ON THE SPATIAL ASYMPTOTICS OF SOLUTIONS OF THE TODA LATTICE

GERALD TESCHL

Abstract. We investigate the spatial asymptotics of decaying solutions of the Toda hierarchy and show that the asymptotic behaviour is preserved by the time evolution. In particular, we show that the leading asymptotic term is time independent. Moreover, we establish infinite propagation speed for the Toda lattice.

1. Introduction

Since the seminal work of Gardner et al. [7] in 1967 it is known that completely integrable wave equations can be solved by virtue of the inverse scattering transform. In particular, this implies that short range perturbations of the free solution remain short range during the time evolution. This raises the question to what extend spatial asymptotical properties are preserved during time evolution. In [1], [2] (see also [9]) Bondareva and Shubin considered the initial value problem for the Korteweg–de Vries (KdV) equation in the class of initial condition which have a prescribed asymptotic expansion in terms of powers of the spatial variable. As part of their analysis they obtained that the leading term of this asymptotic expansion is time independent. Inspired by this intriguing fact, the aim of the present paper is to prove a general result for the Toda equation which contains the analog of this result plus the known results for short range perturbation alluded to before as a special case.

More specifically, recall the Toda lattice [16] (in Flaschka's variables [6])

$$\begin{aligned} \frac{d}{dt}a(n,t) &= a(n,t)\Big(b(n+1,t)-b(n,t)\Big), \\ \frac{d}{dt}b(n,t) &= 2\Big(a(n,t)^2-a(n-1,t)^2\Big), \qquad n\in\mathbb{Z}. \end{aligned} \tag{1.1}$$

It is a well studied physical model and the prototypical discrete integrable wave equation. We refer to the monographs [5], [14], [16] or the review articles [10], [15] for further information.

Then our main result, Theorem 2.5 below, implies for example that

$$a(n,t)=\frac{1}{2}+\frac{\alpha}{n^\delta}+O(\frac{1}{n^{\delta+\varepsilon}}), \quad b(n,t)=\frac{\beta}{n^\delta}+O(\frac{1}{n^{\delta+\varepsilon}}), \quad n\to\infty, \tag{1.2}$$

for all $t\in\mathbb{R}$ provided this holds for the initial condition $t=0$. Here $\alpha,\beta\in\mathbb{R}$ and $\delta\geq 0$, $0<\varepsilon\leq 1$. An analogous result holds for $n\to-\infty$.

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2. The Cauchy problem for the Toda lattice

To set the stage let us recall some basic facts for the Toda lattice. We will only consider bounded solutions and hence require

Hypothesis H.2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z} \times \mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

First of all, to see complete integrability it suffices to find a so-called Lax pair [11], that is, two operators $H(t)$, $P(t)$ in $\ell^2(\mathbb{Z})$ such that the Lax equation

$$\frac{d}{dt} H(t) = P(t)H(t) - H(t)P(t) \tag{2.1}$$

is equivalent to (1.1). Here $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. One can easily convince oneself that the right choice is

$$\begin{aligned} H(t) &= a(t)S^+ + a^-(t)S^- + b(t), \\ P(t) &= a(t)S^+ - a^-(t)S^-, \end{aligned} \tag{2.2}$$

where $(S^\pm f)(n) = f^\pm(n) = f(n \pm 1)$ are the usual shift operators.

Now the Lax equation (2.1) implies that the operators $H(t)$ for different $t \in \mathbb{R}$ are unitarily equivalent (cf. [14, Thm. 12.4]):

Theorem 2.2. *Let $P(t)$ be a family of bounded skew-adjoint operators, such that $t \mapsto P(t)$ is differentiable. Then there exists a family of unitary propagators $U(t,s)$ for $P(t)$, that is,*

$$\frac{d}{dt} U(t,s) = P(t)U(t,s), \qquad U(s,s) = \mathbb{1}. \tag{2.3}$$

Moreover, the Lax equation (2.1) implies

$$H(t) = U(t,s)H(s)U(t,s)^{-1}. \tag{2.4}$$

As pointed out in [12], this result immediately implies global existence of bounded solutions of the Toda lattice as follows: Considering the Banach space of all bounded real-valued coefficients $(a(n), b(n))$ (with the sup norm), local existence is a consequence of standard results for differential equations in Banach spaces. Moreover, Theorem 2.2 implies that the norm $\|H(t)\|$ is constant, which in turn provides a uniform bound on the coefficients of $H(t)$,

$$\|a(t)\|_\infty + \|b(t)\|_\infty \leq 2\|H(t)\| = 2\|H(0)\|. \tag{2.5}$$

Hence solutions of the Toda lattice cannot blow up and are global in time (see [14, Sect. 12.2] for details):

Theorem 2.3. *Suppose $(a_0, b_0) \in M = \ell^\infty(\mathbb{Z}, \mathbb{R}) \oplus \ell^\infty(\mathbb{Z}, \mathbb{R})$. Then there exists a unique integral curve $t \mapsto (a(t), b(t))$ in $C^\infty(\mathbb{R}, M)$ of the Toda lattice (1.1) such that $(a(0), b(0)) = (a_0, b_0)$.*

However, more can be shown. In fact, when considering the inverse scattering transform for the Toda lattice it is desirable to establish existence of solutions within the Marchenko class, that is, solutions satisfying

$$\sum_{n\in\mathbb{Z}} (1 + |n|) \Big(|a(n,t) - \frac{1}{2}| + |b(n,t)| \Big) < \infty \tag{2.6}$$

for all $t \in \mathbb{R}$. That this is indeed true was first established in [13]. Furthermore, the weight $1 + |n|$ can be replaced by an (almost) arbitrary weight function $w(n)$.

Lemma 2.4. *Suppose $a(n,t)$, $b(n,t)$ is some bounded solution of the Toda lattice* (1.1) *satisfying* (2.7) *for one $t_0 \in \mathbb{R}$. Then*

$$\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n,t) - \frac{1}{2}| + |b(n,t)|\Big) < \infty, \tag{2.7}$$

holds for all $t \in \mathbb{R}$, where $w(n) \geq 1$ is some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$.

Moreover, as was demonstrated in [4], one can even replace $|a(n,t) - \frac{1}{2}| + |b(n,t)|$ by $|a(n,t) - \bar{a}(n,t)| + |b(n,t) - \bar{b}(n,t)|$, where $\bar{a}(n,t)$, $\bar{b}(n,t)$ is some other bounded solution of the Toda lattice.

This result shows that the leading asymptotics as $n \to \pm\infty$ is preserved by the Toda lattice. The purpose of this paper is to show that even the leading term is preserved (i.e., time independent) by the time evolution. Set

$$\|(a,b)\|_{w,1} = \sum_{n\in\mathbb{Z}} w(n)\Big(|a(n)| + |b(n)|\Big), \tag{2.8}$$

respectively,

$$\|(a,b)\|_{w,\infty} = \sup_{n\in\mathbb{Z}} w(n)\Big(|a(n)| + |b(n)|\Big). \tag{2.9}$$

Then

Theorem 2.5. *Let $w(n) \geq 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and let $\|(a,b)\|_w = \|(a,b)\|_{w,1}$ or $\|(a,b)\|_w = \|(a,b)\|_{w,\infty}$. Suppose a_0, b_0 and $\tilde{a}_0$, $\tilde{b}_0$ are bounded sequences such that*

$$\|(a_0^+ - a_0, b_0^+ - b_0)\|_w < \infty \quad \text{and} \quad \|(\tilde{a}_0, \tilde{b}_0)\|_w < \infty. \tag{2.10}$$

Suppose $a(t), b(t)$ is the unique solution of the Toda lattice (1.1) *corresponding to the initial conditions*

$$a(0) = a_0 + \tilde{a}_0 \neq 0, \quad b(0) = b_0 + \tilde{b}_0. \tag{2.11}$$

Then this solution is of the form

$$a(t) = a_0 + \tilde{a}(t), \quad b(t) = b_0 + \tilde{b}(t), \quad \text{where} \quad \|(\tilde{a}(t), \tilde{b}(t))\|_w < \infty \tag{2.12}$$

for all $t \in \mathbb{R}$.

Proof. The Toda equation (1.1) implies the differential equation

$$\begin{aligned}
&\frac{d}{dt}\tilde{a}(n,t) = (\tilde{a}(n,t) + a_0(n))\Big(\tilde{b}(n+1,t) - \tilde{b}(n,t) + b_0(n+1) - b_0(n)\Big),\\
&\frac{d}{dt}\tilde{b}(n,t) = 2\Big(\big(\tilde{a}(n,t) + a_0(n)\big)^2 - \big(\tilde{a}(n-1,t) + a_0(n-1)\big)^2\Big), \qquad n \in \mathbb{Z}
\end{aligned} \tag{2.13}$$

for $(\tilde{a}, \tilde{b})$. Since our requirement for $w(n)$ implies that the shift operators are continuous with respect to the norm $\|.\|_w$, we obtain existence of a solution with respect to this norm at least for small t. Moreover, since $w(n) \geq 1$ this solution is bounded and the corresponding coefficients (a,b) coincide with the solution of the

Toda equation from Theorem 2.3. In particular, $(\tilde{a},\tilde{b})$ are uniformly bounded and using the integral form of (2.13) we obtain

$$|\tilde{a}(n,t)| \le |\tilde{a}(n,0)| + const \int_0^t \Big(|\tilde{b}(n+1,s)| + |\tilde{b}(n,s)| + |b_0(n+1,t) - b_0(n)|\Big) ds,$$

$$|\tilde{b}(n,t)| \le |\tilde{b}(n,0)| + const \int_0^t \Big(|\tilde{a}(n,s)| + |\tilde{a}(n-1,s)| + |a_0(n+1,t) - a_0(n)|\Big) ds.$$

Multiplying both inequalities by $w(n)$ and summing over all n, respectively, taking the sup over all n, we infer

$$\|(\tilde{a}(t),\tilde{b}(t))\|_w \le \|(\tilde{a}(0),\tilde{b}(0))\|_w + const \int_0^t \big(\|(\tilde{a}(s),\tilde{b}(s))\|_w + \|(a_0^+ - a_0, b_0^+ - b_0)\|_w\big) ds.$$

Invoking Gronwall's inequality finishes the proof. □

To see the claim (1.2) from the introduction, let

$$a_0(n) = \frac{1}{2} + \frac{\alpha}{n^\delta}, \quad b_0(n) = \frac{\beta}{n^\delta}, \quad \alpha,\beta \in \mathbb{R}, \delta > 0 \tag{2.14}$$

for $n > 0$ and $a_0(n) = b_0(n) = 0$ for $n \le 0$. Now choose $\|(a,b)\|_w = \|(a,b)\|_{w,\infty}$ with

$$w(n) = \begin{cases} (1+|n|)^{\delta+\varepsilon}, & n > 0, \\ 1, & n \le 0. \end{cases} \tag{2.15}$$

and apply the previous theorem. To see Lemma 2.4, just choose $a_0(n) = \frac{1}{2}$, $b_0(n) = 0$ and $\|(a,b)\|_w = \|(a,b)\|_{w,1}$.

Finally, let us remark that the requirement that $w(n)$ does not grow faster than exponentially is important. If it were not present, our result would imply that a compact perturbation of the free solution $a(n,t) = \frac{1}{2}$, $b(n,t) = 0$ remains compact for all time. This is well-known for the KdV equation [17], but we are not aware of a reference for the Toda equation. Hence we include a proof for the sake of completeness.

Theorem 2.6. *Let $a(n,t)$, $b(n,t)$ be a bounded solution of the Toda lattice* (1.1). *If the sequences $a(n,t) - \frac{1}{2}$, $b(n,t)$ are zero for all except for a finite number of $n \in \mathbb{Z}$ for two different times $t_0 \ne t_1$, then they vanish identically.*

Proof. Without loss we can choose $t_0 = 0$ and suppose that the sequences $a(n,0) - \frac{1}{2}$, $b(n,0)$ are zero for all except for a finite number of n. Then the associated reflection coefficients $R_\pm(k,0)$ (see [14] Chapter 10) are rational functions with respect to k and by the inverse scattering transform ([14] Theorem 13.8) we have $R_\pm(k,t) = R_\pm(k,0)\exp(\pm t(k+k^{-1})$, which is not rational for any $t \ne 0$ unless $R_\pm(k,t) \equiv 0$. Hence it must be a pure N soliton solution, which has compact support if and only if it is trivial, $N = 0$. □

3. Extension to the Toda hierarchy

In this section we show that our main result extends to the entire Toda hierarchy. To this end, we introduce the Toda hierarchy using the standard Lax formalism following [3] (see also [8], [14]).

Choose constants $c_0 = 1$, c_j, $1 \le j \le r$, $c_{r+1} = 0$, and set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1}. \end{aligned} \tag{3.1}$$

The sequences g_j, h_j satisfy the recursion relations

$$\begin{aligned} g_0 = 1, \ h_0 = c_1, \\ 2g_{j+1} - h_j - h_j^- - 2bg_j = 0, \quad 0 \le j \le r, \\ h_{j+1} - h_{j+1}^- - 2(a^2 g_j^+ - (a^-)^2 g_j^-) - b(h_j - h_j^-) = 0, \quad 0 \le j < r. \end{aligned} \tag{3.2}$$

Introducing

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t)g_j(t)S^+ - h_j(t))H(t)^{r-j} + g_{r+1}(t), \tag{3.3}$$

a straightforward computation shows that the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{3.4}$$

is equivalent to

$$\mathrm{TL}_r(a(t), b(t)) = \begin{pmatrix} \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) \\ \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) \end{pmatrix} = 0, \tag{3.5}$$

where the dot denotes a derivative with respect to t. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a,b) = 0$.

All results mentioned in the previous section, Theorem 2.2, Theorem 2.3, and Lemma 2.4 remain valid for the entire Toda hierarchy (see [14]) and so does our main result

Theorem 3.1. *Let $w(n) \ge 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and let $\|(a,b)\|_w = \|(a,b)\|_{w,1}$ or $\|(a,b)\|_w = \|(a,b)\|_{w,\infty}$ (see (2.8), respectively, (2.9)). Suppose a_0, b_0 and $\tilde{a}_0$, $\tilde{b}_0$ are bounded sequences such that*

$$\|(a_0^+ - a_0, b_0^+ - b_0)\|_w < \infty \quad \textit{and} \quad \|(\tilde{a}_0, \tilde{b}_0)\|_w < \infty. \tag{3.6}$$

Suppose $a(t), b(t)$ is the unique solution of some equation of the Toda hierarchy, $\mathrm{TL}_r(a,b) = 0$, corresponding to the initial conditions

$$a(0) = a_0 + \tilde{a}_0 > 0, \quad b(0) = b_0 + \tilde{b}_0. \tag{3.7}$$

Then this solution is of the form

$$a(t) = a_0 + \tilde{a}(t), \quad b(t) = b_0 + \tilde{b}(t), \quad \textit{where} \quad \|(\tilde{a}(t), \tilde{b}(t))\|_w < \infty \tag{3.8}$$

for all $t \in \mathbb{R}$.

Proof. The proof is almost identical to the one of Theorem 2.5. All one needs to observe is that

$$\begin{aligned} \|g_{r+1}^+(t) - g_{r+1}(t)\|_w &\le \mathit{const} \|(\tilde{a}(t), \tilde{b}(t))\|_w + \mathit{const} \|(a_0^+ - a_0, b_0^+ - b_0)\|_w, \\ \|h_{r+1}(t) - h_{r+1}^-(t)\|_w &\le \mathit{const} \|(\tilde{a}(t), \tilde{b}(t))\|_w + \mathit{const} \|(a_0^+ - a_0, b_0^+ - b_0)\|_w, \end{aligned}$$

which follows from induction using the recursive definition of $g_{r+1}(t)$ and $h_{r+1}(t)$. $\square$

similarly we also obtain

Theorem 3.2. *Let $a(n,t)$, $b(n,t)$ be a bounded solution of the of some equation of the Toda hierarchy, $\mathrm{TL}_r(a,b)=0$. If the sequences $a(n,t)-\frac{1}{2}$, $b(n,t)$ are zero for all except for a finite number of $n\in\mathbb{Z}$ for two different times $t_0\neq t_1$, then they vanish identically.*

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Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria

E-mail address: `Gerald.Teschl@univie.ac.at`

URL: `http://www.mat.univie.ac.at/~gerald/`