STABILITY OF THE PERIODIC TODA LATTICE: HIGHER ORDER ASYMPTOTICS

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Abstract. In a recent paper we have considered the long time asymptotics of the periodic Toda lattice under a short range perturbation and we have proved that the perturbed lattice asymptotically approaches a modulated lattice. In the present paper we capture the higher order asymptotics, at least away from some resonance regions. In particular we prove that the decay rate is $O(t^{-1/2})$.

Our proof relies on the asymptotic analysis of the associated Riemann–Hilbert factorization problem, which is here set on a hyperelliptic curve. As in previous studies of the free Toda lattice, the higher order asymptotics arise from "local" Riemann–Hilbert factorization problems on small crosses centered on the stationary phase points. We discover that the analysis of such a local problem can be done in a chart around each stationary phase point and reduces to a Riemann–Hilbert factorization problem on the complex plane. This result can then be pulled back to the hyperelliptic curve.

1. Introduction

This article is concerned with the long-time asymptotics of the doubly infinite Toda lattice. In Flaschka's variables (see e.g. [17], [18], or [20]) it reads

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \tag{1.1}$$

$(n,t) \in \mathbb{Z} \times \mathbb{R}$, where the dot denotes differentiation with respect to time. We will consider a quasi-periodic algebro-geometric background solution (a_q, b_q), to be described in the Appendix A, plus a short range perturbation (a, b) satisfying

$$\sum_n (1+|n|)^6 (|a(n,t) - a_q(n,t)| + |b(n,t) - b_q(n,t)|) < \infty. \tag{1.2}$$

If this condition is true for $t = 0$, it is true for all $t \in \mathbb{R}$ (see [5]). The perturbed solution can be computed via the inverse scattering transform. The free case where (a_q, b_q) is constant is classical (see again [17] or [20]) and the more general case we want to investigate here was solved only recently in [5] (see also [14]). The long-time asymptotics in the free case were first computed by Novokshenov and Habibullin [15] and later made rigorous by Kamvissis [8] under the the additional assumption that no solitons are present. The case of solitons was recently solved by Krüger and Teschl in [11]. For a review see Krüger and Teschl [12]. The leading asymptotic in the present situation were given by us in [10] (see also [9] for a short overview).

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Assume for simplicity that the Jacobi operator H_q corresponding to the perturbed problem (1.1) has no eigenvalues, that is, no solitons are present (the case with solitons will be given in [13]). In [10] we have proved that for long times the perturbed Toda lattice is asymptotically close to the following limiting lattice defined by

$$\begin{aligned}\prod_{j=n}^{\infty}\left(\frac{a_l(j,t)}{a_q(j,t)}\right)^2 =& \frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\frac{\theta(\underline{z}(n-1,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(n,t)+\underline{\delta}(n,t))}\times\\ &\times\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\omega_{\infty_+\,\infty_-}\right),\\ \sum_{j=n}^{\infty}\big(b_l(j,t)-b_q(j,t)\big) =& \frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\Omega_0\\ &+\frac{1}{2}\frac{d}{ds}\log\left(\frac{\theta(\underline{z}(n,s)+\underline{\delta}(n,t))}{\theta(\underline{z}(n,s))}\right)\Big|_{s=t},\\ \delta_\ell(n,t) =& \frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\zeta_\ell,\end{aligned} \tag{1.3}$$

where R is the associated reflection coefficient, ζ_ℓ is a canonical basis of holomorphic differentials, $\omega_{\infty_+\,\infty_-}$ is an Abelian differential of the third kind defined in (A.14), Ω_0 is an Abelian differential of the second kind defined in (A.15), $C(n/t) = \pi^{-1}(\sigma(H_q)\cap(-\infty, z_j(n/t)))$, and $z_j(n/t)$ is a special stationary phase point for the phase defined in the beginning of Appendix C.

From the formulas above, one easily recovers $a_l(n,t)$ and $b_l(n,t)$. More precisely, we have the following.

Theorem 1.1 ([10]). *Let C be any (large) positive number and δ be any (small) positive number. Let $E_s\in S$ be the 'resonance points' defined by $S=\{E_s : |R(E_s)|=1\}$. (There are at most $2g+2$ such points, since they are always endpoints E_j of the bands that constitute the spectrum of the Jacobi operator.) Consider the region $D=\{(n,t) : |\frac{n}{t}|<C\}\cap\{(n,t) : |z_j(\frac{n}{t})-E_s|>\delta\}$, where $z_j(\frac{n}{t})$ is the special stationary phase point for the phase defined in the beginning of Appendix C. Then one has*

$$\prod_{j=n}^{\infty}\frac{a_l(j,t)}{a(j,t)}\to 1,\qquad \sum_{j=n}^{\infty}\big(b(j,t)-b_l(j,t)\big)\to 0, \tag{1.4}$$

uniformly in D, as $t\to\infty$.

By dividing in (1.3) one recovers the $a(n,t)$. It follows from the theorem above that

$$|a(n,t)-a_l(n,t)|\to 0 \tag{1.5}$$

uniformly in D, as $t\to\infty$. In other words, the perturbed Toda lattice is asymptotically close to the limiting lattice above. Similarly for the velocities $b(n,t)$.

The question we address here concerns the higher order asymptotics. Namely, what is the rate at which the perturbed lattice approaches the limiting lattice? Even more, what is the exact asymptotic formula?

Theorem 1.2. *Let D_j be the sector $D_j = \{(n,t),: \ z_j(n/t) \in [E_{2j}+\varepsilon, E_{2j+1}-\varepsilon]$ for some $\varepsilon > 0$. Then one has*

$$\prod_{j=n}^{\infty} \left(\frac{a(j,t)}{a_l(j,t)}\right)^2 = 1 + \sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}} 2\mathrm{Re}\left(\overline{\beta(n,t)}\mathrm{i}\Lambda_0(n,t)\right) + O(t^{-\alpha}) \tag{1.6}$$

and

$$\sum_{j=n+1}^{\infty} \big(b(j,t) - b_l(j,t)\big) = \sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}} 2\mathrm{Re}\left(\overline{\beta(n,t)}\mathrm{i}\Lambda_1(n,t)\right) + O(t^{-\alpha}) \tag{1.7}$$

for any $\alpha < 1$ uniformly in D_j, as $t \to \infty$. Here

$$\phi''(z_j)/\mathrm{i} = \frac{\prod_{k=0,k\neq j}^{g}(z_j - z_k)}{\mathrm{i}R_{2g+2}^{1/2}(z_j)} > 0, \tag{1.8}$$

(with $\phi(p, n/t)$ is the phase function defined in (B.13) and $R_{2g+2}^{1/2}(z)$ the square root of the underlying Riemann surface),

$$\Lambda_0(n,t) = \omega_{\infty_-\,\infty_+}(z_j) + \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}(n,t)) \int_{\infty_+}^{\infty_-} \omega_{\hat{\nu}_\ell(n,t),0}\zeta_k(z_j),$$

$$\Lambda_1(n,t) = \omega_{\infty_-,0}(z_j) - \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}(n,t))\omega_{\hat{\nu}_\ell(n,t),0}(\infty_+)\zeta_k(z_j), \tag{1.9}$$

with $c_{k\ell}(\underline{\hat{\nu}}(n,t))$ some constants defined in (2.14), $\omega_{q,0}$ an Abelian differential of the second kind with a second order pole at q (cf. Remark 2.3),

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r(n,t)) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-t\phi(z_j)}t^{-\mathrm{i}\nu}, \tag{1.10}$$

with

$$r(n,t) = R(z_j)\frac{\theta(\underline{z}(z_j,n,t))}{\theta(\underline{z}(z_j,0,0))}\frac{\theta(\underline{z}(z_j^*,0,0))}{\theta(\underline{z}(z_j^*,n,t))}\frac{\overline{e^+(z_j,n,t)}}{e^+(z_j,n,t)}\left(\frac{\phi''(z_j)}{\mathrm{i}}\right)^{\mathrm{i}\nu} \tag{1.11}$$

where $e^{\pm}(p,n,t)$ is defined in (C.13) and

$$\nu = -\frac{1}{2\pi}\log(1 - |R(z_j)|^2) > 0. \tag{1.12}$$

Remark 1.3. *Combining our technique with the one from* [3] *enables one to find a complete asymptotic expansion.*

The necessary changes to include solitons will be given in [13] *(see also* [6], [11], *and* [19]*).*

2. The "local" Riemann-Hilbert problem on the small crosses

In [10] we have shown how the long time asymptotics can be read off from a Riemann–Hilbert problem

$$\begin{aligned}
&m_+^5(p,n,t) = m_-^5(p,n,t)J^5(p,n,t), \quad p \in \Sigma^5,\\
&(m_1^5) \ge -\mathcal{D}_{\underline{\hat{\nu}}(n,t)^*}, \quad (m_2^5) \ge -\mathcal{D}_{\underline{\hat{\nu}}(n,t)},\\
&m^5(p^*,n,t) = m^5(p,n,t)\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\\
&m^5(\infty_+,n,t) = \begin{pmatrix} 1 & * \end{pmatrix}.
\end{aligned} \tag{2.1}$$

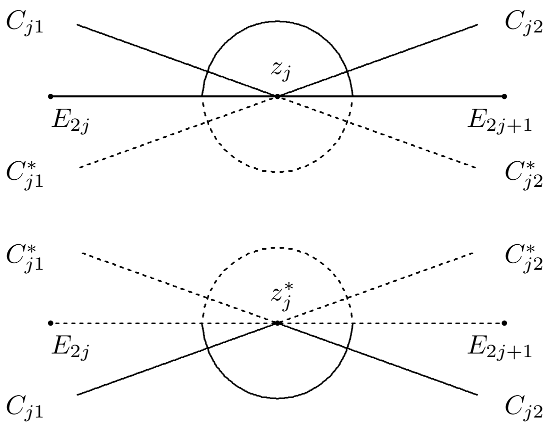

FIGURE 1. The small cross containing the stationary phase point z_j and its flipping image containing z_j^*. Views from the top and bottom sheet. Dotted curves lie in the bottom sheet.

For the exact definition of the associated Riemann surface and for the discussion of the derivation of the Riemann-Hilbert problem (C.24) we refer to the sections of the appendix. In [10] we have only considered the leading asymptotics by showing $m^5(p) \to \begin{pmatrix} 1 & 1 \end{pmatrix}$, asymptotically as $t \to \infty$.

In this paper, however, we are interested in the actual asymptotic rate at which $m^5(p) \to \begin{pmatrix} 1 & 1 \end{pmatrix}$. We have already seen in [10] that the jumps J^5 on the oriented paths C_k, C_k^* for $k \neq j$ are of the form $\mathbb{I} +$ *exponentially small* asymptotically as $t \to \infty$. The same is true for the oriented paths $C_{j1}, C_{j2}, C_{j1}^*, C_{j2}^*$ at least away from the stationary phase points z_j, z_j^*. On these paths, and in particular near the stationary phase points (see Figure 1), the jumps read

$$\begin{aligned}
J^5 &= \tilde{B}_+ = \begin{pmatrix} 1 & -\frac{d}{d^*}\frac{R^*\Theta^*}{1-R^*R}\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \quad p \in C_{j1},\\
J^5 &= \tilde{B}_-^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{d^*}{d}\frac{R\Theta}{1-R^*R}\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}, \quad p \in C_{j1}^*,\\
J^5 &= \tilde{b}_+ = \begin{pmatrix} 1 & 0 \\ \frac{d^*}{d}R\Theta\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}, \quad p \in C_{j2},\\
J^5 &= \tilde{b}_-^{-1} = \begin{pmatrix} 1 & -\frac{d}{d^*}R^*\Theta^*\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \quad p \in C_{j2}^*.
\end{aligned} \tag{2.2}$$

Note that near the stationary phase points the jumps are given by (cf. Lemma C.4)

$$\hat{B}_+ = \begin{pmatrix} 1 & -\left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{2\mathrm{i}\nu} \frac{\overline{r}}{1-|r|^2}\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \quad p \in L_{j1},$$

$$\hat{B}_-^{-1} = \begin{pmatrix} 1 & 0 \\ \left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{-2\mathrm{i}\nu} \frac{r}{1-|r|^2}\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}, \quad p \in L_{j1}^*,$$

$$\hat{b}_+ = \begin{pmatrix} 1 & 0 \\ \left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{-2\mathrm{i}\nu} r\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}, \quad p \in L_{j2},$$

$$\hat{b}_-^{-1} = \begin{pmatrix} 1 & -\left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{2\mathrm{i}\nu} \overline{r}\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \quad p \in L_{j2}^*, \tag{2.3}$$

where (cf. (B.12) and (C.13))

$$r = R(z_j)\Theta(z_j,n,t)\frac{\overline{e^+(z_j)}}{e^+(z_j)}\left(\frac{\phi''(z_j)}{\mathrm{i}}\right)^{\mathrm{i}\nu}. \tag{2.4}$$

The error terms will satisfy appropriate Hölder estimates, that is

$$\|\tilde{B}_+(p) - \hat{B}_+(p)\| \le C|z-z_j|^\alpha, \qquad p=(z,+)\in C_{j1}, \tag{2.5}$$

for any $\alpha < 1$ and similarly for the other matrices.

To reduce our Riemann–Hilbert problem to the one corresponding to the two crosses we proceed as follows: We take a small disc D around $z_j(n/t)$ and project it to the complex plane using the canonical projection π. Now consider the (holomorphic) Riemann–Hilbert problem in the complex plane with the very jump obtained by projection and normalize it to be $\mathbb{I}$ near ∞. Then, as is shown in [2] (see [12, Thm. A.1]), the solution is of the form

$$M(z) = \mathbb{I} + \frac{M_0}{z-z_j}\frac{1}{t^{1/2}} + O(t^{-\alpha}), \tag{2.6}$$

where

$$M_0 = \mathrm{i}\sqrt{\mathrm{i}/\phi''(z_j)}\begin{pmatrix} 0 & -\beta(t) \\ \overline{\beta(t)} & 0 \end{pmatrix},$$

$$\beta(t) = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-\mathrm{i}t\phi(z_j)}t^{-\mathrm{i}\nu}. \tag{2.7}$$

Now we lift this solution back to the small disc on our Riemann-surface by setting $M(p) = M(z)$ for $p \in D$ and $M(p) = \overline{M(\overline{z})}$ for $p \in D^*$. Then

$$m^6(p) = \begin{cases} m^5(p)M^{-1}(p), & p \in D \cup D^* \\ m^5(p), & \text{else.} \end{cases} \tag{2.8}$$

has no jump inside $D \cup D^*$ but jumps on the boundary given by

$$m_+^6(p) = m_-^6(p)M^{-1}(p), \quad p \in \partial D \cup \partial D^*. \tag{2.9}$$

The remaining jumps are unchanged. In summary, all jumps outside $D \cup \partial D^*$. are of the form $\mathbb{I}$ + *exponentially small* and the jump on $\partial D \cup \partial D^*$ is of the form $\mathbb{I} + O(t^{-1/2})$.

In order to identify the leading behaviour it remains to rewrite the Riemann–Hilbert problem for m^6 as a singular integral equation (cf. [10, Sect. 5]). Let the operator $C_{w^6} : L^2(\Sigma^6) \to L^2(\Sigma^6)$ be defined by

$$C_{w^6} f = C_-(f w^6) \tag{2.10}$$

for a vector valued f, where $w^6 = J^6 - \mathbb{I}$ and

$$(C_\pm f)(q) = \lim_{p\to q\in\Sigma}\frac{1}{2\pi\mathrm{i}}\int_\Sigma f\,\underline{\Omega_p^{\hat{\underline{\nu}}}}, \qquad \underline{\Omega_p^{\hat{\underline{\nu}}}} = \begin{pmatrix} \Omega_p^{\hat{\underline{\nu}}^*,\infty_+} & 0 \\ 0 & \Omega_p^{\hat{\underline{\nu}},\infty_-} \end{pmatrix}, \tag{2.11}$$

are the Cauchy operators for our Riemann surface. In particular, $\Omega_p^{\underline{\hat{\nu}},q}$ is the Cauchy kernel given by

$$\Omega_p^{\underline{\hat{\nu}},q} = \omega_{p\,q} + \sum_{j=1}^{g} I_j^{\underline{\hat{\nu}},q}(p)\zeta_j, \tag{2.12}$$

where

$$I_j^{\underline{\hat{\nu}},q}(p) = \sum_{\ell=1}^{g} c_{j\ell}(\underline{\hat{\nu}}) \int_q^p \omega_{\hat{\nu}_\ell,0}. \tag{2.13}$$

Here $\omega_{q,0}$ is the (normalized) Abelian differential of the second kind with a second order pole at q (cf. Remark 2.3 below). Note that $I_j^{\underline{\hat{\nu}},q}(p)$ has first order poles at the points $\underline{\hat{\nu}}$.

The constants $c_{j\ell}(\underline{\hat{\nu}})$ are chosen such that $\Omega_p^{\underline{\hat{\nu}},q}$ is single valued, that is,

$$\int_{b_k} dI_j^{\underline{\hat{\nu}},q} = \sum_{\ell=1}^{g} c_{j\ell} \int_{b_k} \omega_{\hat{\nu}_\ell,0} = \sum_{\ell=1}^{g} c_{j\ell}\eta_k(\hat{\nu}_\ell) = \delta_{jk}, \tag{2.14}$$

where $\zeta_k = \eta_k(z)dz$ is the chart expression in a local chart near $\hat{\nu}_\ell$ (here the b_k periods are evaluated using the usual bilinear relations, see [7, Sect. III.3] or [17, Sect. A.2]).

Remark 2.1. *The Abelian differential $\omega_{p\,q}$ is explicitly given by*

$$\omega_{p\,q} = \left(\frac{R_{2g+2}^{1/2} + R_{2g+2}^{1/2}(p)}{2(\pi - \pi(p))} - \frac{R_{2g+2}^{1/2} + R_{2g+2}^{1/2}(q)}{2(\pi - \pi(q))} + P_{pq}(\pi) \right) \frac{d\pi}{R_{2g+2}^{1/2}},$$

where $P_{pq}(z)$ is a polynomial of degree $g-1$ which has to be determined from the normalization $\int_{a_\ell} \omega_{p\,p^*} = 0$. For $q = \infty_\pm$ we have

$$\omega_{p\,\infty_\pm} = \left(\frac{R_{2g+2}^{1/2} + R_{2g+2}^{1/2}(p)}{2(\pi - \pi(p))} \mp \frac{1}{2}\pi^g + P_{p\infty_\pm}(\pi) \right) \frac{d\pi}{R_{2g+2}^{1/2}}.$$

Consider the solution μ^6 of the singular integral equation

$$\mu = \begin{pmatrix} 1 & 1 \end{pmatrix} + C_{w^6}\mu \quad \text{in} \quad L^2(\Sigma^6). \tag{2.15}$$

Then the solution of our Riemann–Hilbert problem is given by

$$m^6(p) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}} \int_{\Sigma^6} \mu^6\, w^6\, \underline{\Omega_p^{\hat{\nu}}}. \tag{2.16}$$

By $\|w^6\|_\infty = O(t^{-1/2})$ Neumann's formula implies

$$\mu^6(q) = (\mathbb{I} - C_{w^6})^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} + O(t^{-1/2}). \tag{2.17}$$

Moreover,

$$w^6(p) = \begin{cases} -\frac{M_0}{z - z_j} \frac{1}{t^{1/2}} + O(t^{-\alpha}), & p \in \partial D, \\ -\frac{\overline{M_0}}{z - z_j} \frac{1}{t^{1/2}} + O(t^{-\alpha}), & p \in \partial D^*. \end{cases} \tag{2.18}$$

Hence we obtain

$$\begin{aligned} m^6(p) &= \begin{pmatrix}1 & 1\end{pmatrix} - \frac{\begin{pmatrix}1 & 1\end{pmatrix} M_0}{t^{1/2}} \frac{1}{2\pi\mathrm{i}} \int_{\partial D} \frac{1}{\pi - z_j} \underline{\Omega}_p^{\hat{\nu}} \\ &\quad - \frac{\begin{pmatrix}1 & 1\end{pmatrix} \overline{M_0}}{t^{1/2}} \frac{1}{2\pi\mathrm{i}} \int_{\partial D^*} \frac{1}{\pi - z_j} \underline{\Omega}_p^{\hat{\nu}} + O(t^{-\alpha}) \\ &= \begin{pmatrix}1 & 1\end{pmatrix} - \frac{\begin{pmatrix}1 & 1\end{pmatrix} M_0}{t^{1/2}} \underline{\Omega}_p^{\hat{\nu}}(z_j) - \frac{\begin{pmatrix}1 & 1\end{pmatrix} \overline{M_0}}{t^{1/2}} \underline{\Omega}_p^{\hat{\nu}}(z_j^*) + O(t^{-\alpha}) \\ &= \begin{pmatrix}1 & 1\end{pmatrix} - \sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}} \Big(\mathrm{i}\overline{\beta}\Omega_p^{\hat{\nu}^*,\infty_+}(z_j) - \mathrm{i}\beta\Omega_p^{\hat{\nu}^*,\infty_+}(z_j^*) \quad -\mathrm{i}\beta\Omega_p^{\hat{\nu},\infty_-}(z_j) + \mathrm{i}\overline{\beta}\Omega_p^{\hat{\nu},\infty_-}(z_j^*)\Big) \\ &\quad + O(t^{-\alpha}). \end{aligned} \tag{2.19}$$

Note that the right hand side is real-valued for $p \in \pi^{-1}(\mathbb{R})\backslash\Sigma$ since $\overline{\Omega_p^{\hat{\nu},\infty_\pm}(\overline{q})} = \Omega_{\overline{p}}^{\hat{\nu},\infty_\pm}(q)$ implies

$$\Omega_p^{\hat{\nu},\infty_\pm}(z_j^*) = \overline{\Omega_p^{\hat{\nu},\infty_\pm}(z_j)}, \qquad p \in \pi^{-1}(\mathbb{R})\backslash\Sigma. \tag{2.20}$$

Since we need the asymptotic expansions around ∞_- we note

Lemma 2.2. *We have*

$$\Omega_p^{\hat{\nu},\infty_+}(z_j) = \Lambda_0^{\hat{\nu}} + \Lambda_1^{\hat{\nu}} \frac{1}{z} + O(\frac{1}{z^2}) \tag{2.21}$$

for $p = (z, -)$ near ∞_-, where

$$\Lambda_0^{\hat{\nu}} = \Omega_{\infty_-}^{\hat{\nu},\infty_+}(z_j) = \Omega_{\infty_-}^{\hat{\nu}^*,\infty_+}(z_j) = \omega_{\infty_-\,\infty_+}(z_j) + \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}) \int_{\infty_+}^{\infty_-} \omega_{\hat{\nu}_\ell,0}\zeta_k(z_j) \tag{2.22}$$

and

$$\begin{aligned} \Lambda_1^{\hat{\nu}} &= \omega_{\infty_-,0}(z_j) + \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}) \omega_{\hat{\nu}_\ell,0}(\infty_-)\zeta_k(z_j) \\ &= \omega_{\infty_-,0}(z_j) - \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}^*) \omega_{\hat{\nu}_\ell^*,0}(\infty_+)\zeta_k(z_j). \end{aligned} \tag{2.23}$$

Proof. To see $\Omega_{\infty_-}^{\hat{\nu}}(z_j) = \Omega_{\infty_-}^{\hat{\nu}^*}(z_j)$ note $c_{k\ell}(\underline{\hat{\nu}}^*) = -c_{k\ell}(\underline{\hat{\nu}})$ and $\int_{\infty_+}^{\infty_-} \omega_{\hat{\nu}_\ell^*,0} = \int_{\infty_-}^{\infty_+} \omega_{\hat{\nu}_\ell,0}$. □

Observe that since $c_{k\ell}(\underline{\hat{\nu}}) \in \mathbb{R}$ and $\int_{\infty_+}^{\infty_-} \omega_{\hat{\nu}_\ell,0} \in \mathbb{R}$ we have $\Lambda_0^{\hat{\nu}} \in \mathrm{i}\mathbb{R}$.

Remark 2.3. *Note that the Abelian integral appearing in the previous lemma is explicitly given by*

$$\omega_{\infty_-,0} = \frac{-\pi^{g+1} + \frac{1}{2}\sum_{j=0}^{2g+1} E_j \pi^g + P_{\infty_-,0}(\pi) + R_{2g+2}^{1/2}}{R_{2g+2}^{1/2}} d\pi, \tag{2.24}$$

with $P_{\infty_-,0}$ a polynomial of degree $g-1$ which has to be determined from the normalization.

Similarly,

$$\omega_{\hat{\nu},0} = \frac{R_{2g+2}^{1/2} + R_{2g+2}^{1/2}(\hat{\nu}) + \frac{R'_{2g+2}(\hat{\nu})}{2R_{2g+2}^{1/2}(\hat{\nu})}(\pi-\nu) + P_{\hat{\nu},0}(\pi)\cdot(\pi-\nu)^2}{2(\pi-\nu)^2 R_{2g+2}^{1/2}} d\pi, \tag{2.25}$$

with $P_{\hat{\nu},0}$ a polynomial of degree $g-1$ which has to be determined from the normalization.

The asymptotics can be read off by using

$$m^3(p) = d(\infty_-) m^6(p) \begin{pmatrix} \frac{1}{d(p^*)} & 0 \\ 0 & \frac{1}{d(p)} \end{pmatrix} \tag{2.26}$$

for p near ∞_- and comparing with (B.21) using

$$d(p) = 1 + \frac{d_1}{z} + O(\frac{1}{z^2}) \tag{2.27}$$

for $p = (z,+)$ near ∞_+, with ([10])

$$\begin{aligned} d_1 = & -\frac{1}{2\pi\mathrm{i}} \int_{C(n/t)} \log(1-|R|^2)\Omega_0 \\ & - \frac{1}{2}\frac{d}{ds} \log\left(\frac{\theta(\underline{z}(n,s) + \underline{\delta}(n,t))}{\theta(\underline{z}(n,s))}\right)\Big|_{s=t}. \end{aligned}$$

where Ω_0 is the Abelian differential of the second kind defined in (A.14). We obtain

$$A_+(n,t)^2 = \frac{1}{d(\infty_-)}\left(1 + \sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\left(\mathrm{i}\overline{\beta}\Lambda_0^{\hat{\nu}} - \mathrm{i}\beta\overline{\Lambda_0^{\hat{\nu}}}\right)\right) + O(t^{-\alpha}) \tag{2.28}$$

and

$$B_+(n,t) = -d_1 - \sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\left(\mathrm{i}\overline{\beta}\Lambda_1^{\hat{\nu}^*} - \mathrm{i}\beta\overline{\Lambda_1^{\hat{\nu}^*}}\right) + O(t^{-\alpha}). \tag{2.29}$$

Appendix A. Algebro-geometric quasi-periodic finite-gap solutions

As in [10], we state some facts on our background solution (a_q, b_q) which we want to choose from the class of algebro-geometric quasi-periodic finite-gap solutions, that is the class of stationary solutions of the Toda hierarchy, [1]. In particular, this class contains all periodic solutions. We will use the same notation as in [17], where we also refer to for proofs. As a reference for Riemann surfaces in this context we recommend [7].

To set the stage let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z), \qquad R_{2g+2}(z) = \prod_{j=0}^{2g+1}(z-E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{A.1}$$

$g \in \mathbb{N}$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. We will choose $R_{2g+2}^{1/2}(z)$ as the fixed branch

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1}\sqrt{z-E_j}, \tag{A.2}$$

where $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty, 0)$.

A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = (\infty, \pm) = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The points $\{(E_j, 0), 0 \le j \le 2g+1\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C} \setminus \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]\} \subset \mathbb{M} \tag{A.3}$$

are called upper, lower sheet, respectively.

Let $\{a_j, b_j\}_{j=1}^g$ be loops on the surface $\mathbb{M}$ representing the canonical generators of the fundamental group $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counterclockwise on the upper sheet, with pairwise intersection indices given by

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{i,j}, \qquad 1 \le i, j \le g. \tag{A.4}$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^g$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}}, \tag{A.5}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}.$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \qquad \int_{b_j} \zeta_k = \tau_{j,k}, \qquad \tau_{j,k} = \tau_{k,j}, \qquad 1 \le j, k \le g. \tag{A.6}$$

Now pick g numbers (the Dirichlet eigenvalues)

$$(\hat{\mu}_j)_{j=1}^g = (\mu_j, \sigma_j)_{j=1}^g \tag{A.7}$$

whose projections lie in the spectral gaps, that is, $\mu_j \in [E_{2j-1}, E_{2j}]$. Associated with these numbers is the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ which is one at the points $\hat{\mu}_j$ and zero else. Using this divisor we introduce

$$\begin{aligned} \underline{z}(p, n, t) &= \underline{\hat{A}}_{p_0}(p) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - n \underline{\hat{A}}_{\infty_-}(\infty_+) + t \underline{U}_0 - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \\ \underline{z}(n, t) &= \underline{z}(\infty_+, n, t), \end{aligned} \tag{A.8}$$

where $\underline{\Xi}_{p_0}$ is the vector of Riemann constants

$$\hat{\Xi}_{p_0, j} = \frac{1 - \sum_{k=1}^{g} \tau_{j,k}}{2}, \qquad p_0 = (E_0, 0), \tag{A.9}$$

$\underline{U}_0$ are the b-periods of the Abelian differential Ω_0 defined below, and $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{\mathbb{M}}$ (the fundamental polygon associated with $\mathbb{M}$ by cutting along the a and b cycles) to $\mathbb{C}^g$. We recall that the function $\theta(\underline{z}(p, n))$ has precisely g zeros $\hat{\nu}_j(n)$ (with $\hat{\nu}_j(0) = \hat{\nu}_j$), where $\theta(\underline{z})$ is the Riemann theta function of $\mathbb{M}$.

Then our background solution is given by

$$\begin{aligned} a_q(n,t)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2}, \\ b_q(n,t) &= \tilde{b} + \frac{1}{2}\frac{d}{dt}\log\Big(\frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\Big). \end{aligned} \tag{A.10}$$

The constants $\tilde{a}$, $\tilde{b}$ depend only on the Riemann surface (see [17, Section 9.2]).

Introduce the time dependent Baker-Akhiezer function

$$\psi_q(p,n,t) = C(n,0,t)\frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))}\exp\Big(n\int_{E_0}^p \omega_{\infty_+\,\infty_-} + t\int_{E_0}^p \Omega_0\Big), \tag{A.11}$$

where $C(n,0,t)$ is real-valued,

$$C(n,0,t)^2 = \frac{\theta(\underline{z}(0,0))\theta(\underline{z}(-1,0))}{\theta(\underline{z}(n,t))\theta(\underline{z}(n-1,t))}, \tag{A.12}$$

and the sign has to be chosen in accordance with $a_q(n,t)$. Here

$$\theta(\underline{z}) = \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\left(\langle\underline{m},\underline{z}\rangle + \frac{\langle\underline{m},\underline{\tau}\,\underline{m}\rangle}{2}\right), \qquad \underline{z}\in\mathbb{C}^g, \tag{A.13}$$

is the Riemann theta function associated with $\mathbb{M}$,

$$\omega_{\infty_+\,\infty_-} = \frac{\prod_{j=1}^g(\pi-\lambda_j)}{R_{2g+2}^{1/2}}d\pi \tag{A.14}$$

is the Abelian differential of the third kind with poles at ∞_+ and ∞_- and

$$\Omega_0 = \frac{\prod_{j=0}^g(\pi-\tilde{\lambda}_j)}{R_{2g+2}^{1/2}}d\pi, \qquad \sum_{j=0}^g \tilde{\lambda}_j = \frac{1}{2}\sum_{j=0}^{2g+1} E_j, \tag{A.15}$$

is the Abelian differential of the second kind with second order poles at ∞_+ respectively ∞_- (see [17, Sects. 13.1, 13.2]). All Abelian differentials are normalized to have vanishing a_j periods.

The Baker-Akhiezer function is a meromorphic function on $\mathbb{M}\setminus\{\infty_\pm\}$ with an essential singularity at $\infty_\pm$. The two branches are denoted by

$$\psi_{q,\pm}(z,n,t) = \psi_q(p,n,t), \qquad p=(z,\pm) \tag{A.16}$$

and it satisfies

$$\begin{aligned} H_q(t)\psi_q(p,n,t) &= \pi(p)\psi_q(p,n,t), \\ \frac{d}{dt}\psi_q(p,n,t) &= P_{q,2}(t)\psi_q(p,n,t), \end{aligned} \tag{A.17}$$

where H_q, $P_{q,2}$ are the operators from the Lax pair for the Toda lattice.

It is well known that the spectrum of $H_q(t)$ is time independent and consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^g [E_{2j}, E_{2j+1}]. \tag{A.18}$$

For further information and proofs we refer to [17, Chap. 9 and Sect. 13.2].

Appendix B. The Inverse scattering transform and the Riemann–Hilbert problem

In this section our notation and results are taken from [4] and [5]. We will follow essentially [10] with the only difference that we will work directly with the vector Riemann-Hilbert problem. This will avoid the problems arising when the transmission coefficient vanishes at an band edge and somewhat simplify our analysis.

Let $\psi_{q,\pm}(z,n,t)$ be the branches of the Baker-Akhiezer function defined in the previous section. Let $\psi_{\pm}(z,n,t)$ be the Jost functions for the perturbed problem defined by

$$\lim_{n\to\pm\infty} w(z)^{\mp n}(\psi_{\pm}(z,n,t)-\psi_{q,\pm}(z,n,t))=0, \tag{B.1}$$

where $w(z)$ is the quasimomentum map

$$w(z)=\exp\Big(\int_{E_0}^{p}\omega_{\infty_+\,\infty_-}\Big),\quad p=(z,+). \tag{B.2}$$

The asymptotics of the two projections of the Jost function are

$$\begin{aligned}\psi_{\pm}(z,n,t)=&\frac{z^{\mp n}\Big(\prod_{j=0}^{n-1}a_q(j,t)\Big)^{\pm 1}}{A_{\pm}(n,t)}\times\\ &\times\Big(1+\Big(B_{\pm}(n,t)\pm\sum_{j=1}^{n}b_q(j-{\scriptstyle 0 \atop \scriptstyle 1},t)\Big)\frac{1}{z}+O(\frac{1}{z^2})\Big),\end{aligned} \tag{B.3}$$

as $z\to\infty$, where

$$\begin{aligned}A_{+}(n,t)&=\prod_{j=n}^{\infty}\frac{a(j,t)}{a_q(j,t)},\quad B_{+}(n,t)=\sum_{j=n+1}^{\infty}(b_q(j,t)-b(j,t)),\\ A_{-}(n,t)&=\prod_{j=-\infty}^{n-1}\frac{a(j,t)}{a_q(j,t)},\quad B_{-}(n,t)=\sum_{j=-\infty}^{n-1}(b_q(j,t)-b(j,t)).\end{aligned} \tag{B.4}$$

One has the scattering relations

$$T(z)\psi_{\mp}(z,n,t)=\overline{\psi_{\pm}(z,n,t)}+R_{\pm}(z)\psi_{\pm}(z,n,t),\qquad z\in\sigma(H_q), \tag{B.5}$$

where $T(z)$, $R_{\pm}(z)$ are the transmission respectively reflection coefficients. Here $\psi_{\pm}(z,n,t)$ is defined such that $\psi_{\pm}(z,n,t)=\lim_{\varepsilon\downarrow 0}\psi_{\pm}(z+\mathrm{i}\varepsilon,n,t)$, $z\in\sigma(H_q)$. If we take the limit from the other side we have $\overline{\psi_{\pm}(z,n,t)}=\lim_{\varepsilon\downarrow 0}\psi_{\pm}(z-\mathrm{i}\varepsilon,n,t)$.

The transmission $T(z)$ and reflection $R_{\pm}(z)$ coefficients satisfy

$$T(z)\overline{R_{+}(z)}+\overline{T(z)}R_{-}(z)=0,\qquad |T(z)|^2+|R_{\pm}(z)|^2=1. \tag{B.6}$$

In particular one reflection coefficient, say $R(z)=R_{+}(z)$, suffices.

We will define a Riemann–Hilbert problem on the Riemann surface $\mathbb{M}$ as follows:

$$m(p,n,t)=\begin{cases}\Big(T(z)\frac{\psi_{-}(z,n,t)}{\psi_{q,-}(z,n,t)}\quad \frac{\psi_{+}(z,n,t)}{\psi_{q,+}(z,n,t)}\Big), & p=(z,+)\\ \Big(\frac{\psi_{+}(z,n,t)}{\psi_{q,+}(z,n,t)}\quad T(z)\frac{\psi_{-}(z,n,t)}{\psi_{q,-}(z,n,t)}\Big), & p=(z,-)\end{cases}. \tag{B.7}$$

We are interested in the jump condition of $m(p,n,t)$ on Σ, the boundary of $\Pi_{\pm}$ (oriented counterclockwise when viewed from top sheet Π_{+}). It consists of two copies $\Sigma_{\pm}$ of $\sigma(H_q)$ which correspond to non-tangential limits from $p=(z,+)$ with $\pm\mathrm{Im}(z)>0$, respectively to non-tangential limits from $p=(z,-)$ with $\mp\mathrm{Im}(z)>0$.

To formulate our jump condition we use the following convention: When representing functions on Σ, the lower subscript denotes the non-tangential limit from Π_+ or Π_-, respectively,

$$m_\pm(p_0) = \lim_{\Pi_\pm \ni p \to p_0} m(p), \qquad p_0 \in \Sigma. \tag{B.8}$$

Using the notation above implicitly assumes that these limits exist in the sense that $m(p)$ extends to a continuous function on the boundary away from the band edges.

Moreover, we will also use symmetries with respect to the the sheet exchange map

$$p^* = \begin{cases} (z,\mp) & \text{for } p=(z,\pm), \\ \infty_\mp & \text{for } p=\infty_\pm, \end{cases} \tag{B.9}$$

and complex conjugation

$$\overline{p} = \begin{cases} (\overline{z},\pm) & \text{for } p=(z,\pm)\not\in\Sigma, \\ (z,\mp) & \text{for } p=(z,\pm)\in\Sigma, \\ \infty_\pm & \text{for } p=\infty_\pm. \end{cases} \tag{B.10}$$

In particular, we have $\overline{p}=p^*$ for $p\in\Sigma$.

Note that we have $\tilde{m}_\pm(p) = m_\mp(p^*)$ for $\tilde{m}(p) = m(p^*)$ (since $*$ reverses the orientation of Σ) and $\tilde{m}_\pm(p) = \overline{m_\pm(p^*)}$ for $\tilde{m}(p) = \overline{m(\overline{p})}$.

With this notation, using (B.5) and (B.6), we obtain

$$\begin{gathered} m_+(p,n,t) = m_-(p,n,t) J(p,n,t) \\ J(p,n,t) = \begin{pmatrix} 1-|R(p)|^2 & -\overline{R(p)\Theta(p,n,t)}\mathrm{e}^{-t\phi(p)} \\ R(p)\Theta(p,n,t)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}, \end{gathered} \tag{B.11}$$

where

$$\Theta(p,n,t) = \frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))} \frac{\theta(\underline{z}(p^*,0,0))}{\theta(\underline{z}(p^*,n,t))} \tag{B.12}$$

and

$$\phi(p,\frac{n}{t}) = 2\int_{E_0}^p \Omega_0 + 2\frac{n}{t}\int_{E_0}^p \omega_{\infty_+\,\infty_-} \in \mathrm{i}\mathbb{R} \tag{B.13}$$

for $p\in\Sigma$. Note

$$\frac{\psi_q(p,n,t)}{\psi_q(p^*,n,t)} = \Theta(p,n,t)\mathrm{e}^{t\phi(p)}.$$

Here we have extend our definition of T to Σ such that it is equal to $T(z)$ on Σ_+ and equal to $\overline{T(z)}$ on Σ_-. Similarly for $R(z)$. In particular, the condition on Σ_+ is just the complex conjugate of the one on Σ_- since we have $R(p^*) = \overline{R(p)}$ and $m_\pm(p^*,n,t) = \overline{m_\pm(p,n,t)}$ for $p\in\Sigma$.

Furthermore,

$$m(p,n,t) = \Big(A_+(n,t)(1-B_+(n-1,t)\tfrac{1}{z}) \quad \tfrac{1}{A_+(n,t)}(1+B_+(n,t)\tfrac{1}{z}) \Big) + O(\frac{1}{z^2}), \tag{B.14}$$

for $p=(z,+)\to\infty_+$, with $A_\pm(n,t)$ and $B_\pm(n,t)$ are defined in (B.4). The formula near ∞_- follows by flipping the columns. Here we have used

$$T(z) = A_-(n,t)A_+(n,t)\big(1 - \frac{B_+(n,t)+b_q(n,t)-b(n,t)+B_-(n,t)}{z} + O(\frac{1}{z^2}\big). \tag{B.15}$$

Using the properties of $\psi_\pm(z,n,t)$ and $\psi_{q,\pm}(z,n,t)$ one checks that its divisor satisfies

$$(m_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)^*}, \qquad (m_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)}. \tag{B.16}$$

Theorem B.1. *The function*

$$m^3(z) = \frac{1}{A_+(n,t)} m(z,n,t) \tag{B.17}$$

with $m(z,n,t)$ *defined in (B.7) is meromorphic away from* Σ *and satisfies:*

$$\begin{aligned} & m^3_+(p) = m^3_-(p) J^3(p), \quad p \in \Sigma, \\ & (m^3_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)^*}, \quad (m^3_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)}, \\ & m^3(p^*) = m^3(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{B.18}$$

$$m^3(\infty_+) = \begin{pmatrix} 1 & * \end{pmatrix}, \tag{B.19}$$

where the jump is given by

$$J^3(p,n,t) = \begin{pmatrix} 1 - |R(p)|^2 & -\overline{R(p)\Theta(p,n,t)}\mathrm{e}^{-t\phi(p)} \\ R(p)\Theta(p,n,t)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}. \tag{B.20}$$

Note also

$$m^3(p) = \begin{pmatrix} \frac{1}{A_+(n,t)^2} & 1 \end{pmatrix} + \begin{pmatrix} \frac{B_+(n,t)}{A_+(n,t)^2} & -B_+(n-1,t) \end{pmatrix} \frac{1}{z} + O(\frac{1}{z^2}). \tag{B.21}$$

for p near ∞_-.

Appendix C. The stationary phase points and corresponding contour deformations

The phase in the factorization problem (B.11) is $t\,\phi$ where ϕ was defined in (B.13). Invoking (A.14) and (A.15), we see that the stationary phase points are given by

$$\prod_{j=0}^{g} (z - \tilde{\lambda}_j) + \frac{n}{t} \prod_{j=1}^{g} (z - \lambda_j) = 0. \tag{C.1}$$

Due to the normalization of our Abelian differentials, the numbers λ_j, $0 \leq j \leq g$, are real and different with precisely one lying in each spectral gap, say λ_j in the j'th gap. Similarly, $\tilde{\lambda}_j$, $0 \leq j \leq g$, are real and different and $\tilde{\lambda}_j$, $1 \leq j \leq g$, sits in the j'th gap. However $\tilde{\lambda}_0$ can be anywhere (see [17, Sect. 13.5]).

In [10] we examined the dependence of the stationary phase points on $\frac{n}{t}$. We proved the following.

Lemma C.1. *Denote by* $z_j(\eta)$, $0 \leq j \leq g$, *the stationary phase points, where* $\eta = \frac{n}{t}$. *Set* $\lambda_0 = -\infty$ *and* $\lambda_{g+1} = \infty$, *then*

$$\lambda_j < z_j(\eta) < \lambda_{j+1} \tag{C.2}$$

and there is always at least one stationary phase point in the j*'th spectral gap. Moreover,* $z_j(\eta)$ *is monotone decreasing with*

$$\lim_{\eta \to -\infty} z_j(\eta) = \lambda_{j+1} \quad \textit{and} \quad \lim_{\eta \to \infty} z_j(\eta) = \lambda_j. \tag{C.3}$$

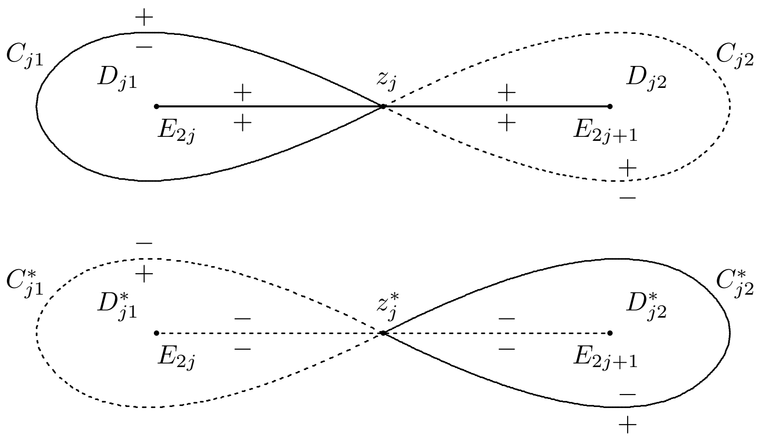

FIGURE 2. The lens contour near a band containing a stationary phase point z_j and its flipping image containing z_j^*. Views from the top and bottom sheet. Dotted curves lie in the bottom sheet.

So, depending on n/t there is at most one single stationary phase point belonging to the union of the bands $\sigma(H_q)$, say $z_j(n/t)$. On the Riemann surface, there are two such points z_j and its flipping image z_j^* which may (depending on n/t) lie in Σ.

There are three possible cases.

(i) One stationary phase point, say z_j, belongs to the interior of a band $[E_{2j}, E_{2j+1}]$ and all other stationary phase points lie in open gaps.
(ii) $z_j = z_j^* = E_j$ for some j and all other stationary phase points lie in open gaps.
(iii) No stationary phase point belongs to $\sigma(H_q)$.

In this paper we consider the first case. Note that in this case

$$\phi''(z_j)/\mathrm{i} = \frac{\prod_{k=0,k\neq j}^{g}(z_j - z_k)}{\mathrm{i}R_{2g+2}^{1/2}(z_j)} > 0. \tag{C.4}$$

Let us introduce the following "lens" contour near the band $[E_{2j}, E_{2j+1}]$ as shown in Figure 2. The oriented paths $C_j = C_{j1} \cup C_{j2}$, $C_j^* = C_{j1}^* \cup C_{j2}^*$ are meant to be close to the band $[E_{2j}, E_{2j+1}]$.

We have

$$\mathrm{Re}(\phi) > 0, \quad \text{in } D_{j1}, \qquad \mathrm{Re}(\phi) < 0, \quad \text{in } D_{j2}.$$

Indeed

$$\mathrm{Im}(\phi') < 0, \quad \text{in } [E_{2j}, z_j], \qquad \mathrm{Im}(\phi') > 0, \quad \text{in } [z_j, E_{2j+1}] \tag{C.5}$$

noting that ϕ is imaginary in $[E_{2j}, E_{2j+1}]$ and writing $\phi' = d\phi/dz$. Using the Cauchy-Riemann equations we find that the above inequalities are true, as long as C_{j1}, C_{j2} are close enough to the band $[E_{2j}, E_{2j+1}]$. A similar picture appears in the lower sheet.

Concerning the other bands, one simply constructs a "lens" contour near each of the other bands $[E_{2k}, E_{2k+1}]$ and $[E_{2k}^*, E_{2k+1}^*]$ as shown in Figure 3. The oriented paths C_k, C_k^* are meant to be close to the band $[E_{2k}, E_{2k+1}]$. The appropriate

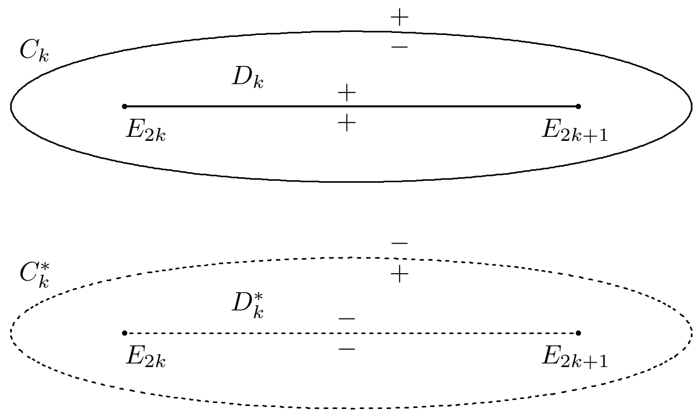

FIGURE 3. The lens contour near a band not including any stationary phase point. Views from the top and bottom sheet.

transformation is now obvious. Arguing as before, for all bands $[E_{2k}, E_{2k+1}]$ we will have

$$\mathrm{Re}(\phi) < (>)0, \quad \text{in } D_k, \quad k > (<)j.$$

Now observe that our jump condition (B.20) has the following important factorization

$$J^3 = (b_-)^{-1} b_+, \tag{C.6}$$

where

$$b_- = \begin{pmatrix} 1 & \overline{R\Theta}\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \qquad b_+ = \begin{pmatrix} 1 & 0 \\ R\Theta\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}.$$

This is the right factorization for $z > z_j(n/t)$. Similarly, we have

$$J^3 = (B_-)^{-1} \begin{pmatrix} 1-|R|^2 & 0 \\ 0 & \frac{1}{1-|R|^2} \end{pmatrix} B_+, \tag{C.7}$$

where

$$B_- = \begin{pmatrix} 1 & 0 \\ -\frac{R\Theta\mathrm{e}^{t\,\phi}}{1-|R|^2} & 1 \end{pmatrix}, \qquad B_+ = \begin{pmatrix} 1 & -\frac{\overline{R\Theta}\mathrm{e}^{-t\,\phi}}{1-|R|^2} \\ 0 & 1 \end{pmatrix}.$$

This is the right factorization for $z < z_j(n/t)$. To get rid of the diagonal part we need to solve the corresponding scalar Riemann–Hilbert problem. Again we have to search for a meromorphic solution. This means that the poles of the scalar Riemann–Hilbert problem will be added to the resulting Riemann–Hilbert problem. On the other hand, a pole structure similar to the one of m is crucial for uniqueness. We will address this problem by choosing the poles of the scalar problem in such a way that its zeros cancel the poles of m. The right choice will turn out to be $\mathcal{D}_{\underline{\hat{\nu}}}$ (that is, the Dirichlet divisor corresponding to the limiting lattice defined in (1.3)).

Lemma C.2. [10] *Define a divisor* $\mathcal{D}_{\underline{\hat{\nu}}(n,t)}$ *of degree* g *via*

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\nu}}(n,t)}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t)}) + \underline{\delta}(n,t), \tag{C.8}$$

where

$$\delta_\ell(n,t) = \frac{1}{2\pi\mathrm{i}} \int_{C(n/t)} \log(1-|R|^2)\zeta_\ell. \tag{C.9}$$

Then $\mathcal{D}_{\underline{\hat{\nu}}(n,t)}$ *is nonspecial and* $\pi(\hat{\nu}_j(n,t)) = \nu_j(n,t) \in \mathbb{R}$ *with precisely one in each spectral gap.*

Now we can formulate the scalar Riemann–Hilbert problem required to eliminate the diagonal part in the factorization (C.7):

$$\begin{aligned} & d_+(p,n,t) = d_-(p,n,t)(1-|R(p)|^2), \quad p \in C(n/t),\\ & (d) \geq -\mathcal{D}_{\underline{\hat{\nu}}(n,t)},\\ & d(\infty_+,n,t) = 1, \end{aligned} \tag{C.10}$$

where $C(n/t) = \Sigma \cap \pi^{-1}((-\infty, z_j(n/t))$. Since the index of the (regularized) jump is zero (see remark below), there will be no solution in general unless we admit g additional poles (see e.g. [16, Thm. 5.2]).

Theorem C.3. *The unique solution of (C.10) is given by*

$$\begin{aligned} d(p,n,t) =& \frac{\theta(\underline{z}(n,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(n,t))} \frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,n,t)+\underline{\delta}(n,t))} \times\\ & \times \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{C(n/t)} \log(1-|R|^2)\omega_{p\,\infty_+}\right), \end{aligned} \tag{C.11}$$

where $\underline{\delta}(n,t)$ *is defined in (C.9) and* $\omega_{p\,q}$ *is the Abelian differential of the third kind with poles at* p *and* q*.*

The function $d(p)$ *is meromorphic in* $\mathbb{M}\setminus C(n/t)$ *with first order poles at* $\hat{\nu}_j(n,t)$ *and first oder zeros at* $\hat{\mu}_j(n,t)$*. Also* $d(p)$ *is uniformly bounded in* n,t *away from the poles.*

In addition, we have $d(p) = \overline{d(\overline{p})}$*.*

In particular,

$$\begin{aligned} d(\infty_-,n,t) =& \frac{\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))} \frac{\theta(\underline{z}(n,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(n-1,t)+\underline{\delta}(n,t))} \times\\ & \times \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{C(n/t)} \log(1-|R|^2)\omega_{\infty_-\,\infty_+}\right), \end{aligned} \tag{C.12}$$

since $\underline{z}(\infty_-,n,t) = \underline{z}(\infty_+,n-1,t) = \underline{z}(n-1,t)$. Note that $\overline{d(\infty_-,n,t)} = d(\overline{\infty_-},n,t) = d(\infty_-,n,t)$ shows that $d(\infty_-,n,t)$ is real-valued. Using (A.14) one can even show that it is positive.

Also, we can give more information on the singularities of $d(p)$ near the stationary phase points and the band edges.

Lemma C.4. *For* p *near a stationary phase point* z_j *or* z_j^* *(not equal to a band edge) we have*

$$d(p) = (z-z_j)^{\pm\mathrm{i}\nu} e^{\pm}(z), \quad p = (z,\pm), \tag{C.13}$$

where $e^{\pm}(z)$ *has continuous limits near* z_j *and*

$$\nu = -\frac{1}{2\pi}\log(1-|R(z_j)|^2) > 0. \tag{C.14}$$

Here $(z-z_j)^{\pm \mathrm{i}\nu} = \exp(\pm \mathrm{i}\nu \log(z-z_j))$, where the branch cut of the logarithm is along the negative real axis.

For p near a band edge $E_k \in C(n/t)$ we have

$$d(p) = T^{\pm 1}(z)\tilde{e}^{\pm}(z), \quad p=(z,\pm), \tag{C.15}$$

where $\tilde{e}^{\pm}(z)$ is holomorphic near E_k if none of the ν_j is equal to E_k and $\tilde{e}_{\pm}(z)$ has a first order pole at $E_k=\nu_j$ else.

Note that we have

$$e^{\pm}(z) = \overline{e^{\mp}(z)}, \qquad p=(z,\pm)\in\Sigma\backslash C(n/t),$$

since $d(p^*) = \overline{d(p)}$ for $p\in\Sigma\backslash C(n/t)$.

Having solved the scalar problem above for d we can introduce the new Riemann–Hilbert problem

$$m^4(p) = d(\infty_-)^{-1} m^3(p) D(p), \quad D(p) = \begin{pmatrix} d(p^*) & 0 \\ 0 & d(p)\end{pmatrix}. \tag{C.16}$$

Then a straightforward calculation shows that m^4 satisfies

$$\begin{aligned} & m^4_+(p) = m^4_-(p) J^4(p), \quad p\in\Sigma,\\ & (m^4_1) \ge -\mathcal{D}_{\underline{\hat{\nu}}(n,t)^*}, \quad (m^4_2) \ge -\mathcal{D}_{\underline{\hat{\nu}}(n,t)},\\ & m^4(p^*) = m^4(p)\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\\ & m^4(\infty_+) = \begin{pmatrix} 1 & *\end{pmatrix}, \end{aligned} \tag{C.17}$$

where the jump is given by

$$J^4(p) = D_-(p)^{-1} J^3(p) D_+(p), \quad p\in\Sigma. \tag{C.18}$$

In particular, m^4 has its poles shifted from $\hat{\mu}_j(n,t)$ to $\hat{\nu}_j(n,t)$.

Furthermore, J^4 can be factorized as

$$J^4 = \begin{pmatrix} 1-|R|^2 & -\frac{d}{d^*}\overline{R\Theta}\mathrm{e}^{-t\,\phi}\\ \frac{d^*}{d} R\Theta \mathrm{e}^{t\,\phi} & 1\end{pmatrix} = (\tilde{b}_-)^{-1}\tilde{b}_+, \quad p\in\Sigma\setminus C(n/t), \tag{C.19}$$

where $\tilde{b}_\pm = D^{-1} b_\pm D$, that is,

$$\tilde{b}_- = \begin{pmatrix} 1 & \frac{d}{d^*}\overline{R\Theta}\mathrm{e}^{-t\,\phi}\\ 0 & 1\end{pmatrix}, \qquad \tilde{b}_+ = \begin{pmatrix} 1 & 0\\ \frac{d^*}{d}R\Theta\mathrm{e}^{t\,\phi} & 1\end{pmatrix},$$

for $\pi(p) < z_j(n/t)$ and

$$J^4 = \begin{pmatrix} 1 & -\frac{d_+}{d_-^*}\overline{R\Theta}\mathrm{e}^{-t\,\phi}\\ \frac{d_-^*}{d_+}R\Theta\mathrm{e}^{t\,\phi} & 1-|R|^2\end{pmatrix} = (\tilde{B}_-)^{-1}\tilde{B}_+, \quad p\in C(n/t), \tag{C.20}$$

where $\tilde{B}_\pm = D_-^{-1} B_\pm D_-$, that is,

$$\tilde{B}_- = \begin{pmatrix} 1 & 0\\ -\frac{d_-^*}{d_-}\frac{R\Theta}{1-|R|^2}\mathrm{e}^{t\,\phi} & 1\end{pmatrix}, \qquad \tilde{B}_+ = \begin{pmatrix} 1 & -\frac{d_+}{d_+^*}\frac{\overline{R\Theta}}{1-|R|^2}\mathrm{e}^{-t\,\phi}\\ 0 & 1\end{pmatrix},$$

for $\pi(p) > z_j(n/t)$.

Note that by $\overline{d(p)} = d(\overline{p})$ we have

$$\frac{d_-^*(p)}{d_+(p)} = \frac{d_-^*(p)}{d_-(p)}\frac{1}{1-|R(p)|^2} = \frac{\overline{d_+(p)}}{d_+(p)}, \qquad p\in C(n/t), \tag{C.21}$$

respectively

$$\frac{d_+(p)}{d_-^*(p)} = \frac{d_+(p)}{d_+^*(p)} \frac{1}{1-|R(p)|^2} = \frac{\overline{d_-^*(p)}}{d_-^*(p)}, \qquad p \in C(n/t). \tag{C.22}$$

We finally define m^5 by

$$\begin{aligned}
m^5 &= m^4 \tilde{B}_+^{-1}, \quad p \in D_k,\ k < j,\\
m^5 &= m^4 \tilde{B}_-^{-1}, \quad p \in D_k^*,\ k < j,\\
m^5 &= m^4 \tilde{B}_+^{-1}, \quad p \in D_{j1},\\
m^5 &= m^4 \tilde{B}_-^{-1}, \quad p \in D_{j1}^*,\\
m^5 &= m^4 \tilde{b}_+^{-1}, \quad p \in D_{j2},\\
m^5 &= m^4 \tilde{b}_-^{-1}, \quad p \in D_{j2}^*,\\
m^5 &= m^4 \tilde{b}_+^{-1}, \quad p \in D_k,\ k > j,\\
m^5 &= m^4 \tilde{b}_-^{-1}, \quad p \in D_k^*,\ k > j,\\
m^5 &= m^4, \quad \text{otherwise},
\end{aligned} \tag{C.23}$$

where we assume that the deformed contour is sufficiently close to the original one. The new jump matrix is given by

$$\begin{aligned}
m_+^5(p,n,t) &= m_-^5(p,n,t) J^5(p,n,t),\\
J^5 &= \tilde{B}_+, \quad p \in C_k,\ k < j,\\
J^5 &= \tilde{B}_-^{-1}, \quad p \in C_k^*,\ k < j,\\
J^5 &= \tilde{B}_+, \quad p \in C_{j1},\\
J^5 &= \tilde{B}_-^{-1}, \quad p \in C_{j1}^*,\\
J^5 &= \tilde{b}_+, \quad p \in C_{j2},\\
J^5 &= \tilde{b}_-^{-1}, \quad p \in C_{j2}^*,\\
J^5 &= \tilde{b}_+, \quad p \in C_k,\ k > j,\\
J^5 &= \tilde{b}_-^{-1}, \quad p \in C_k^*,\ k > j.
\end{aligned} \tag{C.24}$$

Here we have assumed that the function $R(p)$ admits an analytic extension in the corresponding regions. Of course this is not true in general, but we can always evade this obstacle by approximating $R(p)$ by analytic functions in the spirit of [2]. In fact, as in [12, Sect. 5] one sees that it indeed suffices to find an analytic approximation for the left and right reflection coefficients. Moreover, for each spectral band (viewed as a circle on the Riemann surface) one can take the imaginary part of the phase as a coordinate transform and then use the usual Fourier transform with respect to this coordinate (compare [12, Lem. 5.3]). In the band which contains the stationary phase point one has to split $R(p)$ in a polynomial part plus a sufficiently smooth remainder with compact support (see again [12, Lem. 5.3]).

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