infty$

Consider the limit $p \to \infty$ in the expression for K_v. We need $\varrho p - q = \mathcal{O}(1)$ otherwise the Gaussian gives a negligible contribution. In other words, $q \sim \varrho p$, and we are going to assume from now on that $\varrho > 0$ (the other case can be treated analogously). This implies $p - \varrho q \sim (1 - \varrho^2)p$ which also tends to infinity (the same infinity). Therefore,

$$K_v u(\pm\infty) = \frac{\sqrt{\pi}\, v(\pm\infty)\, u(\pm\infty)}{\int e^{-p^2} v(p) \mathrm{d}p} \, .$$

In particular, if $K_v u = u$ and since we assumed

$$\frac{\sqrt{\pi}\, v(\pm\infty)}{\int e^{-p^2} v(p) \mathrm{d}p} \neq 1$$

we get $u(\pm\infty) = 0$.

For the v equation, we have for large p, $q \sim p(1+\varrho)$ and $(1-\varrho)q - p \sim -\varrho^2 p$. Therefore (inverting limit and derivative) we get

$$\partial_x v(\pm\infty) = \mathrm{sign}(\pm\infty) \left[\sqrt{\pi} \sqrt{\frac{1+\varrho}{1-\varrho}} u(\pm\infty)\, v(\mp\infty) - v(\pm\infty) \int e^{-\mu q^2} u(q) \mathrm{d}q \right] \, .$$

Note that the first term vanishes since $u(\pm\infty) = 0$. Since the integral $C(x) = \int e^{-\mu q^2} u(q, x)$ is positive, we conclude that formally,

$$\partial_x v(\pm\infty) = \mp v(\pm\infty) C(x) \, .$$

8.2 Essential spectrum

Conjecture. *The essential spectrum of K_v is the interval $[0, \sigma(v)]$ with*

$$\sigma(v) = \max \frac{\sqrt{\pi}\, v(\pm\infty)}{\int e^{-p^2} v(p) \mathrm{d}p} \, .$$

If $\sigma(v) < 1$ we are looking for an eigenvalue 1 outside the essential spectrum, which is the case we have treated. If $\sigma(v) > 1$ we would be looking for an eigenvalue 1 inside the essential spectrum which would be a much more difficult task, since it may well not exist.

Idea of proof: Similar to the above estimates, the operator K_v should be written as something small plus something compact plus something whose essential spectrum can be computed. This last part is likely to be the limit operator at infinity.

8.3 Dependence on N

It should be noted that the equation for $\partial_x F$ has, in fact a scaling of the form

$$N^{-1}\partial_x F = \mathcal{O}(1) + \mathcal{O}(N^{-1}) \ .$$

This means that in the main theorem (Theorem 6.3), the limit x_{v_0} of x for which we have a result is quite probably bounded by a quantity of the form $1/(N \cdot \Delta(v_0))$, where $\Delta(v_0)$ measures the deviation of the initial condition v_0 from a Gaussian. Thus, either x_{v_0} is very small when N is large, or one has to take v_0 very close to a Gaussian.

Another way to look at this scaling is to introduce a scattering probability $\gamma = b/N$ where $b > 0$ is a constant independent of N. In other words, a particle entering the array of cells from the left has for large N a probability e^{-b} to traverse all the N cells (and leave on the right) without having experienced any scattering. This is analogous to a rarefied gas. It is easy to verify that equation (1.4) is modified by a factor b/N multiplying the right hand side, and hence equation (1.5) is unchanged. The stationary equations (3.5) become

$$\begin{aligned}\tfrac{|\tilde p|}{m} f_{\mathrm{L}}^{-}(t,\tilde p) &= \theta(-\tilde p)\left(1-\frac{b}{N}\right)\tfrac{|\tilde p|}{m} f(t,\tilde p)\\ &+\frac{b}{N}\frac{\theta(-\tilde p)}{1-\varrho}\int_{\mathbb{R}} \mathrm{d}p\, g(t-mL/|\tilde p|, \tfrac{\tilde p+\varrho p}{1-\varrho})\tfrac{|p|}{m}\, f(t-\tfrac{m}{|\tilde p|}L-\tfrac{m}{|p|}L,p)\ ,\end{aligned}$$

and

$$\begin{aligned}\tfrac{|\tilde p|}{m} f_{\mathrm{R}}^{+}(t,\tilde p) &= \theta(+\tilde p)\left(1-\frac{b}{N}\right)\tfrac{|\tilde p|}{m} f(t,\tilde p)\\ &+\frac{b}{N}\frac{\theta(+\tilde p)}{1-\varrho}\int_{\mathbb{R}} \mathrm{d}p\, g(t-mL/|\tilde p|, \tfrac{\tilde p+\varrho p}{1-\varrho})\tfrac{|p|}{m}\, f(t-\tfrac{m}{|\tilde p|}L-\tfrac{m}{|p|}L,p)\ .\end{aligned}$$

Equation (3.5a) follows as explained in Section 3 after a rescaling of space by a factor b.

8.4 Discussion

The model presented in this paper has the nice property that one can control the existence of a solution out of equilibrium. In particular, this means that there is no heating up of the scatterers in the "chain", when the system is out of equilibrium.

The reader should note, however, that the initial condition at the boundary, does not allow for different temperatures in the strict sense, only for different distributions at the ends. For example, a function of the form

$$F(p,0) = \begin{cases}\exp(-\alpha p^2), & \text{if } p > 0,\\ \exp(-\alpha' p^2), & \text{if } p < 0,\end{cases}$$

with $\alpha \neq \alpha'$ is not covered by Theorem 6.3. The reason for this failure is that we could not find an adequate analog of Lemma A.4 for initial conditions of this type, and therefore the bounds on the kernel $K(p,q)$ are not good enough.

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A Appendix: Bounds on $K(p,q)$

We study here the kernel K of (6.2), which equals

$$K(p,q) = e^{E(p,q)} ,$$

with

$$E(p,q) \;=\; \mu p^2 - \mu q^2 - (\tfrac{p-\varrho q}{1+\varrho})^2 = -(\varrho p - q)^2/(1+\varrho)^2 \; . \tag{A.1}$$

Lemma A.1. *Assume* $|q| < L$. *There are constants* $C = C(L,\varrho)$ *and* $D = D(L,\varrho) > 0$ *such that for all* p,

$$K(p,q) < Ce^{-Dp^2} , \tag{A.2}$$

and

$$|\partial_p K(p,q)| < Ce^{-Dp^2}, \tag{A.3}$$

Proof. Obvious. □

Lemma A.2. *Assume* $|p - \varrho q| < L$. *There are constants* $C = C(L,\varrho)$ *and* $D = D(L,\varrho) > 0$ *such that for all* q,

$$K(p,q) < Ce^{-Dq^2} , \tag{A.4}$$

and

$$|\partial_p K(p,q)| < Ce^{-Dq^2}, \tag{A.5}$$

Proof. The proof is as in Lemma A.1, with the difference that now $|p-\varrho q| < L$. □

Lemma A.3. *Consider the domain $\mathcal{D}$ defined by*

$$\mathcal{D} = \{(p,q) \in \mathbb{R}^2 \ : \ |p-\varrho q| < L_2 + \tfrac{1}{2} \text{ and } |q| > L_1 - \tfrac{1}{2}\} \ , \tag{A.6}$$

with

$$L_1 = \frac{3\varrho}{1-\varrho^2} L_2 \ . \tag{A.7}$$

For fixed $\varrho \in (0,1)$ and sufficiently large L_2 there are positive constants C_1 and C_2 such that for $(p,q) \in \mathcal{D}$ one has the bound

$$K(p,q) < \exp\left(-C_1(\varrho p - q)^2 - C_2 L_2^2\right) \ .$$

Proof. From the definition of $\mathcal{D}$ and $(1-\varrho^2)q = (\varrho p - q) - \varrho(p - \varrho q)$, we find (for sufficiently large L_2):

$$|\varrho p - q| \geq (1-\varrho^2)|q| - \varrho|p-\varrho q| \geq (1-\varrho^2)(L_1 - \tfrac{1}{2}) - \varrho(L_2 + \tfrac{1}{2}) > \varrho L_2 \ . \tag{A.8}$$

Using the form

$$(\varrho p - q)^2 > \tfrac{1}{4}(\varrho p - q)^2 + \tfrac{1}{4}\varrho^2 L_2^2 \ , \tag{A.9}$$

the assertion follows immediately. □

We next study the region

$$\mathcal{D}' = \{(p,q) \ : \ |q| > L_1 \text{ and } |p - \varrho q| > L_2\} \ . \tag{A.10}$$

In this region, we have the obvious bound

Lemma A.4. *For $(p,q) \in \mathcal{D}'$, one has the bound*

$$E(p,q) = -(\tfrac{\varrho p - q}{1+\varrho})^2 \ .$$