Dimer $\lambda_3 = .453 \pm .001$ and Some Other Very Intelligent Guesses

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Abstract

Working with a presumed asymptotic series for λ_d developed in previous work, we make some intelligent guesses for λ_d with $d = 3, 4, 5$; and estimates for the corresponding errors. We present arguments in favor of these guesses, we earnestly believe they will turn out to be correct. Such approximate values may help stimulate people working on rigorous bounds. In addition to suggesting bounds to prove, there will be the strong motivation to prove me wrong.

In a previous paper, [1], we developed an asymptotic expansion for λ_d of the dimer problem in powers of $1/d$. In [2] computer calculations were performed to obtain some terms in this expansion, and also related quantities arising in the theory. Using results in [2] we herein will argue for the following estimates.

$$\begin{aligned} \lambda_2 &= .296 \pm .007 \qquad (1)\\ \lambda_3 &= .453 \pm .001 \qquad (2)\\ \lambda_4 &= .5748 \pm .0006 \qquad (3)\\ \lambda_5 &= .6785 \pm .0001 \qquad (4) \end{aligned}$$

This entire note is to argue for (2), (3), and (4). The resut for λ_2 is exactly known, consistent with (1), which is included as the bellwether example of our algorithm to obtain (1) - (4).

In [2] we obtained in dimensions 2 and 3 a series $B_0, ..., B_5$, eq. (41)-(52) of [2]. Using eq. (22)-(27) of [2] and the theory developed in [1] one can obtain such a series $B_0, ..., B_5$ in any dimension. Our algorithm to obtain (1)-(4) above, in given dimension d, is to seek the two successive $B's$, B_g and B_{g+1}, with minimum value of $|B_g - B_{g+1}|$. Then with

$$a = \frac{1}{2}(B_g + B_{g+1}) \tag{5}$$

and

$$b = |B_g - B_{g+1}| \tag{6}$$

our estimate is

$$\lambda_d = a \pm b. \tag{7}$$

The following table encapsulates the B series for $d = 2, 3, 4, 5$.

	$d = 2$	$d = 3$	$d = 4$	$d = 5$
B_0	.1931	.3959	.5397	.6513
B_1	.2556	.4375	.5710	.6763
B_2	.2921	.4538	.5801	.6821
B_3	.2993	.4524	.5781	.6803
B_4	.2906	.4468	.5751	.6786
B_5	.2814	.4445	.5745	.6785

It is generally believed that it is hard to compute the λ_d, $d > 2$, with great precision. We note that the elements B_5 in our table required weeks of computer time to evaluate, and we question whether the B_6 terms will ever be computed. We are only claiming λ_3, say, with a certain accuracy; we do not know how to get λ_3 with greater accuracy. Because we do not have the B_6 terms, it might be better to double the error bounds of (3) and (4) above.

We give the following arguments in favor of our algorithm, (5)-(7) above.

1) The B_i term in the series of $B's$ is the sum of the first i terms in a power series in x (set equal to 1) that is presumed asymptotic. Equations (5)-(7) encode the general rule of thumb wisdom for extracting a 'best value' of the sum of an asymptotic expansion in a simple way.

2) This algorithm gives a correct estimate for λ_2, eq. (1), which is known exactly. We take this as rather compelling.

3) B_g as a function of d is an expansion in powers $1/d$ up to power $1/d^g$. The asymptotic expansion of [1] for λ_d in powers of $1/d$ yields a 'best approximation' in high dimensions that is a power series in $1/d$ to a high power. For the approximation of our algorithm to match this approximation, *g must increase with dimension.*

We have found this last argument hard to explain here, and it would also be hard to write down a precise statement corresponding to the discussion (though it could be done). But for us the fact that $g = 2$ for $d = 2$ and $d = 3$, and $g = 4$ for $d = 4$ and $d = 5$, was as important as arguments 1) and 2) above. It gives the algorithm a final ring of truth.

References

[1] Paul Federbush, Hidden Structure in Tilings, Conjectured Asymptotic Expansion for λ_d in Multidimensional Dimer Problem, arXiv : 0711.1092V9 [math-ph].

[2] Paul Federbush, Dimer λ_d Expansion Computer Computations, arXiv : 0804.4220V1 [math-ph].