SELF-ADJOINT EXTENSIONS OF THE LAPLACIAN AND KREIN-TYPE RESOLVENT FORMULAS IN NONSMOOTH DOMAINS

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Dedicated to the memory of M. Sh. Birman (1928–2009)

ABSTRACT. This paper has two main goals. First, we are concerned with the classification of self-adjoint extensions of the Laplacian $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$. Here, the domain Ω belongs to a subclass of bounded Lipschitz domains (which we term quasi-convex domains), which contain all convex domains, as well as all domains of class $C^{1,r}$, for $r \in (1/2, 1)$. Second, we establish Krein-type formulas for the resolvents of the various self-adjoint extensions of the Laplacian in quasi-convex domains and study the properties of the corresponding Weyl–Titchmarsh operators (or energy-dependent Dirichlet-to-Neumann maps).

One significant technical innovation in this paper is an extension of the classical boundary trace theory for functions in spaces which lack Sobolev regularity in a traditional sense, but are suitably adapted to the Laplacian.

CONTENTS

1. INTRODUCTION

One fundamental problem, with far-reaching implications in functional analysis, spectral theory, mathematical physics, etc., is that of characterizing all self-adjoint extensions of a given symmetric, densely defined, closed, unbounded operator S in a separable, complex Hilbert space $\mathcal{H}$. Historically, J. von Neumann pioneered the theory of self-adjoint extensions of densely defined, closed symmetric operators. Interestingly in [127], he was also the first to produce what we now call the Krein–von-Neumann self-adjoint extension of a given densely defined and strictly positive operator S. The construction of this extension in the general case, where the underlying densely defined operator S is only nonnegative, was given by M. Krein [75] in 1947. Krein's construction turned out to be extremal among all nonnegative self-adjoint extensions in the following sense: A second distinguished extension had earlier been found by K. Friedrichs (cf. [48]) in 1934, and Krein (cf. [75]) proved in 1947 that all other nonnegative self-adjoint extensions of S necessarily lie in between the Krein–von-Neumann and Friedrichs extensions in the sense of order between semibounded self-adjoint operators in $\mathcal{H}$ (cf., the comments surrounding (9.1)), illustrating the extremal roles played by these two extensions.

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Building on the earlier work of Birman, Vishik, and Lions-Magenes, G. Grubb in [60] characterized all self-adjoint extensions of the minimal realization of a symmetric properly elliptic even-order differential operator in $L^2(\Omega; d^n x)$. Actually, the results in [60] also cover the non-symmetric case (cf. also [61] for related results), but in this paper we will focus on the symmetric case. The setting in [60] is that of a differential operator of order $2m$, $m \in \mathbb{N}$,

$$\mathcal{A}u = \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha u, \tag{1.1}$$

whose coefficients belong to $C^\infty(\overline{\Omega})$, and where $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain. In addition, for some of the results derived in [60] it is assumed that $\mathcal{A}$ is symmetric, that is,

$$(\mathcal{A}u, v)_{L^2(\Omega; d^n x)} = (u, \mathcal{A}v)_{L^2(\Omega; d^n x)}, \quad u, v \in C_0^\infty(\Omega), \tag{1.2}$$

has a strictly positive lower bound, that is, there exists $\kappa_0 > 0$ such that

$$(\mathcal{A}u, u)_{L^2(\Omega; d^n x)} \geq \kappa_0 \, \|u\|^2_{L^2(\Omega; d^n x)}, \quad u \in C_0^\infty(\Omega), \tag{1.3}$$

and is strongly elliptic, in the sense that there exists $\kappa_1 > 0$ such that

$$a^0(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha(x) (i\xi)^\alpha \geq \kappa_1 \, |\xi|^{2m}, \quad x \in \overline{\Omega}, \ \xi \in \mathbb{R}^n. \tag{1.4}$$

Denote by $A_{min,\Omega}$, $A_{max,\Omega}$, and $A_{c,\Omega}$, respectively, the $L^2(\Omega; d^n x)$-realizations of $\mathcal{A}$ with domains

$$\operatorname{dom}(A_{min,\Omega}) := H_0^{2m}(\Omega), \quad \operatorname{dom}(A_{c,\Omega}) := C_0^\infty(\Omega), \tag{1.5}$$

$$\operatorname{dom}(A_{max,\Omega}) := \big\{ u \in L^2(\Omega; d^n x) \, \big| \, \mathcal{A}u \in L^2(\Omega; d^n x) \big\}, \tag{1.6}$$

where $H^s(\Omega)$ stands for the L^2-based Sobolev space of order $s \in \mathbb{R}$ in Ω, and $H_0^s(\Omega)$ is the subspace of $H^s(\mathbb{R}^n)$ consisting of distributions supported in $\overline{\Omega}$. For a domain Ω which is smooth, elliptic regularity implies

$$(A_{min,\Omega})^* = A_{max,\Omega} \text{ and } \overline{A_{c,\Omega}} = A_{min,\Omega}. \tag{1.7}$$

This is a crucial ingredient in Grubb's work. Another basic result readily available in the context of smooth domains is a powerful, well-developed Sobolev boundary trace theory which, among many other things, allows for the characterization

$$H_0^{2m}(\Omega) = \big\{ u \in H^{2m}(\Omega) \, \big| \, \gamma_D^{2m-1} u = 0 \big\}, \tag{1.8}$$

where $\gamma_D^{2m-1} u := \big(\gamma_N^j u\big)_{0 \leq j \leq 2m-1}$ is the Dirichlet trace operator of order $2m-1$ (with γ_N^j denoting the j-th normal derivative on $\partial\Omega$).

Our paper has several principal aims, namely the classification of all self-adjoint extensions of the Laplacian $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$, establishing Krein-type formulas for the resolvents of these self-adjoint extensions, and studying some properties of the corresponding Weyl–Titchmarsh operators. In contrast with Grubb's work mentioned earlier, our goal is to go beyond the smooth setting and allow domains with irregular boundaries. In part, this is motivated by the applications of this theory to spectral analysis. For example, in the paper [18], we make essential use of the results developed here in order to prove Weyl-type asymptotic formulas for the eigenvalue counting function of perturbed Krein Laplacians in nonsmooth domains. In this connection it is worth pointing out that, generally speaking, the smoothness of the boundary of the underlying domain Ω affects both the nature of the remainder in the Weyl asymptotics, as well as the types of differential operators and boundary conditions for which such an asymptotic formula holds. Indeed, understanding the interplay between these structures has became a central theme of research in this area. Before proceeding to describing our main results, we will now highlight some of the subtleties and difficulties which the presence of boundary singularities entail.

In the case of an irregular domain $\Omega \subset \mathbb{R}^n$, several new significant difficulties are encountered:

• First, there is the issue of the (global) Sobolev-regularity exhibited by functions belonging to the domains of $-\Delta_{D,\Omega}$, $-\Delta_{N,\Omega}$, $-\Delta_{K,\Omega}$ (the Dirichlet, Neumann and Krein Laplacian, respectively). If Ω has a C^∞-smooth boundary, then it is well-known that the domains of the aforementioned operators are subspaces of $H^2(\Omega)$. For this particular regularity result the smoothness requirement on $\partial\Omega$ can be substantially reduced: e.g., $C^{1,r}$ with $r > 1/2$ will do. On the other hand, if Ω has a Lipschitz boundary then $\operatorname{dom}(-\Delta_{D,\Omega}) \subset H^{3/2}(\Omega)$ and this result is optimal, as simple examples of (two-dimensional) domains with re-entrant corners show. At a more sophisticated level, B. Dahlberg, D. Jerison and C. Kenig (cf. the discussion in [72]) have constructed a bounded C^1 domain $\Omega \subset \mathbb{R}^n$ and $f \in C^\infty(\overline{\Omega})$ such that

$$(-\Delta_{D,\Omega})^{-1} f \notin W^{2,1}(\Omega), \tag{1.9}$$

where $W^{k,p}(\Omega)$, $1 \leq p \leq \infty$, stands for the L^p-based Sobolev space of order $k \in \mathbb{N}$. This shows that the endpoint $p = 1$ of the implication

$$\begin{aligned} &\Omega \text{ a bounded } C^{1,r} \text{ domain with } 1 > r > 1 - 1/p \\ &\text{implies } \ (-\Delta_{D,\Omega})^{-1} \in \mathcal{B}\big(L^p(\Omega; d^n x), W^{2,p}(\Omega)\big), \end{aligned} \tag{1.10}$$

which (as our arguments show) is valid whenever $p \in (1, \infty)$, is sharp. In fact, the results in Section 7 of [90] show that (1.10) is in the nature of best possible (as far as the regularity of $\partial\Omega$, measured on the Hölder scale, is concerned). Nonetheless, if a bounded Lipschitz domain has all its singularities directed outwardly (more precisely, locally, it satisfies either a uniform exterior ball condition, or is of class $C^{1,r}$ for some $r > 1/2$), then the inclusion $\mathrm{dom}(-\Delta_{D,\Omega}) \subset H^2(\Omega)$ holds.

In this paper we shall work with (what we term) the class of quasi-convex domains, which is a hybrid of the two categories of domains mentioned above. More specifically, an open subset of $\mathbb{R}^n$ is called a quasi-convex domain if it behaves locally either like a $C^{1,r}$ domain (with $r > 1/2$) or like a Lipschitz domain satisfying a uniform exterior ball condition.

• Second, the nature of the Dirichlet and Neumann trace operators, γ_D and γ_N, changes fundamentally in the presence of boundary irregularities. To shed some light on this phenomenon, we recall that if Ω is a C^∞ domain, then the second-order boundary trace operator

$$\gamma_2 = (\gamma_D, \gamma_N) : H^2(\Omega) \to H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \tag{1.11}$$

is well-defined, bounded, and has a linear, continuous right-inverse. The problem with the case where Ω is only Lipschitz is that the range of γ_2 acting on $H^2(\Omega)$ no longer decouples into a Cartesian product of boundary Sobolev spaces. In fact, it was rather recently that $\gamma_2(H^2(\Omega))$ has been identified in [89] (where, in fact, higher smoothness Sobolev spaces are considered) as

$$\big\{(g_0, g_1) \in H^1(\partial\Omega) + L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1 \nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\}, \tag{1.12}$$

where ∇_{tan} and ν are the tangential gradient and outward unit normal on $\partial\Omega$, respectively. As opposed to the case of smoother domains, in the situation where Ω is merely Lipschitz, no optimal (Sobolev) smoothness conditions on the individual functions g_0, g_1 can be inferred from the knowledge that $g_0 \in H^1(\partial\Omega)$ and $g_1 \in L^2(\partial\Omega; d^{n-1}\omega)$ are such that $\nabla_{tan} g_0 + g_1 \nu \in \big(H^{1/2}(\partial\Omega)\big)^n$.

In addition to the aforementioned issue, we are now forced to consider Dirichlet and Neumann traces for functions in $\mathrm{dom}(-\Delta_{max}) := \big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}$. When Ω is a bounded Lipschitz domain, it can be shown that γ_D and γ_N extend as linear bounded maps

$$\widetilde{\gamma}_D : \big\{u \in H^{1/2}(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to L^2(\partial\Omega; d^{n-1}\omega), \tag{1.13}$$

$$\widetilde{\gamma}_N : \big\{u \in H^{3/2}(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to L^2(\partial\Omega; d^{n-1}\omega). \tag{1.14}$$

However, it should be noted that in general,

$$\text{the inclusion } \ \mathrm{dom}(-\Delta_{max}) \subseteq H^s(\Omega) \ \text{ fails for every } \ s > 0. \tag{1.15}$$

Indeed, the function $u(z) := \mathrm{Re}\,(z^{-\alpha}) = r^{-\alpha} \cos(\alpha\,\theta)$ if $z = re^{i\theta}$ is harmonic and square integrable in the (smooth) domain $\Omega := \{z \in \mathbb{C} \,|\, |z-1| < 1\}$ if $\alpha < 1$, but fails to be in $H^s(\Omega)$ if $\alpha > 1-s$. Thus, in light of the fact that functions in $\mathrm{dom}(-\Delta_{max})$ do not, generally speaking, exhibit any global Sobolev regularity besides mere square integrability, the trace results (1.13), (1.14) are not satisfactory for our goals. Nonetheless, we are able to augment (1.13), (1.14) by proving that $\widetilde{\gamma}_D$, $\widetilde{\gamma}_N$ above, further extend as bounded maps in the following contexts:

$$\widehat{\gamma}_D : \mathrm{dom}(-\Delta_{max}) \to \big(N^{1/2}(\partial\Omega)\big)^*, \tag{1.16}$$

$$\widehat{\gamma}_N : \mathrm{dom}(-\Delta_{max}) \to \big(N^{3/2}(\partial\Omega)\big)^*, \tag{1.17}$$

where

$$N^{1/2}(\partial\Omega) := \big\{g \in L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, g\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\}, \tag{1.18}$$

$$N^{3/2}(\partial\Omega) := \big\{g \in H^1(\partial\Omega) \,\big|\, \nabla_{tan} g \in \big(H^{1/2}(\partial\Omega)\big)^n\big\}. \tag{1.19}$$

These "exotic" spaces provide the natural context for describing the mapping properties for the Dirichlet and Neumann trace operators acting on $\mathrm{dom}(-\Delta_{max})$. They are natural, in the sense that $N^{1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)$ and $N^{3/2}(\partial\Omega) = H^{3/2}(\partial\Omega)$ if Ω is a $C^{1,r}$ domain for some $r > 1/2$.

Dealing with the aforementioned topics occupies the bulk of the next six sections of the paper, where we develop a trace theory which goes considerably beyond the scope of the traditional treatment of (reasonably) smooth domains. Having dealt with this host of issues in Sections 2–8, we then proceed to the next item on our agenda, namely the task of characterizing all self-adjoint extensions of the minimal Laplacian in quasi-convex domains. Based on our trace theory and an abstract, functional analytic result of Grubb [60], we prove in Section 14 the following theorem, which provides a universal parametrization of all self-adjoint extensions of $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega;d^nx)$:

Theorem 1.1. *Assume that $\Omega\subseteq\mathbb{R}^n$ is a quasi-convex domain and let $z\in\mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$. Suppose that X is a closed subspace of $\big(N^{1/2}(\partial\Omega)\big)^*$ and denote by X^* the conjugate dual space of X. In addition, consider a self-adjoint operator*

$$L:\mathrm{dom}(L)\subseteq X\to X^*, \tag{1.20}$$

and define the linear operator $-\Delta_{X,L,z}^D:\mathrm{dom}(-\Delta_{X,L,z}^D)\subset L^2(\Omega;d^nx)\to L^2(\Omega;d^nx)$, by taking

$$\begin{aligned} &-\Delta_{X,L,z}^D u:=(-\Delta-z)u,\\ &u\in\mathrm{dom}(-\Delta_{X,L,z}^D):=\big\{v\in\mathrm{dom}(-\Delta_{max})\,\big|\,\widehat{\gamma}_D v\in\mathrm{dom}(L),\ \tau_z^N v\big|_X=L\big(\widehat{\gamma}_D v\big)\big\}. \end{aligned} \tag{1.21}$$

Above, τ_z^N is a regularized Neumann trace operator (for details see (12.1), (12.2)), and the boundary condition $\tau_z^N u\big|_X=L\big(\widehat{\gamma}_D u\big)$ is interpreted as

$${}_{N^{1/2}(\partial\Omega)}\langle\tau_z^N u,f\rangle_{(N^{1/2}(\partial\Omega))^*}=\overline{{}_X\langle f,L(\widehat{\gamma}_D u)\rangle_{X^*}},\quad f\in X. \tag{1.22}$$

Then

$$-\Delta_{X,L,z}^D\ \text{ is self-adjoint in } L^2(\Omega;d^nx), \tag{1.23}$$

and

$$-\Delta_{min}-zI_\Omega\subsetneqq-\Delta_{X,L,z}^D\subsetneqq-\Delta_{max}-zI_\Omega. \tag{1.24}$$

Conversely, if

$$\widetilde{S}:\mathrm{dom}\big(\widetilde{S}\big)\subseteq L^2(\Omega;d^nx)\to L^2(\Omega;d^nx) \tag{1.25}$$

is a self-adjoint operator with the property that

$$-\Delta_{min}-zI_\Omega\subseteq\widetilde{S}\subseteq-\Delta_{max}-zI_\Omega, \tag{1.26}$$

then there exist X, a closed subspace of $\big(N^{1/2}(\partial\Omega)\big)^*$, and $L:\mathrm{dom}(L)\subseteq X\to X^*$, a self-adjoint operator, such that

$$\widetilde{S}=-\Delta_{X,L,z}^D. \tag{1.27}$$

In the above scheme, the operator $\widetilde{S}$ and the pair X,L correspond uniquely to each other. In fact,

$$\mathrm{dom}(L)=\widehat{\gamma}_D\big(\mathrm{dom}\big(\widetilde{S}\big)\big),\quad X=\overline{\widehat{\gamma}_D\big(\mathrm{dom}\big(\widetilde{S}\big)\big)}\quad\big(\textit{with closure in }\big(N^{1/2}(\partial\Omega)\big)^*\big). \tag{1.28}$$

In the context of Theorem 15.2, it is illuminating to indicate how various distinguished self-adjoint extensions of the Laplacian occur in this scheme. For example, for every $z_0\in\mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$ one has

$$X:=\mathrm{dom}(L):=\{0\}\ \text{ and }\ L:=0\ \text{ imply }\ -\Delta_{X,L,z_0}^D+z_0I_\Omega=-\Delta_{D,\Omega}, \tag{1.29}$$

the Dirichlet Laplacian, and

$$\left.\begin{array}{c} X:=\big(N^{1/2}(\partial\Omega)\big)^*\text{ and }L:=M_{D,N,\Omega}^{(0)}(z_0)\\ \text{with }\mathrm{dom}(L):=N^{3/2}(\partial\Omega)\end{array}\right\}\ \text{imply }\ -\Delta_{X,L,z_0}^D+z_0I_\Omega=-\Delta_{N,\Omega}, \tag{1.30}$$

the Neumann Laplacian. Furthermore,

$$X:=\mathrm{dom}(L):=\big(N^{1/2}(\partial\Omega)\big)^*,\ \text{ and }\ L:=0\ \text{ imply }\ -\Delta_{X,L,z_0}^D=-\Delta_{K,\Omega,z_0}, \tag{1.31}$$

the Krein Laplacian. Related versions of these results (starting from a different "reference" operator) can be found in Theorem 15.2 and Corollary 15.4.

As an interesting application of the material developed in this paper, the final Section 16 is devoted to proving Krein-type formulas for the resolvents of various self-adjoint extensions of the Laplacian in quasi-convex domains, and to studying the properties of the corresponding spectral parameter dependent Dirichlet-to-Neumann maps (i.e., Weyl–Titchmarsh operators). To put this application in a proper perspective requires some preparation and so we will next recall some pertinent facts derived in [52] (see also [50] and [57]) on abstract Krein-type resolvent formulas. For hints to the literature on this vast and currently very active area of research, we refer to Section 16.

Let S be a closed, symmetric operator in a separable, complex Hilbert space $\mathcal{H}$ with equal deficiency indices $\operatorname{def}(S)=(r,r)$, $r\in\mathbb{N}\cup\{\infty\}$, and suppose that S_ℓ, $\ell=1,2$, are two distinct, relatively prime self-adjoint extensions of S, that is, $\operatorname{dom}(S_1)\cap\operatorname{dom}(S_2)=\operatorname{dom}(S)$. At this instance none of S and S_j, $j=1,2$, need to be bounded from below and we emphasize that S_1 and S_2 are chosen to be relatively prime for simplicity only. Then the deficiency subspaces $\mathcal{N}_\pm$ of S are given by

$$\mathcal{N}_\pm=\ker(S^*\mp iI_{\mathcal{H}}),\quad \dim(\mathcal{N}_\pm)=\operatorname{def}(S)=r, \tag{1.32}$$

and for any self-adjoint extension $\widetilde{S}$ of S in $\mathcal{H}$, the corresponding Cayley transform $C_{\widetilde{S}}$ in $\mathcal{H}$ is defined by

$$C_{\widetilde{S}}=\big(\widetilde{S}+iI_{\mathcal{H}}\big)\big(\widetilde{S}-iI_{\mathcal{H}}\big)^{-1}, \tag{1.33}$$

implying

$$C_{\widetilde{S}}\mathcal{N}_-=\mathcal{N}_+. \tag{1.34}$$

Given a self-adjoint extension $\widetilde{S}$ of S and a closed linear subspace $\mathcal{N}$ of $\mathcal{N}_+$, $\mathcal{N}\subseteq\mathcal{N}_+$, the Donoghue-type Weyl–Titchmarsh operator $M_{\widetilde{S},\mathcal{N}}(z)\in\mathcal{B}(\mathcal{N})$ associated with the pair $(\widetilde{S},\mathcal{N})$ is defined by

$$\begin{aligned} M_{\widetilde{S},\mathcal{N}}(z)&=P_{\mathcal{N}}\big(z\widetilde{S}+I_{\mathcal{H}}\big)\big(\widetilde{S}-zI_{\mathcal{H}}\big)^{-1}P_{\mathcal{N}}\big|_{\mathcal{N}}\\ &=zI_{\mathcal{N}}+(1+z^2)P_{\mathcal{N}}\big(\widetilde{S}-zI_{\mathcal{H}}\big)^{-1}P_{\mathcal{N}}\big|_{\mathcal{N}},\quad z\in\mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{1.35}$$

with $I_{\mathcal{N}}$ the identity operator in $\mathcal{N}$, and $P_{\mathcal{N}}$ the orthogonal projection in $\mathcal{H}$ onto $\mathcal{N}$.

Following Saakjan [112] (in a version presented in Theorem 5 and Corollary 6 in [52]), and using the notions introduced in (1.32)–(1.35), Krein's formula for the difference of the resolvents of S_1 and S_2 then reads as follows (cf. [52], [112]):

$$\begin{aligned} (S_2-zI_{\mathcal{H}})^{-1}&=(S_1-zI_{\mathcal{H}})^{-1}+(S_1-iI_{\mathcal{H}})(S_1-zI_{\mathcal{H}})^{-1}P_{1,2}(z)(S_1+iI_{\mathcal{H}})(S_1-zI_{\mathcal{H}})^{-1}\\ &=(S_1-zI_{\mathcal{H}})^{-1}+(S_1-iI_{\mathcal{H}})(S_1-zI_{\mathcal{H}})^{-1}P_{\mathcal{N}_+}\\ &\quad\times(\tan(\alpha_{1,2})-M_{S_1,\mathcal{N}_+}(z))^{-1}P_{\mathcal{N}_+}(S_1+iI_{\mathcal{H}})(S_1-zI_{\mathcal{H}})^{-1},\quad z\in\rho(S_1)\cap\rho(S_2), \end{aligned} \tag{1.36}$$

where

$$e^{-2i\alpha_{1,2}}=-C_{S_2}C_{S_1}^{-1}\big|_{\mathcal{N}_+}. \tag{1.37}$$

One can show that $M_{S_1,\mathcal{N}_+}(\cdot)$ (and hence $(\tan(\alpha_{1,2})-M_{S_1,\mathcal{N}_+}(\cdot))^{-1}$) is an operator-valued Herglotz function, that is, $M_{S_1,\mathcal{N}_+}(\cdot)$ is analytic on $\mathbb{C}_+=\{z\in\mathbb{C}\,|\,\operatorname{Im}(z)>0\}$ and $\operatorname{Im}(M_{S_1,\mathcal{N}_+}(z))\geq 0$, $z\in\mathbb{C}_+$. In addition, $M_{S_1,\mathcal{N}_+}(\cdot)$ has the symmetry property,

$$[M_{S_1,\mathcal{N}_+}(z)]^*=M_{S_1,\mathcal{N}_+}(\overline{z}),\quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{1.38}$$

Next, assume in addition that $S\geqslant\varepsilon I_{\mathcal{H}}$ for some $\varepsilon>0$, and denote by S_K the Krein–von Neumann extension, and by S_F the Friedrichs extension of S, respectively (cf. Section 9 for more details). Since S_K and S_F are relatively prime (see, e.g., [18, Lemma 2.8]), (1.37) yields the following version of Krein's formula connecting the resolvents of S_K and S_F:

$$\begin{aligned} (S_K-zI_{\mathcal{H}})^{-1}&=(S_F-zI_{\mathcal{H}})^{-1}\\ &\quad+(S_F-iI_{\mathcal{H}})(S_F-zI_{\mathcal{H}})^{-1}P_{\mathcal{N}_+}\big[M_{S_F,\mathcal{N}_+}(0)-M_{S_F,\mathcal{N}_+}(z)\big]^{-1}\\ &\quad\times P_{\mathcal{N}_+}(S_F+iI_{\mathcal{H}})(S_F-zI_{\mathcal{H}})^{-1},\quad z\in\rho(S_K)\cap\rho(S_F). \end{aligned} \tag{1.39}$$

Equation (1.39) follows by combining Theorem 4.7 and Corollary 4.8 in [50] and will be further discussed in more detail elsewhere.

In a nutshell, this represents the approach to Krein-type resolvent formulas with emphasis on the deficiency subspace $\mathcal{N}_+$, and so the only Hilbert spaces involved are $\mathcal{N}_+$ and $\mathcal{H}$ (with $\mathcal{N}_+\subset\mathcal{H}$). However, in connection with PDE applications, where, for example, S is generated by a suitable second-order strongly elliptic differential operator on the domain $\Omega\subset\mathbb{R}^n$, one would naturally prefer to replace the pair of spaces $(\mathcal{N}_+,\mathcal{H})$ by the pair $\big(L^2(\partial\Omega;d^{n-1}\omega),L^2(\Omega;d^nx)\big)$, and at the same time, replace the abstract Donoghue-type Weyl–Titchmarsh operator $M_{S_1,\mathcal{N}_+}(\cdot)$ by appropriate energy-dependent Dirichlet-to-Neumann maps. This change of emphasis then also necessitates the introduction of appropriate (extensions of) Dirichlet and Neumann boundary trace operators γ_D and γ_N.

We note that in the PDE context these ideas were realized only very recently in the work of Amrein and Pearson [9], Behrndt and Langer [22], Brown, Grubb, and Wood [31], Brown, Hinchliffe, Marletta, Naboko, and Wood [32], Brown, Marletta, Naboko, and Wood [34], Gesztesy and Mitrea [53], [54], Grubb [64] (including a discussion of non-self-adjoint extensions), Posilicano [105], Posilicano and Raimondi [106], and Ryzhov [111]. With the exception of Grubb and Ryzhov, who treat $C^{1,1}$ domains Ω, the remaining authors are dealing with the case of C^∞ smooth domains Ω.

One of our motivations for introducing the class of nonsmooth, quasi-convex domains Ω, and the associated boundary trace theory, was precisely to be able to prove Krein-type resolvent formulas for Laplacians on such nonsmooth domains. As a typical result we prove in Section 16 we thus mention the following theorem:

Theorem 1.2. *Assume that Ω is a quasi-convex domain and suppose that $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$. In addition, consider two bounded, self-adjoint operators*

$$L_j : \big(N^{1/2}(\partial\Omega)\big)^* \to N^{1/2}(\partial\Omega), \quad j = 1, 2. \tag{1.40}$$

Associated with these, define the operators $-\Delta^D_{X_j,L_j,z_0}$, $j = 1, 2$, as in (1.21) corresponding to $z = z_0$ and $X_1 = X_2 = \big(N^{1/2}(\partial\Omega)\big)^*$. Finally, fix a complex number $z \in \mathbb{C}\backslash\big(\sigma(-\Delta^D_{X_1,L_1,z_0}) \cup \sigma(-\Delta^D_{X_2,L_2,z_0})\big)$. Then the following Krein-type resolvent formula holds on $L^2(\Omega; d^n x)$,

$$\begin{aligned} \big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1} &= \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} \\ &\quad + \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big]^* M^D_{L_1,L_2,z_0}(z)\big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big], \end{aligned} \tag{1.41}$$

where

$$\begin{aligned} M^D_{L_1,L_2,z_0}(z) &:= (L_2 - L_1)\big[I + \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z + z_0) - L_2\big]^{-1}(L_2 - L_1)\big] \\ &= (L_2 - L_1)\big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z + z_0) - L_2\big]^{-1} \\ &\quad \times \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z + z_0) - L_1\big], \quad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{1.42}$$

has the property that

$$\mathbb{C}_+ \ni z \mapsto M^D_{L_1,L_2,z_0}(z) \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, N^{1/2}(\partial\Omega)\big) \tag{1.43}$$

is an operator-valued Herglotz function which satisfies

$$\big[M^D_{L_1,L_2,z_0}(z)\big]^* = M^D_{L_1,L_2,z_0}(\overline{z}). \tag{1.44}$$

Here $M^{(0)}_{D,N,\Omega}(\cdot)$ denotes the Dirichlet-to-Neumann map for the (unperturbed) Laplacian in the domain Ω described in detail in Section 5.

Finally, we emphasize that an application of the principal results of this paper to Weyl-type spectral asymptotics for perturbed Krein Laplacian in nonsmooth domains Ω has been presented in [18].

We close this section by briefly elaborating on the most basic notational conventions used throughout this paper. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\mathrm{dom}(T)$, $\mathrm{ran}(T)$, and $\ker(T)$ denoting the domain, range, and the kernel (null space) of T. The closure of a closable operator S is denoted by $\overline{S}$. The spectrum of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$. The Banach spaces of bounded and compact linear operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, $\mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathcal{B}_\infty(\mathcal{X}_1, \mathcal{X}_2)$ will be used for bounded and compact operators between two Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$. In addition, $U_1 \dotplus U_2$ denotes the direct sum of the subspaces U_1 and U_2 of a Banach space $\mathcal{X}$; and $V_1 \oplus V_2$ represents the orthogonal direct sum of the subspaces V_j, $j = 1, 2$, of a Hilbert space $\mathcal{H}$.

Throughout this manuscript, if X denotes a Banach space, X^* denotes the *adjoint space* of continuous conjugate linear functionals on X, that is, the *conjugate dual space* of X (rather than the usual dual space of continuous linear functionals on X). This avoids the well-known awkward distinction between adjoint operators in Banach and Hilbert spaces (cf., e.g., the pertinent discussion in [41, p. 3, 4]).

Finally, a notational comment: For obvious reasons in connection with quantum mechanical applications, we will, with a slight abuse of notation, dub $-\Delta$ (rather than Δ) as the "Laplacian" in this paper.

2. Sobolev Spaces in Lipschitz and $C^{1,r}$ Domains

In this section we summarize some fundamental results on Lipschitz and $C^{1,r}$ domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and on the corresponding Sobolev spaces $H^s(\Omega)$ and $H^r(\partial\Omega)$ needed in the remainder of this paper.

Before we focus primarily on bounded Lipschitz domains, we briefly recall some basic facts in connection with Sobolev spaces corresponding to open sets $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$: For an arbitrary $m \in \mathbb{N} \cup \{0\}$, we follow the customary way of defining L^2-Sobolev spaces of order $\pm m$ in Ω as

$$H^m(\Omega) := \big\{u \in L^2(\Omega; d^n x) \,\big|\, \partial^\alpha u \in L^2(\Omega; d^n x),\, 0 \leq |\alpha| \leq m\big\}, \tag{2.1}$$

$$H^{-m}(\Omega) := \bigg\{u \in \mathcal{D}'(\Omega) \,\bigg|\, u = \sum_{0\leq|\alpha|\leq m} \partial^\alpha u_\alpha, \text{ with } u_\alpha \in L^2(\Omega; d^n x),\, 0 \leq |\alpha| \leq m\bigg\}, \tag{2.2}$$

equipped with natural norms (cf., e.g., [2, Ch. 3], [88, Ch. 1]). Here $\mathcal{D}'(\Omega)$ denotes the usual set of distributions on $\Omega \subseteq \mathbb{R}^n$ (i.e., the dual of $\mathcal{D}(\Omega)$, the space of test functions $C_0^\infty(\Omega)$, equipped with the usual inductive limit topology). Then we set

$$H_0^m(\Omega) := \text{ the closure of } C_0^\infty(\Omega) \text{ in } H^m(\Omega), \quad m \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

As is well-known, all three spaces above are Banach, reflexive and, in addition,

$$\big(H_0^m(\Omega)\big)^* = H^{-m}(\Omega). \tag{2.4}$$

Again, see, for instance, [2, Ch. 3], [88, Ch. 1].

We recall that an open, nonempty set $\Omega \subseteq \mathbb{R}^n$ is called a *Lipschitz domain* if the following property holds: There exists an open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of the boundary $\partial\Omega$ of Ω such that for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (considered in a new system of coordinates obtained from the original one via a rigid motion). The number $\max\{\|\nabla\varphi_j\|_{L^\infty(\mathbb{R}^{n-1}; d^{n-1}x')^{n-1}} \,|\, 1 \leq j \leq N\}$ is said to represent the *Lipschitz character* of Ω.

As regards L^2-based Sobolev spaces of fractional order $s \in \mathbb{R}$, on arbitrary Lipschitz domains $\Omega \subseteq \mathbb{R}^n$, we introduce

$$H^s(\mathbb{R}^n) := \bigg\{U \in \mathcal{S}'(\mathbb{R}^n) \,\bigg|\, \|U\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} d^n\xi\, \big|\widehat{U}(\xi)\big|^2 \big(1 + |\xi|^{2s}\big) < \infty\bigg\}, \tag{2.5}$$

$$H^s(\Omega) := \big\{u \in \mathcal{D}'(\Omega) \,\big|\, u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\big\} = R_\Omega\, H^s(\mathbb{R}^n), \tag{2.6}$$

where R_Ω denotes the restriction operator (i.e., $R_\Omega\, U = U|_\Omega$, $U \in H^s(\mathbb{R}^n)$), $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$, and $\widehat{U}$ denotes the Fourier transform of $U \in \mathcal{S}'(\mathbb{R}^n)$. These definitions are consistent with (2.1), (2.2). Next, retaining that $\Omega \subseteq \mathbb{R}^n$ is an arbitrary Lipschitz domain, we introduce

$$H_0^s(\Omega) := \big\{u \in H^s(\mathbb{R}^n) \,\big|\, \operatorname{supp}(u) \subseteq \overline{\Omega}\big\}, \quad s \in \mathbb{R}, \tag{2.7}$$

equipped with the natural norm induced by $H^s(\mathbb{R}^n)$. The space $H_0^s(\Omega)$ is reflexive, being a closed subspace of $H^s(\mathbb{R}^n)$. Finally, we introduce for all $s \in \mathbb{R}$,

$$\mathring{H}^s(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in } H^s(\Omega), \tag{2.8}$$

$$H_z^s(\Omega) = R_\Omega\, H_0^s(\Omega). \tag{2.9}$$

Assuming from now on that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with a compact boundary, we recall the existence of a universal linear extension operator $E_\Omega : \mathcal{D}'(\Omega) \to \mathcal{S}'(\mathbb{R}^n)$ such that $E_\Omega : H^s(\Omega) \to H^s(\mathbb{R}^n)$ is bounded for all $s \in \mathbb{R}$, and $R_\Omega E_\Omega = I_{H^s(\Omega)}$ (cf. [108]). If $\widetilde{C_0^\infty(\Omega)}$ denotes the set of $C_0^\infty(\Omega)$-functions extended to all of $\mathbb{R}^n$ by setting functions zero outside of Ω, then for all $s \in \mathbb{R}$, $\widetilde{C_0^\infty(\Omega)} \hookrightarrow H_0^s(\Omega)$ densely.

Moreover, one has

$$\big(H_0^s(\Omega)\big)^* = H^{-s}(\Omega), \quad s \in \mathbb{R}. \tag{2.10}$$

(cf., e.g., [72]) consistent with (2.3), and also,

$$\big(H^s(\Omega)\big)^* = H_0^{-s}(\Omega), \quad s \in \mathbb{R}, \tag{2.11}$$

in particular, $H^s(\Omega)$ is a reflexive Banach space. We shall also use the fact that for a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with compact boundary, the space $\mathring{H}^s(\Omega)$ satisfies

$$\mathring{H}^s(\Omega) = H_z^s(\Omega) \text{ if } s > -1/2,\ s \notin \big\{\tfrac{1}{2} + \mathbb{N}_0\big\}. \tag{2.12}$$

For a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with compact boundary it is also known that

$$\big(H^s(\Omega)\big)^* = H^{-s}(\Omega), \quad -1/2 < s < 1/2. \tag{2.13}$$

See [121] for this and other related properties. Throughout this paper we agree to use the *adjoint* (rather than the dual) space X^* of a Banach space X.

From this point on (and unless explicitly stated otherwise) we will always make at least the following assumption on the set Ω. (This will be strengthened in the second part of the paper):

Hypothesis 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain.*

At times we will invoke also the notion of $C^{1,r}$ domains and then introduce the following stronger hypothesis on Ω:

Hypothesis 2.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{1,r}$, $r \in (1/2, 1)$.*

The definition of a domain of class $C^{1,r}$, $0 < r < 1$, is similar to that of a Lipschitz domain, except that, this time, the functions φ_j used to locally describe the boundary of Ω are such that $\nabla\varphi_j$ is Hölder continuous of order r.

Parenthetically, we note that a $C^{1,r}$ domain $\Omega \subset \mathbb{R}^n$, with $r \in (0, 1)$, is characterized by the following set of conditions:

$$\begin{aligned} &(i)\ \Omega \text{ is of finite local perimeter, (i.e., } \nabla\chi_\Omega \text{ is a Radon measure in } \mathbb{R}^n),\\ &(ii)\ \Omega \text{ lies on only one side of its topological boundary, (i.e., } \partial\overline{\Omega} = \partial\Omega),\\ &(iii) \text{ the outward unit normal (in the sense of Federer [46]) belongs to } C^r(\partial\Omega). \end{aligned} \tag{2.14}$$

For a more detailed discussion in this regard, the reader is referred to [70].

The classical theorem of Rademacher on almost everywhere differentiability of Lipschitz functions ensures that for any Lipschitz domain Ω, the surface measure $d^{n-1}\omega$ is well-defined on $\partial\Omega$ and that there exists an outward pointing unit normal vector ν at almost every point of $\partial\Omega$.

In the case where $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of a function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ of class $C^{1,r}$, $r \in (0, 1)$, we define the Sobolev space $H^s(\partial\Omega)$ for $0 \leq s < 1+r$, as the space of functions $f \in L^2(\partial\Omega; d^{n-1}\omega)$ with the property that $f(x', \varphi(x'))$, as a function of $x' \in \mathbb{R}^{n-1}$, belongs to $H^s(\mathbb{R}^{n-1})$. This definition is easily adapted to the case where Ω is a domain of class $C^{1,r}$, $r \in (0, 1)$, whose boundary is compact, by using a smooth partition of unity. Finally, for $-1 - r < s < 0$, we set $H^s(\partial\Omega) = \big(H^{-s}(\partial\Omega)\big)^*$. The same construction concerning $H^s(\partial\Omega)$ applies in the case where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain (i.e., $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is only Lipschitz) provided $0 \leqslant s \leqslant 1$. In this scenario we set

$$H^s(\partial\Omega) = \big(H^{-s}(\partial\Omega)\big)^*, \quad -1 \leqslant s \leqslant 0. \tag{2.15}$$

It is useful to observe that this entails ($\approx$ denoting equivalent norms)

$$\|f\|_{H^{-s}(\partial\Omega)} \approx \|\sqrt{1 + |\nabla\varphi(\cdot)|^2} f(\cdot, \varphi(\cdot))\|_{H^{-s}(\mathbb{R}^{n-1})}, \quad 0 \leq s \leq 1. \tag{2.16}$$

To define $H^s(\partial\Omega)$, $0 \leq s \leqslant 1$, when Ω is a Lipschitz domain with compact boundary, we use a smooth partition of unity to reduce matters to the graph case. More precisely, if $0 \leq s \leq 1$ then $f \in H^s(\partial\Omega)$ if and only if the assignment $\mathbb{R}^{n-1} \ni x' \mapsto (\psi f)(x', \varphi(x'))$ is in $H^s(\mathbb{R}^{n-1})$ whenever $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with the property that if Σ is an appropriate rotation and translation of $\{(x', \varphi(x')) \in \mathbb{R}^n \,|\, x' \in \mathbb{R}^{n-1}\}$, then $(\operatorname{supp}(\psi) \cap \partial\Omega) \subset \Sigma$ (this appears to be folklore, but a proof will appear in [94, Proposition 2.4]). Then Sobolev spaces with a negative amount of smoothness are defined as in (2.15) above.

From the above characterization of $H^s(\partial\Omega)$ it follows that any property of Sobolev spaces (of order $s \in [-1, 1]$) defined in Euclidean domains, which are invariant under multiplication by smooth, compactly supported functions as well as composition by bi-Lipschitz diffeomorphisms, readily extends to the setting of $H^s(\partial\Omega)$ (via localization and pullback). As a concrete example, for each Lipschitz domain Ω with compact boundary, one has

$$H^s(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega) \text{ compactly if } 0 < s \leq 1. \tag{2.17}$$

For additional background information in this context we refer, for instance, to [20], [21], [41, Chs. V, VI], [59, Ch. 1], [92, Ch. 3], [128, Sect. I.4.2].

For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with compact boundary, an equivalent definition of the Sobolev space $H^1(\partial\Omega)$ is the collection of functions in $L^2(\partial\Omega; d^{n-1}\omega)$ with the property that the (pointwise, Euclidean) norm of their tangential gradient belongs to $L^2(\partial\Omega; d^{n-1}\omega)$. To make this precise, consider the first-order tangential derivative operators $\partial/\partial\tau_{j,k}$, $1 \leq j, k \leq n$, acting on a function ψ of class C^1 in a neighborhood of $\partial\Omega$ by

$$\partial\psi/\partial\tau_{j,k} = \nu_j(\partial_k\psi)\Big|_{\partial\Omega} - \nu_k(\partial_j\psi)\Big|_{\partial\Omega}. \tag{2.18}$$

For every $f \in L^1(\partial\Omega)$ define the functional $\partial f/\partial\tau_{j,k}$ by setting

$$\partial f/\partial\tau_{j,k} : C^1(\mathbb{R}^n) \ni \psi \mapsto \int_{\partial\Omega} d^{n-1}\omega\, f\, (\partial\psi/\partial\tau_{k,j}). \tag{2.19}$$

When $f \in L^1(\partial\Omega; d^{n-1}\omega)$ has $\partial f/\partial\tau_{j,k} \in L^1(\partial\Omega; d^{n-1}\omega)$, the following integration by parts formula holds:

$$\int_{\partial\Omega} d^{n-1}\omega\, f\,(\partial\psi/\partial\tau_{k,j}) = \int_{\partial\Omega} d^{n-1}\omega\,(\partial f/\partial\tau_{j,k})\,\psi, \quad \psi \in C^1(\mathbb{R}^n). \tag{2.20}$$

One then has the Sobolev-type description of $H^1(\partial\Omega)$:

$$H^1(\partial\Omega) = \big\{f \in L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \partial f/\partial\tau_{j,k} \in L^2(\partial\Omega; d^{n-1}\omega),\ j,k = 1,\dots,n\big\}, \tag{2.21}$$

with

$$\|f\|_{H^1(\partial\Omega)} \approx \|f\|_{L^2(\partial\Omega;d^{n-1}\omega)} + \sum_{j,k=1}^{n} \|\partial f/\partial\tau_{j,k}\|_{L^2(\partial\Omega;d^{n-1}\omega)}, \tag{2.22}$$

or equivalently,

$$\begin{aligned} H^1(\partial\Omega) = \bigg\{f \in L^2(\partial\Omega; d^{n-1}\omega) \,\bigg|\, &\text{ there exists a constant } c>0 \text{ such that} \\ &\qquad\qquad\qquad \text{for every } v \in C_0^\infty(\mathbb{R}^n), \\ \bigg|\int_{\partial\Omega} d^{n-1}\omega f\,\partial v/\partial\tau_{j,k}\bigg| &\leq c\,\|v\|_{L^2(\partial\Omega;d^{n-1}\omega)},\ j,k = 1,\dots,n\bigg\}. \end{aligned} \tag{2.23}$$

We also point out that if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, then for any $j,k \in \{1,...,n\}$, the operator

$$\partial/\partial\tau_{j,k} : H^s(\partial\Omega) \to H^{s-1}(\partial\Omega), \quad 0 \leq s \leq 1, \tag{2.24}$$

is well-defined, linear, and bounded. This is proved by interpolating the case $s = 1$ and its dual version. In fact, the following more general result (extending (2.21)) holds:

Lemma 2.3. *Assume Hypothesis* 2.1. *Then for every* $s \in [0,1]$,

$$H^s(\partial\Omega) = \{f \in L^2(\partial\Omega; d^{n-1}\omega) \,|\, \partial f/\partial\tau_{j,k} \in H^{s-1}(\partial\Omega),\ 1 \leq j,k \leq n\} \tag{2.25}$$

and

$$\|f\|_{H^s(\partial\Omega)} \approx \|f\|_{L^2(\partial\Omega;d^{n-1}\omega)} + \sum_{j,k=1}^{n} \|\partial f/\partial\tau_{j,k}\|_{H^{s-1}(\partial\Omega)}. \tag{2.26}$$

A proof can be found in [53], where the following result is also established.

Lemma 2.4. *Assume Hypothesis* 2.2. *Then*

$$H^{3/2}(\partial\Omega) = \{f \in H^1(\partial\Omega) \,|\, \partial f/\partial\tau_{j,k} \in H^{1/2}(\partial\Omega)\ 1 \leq j,k \leq n\} \tag{2.27}$$

and

$$\|f\|_{H^{3/2}(\partial\Omega)} \approx \|f\|_{H^1(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial f/\partial\tau_{j,k}\|_{H^{1/2}(\partial\Omega)}. \tag{2.28}$$

In the sequel, the sesquilinear form

$$\langle\,\cdot\,,\,\cdot\,\rangle_s = {}_{H^s(\partial\Omega)}\langle\,\cdot\,,\,\cdot\,\rangle_{(H^s(\partial\Omega))^*} : H^s(\partial\Omega) \times \big(H^s(\partial\Omega)\big)^* \to \mathbb{C}, \tag{2.29}$$

(antilinear in the first, linear in the second factor), will occasionally denote the duality pairing between $H^s(\partial\Omega)$ and $\big(H^s(\partial\Omega)\big)^*$ for appropriate $s \geq 0$. In particular,

$$\begin{aligned} &\langle f,g\rangle_s = \int_{\partial\Omega} d^{n-1}\omega(\xi)\,\overline{f(\xi)}g(\xi) = (f,g)_{L^2(\partial\Omega;d^{n-1}\omega)}, \\ &f \in H^s(\partial\Omega),\ g \in L^2(\partial\Omega; d^{n-1}\omega) \hookrightarrow \big(H^s(\partial\Omega)\big)^* = H^{-s}(\partial\Omega),\ s \in [0,1], \end{aligned} \tag{2.30}$$

where, as before, $d^{n-1}\omega$ stands for the surface measure on $\partial\Omega$.

3. The Dirichlet and Neumann Trace Operators in Lipschitz Domains

We briefly recall basic properties of Dirichlet and Neumann trace operators in Lipschitz domains. Assuming Hypothesis 2.1, we introduce the boundary trace operator γ_D^0 (the Dirichlet trace) by

$$\gamma_D^0 \colon C(\overline{\Omega}) \to C(\partial\Omega), \quad \gamma_D^0 u = u|_{\partial\Omega}. \tag{3.1}$$

Then there exists a bounded, linear operator γ_D (cf., e.g., [92, Theorem 3.38]),

$$\begin{aligned} &\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \\ &\gamma_D \colon H^{3/2}(\Omega) \to H^{1-\varepsilon}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad \varepsilon \in (0,1), \end{aligned} \tag{3.2}$$

whose action is compatible with that of γ_D^0. That is, the two Dirichlet trace operators coincide on the intersection of their domains. Moreover, we recall that

$$\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \ \text{ is onto for } 1/2 < s < 3/2, \tag{3.3}$$

and point out that the adjoint of (3.2) maps boundedly as follows

$$\gamma_D^* \colon \big(H^{s-1/2}(\partial\Omega)\big)^* \to \big(H^s(\Omega)\big)^*, \quad 1/2 < s < 3/2. \tag{3.4}$$

While in the class of bounded Lipschitz subdomains in $\mathbb{R}^n$ the endpoint cases $s = 1/2$ and $s = 3/2$ of $\gamma_D \in \mathcal{B}\big(H^s(\Omega), H^{s-(1/2)}(\partial\Omega)\big)$ fail, we nonetheless have

$$\gamma_D \in \mathcal{B}\big(H^{(3/2)+\varepsilon}(\Omega), H^1(\partial\Omega)\big), \quad \varepsilon > 0. \tag{3.5}$$

See [53] for a proof. It is useful to augment this with the following result, also proved in [53]:

Lemma 3.1. *Assume Hypothesis* 2.1. *Then for each $s > -3/2$, γ_D^0 in* (3.1) *extends to a linear operator*

$$\gamma_D : \big\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\big\} \to L^2(\partial\Omega; d^{n-1}\omega), \tag{3.6}$$

is compatible with (3.2), *and is bounded when $\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{1/2}(\Omega)} + \|\Delta u\|_{H^s(\Omega)}$. In addition, this operator has a linear, bounded right-inverse* (*thus, in particular, it is onto*).

Furthermore, for each $s > -3/2$, γ_D^0 in (3.1) *also extends to a linear operator*

$$\gamma_D : \big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in H^{1+s}(\Omega)\big\} \to H^1(\partial\Omega), \tag{3.7}$$

which again is compatible with (3.2), *and is bounded when $\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in H^{1+s}(\Omega)\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{H^{1+s}(\Omega)}$. Once again, this operator has a linear, bounded right-inverse* (*hence, in particular, it is onto*).

For later purposes, let us record here the following useful version of the Divergence Theorem:

$$\left.\begin{array}{l} \Omega \ \text{ as in Hypothesis 2.1,} \\ G \in \big\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\big\}^n, \ s > -\frac{3}{2}, \\ \text{with the property that } \operatorname{div}(G) \in L^1(\Omega; d^n x) \end{array}\right\} \ \text{imply} \ \int_\Omega dx^n \operatorname{div}(G) = \int_{\partial\Omega} d^{n-1}\omega\, \nu \cdot \widetilde{\gamma}_D G. \tag{3.8}$$

A proof can be found in [53].

Next, retaining Hypothesis 2.1, we introduce the operator γ_N (the strong Neumann trace) by

$$\gamma_N = \nu \cdot \gamma_D \nabla \colon H^{s+1}(\Omega) \to L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \tag{3.9}$$

where ν denotes the outward pointing normal unit vector to $\partial\Omega$. It follows from (3.2) that γ_N is also a bounded operator. We seek to extend the action of the Neumann trace operator (3.9) to other (related) settings. To set the stage, assume Hypothesis 2.1 and observe that the inclusion

$$\iota : H^{s_0}(\Omega) \hookrightarrow \big(H^r(\Omega)\big)^*, \quad s_0 > -1/2, \ r > 1/2, \tag{3.10}$$

is well-defined and bounded. One then introduces the weak Neumann trace operator

$$\widetilde{\gamma}_N \colon \big\{u \in H^{s+1/2}(\Omega) \,\big|\, \Delta u \in H^{s_0}(\Omega)\big\} \to H^{s-1}(\partial\Omega), \quad s \in (0,1), \ s_0 > -1/2, \tag{3.11}$$

as follows: Given $u \in H^{s+1/2}(\Omega)$ with $\Delta u \in H^{s_0}(\Omega)$ for some $s \in (0,1)$ and $s_0 > -1/2$, we set (with ι as in (3.10) for $r := 3/2 - s > 1/2$)

$$\langle \phi, \widetilde{\gamma}_N u \rangle_{1-s} = {}_{H^{1/2-s}(\Omega)}\langle \nabla\Phi, \nabla u \rangle_{(H^{1/2-s}(\Omega))^*} + {}_{H^{3/2-s}(\Omega)}\langle \Phi, \iota(\Delta u) \rangle_{(H^{3/2-s}(\Omega))^*}, \tag{3.12}$$

for all $\phi \in H^{1-s}(\partial\Omega)$ and $\Phi \in H^{3/2-s}(\Omega)$ such that $\gamma_D\Phi = \phi$. We note that the first pairing on the right-hand side above is meaningful since

$$\big(H^{1/2-s}(\Omega)\big)^* = H^{s-1/2}(\Omega), \quad s \in (0,1), \tag{3.13}$$

that the definition (3.12) is independent of the particular extension Φ of ϕ, and that $\widetilde{\gamma}_N$ is a bounded extension of the Neumann trace operator γ_N defined in (3.9).

Corresponding to the endpoint cases $s = 0, 1$ of (3.11), the following result has been established in [53]:

Lemma 3.2. *Assume Hypothesis* 2.1. *Then the Neumann trace operator* (3.9) *also extends to*

$$\widetilde{\gamma}_N : \big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to L^2(\partial\Omega; d^{n-1}\omega) \tag{3.14}$$

in a bounded fashion when the space $\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$. *This extension is compatible with* (3.11) *and has a linear, bounded right-inverse* (*hence, it is onto*).

Moreover, the Neumann trace operator (3.9) *further extends to*

$$\widetilde{\gamma}_N : \big\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to H^{-1}(\partial\Omega) \tag{3.15}$$

in a bounded fashion when the space $\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{1/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$. *Once again, this extension is compatible with* (3.11) *and has a linear, bounded right-inverse* (*thus, in particular, it is onto*).

For future purposes, we shall need yet another extension of the concept of a Neumann trace. This requires some preparation (throughout, Hypothesis 2.1 is enforced). First, we recall that, as is well-known (see, e.g., [72]), one has the natural identification

$$\big(H^1(\Omega)\big)^* \equiv \big\{u \in H^{-1}(\mathbb{R}^n) \,\big|\, \operatorname{supp}(u) \subseteq \overline{\Omega}\big\}. \tag{3.16}$$

We note that the latter is a closed subspace of $H^{-1}(\mathbb{R}^n)$. In particular, if $R_\Omega u := u|_\Omega$ denotes the operator of restriction to Ω (considered in the sense of distributions), then

$$R_\Omega : \big(H^1(\Omega)\big)^* \to H^{-1}(\Omega) \tag{3.17}$$

is well-defined, linear, and bounded. Furthermore, the composition of R_Ω in (3.17) with ι in (3.10) is the natural inclusion of $H^s(\Omega)$ into $H^{-1}(\Omega)$. Next, given $z \in \mathbb{C}$, set

$$W_z(\Omega) := \big\{(u,f) \in H^1(\Omega) \times \big(H^1(\Omega)\big)^* \,\big|\, (-\Delta - z)u = f|_\Omega \text{ in } \mathcal{D}'(\Omega)\big\}, \tag{3.18}$$

equipped with the norm inherited from $H^1(\Omega) \times \big(H^1(\Omega)\big)^*$. We then denote by

$$\widetilde{\gamma}_{\mathcal{N}} \colon W_z(\Omega) \to H^{-1/2}(\partial\Omega) \tag{3.19}$$

the ultra weak Neumann trace operator defined by

$$\begin{aligned} \langle \phi, \widetilde{\gamma}_{\mathcal{N}}(u,f)\rangle_{1/2} := &\int_\Omega d^n x\, \overline{\nabla\Phi(x)} \cdot \nabla u(x) \\ &- z \int_\Omega d^n x\, \overline{\Phi(x)} u(x) - {}_{H^1(\Omega)}\langle \Phi, f\rangle_{(H^1(\Omega))^*}, \quad (u,f) \in W_z(\Omega), \end{aligned} \tag{3.20}$$

for all $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D\Phi = \phi$. Once again, this definition is independent of the particular extension Φ of ϕ. As a corollary, the following Green's formula

$$\langle \gamma_D\Phi, \widetilde{\gamma}_{\mathcal{N}}(u,f)\rangle_{1/2} = (\nabla\Phi, \nabla u)_{L^2(\Omega;d^n x)^n} + {}_{H^1(\Omega)}\langle \Phi, f\rangle_{(H^1(\Omega))^*}, \tag{3.21}$$

is valid, granted Hypothesis 2.1, for any $u \in H^1(\Omega)$, $f \in \big(H^1(\Omega)\big)^*$ with $\Delta u = f|_\Omega$, and any $\Phi \in H^1(\Omega)$. The pairing on the left-hand side of (3.21) is between functionals in $\big(H^{1/2}(\partial\Omega)\big)^*$ and elements in $H^{1/2}(\partial\Omega)$, whereas the last pairing on the right-hand side is between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$. Furthermore, as in the case of the Dirichlet trace, the ultra weak Neumann trace operator (3.19), (3.20) is onto (this is a corollary of Theorem 5.5).

The relationship between the ultra weak Neumann trace operator (3.19), (3.20) and the weak Neumann trace operator (3.11), (3.12) can be described as follows: Given $s > -1/2$ and $z \in \mathbb{C}$, denote by

$$j_z : \{u \in H^1(\Omega) \,\big|\, \Delta u \in H^s(\Omega)\} \to W_z(\Omega) \tag{3.22}$$

the injection

$$j_z(u) := (u, \iota(-\Delta u - zu)), \quad u \in H^1(\Omega),\ \Delta u \in H^s(\Omega), \tag{3.23}$$

where ι is as in (3.10). Then

$$\widetilde{\gamma}_{\mathcal{N}} j_z = \widetilde{\gamma}_N. \tag{3.24}$$

Thus, from this particular perspective, $\widetilde{\gamma}_{\mathcal{N}}$ can also be regarded as a bounded extension of the Neumann trace operator γ_N defined in (3.9).

Remark 3.3. Since the ultra weak Neumann trace $\widetilde{\gamma}_{\mathcal{N}}(u, f)$ is defined for a class of (pairs of) functions $(u, f) \in W_z(\Omega)$ for which the notion of the strong Neumann trace $\gamma_N u$ is utterly ill-defined, it is appropriate to remark that $(u, f) \mapsto \widetilde{\gamma}_{\mathcal{N}}(u, f)$ is not an extension of the operation of taking the trace $u \mapsto \gamma_N u$ in an ordinary sense. In fact, it is more appropriate to regard the former as a "renormalization" of the latter trace, in a fashion that depends strongly on the choice of f.

To further shed light on this issue, we recall that for $u \in H^1(\Omega)$, Δu is naturally defined as a linear functional in $(H_0^1(\Omega))^*$, where $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The choice of f is the choice of an extension of this linear functional to a functional in $(H^1(\Omega))^*$ (a space which is naturally identified with $H_0^{-1}(\Omega) := \{g \in H^{-1}(\mathbb{R}^n) \,|\, \mathrm{supp}(g) \subseteq \overline{\Omega}\}$). As an example, consider $u \in H^1(\Omega)$ and suppose that actually $u \in H^2(\Omega)$, so $\gamma_N u \in L^2(\partial\Omega; d^{n-1}\omega)$ is well defined. In this case, $\Delta u \in L^2(\Omega; d^n x)$ has a "natural" extension $f_0 \in H_0^{-1}(\Omega)$ (i.e., f_0 is the extension of Δu to $\mathbb{R}^n$ by setting this equal zero outside Ω). Any other extension $f_1 \in H_0^{-1}(\Omega)$ differs from f_0 by a distribution $\eta \in H^{-1}(\mathbb{R}^n)$ supported on $\partial\Omega$. We have

$$\widetilde{\gamma}_{\mathcal{N}}(u, f_0) = \gamma_N u, \tag{3.25}$$

but if $\eta \neq 0$ then $\widetilde{\gamma}_{\mathcal{N}}(u, f_1)$ is not equal to $\widetilde{\gamma}_{\mathcal{N}}(u, f_0)$. Indeed, by linearity one concludes that $\widetilde{\gamma}_{\mathcal{N}}(u, f_1) = \widetilde{\gamma}_{\mathcal{N}}((u, f_0) + (0, \eta)) = \widetilde{\gamma}_{\mathcal{N}}(u, f_0) + \widetilde{\gamma}_{\mathcal{N}}(0, \eta)$ and (3.20) shows that

$$\langle \phi, \widetilde{\gamma}_{\mathcal{N}}(0, \eta)\rangle_{1/2} = -{}_{H^1(\Omega)}\langle \Phi, \eta\rangle_{(H^1(\Omega))^*}, \tag{3.26}$$

for each $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D \Phi = \phi$. Consequently, $\widetilde{\gamma}_{\mathcal{N}}(0, \eta) \neq 0$ if $\eta \neq 0$.

We conclude this section by recording the following result, proved in [53]:

Lemma 3.4. *Assume Hypothesis* 2.1. *Then for each $r \in (1/2, 1)$, the Hölder space $C^r(\partial\Omega)$ is a module over $H^{1/2}(\partial\Omega)$. More precisely, if M_f denotes the operator of multiplication by f, then there exists $C = C(\Omega, r) > 0$ such that*

$$M_f \in \mathcal{B}\big(H^{1/2}(\partial\Omega)\big) \text{ and } \|M_f\|_{\mathcal{B}\left(H^{1/2}(\partial\Omega)\right)} \leq C\|f\|_{C^r(\partial\Omega)}, \quad f \in C^r(\partial\Omega). \tag{3.27}$$

As a consequence, if Hypothesis 2.2 *is enforced, then the Neumann and Dirichlet trace operators γ_N, γ_D satisfy*

$$\gamma_N \in \mathcal{B}\big(H^2(\Omega), H^{1/2}(\partial\Omega)\big), \tag{3.28}$$

$$\gamma_D \in \mathcal{B}\big(H^2(\Omega), H^{3/2}(\partial\Omega)\big). \tag{3.29}$$

4. Boundary Layer Potential Operators in Lipschitz Domains

We recall the fundamental solution of the Helmholtz equation and some fundamental properties of boundary layer potential operators in Lipschitz domains.

Let $E_n(z; x)$ be the fundamental solution associated with the Helmholtz differential operator $(-\Delta - z)$ in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, that is,

$$E_n(z; x) = \begin{cases} (i/4)\big(2\pi|x|/z^{1/2}\big)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x|\big), & n \geq 2,\ z \in \mathbb{C}\backslash\{0\}, \\ \frac{-1}{2\pi}\ln(|x|), & n = 2,\ z = 0, \\ \frac{1}{(n-2)\omega_{n-1}}|x|^{2-n}, & n \geq 3,\ z = 0, \end{cases}$$
$$\mathrm{Im}\big(z^{1/2}\big) \geq 0,\ x \in \mathbb{R}^n\backslash\{0\}. \tag{4.1}$$

Here $H^{(1)}_\nu(\cdot)$ denotes the Hankel function of the first kind with index $\nu \geq 0$ (cf. [1, Sect. 9.1]).

Assuming Hypothesis 2.1, we next introduce the single layer potential by setting

$$(\mathcal{S}_z g)(x) = \int_{\partial\Omega} d^{n-1}\omega(y)\, E_n(z; x - y) g(y), \quad x \in \Omega,\ z \in \mathbb{C}, \tag{4.2}$$

where g is an arbitrary measurable function on $\partial\Omega$. We shall also be interested in the adjoint double layer on $\partial\Omega$, given by

$$(K_z^{\#} g)(x) = \text{p.v.} \int_{\partial\Omega} d^{n-1}\omega(y)\, \partial_{\nu_x} E_n(z; x-y) g(y), \quad x \in \partial\Omega,\ z \in \mathbb{C}. \tag{4.3}$$

We denote by $I_{\partial\Omega}$ the identity operator in the various spaces of functions (and distributions) on $\partial\Omega$. In [53], the following results boundedness and jump-relations have been established:

Lemma 4.1. *Assume Hypothesis* 2.1 *and fix $z \in \mathbb{C}$. Then*

$$K_z^{\#} \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \tag{4.4}$$

and

$$\gamma_D \mathcal{S}_z \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^1(\partial\Omega)\big), \tag{4.5}$$

$$\widetilde{\gamma}_N \mathcal{S}_z g = \big(-\tfrac{1}{2} I_{\partial\Omega} + K_z^{\#}\big) g, \quad g \in L^2(\partial\Omega; d^{n-1}\omega). \tag{4.6}$$

Lemma 4.2. *Assume Hypothesis* 2.2*. Then*

$$K_z^{\#} \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega)\big), \quad z \in \mathbb{C}. \tag{4.7}$$

Furthermore,

$$\mathcal{S}_z \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^2(\Omega)\big), \quad z \in \mathbb{C}. \tag{4.8}$$

5. Dirichlet and Neumann Boundary Value Problems in Lipschitz Domains

This section is devoted to Dirichlet and Neumann Laplacians and to Dirichlet and Neumann boundary value problems. We also briefly recall some results on nonlocal Robin Laplacians.

We start by reviewing the Dirichlet Laplacian $-\Delta_{D,\Omega}$ associated with a domain Ω in $\mathbb{R}^n$.

Theorem 5.1. *Assume Hypothesis* 2.1*. Then the Dirichlet Laplacian, $-\Delta_{D,\Omega}$, defined by*

$$\begin{aligned} &-\Delta_{D,\Omega} = -\Delta, \\ &\operatorname{dom}(-\Delta_{D,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \gamma_D u = 0 \text{ in } H^{1/2}(\partial\Omega)\big\} \\ &\qquad\qquad\quad = \big\{u \in H_0^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}, \end{aligned} \tag{5.1}$$

is self-adjoint and strictly positive in $L^2(\Omega; d^n x)$. Moreover,

$$\operatorname{dom}\big((-\Delta_{D,\Omega})^{1/2}\big) = H_0^1(\Omega). \tag{5.2}$$

This is essentially known; we refer to [53] for a discussion where also the following result concerning the Neumann Laplacian $-\Delta_{N,\Omega}$ associated with Ω can be found:

Theorem 5.2. *Assume Hypothesis* 2.1*. Then the Neumann Laplacian, $-\Delta_{N,\Omega}$, defined by*

$$\begin{aligned} &-\Delta_{N,\Omega} = -\Delta, \\ &\operatorname{dom}(-\Delta_{N,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \widetilde{\gamma}_N u = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}, \end{aligned} \tag{5.3}$$

is self-adjoint and bounded from below in $L^2(\Omega; d^n x)$. Moreover,

$$\operatorname{dom}\big(|-\Delta_{N,\Omega}|^{1/2}\big) = H^1(\Omega). \tag{5.4}$$

In our next two theorems, we review well-posedness results for the Dirichlet and Neumann problems. To state these we will denote the identity operator in the various spaces of functions (and distributions) in Ω as I_Ω.

Theorem 5.3. *Assume Hypothesis* 2.1 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Then for every $f \in H^s(\partial\Omega)$, $0 \le s \le 1$, the following Dirichlet boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^{s+1/2}(\Omega), \\ \gamma_D u = f \text{ on } \partial\Omega, \end{cases} \tag{5.5}$$

has a unique solution $u = u_D$. This solution u_D satisfies

$$\widetilde{\gamma}_N u_D \in H^{s-1}(\partial\Omega) \text{ and } \|\widetilde{\gamma}_N u_D\|_{H^{s-1}(\partial\Omega)} \le C_D \|f\|_{H^s(\partial\Omega)}, \tag{5.6}$$

for some constant $C_D = C_D(\Omega, s, z) > 0$. Moreover,

$$\|u_D\|_{H^{s+1/2}(\Omega)} \le C_D \|f\|_{H^s(\partial\Omega)}. \tag{5.7}$$

Finally,
$$\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\in\mathcal{B}\big(H^s(\partial\Omega),H^{s+1/2}(\Omega)\big), \tag{5.8}$$
and the solution u_D is given by the formula
$$u_D=-\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*f. \tag{5.9}$$

Theorem 5.4. *Assume Hypothesis 2.1 and suppose that $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. Then for every $g\in H^{s-1}(\partial\Omega)$, $0\le s\le 1$, the following Neumann boundary value problem,*
$$\begin{cases}(-\Delta-z)u=0 \text{ in } \Omega, \quad u\in H^{s+1/2}(\Omega),\\ \widetilde{\gamma}_N u=g \text{ on } \partial\Omega,\end{cases} \tag{5.10}$$
has a unique solution $u=u_N$. This solution u_N satisfies
$$\gamma_D u_N\in H^s(\partial\Omega) \text{ and } \|\gamma_D u_N\|_{H^s(\partial\Omega)}\le C\|g\|_{H^{s-1}(\partial\Omega)}, \tag{5.11}$$
as well as
$$\|u_N\|_{H^{s+1/2}(\Omega)}\le C\|g\|_{H^{s-1}(\partial\Omega)}, \tag{5.12}$$
for some constant constant $C=C(\Omega,s,z)>0$. Finally,
$$\big[\gamma_D(-\Delta_{N,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\in\mathcal{B}\big(H^{s-1}(\partial\Omega),H^{s+1/2}(\Omega)\big), \tag{5.13}$$
and the solution u_N is given by the formula
$$u_N=\big(\gamma_D(-\Delta_{N,\Omega}-\overline{z}I_\Omega)^{-1}\big)^*g. \tag{5.14}$$

When $s\in\{0,1\}$, the above results have been established in [53]. The more general case $0\le s\le 1$ is, however, proved along very similar lines.

We shall now review well-posedness results for the inhomogeneous Dirichlet and Neumann problems (again, see [53] for proofs). In the following we denote by $\widetilde{I}_\Omega$ the continuous inclusion (embedding) map of $H^1(\Omega)$ into $\big(H^1(\Omega)\big)^*$. By a slight abuse of notation, we also denote the continuous inclusion map of $H^1_0(\Omega)$ into $\big(H^1_0(\Omega)\big)^*$ by the same symbol $\widetilde{I}_\Omega$. We recall the ultra weak Neumann trace operator $\widetilde{\gamma}_{\mathcal{N}}$ in (3.19), (3.20). Finally, assuming Hypothesis 2.1, we denote by
$$-\widetilde{\Delta}_{N,\Omega}\in\mathcal{B}\big(H^1(\Omega),\big(H^1(\Omega)\big)^*\big) \tag{5.15}$$
the operator defined uniquely by the requirement that
$${}_{H^1(\Omega)}\langle u,-\widetilde{\Delta}_{N,\Omega}v\rangle_{(H^1(\Omega))^*}=\int_\Omega d^nx\,\overline{\nabla u(x)}\cdot\nabla v(x)=(\nabla u,\nabla v)_{L^2(\Omega;d^nx)}, \quad u,v\in H^1(\Omega). \tag{5.16}$$
We note that $-\Delta_{N,\Omega}$ then becomes the part of $-\widetilde{\Delta}_{N,\Omega}$ in $L^2(\Omega;d^nx)$.

Theorem 5.5. *Assume Hypothesis 2.1 and suppose that $z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$. Then for every $w\in(H^1(\Omega))^*$, the following generalized inhomogeneous Neumann problem,*
$$\begin{cases}(-\Delta-z)u=w|_\Omega \text{ in } \mathcal{D}'(\Omega), \quad u\in H^1(\Omega),\\ \widetilde{\gamma}_{\mathcal{N}}(u,w)=0 \text{ on } \partial\Omega,\end{cases} \tag{5.17}$$
has a unique solution $u=u_{N,w}$. Moreover, there exists a constant $C=C(\Omega,z)>0$ such that
$$\|u_{N,w}\|_{H^1(\Omega)}\le C\|w\|_{(H^1(\partial\Omega))^*}. \tag{5.18}$$
In particular, the operator $(-\Delta_{N,\Omega}-zI_\Omega)^{-1}$, $z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, originally defined as a bounded operator on $L^2(\Omega;d^nx)$,
$$(-\Delta_{N,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx)\big), \tag{5.19}$$
can be extended to a map in $\mathcal{B}\big(\big(H^1(\Omega)\big)^*,H^1(\Omega)\big)$, which in fact coincides with
$$\big(-\widetilde{\Delta}_{N,\Omega}-z\widetilde{I}_\Omega\big)^{-1}\in\mathcal{B}\big(\big(H^1(\Omega)\big)^*,H^1(\Omega)\big). \tag{5.20}$$

Continuing to retain Hypothesis 2.1, we denote by
$$-\widetilde{\Delta}_{D,\Omega}\in\mathcal{B}\big(H^{-1}(\Omega),H^1_0(\Omega)\big) \tag{5.21}$$
the operator defined uniquely by the requirement that
$${}_{H^1_0(\Omega)}\langle u,-\widetilde{\Delta}_{D,\Omega}v\rangle_{(H^1_0(\Omega))^*}=\int_\Omega d^nx\,\overline{\nabla u(x)}\cdot\nabla v(x)=(\nabla u,\nabla v)_{L^2(\Omega;d^nx)}, \quad u,v\in H^1_0(\Omega). \tag{5.22}$$

In this context, $-\Delta_{D,\Omega}$ then becomes the part of $-\widetilde{\Delta}_{D,\Omega}$ in $L^2(\Omega; d^n x)$.

Theorem 5.6. *Assume Hypothesis* 2.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. *Then for every* $w \in H^{-1}(\Omega)$, *the following inhomogeneous Dirichlet problem,*

$$(-\Delta - z)u = w \ \text{in} \ \Omega, \quad u \in H_0^1(\Omega), \tag{5.23}$$

has a unique solution $u = u_{D,w}$. *Moreover, there exists a constant* $C = C(\Omega, z) > 0$ *such that*

$$\|u_{D,w}\|_{H^1(\Omega)} \leq C\|w\|_{H^{-1}(\partial\Omega)}. \tag{5.24}$$

In particular, the operator $(-\Delta_{D,\Omega} - zI_\Omega)^{-1}$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, *originally defined as a bounded operator on* $L^2(\Omega; d^n x)$,

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \tag{5.25}$$

can be extended to a map in $\mathcal{B}\big(H^{-1}(\Omega), H_0^1(\Omega)\big)$, *which in fact coincides with*

$$\big(-\widetilde{\Delta}_{N,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \in \mathcal{B}\big(H^{-1}(\Omega), H_0^1(\Omega)\big). \tag{5.26}$$

Assuming Hypothesis 2.1, we now introduce the Dirichlet-to-Neumann map $M_{D,N,\Omega}^{(0)}(z)$ associated with $(-\Delta - z)$ on Ω, as follows,

$$M_{D,N,\Omega}^{(0)}(z)\colon \begin{cases} H^1(\partial\Omega) \to L^2(\partial\Omega; d^{n-1}\omega), \\ \qquad f \mapsto -\widetilde{\gamma}_N(u_D), \end{cases} \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{5.27}$$

where u_D is the unique solution of

$$(-\Delta - z)u = 0 \ \text{ in } \Omega, \quad u \in H^{3/2}(\Omega), \quad \gamma_D u = f \ \text{ on } \partial\Omega. \tag{5.28}$$

Still assuming Hypothesis 2.1, we next introduce the Neumann-to-Dirichlet map $M_{N,D,\Omega}^{(0)}(z)$ associated with $(-\Delta - z)$ on Ω, as follows,

$$M_{N,D,\Omega}^{(0)}(z)\colon \begin{cases} L^2(\partial\Omega; d^{n-1}\omega) \to H^1(\partial\Omega), \\ \qquad g \mapsto \gamma_D u_N, \end{cases} \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}), \tag{5.29}$$

where u_N is the unique solution of

$$(-\Delta - z)u = 0 \ \text{ in } \Omega, \quad u \in H^{3/2}(\Omega), \quad \widetilde{\gamma}_N u = g \ \text{ on } \partial\Omega. \tag{5.30}$$

Theorem 5.7. *Assume Hypothesis* 2.1. *Then*

$$M_{D,N,\Omega}^{(0)}(z) \in \mathcal{B}\big(H^1(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{5.31}$$

and

$$M_{D,N,\Omega}^{(0)}(z) = \widetilde{\gamma}_N\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{5.32}$$

Moreover,

$$M_{N,D,\Omega}^{(0)}(z) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^1(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}), \tag{5.33}$$

and, in fact,

$$M_{N,D,\Omega}^{(0)}(z) \in \mathcal{B}_\infty\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}). \tag{5.34}$$

In addition,

$$M_{N,D,\Omega}^{(0)}(z) = \gamma_D\big[\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}). \tag{5.35}$$

Finally, let $z \in \mathbb{C}\backslash(\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{N,\Omega}))$. *Then*

$$M_{N,D,\Omega}^{(0)}(z) = -M_{D,N,\Omega}^{(0)}(z)^{-1}. \tag{5.36}$$

For closely related recent work on Weyl–Titchmarsh operators associated with nonsmooth domains we refer to [53], [54], [55], and [56]. For an extensive list of references on z-dependent Dirichlet-to-Neumann maps predominantly associated with smooth domains we also refer, for instance, to [4], [6], [9], [22], [29], [31], [32], [34], [35], [39], [40], [51]–[56], [64], [105], [109], [110], [111]. We will return to this topic in Section 11 in the context of quasi-convex domains.

In the last part of this section we include a brief discussion of nonlocal Robin Laplacians in bounded Lipschitz subdomains of $\mathbb{R}^n$. Concretely, we describe a family of self-adjoint Laplace operators $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ indexed by the boundary operator Θ. We will refer to $-\Delta_{\Theta,\Omega}$ as the (nonlocal) Robin Laplacian. To facilitate the presentation, we isolate a technical condition in the hypothesis below:

Hypothesis 5.8. *Assume Hypothesis* 2.1*, suppose that* $\delta > 0$ *is a given number, and assume that* $\Theta \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ *is a self-adjoint operator which can be written as*

$$\Theta = \Theta_1 + \Theta_2 + \Theta_3, \tag{5.37}$$

where the operators Θ_j*,* $j = 1, 2, 3$*, have the following properties: There exists a closed sesquilinear form* a_{Θ_0} *in* $L^2(\partial\Omega; d^{n-1}\omega)$*, with domain* $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$*, bounded from below by* $c_{\Theta_0} \in \mathbb{R}$ *(hence,* a_{Θ_0} *is symmetric) such that if* $\Theta_0 \geqslant c_\Theta I_{\partial\Omega}$ *denotes the self-adjoint operator in* $L^2(\partial\Omega; d^{n-1}\omega)$ *uniquely associated with* a_{Θ_0}*, then* Θ_1 *is the extension of* Θ_0 *to an operator in* $\mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$*. In addition,*

$$\Theta_2 \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big), \tag{5.38}$$

whereas $\Theta_3 \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ *satisfies*

$$\|\Theta_3\|_{\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} < \delta. \tag{5.39}$$

The following result has been proved in [55]:

Theorem 5.9. *Assume Hypothesis* 5.8*, where the number* $\delta > 0$ *is taken to be sufficiently small relative to the Lipschitz character of* Ω*. Then the nonlocal Robin Laplacian,* $-\Delta_{\Theta,\Omega}$*, defined by*

$$\begin{aligned} &-\Delta_{\Theta,\Omega} = -\Delta, \\ &\mathrm{dom}(-\Delta_{\Theta,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x),\ \big(\widetilde\gamma_N + \Theta\gamma_D\big)u = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\} \end{aligned} \tag{5.40}$$

is self-adjoint and bounded from below in $L^2(\Omega; d^n x)$*. Moreover,*

$$\mathrm{dom}\big(|-\Delta_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega), \tag{5.41}$$

and $-\Delta_{\Theta,\Omega}$*, has purely discrete spectrum bounded from below, in particular,*

$$\sigma_{\rm ess}(-\Delta_{\Theta,\Omega}) = \emptyset. \tag{5.42}$$

6. Higher-Order Smoothness Spaces on Lipschitz Surfaces

This section presents some new results on higher-order smoothness spaces on Lipschitz surfaces.

Assuming Hypothesis 2.1 we recall the tangential derivative operators $\partial/\partial\tau_{j,k}$ in (2.24). We can then define the tangential gradient operator

$$\nabla_{tan} : \begin{cases} H^1(\partial\Omega) \to \big(L^2(\partial\Omega; d^{n-1}\omega)\big)^n \\ \qquad f \mapsto \nabla_{tan} f = \Big(\sum_{k=1}^n \nu_k \frac{\partial f}{\partial\tau_{k,j}}\Big)_{1\leq j\leq n}. \end{cases} \tag{6.1}$$

The following result has been proved in [89].

Theorem 6.1. *Assume Hypothesis* 2.1 *and denote by* ν *the outward unit normal to* $\partial\Omega$*. Then the operator*

$$\gamma_2 : \begin{cases} H^2(\Omega) \to \big\{(g_0, g_1) \in H^1(\partial\Omega) \dotplus L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\} \\ \qquad u \mapsto \gamma_2 u = \big(\gamma_D u, \gamma_N u\big) \end{cases} \tag{6.2}$$

is well-defined, linear, bounded, onto, and has a linear, bounded right-inverse. In (6.2)*, the space* $\big\{(g_0, g_1) \in H^1(\partial\Omega) \dotplus L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\}$ *is considered equipped with the natural norm*

$$(g_0, g_1) \mapsto \|g_0\|_{H^1(\partial\Omega)} + \|g_1\|_{L^2(\partial\Omega; d^{n-1}\omega)} + \|\nabla_{tan} g_0 + g_1\nu\|_{(H^{1/2}(\partial\Omega))^n}. \tag{6.3}$$

Furthermore, the null space of the operator (6.2) *is given by*

$$\ker(\gamma_2) = \big\{u \in H^2(\Omega) \,\big|\, \gamma_D u = \gamma_N u = 0\big\} = H^2_0(\Omega), \tag{6.4}$$

with the latter space denoting the closure of $C_0^\infty(\Omega)$ *in* $H^2(\Omega)$*.*

Assuming Hypothesis 2.1, we now introduce

$$N^{1/2}(\partial\Omega; d^{n-1}\omega) = N^{1/2}(\partial\Omega) := \big\{g \in L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, g\nu_j \in H^{1/2}(\partial\Omega),\ 1 \leq j \leq n\big\}, \tag{6.5}$$

where the ν_j's are the components of ν. We equip this space with the natural norm

$$\|g\|_{N^{1/2}(\partial\Omega)} := \sum_{j=1}^n \|g\nu_j\|_{H^{1/2}(\partial\Omega)}. \tag{6.6}$$

Lemma 6.2. *Assuming Hypothesis* 2.1. *Then $N^{1/2}(\partial\Omega)$ is a reflexive Banach space which embeds continuously into $L^2(\partial\Omega; d^{n-1}\omega)$. Furthermore,*

$$N^{1/2}(\partial\Omega) = H^{1/2}(\partial\Omega), \quad \textit{whenever } \Omega \textit{ is a bounded } C^{1,r} \textit{ domain with } r > 1/2. \tag{6.7}$$

Proof. Obviously we have

$$g = \sum_{j=1}^{n} \nu_j (g\nu_j) \text{ for any function } g \in L^2(\partial\Omega; d^{n-1}\omega) \tag{6.8}$$

so that, in particular, $\|g\|_{L^2(\partial\Omega;d^{n-1}\omega)} \leq n\|g\|_{N^{1/2}(\partial\Omega)}$. This proves that the natural inclusion $N^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega)$ is bounded. If $\{g_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $N^{1/2}(\partial\Omega)$ then, for each $j \in \{1, ..., n\}$, $\{g_k\nu_j\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^{1/2}(\partial\Omega)$ and, from what we have proved so far, $\{g_k\}_{k\in\mathbb{N}}$ converges in $L^2(\partial\Omega; d^{n-1}\omega)$ to some $g \in L^2(\partial\Omega; d^{n-1}\omega)$. It follows that $\{g_k\nu_j\}_{k\in\mathbb{N}}$ converges in $L^2(\partial\Omega; d^{n-1}\omega)$ to $g\nu_j$ for each $j \in \{1, ..., n\}$. With this at hand, it is then easy to conclude that g is the limit of $\{g_k\}_{k\in\mathbb{N}}$ in $N^{1/2}(\partial\Omega)$. This proves that $N^{1/2}(\partial\Omega)$ is a Banach space.

Next, by relying on the same simple identity in (6.8) and Lemma 3.4, we also see that (6.7) holds. If we now consider

$$\Phi : N^{1/2}(\partial\Omega) \to \big[H^{1/2}(\partial\Omega)\big]^n, \quad \Phi(g) := (g\nu_j)_{1\leq j\leq n}, \tag{6.9}$$

it follows that Φ is an isometric embedding, which allows identifying $N^{1/2}(\partial\Omega)$ with a closed subspace of the reflexive space $\big[H^{1/2}(\partial\Omega)\big]^n$, implying that $N^{1/2}(\partial\Omega)$ is also reflexive. □

It should be mentioned that, in spite of (6.7), the spaces $H^{1/2}(\partial\Omega)$ and $N^{1/2}(\partial\Omega)$ can be quite different for an arbitrary Lipschitz domain Ω. Our interest in the latter space stems from the fact that this arises naturally when considering the Neumann trace operator acting on

$$\big\{u \in H^2(\Omega) \,\big|\, \gamma_D u = 0\big\} = H^2(\Omega) \cap H^1_0(\Omega), \tag{6.10}$$

considered as a closed subspace of $H^2(\Omega)$ (hence, a Banach space when equipped with the H^2-norm). More specifically, we have the following result:

Lemma 6.3. *Assume Hypothesis* 2.1. *Then the Neumann trace operator γ_N considered in the context*

$$\gamma_N : H^2(\Omega) \cap H^1_0(\Omega) \to N^{1/2}(\partial\Omega) \tag{6.11}$$

is well-defined, linear, bounded, onto, and with a linear, bounded right-inverse. In addition, the null space of γ_N in (6.11) *is precisely $H^2_0(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^2(\Omega)$.*

Proof. To prove that (6.11) is well-defined, we note that if $u \in H^2(\Omega) \cap H^1_0(\Omega)$, then Theorem 6.1 yields

$$(0, \gamma_N u) = \gamma_2 u \in \big\{(g_0, g_1) \in H^1(\partial\Omega) \dotplus L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\}, \tag{6.12}$$

implying that $\gamma_N u$ belongs to $N^{1/2}(\partial\Omega)$ and $\|\gamma_N u\|_{N^{1/2}(\partial\Omega)} \leq C\|u\|_{H^2(\Omega)}$ for some $C = C(\Omega) > 0$ independent of u. This shows that (6.11) is well-defined, linear, and bounded. Next, denote by $\mathcal{E}_2$ a linear, bounded right-inverse for γ_2 in (6.2). Then, if

$$\iota : N^{1/2}(\partial\Omega) \to \big\{(g_0, g_1) \in H^1(\partial\Omega) \dotplus L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\} \tag{6.13}$$

is the injection given by $\iota(g) := (0, g)$, for every $g \in N^{1/2}(\partial\Omega)$, it follows that the composition $\mathcal{E}_2\iota : N^{1/2}(\partial\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)$ is a linear, bounded right-inverse for the operator γ_N in (6.11). Consequently, this operator is onto. Finally, the fact that the null space of γ_N in (6.11) is precisely $H^2_0(\Omega)$ follows from its definition and the last part in the statement of Theorem 6.1. □

Our goal is to use the above Neumann trace result in order to extend the action of the Dirichlet trace operator (3.2) to $\mathrm{dom}(-\Delta_{max})$, the domain of the maximal Laplacian, that is, $\big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}$, which we consider equipped with the graph norm $u \mapsto \|u\|_{L^2(\Omega;d^n x)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$. Specifically, with $\big(N^{1/2}(\partial\Omega)\big)^*$ denoting the conjugate dual space of $N^{1/2}(\partial\Omega)$, we have the following result:

Theorem 6.4. *Assume Hypothesis* 2.1. *Then there exists a unique linear, bounded operator*

$$\widehat{\gamma}_D : \big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to \big(N^{1/2}(\partial\Omega)\big)^* \tag{6.14}$$

which is compatible with the Dirichlet trace introduced in (3.2), *in the sense that, for each $s > 1/2$, one has*

$$\widehat{\gamma}_D u = \gamma_D u \textit{ for every } u \in H^s(\Omega) \textit{ with } \Delta u \in L^2(\Omega; d^n x). \tag{6.15}$$

Furthermore, this extension of the Dirichlet trace operator in (3.2) *allows for the following generalized integration by parts formula*

$$ {}_{N^{1/2}(\partial\Omega)}\langle \gamma_N w, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*} = (\Delta w, u)_{L^2(\Omega;d^nx)} - (w, \Delta u)_{L^2(\Omega;d^nx)}, \tag{6.16} $$

valid for every $u \in L^2(\Omega;d^nx)$ *with* $\Delta u \in L^2(\Omega;d^nx)$ *and every* $w \in H^2(\Omega)\cap H^1_0(\Omega)$.

Proof. Let $u \in L^2(\Omega;d^nx)$ be such that $\Delta u \in L^2(\Omega;d^nx)$. We attempt to define a functional $\widehat{\gamma}_D u \in \big(N^{1/2}(\partial\Omega)\big)^*$ as follows: Assume that $g \in N^{1/2}(\partial\Omega)$ has been given arbitrarily. By Lemma 6.3 there exists a $w \in H^2(\Omega)\cap H^1_0(\Omega)$ such that $\gamma_N w = g$ and $\|w\|_{H^2(\Omega)} \leq C\|g\|_{N^{1/2}(\partial\Omega)}$ for some finite constant $C = C(\Omega) > 0$, independent of g. We then set

$$ {}_{N^{1/2}(\partial\Omega)}\langle g, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*} := (\Delta w, u)_{L^2(\Omega;d^nx)} - (w, \Delta u)_{L^2(\Omega;d^nx)}. \tag{6.17} $$

First we need to show that the above definition does not depend on the particular choice of w, with the properties listed above. By linearity, this comes down to proving the following claim: If u is as before and $w \in H^2(\Omega)\cap H^1_0(\Omega)$ is such that $\gamma_N w = 0$ then

$$ (\Delta w, u)_{L^2(\Omega;d^nx)} = (w, \Delta u)_{L^2(\Omega;d^nx)}. \tag{6.18} $$

However, since Lemma 6.3 yields that $w \in H^2_0(\Omega)$, formula (6.18) readily follows by approximating w with functions form $C_0^\infty(\Omega)$ in the norm of $H^2(\Omega)$. Thus, formula (6.17) yields a well-defined, linear, and bounded operator in the context of (6.17). By definition, this operator will satisfy (6.16).

Next, we will show that (6.15) is valid for each $s > 1/2$. Fix $s \in (1/2, 1)$ along with some function $u \in H^s(\Omega)$ with $\Delta u \in L^2(\Omega;d^nx)$. In particular, $\gamma_D u \in H^{s-(1/2)}(\partial\Omega)$ and we select a sequence $\{f_j\}_{j\in\mathbb{N}}$ such that

$$ f_j \in H^{1/2}(\partial\Omega),\ j\in\mathbb{N}, \ \text{ and } \ f_j \to \gamma_D u \ \text{ in } H^{s-(1/2)}(\partial\Omega) \text{ as } j\to\infty. \tag{6.19} $$

Next, for each $j \in \mathbb{N}$, select a function u_j such that

$$ \Delta u_j = \Delta u \ \text{ in } \Omega, \quad u_j \in H^1(\Omega),\ \gamma_D u_j = f_j \ \text{ on } \partial\Omega. \tag{6.20} $$

From (6.19) and the continuous dependence of the solution on the data, we may conclude that

$$ u_j \to u \ \text{ in } H^s(\partial\Omega) \text{ as } j \to \infty. \tag{6.21} $$

In addition, we also select a sequence $\{w_k\}_{k\in\mathbb{N}}$ such that

$$ w_k \in C^\infty(\overline{\Omega}),\ k\in\mathbb{N}, \ \text{ and } \ w_k \to w \ \text{ in } H^2(\Omega) \text{ as } k\to\infty. \tag{6.22} $$

With j fixed, we now introduce the vector fields $G_k := \overline{u_j}\nabla w_k \in \big(H^1(\Omega)\big)^n$, $k\in\mathbb{N}$, and obtain

$$ \begin{aligned} &\mathrm{div}(G_k) = \overline{u_j}\Delta w_k + \overline{\nabla u_j}\cdot\nabla w_k \in L^2(\Omega;d^n) \hookrightarrow L^1(\Omega;d^nx), \\ &\Delta G_k = \overline{\Delta u_j}\nabla w_k + \overline{u_j}\nabla\Delta w_k + 2\textstyle\sum_{i=1}^n \overline{\nabla u_j}\cdot\nabla\partial_i w_k \in L^2(\Omega;d^nx), \\ &\nu\cdot\widetilde{\gamma}_D(G_k) = \nu\cdot\gamma_D(G_k) = \overline{\gamma_D u_j}\gamma_N w_k = \overline{f_j}\gamma_N w_k. \end{aligned} \tag{6.23} $$

Based on this and (3.8) we may then write

$$ \begin{aligned} (u_j, \Delta w)_{L^2(\Omega;d^nx)} &= \lim_{k\to\infty}(u_j, \Delta w_k)_{L^2(\Omega;d^nx)} \\ &= \lim_{k\to\infty}\bigg(\int_\Omega d^nx\,\mathrm{div}(G_k) - (\nabla u_j, \nabla w_k)_{(L^2(\Omega;d^nx))^n}\bigg) \\ &= \lim_{k\to\infty}\bigg(\int_{\partial\Omega} d\omega^{n-1}\,\overline{f_j}\gamma_N w_k - (\nabla u_j, \nabla w_k)_{(L^2(\Omega;d^nx))^n}\bigg) \\ &= \int_{\partial\Omega} d\omega^{n-1}\,\overline{f_j}\gamma_N w - (\nabla u_j, \nabla w)_{(L^2(\Omega;d^nx))^n}. \end{aligned} \tag{6.24} $$

In order to continue, we now select a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that

$$ v_k \in C_0^\infty(\Omega),\ k\in\mathbb{N}, \ \text{ and } \ v_k \to w \ \text{ in } H^1(\Omega) \text{ as } k\to\infty. \tag{6.25} $$

Then we can further transform the last integral in (6.24) into

$$ \begin{aligned} (\nabla u_j, \nabla w)_{(L^2(\Omega;d^nx))^n} &= \lim_{k\to\infty}(\nabla u_j, \nabla v_k)_{(L^2(\Omega;d^nx))^n} \\ &= -\lim_{k\to\infty}(\Delta u_j, v_k)_{L^2(\Omega;d^nx)} \\ &= -(\Delta u_j, w)_{L^2(\Omega;d^nx)} = (\Delta u, w)_{L^2(\Omega;d^nx)}, \end{aligned} \tag{6.26} $$

where the last equality utilizes the fact that $\Delta u_j = \Delta u$ (cf.(6.20)). Together, (6.24) and (6.26) yield that

$$(u_j, \Delta w)_{L^2(\Omega;d^nx)} = \int_{\partial\Omega} d\omega^{n-1}\, \overline{f_j}\gamma_N w + (\Delta u, w)_{L^2(\Omega;d^nx)}, \tag{6.27}$$

for every $j \in \mathbb{N}$. By passing to the limit as $j \to \infty$ in (6.27), we arrive, on account of (6.19) and (6.21), at

$$(u, \Delta w)_{L^2(\Omega;d^nx)} = \int_{\partial\Omega} d\omega^{n-1}\, \overline{\gamma_D u}\gamma_N w + (\Delta u, w)_{L^2(\Omega;d^nx)}. \tag{6.28}$$

Consequently,

$$\begin{aligned}\int_{\partial\Omega} d\omega^{n-1}\, \overline{\gamma_D u}\gamma_N w &= (u, \Delta w)_{L^2(\Omega;d^nx)} - (\Delta u, w)_{L^2(\Omega;d^nx)} \\ &= {}_{N^{1/2}(\partial\Omega)}\langle \gamma_N w, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}.\end{aligned} \tag{6.29}$$

We recall that this formula is valid for any $w \in H^2(\Omega) \cap H^1_0(\Omega)$ and the operator (6.11) maps this space onto $N^{1/2}(\partial\Omega)$. We may therefore deduce from (6.29) that $\widehat{\gamma}_D u$, originally viewed as a functional on $\big(N^{1/2}(\partial\Omega)\big)^*$, is given by integration against $\gamma_D u \in L^2(\partial\Omega; d^{n-1}\omega)$ in the case where $u \in H^s(\Omega)$ satisfies $\Delta u \in L^2(\Omega; d^nx)$. This concludes the proof of (6.15).

It remains to establish the uniqueness of an operator satisfying (6.14), (6.15). For this it suffices to show that $C^\infty(\overline{\Omega})$ embeds densely into $\big\{u \in L^2(\Omega; d^nx) \,\big|\, \Delta u \in L^2(\Omega; d^nx)\big\}$. In fact, a more general result holds, namely,

$$C^\infty(\overline{\Omega}) \hookrightarrow \big\{u \in H^s(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^nx)\big\} \text{ densely, whenever } s < 2, \tag{6.30}$$

where the latter space is equipped with the natural graph norm $u \mapsto \|u\|_{H^s(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^nx)}$. When $s = 1$, this appears as Lemma 1.5.3.9 on p. 60 of [59], and the extension to $s < 2$ has been worked out, along similar lines, in [36]. □

Corollary 6.5. *Assume Hypothesis* 2.1. *Then*

$$\big\{u|_{\partial\Omega} \,\big|\, u \in C^\infty(\overline{\Omega})\big\} \text{ is a dense subspace of } \big(N^{1/2}(\partial\Omega)\big)^*. \tag{6.31}$$

As a consequence,

$$H^1(\partial\Omega) \hookrightarrow \big(N^{1/2}(\partial\Omega)\big)^* \text{ densely.} \tag{6.32}$$

Proof. This is an immediate consequence of (6.30) and the ontoness of the map (6.14). □

For smoother domains, Theorem 6.4 takes a somewhat more familiar form (compare with [83] and [60]):

Corollary 6.6. *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded $C^{1,r}$ domain with $r > 1/2$. Then the Dirichlet trace γ_D in* (3.2) *exists in a unique fashion to a linear, bounded operator*

$$\widehat{\gamma}_D : \big\{u \in L^2(\Omega; d^nx) \,\big|\, \Delta u \in L^2(\Omega; d^nx)\big\} \to H^{-1/2}(\partial\Omega). \tag{6.33}$$

Moreover, for this extension of the Dirichlet trace operator one has the following generalized integration by parts formula

$$\langle \gamma_N w, \widehat{\gamma}_D u\rangle_{1/2} = (\Delta w, u)_{L^2(\Omega;d^nx)} - (w, \Delta u)_{L^2(\Omega;d^nx)}, \tag{6.34}$$

whenever $u \in L^2(\Omega; d^nx)$ satisfies $\Delta u \in L^2(\Omega; d^nx)$, and $w \in H^2(\Omega) \cap H^1_0(\Omega)$.

Proof. This is an immediate consequence of Theorem 6.4 and equation (6.7). □

We next turn our attention to case of the Neumann trace, whose action we would like to extend to $\mathrm{dom}(-\Delta_{max})$. To this end, we need to address a number of preliminary matters. First, assuming Hypothesis 2.1, we make the following definition (compare with (6.5)):

$$N^{3/2}(\partial\Omega; d^{n-1}\omega) = N^{3/2}(\partial\Omega) := \big\{g \in H^1(\partial\Omega) \,\big|\, \nabla_{tan} g \in \big(H^{1/2}(\partial\Omega)\big)^n\big\}, \tag{6.35}$$

equipped with the natural norm

$$\|g\|_{N^{3/2}(\partial\Omega)} := \|g\|_{L^2(\partial\Omega;d^{n-1}\omega)} + \|\nabla_{tan} g\|_{(H^{1/2}(\partial\Omega))^n}. \tag{6.36}$$

Lemma 6.7. *Assume Hypothesis* 2.1. *Then $N^{3/2}(\partial\Omega)$ is a reflexive Banach space which embeds continuously into $H^1(\partial\Omega)$.*

Proof. That the natural injection $N^{3/2}(\partial\Omega) \hookrightarrow H^1(\partial\Omega)$ is bounded is clear from (6.36), (6.1) and (2.26). Next, let $\{g_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence in $N^{3/2}(\partial\Omega)$. Then $\{g_m\}_{m\in\mathbb{N}}$ converges in $H^1(\partial\Omega)$ to some $g \in H^1(\partial\Omega)$. Consequently, for each $j,k \in \{1,...,n\}$, the sequence $\{\partial g_m/\partial\tau_{j,k}\}_{m\in\mathbb{N}}$ converges to $\partial g/\partial\tau_{j,k}$ in $L^2(\partial\Omega; d^{n-1}\omega)$. This and (6.1) then imply that $\{\nabla_{tan} g_m\}_{m\in\mathbb{N}}$ converges to ∇g in $\big[L^2(\partial\Omega; d^{n-1}\omega)\big]^n$. Since $\{\nabla_{tan} g_m\}_{m\in\mathbb{N}}$ is also known to be Cauchy in $H^1(\partial\Omega)$, we may finally conclude that $g \in N^{3/2}(\partial\Omega)$ and $\{g_m\}_{m\in\mathbb{N}}$ converges to g in $N^{3/2}(\partial\Omega)$. This proves that $N^{3/2}(\partial\Omega)$ is a Banach space. Next, one observes that

$$\Phi : N^{3/2}(\partial\Omega) \to L^2(\partial\Omega; d^{n-1}\omega) \dotplus \big[H^{1/2}(\partial\Omega)\big]^n, \quad \Phi(g) := (g, \nabla_{tan} g), \tag{6.37}$$

is an isometric embedding, allowing for the identification of $N^{3/2}(\partial\Omega)$ with a closed subspace of the reflexive space $L^2(\partial\Omega; d^{n-1}\omega) \dotplus \big[H^{1/2}(\partial\Omega)\big]^n$. Consequently, $N^{3/2}(\partial\Omega)$ is also reflexive. $\square$

Our next result shows that this is a natural substitute for the more familiar space $H^{3/2}(\partial\Omega)$ in the case where Ω is sufficiently smooth. Concretely, we have the following result:

Lemma 6.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{1,r}$ domain with $r > 1/2$. Then*

$$N^{3/2}(\partial\Omega) = H^{3/2}(\partial\Omega), \tag{6.38}$$

as vector spaces with equivalent norms.

Proof. Denote by $(\nu_1, ..., \nu_n)$ the components of the outward unit normal ν to $\partial\Omega$. If $g \in H^1(\partial\Omega)$ then (6.1) and elementary algebra yield that

$$\frac{\partial g}{\partial\tau_{j,k}} = \nu_j(\nabla_{tan} g)_k - \nu_k(\nabla_{tan} g)_j, \quad j,k = 1,...,n, \tag{6.39}$$

where $(\nabla_{tan} g)_r$ stands for the r-th component of the vector field $\nabla_{tan} g$. As a consequence of this identity and Lemma 3.4, if $g \in N^{3/2}(\partial\Omega)$, then $\partial g/\partial\tau_{j,k} \in H^{1/2}(\partial\Omega)$. Thus, Lemma 2.4 yields that $g \in H^{3/2}(\partial\Omega)$ with $\|g\|_{H^{3/2}(\partial\Omega)} \approx \|g\|_{N^{3/2}(\partial\Omega)}$, whenever $g \in N^{3/2}(\partial\Omega)$. This proves the left-to-right inclusion in (6.38).

Conversely, if $g \in H^{3/2}(\partial\Omega)$, then $g \in H^1(\partial\Omega)$ and $\partial g/\partial\tau_{j,k} \in H^{1/2}(\partial\Omega)$ for $1 \leq j,k \leq n$, by Lemma 2.4 (and a natural equivalence of norms). Based on this and (6.1), we may then conclude that $g \in N^{3/2}(\partial\Omega)$ with $\|g\|_{N^{3/2}(\partial\Omega)} \approx \|g\|_{H^{3/2}(\partial\Omega)}$, whenever $g \in H^{3/2}(\partial\Omega)$. This proves the right-to-left inclusion in (6.38). $\square$

It should be pointed out that, generally speaking, the spaces $H^{3/2}(\partial\Omega)$ and $N^{3/2}(\partial\Omega)$ can be quite different in the class of Lipschitz domains.

The reason we are interested in $N^{3/2}(\partial\Omega)$ is that this space arises naturally when considering the Dirichlet trace operator acting on

$$\big\{u \in H^2(\Omega) \,\big|\, \gamma_N u = 0\big\}, \tag{6.40}$$

considered as a closed subspace of $H^2(\Omega)$ (thus, a Banach space when equipped with the norm inherited from $H^2(\Omega)$). More precisely, we have the following result:

Lemma 6.9. *Assume Hypothesis* 2.1. *Then the Dirichlet trace operator γ_D considered in the context*

$$\gamma_D : \big\{u \in H^2(\Omega) \,\big|\, \gamma_N u = 0\big\} \to N^{3/2}(\partial\Omega) \tag{6.41}$$

is well-defined, linear, bounded, onto, and with a linear, bounded right-inverse. In addition, the null space of γ_D in (6.41) *is precisely $H_0^2(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^2(\Omega)$.*

Proof. If $u \in H^2(\Omega)$ is such that $\gamma_N u = 0$, then Theorem 6.1 yields

$$(\gamma_D u, 0) = \gamma_2 u \in \big\{(g_0, g_1) \in H^1(\partial\Omega) \dotplus L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\} \tag{6.42}$$

which entails $\gamma_D u \in N^{3/2}(\partial\Omega)$. Furthermore, there exists $C = C(\Omega) > 0$, independent of u, such that $\|\gamma_D u\|_{N^{3/2}(\partial\Omega)} \leq C\|u\|_{H^2(\Omega)}$. As a consequence, the operator (6.11) is well-defined, linear, and bounded. Next, we recall that $\mathcal{E}_2$ stands for a linear, bounded right-inverse for γ_2 in (6.2). Then, if

$$\iota' : N^{3/2}(\partial\Omega) \to \big\{(g_0, g_1) \in H^1(\partial\Omega) \dotplus L^2(\partial\Omega; d^{n-1}\omega) \,\big|\, \nabla_{tan} g_0 + g_1\nu \in \big(H^{1/2}(\partial\Omega)\big)^n\big\} \tag{6.43}$$

is the injection given by $\iota'(g) := (g, 0)$, for every $g \in N^{3/2}(\partial\Omega)$, it follows that the composition $\mathcal{E}_2\iota' : N^{3/2}(\partial\Omega) \to \big\{u \in H^2(\Omega) \,\big|\, \gamma_N u = 0\big\}$ is a linear, bounded right-inverse for the operator γ_D in (6.41). Hence, this operator is also onto. Finally, the fact that the null space of γ_D in (6.41) is precisely $H_0^2(\Omega)$ follows from its definition and the last part in the statement of Theorem 6.1. $\square$

Next, we shall use the Neumann trace result in Lemma 6.9 to extend the action of the Neumann trace operator (3.9) to $\mathrm{dom}(-\Delta_{max}) = \{u \in L^2(\Omega; d^n x) \,|\, \Delta u \in L^2(\Omega; d^n x)\}$. As before, this space is equipped with the natural graph norm. We denote by $\big(N^{3/2}(\partial\Omega)\big)^*$ the conjugate dual space of $N^{3/2}(\partial\Omega)$.

Theorem 6.10. *Assume Hypothesis* 2.1. *Then there exists a unique linear and bounded operator*

$$\widehat{\gamma}_N : \{u \in L^2(\Omega; d^n x) \,|\, \Delta u \in L^2(\Omega; d^n x)\} \to \big(N^{3/2}(\partial\Omega)\big)^* \tag{6.44}$$

which is compatible with the Neumann trace introduced in (3.9)*, in the sense that, for each $s > 3/2$, one has*

$$\widehat{\gamma}_N u = \gamma_N u \ \text{ for every } u \in H^s(\Omega) \text{ with } \Delta u \in L^2(\Omega; d^n x). \tag{6.45}$$

Furthermore, this extension of the Neumann trace operator from (3.9) *allows for the following generalized integration by parts formula*

$${}_{N^{3/2}(\partial\Omega)}\langle \gamma_D w, \widehat{\gamma}_N u\rangle_{(N^{3/2}(\partial\Omega))^*} = (w, \Delta u)_{L^2(\Omega; d^n x)} - (\Delta w, u)_{L^2(\Omega; d^n x)}, \tag{6.46}$$

valid for every $u \in L^2(\Omega; d^n x)$ with $\Delta u \in L^2(\Omega; d^n x)$ and every $w \in H^2(\Omega)$ with $\gamma_N w = 0$.

Proof. Consider $u \in L^2(\Omega; d^n x)$ arbitrary such that $\Delta u \in L^2(\Omega; d^n x)$. We shall then define a functional $\widehat{\gamma}_N u \in \big(N^{3/2}(\partial\Omega)\big)^*$ in the following fashion: Given an arbitrary function $g \in N^{3/2}(\partial\Omega)$, we invoke Lemma 6.9 in order to find $w \in H^2(\Omega)$ such that $\gamma_N w = 0$ and $\gamma_D w = g$. In addition, matters can be arranged so that $\|w\|_{H^2(\Omega)} \leq C\|g\|_{N^{3/2}(\partial\Omega)}$ for some finite constant $C = C(\Omega) > 0$, independent of g. We then define

$${}_{N^{3/2}(\partial\Omega)}\langle g, \widehat{\gamma}_N u\rangle_{(N^{3/2}(\partial\Omega))^*} := (w, \Delta u)_{L^2(\Omega; d^n x)} - (\Delta w, u)_{L^2(\Omega; d^n x)}. \tag{6.47}$$

We claim that the above definition is unambiguous in the sense that the action of the functional $\widehat{\gamma}_N u$ does not depend on the particular choice of w, with the properties listed above. By linearity, this comes down to proving the following claim: If u is as before and $w \in H^2(\Omega)$ is such that $\gamma_D w = \gamma_N w = 0$, then

$$(\Delta u, w)_{L^2(\Omega; d^n x)} = (u, \Delta w)_{L^2(\Omega; d^n x)}. \tag{6.48}$$

However, since Theorem 6.1 ensures that $w \in H^2_0(\Omega)$, formula (6.48) follows similarly to (6.18), by approximating w with functions form $C_0^\infty(\Omega)$ in the norm of $H^2(\Omega)$.

The above reasoning shows that formula (6.47) yields a well-defined, linear, and bounded operator in the context of (6.47). By definition, this operator will satisfy (6.46).

Next, we will show that (6.45) is valid for each $s > 3/2$. Fix $s > 3/2$ and let $u \in H^s(\Omega)$ with $\Delta u \in L^2(\Omega; d^n x)$. Then for every $w \in H^2(\Omega)$ with $\gamma_N w = 0$, Green's formula (3.21) yields

$$\begin{aligned}(\Delta w, u)_{L^2(\Omega; d^n x)} &= {}_{H^1(\Omega)}\langle w, \Delta u\rangle_{(H^1(\Omega))^*} \\ &= -(\nabla w, \nabla u)_{(L^2(\Omega; d^n x))^n} + \overline{\langle \gamma_D u, \widetilde{\gamma}_N w\rangle_{1/2}} \\ &= -(\nabla w, \nabla u)_{(L^2(\Omega; d^n x))^n}.\end{aligned} \tag{6.49}$$

On the other hand, we may once again employ (3.21) in order to rewrite the last integral above as

$$\begin{aligned}(\nabla w, \nabla u)_{(L^2(\Omega; d^n x))^n} &= \langle \gamma_D w, \widetilde{\gamma}_N u\rangle_{1/2} - {}_{H^1(\Omega)}\langle w, \Delta u\rangle_{(H^1(\Omega))^*} \\ &= \int_{\partial\Omega} d^{n-1}\omega\, \overline{\gamma_D w}\gamma_N u - (w, \Delta u)_{L^2(\Omega; d^n x)},\end{aligned} \tag{6.50}$$

given the smoothness properties of the functions involved. Altogether, (6.49) and (6.50) imply that

$$\int_{\partial\Omega} d^{n-1}\omega\, \overline{\gamma_D w}\gamma_N u = (w, \Delta u)_{L^2(\Omega; d^n x)} - (\Delta w, u)_{L^2(\Omega; d^n x)}. \tag{6.51}$$

This formula is valid for any $w \in H^2(\Omega)$ with $\gamma_N w = 0$, and the operator (6.41) maps this space onto $N^{3/2}(\partial\Omega)$. We may therefore deduce from (6.51) that $\widehat{\gamma}_N u$, originally viewed as a functional on $\big(N^{3/2}(\partial\Omega)\big)^*$, is given by integration against $\gamma_N u \in L^2(\partial\Omega; d^{n-1}\omega)$ in the case where $u \in H^s(\Omega)$ satisfies $\Delta u \in L^2(\Omega; d^n x)$. This concludes the proof of (6.45).

Finally, the uniqueness of an operator satisfying (6.44) and (6.45) is guaranteed by what we have proved thus far and the density result in (6.30). $\square$

For smoother domains, Theorem 6.10 takes a somewhat more familiar form (compare with Lions-Magenes [83] and Grubb [60], as before). We recall that $H^{-3/2}(\partial\Omega) = \big(H^{3/2}(\partial\Omega)\big)^*$.

Corollary 6.11. *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded $C^{1,r}$ domain with $r > 1/2$. Then the Neumann trace in* (3.9) *exists in a unique fashion to a linear, bounded operator*

$$\widehat{\gamma}_N : \{u \in L^2(\Omega; d^n x) \,|\, \Delta u \in L^2(\Omega; d^n x)\} \to H^{-3/2}(\partial\Omega). \tag{6.52}$$

Moreover, for this extension of the Neumann trace operator one has the following generalized integration by parts formula

$$_{H^{3/2}(\partial\Omega)}\langle \gamma_D w, \widehat{\gamma}_N u\rangle_{(H^{3/2}(\partial\Omega))^*} = (w, \Delta u)_{L^2(\Omega;d^n x)} - (\Delta w, u)_{L^2(\Omega;d^n x)}, \tag{6.53}$$

whenever $u \in L^2(\Omega; d^n x)$ satisfies $\Delta u \in L^2(\Omega; d^n x)$, and $w \in H^2(\Omega)$ has $\gamma_N w = 0$.

Proof. This is a direct consequence of Theorem 6.10 and (6.38). □

Corollary 6.12. *Assume Hypothesis* 2.1. *Then*

$$\{\nu \cdot (\nabla u)|_{\partial\Omega} \,|\, u \in C^\infty(\overline{\Omega})\} \text{ is a dense subspace of } \big(N^{3/2}(\partial\Omega)\big)^*. \tag{6.54}$$

Consequently,

$$L^2(\partial\Omega; d^{n-1}\omega) \hookrightarrow \big(N^{3/2}(\partial\Omega)\big)^* \text{ densely.} \tag{6.55}$$

Proof. The claim in (6.54) is an immediate consequence of (6.30) and the ontoness of the map (6.44). Then (6.55) follows easily from this. □

Lemma 6.13. *Assume Hypothesis* 2.1. *Then*

$$N^{3/2}(\partial\Omega) \subseteq \big(N^{1/2}(\partial\Omega)\big)^*, \quad N^{1/2}(\partial\Omega) \subseteq \big(N^{3/2}(\partial\Omega)\big)^*. \tag{6.56}$$

Moreover, in each case, the inclusion is given by canonical continuous injections.

Proof. The first inclusion above is a consequence of Lemma 6.9 and Theorem 6.4. More specifically, let

$$\iota : H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow \{u \in L^2(\Omega; d^n x) \,|\, \Delta u \in L^2(\Omega; d^n x)\} \tag{6.57}$$

be the natural embedding map, and denote by

$$\mathcal{E}_D : N^{3/2}(\partial\Omega) \to \{u \in H^2(\Omega) \,|\, \gamma_N u = 0\} \tag{6.58}$$

a linear, bounded right-inverse for γ_D in (6.41) (cf. Lemma 6.9). Then, due to Theorem 6.4, the composition

$$j := \widehat{\gamma}_D \iota \mathcal{E}_D : N^{3/2}(\partial\Omega) \to \big(N^{1/2}(\partial\Omega)\big)^* \tag{6.59}$$

is a linear, bounded operator mapping $N^{3/2}(\partial\Omega)$ into $\big(N^{1/2}(\partial\Omega)\big)^*$. We note that j does not, in fact, depend on the choice of a particular right-inverse $\mathcal{E}_D$ for γ_D in (6.41). Indeed, the difference of any two such right-inverses, call them $\mathcal{E}'_D$ and $\mathcal{E}''_D$, is (by (6.4)) an operator mapping into $H^2_0(\Omega)$. As such, this is sent to zero when further composed to the left by $\widehat{\gamma}_D \iota$. Finally, to prove that j is injective, it suffices to observe that if $u \in H^2(\Omega)$ has $\widehat{\gamma}_D u = 0$ then also $\gamma_D u = 0$; cf. (6.15). This shows that j is a canonical injection of $N^{3/2}(\partial\Omega)$ into $\big(N^{1/2}(\partial\Omega)\big)^*$. The case of the second inclusion in (6.56) is treated similarly, based on Lemma 6.3 and Theorem 6.10. □

7. The Minimal and Maximal Laplacians on Lipschitz Domains

Minimal and maximal $L^2(\Omega; d^n x)$-realizations of the Laplacian on Lipschitz domains are considered in this short section.

Given an open set $\Omega \subset \mathbb{R}^n$, consider the maximal Laplacian $-\Delta_{max}$ in $L^2(\Omega; d^n x)$ defined by

$$\begin{aligned} &-\Delta_{max} u := -\Delta u, \\ &u \in \operatorname{dom}(-\Delta_{max}) := \{v \in L^2(\Omega; d^n x) \,|\, \Delta v \in L^2(\Omega; d^n x)\}. \end{aligned} \tag{7.1}$$

Lemma 7.1. *Assume Hypothesis* 2.1. *Then the maximal Laplacian associated with Ω is a closed, densely defined operator for which*

$$\begin{aligned} H^2_0(\Omega) &\subseteq \operatorname{dom}((-\Delta_{max})^*) \\ &\subseteq \{u \in L^2(\Omega; d^n x) \,|\, \Delta u \in L^2(\Omega; d^n x),\, \widehat{\gamma}_D u = \widehat{\gamma}_N u = 0\}. \end{aligned} \tag{7.2}$$

Proof. If $u_j \in L^2(\Omega; d^n x)$, $j \in \mathbb{N}$, are functions such that $\Delta u_j \in L^2(\Omega; d^n x)$ for each j and

$$u_j \to u, \quad \Delta u_j \to v \text{ in } L^2(\Omega; d^n x) \text{ as } j \to \infty, \tag{7.3}$$

then for every $w \in C_0^\infty(\Omega)$ one has

$$\begin{aligned}(u, \Delta w)_{L^2(\Omega;d^n x)} &= \lim_{j\to\infty} (u_j, \Delta w)_{L^2(\Omega;d^n x)} = \lim_{j\to\infty} (\Delta u_j, w)_{L^2(\Omega;d^n x)} \\ &= \lim_{j\to\infty} (v, w)_{L^2(\Omega;d^n x)}.\end{aligned} \tag{7.4}$$

This shows that $\Delta u = v \in L^2(\Omega; d^n x)$ in the sense of distributions. Hence, $u \in \mathrm{dom}(-\Delta_{max})$ and $\Delta_{max} u = v$, proving that the operator (7.1) is closed. By (6.30), this operator is also densely defined.

Consider next (7.2). If $w \in H_0^2(\Omega)$ then for every $u \in \mathrm{dom}(-\Delta_{max})$ Theorem 6.4 yields

$$(\Delta w, u)_{L^2(\Omega;d^n x)} - (w, \Delta u)_{L^2(\Omega;d^n x)} = {}_{N^{1/2}(\partial\Omega)}\langle \gamma_N w, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*} = 0. \tag{7.5}$$

This shows that $w \in \mathrm{dom}((-\Delta_{max})^*)$ and $(-\Delta_{max})^*(w) = -\Delta w$, justifying the first inclusion in (7.2).

Next, if $u \in \mathrm{dom}((-\Delta_{max})^*)$ then $u \in L^2(\Omega; d^n x)$ and, in addition, there exists $v \in L^2(\Omega; d^n x)$ such that

$$(u, \Delta w)_{L^2(\Omega;d^n x)} = (v, w)_{L^2(\Omega;d^n x)} \tag{7.6}$$

for every $w \in L^2(\Omega; d^n x)$ with $\Delta w \in L^2(\Omega; d^n x)$. We shall now specialize this to three distinguished classes of functions w. First, taking $w \in C_0^\infty(\Omega)$ arbitrarily, it follows that $\Delta u = v \in L^2(\Omega; d^n x)$ in the sense of distributions. With this at hand, and taking $w \in H^2(\Omega) \cap H_0^1(\Omega)$ arbitrarily, it follows from (6.16) that

$${}_{N^{1/2}(\partial\Omega)}\langle \gamma_N w, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*} = (\Delta w, u)_{L^2(\Omega;d^n x)} - (w, \Delta u)_{L^2(\Omega;d^n x)} = 0. \tag{7.7}$$

Hence, $\widehat{\gamma}_D u = 0$ by Lemma 6.3. Finally, consider arbitrary functions $w \in H^2(\Omega)$ satisfying $\widehat{\gamma}_N w = 0$. In this scenario, it follows from (6.46) that

$${}_{N^{3/2}(\partial\Omega)}\langle \gamma_D w, \widehat{\gamma}_N u\rangle_{(N^{3/2}(\partial\Omega))^*} = (w, \Delta u)_{L^2(\Omega;d^n x)} - (\Delta w, u)_{L^2(\Omega;d^n x)} = 0. \tag{7.8}$$

Thus, by Lemma 6.9, we also have $\widehat{\gamma}_N u = 0$. This concludes the justification of the second inclusion in (7.2). □

For an open set $\Omega \subset \mathbb{R}^n$, we also bring in the minimal Laplacian in $L^2(\Omega; d^n x)$, that is,

$$-\Delta_{min} u := -\Delta u, \quad u \in \mathrm{dom}(-\Delta_{min}) := H_0^2(\Omega). \tag{7.9}$$

Corollary 7.2. *Assume Hypothesis* 2.1. *Then $-\Delta_{min}$ is a densely defined, symmetric operator which satisfies*

$$-\Delta_{min} \subseteq (-\Delta_{max})^* \text{ and } -\Delta_{max} \subseteq (-\Delta_{min})^*. \tag{7.10}$$

Equality holds in one (*and hence in both*) *inclusions in* (7.10) *if and only if*

$$H_0^2(\Omega) \text{ equals } \big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x),\, \widehat{\gamma}_D u = \widehat{\gamma}_N u = 0\big\}. \tag{7.11}$$

Proof. The fact that $-\Delta_{min}$ is a densely defined, symmetric operator is obvious. As far as equality in (7.10) is concerned, it suffices to prove only the first inclusion (since the second one is a consequence of this and duality). This, however, is implied by Lemma 7.1. This lemma also shows that equality holds if and only if one has equality in (7.11). □

8. The Class of Quasi-Convex Domains

This section is devoted to one of our principal new results, the introduction and study of the class of quasi-convex domains.

In the class of Lipschitz domains, the two spaces appearing in (7.11) could be quite different. Obviously, the left-to-right inclusion always holds,

$$H_0^2(\Omega) \subseteq \big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x),\, \widehat{\gamma}_D u = \widehat{\gamma}_N u = 0\big\}. \tag{8.1}$$

The question now arises: What extra qualities of the Lipschitz domain will guarantee the equality in (8.1)? To address this issue, we need some preparations: Given $n \geq 1$, denote by $MH^{1/2}(\mathbb{R}^n)$ the class of pointwise multipliers of the Sobolev space $H^{1/2}(\mathbb{R}^n)$. That is,

$$MH^{1/2}(\mathbb{R}^n) := \big\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,\big|\, M_f \in \mathcal{B}\big(H^{1/2}(\mathbb{R}^n)\big)\big\}, \tag{8.2}$$

where M_f is the operator of pointwise multiplication by f. This space is equipped with the natural norm, that is,

$$\|f\|_{MH^{1/2}(\mathbb{R}^n)} := \|M_f\|_{\mathcal{B}(H^{1/2}(\mathbb{R}^n))}. \tag{8.3}$$

Definition 8.1. *Given $\delta > 0$, a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is called to be of class $MH^{1/2}_\delta$, and one writes*

$$\partial\Omega \in MH^{1/2}_\delta, \tag{8.4}$$

provided the following holds: There exists a finite open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of the boundary $\partial\Omega$ of Ω such that for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (considered in a new system of coordinates obtained from the original one via a rigid motion) which has the property that

$$\varphi_j \in MH^{1/2}(\mathbb{R}^{n-1}) \text{ and } \quad \|\varphi_j\|_{MH^{1/2}(\mathbb{R}^{n-1})} \leq \delta. \tag{8.5}$$

Continuing, we consider the following classes of domains

$$MH^{1/2}_\infty := \bigcup_{\delta>0} MH^{1/2}_\delta, \quad MH^{1/2}_0 := \bigcap_{\delta>0} MH^{1/2}_\delta. \tag{8.6}$$

Finally, we also introduce the following definition:

Definition 8.2. *A bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is called square-Dini, and one writes*

$$\partial\Omega \in \mathrm{SD}, \tag{8.7}$$

provided the following holds: There exists a finite open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of the boundary $\partial\Omega$ of Ω such that for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (considered in a new system of coordinates obtained from the original one via a rigid motion) which, additionally, has the property that

$$\int_0^1 \left(\frac{\omega(\nabla\varphi_j; t)}{t^{1/2}}\right)^2 \frac{dt}{t} < \infty. \tag{8.8}$$

Here, given a (possibly vector-valued) function f in $\mathbb{R}^{n-1}$,

$$\omega(f; t) := \sup\{|f(x) - f(y)| \,|\, x, y \in \mathbb{R}^{n-1}, |x - y| \leq t\}, \quad t \in (0, 1), \tag{8.9}$$

is the modulus of continuity of f, at scale t.

From the work of V. Maz'ya and T. Shaposhnikova [90], [91], it is known that if $r > 1/2$ then

$$C^{1,r} \subseteq \mathrm{SD} \subseteq MH^{1/2}_0 \subseteq MH^{1/2}_\infty. \tag{8.10}$$

As pointed out in [91], domains of class $MH^{1/2}_\infty$ can have certain types of vertices and edges when $n \geq 3$.

Next, we recall that a domain is said to satisfy a uniform exterior ball condition provided there exists a number $r > 0$ with the property that

$$\text{for every } x \in \partial\Omega \text{ there exists } y \in \mathbb{R}^n \text{ such that} \tag{8.11}$$

$$B(y, r) \cap \Omega = \emptyset \text{ and } x \in \partial B(y, r) \cap \partial\Omega. \tag{8.12}$$

Next, we review the class of almost-convex domains introduced in [96].

Definition 8.3. *A bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is called an almost-convex domain provided there exists a family $\{\Omega_\ell\}_{\ell\in\mathbb{N}}$ of open sets in $\mathbb{R}^n$ with the following properties:*

(i) *$\partial\Omega_\ell \in C^2$ and $\overline{\Omega_\ell} \subset \Omega$ for every $\ell \in \mathbb{N}$.*

(ii) *$\Omega_\ell \nearrow \Omega$ as $\ell \to \infty$, in the sense that $\overline{\Omega_\ell} \subset \Omega_{\ell+1}$ for each $\ell \in \mathbb{N}$ and $\bigcup_{\ell\in\mathbb{N}} \Omega_\ell = \Omega$.*

(iii) *There exists a neighborhood U of $\partial\Omega$ and, for each $\ell \in \mathbb{N}$, a C^2 real-valued function ρ_ℓ defined in U with the property that $\rho_\ell < 0$ on $U \cap \Omega_\ell$, $\rho_\ell > 0$ in $U\backslash\overline{\Omega_\ell}$, and which vanishes on $\partial\Omega_\ell$. In addition, it is assumed that there exists some constant $C_1 \in (1, \infty)$ such that*

$$C_1^{-1} \leq |\nabla\rho_\ell(x)| \leq C_1, \quad \forall\, x \in \partial\Omega_\ell, \quad \forall\, \ell \in \mathbb{N}. \tag{8.13}$$

(iv) *There exists $C_2 \geq 0$ such that for every number $\ell \in \mathbb{N}$, every point $x \in \partial\Omega_\ell$, and every vector $\xi \in \mathbb{R}^n$ which is tangent to $\partial\Omega_\ell$ at x, there holds*

$$\big\langle \mathrm{Hess}\,(\rho_\ell)\xi, \xi\big\rangle \geq -C_2|\xi|^2, \tag{8.14}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the standard inner product in $\mathbb{R}^n$ and

$$\mathrm{Hess}\,(\rho_\ell) := \left(\frac{\partial^2\rho_\ell}{\partial x_i\partial x_j}\right)_{1\leq i,j\leq n}, \tag{8.15}$$

is the Hessian of ρ_ℓ.

A few remarks are in order here: First, it is not difficult to see that (8.13) ensures that each domain Ω_ℓ is Lipschitz, with Lipschitz constant bounded uniformly in ℓ. Second, (8.14) simply says that, as quadratic forms on the tangent bundle $T\partial\Omega_\ell$ to $\partial\Omega_\ell$, we have

$$\mathrm{Hess}\,(\rho_\ell) \geq -C_2\, I_n, \tag{8.16}$$

where I_n is the identity matrix in $\mathbb{R}^n$. Hence, another equivalent formulation of (8.14) is the following requirement:

$$\sum_{i,j=1}^n \frac{\partial^2 \rho_\ell}{\partial x_i \partial x_j} \xi_i \xi_j \geq -C_2 \sum_{i=1}^n \xi_i^2, \text{ whenever } \rho_\ell = 0 \text{ and } \sum_{i=1}^n \frac{\partial \rho_\ell}{\partial x_i} \xi_i = 0. \tag{8.17}$$

We note that since the second fundamental form II_ℓ on $\partial\Omega_\ell$ is given by $II_\ell = \mathrm{Hess}\,\rho_\ell/|\nabla\rho_\ell|$, almost-convexity is, in view of (8.13), equivalent to requiring that II_ℓ be bounded below, uniformly in ℓ.

We now discuss some important special classes of almost-convex domains.

Definition 8.4. *A bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ satisfies a local exterior ball condition, henceforth referred to as LEBC, if every boundary point $x_0 \in \partial\Omega$ has an open neighborhood $\mathcal{O}$ which satisfies the following two conditions:*

- (i) *There exists a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0$ and such that if D is the domain above the graph of φ then D satisfies a UEBC.*
- (ii) *There exists a $C^{1,1}$ diffeomorphism Υ mapping $\mathcal{O}$ onto the unit ball $B(0,1)$ in $\mathbb{R}^n$ and such that $\Upsilon(x_0) = 0$, $\Upsilon(\mathcal{O} \cap \Omega) = B(0,1) \cap D$, $\Upsilon(\mathcal{O} \backslash \overline{\Omega}) = B(0,1) \backslash \overline{D}$.*

It is clear from Definition 8.4 that the class of bounded domains satisfying a LEBC is invariant under $C^{1,1}$ diffeomorphisms. This makes this class of domains amenable to working on manifolds. This is the point of view adopted in [96], where the following result is also proved:

Lemma 8.5. *If the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ satisfies a LEBC then it is almost-convex.*

Hence, in the class of bounded Lipschitz domains in $\mathbb{R}^n$, we have

$$\text{convex} \Longrightarrow \text{UEBC} \Longrightarrow \text{LEBC} \Longrightarrow \text{almost-convex}. \tag{8.18}$$

For a vector field $w = (w_1, w_2, \dots, w_n)$ whose components are distributions in an open set $\Omega \subset \mathbb{R}^n$, we define its curl, $\mathrm{curl}(w)$, to be the tensor field with n^2 components (entries) given by

$$(\mathrm{curl}(w))_{j,k} = \partial_j w_k - \partial_k w_j, \quad j,k = 1, \dots, n. \tag{8.19}$$

In addition, if $\Psi = (\Psi_{j,k})_{1 \leq j,k \leq n}$ is a tensor field whose n^2 components are distributions in Ω, we set

$$(\mathrm{div}(\Psi))_k = \sum_{j=1}^n \partial_j (\Psi_{j,k} - \Psi_{k,j}), \quad k = 1, \dots, n. \tag{8.20}$$

We also use the notation

$$\langle A, B \rangle_{\mathbb{C}^{n^2}} = \sum_{j,k=1}^n \overline{A_{k,j}}\, B_{k,j} = \mathrm{tr}(A^* B) \tag{8.21}$$

for two tensor fields A and B with n^2 components. We then introduce the following definition:

Definition 8.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with outward unit normal ν. Let $w = (w_1, \dots, w_n)$ be a vector field with components in $L^2(\Omega; d^n x)$ such that $\mathrm{curl}(w)$ also has components in $L^2(\Omega; d^n x)$. Then $\nu \times w$ is the unique tensor field with n^2 components in $H^{-1/2}(\partial\Omega) = \big(H^{1/2}(\partial\Omega)\big)^*$ which satisfies the following property: If Ψ is any tensor field with n^2 components in $H^1(\Omega)$ and $\psi = \gamma_D \Psi$, with the Dirichlet trace taken componentwise, then*

$$_{H^{1/2}(\partial\Omega)^{n^2}}\langle \psi, \nu \times w \rangle_{H^{-1/2}(\partial\Omega)^{n^2}} = \int_\Omega d^n x\, \langle \Psi, \mathrm{curl}(w) \rangle_{\mathbb{C}^{n^2}} + \int_\Omega d^n x\, \overline{\mathrm{div}(\Psi)} \cdot w. \tag{8.22}$$

It is not difficult to check (using the fact that $\gamma_D : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is onto, and that $C_0^\infty(\Omega)$ is dense in the space $H^1(\Omega)$ with vanishing Dirichlet trace) that (8.22) uniquely defines $\nu \times w$ as a functional in $H^{-1/2}(\partial\Omega)^{n^2}$.

We shall also need the following companion of Definition 8.6:

Definition 8.7. *Assume that $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain with outward unit normal ν. Let $w=(w_1,\dots,w_n)$ be a vector field with components in $L^2(\Omega;d^nx)$ such that* $\operatorname{div}(w)\in L^2(\Omega;d^nx)$. *Then $\nu\cdot w$ is the unique functional in $H^{-1/2}(\partial\Omega)=\big(H^{1/2}(\partial\Omega)\big)^*$ which satisfies the following property. If $\Phi\in H^1(\Omega)$ and $\phi=\gamma_D\Phi$ then*

$$ {}_{H^{1/2}(\partial\Omega)}\langle\phi,\nu\cdot w\rangle_{H^{-1/2}(\partial\Omega)}=\int_\Omega d^nx\,\overline{\Phi}\operatorname{div}(w)+\int_\Omega d^nx\,\overline{\nabla\Phi}\cdot w. \tag{8.23}$$

As before, one can check that this uniquely defines $\nu\cdot w$ as a functional in $H^{-1/2}(\partial\Omega)$.

Next, we note the following regularity result, which is a consequence of Theorem 4.1 on p. 1458 in [96] (which contains a more general result, formulated in the language of differential forms).

Lemma 8.8. *Assume that $\Omega\subset\mathbb{R}^n$ is an almost-convex domain with outward unit normal ν. Then,*

$$\begin{aligned}&\big\{w\in L^2(\Omega;d^nx)^n\,\big|\operatorname{div}(w)\in L^2(\Omega;d^nx),\ \operatorname{curl}(w)\in L^2(\Omega;d^nx)^{n^2},\ \nu\times w=0\big\}\\&\quad=\big\{w\in H^1(\Omega)^n\,\big|\,\nu\times w=0\big\}\end{aligned}\tag{8.24}$$

and, in addition, there exists a finite constant $C=C(\Omega)>0$ such that

$$\|w\|_{H^1(\Omega)^n}\le C\big(\|w\|_{L^2(\Omega;d^nx)^n}+\|\operatorname{div}(w)\|_{L^2(\Omega;d^n)}+\|\operatorname{curl}(w)\|_{L^2(\Omega;d^nx)^{n^2}}\big),\tag{8.25}$$

whenever $\nu\times w=0$. Furthermore,

$$\begin{aligned}&\big\{w\in L^2(\Omega;d^nx)^n\,\big|\operatorname{div}(w)\in L^2(\Omega;d^nx),\ \operatorname{curl}(w)\in L^2(\Omega;d^nx)^{n^2},\ \nu\cdot w=0\big\}\\&\quad=\big\{w\in H^1(\Omega)^n\,\big|\,\nu\cdot w=0\big\},\end{aligned}\tag{8.26}$$

and there exists a finite constant $C=C(\Omega)>0$ such that

$$\|w\|_{H^1(\Omega)^n}\le C\big(\|w\|_{L^2(\Omega;d^nx)^n}+\|\operatorname{div}(w)\|_{L^2(\Omega;d^n)}+\|\operatorname{curl}(w)\|_{L^2(\Omega;d^nx)^{n^2}}\big),\tag{8.27}$$

whenever $\nu\cdot w=0$.

We are now in a position to specify the class of domains in which most of our subsequent analysis will be carried out.

Definition 8.9. *Let $n\in\mathbb{N}$, $n\ge 2$, and assume that $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain. Then Ω is called a quasi-convex domain if there exists $\delta>0$ sufficiently small (relative to n and the Lipschitz character of Ω), with the following property that for every $x\in\partial\Omega$ there exists an open subset Ω_x of Ω such that $\partial\Omega\cap\partial\Omega_x$ is an open neighborhood of x in $\partial\Omega$, and for which one of the following two conditions holds:*
(i) Ω_x is of class $MH^{1/2}_\delta$ if $n\ge3$, and of class $C^{1,r}$ for some $1/2<r<1$ if $n=2$.
(ii) Ω_x is an almost-convex domain.

Given Definition 8.9, we thus introduce the following basic assumption:

Hypothesis 8.10. *Let $n\in\mathbb{N}$, $n\ge2$, and assume that $\Omega\subset\mathbb{R}^n$ is a quasi-convex domain.*

The following lemma will play a basic role in our work:

Lemma 8.11. *Assume Hypothesis* 8.10. *Then,*

$$\operatorname{dom}\big(-\Delta_{D,\Omega}\big)\subset H^2(\Omega),\quad \operatorname{dom}\big(-\Delta_{N,\Omega}\big)\subset H^2(\Omega).\tag{8.28}$$

Proof. When Ω is a domain of class $C^{1,r}$ for some $1/2<r<1$ this result appeared in [56]. Assume now that Ω is an almost-convex domain. Then, if $u\in\operatorname{dom}\big(-\Delta_{D,\Omega}\big)$ or $u\in\operatorname{dom}\big(-\Delta_{N,\Omega}\big)$, Lemma 8.8 applied to $w:=\nabla u$ yields that $w\in H^1(\Omega)^n$ so that, ultimately, $u\in H^2(\Omega)$, as desired. Next, for domains of class $MH^{1/2}_\delta$ with $\delta>0$ sufficiently small (relative to dimension and the Lipschitz character), (8.28) is a consequence of work done in [90] and [91]. The more general types of domains considered here can be treated by using this and a localization argument: More specifically, assume that $u\in H^1_0(\Omega)$ is such that $f:=\Delta u\in L^2(\Omega;d^nx)$. Fix an arbitrary boundary point $x_0\in\partial\Omega$ and let Ω_{x_0} be the associated domain as in Hypothesis 8.10. Finally, fix $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\partial\Omega\cap\operatorname{supp}(\psi)\subset\partial\Omega\cap\partial\Omega_{x_0}$. Then $v:=(\psi u)|_{\Omega_x}\in H^1_0(\Omega_x)$ has the property that $\Delta v=[(\Delta\psi)u+2\nabla\psi\cdot\nabla u+\psi f]|_{\Omega_{x_0}}\in L^2(\Omega_{x_0};d^nx)$. Thus, by the result recalled at the beginning of the current proof, $v\in H^2(\Omega_{x_0})$. Using a partition of unity argument (and interior elliptic regularity), one deduces from this that $u\in H^2(\Omega)$. This proves the first inclusion in (8.28).

The second inclusion in (8.28) is a bit more delicate since, as opposed to the Dirichlet case, Neumann boundary conditions are not stable under truncations by smooth functions. We shall, nonetheless, overcome this difficulty by proceeding as follows: Let $u\in H^1(\Omega)$ be such that $f:=\Delta u\in L^2(\Omega;d^nx)$ and $\widetilde\gamma_N u=0$.

As before, fix $x_0 \in \partial\Omega$, denote by Ω_{x_0} the domain associated with x_0 as in Hypothesis 8.10, and select $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\partial\Omega \cap \operatorname{supp}(\psi) \subset \partial\Omega \cap \partial\Omega_{x_0}$. Our goal is to show that $\psi u \in H^2(\Omega_{x_o})$. From this and a partition of unity argument we may then once again conclude that $u \in H^2(\Omega)$, as desired. At this point, the discussion branches out into several cases, which we treat separately below.

Case I: *Assume that Ω_{x_0} is a domain of class $C^{1,r}$, for some $r > 1/2$.* In this scenario, we consider the function $v := (\psi u)|_{\Omega_{x_0}} \in H^1(\Omega_{x_o})$ and note that, as before, $\Delta v \in L^2(\Omega_{x_0}; d^n x)$. We wish to show that $v \in H^2(\Omega_{x_o})$. To this end, one observes that the support of $\widetilde{\gamma}_N v$ is a relatively compact subset of $\partial\Omega_{x_0} \cap \partial\Omega$ and that

$$\begin{aligned} \widetilde{\gamma}_N v|_{\partial\Omega_{x_0}\cap\partial\Omega} &= [(\gamma_D\psi)(\widetilde{\gamma}_N u) + (\widetilde{\gamma}_N\psi)(\gamma_D u)]|_{\partial\Omega_{x_0}\cap\partial\Omega} \\ &= [(\gamma_N\psi)(\gamma_D u)]|_{\partial\Omega_{x_0}\cap\partial\Omega} \in H^{1/2}(\partial\Omega_{x_o} \cap \partial\Omega), \end{aligned} \tag{8.29}$$

by Lemma 3.4 and the fact that $\widetilde{\gamma}_N u = 0$ on $\partial\Omega$. Consequently, in order to conclude that $v \in H^2(\Omega_{x_o})$, it suffices to show the following:

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ bounded } C^{1,r} \text{ domain, for some } 1/2 < r < 1 \\ w \in H^1(\Omega), \quad \Delta w \in L^2(\Omega; d^n x), \quad \widetilde{\gamma}_N w \in H^{1/2}(\partial\Omega) \end{array}\right\} \text{ imply } w \in H^2(\Omega). \tag{8.30}$$

To prove (8.30), we shall employ an integral representation of w in terms of layer potentials. Concretely, set $f := (-\Delta + 1)w \in L^2(\Omega; d^n x)$, and denote by w_0 the restriction to Ω of the convolution of f (extended by zero outside Ω, to the entire $\mathbb{R}^n$) with $E_n(-1; \cdot)$. Then $w_0 \in H^2(\Omega)$ and $(-\Delta + 1)w_0 = f$ in Ω. Then $w_1 := w - w_0$ satisfies

$$w_1 \in H^1(\Omega), \quad (-\Delta + 1)w_1 = 0 \text{ in } \Omega, \quad \widetilde{\gamma}_N w_1 \in H^{1/2}(\partial\Omega), \tag{8.31}$$

where the last membership makes use of Lemma 3.4 once again. Bring in the adjoint double layer potential operator $K^{\#}_{-1}$ in (4.3). From [95] we know that

$$-\tfrac{1}{2}I_{\partial\Omega} + K^{\#}_{-1} \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big) \text{ is an isomorphism.} \tag{8.32}$$

In addition, by Lemma 4.2, one concludes that

$$-\tfrac{1}{2}I_{\partial\Omega} + K^{\#}_{-1} \in \mathcal{B}\big(H^{1/2}(\partial\Omega)\big) \text{ is a Fredholm operator with index zero.} \tag{8.33}$$

Together, (8.32), (8.33) then show that

$$-\tfrac{1}{2}I_{\partial\Omega} + K^{\#}_{-1} \in \mathcal{B}\big(H^{1/2}(\partial\Omega)\big) \text{ is an isomorphism.} \tag{8.34}$$

Thus, w_1 permits the representation

$$w_1 = \mathcal{S}_{-1}\big[\big(-\tfrac{1}{2}I_{\partial\Omega} + K^{\#}_{-1}\big)^{-1}(\widetilde{\gamma}_N(w_1))\big] \in H^2(\Omega), \tag{8.35}$$

by (4.8) and (8.34), and hence, $w = w_0 + w_1 \in H^2(\Omega)$, completing the treatment of Case I.

Case II: *Assume that $n \geq 3$ and Ω_{x_0} is a domain of class $MH^{1/2}_\delta$, for some sufficiently small $\delta > 0$.* In this situation we proceed as in Case I, since the main ingredients in the proof carried out there, that is, (8.33) and (8.35), continue to hold in this setting due to the work in [91].

Case III: *Assume that Ω_{x_0} is an almost-convex domain.* In this situation, if ψ is as before, we consider the vector field $w := (\psi\nabla u)|_{\Omega_{x_o}} \in (L^2(\Omega_{x_o}; d^n x))^n$. This vector field satisfies

$$\begin{aligned} &\operatorname{curl}(w) = (\partial_j w_k - \partial_k w_j)_{1\leq j,k\leq n} \in L^2(\Omega_{x_0}; d^n x)^{n^2}, \\ &\operatorname{div}(w) \in L^2(\Omega; d^n x) \text{ and } \nu \cdot w = 0 \text{ in } H^{-1/2}(\partial\Omega_{x_0}). \end{aligned} \tag{8.36}$$

Hence, Lemma 8.8 yields $w \in (H^1(\Omega_{x_0}))^n$, implying $\psi u \in H^2(\Omega_{x_o})$. □

Remark 8.12. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with at least one re-entrant corner. Let $\omega_1, ..., \omega_N$ be the internal angles of Ω satisfying $\pi < \omega_j < 2\pi$, $1 \leq j \leq N$, and denote by $P_1, ..., P_N$ the corresponding vertices. Then the solution to the Poisson problem

$$-\Delta u = f \in L^2(\Omega; d^2 x), \quad u \in H^1_0(\Omega), \tag{8.37}$$

permits the representation

$$u = \sum_{j=1}^N \lambda_j v_j + w, \quad \lambda_j \in \mathbb{R}, \tag{8.38}$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and, for each j, v_j is a function exhibiting a singular behavior at P_j of the following nature: Given $j \in \{1, ..., N\}$, choose polar coordinates (r_j, θ_j) taking P_j as the origin and so that the internal angle is spanned by the half-lines $\theta_j = 0$ and $\theta_j = \omega_j$. Then

$$v_j(r_j, \theta_j) = \phi_j(r_j) r_j^{\pi/\omega_j} \sin(\pi\theta_j/\omega_j), \quad 1 \leq j \leq N, \tag{8.39}$$

where ϕ_j is a C^∞-smooth cut-off function of small support, which is identically one near P_j.

In this scenario, $v_j \in H^s(\Omega)$ for every $s < 1 + (\pi/\omega_j)$, though $v_j \notin H^{1+(\pi/\omega_j)}(\Omega)$. This implies that the best regularity statement regarding the solution of (8.37) is

$$u \in H^s(\Omega) \text{ for every } s < 1 + \frac{\pi}{\max\{\omega_1, ..., \omega_N\}} \tag{8.40}$$

and this fails for the critical value of s. In particular, this provides a quantifiable way of measuring the failure of the conclusion in Lemma 8.11 for Lipschitz, piecewise C^∞ domains exhibiting inwardly directed irregularities.

We wish to augment Lemma 8.11 with an analogous result for the Robin Laplacian $-\Delta_{\Theta,\Omega}$ (cf. Theorem 5.9) corresponding to the case where Θ is a nonnegative, constant function on $\partial\Omega$, that is,

$$\Theta(x) \equiv \theta \geq 0, \text{ for } x \in \partial\Omega. \tag{8.41}$$

From Theorem 5.9 we know that assuming Hypothesis 2.1, this is a self-adjoint operator in $L^2(\Omega; d^n x)$, with domain

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x), \big(\widetilde{\gamma}_N + \theta\gamma_D\big)u = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}. \tag{8.42}$$

In the case where Hypothesis 8.10 is assumed, the following regularity result has been recently established in [71] (see also Theorem 3.2.3.1 on p 156 of [59] for the case of convex domains).

Lemma 8.13. *Assume Hypothesis* 8.10 *and* (8.41)*. Then the Robin Laplacian satisfies*

$$\mathrm{dom}(-\Delta_{\Theta,\Omega}) \subset H^2(\Omega). \tag{8.43}$$

We are now ready to address the issue raised at the beginning of Section 8:

Theorem 8.14. *Assume Hypothesis* 8.10*. Then equality holds in* (7.11)*, that is, one has*

$$H_0^2(\Omega) = \big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x),\, \widehat{\gamma}_D u = \widehat{\gamma}_N u = 0\big\}. \tag{8.44}$$

In particular,

$$-\Delta_{min} = (-\Delta_{max})^* \text{ and } -\Delta_{max} = (-\Delta_{min})^*, \tag{8.45}$$

with

$$\mathrm{dom}(-\Delta_{min}) = H_0^2(\Omega). \tag{8.46}$$

Proof. As already pointed out before, the left-to-right inclusion in (8.44) holds under the mere assumption of Lipschitzianity for Ω. To prove the opposite inclusion, consider $u \in L^2(\Omega; d^n x)$ satisfying $\Delta u \in L^2(\Omega; d^n x)$ and $\widehat{\gamma}_D u = \widehat{\gamma}_N u = 0$. Next, let $w \in H_0^1(\Omega)$ solve $\Delta w = \Delta u$ in Ω. Then $w \in \mathrm{dom}(-\Delta_{D,\Omega}) \subset H^2(\Omega)$ by Lemma 8.11 and the current assumptions. Set $v := u - w$ in Ω and notice that

$$v \in L^2(\Omega; d^n x), \quad \Delta v = 0 \text{ in } \Omega, \quad \widehat{\gamma}_D v = 0 \text{ on } \partial\Omega. \tag{8.47}$$

We claim that these conditions imply that $v = 0$. To show that any v as in (8.47) necessarily vanishes, consider an arbitrary $f \in L^2(\Omega; d^n x)$ and let u_f be the unique solution of

$$u_f \in H_0^1(\Omega), \quad \Delta u_f = f \text{ in } \Omega. \tag{8.48}$$

Then $u_f \in \mathrm{dom}(-\Delta_{D,\Omega}) \subset H^2(\Omega)$ by Lemma 8.11, so that $u_f \in H^2(\Omega) \cap H_0^1(\Omega)$. Then the integration by parts formula (6.16) yields

$$\begin{aligned} (f, v)_{L^2(\Omega; d^n x)} &= (\Delta u_f, v)_{L^2(\Omega; d^n x)} \\ &= (u_f, \Delta v)_{L^2(\Omega; d^n x)} + {}_{N^{1/2}(\partial\Omega)}\langle \gamma_N u_f, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} \\ &= 0. \end{aligned} \tag{8.49}$$

Since $f \in L^2(\Omega; d^n x)$ was arbitrary, it follows that $v = 0$, as wanted. In turn, this implies that $u = w \in H^2(\Omega)$. We recall that u also satisfies $\widehat{\gamma}_D u = \widehat{\gamma}_N u = 0$, or $\gamma_D u = \gamma_N u = 0$, by Theorem 6.4 and Theorem 6.10. Invoking Theorem 6.1 we may then finally conclude that $u \in \ker(\gamma_2) = H_0^2(\Omega)$. □

It is easy to see that, for any given open set $\Omega \subseteq \mathbb{R}^n$, and any $z \in \mathbb{C}$, the orthogonal complement of $\{(-\Delta - \overline{z})\varphi \,|\, \varphi \in C_0^\infty(\Omega)\}$ in $L^2(\Omega; d^n x)$ is $\ker(-\Delta_{max} - zI_\Omega)$. This implies that for every $z \in \mathbb{C}$ one has

$$L^2(\Omega; d^n x) = \ker(-\Delta_{max} - zI_\Omega) \oplus \overline{\{(-\Delta - \overline{z})\varphi \,|\, \varphi \in C_0^\infty(\Omega)\}}, \tag{8.50}$$

where the closure is taken in $L^2(\Omega; d^n x)$. Below, we provide a more precise version of this result under stronger assumptions on Ω and z.

Theorem 8.15. *Assume Hypothesis* 8.10 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. *Then* $\mathrm{ran}(-\Delta_{min} - \overline{z}I_\Omega)$ *is closed and*

$$L^2(\Omega; d^n x) = \ker(-\Delta_{max} - zI_\Omega) \oplus \big\{(-\Delta - \overline{z})u \,\big|\, u \in H_0^2(\Omega)\big\}. \tag{8.51}$$

Proof. By the previous discussion it suffices to check that

$$\overline{\{(-\Delta - \overline{z})\varphi \,|\, \varphi \in C_0^\infty(\Omega)\}} = \big\{(-\Delta - \overline{z})u \,\big|\, u \in H_0^2(\Omega)\big\}. \tag{8.52}$$

Pick an arbitrary $u \in H_0^2(\Omega)$ and select a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ such that $\varphi_j \in C_0^\infty(\Omega)$ for every $j \in \mathbb{N}$ and $\varphi_j \to u$ in $H^2(\Omega)$ as $j \to \infty$. Then $(-\Delta - \overline{z})\varphi_j \to (-\Delta - \overline{z})u$ as $j \to \infty$, proving the right-to-left inclusion in (8.52).

In the opposite direction, assume that $w \in L^2(\Omega; d^n x)$ has the property that there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ such that $\varphi_j \in C_0^\infty(\Omega)$ for every $j \in \mathbb{N}$ and $(-\Delta - \overline{z})\varphi_j \to w$ in $L^2(\Omega; d^n x)$ as $j \to \infty$. Then

$$\varphi_j = (-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}[(-\Delta - \overline{z})\varphi_j] \to (-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}w \ \text{ in } \ L^2(\Omega; d^n x) \tag{8.53}$$

as $j \to \infty$. Thus, if we set $u := (-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}w \in \mathrm{dom}(-\Delta_{D\Omega})$, then

$$\varphi_j \to u \ \text{ and } \ (-\Delta - \overline{z})\varphi_j \to (-\Delta - \overline{z})u \ \text{ in } \ L^2(\Omega; d^n x) \ \text{ as } \ j \to \infty. \tag{8.54}$$

Next, from (8.54) and the continuity of the map in (6.44) we may deduce that

$$0 = \widehat{\gamma}_N\varphi_j \to \widehat{\gamma}_N u \ \text{ in } \ \big(N^{3/2}(\partial\Omega)\big)^* \ \text{ as } \ j \to \infty. \tag{8.55}$$

This shows that $\widehat{\gamma}_N u = 0$, hence $u \in H_0^2(\Omega)$ by Theorem 8.14. Consequently, $w = (-\Delta - \overline{z})u$ for some $u \in H_0^2(\Omega)$, proving the left-to-right inclusion in (8.52). $\square$

Remark 8.16. An alternative proof of Theorem 8.15 is to observe that, for every $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the solvability of the Poisson problem with a homogeneous Dirichlet boundary condition for $-\Delta - z$ implies that $\mathrm{ran}(-\Delta_{max} - zI_\Omega)$ equals $L^2(\Omega; d^n x)$ and hence is closed. Then Theorem 8.14 along with Banach's Closed Range Mapping Theorem (cf. [73, Theorem IV.2.5.13]) yield that the range of $-\Delta_{min} - \overline{z}I_\Omega = (-\Delta_{max} - zI_\Omega)^*$ is also closed.

Remark 8.17. As a corollary of Theorem 8.15, for every $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the operator $-\Delta_{min} - \overline{z}I_\Omega$ maps its domain $H_0^2(\Omega)$ isomorphically onto its range.

Lemma 8.18. *Assume Hypothesis* 8.10 *and that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. *Then*

$$H^2(\Omega) = \big[H^2(\Omega) \cap H_0^1(\Omega)\big] \dotplus \big[H^2(\Omega) \cap \ker(-\Delta_{max} - zI_\Omega)\big]. \tag{8.56}$$

Proof. Indeed, the membership of z to $\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$ ensures that the sum in (8.56) is direct. In addition, if $u \in H^2(\Omega)$ is arbitrary and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies $(-\Delta - z)v = (-\Delta - z)u$ in Ω, then $u = v + (u - v)$ and $u - v \in H^2(\Omega) \cap \ker(-\Delta_{max} - zI_\Omega)$. From this, the desired conclusion follows easily. $\square$

We recall that, given a Hilbert space $\mathcal{H}$ with inner product $(\,\cdot\,,\,\cdot\,)_\mathcal{H}$, a linear operator $S : \mathrm{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called *nonnegative* provided $(Su, u)_\mathcal{H} \geq 0$ for every $u \in \mathrm{dom}(S)$. A simple argument based on polarization formula then yields (the well-known fact) that any nonnegative operator S is *symmetric*, that is, $(Su, v)_\mathcal{H} = (u, Sv)_\mathcal{H}$ for every $u, v \in \mathrm{dom}(S)$.

Corollary 8.19. *Assume Hypothesis* 8.10. *Then* $-\Delta_{min}$ *is a densely defined, closed, nonnegative* (*and hence, symmetric*) *operator. Furthermore,*

$$\overline{-\Delta\big|_{C_0^\infty(\Omega)}} = -\Delta_{min}. \tag{8.57}$$

Proof. The first claim in the statement is a direct consequence of Theorem 8.14. As for (8.57), let us temporarily denote $-\Delta_0 = -\Delta\big|_{C_0^\infty(\Omega)}$. Then

$u \in \mathrm{dom}(-\Delta_0)$

$$\text{if and only if } \begin{cases} \text{there exist } v \in L^2(\Omega; d^n x) \text{ and } u_j \in C_0^\infty(\Omega),\ j \in \mathbb{N}, \text{ such that} \\ u_j \to u \text{ and } \Delta u_j \to v \text{ in } L^2(\Omega; d^n x) \text{ as } j \to \infty. \end{cases} \tag{8.58}$$

Thus, if $u \in \operatorname{dom}(-\Delta_0)$ and v, $\{u_j\}_{j\in\mathbb{N}}$ are as in the right-hand side of (8.58), then $\Delta u = v$ in the sense of distributions in Ω, and

$$\begin{aligned} 0 = \widehat{\gamma}_D u_j \to \widehat{\gamma}_D u \text{ in } \big(N^{1/2}(\partial\Omega)\big)^* \text{ as } j \to \infty, \\ 0 = \widehat{\gamma}_N u_j \to \widehat{\gamma}_N u \text{ in } \big(N^{1/2}(\partial\Omega)\big)^* \text{ as } j \to \infty, \end{aligned} \tag{8.59}$$

by Theorems 6.4 and 6.10. Consequently, $u \in \operatorname{dom}(-\Delta_{max})$ satisfies $\widehat{\gamma}_D u = 0$ and $\widehat{\gamma}_N u = 0$. Hence, $u \in H_0^2(\Omega) = \operatorname{dom}(-\Delta_{min})$ by Theorem 8.14 and the current assumptions on Ω. This shows that $-\Delta_0 \subseteq -\Delta_{min}$. The converse inclusion readily follows from the fact that any $u \in H_0^2(\Omega)$ is the limit in $H^2(\Omega)$ of a sequence of test functions in Ω. □

Lemma 8.20. *Assume Hypothesis* 8.10. *Then the domain of $-\Delta_{max}$ has the description*

$$\operatorname{dom}(-\Delta_{max}) = \big[H^2(\Omega) \cap H_0^1(\Omega)\big] \dotplus \ker(-\Delta_{max}). \tag{8.60}$$

Proof. The right-to-left inclusion in (8.60) is obvious. As for the opposite one, if $u \in \operatorname{dom}(-\Delta_{max})$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ solves the inhomogeneous Dirichlet problem $\Delta v = \Delta u$ in Ω, then $u = v + (u - v)$ with $(u - v) \in \ker(-\Delta_{max})$. Finally, the fact that the sum in (8.60) is direct is a consequence of the fact that $0 \notin \sigma(-\Delta_{D,\Omega})$. □

9. The Friedrichs and Krein–von Neumann Extensions: Some Abstract Results

This short section is devoted to a brief summary of abstract results on the Friedrichs and Krein–von Neumann extensions of a closed symmetric operator S in some separable, complex Hilbert space $\mathcal{H}$.

A linear operator $S : \operatorname{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$, is called *symmetric*, if $(u, Sv)_{\mathcal{H}} = (Su, v)_{\mathcal{H}}$, $u, v \in \operatorname{dom}(S)$. If $\operatorname{dom}(S) = \mathcal{H}$, the classical Hellinger–Toeplitz theorem guarantees that $S \in \mathcal{B}(\mathcal{H})$, in which situation S is readily seen to be self-adjoint. In general, however, symmetry is a considerably weaker property than self-adjointness and a classical problem in functional analysis is that of determining all self-adjoint extensions of a given closed symmetric operator of equal and nonzero deficiency indices. In this manuscript we will be interested in this question within the class of densely defined (i.e., $\overline{\operatorname{dom}(S)} = \mathcal{H}$), strictly positive, closed operators S. In this section we focus exclusively on self-adjoint extensions of S that are nonnegative operators. In the latter scenario, there are two distinguished (and extremal) constructions which we will briefly review next.

To set the stage, we recall that a linear operator $S : \operatorname{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called *nonnegative* provided $(u, Su)_{\mathcal{H}} \geq 0$, $u \in \operatorname{dom}(S)$. (In particular, S is symmetric in this case.) S is called *strictly positive*, if for some $\varepsilon > 0$, $(u, Su)_{\mathcal{H}} \geq \varepsilon \|u\|_{\mathcal{H}}$, $u \in \operatorname{dom}(S)$. Next, we recall that $A \leq B$ for two self-adjoint operators in $\mathcal{H}$ if

$$\begin{aligned} &\operatorname{dom}\big(|A|^{1/2}\big) \supseteq \operatorname{dom}\big(|B|^{1/2}\big) \text{ and} \\ &\big(|A|^{1/2}u, U_A|A|^{1/2}u\big)_{\mathcal{H}} \leq \big(|B|^{1/2}u, U_B|B|^{1/2}u\big)_{\mathcal{H}}, \quad u \in \operatorname{dom}\big(|B|^{1/2}\big), \end{aligned} \tag{9.1}$$

where U_C denotes the partial isometry in $\mathcal{H}$ in the polar decomposition of a densely defined closed operator C in $\mathcal{H}$, $C = U_C|C|$, $|C| = (C^*C)^{1/2}$. (If in addition, C is self-adjoint, then U_C and $|C|$ commute.) We also recall ([45, Part II], [73, Theorem VI.2.21]) that if A and B are both self-adjoint and nonnegative in $\mathcal{H}$, then

$$\begin{aligned} &0 \leq A \leq B \text{ if and only if } 0 \leq A^{1/2} \leq B^{1/2}, \\ &\quad \text{equivalently, if and only if } (B + aI_{\mathcal{H}})^{-1} \leq (A + aI_{\mathcal{H}})^{-1} \text{ for all } a > 0, \end{aligned} \tag{9.2}$$

and

$$\ker(A) = \ker\big(A^{1/2}\big) \tag{9.3}$$

(with $C^{1/2}$ the unique nonnegative square root of a nonnegative self-adjoint operator C in $\mathcal{H}$).

For simplicity we will always adhere in this section to the conventions that S is a linear, unbounded, densely defined, nonnegative (i.e., $S \geq 0$) operator in $\mathcal{H}$, and that S has nonzero deficiency indices. In particular

$$\operatorname{def}(S) = \dim(\ker(S^* - zI_{\mathcal{H}})) \in \mathbb{N} \cup \{\infty\}, \quad z \in \mathbb{C}\backslash[0, \infty), \tag{9.4}$$

is well-known to be independent of z. Moreover, since S and its closure $\overline{S}$ have the same self-adjoint extensions in $\mathcal{H}$, we will without loss of generality assume that S is closed in the remainder of this section.

The following is a fundamental result to be found in M. Krein's celebrated 1947 paper [75] (cf. also Theorems 2 and 5–7 in the English summary on page 492):

Theorem 9.1. *Assume that S is a densely defined, closed, nonnegative operator in $\mathcal{H}$. Then, among all nonnegative self-adjoint extensions of S, there exist two distinguished ones, S_K and S_F, which are, respectively, the smallest and largest (in the sense of order between linear operators, cf.* (9.1)*) such extension. Furthermore, a nonnegative self-adjoint operator $\widetilde{S}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies*

$$S_K \leq \widetilde{S} \leq S_F. \tag{9.5}$$

In particular, (9.5) *determines S_K and S_F uniquely.*

In addition, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, one has

$$\operatorname{dom}(S_F) = \operatorname{dom}(S) \dotplus (S_F)^{-1}\ker(S^*), \tag{9.6}$$

$$\operatorname{dom}(S_K) = \operatorname{dom}(S) \dotplus \ker(S^*), \tag{9.7}$$

$$\begin{aligned} \operatorname{dom}(S^*) &= \operatorname{dom}(S) \dotplus (S_F)^{-1}\ker(S^*) \dotplus \ker(S^*) \\ &= \operatorname{dom}(S_F) \dotplus \ker(S^*), \end{aligned} \tag{9.8}$$

In particular,

$$\ker(S_K) = \ker\big((S_K)^{1/2}\big) = \ker(S^*) = \operatorname{ran}(S)^{\perp}. \tag{9.9}$$

Here the operator inequalities in (9.5) are understood in the sense of (9.1) and hence they can equivalently be written as

$$(S_F + aI_{\mathcal{H}})^{-1} \leqslant \big(\widetilde{S} + aI_{\mathcal{H}}\big)^{-1} \leqslant (S_K + aI_{\mathcal{H}})^{-1} \text{ for some (and hence for all) } a > 0. \tag{9.10}$$

We also refer to Birman [26], [27], Friedrichs [48], Freudenthal [47], Grubb [60], [61], Krein [76], Štraus [120], and Višik [126] (see also the monographs by Akhiezer and Glazman [5, Sect. 109], Faris [45, Part III], and the recent book by Grubb [65, Sect. 13.2]) for classical references on the subject of self-adjoint extensions of semibounded operators.

We will call the operator S_K the *Krein–von Neumann extension* of S. See [75] and also the discussion in [8], [15], [16]. It should be noted that the Krein–von Neumann extension was first considered by von Neumann [127] in 1929 in the case where S is strictly positive, that is, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$. (His construction appears in the proof of Theorem 42 on pages 102–103.) However, von Neumann did not isolate the extremal property of this extension as described in (9.5) and (9.10). M. Krein [75], [76] was the first to systematically treat the general case $S \geq 0$ and to study all nonnegative self-adjoint extensions of S, illustrating the special role of the *Friedrichs extension* (i.e., the "hard" extension) S_F of S and the Krein–von Neumann (i.e., the "soft") extension S_K of S as extremal cases when considering all nonnegative extensions of S. For a recent exhaustive treatment of self-adjoint extensions of semibounded operators we refer to [14]–[17].

An intrinsic description of the Friedrichs extension S_F of $S \geq 0$ due to Freudenthal [47] in 1936 describes S_F as the operator $S_F : \operatorname{dom}(S_F) \subset \mathcal{H} \to \mathcal{H}$ given by

$$\begin{aligned} &S_F u := S^* u, \\ &u \in \operatorname{dom}(S_F) := \big\{v \in \operatorname{dom}(S^*) \,\big|\, \text{there exists} \{v_j\}_{j\in\mathbb{N}} \subset \operatorname{dom}(S), \\ &\quad \text{with } \lim_{j\to\infty} \|v_j - v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k))_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}. \end{aligned} \tag{9.11}$$

Then, as is well-known,

$$S_F \geq 0, \tag{9.12}$$

$$\begin{aligned} &\operatorname{dom}\big((S_F)^{1/2}\big) = \big\{v \in \mathcal{H} \,\big|\, \text{there exists} \{v_j\}_{j\in\mathbb{N}} \subset \operatorname{dom}(S), \\ &\quad \text{with } \lim_{j\to\infty} \|v_j - v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k))_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}, \end{aligned} \tag{9.13}$$

and

$$S_F = S^*|_{\operatorname{dom}(S^*)\cap\operatorname{dom}((S_F)^{1/2})}. \tag{9.14}$$

An intrinsic description of the Krein–von Neumann extension S_K of $S \geq 0$ has been given by Ando and Nishio [10] in 1970, where S_K has been characterized as the operator $S_K : \operatorname{dom}(S_K) \subset \mathcal{H} \to \mathcal{H}$ given by

$$\begin{aligned} &S_K u := S^* u, \\ &u \in \operatorname{dom}(S_K) := \big\{v \in \operatorname{dom}(S^*) \,\big|\, \text{there exists} \{v_j\}_{j\in\mathbb{N}} \subset \operatorname{dom}(S), \\ &\quad \text{with } \lim_{j\to\infty} \|Sv_j - S^*v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k)_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}. \end{aligned} \tag{9.15}$$

For a variety of recent new results on S_K in connection with Weyl-type spectral asymptotics for perturbed Laplacians on nonsmooth bounded open domains in $\mathbb{R}^n$, and for a connection to an abstract buckling problem that illustrates the relevance of S_K in elasticity theory, we refer to [18] and [19].

We continue by recording an abstract result regarding the parametrization of all nonnegative self-adjoint extensions of a given strictly positive, densely defined, symmetric operator. The following results were developed from Krein [75], Višik [126], and Birman [26], by Grubb [60], [61]. Subsequent expositions are due to Faris [45, Sect. 15] and Alonso and Simon [8].

Theorem 9.2. *Suppose that S is a densely defined, closed operator in $\mathcal{H}$, and $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$. Then there exists a one-to-one correspondence between nonnegative self-adjoint operators $0 \leq B : \operatorname{dom}(B) \subseteq \mathcal{W} \to \mathcal{W}$, $\overline{\operatorname{dom}(B)} = \mathcal{W}$, where $\mathcal{W}$ is a closed subspace of $\mathcal{N}_0 := \ker(S^*)$, and nonnegative self-adjoint extensions $S_{B,\mathcal{W}} \geq 0$ of S. More specifically, S_F is invertible, $S_F \geq \varepsilon I_{\mathcal{H}}$, and one has*

$$\begin{aligned}
&\operatorname{dom}(S_{B,\mathcal{W}}) \\
&\quad = \big\{ f + (S_F)^{-1}(Bw + \eta) + w \,\big|\, f \in \operatorname{dom}(S),\ w \in \operatorname{dom}(B),\ \eta \in \mathcal{N}_0 \cap \mathcal{W}^{\perp} \big\}, \\
&S_{B,\mathcal{W}} = S^*|_{\operatorname{dom}(S_{B,\mathcal{W}})},
\end{aligned} \tag{9.16}$$

where $\mathcal{W}^{\perp}$ denotes the orthogonal complement of $\mathcal{W}$ in $\mathcal{N}_0$. In addition,

$$\operatorname{dom}\big((S_{B,\mathcal{W}})^{1/2}\big) = \operatorname{dom}\big((S_F)^{1/2}\big) \dotplus \operatorname{dom}\big(B^{1/2}\big), \tag{9.17}$$

$$\begin{gathered}
\big\|(S_{B,\mathcal{W}})^{1/2}(u+g)\big\|_{\mathcal{H}}^2 = \big\|(S_F)^{1/2}u\big\|_{\mathcal{H}}^2 + \big\|B^{1/2}g\big\|_{\mathcal{H}}^2, \\
u \in \operatorname{dom}\big((S_F)^{1/2}\big),\ g \in \operatorname{dom}\big(B^{1/2}\big),
\end{gathered} \tag{9.18}$$

implying,

$$\ker(S_{B,\mathcal{W}}) = \ker(B). \tag{9.19}$$

Moreover,

$$B \leq \widetilde{B} \ \text{ implies } \ S_{B,\mathcal{W}} \leq S_{\widetilde{B},\widetilde{\mathcal{W}}}, \tag{9.20}$$

where

$$\begin{aligned}
&B\colon \operatorname{dom}(B) \subseteq \mathcal{W} \to \mathcal{W}, \quad \widetilde{B}\colon \operatorname{dom}\big(\widetilde{B}\big) \subseteq \widetilde{\mathcal{W}} \to \widetilde{\mathcal{W}}, \\
&\overline{\operatorname{dom}\big(\widetilde{B}\big)} = \widetilde{\mathcal{W}} \subseteq \mathcal{W} = \overline{\operatorname{dom}(B)}.
\end{aligned} \tag{9.21}$$

In the above scheme, the Krein–von Neumann extension S_K of S corresponds to the choice $\mathcal{W} = \mathcal{N}_0$ and $B = 0$ (with $\operatorname{dom}(B) = \operatorname{dom}(B^{1/2}) = \mathcal{N}_0 = \ker(S^*)$). In particular, one thus recovers (9.7)*, and* (9.9)*, and also obtains*

$$\operatorname{dom}\big((S_K)^{1/2}\big) = \operatorname{dom}\big((S_F)^{1/2}\big) \dotplus \ker(S^*), \tag{9.22}$$

$$\big\|(S_K)^{1/2}(u+g)\big\|_{\mathcal{H}}^2 = \big\|(S_F)^{1/2}u\big\|_{\mathcal{H}}^2, \quad u \in \operatorname{dom}\big((S_F)^{1/2}\big),\ g \in \ker(S^*). \tag{9.23}$$

Finally, the Friedrichs extension S_F corresponds to the choice $\operatorname{dom}(B) = \{0\}$ (i.e., formally, $B \equiv \infty$), in which case one recovers (9.6)*.*

The relation $B \leq \widetilde{B}$ in the case where $\widetilde{\mathcal{W}} \subsetneqq \mathcal{W}$ requires an explanation: In analogy to (9.1) we mean

$$\big(|B|^{1/2}u, U_B|B|^{1/2}u\big)_{\mathcal{W}} \leq \big(|\widetilde{B}|^{1/2}u, U_{\widetilde{B}}|\widetilde{B}|^{1/2}u\big)_{\mathcal{W}}, \quad u \in \operatorname{dom}\big(|\widetilde{B}|^{1/2}\big) \tag{9.24}$$

and (following [8]) we put

$$\big(|\widetilde{B}|^{1/2}u, U_{\widetilde{B}}|\widetilde{B}|^{1/2}u\big)_{\mathcal{W}} = \infty \ \text{ for } \ u \in \mathcal{W}\backslash\operatorname{dom}\big(|\widetilde{B}|^{1/2}\big). \tag{9.25}$$

We also note that under the assumptions on S in Theorem 9.2, one has

$$\dim(\ker(S^* - zI_{\mathcal{H}})) = \dim(\ker(S^*)) = \dim(\mathcal{N}_0) = \operatorname{def}(S), \quad z \in \mathbb{C}\backslash[\varepsilon,\infty). \tag{9.26}$$

For now, our goal is to prove the following result:

Lemma 9.3. *Assume Hypothesis* 2.1*. Then the Friedrichs extension of the operator $-\Delta\big|_{C_0^\infty(\Omega)}$, and hence that of $\overline{-\Delta\big|_{C_0^\infty(\Omega)}}$ in $L^2(\Omega; d^n x)$, is precisely the Dirichlet Laplacian $-\Delta_{D,\Omega}$. As a consequence, if Hypothesis* 8.10 *is assumed, the Friedrichs extension of $-\Delta_{min}$ in* (7.9) *is the Dirichlet Laplacian $-\Delta_{D,\Omega}$.*

Proof. We recall that $\mathrm{dom}(-\Delta_{D,\Omega}) = \big\{u \in H_0^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}$. Unravelling definitions, it can then easily be checked that, with $\dot{S}$ denoting $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$, we have $\mathrm{dom}(-\Delta_{D,\Omega}) \subseteq \mathrm{dom}\big((\dot{S})^*\big) = \mathrm{dom}\Big(\Big(\overline{\dot{S}}\Big)^*\Big)$. Furthermore, if $u \in \mathrm{dom}(-\Delta_{D,\Omega}) \hookrightarrow H_0^1(\Omega)$, then there exist $u_j \in C_0^\infty(\Omega)$, $j \in \mathbb{N}$, such that $u_j \to u$ in $L^2(\Omega; d^n x)$ as $j \to \infty$ and $\langle \dot{S}(u_j - u_k), u_j - u_k\rangle_{L^2(\Omega;d^n x)} = \|\nabla u_j - \nabla u_k\|_{L^2(\Omega;d^n x)} \to 0$ as $j, k \to \infty$. Thus, if $\Big(\overline{\dot{S}}\Big)_F : \mathrm{dom}\Big(\Big(\overline{\dot{S}}\Big)_F\Big) \subseteq L^2(\Omega; d^n x) \to L^2(\Omega; d^n x)$ is defined according to (9.11), it follows that $-\Delta_{D,\Omega} \subseteq \Big(\overline{\dot{S}}\Big)_F$. Hence, $-\Delta_{D,\Omega} = \Big(\overline{\dot{S}}\Big)_F$ by the maximality of the self-adjoint operator $-\Delta_{D,\Omega}$. □

Concluding this section, we point out that a great variety of additional results for the Krein–von Neumann extension can be found, for instance, in [3], [5, Sect. 109], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [30], [45, Part III], [49, Sect. 3.3], [54], [62], [65, Sect. 13.2], [66], [67], [68], [69], [77], [78], [84], [98], [107], [113], [114], [115], [116], [122], [123], [124], and the references therein. We also mention the references [42], [43], [44] (these authors, apparently unaware of the work of von Neumann, Krein, Višhik, Birman, Grubb, Štrauss, etc., in this context, introduced the Krein Laplacian and called it the harmonic operator; see also [63]).

10. Trace Operators and Boundary Value Problems on Quasi-Convex Domains

In this section we revisit the trace theory discussed in Section 6, this time assuming that the underlying domain is either smooth or quasi-convex. Such a context allows for more refined results, such as the ontoness of the trace operators in question. In the case of quasi-convex domains, these results are establish as a corollary of the well-posedness of certain basic boundary value problems for the Laplacian.

Lemma 10.1. *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded $C^{1,r}$ domain for some $r \in (1/2, 1)$. Then the Dirichlet trace operator*

$$\gamma_D : H^2(\Omega) \to H^{3/2}(\partial\Omega), \tag{10.1}$$

along with the Neumann trace operator

$$\gamma_N : H^2(\Omega) \to H^{1/2}(\partial\Omega), \tag{10.2}$$

are well-defined, bounded, and onto. In fact, each operator has a linear, bounded right-inverse.

Proof. That the operator (10.1) is well-defined and bounded follows from the observation that for each $u \in H^2(\Omega)$ one has

$$\nabla_{tan}(\gamma_D u) = \left(\sum_{k=1}^n \nu_k \frac{\partial \gamma_D u}{\partial \tau_{kj}}\right)_{1\leq j\leq n} = \left(\gamma_D(\partial_j u) - \sum_{k=1}^n \nu_k \nu_j \gamma_D(\partial_k u)\right)_{1\leq j\leq n}, \tag{10.3}$$

and from Lemmas 2.4 and 3.4. That the operator (10.1) has a linear, bounded right-inverse is a consequence of Lemmas 6.8 and 6.9.

With the goal of proving that the operator introduced in (10.2) is well-defined and bounded, one first observes that for each $u \in H^2(\Omega)$ one has

$$\gamma_N u = \nu \cdot \gamma_D(\nabla u). \tag{10.4}$$

Then the desired conclusion follows from (3.2) and Lemma 3.4. That the operator (10.2) also has a linear, bounded right-inverse is a consequence of (6.7) and Lemma 6.3. □

Theorem 10.2. *Assume Hypothesis* 8.10. *Then the Dirichlet trace operator* (6.14) *is onto. More specifically, for every $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$ there exists $C = C(\Omega, z) > 0$ with the property that for any functional $\Lambda \in \big(N^{1/2}(\partial\Omega)\big)^*$ there exists $u_\Lambda \in L^2(\Omega; d^n x)$ with $(-\Delta - z)u_\Lambda = 0$ in Ω, such that*

$$\text{the assignment } \big(N^{1/2}(\partial\Omega)\big)^* \ni \Lambda \mapsto u_\Lambda \in L^2(\Omega; d^n x) \text{ is linear}, \tag{10.5}$$

and

$$\widehat{\gamma}_D u_\Lambda = \Lambda \text{ and } \|u_\Lambda\|_{L^2(\Omega;d^n x)} \leq C\|\Lambda\|_{(N^{1/2}(\partial\Omega))^*}. \tag{10.6}$$

Proof. Fix $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. By Lemma 8.11, the inverse $(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^n x) \to H^2(\Omega) \cap H_0^1(\Omega)$ is well-defined, linear, and bounded. Given $\Lambda \in \big(N^{1/2}(\partial\Omega)\big)^*$, consider the bounded linear functional

$$\Lambda\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \mathbb{C}. \tag{10.7}$$

Then, by the Riesz representation theorem, there exists a unique $u_\Lambda \in L^2(\Omega; d^n x)$ with the property that

$$\Lambda\big(\gamma_N((-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}f)\big) = (f, u_\Lambda)_{L^2(\Omega;d^n x)}, \tag{10.8}$$

and such that

$$\begin{aligned} \|u_\Lambda\|_{L^2(\Omega;d^n x)} &\leq C\|\Lambda\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\|_{\mathcal{B}(L^2(\Omega;d^n x),\mathbb{C})} \\ &\leq C\|\Lambda\|_{(N^{1/2}(\partial\Omega))^*} \end{aligned} \tag{10.9}$$

for some constant $C = C(\Omega, z) > 0$, independent of Λ. Furthermore, by the uniqueness of u_Λ, the assignment $\big(N^{1/2}(\partial\Omega)\big)^* \ni \Lambda \mapsto u_\Lambda \in L^2(\Omega; d^n x)$ is linear.

Specializing (10.8) to the case where $f := (-\Delta - \overline{z})w$, with $w \in H^2(\Omega) \cap H^1_0(\Omega)$ then yields

$$\Lambda\big(\gamma_N w\big) = ((-\Delta - \overline{z})w, u_\Lambda)_{L^2(\Omega;d^n x)}, \quad w \in H^2(\Omega) \cap H^1_0(\Omega). \tag{10.10}$$

In particular, choosing $w \in C_0^\infty(\Omega)$ yields

$$0 = \Lambda(0) = \Lambda\big(\gamma_N w\big) = ((-\Delta - \overline{z})w, u_\Lambda)_{L^2(\Omega;d^n x)}, \quad w \in C_0^\infty(\Omega). \tag{10.11}$$

Thus, one concludes that $(-\Delta - z)u_\Lambda = 0$ in Ω, in the sense of distributions. Using this in (10.8) one then obtains

$$\begin{aligned} \Lambda\big(\gamma_N w\big) &= ((-\Delta - \overline{z})w, u_\Lambda)_{L^2(\Omega;d^n x)} - (w, (-\Delta - z)u_\Lambda)_{L^2(\Omega;d^n x)} \\ &= -{}_{N^{1/2}(\partial\Omega)}\langle \gamma_N w, \widehat{\gamma}_D u_\Lambda\rangle_{(N^{1/2}(\partial\Omega))^*}, \quad w \in H^2(\Omega) \cap H^1_0(\Omega). \end{aligned} \tag{10.12}$$

Since γ_N maps $H^2(\Omega) \cap H^1_0(\Omega)$ onto $N^{1/2}(\partial\Omega)$, (10.12) implies that $\widehat{\gamma}_D u_\Lambda = -\Lambda$. This shows that the operator (6.14) has a linear, bounded right-inverse. In particular, it is onto. $\square$

Corollary 10.3. *Assume Hypothesis* 8.10. *Then for every $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the map*

$$\widehat{\gamma}_D : \big\{u \in L^2(\Omega; d^n x) \,\big|\, (-\Delta - z)u = 0 \text{ in } \Omega\big\} \to \big(N^{1/2}(\partial\Omega)\big)^* \tag{10.13}$$

is onto.

Proof. This is a direct consequence of Theorem 10.2. $\square$

Theorem 10.4. *Assume Hypothesis* 8.10 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Then for any $f \in L^2(\Omega; d^n x)$ and $g \in (N^{1/2}(\partial\Omega))^*$ the following inhomogeneous Dirichlet boundary value problem*

$$\begin{cases} (-\Delta - z)u = f \text{ in } \Omega, \\ u \in L^2(\Omega; d^n x), \\ \widehat{\gamma}_D u = g \text{ on } \partial\Omega, \end{cases} \tag{10.14}$$

has a unique solution $u = u_D$. This solution satisfies

$$\|u_D\|_{L^2(\Omega;d^n x)} + \|\widehat{\gamma}_N u_D\|_{(N^{3/2}(\partial\Omega))^*} \leq C_D(\|f\|_{L^2(\Omega;d^n x)} + \|g\|_{(N^{1/2}(\partial\Omega))^*}) \tag{10.15}$$

for some constant $C_D = C_D(\Omega, z) > 0$, and the following regularity results hold:

$$g \in H^1(\partial\Omega) \text{ implies } u_D \in H^{3/2}(\Omega), \tag{10.16}$$

$$g \in \gamma_D\big(H^2(\Omega)\big) \text{ implies } u_D \in H^2(\Omega). \tag{10.17}$$

In particular,

$$g = 0 \text{ implies } u_D \in H^2(\Omega) \cap H^1_0(\Omega). \tag{10.18}$$

Natural estimates are valid in each case.

Moreover,

$$\big[\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, L^2(\Omega; d^n x)\big), \tag{10.19}$$

and the solution of (10.14) *is given by the formula*

$$u_D = (-\Delta_{D,\Omega} - zI_\Omega)^{-1}f - \big[\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g. \tag{10.20}$$

Proof. By Theorem 10.2 there exists a function $v \in L^2(\Omega; d^n x)$ satisfying $\Delta v = 0$ in Ω, $\widehat{\gamma}_D v = g$ and $\|v\|_{L^2(\Omega;d^n x)} \leq C\|g\|_{(N^{1/2}(\partial\Omega))^*}$ for some finite $C = C(\Omega) > 0$. Setting

$$w := (-\Delta_{D,\Omega} - zI_\Omega)^{-1}(f + zv) \tag{10.21}$$

so that

$$w \in \mathrm{dom}(-\Delta_{D,\Omega}) \hookrightarrow H^2(\Omega), \tag{10.22}$$

by our hypotheses and by Lemma 8.11, and

$$\|w\|_{H^2(\Omega)} \leq C\|f + zv\|_{L^2(\Omega,d^n x)} \leq C\big(\|f\|_{L^2(\Omega;d^n x)} + \|g\|_{(N^{1/2}(\partial\Omega))^*}\big), \tag{10.23}$$

for some finite constant $C = C(\Omega, z) > 0$. Then $u := v + w$ solves (10.14) and satisfies (10.15).

To prove uniqueness, we assume that u satisfies the homogeneous version of (10.14), and consider $w := -z(-\Delta_{D,\Omega} - zI_\Omega)^{-1}u \in H_0^1(\Omega)$. Then $v := u - w$ satisfies

$$v \in L^2(\Omega; d^n x), \quad (-\Delta - z)v = 0 \text{ in } \Omega, \quad \widehat{\gamma}_D v = 0 \text{ on } \partial\Omega. \tag{10.24}$$

We claim that these conditions imply that $v = 0$. To show this, consider an arbitrary $f \in L^2(\Omega; d^n x)$ and let u_f be the unique solution of

$$u_f \in H_0^1(\Omega), \quad (-\Delta - \overline{z})u_f = f \text{ in } \Omega. \tag{10.25}$$

Then $u_f \in \mathrm{dom}(-\Delta_{D,\Omega}) \subset H^2(\Omega)$ by Lemma 8.11, so that $u_f \in H^2(\Omega) \cap H_0^1(\Omega)$. Then the integration by parts formula (6.16) yields

$$\begin{aligned}(f, v)_{L^2(\Omega;d^n x)} &= ((-\Delta - \overline{z})u_f, v)_{L^2(\Omega;d^n x)} \\ &= (u_f, (-\Delta - z)v)_{L^2(\Omega;d^n x)} - {}_{N^{1/2}(\partial\Omega)}\langle \gamma_N u_f, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} \\ &= 0.\end{aligned} \tag{10.26}$$

Since $f \in L^2(\Omega; d^n x)$ was arbitrary it follows that $v = 0$ and consequently, $u \in H_0^1(\Omega)$. Given that u solves the homogeneous version of (10.14) and the fact that $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, we may therefore conclude that $u = 0$ in Ω. The regularity result (10.16) is a consequence of Theorem 5.3 and the uniqueness for (10.14), while (10.17) follows from Lemma 8.11 and the uniqueness for (10.14). Finally, (10.18) is a direct consequence of (10.17).

Next, consider (10.19). On one hand, by Lemma 8.11 and the current assumptions, one concludes that

$$(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^n x) \to H^2(\Omega) \cap H_0^1(\Omega) \tag{10.27}$$

is a well-defined linear, bounded operator. On the other hand, the operator γ_N in (6.11) is also well-defined, linear, and bounded. Consequently,

$$\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), N^{1/2}(\partial\Omega)\big), \tag{10.28}$$

and (10.19) follows by dualizing in (10.28). We note that, with $\widetilde{\gamma}_N$ as in (3.11), this also implies that

$$\big[\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* = \big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, L^2(\Omega; d^n x)\big). \tag{10.29}$$

This will play a role shortly.

Next, let u_g be the (unique) solution of (10.14) with $f = 0$ and $g \in \big(N^{1/2}(\partial\Omega)\big)^*$ arbitrary. In addition, set $u := \big[\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g \in L^2(\Omega; d^n x)$. Our goal is to show that $u = u_g$. Since $H^1(\partial\Omega)$ is densely embedded into $\big(N^{1/2}(\partial\Omega)\big)^*$ by (6.32), it follows that there exists a sequence $\{g_j\}_{j\in\mathbb{N}}$ such that $g_j \in H^1(\partial\Omega)$, $j \in \mathbb{N}$, and $g_j \to g$ in $\big(N^{1/2}(\partial\Omega)\big)^*$ as $j \to \infty$. For each fixed j, we know from Theorem 5.3 that $u_{g_j} = -\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g_j$. Due to (10.29), this implies that $u_{g_j} \to u$ in $L^2(\Omega; d^n x)$. Hence, $\Delta u_{g_j} = -zu_{g_j} \to -zu$ in $L^2(\Omega; d^n x)$. With these at hand, it is then clear that $(-\Delta - z)u = 0$ in the sense of distributions in Ω. Next, given that

$$u_{g_j} \to u \text{ in } L^2(\Omega; d^n x) \text{ as } j \to \infty, \text{ and } \Delta u_{g_j} \to \Delta u \text{ in } L^2(\Omega; d^n x) \text{ as } j \to \infty, \tag{10.30}$$

it follows from this and Theorem 6.4 that

$$g_j = \widehat{\gamma}_D u_{g_j} \to \widehat{\gamma}_D u \text{ in } \big(N^{1/2}(\partial\Omega)\big)^* \text{ as } j \to \infty. \tag{10.31}$$

However, since $g_j \to g$ in $\big(N^{1/2}(\partial\Omega)\big)^*$ as $j \to \infty$, one finally concludes that $\widehat{\gamma}_D u = g$. This proves that $u \in L^2(\Omega; d^n x)$ satisfies $(-\Delta - z)u = 0$ in Ω and $\widehat{\gamma}_D u = g$ and hence by the uniqueness in (10.14) one obtains $u = u_g$.

This concludes the proof of the fact that the unique solution of (10.14), with data $f = 0$ and $g \in \big(N^{1/2}(\partial\Omega)\big)^*$ arbitrary, is given by $-\big[\gamma_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g$. Having established this, it is then clear that (10.20) solves (10.14), as originally stated. □

Corollary 10.5. *Assume Hypothesis* 8.10. *Then for every* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, *the map* $\widehat{\gamma}_D$ *in* (10.13) *is an isomorphism* (*i.e., bijective and bicontinuous*).

Proof. This follows from Corollary 10.3 and Theorem 10.4. □

Theorem 10.6. *Assume Hypothesis* 8.10. *Then the Neumann trace operator* (6.44) *is onto. More specifically, for every* $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$ *there exists* $C = C(\Omega, z) > 0$ *with the property that for any functional* $\Lambda \in \big(N^{3/2}(\partial\Omega)\big)^*$ *there exists* $u_\Lambda \in L^2(\Omega; d^n x)$ *with* $(-\Delta - z)u_\Lambda = 0$ *in* Ω, *such that*

$$\text{the assignment } \big(N^{3/2}(\partial\Omega)\big)^* \ni \Lambda \mapsto u_\Lambda \in L^2(\Omega; d^n x) \text{ is linear,} \tag{10.32}$$

and

$$\widehat{\gamma}_N u_\Lambda = \Lambda \ \text{ and } \ \|u_\Lambda\|_{L^2(\Omega;d^n x)} \leq C\|\Lambda\|_{(N^{3/2}(\partial\Omega))^*}. \tag{10.33}$$

Proof. Fix $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$. By Lemma 8.11, the operator $(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^n x) \to H^2(\Omega)$ is well-defined, linear, and bounded. Given $\Lambda \in \big(N^{3/2}(\partial\Omega)\big)^*$, we consider the bounded linear functional

$$\Lambda\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \mathbb{C}. \tag{10.34}$$

Then, by the Riesz representation theorem, there exists a unique $u_\Lambda \in L^2(\Omega; d^n x)$ with the property that

$$\Lambda\big(\gamma_D((-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1} f)\big) = \langle f, u_\Lambda\rangle_{L^2(\Omega;d^n x)}, \tag{10.35}$$

and such that

$$\begin{aligned} \|u_\Lambda\|_{L^2(\Omega;d^n x)} &\leq C\|\Lambda\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\|_{\mathcal{B}(L^2(\Omega;d^n x),\mathbb{C})} \\ &\leq C\|\Lambda\|_{(N^{3/2}(\partial\Omega))^*}, \end{aligned} \tag{10.36}$$

for some constant $C = C(\Omega, z) > 0$, independent of Λ. Furthermore, by the uniqueness of u_Λ, the assignment $\big(N^{3/2}(\partial\Omega)\big)^* \ni \Lambda \mapsto u_\Lambda \in L^2(\Omega; d^n x)$ is linear.

Specializing (10.35) to the case where $f := (-\Delta - \overline{z})w$, with $w \in H^2(\Omega)$ satisfying $\gamma_N w = 0$, then yields

$$\Lambda\big(\gamma_D w\big) = ((-\Delta - \overline{z})w, u_\Lambda)_{L^2(\Omega;d^n x)} \ \text{ for all } \ w \in H^2(\Omega) \ \text{ with } \ \gamma_N w = 0. \tag{10.37}$$

In particular, choosing $w \in C_0^\infty(\Omega)$ yields

$$0 = \Lambda(0) = \Lambda\big(\gamma_D w\big) = ((-\Delta - \overline{z})w, u_\Lambda)_{L^2(\Omega;d^n x)}, \quad w \in C_0^\infty(\Omega). \tag{10.38}$$

Thus, one obtains $(-\Delta - z)u_\Lambda = 0$ in Ω, in the sense of distributions. Inserting this into (10.35) one then obtains

$$\begin{aligned} \Lambda\big(\gamma_D w\big) &= ((-\Delta - \overline{z})w, u_\Lambda)_{L^2(\Omega;d^n x)} - (w, (-\Delta - z)u_\Lambda)_{L^2(\Omega;d^n x)} \\ &= -{}_{N^{3/2}(\partial\Omega)}\langle\gamma_D w, \widehat{\gamma}_N u_\Lambda\rangle_{(N^{3/2}(\partial\Omega))^*}, \end{aligned} \tag{10.39}$$

for every w in $\{w \in H^2(\Omega) \,|\, \gamma_N w = 0\}$. Since γ_D maps the latter space onto $N^{3/2}(\partial\Omega)$, (10.39) implies that $\widehat{\gamma}_N u_\Lambda = -\Lambda$. This shows that the operator (6.44) has a linear, bounded right-inverse. In particular, it is onto. □

Corollary 10.7. *Assume Hypothesis* 8.10. *Then for every* $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, *the map*

$$\widehat{\gamma}_N : \big\{u \in L^2(\Omega; d^n x) \,\big|\, (-\Delta - z)u = 0 \text{ in } \Omega\big\} \to \big(N^{3/2}(\partial\Omega)\big)^* \tag{10.40}$$

is onto.

Proof. This is a direct consequence of Theorem 10.6. □

Theorem 10.8. *Assume Hypothesis* 8.10 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$. *Then for any* $f \in L^2(\Omega; d^n x)$ *and* $g \in (N^{3/2}(\partial\Omega))^*$ *the following inhomogeneous Neumann boundary value problem*

$$\begin{cases} (-\Delta - z)u = f \text{ in } \Omega, \\ u \in L^2(\Omega; d^n x), \\ \widehat{\gamma}_N u = g \text{ on } \partial\Omega, \end{cases} \tag{10.41}$$

has a unique solution $u = u_N$. This solution satisfies

$$\|u_N\|_{L^2(\Omega;d^nx)} + \|\widehat{\gamma}_D u_N\|_{(N^{1/2}(\partial\Omega))^*} \leq C_N\big(\|f\|_{L^2(\Omega;d^nx)} + \|g\|_{(N^{3/2}(\partial\Omega))^*}\big) \tag{10.42}$$

for some constant $C_N = C_N(\Omega, z) > 0$, and the following regularity results hold:

$$g \in L^2(\partial\Omega; d^{n-1}\omega) \ \text{ implies } \ u_N \in H^{3/2}(\Omega), \tag{10.43}$$

$$g \in \gamma_N\big(H^2(\Omega)\big) \ \text{ implies } \ u_N \in H^2(\Omega). \tag{10.44}$$

Natural estimates are valid in each case.

Moreover,

$$\big[\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big((N^{3/2}(\partial\Omega))^*, L^2(\Omega; d^nx)\big), \tag{10.45}$$

and the solution of (10.41) *is given by the formula*

$$u_N = (-\Delta_{N,\Omega} - zI_\Omega)^{-1}f + \big[\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g. \tag{10.46}$$

Proof. By Theorem 10.6 there exists a function $v \in L^2(\Omega; d^nx)$ satisfying $\Delta v = 0$ in Ω, $\widehat{\gamma}_N v = g$ and $\|v\|_{L^2(\Omega;d^nx)} \leq C\|g\|_{(N^{3/2}(\partial\Omega))^*}$ for some finite $C = C(\Omega) > 0$. Setting

$$w := (-\Delta_{N,\Omega} - zI_\Omega)^{-1}(f + zv) \tag{10.47}$$

so that

$$w \in \mathrm{dom}(-\Delta_{N,\Omega}) \hookrightarrow H^2(\Omega), \tag{10.48}$$

by our hypotheses and by Lemma 8.11, and

$$\|w\|_{H^2(\Omega)} \leq C\|f + zv\|_{L^2(\Omega,d^nx)} \leq C\big(\|f\|_{L^2(\Omega;d^nx)} + \|g\|_{(N^{3/2}(\partial\Omega))^*}\big), \tag{10.49}$$

for some finite constant $C = C(\Omega, z) > 0$. Then $u := v + w$ solves (10.41) and satisfies (10.42).

To prove uniqueness, we assume that u satisfies the homogeneous version of (10.41), and consider $w := -z(-\Delta_{N,\Omega} - zI_\Omega)^{-1}u \in H^1_0(\Omega)$. Then $v := u - w$ satisfies

$$v \in L^2(\Omega; d^nx), \quad (-\Delta - z)v = 0 \ \text{ in } \ \Omega, \quad \widehat{\gamma}_N v = 0 \ \text{ on } \ \partial\Omega. \tag{10.50}$$

We claim that these conditions imply that $v = 0$. To show this, consider an arbitrary $f \in L^2(\Omega; d^nx)$ and let u_f be the unique solution of

$$u_f \in H^1(\Omega), \quad \widetilde{\gamma}_N w = 0 \ \text{ on } \ \partial\Omega, \quad (-\Delta - z)u_f = f \ \text{ in } \ \Omega. \tag{10.51}$$

Then $u_f \in \mathrm{dom}(-\Delta_{N,\Omega}) \subset H^2(\Omega)$ by Lemma 8.11, so that $u_f \in H^2(\Omega)$ and $\gamma_N w = 0$. The integration by parts formula (6.46) then yields

$$\begin{aligned}
(f, v)_{L^2(\Omega;d^nx)} &= ((-\Delta - z)u_f, v)_{L^2(\Omega;d^nx)} \\
&= (u_f, (-\Delta - z)v)_{L^2(\Omega;d^nx)} + {}_{N^{3/2}(\partial\Omega)}\langle \gamma_D u_f, \widehat{\gamma}_N v\rangle_{(N^{3/2}(\partial\Omega))^*} \\
&= 0.
\end{aligned} \tag{10.52}$$

Since $f \in L^2(\Omega; d^nx)$ was arbitrary, it follows that $v = 0$ and consequently, $u \in H^1(\Omega)$. Given that u solves the homogeneous version of (10.41) and the fact that $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, we may therefore conclude that $u = 0$ in Ω. The regularity result (10.43) is a consequence of Theorem 5.5 and the uniqueness for (10.41). Finally, (10.44) follows from (10.43) and Lemma 8.11.

Next, consider(10.45). On one hand, by Lemma 8.11 and the current assumptions, one concludes that

$$(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^nx) \to \big\{w \in H^2(\Omega) \,\big|\, \gamma_N w = 0\big\} \tag{10.53}$$

is a well-defined, linear, bounded operator. On the other hand, the operator γ_D in (6.41) is also well-defined, linear, and bounded. Consequently,

$$\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^nx), N^{3/2}(\partial\Omega)\big), \tag{10.54}$$

and (10.45) follows dualizing in (10.54). We note that this and Lemma 6.9 also imply that

$$\big[\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* = \big[\widehat{\gamma}_D(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big((N^{3/2}(\partial\Omega))^*, L^2(\Omega; d^nx)\big). \tag{10.55}$$

This will play a role shortly.

Next, let u_g be the (unique) solution of (10.41) with $f = 0$ and $g \in \big(N^{3/2}(\partial\Omega)\big)^*$ arbitrary. In addition, set $u := \big[\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g \in L^2(\Omega; d^nx)$. Our goal is to show that $u = u_g$. To this end, we note that (6.55) implies the existence of a sequence $\{g_j\}_{j\in\mathbb{N}}$ such that $g_j \in L^2(\partial\Omega; d^{n-1}\omega)$, $j \in \mathbb{N}$, and $g_j \to g$ in $\big(N^{3/2}(\partial\Omega)\big)^*$ as $j \to \infty$. For each fixed j, we know from Theorem 5.5 that $u_{g_j} = \big[\gamma_D(-\Delta_{N,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* g_j$.

By (10.55), this implies that $u_{g_j} \to u$ in $L^2(\Omega; d^n x)$. Hence, $\Delta u_{g_j} = -z u_{g_j} \to -zu$ in $L^2(\Omega; d^n x)$. With these at hand, it is then clear that $(-\Delta - z)u = 0$ in the sense of distributions in Ω. Next, given that

$$u_{g_j} \to u \text{ in } L^2(\Omega; d^n x) \text{ as } j \to \infty, \text{ and } \Delta u_{g_j} \to \Delta u \text{ in } L^2(\Omega; d^n x) \text{ as } j \to \infty, \tag{10.56}$$

it follows from this and Theorem 6.10 that

$$g_j = \widehat{\gamma}_N u_{g_j} \to \widehat{\gamma}_N u \text{ in } \big(N^{3/2}(\partial\Omega)\big)^* \text{ as } j \to \infty. \tag{10.57}$$

However, since $g_j \to g$ in $\big(N^{3/2}(\partial\Omega)\big)^*$ as $j \to \infty$, one finally concludes that $\widehat{\gamma}_N u = g$. This proves that $u \in L^2(\Omega; d^n x)$ satisfies $(-\Delta - z)u = 0$ in Ω and $\widehat{\gamma}_N u = g$ hence, by the uniqueness in (10.41), one obtains $u = u_g$.

This concludes the proof of the fact that the unique solution of (10.41), with $f = 0$ and $g \in \big(N^{3/2}(\partial\Omega)\big)^*$ arbitrary, is given by $\big[\gamma_D(-\Delta_{N,\Omega} - \overline{z} I_\Omega)^{-1}\big]^* g$. Having established this, it is then clear that (10.46) solves (10.41), as originally stated. $\square$

Corollary 10.9. *Assume Hypothesis* 8.10. *Then for any* $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, *the map* $\widehat{\gamma}_N$ *in* (10.40) *is an isomorphism* (*i.e., bijective and bicontinuous*).

Proof. This is a consequence of Corollary 10.9 and Theorem 10.8. $\square$

Next, we wish to augment Lemma 6.13 with the following result:

Lemma 10.10. *Assume Hypothesis* 8.10. *Then*

$$N^{1/2}(\partial\Omega) \text{ is a dense subspace of } \big(N^{3/2}(\partial\Omega)\big)^*, \tag{10.58}$$

and

$$N^{3/2}(\partial\Omega) \text{ is a dense subspace of } \big(N^{1/2}(\partial\Omega)\big)^*. \tag{10.59}$$

Proof. Consider (10.58): By relying on Lemmas 6.13 and 6.7 and the Hahn-Banach theorem, it suffices to show that if $g \in N^{3/2}(\partial\Omega)$ is such that $(g, f)_{L^2(\partial\Omega; d^{n-1}\omega)} = 0$ for every $f \in N^{1/2}(\partial\Omega)$, then necessarily $g = 0$. With this goal in mind, using the ontoness of the maps γ_N and γ_D in (6.11) and (6.41), we further rephrase the task at hand as

$$\left.\begin{array}{l} \text{if } w \in H^2(\Omega) \text{ is a function such that } \gamma_N w = 0 \text{ and} \\ (\gamma_D w, \gamma_N u)_{L^2(\partial\Omega; d^{n-1}\omega)} = 0 \text{ for all } u \in H^2(\Omega) \cap H^1_0(\Omega) \end{array}\right\} \text{ then } \gamma_D w = 0. \tag{10.60}$$

To justify this implication, consider $u := \Delta_{D,\Omega}^{-1} v \in H^2(\Omega) \cap H^1_0(\Omega)$ for an arbitrary $v \in L^2(\Omega; d^n x)$ so that

$$\begin{aligned} 0 &= (\gamma_D w, \gamma_N u)_{L^2(\partial\Omega; d^{n-1}\omega)} = (w, \Delta u)_{L^2(\Omega; d^n x)} - (\Delta w, u)_{L^2(\Omega; d^n x)} \\ &= (w, v)_{L^2(\Omega; d^n x)} - (\Delta w, \Delta_{D,\Omega}^{-1} v)_{L^2(\Omega; d^n x)} \\ &= (w, v)_{L^2(\Omega; d^n x)} - (\Delta_{D,\Omega}^{-1}(\Delta w), v)_{L^2(\Omega; d^n x)} \\ &= ([w - \Delta_{D,\Omega}^{-1}(\Delta w)], v)_{L^2(\Omega; d^n x)}, \end{aligned} \tag{10.61}$$

by (10.60), Green's formula, the fact that $\gamma_D u = \gamma_N w = 0$, the fact that $\Delta u = v$, and the self-adjointness of $-\Delta_{D,\Omega}^{-1}$ in $L^2(\Omega; d^n x)$. Thus, since $v \in L^2(\Omega; d^n x)$ was arbitrary, (10.61) implies $w = \Delta_{D,\Omega}^{-1}(\Delta w) \in H^1_0(\Omega)$. In particular, $\gamma_D w = 0$, completing the proof of (10.58). The justification of (10.59) follows along similar lines. $\square$

11. Dirichlet-to-Neumann Operators on Quasi-Convex Domains

In this section we discuss spectral parameter dependent Dirichlet-to-Neumann maps, also known in the literature as Weyl–Titchmarsh operators (in particular, they can be viewed as extensions of the Poincaré–Steklov operator).

Assuming Hypothesis 8.10, we introduce the Dirichlet-to-Neumann map $M^{(0)}_{D,N,\Omega}(z)$ associated with $(-\Delta - z)$ on Ω, as follows,

$$M^{(0)}_{D,N,\Omega}(z) \colon \begin{cases} \big(N^{1/2}(\partial\Omega)\big)^* \to \big(N^{3/2}(\partial\Omega)\big)^* \\ \qquad\qquad f \mapsto -\widehat{\gamma}_N u_D, \end{cases} \qquad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{11.1}$$

where u_D is the unique solution of

$$(-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in L^2(\Omega; d^n x), \quad \widehat{\gamma}_D u = f \text{ on } \partial\Omega. \tag{11.2}$$

Assuming Hypothesis 8.10, we introduce the Neumann-to-Dirichlet map $M_{N,D,\Omega}^{(0)}(z)$, associated with $(-\Delta-z)$ on Ω, as follows,

$$M_{N,D,\Omega}^{(0)}(z)\colon \begin{cases} \big(N^{3/2}(\partial\Omega)\big)^* \to \big(N^{1/2}(\partial\Omega)\big)^*, \\ \qquad\qquad g \mapsto \widehat{\gamma}_D u_N, \end{cases} \qquad z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}), \tag{11.3}$$

where u_N is the unique solution of

$$(-\Delta-z)u=0 \text{ in } \Omega, \quad u\in L^2(\Omega;d^nx), \quad \widehat{\gamma}_N u=g \text{ on } \partial\Omega. \tag{11.4}$$

The following result is the natural counterpart of Theorem 5.7 in the setting just introduced:

Theorem 11.1. *Assume Hypothesis* 8.10. *Then, with the above notation,*

$$M_{D,N,\Omega}^{(0)}(z)\in\mathcal{B}\big((N^{1/2}(\partial\Omega))^*,(N^{3/2}(\partial\Omega))^*\big), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{11.5}$$

its action is compatible with that of $M_{D,N,\Omega}^{(0)}(z)$ introduced in (5.27) (*thus, justifying retaining the same notation*), *and*

$$M_{D,N,\Omega}^{(0)}(z)=\widehat{\gamma}_N\big[\gamma_N(-\Delta_{D,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*, \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{11.6}$$

Similarly,

$$M_{N,D,\Omega}^{(0)}(z)\in\mathcal{B}\big((N^{3/2}(\partial\Omega))^*,(N^{1/2}(\partial\Omega))^*\big), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}), \tag{11.7}$$

its action is compatible with that of $M_{N,D\Omega}^{(0)}(z)$ introduced in (5.29) (*once again justifying retaining the same notation*), *and*

$$M_{N,D,\Omega}^{(0)}(z)=\widehat{\gamma}_D\big[\gamma_D(-\Delta_{N,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*, \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}). \tag{11.8}$$

Moreover,

$$M_{N,D,\Omega}^{(0)}(z)=-M_{D,N,\Omega}^{(0)}(z)^{-1}, \quad z\in\mathbb{C}\backslash(\sigma(-\Delta_{D,\Omega})\cup\sigma(-\Delta_{N,\Omega})), \tag{11.9}$$

and

$$\big[M_{D,N,\Omega}^{(0)}(z)\big]^*=M_{D,N,\Omega}^{(0)}(\overline{z}), \quad \big[M_{N,D,\Omega}^{(0)}(z)\big]^*=M_{N,D,\Omega}^{(0)}(\overline{z}). \tag{11.10}$$

As a consequence, one also has

$$M_{D,N,\Omega}^{(0)}(z)\in\mathcal{B}\big(N^{3/2}(\partial\Omega),N^{1/2}(\partial\Omega)\big), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{11.11}$$

$$M_{N,D,\Omega}^{(0)}(z)\in\mathcal{B}\big(N^{1/2}(\partial\Omega),N^{3/2}(\partial\Omega)\big), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}). \tag{11.12}$$

Finally, for every $z\le 0$ one has

$${}_{N^{1/2}(\partial\Omega)}\big\langle M_{D,N,\Omega}^{(0)}(z)f,f\big\rangle_{(N^{1/2}(\partial\Omega))^*}\le 0, \quad f\in N^{3/2}(\partial\Omega)\hookrightarrow\big(N^{1/2}(\partial\Omega)\big)^*, \tag{11.13}$$

whereas for every $z<0$

$${}_{N^{3/2}(\partial\Omega)}\big\langle M_{N,D,\Omega}^{(0)}(z)f,f\big\rangle_{(N^{3/2}(\partial\Omega))^*}\le 0, \quad f\in N^{1/2}(\partial\Omega)\hookrightarrow\big(N^{3/2}(\partial\Omega)\big)^*. \tag{11.14}$$

Proof. The membership in (11.5) is a consequence of Theorem 10.4, whereas (11.6) follows from (10.20). In a similar fashion, the membership in (11.7) is a consequence of Theorem 10.8, while (11.6) follows from (10.46). In addition, the fact that the operators in (11.1), (11.3) act in a compatible fashion with their counterparts from (5.27), (5.27) (in the latter case, assuming $\Theta=0$), is implied by the regularity statements (10.16) and (10.43). Going further, formula (11.9) follows from this compatibility result, (5.36), (11.5), (11.7), and the density results in (6.32) and (6.55). The duality formula (11.10) is a consequence of Lemma 4.12 in [54] (considered with $\Theta=0$) and a density argument, as before. Next, (11.11), (11.12) follow from duality, (11.10), (11.5), (11.7), and Lemmas 6.2 and 6.7.

As far as (11.13) is concerned, the density and compatibility results mentioned above show that it suffices to prove that

$$\big\langle f,M_{D,N,\Omega}^{(0)}(z)f\big\rangle_{1/2}\le 0, \quad f\in H^{1/2}(\partial\Omega), \tag{11.15}$$

given that $z\le 0$. However, Green's formula shows that, if $f\in H^{1/2}(\partial\Omega)$ and $u\in H^1(\Omega)$ satisfies $(-\Delta-z)u=0$ in Ω and $\gamma_D u=f$ on $\partial\Omega$ then

$$\begin{aligned}\big\langle f,M_{D,N,\Omega}^{(0)}(z)f\big\rangle_{1/2}&=-\langle\gamma_D u,\widetilde{\gamma}_N u\rangle_{1/2}\\&=-\|\nabla u\|_{(L^2(\Omega;d^nx))^n}^2+z\|u\|_{L^2(\Omega;d^nx)}^2\le 0.\end{aligned} \tag{11.16}$$

Finally, (11.14) is proved analogously. $\square$

12. The Regularized Neumann Trace Operator on Quasi-Convex Domains

In this section we introduce a regularized version of the Neumann trace operator (cf. (6.45)) on quasi-convex domains, and study some of its basic properties (such as a useful variant of Green's formula; cf. (12.8) below).

Theorem 12.1. *Assume Hypothesis* 8.10. *Then, for every $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the map*

$$\tau_z^N : \big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to N^{1/2}(\partial\Omega) \tag{12.1}$$

given by

$$\tau_z^N u := \widehat{\gamma}_N u + M_{D,N,\Omega}^{(0)}(z)\big(\widehat{\gamma}_D u\big), \quad u \in L^2(\Omega; d^n x),\ \Delta u \in L^2(\Omega; d^n x), \tag{12.2}$$

is well-defined, linear, and bounded when the space

$$\big\{u \in L^2(\Omega; d^n x) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} = \mathrm{dom}(-\Delta_{max}) \tag{12.3}$$

is endowed with the natural graph norm $u \mapsto \|u\|_{L^2(\Omega;d^n x)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$. Moreover, this operator satisfies the following additional properties:

(*i*) *For each $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the map τ_z^N in* (12.1), (12.2) *is onto* (*i.e., $\tau_z^N(\mathrm{dom}(-\Delta_{max})) = N^{1/2}(\partial\Omega)$*). *In fact,*

$$\tau_z^N\big(H^2(\Omega) \cap H_0^1(\Omega)\big) = N^{1/2}(\partial\Omega), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{12.4}$$

(*ii*) *One has*

$$\tau_z^N = \gamma_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}(-\Delta - z), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{12.5}$$

(*iii*) *For each $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the kernel of the map τ_z^N in* (12.1), (12.2) *is given by*

$$\ker\big(\tau_z^N\big) = H_0^2(\Omega) \dotplus \big\{u \in L^2(\Omega; d^n x) \,\big|\, (-\Delta - z)u = 0 \text{ in } \Omega\big\}. \tag{12.6}$$

In particular, if $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, then

$$\tau_z^N u = 0 \text{ for every } u \in \ker(-\Delta_{max} - zI_\Omega). \tag{12.7}$$

(*iv*) *For each $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the following Green formula holds for every $u, v \in \mathrm{dom}(-\Delta_{max})$,*

$$\begin{aligned}&((-\Delta - z)u, v)_{L^2(\Omega;d^n x)} - (u, (-\Delta - \overline{z})v)_{L^2(\Omega;d^n x)} \\ &\quad = -{}_{N^{1/2}(\partial\Omega)}\langle \tau_z^N u, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} + \overline{{}_{N^{1/2}(\partial\Omega)}\langle \tau_{\overline{z}}^N v, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}}.\end{aligned} \tag{12.8}$$

In particular, for every $u, v \in \mathrm{dom}(-\Delta_{max})$, $z \in \mathbb{C}$, and $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$, one has

$$\begin{aligned}&((-\Delta - (z + z_0))u, v)_{L^2(\Omega;d^n x)} - (u, (-\Delta - (\overline{z} + z_0))v)_{L^2(\Omega;d^n x)} \\ &\quad = -{}_{N^{1/2}(\partial\Omega)}\langle \tau_{z_0}^N u, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} + \overline{{}_{N^{1/2}(\partial\Omega)}\langle \tau_{z_0}^N v, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}}.\end{aligned} \tag{12.9}$$

Moreover, as a consequence of (12.8) *and* (12.7), *for every $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$ one infers that*

$$\begin{aligned}&u \in \mathrm{dom}(-\Delta_{max}) \text{ and } v \in \ker(-\Delta_{max} - \overline{z}I_\Omega) \\ &\quad \text{imply } ((-\Delta - z)u, v)_{L^2(\Omega;d^n x)} = -{}_{N^{1/2}(\partial\Omega)}\langle \tau_z^N u, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*}.\end{aligned} \tag{12.10}$$

Proof. Let$z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Consider an arbitrary $u \in L^2(\Omega; d^n x)$ satisfying $\Delta u \in L^2(\Omega; d^n x)$, and let v solve

$$(-\Delta - z)v = 0 \text{ in } \Omega, \quad v \in L^2(\Omega; d^n x), \quad \widehat{\gamma}_D v = \widehat{\gamma}_D u \text{ on } \partial\Omega. \tag{12.11}$$

Theorems 6.4 and 10.4 ensure that this is possible and guarantee the existence of a finite constant $C = C(\Omega, z) > 0$ for which

$$\|v\|_{L^2(\Omega;d^n x)} \leq C(\|u\|_{L^2(\Omega;d^n x)} + \|\Delta u\|_{L^2(\Omega;d^n x)}). \tag{12.12}$$

Then $w := u - v$ satisfies

$$(-\Delta - z)w = (-\Delta - z)u \in L^2(\Omega; d^n x), \quad w \in L^2(\Omega; d^n x), \quad \widehat{\gamma}_D w = 0 \text{ on } \partial\Omega, \tag{12.13}$$

so that, by (10.18), there exists $C = C(\Omega, z) > 0$ such that

$$w \in H^2(\Omega) \cap H_0^1(\Omega) \text{ and } \|w\|_{H^2(\Omega)} \leq C(\|u\|_{L^2(\Omega;d^n x)} + \|\Delta u\|_{L^2(\Omega;d^n x)}). \tag{12.14}$$

Since $M_{D,N,\Omega}^{(0)}(z)\big(\widehat{\gamma}_D u\big) = -\widehat{\gamma}_N v$, it follows from (12.14), the compatibility part of Theorem 6.10, and Lemma 6.3 that

$$\tau_z^N u = \widehat{\gamma}_N u - \widehat{\gamma}_N v = \widehat{\gamma}_N (u-v) = \widehat{\gamma}_N w = \gamma_N w \in N^{1/2}(\partial\Omega). \tag{12.15}$$

Moreover,

$$\begin{aligned}\|\tau_z^N u\|_{N^{1/2}(\partial\Omega)} &= \|\gamma_N w\|_{N^{1/2}(\partial\Omega)} \leq C\|w\|_{H^2(\Omega)} \\ &\leq C(\|u\|_{L^2(\Omega;d^nx)} + \|\Delta u\|_{L^2(\Omega;d^nx)}).\end{aligned} \tag{12.16}$$

This shows that the operator τ_z^N in (12.1), (12.2) is indeed well-defined and bounded.

Incidentally, the above argument also shows that (12.5) holds. That τ_z^N defined in (12.1), (12.2) is onto, is a direct consequence of the fact that γ_N in (6.11) is onto (cf. Lemma 6.3). This also justifies (12.4).

Next, observe that due to Theorem 10.4 the sum in (12.6) is direct. Moreover, $H_0^2(\Omega) \subseteq \ker\big(\tau_z^N\big)$ by (12.2) and $\big\{u \in L^2(\Omega; d^nx) \,\big|\, (-\Delta - z)u = 0 \text{ in } \Omega\big\} \subseteq \ker\big(\tau_z^N\big)$ by (12.5). This proves the right-to-left inclusion in (12.6). To prove the opposite one, consider $u \in L^2(\Omega; d^nx)$ with $\Delta u \in L^2(\Omega; d^nx)$ for which $\tau_z^N u = 0$. If we now set $w := (-\Delta_{D,\Omega} - zI_\Omega)^{-1}(-\Delta - z)u$, then $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\gamma_N w = \tau_z^N u = 0$, by (12.5). Hence, $w \in H_0^2(\Omega)$ by Theorem 8.14. Since $u = w + (u - w)$ and $(u - w) \in L^2(\Omega; d^nx)$ satisfies $(-\Delta - z)(u - w) = 0$ in $L^2(\Omega; d^nx)$, the proof of (12.6) is complete.

Next, consider the Green formulas in (iv). Fix $z \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$, let $u, v \in \operatorname{dom}(-\Delta_{min})$, and set $\widetilde{u} := (-\Delta_{D,\Omega} - zI_\Omega)^{-1}((-\Delta - z)u)$, $\widetilde{v} := (-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}((-\Delta - \overline{z})v)$. Then

$$\begin{aligned}&\widetilde{u}, \widetilde{v} \in H^2(\Omega) \cap H_0^1(\Omega), \quad (-\Delta - z)\widetilde{u} = (-\Delta - z)u, \quad (-\Delta - \overline{z})\widetilde{v} = (-\Delta - \overline{z})v, \\ &\gamma_N \widetilde{u} = \tau_z^N u, \quad \gamma_N \widetilde{v} = \tau_{\overline{z}}^N v,\end{aligned} \tag{12.17}$$

by (12.5). Based on these observations and repeated applications of Green's formula (6.16) one can then write

$$\begin{aligned}&((-\Delta - z)u, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})v)_{L^2(\Omega;d^nx)} \\ &\quad = ((-\Delta - z)\widetilde{u}, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})\widetilde{v})_{L^2(\Omega;d^nx)} \\ &\quad = (\widetilde{u}, (-\Delta - \overline{z})v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})\widetilde{v})_{L^2(\Omega;d^nx)} \\ &\qquad - {}_{N^{1/2}(\partial\Omega)}\langle \gamma_N \widetilde{u}, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} \\ &\quad = (\widetilde{u} - u, (-\Delta - \overline{z})\widetilde{v})_{L^2(\Omega;d^nx)} - {}_{N^{1/2}(\partial\Omega)}\langle \tau_z^N u, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} \\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\langle \tau_{\overline{z}}^N v, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}} - {}_{N^{1/2}(\partial\Omega)}\langle \tau_z^N u, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*},\end{aligned} \tag{12.18}$$

where in the last step we have used the fact that $(-\Delta - z)(\widetilde{u} - u) = 0$ and $\widehat{\gamma}_D(\widetilde{u} - u) = -\widehat{\gamma}_D u$. This justifies (12.8) and completes the proof of the theorem. □

Recalling equation (10.58), we also state the following consequence of Theorem 12.1:

Corollary 12.2. *Assume Hypothesis* 8.10. *Then, for every $z_1, z_2 \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, the operator*

$$\big[M_{D,N,\Omega}^{(0)}(z_1) - M_{D,N,\Omega}^{(0)}(z_2)\big] \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, (N^{3/2}(\partial\Omega))^*\big) \tag{12.19}$$

extends to

$$\big[M_{D,N,\Omega}^{(0)}(z_1) - M_{D,N,\Omega}^{(0)}(z_2)\big] \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, N^{1/2}(\partial\Omega)\big). \tag{12.20}$$

Proof. From Theorem 10.2 we know that there exists a constant $C > 0$ with the property that for every $f \in (N^{1/2}(\partial\Omega))^*$ one can find $u \in \operatorname{dom}(-\Delta_{max})$ such that

$$\widehat{\gamma}_D u = f \text{ and } \|u\|_{L^2(\Omega;d^nx)} + \|\Delta u\|_{L^2(\Omega;d^nx)} \leq C\|f\|_{(N^{1/2}(\partial\Omega))^*}. \tag{12.21}$$

With the help of (12.1) we may then write

$$\begin{aligned}\|[M_{D,N,\Omega}^{(0)}(z_1) - M_{D,N,\Omega}^{(0)}(z_2)]f\|_{N^{1/2}(\partial\Omega)} &= \big\|\tau_{z_1}^N u - \tau_{z_2}^N u\big\|_{N^{1/2}(\partial\Omega)} \\ &\leq C\big[\|u\|_{L^2(\Omega;d^nx)} + \|\Delta u\|_{L^2(\Omega;d^nx)}\big] \\ &\leq C\|f\|_{(N^{1/2}(\partial\Omega))^*},\end{aligned} \tag{12.22}$$

proving (12.20). □

13. The Krein Laplacian in Quasi-Convex Domains

We now discuss the Krein–von Neumann extension $-\Delta_{K,\Omega}$ of Laplacian $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega;d^nx)$, with $\Omega\subset\mathbb{R}^n$ a bounded Lipschitz domain satisfying Hypothesis 8.10.

In this situation, equations (9.7) and (8.45) yield the following:

$$\begin{aligned} \operatorname{dom}(-\Delta_{K,\Omega}) &= \operatorname{dom}(-\Delta_{min})\dot{+}\ker((-\Delta_{min})^*) \\ &= \operatorname{dom}(-\Delta_{min})\dot{+}\ker(-\Delta_{max}) \\ &= H_0^2(\Omega)\dot{+}\big\{u\in L^2(\Omega;d^nx)\,\big|\,\Delta u=0 \text{ in } \Omega\big\}. \end{aligned}\tag{13.1}$$

Nonetheless, we shall adopt a different point of view which better elucidates the nature of the boundary condition associated with the Krein Laplacian. Our construction was originally inspired by the discussion in [8]. In Example 5.3 of that paper, the authors consider the Laplacian with the boundary condition

$$\frac{\partial f}{\partial n}(x)=\frac{\partial H(f)}{\partial n}(x),\quad x\in\partial\Omega,\tag{13.2}$$

where, given $f:\Omega\to\mathbb{R}$, $H(f)$ is the harmonic extension of $f|_{\partial\Omega}$ to Ω, and $\partial/\partial n$ denotes the normal derivative. The algebraic manipulations in [8] leading up to (13.2) are somewhat formal, and Alonso and Simon mention that *"it seems to us that the Krein extension of $-\Delta$, that is, $-\Delta$ with the boundary condition* (13.2)*, is a natural object and therefore worthy of further study."*

Theorem 13.1. *Assume Hypothesis* 8.10 *and $z\in\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Then the operator $-\Delta_{K,\Omega,z}$ in $L^2(\Omega;d^nx)$ given by*

$$\begin{aligned} &-\Delta_{K,\Omega,z}u:=(-\Delta-z)u,\\ &u\in\operatorname{dom}(-\Delta_{K,\Omega,z}):=\big\{v\in\operatorname{dom}(-\Delta_{max})\,\big|\,\tau_z^Nv=0\big\}, \end{aligned}\tag{13.3}$$

satisfies

$$\big(-\Delta_{K,\Omega,z}\big)^*=-\Delta_{K,\Omega,\overline{z}}.\tag{13.4}$$

In particular, if $z\in\mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$ then $-\Delta_{K,\Omega,z}$ is self-adjoint. Moreover, if $z\leq 0$ then $-\Delta_{K,\Omega,z}\geq 0$.

In the following, $-\Delta_{K,\Omega,z}$ will be referred to as the Krein Laplacian in Ω (associated with z). Hence, the Krein Laplacian $-\Delta_{K,\Omega}:=-\Delta_{K,\Omega,0}$ is a self-adjoint operator in $L^2(\Omega;d^nx)$ which satisfies

$$-\Delta_{K,\Omega}\geq 0,\ \text{ and }\ -\Delta_{min}\subseteq-\Delta_{K,\Omega}\subseteq-\Delta_{max}.\tag{13.5}$$

Furthermore, $\ker(-\Delta_{K,\Omega})=\big\{u\in L^2(\Omega;d^nx)\,\big|\,\Delta u=0\big\}$,

$$\text{with the possible exception of the origin, } -\Delta_{K,\Omega} \text{ has a discrete, real spectrum,}\tag{13.6}$$

and for any nonnegative self-adjoint extension $\widetilde{S}$ of $-\Delta\big|_{C_0^\infty(\Omega)}$ one has

$$-\Delta_{K,\Omega}\leq\widetilde{S}\leq-\Delta_{D,\Omega}.\tag{13.7}$$

Proof. It is clear that the operator (13.3) is densely defined. In addition, due to (12.8), this operator satisfies $-\Delta_{K,\Omega,\overline{z}}\subseteq(-\Delta_{K,\Omega,z})^*$. Next, consider $w\in\operatorname{dom}(-\Delta_{K,\Omega,z})^*$. Then $w\in L^2(\Omega;d^nx)$, and there exists a function $v\in L^2(\Omega;d^nx)$ such that

$$(v,u)_{L^2(\Omega;d^nx)}=(w,(\Delta+z)u)_{L^2(\Omega;d^nx)},\quad u\in\operatorname{dom}(-\Delta_{K,\Omega,z}).\tag{13.8}$$

Choosing $u\in C_0^\infty(\Omega)$ then shows that $w\in\operatorname{dom}(-\Delta_{max})$ and $(\Delta+\overline{z})w=v$. Thus, (13.8) becomes

$$\begin{aligned} 0&=((\Delta+\overline{z})w,u)_{L^2(\Omega;d^nx)}-(w,(\Delta+z)u)_{L^2(\Omega;d^nx)}\\ &={}_{N^{1/2}(\partial\Omega)}\langle\tau_{\overline{z}}^Nw,\widehat{\gamma}_Du\rangle_{(N^{1/2}(\partial\Omega))^*},\quad u\in\operatorname{dom}(-\Delta_{K,\Omega,z}), \end{aligned}\tag{13.9}$$

by (12.8) and the fact that $\tau_z^Nu=0$. In order to continue, we remark that

$$\begin{aligned} \widehat{\gamma}_D(\operatorname{dom}(-\Delta_{K,\Omega,z}))&=\widehat{\gamma}_D(\ker(\tau_z^N))\\ &=\widehat{\gamma}_D\big(\big\{u\in L^2(\Omega;d^nx)\,\big|\,(-\Delta-z)u=0\,\text{in}\,\Omega\big\}\big)\\ &=\big(N^{1/2}(\partial\Omega)\big)^*, \end{aligned}\tag{13.10}$$

by virtue of (12.6) and Corollary 10.3. Using this in (13.9) then yields $\tau_{\overline{z}}^Nw=0$ in $N^{1/2}(\partial\Omega)$ and hence, $w\in\operatorname{dom}(-\Delta_{K,\Omega,\overline{z}})$. This shows that $-\Delta_{K,\Omega,z}^*\subseteq-\Delta_{K,\Omega,\overline{z}}$, which completes the proof of (13.4). The subsequent commentary in the statement of the theorem is then justified by this.

Next, we will show that $-\Delta_{K,\Omega,z} \geq 0$ whenever $z \leq 0$. To this end, fix such a z and, given an arbitrary $u \in \mathrm{dom}(-\Delta_{K,\Omega,0})$, consider $v \in \widetilde{H}_0^2(\Omega)$ and $w \in L^2(\Omega; d^n x)$ with $(-\Delta - z)w = 0$ such that $u = v + w$. That this is possible is ensured by (13.3) and (12.6). We may then write

$$\begin{aligned}(u, (-\Delta_{K,\Omega,z})u)_{L^2(\Omega;d^nx)} &= (v + w, (-\Delta - z)v)_{L^2(\Omega;d^nx)} \\ &= (v, (-\Delta - z)v)_{L^2(\Omega;d^nx)} + ((-\Delta - z)w, v)_{L^2(\Omega;d^nx)} \\ &= 2\mathrm{Re}[(v, (-\Delta - z)v)_{L^2(\Omega;d^nx)}] \\ &= 2[(-z)\,\|v\|^2_{L^2(\Omega;d^nx)} + \|\nabla v\|^2_{(L^2(\Omega;d^nx))^n}] \geq 0, \end{aligned} \tag{13.11}$$

as required.

Next, we note that by (13.3) and (12.6), the domain of $-\Delta_{K,\Omega}$ has the description given in (13.1), justifying the terminology of Krein Laplacian used in connection with this operator. Finally, that the kernel of $-\Delta_{K,\Omega}$ consists of all harmonic, square integrable functions in Ω follows from (9.9) and (8.45), whereas (13.6) is a consequence of Lemma 9.3 and Theorem 5.1 in [8]. $\square$

14. Self-adjoint Extensions with the Dirichlet Laplacian as Reference Operator

Having discussed the Friedrichs and the Krein–von Neumann extensions of $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$, our goal in this section is now to identify *all* self-adjoint extensions of this operator (nonnegative or not).

To set the stage, we first recall an abstract functional analytic result which is pertinent to the task at hand. The following is essentially Theorem (II) 2.1 on p. 448 of [60]:

Theorem 14.1. *Let $\mathcal{H}$ be a Hilbert space, with inner product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$, and assume that*

$$A_0 : \mathrm{dom}(A_0) \subseteq \mathcal{H} \to \mathcal{H} \tag{14.1}$$

is a closed, densely defined, symmetric unbounded linear operator. Set $A_1 := (A_0)^*$, and let

$$A_\beta : \mathrm{dom}(A_\beta) \subseteq \mathcal{H} \to \mathcal{H} \tag{14.2}$$

be a self-adjoint extension of A_0 with $0 \notin \sigma(A_\beta)$ (A_β is also called the reference operator).

(*i*) *Then*

$$\begin{aligned}\mathrm{pr}_\beta &:= A_\beta^{-1}A_1 : \mathrm{dom}(A_1) \to \mathrm{dom}(A_\beta), \\ \mathrm{pr}_\zeta &:= I_{\mathcal{H}} - \mathrm{pr}_\beta : \mathrm{dom}(A_1) \to \ker(A_\beta),\end{aligned} \tag{14.3}$$

are complementary projections which induce the decomposition

$$\mathrm{dom}(A_1) = \mathrm{dom}(A_\beta) \dotplus \ker(A_1). \tag{14.4}$$

In the sequel, we denote the above decomposition schematically by writing $u = u_\beta + u_\zeta$ for each $u \in \mathrm{dom}(A_1)$, where $u_\beta := \mathrm{pr}_\beta(u) \in \mathrm{dom}(A_\beta)$ and $u_\zeta := \mathrm{pr}_\zeta(u) \in \ker(A_1)$.

(*ii*) *Let V be an arbitrary closed subspace of $\ker(A_1)$ and let $T : \mathrm{dom}(T) \subseteq V \to V^*$ be an arbitrary self-adjoint operator. Then the operator $A^{V,T} \subseteq A_1$ given by $A^{V,T} : \mathrm{dom}\big(A^{V,T}\big) \subseteq \mathcal{H} \to \mathcal{H}$,*

$$\begin{aligned}&A^{V,T}u := A_1 u, \\ &u \in \mathrm{dom}\big(A^{V,T}\big) := \{v \in dom(A_1)\,|\, v_\zeta \in \mathrm{dom}(T) \text{ and} \\ &\qquad\qquad (w, A_1 v)_{\mathcal{H}} = {}_V\langle w, T v_\zeta\rangle_{V^*},\, w \in V\},\end{aligned} \tag{14.5}$$

is a self-adjoint extension of A_0.

(*iii*) *Conversely, let $\widetilde{A} : \mathrm{dom}(\widetilde{A}) \subseteq \mathcal{H} \to \mathcal{H}$ be a self-adjoint extension of A_0 (so that, necessarily, $\widetilde{A} \subseteq A_1$) and define*

$$V := \overline{\{u_\zeta \,|\, u \in \mathrm{dom}(\widetilde{A})\}} \quad (\text{with closure in } \mathcal{H}), \tag{14.6}$$

and consider the operator $T : \mathrm{dom}(T) \subseteq V \to V^*$ given by

$$T u_\zeta := (\,\cdot\,, A_1 u)_{\mathcal{H}} \in V^*, \quad u_\zeta \in \mathrm{dom}(T) := \big\{v_\zeta \,\big|\, v \in \mathrm{dom}\big(\widetilde{A}\big)\big\}. \tag{14.7}$$

Then T is a self-adjoint operator, and $\widetilde{A} = A^{V,T}$, where $A^{V,T}$ is associated with V and T as in item (ii).

Consequently, the constructions in items (ii), (iii) establish a one-to-one correspondence between self-adjoint extensions $\widetilde{A}$ of A_0 and self-adjoint operators $T : \mathrm{dom}(T) \subseteq V \to V^*$ with V a closed subspace of $\ker(A_1)$.

Remark 14.2. Assume that $\mathcal{H}$ is a Hilbert space (with inner product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$), and suppose that V is a closed subspace of $\mathcal{H}$. In addition, denote by π_V the orthogonal projection of $\mathcal{H}$ onto V. Then

$$\pi_V \pi_V^* : V^* \to V \text{ isomorphically,} \tag{14.8}$$

with inverse $V \ni v \mapsto (v,\,\cdot\,)_{\mathcal{H}} \in V^*$. In the context of Theorem 14.1 (i), and keeping the identification $V^* \equiv V$ in mind given by (14.8), an alternative description of the domain of the operator $A^{V,T}$, originally introduced in (14.5) is

$$u \in \operatorname{dom}\big(A^{V,T}\big) \text{ if and only if } \begin{cases} u = v + w + A_\beta^{-1}(Tw + \eta), \\ v \in \operatorname{dom}(A_0),\ w \in \operatorname{dom}(T), \\ \quad \text{and } \ \eta \in \ker(A_1) \cap V^\top. \end{cases} \tag{14.9}$$

See Lemma 1.4 on p. 444 of [60]. Furthermore, with the above identification understood,

$$\ker\big(A^{V,T}\big) = \ker(T), \quad \operatorname{ran}\big(A^{V,T}\big) = \operatorname{ran}(T) + V^\top. \tag{14.10}$$

In particular, $A^{V,T}$ is Fredholm if and only if T is Fredholm, with the same kernel and cokernel. If $A^{V,T}$ and hence, also T, is injective then the inverse satisfies

$$\big(A^{V,T}\big)^{-1} = A_\beta^{-1} + T^{-1}\pi_V, \text{ defined on } \operatorname{ran}\big(A^{V,T}\big). \tag{14.11}$$

A related result is [31, Theorem 2.1].

Theorem 14.1 provides a universal parametrization of all self-adjoint extensions of A_0, and our aim is to implement this abstract scheme in the case of $-\Delta\big|_{C_0^\infty(\Omega)}$. In the theorem below we choose the Dirichlet Laplacian as the reference operator.

Theorem 14.3. *Assume Hypothesis* 8.10 *and let $z \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$. Suppose that X is a closed subspace of $\big(N^{1/2}(\partial\Omega)\big)^*$ and denote by X^* the conjugate dual space of X. In addition, consider a self-adjoint operator*

$$L : \operatorname{dom}(L) \subseteq X \to X^*, \tag{14.12}$$

and define the linear operator $-\Delta^D_{X,L,z} : \operatorname{dom}\big(-\Delta^D_{X,L,z}\big) \subset L^2(\Omega; d^nx) \to L^2(\Omega; d^nx)$ by

$$\begin{aligned} & -\Delta^D_{X,L,z} u := (-\Delta - z)u, \\ & u \in \operatorname{dom}\big(-\Delta^D_{X,L,z}\big) := \big\{v \in \operatorname{dom}(-\Delta_{max}) \,\big|\, \widehat{\gamma}_D v \in \operatorname{dom}(L),\ \tau_z^N v\big|_X = L\big(\widehat{\gamma}_D v\big)\big\}. \end{aligned} \tag{14.13}$$

Above, τ_z^N is the map introduced in (12.1), (12.2), *and the boundary condition $\tau_z^N u\big|_X = L\big(\widehat{\gamma}_D u\big)$ is interpreted as*

$${}_{N^{1/2}(\partial\Omega)}\big\langle \tau_z^N u, f\big\rangle_{(N^{1/2}(\partial\Omega))^*} = \overline{{}_X\langle f, L(\widehat{\gamma}_D u)\rangle_{X^*}}, \quad f \in X. \tag{14.14}$$

Then

$$-\Delta^D_{X,L,z} \text{ is self-adjoint in } L^2(\Omega; d^nx), \tag{14.15}$$

and

$$-\Delta_{min} - zI_\Omega \subseteq -\Delta^D_{X,L,z} \subseteq -\Delta_{max} - zI_\Omega. \tag{14.16}$$

Conversely, if

$$\widetilde{S} : \operatorname{dom}\big(\widetilde{S}\big) \subseteq L^2(\Omega; d^nx) \to L^2(\Omega; d^nx) \tag{14.17}$$

is a self-adjoint operator with the property that

$$-\Delta_{min} - zI_\Omega \subseteq \widetilde{S} \subseteq -\Delta_{max} - zI_\Omega, \tag{14.18}$$

then there exist X, a closed subspace of $\big(N^{1/2}(\partial\Omega)\big)^*$, and $L : \operatorname{dom}(L) \subseteq X \to X^*$, a self-adjoint operator, such that

$$\widetilde{S} = -\Delta^D_{X,L,z}. \tag{14.19}$$

In the above scheme, the operator $\widetilde{S}$ and the pair X, L correspond uniquely to each other. In fact,

$$\operatorname{dom}(L) = \widehat{\gamma}_D\big(\operatorname{dom}\big(\widetilde{S}\big)\big), \quad X = \overline{\widehat{\gamma}_D\big(\operatorname{dom}\big(\widetilde{S}\big)\big)} \quad \big(\text{with closure in } \big(N^{1/2}(\partial\Omega)\big)^*\big). \tag{14.20}$$

Proof. Let $z \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$. We will employ Theorem 14.1 in the following context:

$$A_0 := -\Delta_{min} - zI_\Omega, \quad A_1 := -\Delta_{max} - zI_\Omega, \quad A_\beta := -\Delta_{D,\Omega} - zI_\Omega. \tag{14.21}$$

In particular, $0 \in \mathbb{C}\backslash\sigma(A_\beta)$.

Throughout the proof, we shall make use of the following results and notation:

$$u \in \operatorname{dom}(-\Delta_{max}) \text{ implies } \begin{cases} u_\beta := (-\Delta_{D,\Omega} - zI_\Omega)^{-1}(-\Delta - z)u \in H^2(\Omega) \cap H_0^1(\Omega), \\ u_\zeta := u - (-\Delta_{D,\Omega} - zI_\Omega)^{-1}(-\Delta - z)u \in \ker(-\Delta_{max} - zI_\Omega), \\ \widehat{\gamma}_N u_\zeta = \tau_z^N u \in N^{1/2}(\partial\Omega), \\ \widehat{\gamma}_D u_\zeta = \widehat{\gamma}_D u \in \big(N^{1/2}(\partial\Omega)\big)^*. \end{cases} \tag{14.22}$$

We now assume that a closed subspace X of $\big(N^{1/2}(\partial\Omega)\big)^*$ has been specified, that L is a self-adjoint operator as in (14.12), and that $-\Delta_{X,L,z}^D$ is defined as in (14.13). We will show that $-\Delta_{X,L,z}^D$ is a self-adjoint operator. To this end we define

$$V := \big\{u \in \ker(-\Delta_{max} - zI_\Omega) \,\big|\, \widehat{\gamma}_D u \in X\big\}, \tag{14.23}$$

and observe that

$$\begin{aligned} V = \ker(-\Delta_{max} - zI_\Omega) \cap \widehat{\gamma}_D^{-1}(X) \\ \text{is a closed subspace of } \ker(-\Delta_{max} - zI_\Omega), \end{aligned} \tag{14.24}$$

and that

$$\widehat{\gamma}_D : V \to X \text{ isomorphically, with } \widehat{\gamma}_D \in \mathcal{B}(V, X) \text{ and } \widehat{\gamma}_D^{-1} \in \mathcal{B}(X, V), \tag{14.25}$$

by (14.23) and Corollary 10.3.

Next, we also introduce the operator $T : \operatorname{dom}(T) \subseteq V \to V^*$ by setting

$$\begin{aligned} &Tu := -{}_X\langle\widehat{\gamma}_D(\cdot), L(\widehat{\gamma}_D u)\rangle_{X^*}, \\ &u \in \operatorname{dom}(T) := \{v \in V \,|\, \widehat{\gamma}_D v \in \operatorname{dom}(L)\}. \end{aligned} \tag{14.26}$$

For every $u, v \in \operatorname{dom}(T)$, using the self-adjointness of L, we may write

$$\begin{aligned} {}_V\langle v, Tu\rangle_{V^*} = -{}_X\langle\widehat{\gamma}_D v, L(\widehat{\gamma}_D u)\rangle_{X^*} &= -\overline{{}_X\langle\widehat{\gamma}_D u, L(\widehat{\gamma}_D v)\rangle_{X^*}} \\ &= \overline{{}_V\langle u, Tv\rangle_{V^*}}. \end{aligned} \tag{14.27}$$

This shows that T is symmetric, that is, $T \subseteq T^*$. To prove the converse inclusion, consider $u \in \operatorname{dom}(T^*)$. Then $u \in V$ and there exists $\Lambda \in V^*$ such that

$${}_V\langle w, \Lambda\rangle_{V^*} = \overline{{}_V\langle u, Tw\rangle_{V^*}} = -\overline{{}_X\langle\widehat{\gamma}_D u, L(\widehat{\gamma}_D w)\rangle_{X^*}}, \quad w \in \operatorname{dom}(T). \tag{14.28}$$

Our goal is to show that $u \in \operatorname{dom}(T)$ which, by (14.26), comes down to proving that $\widehat{\gamma}_D u \in \operatorname{dom}(L)$, or equivalently, that $\widehat{\gamma}_D u \in \operatorname{dom}(L^*)$. In turn, the veracity of the latter condition is established as soon as we show that there exists a finite constant $C > 0$ with the property that

$$|{}_X\langle\widehat{\gamma}_D u, L(f)\rangle_{X^*}| \le C\|f\|_X, \quad f \in \operatorname{dom}(L). \tag{14.29}$$

Since (14.25) entails that

$$\widehat{\gamma}_D : \operatorname{dom}(T) \to \operatorname{dom}(L) \text{ boundedly, with a bounded inverse,} \tag{14.30}$$

the estimate (14.29) is going to be implied by

$${}_X\langle\widehat{\gamma}_D u, L(\widehat{\gamma}_D w)\rangle_{X^*}| \le C\|w\|_V, \quad w \in \operatorname{dom}(T). \tag{14.31}$$

This, however, for the choice $C = \|\Lambda\|_{V^*}$, is a direct consequence of (14.28). In summary, the above reasoning yields that $u \in \operatorname{dom}(T^*)$, completing the proof of the fact that T is self-adjoint.

With this at hand, Theorem 14.1 will imply that $-\Delta_{X,L,z}^D$ is self-adjoint as soon as we establish that

$$\begin{aligned} \operatorname{dom}\big(-\Delta_{X,L,z}^D\big) = \{u \in \operatorname{dom}(-\Delta_{max}) \,|\, u_\zeta \in \operatorname{dom}(T), \\ (w, (-\Delta - z)u)_{L^2(\Omega; d^n x)} = {}_V\langle w, Tu_\zeta\rangle_{V^*}, \, w \in V\}. \end{aligned} \tag{14.32}$$

We note that for each $u \in \operatorname{dom}(-\Delta_{max})$,

$$u_\zeta \in \operatorname{dom}(T) \text{ if and only if } \widehat{\gamma}_D u \in \operatorname{dom}(L) \tag{14.33}$$

by (14.22). Consequently, it remains to show that

$$\tau_z^N u\big|_X = L(\widehat{\gamma}_D u) \text{ if and only if } (w, (-\Delta - z)u)_{L^2(\Omega; d^n x)} = {}_V\langle w, Tu_\zeta\rangle_{V^*}, \quad w \in V, \tag{14.34}$$

whenever $u\in\mathrm{dom}(-\Delta_{max})$ has $\widehat{\gamma}_D u\in\mathrm{dom}(L)$. Fix such a u and recall from (14.22) that $\widehat{\gamma}_D u_\zeta=\widehat{\gamma}_D u$. Unraveling definitions, the task at hand becomes showing that

$$\begin{aligned}&\tau_z^N u\big|_X = L(\widehat{\gamma}_D u)\\ &\quad\text{if and only if } (w,(-\Delta-z)u)_{L^2(\Omega;d^nx)} = -{}_X\langle\widehat{\gamma}_D w, L(\widehat{\gamma}_D u)\rangle_{X^*}, \quad w\in V.\end{aligned} \tag{14.35}$$

One observes that for every $w\in V\subseteq\ker(-\Delta_{max}-zI_\Omega)$ one has (recalling that $z\in\mathbb{R}$)

$$(w,(-\Delta-z)u)_{L^2(\Omega;d^nx)} = -\overline{{}_{N^{1/2}(\partial\Omega)}\langle\tau_z^N u,\widehat{\gamma}_D w\rangle_{(N^{1/2}(\partial\Omega))^*}} \tag{14.36}$$

by (12.10). Then (14.35) readily follows from this and (14.25). This concludes the proof of (14.15).

Conversely, assume that $\widetilde{S}$, as in (14.17), is a self-adjoint operator which satisfies (14.18). If we define

$$V:=\overline{\{u_\zeta\,|\,u\in\mathrm{dom}(\widetilde{S})\}} \quad \big(\text{with closure in } L^2(\Omega;d^nx)\big), \tag{14.37}$$

then V is a closed subspace of $\ker(-\Delta_{max}-zI_\Omega)$ for which (14.25) continues to hold. Next, one introduces the operator $T:\mathrm{dom}(T)\subseteq V\to V^*$ by setting

$$\begin{aligned}&Tu_\zeta := -(\,\cdot\,,(-\Delta-z)u)_{L^2(\Omega;d^nx)}\in V^*,\\ &u_\zeta\in\mathrm{dom}(T):=\big\{v_\zeta\,\big|\,v\in\mathrm{dom}(\widetilde{S})\big\}.\end{aligned} \tag{14.38}$$

Then Theorem 14.1 ensures that T is a self-adjoint operator.

Next, consider X, $\mathrm{dom}(L)$ as in (14.20), and introduce an operator L as in (14.12) by requiring that

$${}_X\langle\widehat{\gamma}_D v, L(\widehat{\gamma}_D u)\rangle_{X^*} = {}_V\langle v, Tu\rangle_{V^*}, \quad u\in\mathrm{dom}(T),\ v\in V. \tag{14.39}$$

Since

$$\widehat{\gamma}_D: V\to X \ \text{ and } \ \widehat{\gamma}_D:\mathrm{dom}(T)\to\mathrm{dom}(L) \ \text{ isomorphically,} \tag{14.40}$$

the requirement in (14.39) uniquely defines L. Furthermore, for every $u,v\in\mathrm{dom}(T)$, using the self-adjointness of T we may write

$${}_X\langle\widehat{\gamma}_D v, L(\widehat{\gamma}_D u)\rangle_{X^*} = {}_V\langle v,Tu\rangle_{V^*} = \overline{{}_V\langle u,Tv\rangle_{V^*}} = \overline{{}_X\langle\widehat{\gamma}_D u, L(\widehat{\gamma}_D v)\rangle_{X^*}}. \tag{14.41}$$

Together with (14.40), this shows that L is symmetric, that is, $L\subseteq L^*$. To prove the converse inclusion, consider $f\in\mathrm{dom}(L^*)$. Then $f\in X$ and there exists $\Lambda\in X^*$ such that

$${}_X\langle\widehat{\gamma}_D w,\Lambda\rangle_{X^*} = \overline{{}_X\langle f, L(\widehat{\gamma}_D w)\rangle_{X^*}}, \quad w\in\mathrm{dom}(T). \tag{14.42}$$

Let $u\in V$ be such that $\widehat{\gamma}_D u=f$. Upon recalling (14.39), the above formula becomes

$${}_X\langle\widehat{\gamma}_D w,\Lambda\rangle_{X^*} = \overline{{}_V\langle u,Tw\rangle_{V^*}}, \quad w\in\mathrm{dom}(T). \tag{14.43}$$

In particular,

$$\begin{aligned}|{}_V\langle u,Tw\rangle_{V^*}| &\le \|\Lambda\|_{X^*}\|\widehat{\gamma}_D w\|_{(N^{1/2}(\partial\Omega))^*}\\ &\le \|\Lambda\|_{X^*}\|w\|_{L^2(\Omega;d^nx)}, \quad w\in\mathrm{dom}(T).\end{aligned} \tag{14.44}$$

This shows that $u\in\mathrm{dom}(T^*)=\mathrm{dom}(T)$ and hence, $f=\widehat{\gamma}_D u\in\mathrm{dom}(L)$, by (14.40). Altogether, the above argument proves that L is a self-adjoint operator.

Next, we will prove that

$$\mathrm{dom}\big(\widetilde{S}\big)\subseteq\mathrm{dom}(-\Delta^D_{X,L,z}). \tag{14.45}$$

We note that if $u\in\mathrm{dom}\big(\widetilde{S}\big)$ then $\widehat{\gamma}_D u\in\mathrm{dom}(L)$, by definition. Thus, as far as (14.45) is concerned, it remains to verify that (14.14) holds. To see that this is indeed the case, given an arbitrary $f\in X$, pick $w\in V\subseteq\ker(-\Delta_{max}-zI_\Omega)$ such that $\widehat{\gamma}_D w=f$. Then for each $u\in\mathrm{dom}\big(\widetilde{S}\big)$ we may write

$$\begin{aligned}{}_{N^{1/2}(\partial\Omega)}\langle\tau_z^N u,f\rangle_{(N^{1/2}(\partial\Omega))^*} &= {}_{N^{1/2}(\partial\Omega)}\langle\tau_z^N u,\widehat{\gamma}_D w\rangle_{(N^{1/2}(\partial\Omega))^*}\\ &= -((-\Delta-z)u,w)_{L^2(\Omega;d^nx)}\\ &= \overline{{}_V\langle w,Tu_\zeta\rangle_{V^*}}\\ &= \overline{{}_X\langle\widehat{\gamma}_D w, L(\widehat{\gamma}_D u_\zeta)\rangle_{X^*}}\\ &= \overline{{}_X\langle f, L(\widehat{\gamma}_D u)\rangle_{X^*}}.\end{aligned} \tag{14.46}$$

Above, the second equality is a consequence of (12.10), the third equality follows from (14.38), the fourth equality is implied by (14.39), and the fifth equality is derived with the help of the last line in (14.22). This shows that (14.14) holds, thus completing the proof of (14.45).

Since both $\widetilde{S}$ and $-\Delta^D_{X,L,z}$ are self-adjoint, (14.45) implies $\widetilde{S} = -\Delta^D_{X,L,z}$. $\square$

Several distinguished self-adjoint extensions of $-\Delta\big|_{C_0^\infty(\Omega)}$ are singled out below:

Corollary 14.4. *In the context of Theorem* 14.3, *for every* $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$ *one has the following facts:*

$$X := \mathrm{dom}(L) := \{0\} \ \textit{and} \ L := 0 \ \textit{imply} \ -\Delta^D_{X,L,z_0} + z_0 I_\Omega = -\Delta_{D,\Omega}, \tag{14.47}$$

the Dirichlet Laplacian, and

$$\left.\begin{aligned} X := \big(N^{1/2}(\partial\Omega)\big)^* \ \textit{and} \ L := M^{(0)}_{D,N,\Omega}(z_0) \\ \textit{with} \ \mathrm{dom}(L) := N^{3/2}(\partial\Omega) \end{aligned}\right\} \ \textit{imply} \ -\Delta^D_{X,L,z_0} + z_0 I_\Omega = -\Delta_{N,\Omega}, \tag{14.48}$$

the Neumann Laplacian. Furthermore,

$$X := \mathrm{dom}(L) := \big(N^{1/2}(\partial\Omega)\big)^*, \ \textit{and} \ L := 0 \ \textit{imply} \ -\Delta^D_{X,L,z_0} = -\Delta_{K,\Omega,z_0}, \tag{14.49}$$

the Krein Laplacian introduced in (13.3).

Proof. For the case of the Dirichlet Laplacian $-\Delta_{D,\Omega}$, the regularity result in (10.18) of Theorem 10.4 shows that $\mathrm{dom}(-\Delta_{N,\Omega}) = H^2(\Omega) \cap H^1_0(\Omega)$. Thus,

$$\mathrm{dom}(L) := \widehat{\gamma}_D\big(\mathrm{dom}(-\Delta_{D,\Omega})\big) = \{0\}, \tag{14.50}$$

and further, X, the closure of $\mathrm{dom}(L)$ in $\big(N^{1/2}(\partial\Omega)\big)^*$ is the zero vector. Hence, trivially, $L := 0$ is self-adjoint and the boundary condition (14.14) is always satisfied. Finally, the requirement for $u \in \mathrm{dom}(-\Delta_{max})$ that $\widehat{\gamma}_D u \in \mathrm{dom}(L) = \{0\}$ forces, by Theorem 10.4, that $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Consequently, in this case, $\mathrm{dom}(-\Delta^D_{X,L,z_0}) = H^2(\Omega) \cap H^1_0(\Omega)$, and since $-\Delta_{min} \subseteq -\Delta^D_{X,L,z_0} + z_0 I_\Omega$, this proves (14.47).

For the Neumann Laplacian $-\Delta_{N,\Omega}$, the regularity result in (10.44) of Theorem 10.8 shows that, under the current geometrical assumptions, $\mathrm{dom}(-\Delta_{N,\Omega}) = \big\{u \in H^2(\Omega) \,\big|\, \gamma_N u = 0\big\}$. Hence, by Lemma 6.9,

$$\mathrm{dom}(L) := \widehat{\gamma}_D\big(\mathrm{dom}(-\Delta_{N,\Omega})\big) = N^{3/2}(\partial\Omega). \tag{14.51}$$

Moreover, by Lemma 10.10, we have

$$X := \overline{\mathrm{dom}(L)} \quad \big(\text{with closure in } \big(N^{1/2}(\partial\Omega)\big)^*\big), \tag{14.52}$$

so that $X^* = N^{1/2}(\partial\Omega)$, by Lemma 6.2. In this context, the operator

$$L : \mathrm{dom}(L) \subseteq X \to X^*, \quad L = M^{(0)}_{D,N,\Omega}(z_0) \tag{14.53}$$

is well-defined, linear, and symmetric by Theorem 11.1. Proving that L is in fact self-adjoint then requires establishing the following regularity result:

$$f \in \big(N^{1/2}(\partial\Omega)\big)^* \ \text{has} \ M^{(0)}_{D,N,\Omega}(z_0)f \in N^{1/2}(\partial\Omega), \ \text{implying} \ f \in N^{3/2}(\partial\Omega). \tag{14.54}$$

This, however, is a consequence of the fact that the action of $M^{(0)}_{D,N,\Omega}(z_0) \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, (N^{3/2}(\partial\Omega))^*\big)$ is compatible with that of the operator $M^{(0)}_{D,N,\Omega}(z_0) \in \mathcal{B}\big(N^{3/2}(\partial\Omega), N^{1/2}(\partial\Omega)\big)$ and the latter is an isomorphism. This justifies (14.54) which completes the proof of the fact that L in (14.53) is indeed a self-adjoint operator.

Upon recalling that $\tau^N_{z_0} u = \widehat{\gamma}_N u + M^{(0)}_{D,N,\Omega}(z_0)\big(\widehat{\gamma}_D u\big)$, the boundary condition $\tau^N_{z_0} u\big|_X = L(\widehat{\gamma}_D u)$ reduces to $\widehat{\gamma}_N u = 0$. Now consider the requirement for $u \in \mathrm{dom}(-\Delta_{max})$ that $\widehat{\gamma}_D u \in \mathrm{dom}(L) = N^{3/2}(\partial\Omega)$. Our claim is that this is automatically satisfied whenever u fulfills the boundary condition (14.14). Indeed, as just discussed, the latter entails $\widehat{\gamma}_N u = 0$ and hence u belongs to $\big\{u \in H^2(\Omega) \,\big|\, \gamma_N u = 0\big\}$ by the regularity result (10.44) of Theorem 10.8. Then Lemma 6.9 yields $\widehat{\gamma}_D u \in N^{3/2}(\partial\Omega)$, justifying the claim and hence (14.48).

Next, consider the case where

$$X := \mathrm{dom}(L) := \big(N^{1/2}(\partial\Omega)\big)^* \ \text{and} \ L := 0. \tag{14.55}$$

Trivially, L is self-adjoint. Moreover, for a function $u \in \mathrm{dom}(-\Delta_{max})$, the boundary condition $\tau^N_{z_0} u\big|_X = L(\widehat{\gamma}_D u)$ reduces to $\tau^N_{z_0} u = 0$, whereas the requirement that u satisfies $\widehat{\gamma}_D u \in \mathrm{dom}(L)$ becomes superfluous, by Theorem 6.4. Hence, $\mathrm{dom}\big(-\Delta^D_{X,L,z_0}\big) = \mathrm{dom}\big(-\Delta^D_{K,\Omega,z_0}\big)$ and ultimately, $-\Delta^D_{X,L,z_0} = -\Delta^D_{K,\Omega,z_0}$ since they are both contained in $-\Delta_{max} - z_0 I_\Omega$. This yields (14.49) and completes the proof of the corollary. $\square$

We wish to augment Corollary 14.4 by elaborating on connections with the Robin Laplacian (cf. Theorem 5.9).

Corollary 14.5. *Suppose that $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a bounded $C^{1,r}$ domain with $r>1/2$, and assume* (8.41)*. Then, in the context of Theorem* 14.3*, for every $z_0\in\mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$ one has*

$$\left.\begin{array}{c} X:=\big(N^{1/2}(\partial\Omega)\big)^* \text{ and } L:=M^{(0)}_{D,N,\Omega}(z_0)-\theta I_{\partial\Omega} \\ \text{with } \operatorname{dom}(L):=N^{3/2}(\partial\Omega)\end{array}\right\} \text{ imply } -\Delta^D_{X,L,z_0}+z_0I_\Omega=-\Delta_{\Theta,\Omega}, \tag{14.56}$$

the Robin Laplacian.

Proof. The argument is analogous to the proof of Corollary 14.4. This time, we make use of Lemma 8.13 and the fact that, by (6.7) and Lemma 6.8, one has

$$N^{3/2}(\partial\Omega)\hookrightarrow N^{1/2}(\partial\Omega) \tag{14.57}$$

boundedly. □

Our next result elucidates the circumstances under which a given self-adjoint extension of the Laplacian is a nonnegative operator.

Theorem 14.6. *Retaining the context of Theorem* 14.3*, let $z_0\leq 0$. Then*

$$-\Delta^D_{L,X,z_0}\geq 0 \text{ if and only if } L\leq 0. \tag{14.58}$$

Proof. A direct comparison of (9.16) and (14.9) reveals that, if T is as in (14.26), then

$$-\Delta^D_{L,X,z_0}\geq 0 \text{ if and only if } T\geq 0, \text{ equivalently, if and only if } L\leq 0. \tag{14.59}$$

□

15. Self-Adjoint Extensions with the Neumann Laplacian as Reference Operator

Having used the Dirichlet Laplacian as the reference operator in the previous section, we now illustrate the construction of self-adjoint extensions of $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega;d^nx)$, with the (shifted) Neumann Laplacian as the reference operator. (The shift $-\Delta_{N,\Omega}\longrightarrow -\Delta_{N,\Omega}-z_0I_\Omega$ for some $z_0\in\mathbb{R}$ becomes necessary since $0\in\sigma(-\Delta_{N,\Omega})$.)

As a preamble, we now state and prove a result which is the counterpart of Theorem 12.1 pertaining to the regularized version of the Dirichlet trace operator on quasi-convex domains:

Theorem 15.1. *Assume Hypothesis* 8.10*. Then, for every $z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, the map*

$$\tau^D_z:\big\{u\in L^2(\Omega;d^nx)\,\big|\,\Delta u\in L^2(\Omega;d^nx)\big\}\to N^{3/2}(\partial\Omega) \tag{15.1}$$

given by

$$\tau^D_z u:=\widehat{\gamma}_D u-M^{(0)}_{N,D,\Omega}(z)(\widehat{\gamma}_N u), \quad u\in L^2(\Omega;d^nx),\ \Delta u\in L^2(\Omega;d^nx), \tag{15.2}$$

is well-defined, linear, and bounded when the space

$$\big\{u\in L^2(\Omega;d^nx)\,\big|\,\Delta u\in L^2(\Omega;d^nx)\big\}=\operatorname{dom}(-\Delta_{max}) \tag{15.3}$$

is endowed with the natural graph norm $u\mapsto\|u\|_{L^2(\Omega;d^nx)}+\|\Delta u\|_{L^2(\Omega;d^nx)}$. Moreover, this operator satisfies the following additional properties:

(i) For each $z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, the map τ^D_z in (15.1), (15.2) *is onto (i.e., $\tau^D_z(\operatorname{dom}(-\Delta_{max}))=N^{3/2}(\partial\Omega)$). In fact,*

$$\tau^D_z\big(\big\{u\in H^2(\Omega)\,\big|\,\gamma_N u=0\big\}\big)=N^{1/2}(\partial\Omega), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}). \tag{15.4}$$

(ii) For each $z\in\mathbb{C}\backslash\big[\sigma(-\Delta_{D,\Omega})\cup\sigma(-\Delta_{N,\Omega})\big]$, one has

$$\tau^D_z=-M^{(0)}_{N,D,\Omega}(z)\,\tau^N_z. \tag{15.5}$$

(iii) One has

$$\tau^D_z=\gamma_D(-\Delta_{N,\Omega}-zI_\Omega)^{-1}(-\Delta-z), \quad z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega}). \tag{15.6}$$

(iv) For each $z\in\mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, the kernel of the map τ^D_z in (15.1), (15.2) *is given by*

$$\ker\big(\tau^D_z\big)=H^2_0(\Omega)\dotplus\big\{u\in L^2(\Omega;d^nx)\,\big|\,(-\Delta-z)u=0 \text{ in } \Omega\big\}. \tag{15.7}$$

In particular,

$$\tau^D_z u=0 \text{ for every } u\in\ker(-\Delta_{max}-zI_\Omega). \tag{15.8}$$

(*v*) *For each* $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, *the following Green formula holds for every* $u, v \in \mathrm{dom}(-\Delta_{max})$,

$$\begin{aligned}&((-\Delta - z)u, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})v)_{L^2(\Omega;d^nx)} \\ &\quad = -{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_z^D u, \widehat{\gamma}_N v\big\rangle_{(N^{3/2}(\partial\Omega))^*} + \overline{{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_{\overline{z}}^D v, \widehat{\gamma}_N u\big\rangle_{(N^{3/2}(\partial\Omega))^*}}. \end{aligned} \tag{15.9}$$

In particular, for every $u, v \in \mathrm{dom}(-\Delta_{max})$, $z \in \mathbb{C}$, *and* $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{N,\Omega})$, *one has*

$$\begin{aligned}&((-\Delta - (z+z_0))u, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - (\overline{z}+z_0))v)_{L^2(\Omega;d^nx)} \\ &\quad = -{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_{z_0}^D u, \widehat{\gamma}_N v\big\rangle_{(N^{3/2}(\partial\Omega))^*} + \overline{{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_{z_0}^D v, \widehat{\gamma}_N u\big\rangle_{(N^{3/2}(\partial\Omega))^*}}. \end{aligned} \tag{15.10}$$

Moreover, as a consequence of (15.9) *and* (15.8), *for every* $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$ *one infers that*

$$\begin{aligned}&u \in \mathrm{dom}(-\Delta_{max}) \text{ and } v \in \ker(-\Delta_{max} - \overline{z}I_\Omega) \\ &\quad \text{imply } ((-\Delta - z)u, v)_{L^2(\Omega;d^nx)} = -{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_z^D u, \widehat{\gamma}_N v\big\rangle_{(N^{3/2}(\partial\Omega))^*}. \end{aligned} \tag{15.11}$$

Proof. Let $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$. Consider an arbitrary $u \in L^2(\Omega; d^nx)$ satisfying $\Delta u \in L^2(\Omega; d^nx)$ and let v solve

$$(-\Delta - z)v = 0 \text{ in } \Omega, \quad v \in L^2(\Omega; d^nx), \quad \widehat{\gamma}_N v = \widehat{\gamma}_N u \text{ on } \partial\Omega. \tag{15.12}$$

Theorems 6.10 and 10.8 ensure that this is possible and guarantee the the existence of a finite constant $C = C(\Omega, z) > 0$ for which

$$\|v\|_{L^2(\Omega;d^nx)} \leq C\big(\|u\|_{L^2(\Omega;d^nx)} + \|\Delta u\|_{L^2(\Omega;d^nx)}\big). \tag{15.13}$$

Then $w := u - v$ satisfies

$$(-\Delta - z)w = (-\Delta - z)u \in L^2(\Omega; d^nx), \quad w \in L^2(\Omega; d^nx), \quad \widehat{\gamma}_N w = 0 \text{ on } \partial\Omega, \tag{15.14}$$

so that, by (10.44), there exists $C = C(\Omega, z) > 0$ such that

$$w \in H^2(\Omega) \text{ and } \|w\|_{H^2(\Omega)} \leq C\big(\|u\|_{L^2(\Omega;d^nx)} + \|\Delta u\|_{L^2(\Omega;d^nx)}\big). \tag{15.15}$$

Since $M_{N,D\Omega}^{(0)}(z)\big(\widehat{\gamma}_N u\big) = \widehat{\gamma}_D v$, it follows from (15.15), the compatibility part of Theorem 6.4, and Lemma 6.9 that

$$\tau_z^D u = \widehat{\gamma}_D u - \widehat{\gamma}_D v = \widehat{\gamma}_D(u - v) = \widehat{\gamma}_D w = \gamma_D w \in N^{3/2}(\partial\Omega). \tag{15.16}$$

Moreover,

$$\begin{aligned}\big\|\tau_z^D u\big\|_{N^{3/2}(\partial\Omega)} &= \|\gamma_D w\|_{N^{3/2}(\partial\Omega)} \leq C\|w\|_{H^2(\Omega)} \\ &\leq C(\|u\|_{L^2(\Omega;d^nx)} + \|\Delta u\|_{L^2(\Omega;d^nx)}). \end{aligned} \tag{15.17}$$

This shows that the operator τ_z^D in (15.1), (15.2) is indeed well-defined and bounded.

Incidentally, the above argument also shows that (15.6) holds. That τ_z^D defined in (15.1), (15.2) is onto, is a direct consequence of the fact that γ_D in (6.41) is onto (cf. Lemma 6.9). This also justifies (15.4). Next, (15.5) is a direct consequence of (11.9).

As previously remarked, the sum in (12.6) is direct. Moreover, $H_0^2(\Omega) \subseteq \ker(\tau_z^D)$ by (15.2) and $\big\{u \in L^2(\Omega; d^nx) \,\big|\, (-\Delta - z)u = 0 \text{ in } \Omega\big\} \subseteq \ker\big(\tau_z^D\big)$ by (15.6). This proves the right-to-left inclusion in (15.7). To prove the opposite one, consider $u \in L^2(\Omega; d^nx)$ with $\Delta u \in L^2(\Omega; d^nx)$ for which $\tau_z^D u = 0$. If we now set $w := (-\Delta_{N,\Omega} - zI_\Omega)^{-1}(-\Delta - z)u$, then $w \in H^2(\Omega)$, $\gamma_N w = 0$ and $\gamma_D w = \tau_z^D u = 0$ by (15.6). Hence, $w \in H_0^2(\Omega)$ by Theorem 8.14. Since $u = w + (u - w)$ and $u - w \in L^2(\Omega; d^nx)$ is harmonic, the proof of (15.7) is complete.

Next, consider the Green formulas in (v). Fix $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, let $u, v \in \mathrm{dom}(-\Delta_{min})$, and set $\widetilde{u} := (-\Delta_{N,\Omega} - z)^{-1}(-\Delta - z)u$, $\widetilde{v} := (-\Delta_{N,\Omega} - \overline{z})^{-1}(-\Delta - \overline{z})v$. Then

$$\begin{aligned}&\widetilde{u}, \widetilde{v} \in H^2(\Omega), \quad \gamma_N\widetilde{u} = \gamma_N\widetilde{v} = 0, \\ &(-\Delta - z)\widetilde{u} = (-\Delta - z)u, \quad (-\Delta - \overline{z})\widetilde{v} = (-\Delta - \overline{z})v, \\ &\gamma_D\widetilde{u} = \tau_z^D u, \quad \gamma_D\widetilde{v} = \tau_{\overline{z}}^D v \end{aligned} \tag{15.18}$$

by (15.6). Based on these observations and repeated applications of Green's formula (6.46) one can then write

$$\begin{aligned}&((-\Delta - z)u, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})v)_{L^2(\Omega;d^nx)} \\ &\quad = ((-\Delta - z)\widetilde{u}, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})\widetilde{v})_{L^2(\Omega;d^nx)} \end{aligned}$$

$$
\begin{aligned}
&= (\widetilde{u}, (-\Delta - \overline{z})v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - \overline{z})\widetilde{v})_{L^2(\Omega;d^nx)} \\
&\quad - {}_{N^{3/2}(\partial\Omega)}\langle \gamma_D \widetilde{u}, \widehat{\gamma}_N v\rangle_{(N^{3/2}(\partial\Omega))^*} \\
&= \overline{([\widetilde{u} - u], (-\Delta - z)\widetilde{v})_{L^2(\Omega;d^nx)}} - {}_{N^{3/2}(\partial\Omega)}\langle \tau_0^D u, \widehat{\gamma}_N v\rangle_{(N^{3/2}(\partial\Omega))^*} \\
&= \overline{{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_0^D v, \widehat{\gamma}_N u\big\rangle_{(N^{3/2}(\partial\Omega))^*}} - {}_{N^{3/2}(\partial\Omega)}\big\langle \tau_0^D u, \widehat{\gamma}_N v\big\rangle_{(N^{3/2}(\partial\Omega))^*},
\end{aligned} \tag{15.19}
$$

where in the last step we have used the fact that $(-\Delta - z)(\widetilde{u} - u) = 0$ and $\widehat{\gamma}_N(\widetilde{u} - u) = -\widehat{\gamma}_N u$. This justifies (15.9) and completes the proof of the theorem. ☐

We are now ready to implement Theorem 14.1 in the case where the reference operator is the (shifted) Neumann Laplacian. Specifically, we have the following result:

Theorem 15.2. *Assume Hypothesis* 8.10 *and let* $z \in \mathbb{R}\backslash\sigma(-\Delta_{N,\Omega})$*. Suppose that* X *is a closed subspace of* $\big(N^{3/2}(\partial\Omega)\big)^*$ *and denote by* X^* *the conjugate dual space of* X*. In addition, consider a self-adjoint operator*

$$L : \mathrm{dom}(L) \subseteq X \to X^*, \tag{15.20}$$

and define the linear operator $-\Delta^N_{X,L,z} : \mathrm{dom}(-\Delta^N_{X,L,z}) \subseteq L^2(\Omega; d^nx) \to L^2(\Omega; d^nx)$ *by*

$$
\begin{aligned}
&-\Delta^N_{X,L,z} u := (-\Delta - z)u, \\
&u \in \mathrm{dom}(-\Delta^N_{X,L,z}) := \big\{v \in \mathrm{dom}(-\Delta_{max}) \,\big|\, \widehat{\gamma}_N v \in \mathrm{dom}(L),\ \tau_z^D v|_X = L(\widehat{\gamma}_N v)\big\},
\end{aligned} \tag{15.21}
$$

where τ_z^D *is the map* (15.1), (15.2)*, and the boundary condition* $\tau_z^D u|_X = L(\widehat{\gamma}_N u)$ *is interpreted as*

$${}_{N^{3/2}(\partial\Omega)}\big\langle \tau_z^D u, f\big\rangle_{(N^{3/2}(\partial\Omega))^*} = \overline{{}_X\big\langle f, L(\widehat{\gamma}_N u)\big\rangle_{X^*}}, \quad f \in X. \tag{15.22}$$

Then

$$-\Delta^N_{X,L,z} \text{ is self-adjoint in } L^2(\Omega; d^nx), \tag{15.23}$$

and

$$-\Delta_{min} - zI_\Omega \subseteq -\Delta^N_{X,L,z} \subseteq -\Delta_{max} - zI_\Omega. \tag{15.24}$$

Conversely, if

$$\widetilde{S} : \mathrm{dom}\big(\widetilde{S}\big) \subseteq L^2(\Omega; d^nx) \to L^2(\Omega; d^nx) \tag{15.25}$$

is a self-adjoint operator with the property that

$$-\Delta_{min} - zI_\Omega \subseteq \widetilde{S} \subseteq -\Delta_{max} - zI_\Omega, \tag{15.26}$$

then there exists X*, a closed subspace of* $\big(N^{3/2}(\partial\Omega)\big)^*$*, and* $L : \mathrm{dom}(L) \subseteq X \to X^*$*, a self-adjoint operator, such that*

$$\widetilde{S} = -\Delta^N_{X,L,z}. \tag{15.27}$$

In the above scheme, the operator $\widetilde{S}$ *and the pair* X, L *correspond uniquely to each other. In fact,*

$$\mathrm{dom}(L) = \widehat{\gamma}_N\big(\mathrm{dom}\big(\widetilde{S}\big)\big), \quad X = \overline{\widehat{\gamma}_N\big(\mathrm{dom}\big(\widetilde{S}\big)\big)} \quad \big(\textit{with closure in } \big(N^{3/2}(\partial\Omega)\big)^*\big). \tag{15.28}$$

Proof. Fix $z \in \mathbb{R}\backslash\sigma(-\Delta_{N,\Omega})$. We will employ Theorem 14.1 in the following context:

$$A_0 := -\Delta_{min} - zI_\Omega, \quad A_1 := -\Delta_{max} - zI_\Omega, \quad A_\beta := -\Delta_{N,\Omega} - zI_\Omega. \tag{15.29}$$

Throughout the proof, we shall make use of the following results and notation:

$$
u \in \mathrm{dom}(-\Delta_{max}) \text{ implies } \begin{cases}
u_\beta := (-\Delta_{N,\Omega} - zI_\Omega)^{-1}(-\Delta - z)u \in H^2(\Omega),\ \widehat{\gamma}_N u_\beta = 0, \\
u_\zeta := u - (-\Delta_{N,\Omega} - zI_\Omega)^{-1}(-\Delta - z)u \in \ker(-\Delta_{max} - zI_\Omega), \\
\widehat{\gamma}_D u_\zeta = \tau_z^D u \in N^{3/2}(\partial\Omega), \\
\widehat{\gamma}_N u_\zeta = \widehat{\gamma}_N u \in \big(N^{3/2}(\partial\Omega)\big)^*.
\end{cases} \tag{15.30}
$$

We now assume that a closed subspace X of $\big(N^{3/2}(\partial\Omega)\big)^*$ has been specified, that L is a self-adjoint operator as in (15.20), and that $-\Delta^N_{X,L}$ is defined as in (15.21). We will show that $-\Delta^N_{X,L}$ is a self-adjoint operator. To this end, we define

$$V := \big\{u \in \ker(-\Delta_{max} - zI_\Omega) \,\big|\, \widehat{\gamma}_N u \in X\big\}, \tag{15.31}$$

and observe that

$$V = \ker(-\Delta_{max} - zI_\Omega) \cap \widehat{\gamma}_N^{-1}(X) \text{ is a closed subspace of } \ker(-\Delta_{max}), \tag{15.32}$$

and

$$\widehat{\gamma}_N : V \to X \text{ isomorphically, with } \widehat{\gamma}_N \in \mathcal{B}(V,X) \text{ and } \widehat{\gamma}_N^{-1} \in \mathcal{B}(X,V), \tag{15.33}$$

by (15.31) and Corollary 10.7.

Next, we also introduce the operator $T : \mathrm{dom}(T) \subseteq V \to V^*$ by setting

$$\begin{aligned} &Tu := -{}_X\langle \widehat{\gamma}_D(\cdot), L(\widehat{\gamma}_N u)\rangle_{X^*}, \\ &u \in \mathrm{dom}(T) := \{v \in V \,|\, \widehat{\gamma}_N v \in \mathrm{dom}(L)\}. \end{aligned} \tag{15.34}$$

For every $u, v \in \mathrm{dom}(T)$, using the self-adjointness of L, we may write

$$\begin{aligned} {}_V\langle v, Tu\rangle_{V^*} &= -{}_X\langle \widehat{\gamma}_N v, L(\widehat{\gamma}_N u)\rangle_{X^*} = -\overline{{}_X\langle \widehat{\gamma}_N u, L(\widehat{\gamma}_N v)\rangle_{X^*}} \\ &= \overline{{}_V\langle u, Tv\rangle_{V^*}}. \end{aligned} \tag{15.35}$$

This shows that T is symmetric, that is, $T \subseteq T^*$. To prove the converse inclusion, consider $u \in \mathrm{dom}(T^*)$. Then $u \in V$ and there exists $\Lambda \in V^*$ such that

$${}_V\langle w, \Lambda\rangle_{V^*} = \overline{{}_V\langle u, Tw\rangle_{V^*}} = -\overline{{}_X\langle \widehat{\gamma}_N u, L(\widehat{\gamma}_N w)\rangle_{X^*}}, \quad w \in \mathrm{dom}(T). \tag{15.36}$$

Our goal is to show that $u \in \mathrm{dom}(T)$ which, by (15.34), comes down to proving that $\widehat{\gamma}_N u \in \mathrm{dom}(L)$, or equivalently, that $\widehat{\gamma}_N u \in \mathrm{dom}(L^*)$. In turn, the veracity of the latter condition is established as soon as we show that there exists a finite constant $C > 0$ with the property that

$$\big|{}_X\langle \widehat{\gamma}_N u, L(f)\rangle_{X^*}\big| \leq C\|f\|_X, \quad f \in \mathrm{dom}(L). \tag{15.37}$$

Since (15.33) entails that

$$\widehat{\gamma}_N : \mathrm{dom}(T) \to \mathrm{dom}(L) \text{ boundedly, with a bounded inverse,} \tag{15.38}$$

the estimate (15.37) is going to be implied by

$$\big|{}_X\langle \widehat{\gamma}_N u, L(\widehat{\gamma}_N w)\rangle_{X^*}\big| \leq C\|w\|_V, \quad w \in \mathrm{dom}(T). \tag{15.39}$$

This, however, for the choice $C = \|\Lambda\|_{V^*}$, is a direct consequence of (15.36). In summary, the above reasoning yields that $u \in \mathrm{dom}(T^*)$, completing the proof of the fact that T is self-adjoint.

With this at hand, Theorem 14.1 will imply that $-\Delta^N_{X,L,z}$ is self-adjoint as soon as we establish that

$$\begin{aligned} \mathrm{dom}(-\Delta^N_{X,L,z}) = \big\{&u \in \mathrm{dom}(-\Delta_{max}) \,\big|\, u_\zeta \in \mathrm{dom}(T), \\ &(w, (-\Delta - z)u)_{L^2(\Omega;d^nx)} = {}_V\langle w, Tu_\zeta\rangle_{V^*},\, w \in V\big\}. \end{aligned} \tag{15.40}$$

We note that for each $u \in \mathrm{dom}(-\Delta_{max})$,

$$u_\zeta \in \mathrm{dom}(T) \text{ if and only if } \widehat{\gamma}_N u \in \mathrm{dom}(L) \tag{15.41}$$

by (15.30). Consequently, it remains to show that

$$\tau_z^D u\big|_X = L\big(\widehat{\gamma}_D u\big) \text{ if and only if } (w, (-\Delta - z)u)_{L^2(\Omega;d^nx)} = {}_V\langle w, Tu_\zeta\rangle_{V^*}, \quad w \in V, \tag{15.42}$$

whenever $u \in \mathrm{dom}(-\Delta_{max})$ has $\widehat{\gamma}_N u \in \mathrm{dom}(L)$. Fix such a u and recall from (15.30) that $\widehat{\gamma}_N u_\zeta = \widehat{\gamma}_N u$. Unraveling definitions, the task at hand becomes showing that

$$\begin{aligned} &\tau_z^D u\big|_X = L\big(\widehat{\gamma}_N u\big) \\ &\quad \text{if and only if } (w, (-\Delta - z)u)_{L^2(\Omega;d^nx)} = -{}_X\langle \widehat{\gamma}_N w, L(\widehat{\gamma}_N u)\rangle_{X^*}, \quad w \in V. \end{aligned} \tag{15.43}$$

One observes, however, that for every $w \in V \subseteq \ker(-\Delta_{max} - zI_\Omega)$ one has (recalling $z \in \mathbb{R}$)

$$(w, (-\Delta - z)u)_{L^2(\Omega;d^nx)} = -\overline{{}_{N^{3/2}(\partial\Omega)}\big\langle \tau_z^D u, \widehat{\gamma}_N w\big\rangle_{(N^{3/2}(\partial\Omega))^*}} \tag{15.44}$$

by (15.11). Then (15.43) readily follows from this and (15.33). This concludes the proof of (15.23).

Conversely, assume that $\widetilde{S}$, as in (15.25), is a self-adjoint operator which satisfies (15.26). If we define

$$V := \overline{\big\{u_\zeta \,\big|\, u \in \mathrm{dom}\big(\widetilde{S}\big)\big\}} \quad \big(\text{with closure in } L^2(\Omega; d^nx)\big), \tag{15.45}$$

then V is a closed subspace of $\ker(-\Delta_{max} - zI_\Omega)$ for which (15.33) continues to hold. Next, one introduces the operator $T : \mathrm{dom}(T) \subseteq V \to V^*$ by setting

$$\begin{aligned} &Tu_\zeta := -(\,\cdot\,, (-\Delta - z)u)_{L^2(\Omega;d^nx)} \in V^*, \\ &u_\zeta \in \mathrm{dom}(T) := \big\{v_\zeta \,\big|\, v \in \mathrm{dom}\big(\widetilde{S}\big)\big\}. \end{aligned} \tag{15.46}$$

Then Theorem 14.1 ensures that T is a self-adjoint operator.

Next, consider X, $\mathrm{dom}(L)$ as in (15.28), and introduce an operator L as in (15.20) by requiring that

$$ {}_X\langle\widehat{\gamma}_N v, L(\widehat{\gamma}_N u)\rangle_{X^*} = {}_V\langle v, Tu\rangle_{V^*}, \quad u\in\mathrm{dom}(T),\ v\in V. \tag{15.47}$$

Since

$$\widehat{\gamma}_N : V\to X \ \text{ and } \ \widehat{\gamma}_N : \mathrm{dom}(T)\to\mathrm{dom}(L) \ \text{ isomorphically}, \tag{15.48}$$

the requirement in (15.47) uniquely defines L. Furthermore, for every $u,v\in\mathrm{dom}(T)$, using the self-adjointness of T we may write

$$ {}_X\langle\widehat{\gamma}_N v, L(\widehat{\gamma}_N u)\rangle_{X^*} = {}_V\langle v, Tu\rangle_{V^*} = \overline{{}_V\langle u, Tv\rangle_{V^*}} = \overline{{}_X\langle\widehat{\gamma}_N u, L(\widehat{\gamma}_N v)\rangle_{X^*}}. \tag{15.49}$$

Together with (15.48), this shows that L is symmetric, that is, $L\subseteq L^*$. To prove the converse inclusion, consider $f\in\mathrm{dom}(L^*)$. Then $f\in X$ there exists $\Lambda\in X^*$ such that

$$ {}_X\langle\widehat{\gamma}_N w, \Lambda\rangle_{X^*} = \overline{{}_X\langle f, L(\widehat{\gamma}_N w)\rangle_{X^*}}, \quad w\in\mathrm{dom}(T). \tag{15.50}$$

Let $u\in V$ be such that $\widehat{\gamma}_N u = f$. Upon recalling (15.47), the above formula becomes

$$ {}_X\langle\widehat{\gamma}_N w, \Lambda\rangle_{X^*} = \overline{{}_V\langle u, Tw\rangle_{V^*}}, \quad w\in\mathrm{dom}(T). \tag{15.51}$$

In particular,

$$\begin{aligned} |{}_V\langle u, Tw\rangle_{V^*}| &\le \|\Lambda\|_{X^*}\|\widehat{\gamma}_D w\|_{(N^{1/2}(\partial\Omega))^*} \\ &\le \|\Lambda\|_{X^*}\|w\|_{L^2(\Omega;d^nx)}, \quad w\in\mathrm{dom}(T). \end{aligned} \tag{15.52}$$

This shows that $u\in\mathrm{dom}(T^*)=\mathrm{dom}(T)$ and hence, $f=\widehat{\gamma}_D u\in\mathrm{dom}(L)$, by (15.48). Altogether, the above argument proves that L is a self-adjoint operator.

Next, we will prove that

$$\mathrm{dom}\big(\widetilde{S}\big)\subseteq\mathrm{dom}(-\Delta^D_{X,L,z}). \tag{15.53}$$

We note that if $u\in\mathrm{dom}\big(\widetilde{S}\big)$ then $\widehat{\gamma}_D u\in\mathrm{dom}(L)$, by definition. Thus, as far as (15.53) is concerned, it remains to verify that (15.22) holds. To see that this is indeed the case, given an arbitrary $f\in X$, pick $w\in V\subseteq\ker(-\Delta_{max}-zI_\Omega)$ such that $\widehat{\gamma}_N w = f$. Then for each $u\in\mathrm{dom}\big(\widetilde{S}\big)$ we may write

$$\begin{aligned} {}_{N^{3/2}(\partial\Omega)}\langle\tau^D_z u, f\rangle_{(N^{3/2}(\partial\Omega))^*} &= {}_{N^{3/2}(\partial\Omega)}\langle\tau^D_z u, \widehat{\gamma}_N w\rangle_{(N^{3/2}(\partial\Omega))^*} \\ &= -((-\Delta-z)u, w)_{L^2(\Omega;d^nx)} \\ &= \overline{{}_V\langle w, Tu_\zeta\rangle_{V^*}} \\ &= \overline{{}_X\langle\widehat{\gamma}_N w, L(\widehat{\gamma}_N u_\zeta)\rangle_{X^*}} \\ &= \overline{{}_X\langle f, L(\widehat{\gamma}_N u)\rangle_{X^*}}. \end{aligned} \tag{15.54}$$

Above, the second equality is a consequence of (15.11), the third equality follows from (15.46), the fourth equality is implied by (15.47), and the fifth equality is derived with the help of the last line in (15.30). This shows that (15.22) holds, thus completing the proof of (15.53).

Since both $\widetilde{S}$ and $-\Delta^N_{X,L,z}$ are self-adjoint, (15.53) implies $\widetilde{S}=-\Delta^N_{X,L,z}$. □

The following lemma is useful in the statement of Corollary 15.4 below.

Lemma 15.3. *Assume Hypothesis* 8.10. *Then the operator-valued map*

$$\mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})\ni z\mapsto -\Delta_{K,\Omega,z} \tag{15.55}$$

extends naturally (*i.e., with preservations of properties stated in Theorem* 13.1) *to the larger domain*

$$\mathbb{C}\backslash\big[\sigma(-\Delta_{D,\Omega})\cap\sigma(-\Delta_{N,\Omega})\big]\ni z\mapsto -\Delta_{K,\Omega,z}. \tag{15.56}$$

Proof. Consider the linear operator in $L^2(\Omega;d^nx)$ given by

$$\begin{aligned} &-\widetilde{\Delta}_{K,\Omega,z}(u) := (-\Delta-z)u, \\ &u\in\mathrm{dom}(-\widetilde{\Delta}_{K,\Omega,z}) := \big\{v\in\mathrm{dom}(-\Delta_{max})\,\big|\,\tau^D_z v = 0\big\}, \end{aligned} \tag{15.57}$$

In a similar fashion to Theorem 13.1 it can be shown that $-\widetilde{\Delta}_{K,\Omega,z}$ satisfies

$$\big(-\widetilde{\Delta}_{K,\Omega,z}\big)^* = -\widetilde{\Delta}_{K,\Omega,\overline{z}}. \tag{15.58}$$

Moreover, if $z\in\mathbb{R}\backslash\sigma(-\Delta_{N,\Omega})$, then $-\Delta_{K,\Omega,z}$ is self-adjoint, and if $z<0$ then $-\Delta_{K,\Omega,z}\ge 0$. Finally,

$$-\Delta_{min}\subseteq -\Delta_{K,\Omega,z}+zI_\Omega\subseteq -\Delta_{max}. \tag{15.59}$$

Since, by (12.6) and (15.7), $\ker(\tau_z^D) = \ker(\tau_z^N)$ whenever $z \in \mathbb{C}\backslash\big[\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{N,\Omega})\big]$, one obtains that $-\widetilde{\Delta}_{K,\Omega,z} = -\Delta_{K,\Omega,z}$ for $z \in \mathbb{C}\backslash\big[\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{N,\Omega})\big]$. Since by (12.2) and the representation of $M_{D,N,\Omega}^{(0)}(z)$ in (11.6), τ_z^N is analytic for $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, and similarly, by (15.2) and the representation of $M_{N,D,\Omega}^{(0)}(z)$ in (11.8), τ_z^D is analytic for $z \in \mathbb{C}\backslash\sigma(-\Delta_{N,\Omega})$, it follows that the map (15.55) extends to $\mathbb{C}\backslash\big[\sigma(-\Delta_{D,\Omega}) \cap \sigma(-\Delta_{N,\Omega})\big]$, as claimed in (15.56). $\square$

Analogously to the proof of Corollary 14.4, it is then straightforward to establish the following result:

Corollary 15.4. *In the context of Theorem* 15.2, *for every $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{N,\Omega})$ one has the following facts:*

$$X := \mathrm{dom}(L) := \{0\} \ \text{ and } \ L := 0 \ \text{ imply } \ -\Delta_{X,L,z_0}^N + z_0 I_\Omega = -\Delta_{N,\Omega}, \tag{15.60}$$

the Neumann Laplacian, and

$$\left.\begin{array}{c} X := \big(N^{3/2}(\partial\Omega)\big)^* \ \text{ and } L := -M_{N,D,\Omega}^{(0)}(z_0) \\ \text{with } \mathrm{dom}(L) := N^{1/2}(\partial\Omega) \end{array}\right\} \ \text{imply } \ -\Delta_{X,L,z_0}^N + z_0 I_\Omega = -\Delta_{D,\Omega}, \tag{15.61}$$

the Dirichlet Laplacian. Furthermore,

$$X := \mathrm{dom}(L) := \big(N^{3/2}(\partial\Omega)\big)^* \ \text{ and } \ L := 0 \ \text{ imply } \ -\Delta_{X,L,z_0}^N = -\Delta_{K,\Omega,z_0}, \tag{15.62}$$

the Krein Laplacian (*initially introduced in* (13.3)*, and further extended in Lemma* 15.3)*.*

16. Krein-Type Resolvent Formulas

Having catalogued all self-adjoint extensions of the Laplacian, we establish in this section a variety of Krein-type resolvent formulas, which express the difference of two resolvents for two different self-adjoint extensions of the Laplacian as R^*MR, where M is a suitable Weyl–Titchmarsh operator on the boundary and R is closely related to one of the resolvents in question.

Krein-type resolvent formulas have been studied in a great variety of contexts, far too numerous to account for all in this paper. For instance, they are of fundamental importance in connection with the spectral and inverse spectral theory of ordinary and partial differential operators. Abstract versions of Krein-type resolvent formulas (see also the brief discussion at the end of our introduction), connected to boundary value spaces (boundary triples) and self-adjoint extensions of closed symmetric operators with equal (possibly infinite) deficiency spaces, have received enormous attention in the literature. In particular, we note that Robin-to-Dirichlet maps in the context of ordinary differential operators reduce to the celebrated (possibly, matrix-valued) Weyl–Titchmarsh function, the basic object of spectral analysis in this context. Since it is impossible to cover the literature in this paper, we refer to the rather extensive recent bibliography in [53] and [54]. Here we mention, for instance, [5, Sect. 84], [6], [7], [9], [15], [16], [22], [23], [24], [25], [28], [29], [31], [32], [33], [34], [35], [37], [38], [39], [40], [50], [51], [52], [56], [57], [58, Ch. 3], [64], [65, Ch. 13], [74], [77], [78], [79], [80], [81], [82], [85], [86], [87], [93], [98], [99], [100], [101], [102], [103], [104], [105], [106], [109], [110], [111], [112], [117], [118], [119], [125], and the references cited therein. We add, however, that the case of infinite deficiency indices in the context of partial differential operators (in our concrete case, related to the deficiency indices of the operator closure of $-\Delta\restriction_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$), is much less studied and the results obtained in this section, especially, under the assumption of quasi-convex domains are new.

We start with a couple of preliminary results, contained in the next two lemmas:

Lemma 16.1. *Assume Hypothesis* 8.10 *and suppose that $z_0 \in \mathbb{R}$, $z_0, (z+z_0) \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Let X be a closed subspace of $\big(N^{1/2}(\partial\Omega)\big)^*$ and denote by X^* the conjugate dual space of X. In addition, consider a self-adjoint operator L as in* (14.12) *and define the self-adjoint operator $-\Delta_{X,L,z_0}^D$ as in* (14.13) *corresponding to $z = z_0$. Then the following resolvent relation holds on $L^2(\Omega; d^n x)$,*

$$\begin{aligned} \big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1} &= \big(-\Delta_{D,\Omega} - (z+z_0)I_\Omega\big)^{-1} \\ &\quad + \big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* \big[\tau_{z_0}^N(-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1}\big], \\ &\qquad z \in \mathbb{C}\backslash\sigma(-\Delta_{X,L,z_0}^D). \end{aligned} \tag{16.1}$$

Proof. We first assume that $z \in \mathbb{C}\backslash\mathbb{R}$.

To set the stage, we recall that

$$(-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \mathrm{dom}(-\Delta_{max}), \tag{16.2}$$

$$\tau_{z_0}^N : \mathrm{dom}(-\Delta_{max}) \to N^{1/2}(\partial\Omega), \tag{16.3}$$

$$(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} : L^2(\Omega;d^nx) \to \mathrm{dom}(-\Delta_{max}), \tag{16.4}$$

$$\widehat{\gamma}_D : \mathrm{dom}(-\Delta_{max}) \to \big(N^{1/2}(\partial\Omega)\big)^*, \tag{16.5}$$

are bounded, linear operators. These facts imply

$$\tau_{z_0}^N(-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega;d^nx), N^{1/2}(\partial\Omega)\big), \tag{16.6}$$

$$\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega;d^nx), \big(N^{1/2}(\partial\Omega)\big)^*\big), \tag{16.7}$$

$$\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), L^2(\Omega;d^nx)\big), \tag{16.8}$$

which ensures that the composition of operators appearing on the right-hand side of (16.1) is meaningful. Next, let $u_1, v_1 \in L^2(\Omega;d^nx)$ be arbitrary and define

$$\begin{aligned} u &:= (-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}u_1 \in \mathrm{dom}(-\Delta_{X,L,z_0}^D) \subset \mathrm{dom}(-\Delta_{max}), \\ v &:= (-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1}v_1 \in \mathrm{dom}(-\Delta_{D,\Omega}) = H^2(\Omega)\cap H_0^1(\Omega). \end{aligned} \tag{16.9}$$

Checking (16.1) then reduces to showing that the following identity holds:

$$\begin{aligned} &\big(u_1, \big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1}v_1\big)_{L^2(\Omega;d^nx)} - (u_1, (-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1}v_1)_{L^2(\Omega;d^nx)} \\ &\quad = \big(u_1, \big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^*\big[\tau_{z_0}^N(-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1}\big]v_1\big)_{L^2(\Omega;d^nx)}. \end{aligned} \tag{16.10}$$

We note that according to (16.9) one has,

$$\begin{aligned} (u_1, (-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1}v_1)_{L^2(\Omega;d^nx)} &= ((-\Delta_{D,\Omega} - (\overline{z}+z_0)I_\Omega)u, v)_{L^2(\Omega;d^nx)}, \\ \big(v_1, \big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1}v_1\big)_{L^2(\Omega;d^nx)} &= \big(\big(\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1}\big)^*u_1, v_1\big)_{L^2(\Omega;d^nx)} \\ &= \big(\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}u_1, v_1\big)_{L^2(\Omega;d^nx)} \\ &= (u, (-\Delta_{D,\Omega} - (z+z_0)I_\Omega)v)_{L^2(\Omega;d^nx)}, \end{aligned} \tag{16.11}$$

and

$$= \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N(-\Delta_{D,\Omega} - (z+z_0)I_\Omega)^{-1}v_1, \widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}u_1\big\rangle_{(N^{1/2}(\partial\Omega))^*}}$$

$$= \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, \widehat{\gamma}_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}}.$$

With this, matters have been reduced to proving that

$$\begin{aligned} &\big(\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)u, v\big)_{L^2(\Omega;d^nx)} - (u, (-\Delta_{D,\Omega} - (z+z_0)I_\Omega)v)_{L^2(\Omega;d^nx)} \\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, \widehat{\gamma}_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}}, \end{aligned} \tag{16.12}$$

or equivalently, to

$$\begin{aligned} &((-\Delta - z_0)u, v)_{L^2(\Omega;d^nx)} - (u, (-\Delta - z_0)v)_{L^2(\Omega;d^nx)} \\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, \widehat{\gamma}_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}}, \end{aligned} \tag{16.13}$$

since $\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)u = (-\Delta - (\overline{z}+z_0))u$ and $(-\Delta_{D,\Omega} - (z+z_0)I_\Omega)v = (-\Delta - (z+z_0))v$. However, (16.13) is a consequence of (12.8) and the fact that $\widehat{\gamma}_D v = 0$.

In order to remove the additional hypothesis that $z \in \mathbb{C}\backslash\mathbb{R}$, it now suffices to note that due to the assumptions $z_0 \in \mathbb{R}$ and $z_0, (z+z_0) \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, and due to the self-adjointness of $-\Delta_{X,L,z_0}^D$, both sides of (16.1) extend to all $z \in \mathbb{C}\backslash\sigma(-\Delta_{X,L,z_0}^D)$ by analytic continuation with respect to z. □

Lemma 16.2. *Retain the hypotheses and conventions made in Lemma* 16.1 *and let* $z \in \mathbb{C}\backslash\sigma(-\Delta_{X,L,z_0}^D)$. *Then*

$$\big[\tau_{z_0}^N(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* \in \mathcal{B}\big(\big(N^{1/2}(\partial\Omega)\big)^*, \ker(-\Delta_{max} - (\overline{z}+z_0)I_\Omega)\big), \tag{16.14}$$

and

$$\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \ker(-\Delta_{max} - (\overline{z}+z_0)I_\Omega)\big). \tag{16.15}$$

Proof. From (16.6) we know that

$$\big[\tau_{z_0}^N(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* \in \mathcal{B}\big(\big(N^{1/2}(\partial\Omega)\big)^*, L^2(\Omega; d^n x)\big). \tag{16.16}$$

Thus, it suffices to show that if $f \in \big(N^{1/2}(\partial\Omega)\big)^*$ then, in the sense of distributions,

$$(-\Delta - (\overline{z} + z_0))\big[\tau_{z_0}^N(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* f = 0 \ \text{ in } \ \Omega. \tag{16.17}$$

To this end, pick an arbitrary $\varphi \in C_0^\infty(\Omega)$ and write

$$\begin{aligned}&\big(\big[\tau_{z_0}^N(-\Delta_{X.L,z_0}^D - zI_\Omega)^{-1}\big]^* f, (-\Delta - (z + z_0))\varphi\big)_{L^2(\Omega;d^n x)}\\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\langle \tau_{z_0}^N(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}(-\Delta - (z + z_0))\varphi, f\rangle_{(N^{1/2}(\partial\Omega))^*}}.\end{aligned} \tag{16.18}$$

To continue, we notice that

$$\varphi \in H_0^2(\Omega) = \mathrm{dom}(-\Delta_{min}) \subset \mathrm{dom}(-\Delta_{X,L,z_0}^D). \tag{16.19}$$

Thus, we have $(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}(-\Delta - (z + z_0))\varphi = \varphi$ and also $\tau_{z_0}^N\varphi = 0$ by (12.6). Consequently, the right-hand side of (16.18) vanishes and (16.17) follows.

As far as (16.15) is concerned, (16.7) yields

$$\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), L^2(\Omega; d^n x)\big). \tag{16.20}$$

Consequently, it suffices to show that if $f \in N^{1/2}(\partial\Omega)$ then, in the sense of distributions,

$$(-\Delta - (\overline{z} + z_0))\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* f = 0 \ \text{ in } \ \Omega. \tag{16.21}$$

To verify this, select an arbitrary $\varphi \in C_0^\infty(\Omega)$ and write

$$\begin{aligned}&\big(\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}\big]^* f, (-\Delta - (z + z_0))\varphi\big)_{L^2(\Omega;d^n x)}\\ &\quad = {}_{N^{1/2}(\partial\Omega)}\langle f, \widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}(-\Delta - (z + z_0))\varphi\rangle_{(N^{1/2}(\partial\Omega))^*}.\end{aligned} \tag{16.22}$$

Because of (16.19) one obtains

$$\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - zI_\Omega)^{-1}(-\Delta - (z + z_0))\varphi = \widehat{\gamma}_D\varphi = 0. \tag{16.23}$$

Thus, the right-hand side of (16.22) vanishes, proving (16.21). ☐

Lemmas 16.1 and 16.2 have been inspired by [97, Lemmas 6, 7], where the special case of Dirichlet and Neumann Laplacians on Ω a cubic box in $\mathbb{R}^n$ was studied.

In the context of Lemma 16.1, let us define the boundary operator

$$M_{X,L,z_0}^D(z) := \big(\widehat{\gamma}_D\big[\widehat{\gamma}_D\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1}\big]^*\big)^*, \quad z \in \mathbb{C}\backslash\sigma\big(-\Delta_{X,L,z_0}^D\big), \tag{16.24}$$

so that

$$M_{X,L,z_0}^D(z) \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big) \tag{16.25}$$

by Theorem 6.4 and (16.15). We are now ready to prove a version of Krein 's resolvent formula for arbitrary self-adjoint extensions of the Laplacian. Concretely, we have the following result:

Theorem 16.3. *Assume Hypothesis* 8.10 *and suppose that $z_0 \in \mathbb{R}$, $z_0, (z + z_0) \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Let X be a closed subspace of $\big(N^{1/2}(\partial\Omega)\big)^*$ and denote by X^* the conjugate dual space of X. In addition, consider a self-adjoint operator L as in* (14.12) *and define the self-adjoint operator $-\Delta_{X,L,z_0}^D$ as in* (14.13) *corresponding to $z = z_0$. Then the following Krein formula holds on $L^2(\Omega; d^n x)$,*

$$\begin{aligned}\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1} &= \big(-\Delta_{D,\Omega} - (z + z_0)I_\Omega\big)^{-1}\\ &\quad + \big[\tau_{z_0}^N\big(-\Delta_{D,\Omega} - (\overline{z} + z_0)I_\Omega\big)^{-1}\big]^* M_{X,L,z_0}^D(z)\big[\tau_{z_0}^N\big(-\Delta_{D,\Omega} - (z + z_0)I_\Omega\big)^{-1}\big],\\ &\qquad z \in \mathbb{C}\backslash\sigma(-\Delta_{X,L,z_0}^D).\end{aligned} \tag{16.26}$$

Proof. Applying $\widehat{\gamma}_D$ from the left to both sides of (16.1) yields

$$\widehat{\gamma}_D\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1} = \widehat{\gamma}_D\big[\widehat{\gamma}_D\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1}\big]^*\big[\tau_{z_0}^N\big(-\Delta_{D,\Omega} - (z + z_0)I_\Omega\big)^{-1}\big], \tag{16.27}$$

since $\widehat{\gamma}_D\big(-\Delta_{D,\Omega} - (z + z_0)I_\Omega\big)^{-1} = 0$. Thus, upon recalling (16.24), (16.27) then becomes

$$\widehat{\gamma}_D\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1} = M_{X,L,z_0}^D(\overline{z})^*\tau_{z_0}^N\big(-\Delta_{D,\Omega} - (z + z_0)I_\Omega\big)^{-1}, \tag{16.28}$$

as operators in $\in \mathcal{B}\big(L^2(\Omega; d^n x), \big(N^{1/2}(\partial\Omega)\big)^*\big)$. Taking adjoints in (16.28), written with $\overline{z}$ in place of z, then leads to

$$\big[\widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big]^* = \big[\tau_{z_0}^N\big(- \Delta_{D,\Omega} - (\overline{z}+z_0)I_\Omega\big)^{-1}\big]^* M_{X,L,z_0}^D(z). \tag{16.29}$$

Inserting this into (16.1), one arrives at (16.26). $\square$

Remark 16.4. Formula (16.26) relates the resolvent of an arbitrary self-adjoint extension of $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$ to that of the Dirichlet Laplacian $-\Delta_{D,\Omega}$ in a transparent way which makes it possible to extract information about the spectrum of the extension from information about the associated operator-valued Weyl–Titchmarsh M-function. More details on this will appear elsewhere.

Next we study some properties of the Weyl–Titchmarsh M-function (16.24) in more detail and show that it satisfies a natural symmetry condition. More specifically, we have the following result:

Theorem 16.5. *Retain the hypotheses and conventions made in Theorem* 16.3 *and let* $z \in \mathbb{C}\backslash\sigma(-\Delta_{X,L,z_0}^D)$. *Then*

$$M_{X,L,z_0}^D(z)^* = M_{X,L,z_0}^D(\overline{z}). \tag{16.30}$$

As a consequence,

$$M_{X,L,z_0}^D(z) = \widehat{\gamma}_D\big[\widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big]^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big). \tag{16.31}$$

Proof. resolvent estimate Let $z_0 \in \mathbb{R}$, $z_0, (z+z_0) \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$ and fix arbitrary $f, g \in N^{1/2}(\partial\Omega)$. By Theorem 12.1 (i), it is then possible to find $u, w \in \operatorname{dom}(-\Delta_{max})$ such that $\tau_{z_0}^N u = f$ and $\tau_{z_0}^N w = g$. In fact, by (12.4), we can actually pick $u, w \in H^2(\Omega) \cap H_0^1(\Omega)$ (this is going to be of relevance shortly). One then has

$$\begin{aligned}
&{}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(\overline{z})^* g\big\rangle_{(N^{1/2}(\partial\Omega))^*} = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle g, M_{X,L,z_0}^D(\overline{z}) f\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle f, \widehat{\gamma}_D\big[\widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big]^* g\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u, \widehat{\gamma}_D\big[\widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big]^* g\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = -\big((-\Delta - (\overline{z}+z_0))u, \big[\widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big]^* g\big)_{L^2(\Omega; d^n x)},
\end{aligned} \tag{16.32}$$

by (12.9), (16.15) and the fact that $\widehat{\gamma}_D u = 0$. Continuing, we employ taking adjoints and conjugation and bring in w in order to write

$$\begin{aligned}
&-\big((-\Delta - (\overline{z}+z_0))u, \big[\widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big]^* g\big)_{L^2(\Omega; d^n x)} \\
&\quad = -\overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N w, \widehat{\gamma}_D\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}(-\Delta - (\overline{z}+z_0))u\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\
&\quad = -\big((-\Delta - (\overline{z}+z_0))\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}(-\Delta - (\overline{z}+z_0))u, w\big)_{L^2(\Omega; d^n x)} \\
&\qquad + \big(\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}(-\Delta - (\overline{z}+z_0))u, (-\Delta - (z+z_0))w\big)_{L^2(\Omega; d^n x)},
\end{aligned} \tag{16.33}$$

by (12.9) and the fact that $\widehat{\gamma}_D w = 0$. Since

$$(-\Delta - (\overline{z}+z_0))\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}(-\Delta - (\overline{z}+z_0))u = (-\Delta - (\overline{z}+z_0))u, \tag{16.34}$$

we may summarize the above calculation as follows:

$$\begin{aligned}
{}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(\overline{z})^* g\big\rangle_{(N^{1/2}(\partial\Omega))^*} &= -\big((-\Delta - (\overline{z}+z_0))u, w\big)_{L^2(\Omega; d^n x)} \\
&\quad + \big(\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}(-\Delta - (\overline{z}+z_0))u, (-\Delta - (z+z_0))w\big)_{L^2(\Omega; d^n x)}.
\end{aligned} \tag{16.35}$$

At this stage, we recall once more that $u, w \in H^2(\Omega) \cap H_0^1(\Omega)$. Consequently, we may integrate by parts in the first pairing in the right-hand side of (16.35) (without boundary terms) and obtain

$$\begin{aligned}
&{}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(\overline{z})^* g\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = (z+z_0)(u,w)_{L^2(\Omega; d^n x)} - (\nabla u, \nabla w)_{(L^2(\Omega; d^n x))^n} \\
&\qquad + \big(\big(- \Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}(-\Delta - (\overline{z}+z_0))u, (-\Delta - (z+z_0))w\big)_{L^2(\Omega; d^n x)}.
\end{aligned} \tag{16.36}$$

At this point, the right-hand side of (16.36) is invariant under interchanging u, w, replacing z by $\overline{z}$, and taking conjugation (here, we have also used the fact that the adjoint of $\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}$ is the operator $\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1}$). Given the relationship between u, w on the one hand, and f, g on the other, this observation then translates into

$$ {}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(\overline{z})^* g\big\rangle_{(N^{1/2}(\partial\Omega))^*} = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle g, M_{X,L,z_0}^D(z)^* f\big\rangle_{(N^{1/2}(\partial\Omega))^*}}. \tag{16.37} $$

Since the right-hand side of (16.37) is given by

$$ \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle g, M_{X,L,z_0}^D(z)^* f\big\rangle_{(N^{1/2}(\partial\Omega))^*}} = {}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(z) g\big\rangle_{(N^{1/2}(\partial\Omega))^*}, \tag{16.38} $$

this proves (16.30). □

Next, we recall (12.20) and (16.15) and state the following fact:

Lemma 16.6. *Under the assumptions of Theorem* 16.3*, one has*

$$ \tau_{z_0}^N\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* = \big[M_{D,N,\Omega}^{(0)}(z_0) - M_{D,N,\Omega}^{(0)}(z+z_0)\big] M_{X,L,z_0}^D(z) \tag{16.39} $$

as operators in $\mathcal{B}\big(N^{1/2}(\partial\Omega), N^{1/2}(\partial\Omega)\big)$. *In addition,*

$$ \widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \operatorname{dom}(L) \tag{16.40} $$

is a well-defined, bounded, linear operator, when $\operatorname{dom}(L)$ *is equipped with the natural graph norm, that is,* $g \mapsto \|g\|_{(N^{1/2}(\partial\Omega))^*} + \|L(g)\|_{X^*}$.

Proof. Pick arbitrary $f \in N^{1/2}(\partial\Omega)$, $g \in \big(N^{1/2}(\partial\Omega)\big)^*$, and let $v \in \operatorname{dom}(-\Delta_{max})$ satisfy

$$ (-\Delta - (\overline{z}+z_0))v = 0 \text{ in } \Omega, \quad \widehat{\gamma}_D v = g. \tag{16.41} $$

Then, since $\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* f \in \ker(-\Delta_{max} - (z+z_0)I_\Omega)$ by (16.15), we may write

$$ \begin{aligned} &{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* f, g\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, \widehat{\gamma}_D\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* f\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, M_{X,L,z_0}^D(z) f\big\rangle_{(N^{1/2}(\partial\Omega))^*}}, \end{aligned} \tag{16.42} $$

by (12.9) and (16.31). Next, one observes that

$$ \tau_{z_0}^N v = \widehat{\gamma}_N v + M_{D,N,\Omega}^{(0)}(z_0) g = \big[-M_{D,N\Omega}^{(0)}(\overline{z}+z_0) + M_{D,N,\Omega}^{(0)}(z_0)\big] g \tag{16.43} $$

by (12.2) and (11.1). Thus,

$$ \begin{aligned} &\overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, M_{X,L,z_0}^D(z) f\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\ &\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \big[-M_{D,N\Omega}^{(0)}(\overline{z}+z_0) + M_{D,N,\Omega}^{(0)}(z_0)\big] g, M_{X,L,z_0}^D(z) f\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\ &\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle \big[-M_{D,N\Omega}^{(0)}(\overline{z}+z_0) + M_{D,N,\Omega}^{(0)}(z_0)\big]^* M_{X,L,z_0}^D(z) f, g\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\ &\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle \big[-M_{D,N\Omega}^{(0)}(z+z_0) + M_{D,N,\Omega}^{(0)}(z_0)\big] M_{X,L,z_0}^D(z) f, g\big\rangle_{(N^{1/2}(\partial\Omega))^*}, \end{aligned} \tag{16.44} $$

by (11.10), implying (16.39).

Next, consider the claim made about the operator (16.40). Let $f \in X$ and $u \in L^2(\Omega; d^n x)$ be arbitrary. Then $(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} u \in \operatorname{dom}(-\Delta_{X,L,z_0}^D)$ so that $\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} u \in \operatorname{dom}(L)$ and

$$ \begin{aligned} &\big|{}_X\big\langle f, L\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} u\big\rangle_{X^*}\big| \\ &\quad = \Big|{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} u, f\big\rangle_{(N^{1/2}(\partial\Omega))^*}\Big| \\ &\quad \le C\|\tau_{z_0}^N(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} u\|_{N^{1/2}(\partial\Omega)} \|f\|_{(N^{1/2}(\partial\Omega))^*} \\ &\quad \le C\|u\|_{L^2(\Omega; d^n x)} \|f\|_X, \end{aligned} \tag{16.45} $$

by (14.14) and Theorem 12.1. Since $u \in L^2(\Omega; d^n x)$ was arbitrary, this implies that

$$ \|L\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1} u\|_{X^*} \le C\|f\|_X. \tag{16.46} $$

Together with (16.7), this shows that the operator (16.40) is indeed well-defined, linear, and bounded. □

We continue to study the properties of the boundary "transition" operator (16.24), with the goal of establishing a Herglotz property. We recall that an operator-valued function $M(z) \in \mathcal{B}(\mathcal{X}, \mathcal{X}^*)$, with $z \in \mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ and $\mathcal{X}$ a reflexive Banach space, is called an operator-valued Herglotz function if $M(\cdot)$ is analytic on $\mathbb{C}_+$ and

$$\mathrm{Im}(M(z)) := (M(z) - M(z)^*)/(2i) \geq 0, \quad z \in \mathbb{C}_+. \tag{16.47}$$

It is customary to extend $M(\cdot)$ to $\mathbb{C}_- := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0\}$ by defining $M(z) = M(\overline{z})^*$, $z \in \mathbb{C}_-$, but the latter is generally not an analytic continuation of $M|_{\mathbb{C}_+}$.

Theorem 16.7. *Retain the hypotheses and conventions made in Theorem* 16.3. *Then the assignment*

$$\mathbb{C}_+ \ni z \mapsto M^D_{X,L,z_0}(z) \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big) \tag{16.48}$$

is an operator-valued Herglotz function whenever

$$X = \big(N^{1/2}(\partial\Omega)\big)^*. \tag{16.49}$$

Proof. For the duration of this proof we suppose that (16.49) holds. We first establish the analytical dependence of $M^D_{X,L,z_0}(z)$ on $z \in \mathbb{C}\backslash\sigma\big(-\Delta^D_{X,L,z_0}\big)$: Based on (16.24) and the resolvent equation for $-\Delta^D_{X,L,z_0}$, for $z \in \mathbb{C}\backslash\sigma\big(-\Delta^D_{X,L,z_0}\big)$ and $w \in \mathbb{C}$ sufficiently close to 0, one obtains

$$\frac{1}{w}\big[M^D_{X,L,z_0}(z+w) - M^D_{X,L,z_0}(z)\big] = \big(\widehat{\gamma}_D\big[\widehat{\gamma}_D\big(-\Delta^D_{X,L,z_0} - zI_\Omega\big)^{-1}\big(-\Delta^D_{X,L,z_0} - (z+w)I_\Omega\big)^{-1}\big]^*\big)^*. \tag{16.50}$$

Based on (16.7) and the fact that

$$\big(-\Delta^D_{X,L,z_0} - (z+w)I_\Omega\big)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \tag{16.51}$$

one infers that

$$T_w := \widehat{\gamma}_D\big(-\Delta^D_{X,L,z_0} - zI_\Omega\big)^{-1}\big(-\Delta^D_{X,L,z_0} - (z+w)I_\Omega\big)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), \big(N^{1/2}(\partial\Omega)\big)^*\big) \tag{16.52}$$

and

$$\lim_{w\to 0} T_w = \widehat{\gamma}_D\big(-\Delta^D_{X,L,z_0} - zI_\Omega\big)^{-2} \text{ in } \mathcal{B}\big(L^2(\Omega; d^n x), \big(N^{1/2}(\partial\Omega)\big)^*\big). \tag{16.53}$$

Consequently,

$$T_w^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), L^2(\Omega; d^n x)\big) \text{ and } \lim_{w\to 0} T_w^* \text{ exists in } \mathcal{B}\big(N^{1/2}(\partial\Omega), L^2(\Omega; d^n x)\big). \tag{16.54}$$

Next, we claim that actually,

$$T_w^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \mathrm{dom}(-\Delta_{max})\big) \tag{16.55}$$

and

$$\lim_{w\to 0} T_w^* \text{ exists in } \mathcal{B}\big(N^{1/2}(\partial\Omega), \mathrm{dom}(-\Delta_{max})\big). \tag{16.56}$$

Given (16.54) and (16.15), (16.55) follows once we establish that

$$(-\Delta - (\overline{z} + z_0)I_\Omega)T_w^* = \big[\widehat{\gamma}_D\big(-\Delta^D_{X,L,z_0} - (z+w)I_\Omega\big)^{-1}\big]^*. \tag{16.57}$$

To prove (16.57), we fix $f \in N^{1/2}(\partial\Omega)$ and write for arbitrary $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&{}_{\mathcal{D}'(\Omega)}\langle(-\Delta - (\overline{z} + z_0)I_\Omega)T_w^* f, \varphi\rangle_{\mathcal{D}(\Omega)} = (T_w^* f, (-\Delta - (z+z_0)I_\Omega)\varphi)_{L^2(\Omega; d^n x}\\
&\quad = {}_{N^{1/2}(\partial\Omega)}\langle f, T_w(-\Delta - (z+z_0)I_\Omega)\varphi\rangle_{(N^{1/2}(\partial\Omega))^*}\\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle f, \widehat{\gamma}_D\big(-\Delta^D_{X,L,z_0} - (z+w)I_\Omega\big)^{-1}\varphi\big\rangle_{(N^{1/2}(\partial\Omega))^*}\\
&\quad = \big(\big[\widehat{\gamma}_D\big(-\Delta^D_{X,L,z_0} - (z+w)I_\Omega\big)^{-1}\big]^* f, \varphi\big)_{L^2(\Omega; d^n x)},
\end{aligned} \tag{16.58}$$

using (16.52) in the next to last step. (Here ${}_{\mathcal{D}'(\Omega)}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{D}(\Omega)}$ denotes the usual distributional pairing.)

To prove (16.56) one can argue as follows: First, one notes that if $T_j \in \mathcal{B}(X, \mathrm{dom}(-\Delta_{max}))$, $j \in \mathbb{N}$, where X is a Banach space and $\mathrm{dom}(-\Delta_{max})$ is equipped with the graph norm (cf. (15.3)),

$$\begin{aligned}
&\lim_{j\to\infty} T_j \text{ exists in } \mathcal{B}(X, \mathrm{dom}(-\Delta_{max}))\\
&\qquad \text{if and only if } \lim_{j\to\infty} T_j \text{ and } \lim_{j\to\infty} \Delta T_j \text{ exist in } \mathcal{B}\big(X, L^2(\Omega; d^n x)\big).
\end{aligned} \tag{16.59}$$

While the left-to-right implication in (16.59) is clear from the graph norm employed on $\mathrm{dom}(-\Delta_{max})$, we indicate the proof of the right-to-left implication next, that is, we assume

$$\lim_{j\to\infty} T_j = T \text{ in } \mathcal{B}\big(X, L^2(\Omega; d^n x)\big) \text{ and } \lim_{j\to\infty} \Delta T_j = S \text{ in } \mathcal{B}\big(X, L^2(\Omega; d^n x)\big). \tag{16.60}$$

Let $f \in X$, then $Tf \in L^2(\Omega; d^n x) \hookrightarrow \mathcal{D}'(\Omega)$, and for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
{}_{\mathcal{D}'(\Omega)}\langle \Delta(Tf), \varphi\rangle_{\mathcal{D}(\Omega)} &= (Tf, \Delta\varphi)_{L^2(\Omega;d^nx)} = \lim_{j\to\infty} (T_j f, \Delta\varphi)_{L^2(\Omega;d^nx)} = \lim_{j\to\infty} {}_{\mathcal{D}'(\Omega)}\langle T_j f, \Delta\varphi\rangle_{\mathcal{D}(\Omega)} \\
&= \lim_{j\to\infty} {}_{\mathcal{D}'(\Omega)}\langle \Delta(T_j f), \varphi\rangle_{\mathcal{D}(\Omega)} = \lim_{j\to\infty} (\Delta(T_j f), \varphi)_{L^2(\Omega;d^nx)} = (Sf, \varphi)_{L^2(\Omega;d^nx)} \\
&= {}_{\mathcal{D}'(\Omega)}\langle Sf, \varphi\rangle_{\mathcal{D}(\Omega)}, \qquad (16.61)
\end{aligned}$$

implying

$$\Delta T = S \text{ on } L^2(\Omega; d^n x) \text{ and hence } T \in \mathcal{B}(X, \mathrm{dom}(-\Delta_{max})). \qquad (16.62)$$

Moreover, this yields

$$\lim_{j\to\infty} \|T_j - T\|_{\mathcal{B}(X,\mathrm{dom}(-\Delta_{max}))} = 0. \qquad (16.63)$$

Next, using again the equivalence of norms

$$\|T\|_{\mathcal{B}(X,\mathrm{dom}(-\Delta_{max}))} \approx \|T\|_{\mathcal{B}(X,L^2(\Omega;d^nx))} + \|\Delta T\|_{\mathcal{B}(X,L^2(\Omega;d^nx))}, \qquad (16.64)$$

and the fact that we may change ΔT into $(\Delta + zI_\Omega)T$, $z \in \mathbb{C}$, in (16.64) (at the expense of altering the comparability constants), one finally concludes that $\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big)$ can actually be improved to $\big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), \mathrm{dom}(-\Delta_{max})\big)$ and that the assignment $\mathbb{C}\backslash\sigma\big(-\Delta_{X,L,z_0}^D\big) \ni z \mapsto \big(-\Delta_{X,L,z_0}^D - zI_\Omega\big)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), \mathrm{dom}(-\Delta_{max})\big)$ is continuous.

With the help of (16.55) and (16.56) one concludes from Theorem 16.5 that

$$\widehat{\gamma}_D T_w^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big) \text{ and hence also } \big(\widehat{\gamma}_D T_w^*\big)^* \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big), \qquad (16.65)$$

and

$$\lim_{w\to 0} \big(\widehat{\gamma}_D T_w^*\big)^* \text{ exists in } \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big). \qquad (16.66)$$

Consequently, employing (16.65) and (16.66) in (16.50) one obtains that

$$\lim_{w\to 0} \frac{1}{w}\big[M_{X,L,z_0}^D(z+w) - M_{X,L,z_0}^D(z)\big] \text{ exists in } \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big). \qquad (16.67)$$

Thus, $\mathbb{C}\backslash\sigma\big(-\Delta_{X,L,z_0}^D\big) \ni z \mapsto M_{X,L,z_0}^D(z) \in \mathcal{B}\big(N^{1/2}(\partial\Omega), \big(N^{1/2}(\partial\Omega)\big)^*\big)$ is analytic.

Next, for an arbitrary $f \in N^{1/2}(\partial\Omega)$ set

$$u := \big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* f \in \ker(-\Delta_{max} - (z+z_0)I_\Omega). \qquad (16.68)$$

For each $v \in \mathrm{dom}(-\Delta_{X,L,z_0}^D)$ we may then write

$$\begin{aligned}
&{}_{N^{1/2}(\partial\Omega)}\langle f, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle f, \widehat{\gamma}_D\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)^{-1}\big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)v\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = \big(\big[\widehat{\gamma}_D(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega)^{-1}\big]^* f, \big(-\Delta_{X,L,z_0}^D - \overline{z}I_\Omega\big)v\big)_{L^2(\Omega;d^nx)} \\
&\quad = (u, (-\Delta - (\overline{z} + z_0))v)_{L^2(\Omega;d^nx)} \qquad (16.69) \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u, \widehat{\gamma}_D v\big\rangle_{(N^{1/2}(\partial\Omega))^*} - \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, \widehat{\gamma}_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}},
\end{aligned}$$

by (12.9) and (16.68). Granted (16.49) and given that v belongs to $\mathrm{dom}\big(-\Delta_{X,L,z_0}^D\big)$, we may then transform the last term in (16.69) into

$$\overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N v, \widehat{\gamma}_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}} = \overline{{}_{N^{1/2}(\partial\Omega)}\langle L(\widehat{\gamma}_D v), \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}}. \qquad (16.70)$$

When inserted into (16.69), this yields

$${}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u - f, \widehat{\gamma}_D v\big\rangle_{(N^{1/2}(\partial\Omega))^*} = \overline{{}_{N^{1/2}(\partial\Omega)}\langle L(\widehat{\gamma}_D v), \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}} \qquad (16.71)$$

Since $\mathrm{dom}(L) = \widehat{\gamma}_D \mathrm{dom}\big(-\Delta_{X,L,z_0}^D\big)$ and L is self-adjoint, the above formula yields

$$\widehat{\gamma}_D u \in \mathrm{dom}(L) \text{ and } L(\widehat{\gamma}_D u) = \tau_{z_0}^N u - f. \qquad (16.72)$$

Thus, $M_{X,L,z_0}^D(z)f = \widehat{\gamma}_D u$, (16.72), (12.9), and the fact that $\Delta u = -(z+z_0)u$ imply

$$\begin{aligned}
&{}_{N^{1/2}(\partial\Omega)}\big\langle f, \mathrm{Im}\big(M_{X,L,z_0}^D(z)\big)f\big\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = \frac{1}{2i}\big[{}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(z)f\big\rangle_{(N^{1/2}(\partial\Omega))^*}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle f, M_{X,L,z_0}^D(z)f\big\rangle_{(N^{1/2}(\partial\Omega))^*}}\big]\\
&= \frac{1}{2i}\big[{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u - L(\widehat\gamma_D u), \widehat\gamma_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}\\
&\quad - \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u - L(\widehat\gamma_D u), \widehat\gamma_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}}\big]\\
&= \frac{1}{2i}\big[{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u, \widehat\gamma_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*} - \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle \tau_{z_0}^N u, \widehat\gamma_D u\big\rangle_{(N^{1/2}(\partial\Omega))^*}}\big]\\
&= \frac{1}{2i}\big[(\Delta u, u)_{L^2(\Omega;d^nx)} - (u, \Delta u)_{L^2(\Omega;d^nx)}\big]\\
&= \frac{1}{2i}\big[(-(z+z_0)u, u)_{L^2(\Omega;d^nx)} - (u, -(z+z_0)u)_{L^2(\Omega;d^nx)}\big]\\
&= \mathrm{Im}(z)\|u\|^2_{L^2(\Omega;d^nx)} \geq 0.
\end{aligned}
\tag{16.73}
$$

□

Our next theorem provides a transparent connection between the inverse of $M_{X,L,z_0}^D(z)$ and $M_{D,N,\Omega}^{(0)}(z+z_0)$, $M_{D,N,\Omega}^{(0)}(z_0)$.

Theorem 16.8. *Retain the hypotheses and conventions made in Theorem* 16.3. *In addition, assume that* (16.49) *holds. Then the map*

$$M_{X,L,z_0}^D(z) : N^{1/2}(\partial\Omega) \to \mathrm{dom}(L) \tag{16.74}$$

is well-defined, linear, and bounded, provided $\mathrm{dom}(L)$ *is equipped with the natural graph norm, that is,* $\mathrm{dom}(L) \ni g \mapsto \|g\|_{(N^{1/2}(\partial\Omega))^*} + \|L(g)\|_{N^{1/2}(\partial\Omega)}$ *(in which case* $\mathrm{dom}(L)$ *becomes a Banach space). Its inverse is given by*

$$M_{X,L,z_0}^D(z)^{-1} = -L - M_{D,N,\Omega}^{(0)}(z+z_0) + M_{D,N,\Omega}^{(0)}(z_0) : \mathrm{dom}(L) \to N^{1/2}(\partial\Omega). \tag{16.75}$$

Proof. Fix $f \in N^{1/2}(\partial\Omega)$ arbitrary, and define u as in (16.68). Then $\widehat\gamma_D u \in \mathrm{dom}(L)$ and

$$(-\Delta - (z+z_0))u = 0 \text{ in } \Omega, \quad u \in L^2(\Omega; d^nx), \quad \tau_{z_0}^N u - L(\widehat\gamma_D u) = f, \tag{16.76}$$

because of the argument carried out in (16.68)–(16.72). Then $M_{X,L,z_0}^D(z)f = \widehat\gamma_D u$ so that

$$
\begin{aligned}
&\big\|M_{X,L,z_0}^D(z)f\big\|_{(N^{1/2}(\partial\Omega))^*} + \big\|L(M_{X,L,z_0}^D(z)f)\big\|_{N^{1/2}(\partial\Omega)}\\
&\quad = \|\widehat\gamma_D u\|_{(N^{1/2}(\partial\Omega))^*} + \|L(\widehat\gamma_D u)\|_{N^{1/2}(\partial\Omega)}\\
&\quad = \|\widehat\gamma_D u\|_{(N^{1/2}(\partial\Omega))^*} + \big\|\tau_{z_0}^N u - f\big\|_{N^{1/2}(\partial\Omega)}\\
&\quad \leq C\|u\|_{L^2(\Omega;d^nx)} + C\|f\|_{N^{1/2}(\partial\Omega)}\\
&\quad \leq C\|f\|_{N^{1/2}(\partial\Omega)}.
\end{aligned}
\tag{16.77}
$$

Above, we have used the boundedness of the operators (6.14) and (12.1). The estimate (16.77) proves that the map (16.74) is bounded. Furthermore, the fact that $M_{X,L,z_0}^D(z)f = \widehat\gamma_D u$ also entails

$$
\begin{aligned}
&\big[-L - M_{D,N,\Omega}^{(0)}(z+z_0) + M_{D,N,\Omega}^{(0)}(z_0)\big]M_{X,L,z_0}^D(z)f\\
&\quad = -L(\widehat\gamma_D u) - M_{D,N,\Omega}^{(0)}(z+z_0)(\widehat\gamma_D u) + M_{D,N,\Omega}^{(0)}(z_0)(\widehat\gamma_D u)\\
&\quad = f - \tau_{z_0}^N u - M_{D,N,\Omega}^{(0)}(z+z_0)(\widehat\gamma_D u) + M_{D,N,\Omega}^{(0)}(z_0)(\widehat\gamma_D u)\\
&\quad = f - \widehat\gamma_N u + M_{D,N,\Omega}^{(0)}(z_0)(\widehat\gamma_D u) - M_{D,N,\Omega}^{(0)}(z+z_0)(\widehat\gamma_D u) + M_{D,N,\Omega}^{(0)}(z_0)(\widehat\gamma_D u)\\
&\quad = f,
\end{aligned}
\tag{16.78}
$$

since $M_{D,N,\Omega}^{(0)}(z+z_0)(\widehat\gamma_D u) = -\widehat\gamma_N u$. Conversely, assume that $g \in \mathrm{dom}(L)$ is given and let w be such that

$$(-\Delta - (z+z_0))w = 0 \text{ in } \Omega, \quad w \in L^2(\Omega; d^nx), \quad \widehat\gamma_D w = g \in \big(N^{1/2}(\partial\Omega)\big)^*. \tag{16.79}$$

Then $\big[-L - M_{D,N,\Omega}^{(0)}(z+z_0) + M_{D,N,\Omega}^{(0)}(z_0)\big]g = f$, where we have set $f := \tau_{z_0}^N w - Lg \in N^{1/2}(\partial\Omega)$. Next, $M_{X,L,z_0}^D(z)f = \widehat\gamma_D u$, where u is as in (16.68). However, because of (16.76), the function $v := u - w$ satisfies

$$(-\Delta - (z+z_0))v = 0 \text{ in } \Omega, \quad v \in L^2(\Omega; d^nx), \quad \widehat\gamma_D v = 0, \tag{16.80}$$

and hence necessarily $v = 0$, by the uniqueness part in Theorem 10.4. This yields $u = w$ which implies $\widehat\gamma_D u = \widehat\gamma_D w = g$, and hence, $M_{X,L,z_0}^D(z)f = g$. □

Having established Theorem 16.8, we now proceed to state an alternative version of our earlier Krein formula (cf. Theorem 16.3) on the space $L^2(\Omega; d^n x)$.

Corollary 16.9. *In addition to the hypotheses made in Theorem* 16.3 *assume that* $X = \big(N^{1/2}(\partial\Omega)\big)^*$. *Then the following Krein formula holds on* $L^2(\Omega; d^n x)$*:*

$$\begin{aligned}
&\big(-\Delta^D_{X,L,z_0} - zI_\Omega\big)^{-1} = \big(-\Delta_{D,\Omega} - (z+z_0)I_\Omega\big)^{-1} \\
&\quad + \big[\tau^N_{z_0}\big(-\Delta_{D,\Omega} - (\overline{z}+z_0)I_\Omega\big)^{-1}\big]^* \big[-L - M^{(0)}_{D,N,\Omega}(z+z_0) + M^{(0)}_{D,N,\Omega}(z_0)\big]^{-1} \\
&\quad \times \big[\tau^N_{z_0}\big(-\Delta_{D,\Omega} - (z+z_0)I_\Omega\big)^{-1}\big].
\end{aligned} \tag{16.81}$$

In particular, for the resolvents of the Krein Laplacian, one has

$$\begin{aligned}
&\big(-\Delta_{K,\Omega,z_0} - zI_\Omega\big)^{-1} = \big(-\Delta_{D,\Omega} - (z+z_0)I_\Omega\big)^{-1} \\
&\quad + \big[\tau^N_{z_0}\big(-\Delta_{D,\Omega} - (\overline{z}+z_0)I_\Omega\big)^{-1}\big]^* \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z+z_0)\big]^{-1} \\
&\quad \times \big[\tau^N_{z_0}\big(-\Delta_{D,\Omega} - (z+z_0)I_\Omega\big)^{-1}\big],
\end{aligned} \tag{16.82}$$

as operators in $\mathcal{B}\big(L^2(\Omega; d^n x)\big)$.

Proof. Formula (16.81) is a direct consequence of Theorem 16.3 and Theorem 16.8. Formula (16.82) follows from this and (14.49). □

Naturally, (16.82) (with $z_0 = 0$) resembles the abstract relation (1.39) between the Krein–von Neumann extension S_K and the Friedrichs extension S_F of S, but the actual methods of proof are quite different in either case.

Remark 16.10. Under the assumptions made in Corollary 16.9, the operator-valued function given by

$$z \mapsto \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z+z_0)\big]^{-1} \in \mathcal{B}\big(N^{1/2}(\partial\Omega), (N^{1/2}(\partial\Omega))^*\big), \tag{16.83}$$

appearing in (16.82), has the Herglotz property. Indeed, by Theorem 16.7 and Theorem 16.8, this is the case for the operator-valued function $M^D_{X,L,z_0}(z) = \big[-L - M^{(0)}_{D,N,\Omega}(z+z_0) + M^{(0)}_{D,N,\Omega}(z_0)\big]^{-1}$ and the map in question corresponds to this when $L = 0$.

Our last theorem in this section is a Krein formula involving the resolvents of two arbitrary self-adjoint extensions of the Laplacian. To set the stage, we first prove the following lemma.

Lemma 16.11. *Assume Hypothesis* 8.10 *and suppose that* $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$. *Let* X_1 *be a closed subspace of* $X_2 := \big(N^{1/2}(\partial\Omega)\big)^*$. *In addition, consider two self-adjoint operators*

$$L_j : \mathrm{dom}(L_j) \subseteq X_j \to X_j^*, \quad j = 1, 2, \tag{16.84}$$

with the property that

$$\mathrm{dom}(L_1) \hookrightarrow \mathrm{dom}(L_2) \ \textit{boundedly}. \tag{16.85}$$

Associated with L_j, $j = 1, 2$, *define the self-adjoint operators* $-\Delta^D_{X_j,L_j,z_0}$, $j = 1, 2$, *as in* (14.13) *corresponding to* $z = z_0$. *In addition, let* $z \in \mathbb{C}\backslash(\sigma(-\Delta^D_{X_1,L_1,z_0}) \cup \sigma(-\Delta^D_{X_2,L_2,z_0}))$. *Then the following resolvent relation holds on* $L^2(\Omega; d^n x)$*,*

$$\begin{aligned}
&\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1} = \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} \\
&\quad + \big[\widehat{\gamma}_D\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^* \big(\tau^N_{z_0} - L_2\widehat{\gamma}_D\big)\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}.
\end{aligned} \tag{16.86}$$

Proof. First, we note that

$$\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} : L^2(\Omega; d^n x) \to \mathrm{dom}(L_1) \hookrightarrow \mathrm{dom}(L_2) \tag{16.87}$$

$$\hookrightarrow X_2 = \big(N^{1/2}(\partial\Omega)\big)^*. \tag{16.88}$$

Together with (16.40), (16.84), Lemma 6.2, (12.1) and (16.15), this ensures that the composition of the operators appearing on the right-hand side of (16.86) is well-defined. Next, let $\phi_1, \phi_2 \in L^2(\Omega; d^n x)$ be arbitrary and define

$$\begin{aligned}
&\psi_1 := (-\Delta^D_{X_1,L_1,z_0} - zI_\Omega)^{-1}\phi_1 \in \mathrm{dom}(-\Delta^D_{X_1,L_1,z_0}) \subset \mathrm{dom}(-\Delta_{max}), \\
&\psi_2 := (-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega)^{-1}\phi_2 \in \mathrm{dom}(-\Delta^D_{X_2,L_2,z_0}) \subset \mathrm{dom}(-\Delta_{max}).
\end{aligned} \tag{16.89}$$

Our goal is to show that the following identity holds:

$$\begin{aligned}
&\big(\phi_2, \big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1}\phi_1\big)_{L^2(\Omega;d^nx)} - \big(\phi_2, \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\phi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \big(\phi_2, \big[\widehat{\gamma}_D\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^*\big(\tau^N_{z_0} - L_2\widehat{\gamma}_D\big)\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\phi_1\big)_{L^2(\Omega;d^nx)}.
\end{aligned} \tag{16.90}$$

We note that according to (16.89) one has,

$$\begin{aligned}
&\big(\phi_2, \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\phi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \big(\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)\psi_2, \psi_1\big)_{L^2(\Omega;d^nx)},
\end{aligned} \tag{16.91}$$

$$\begin{aligned}
&\big(\phi_2, \big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1}\phi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \big(\big(\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1}\big)^*\phi_2, \phi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \big(\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)^{-1}\phi_2, \phi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \big(\psi_2, \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)\psi_1\big)_{L^2(\Omega;d^nx)},
\end{aligned} \tag{16.92}$$

and

$$\begin{aligned}
&\big(\phi_2, \big[\widehat{\gamma}_D\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^*\big(\tau^N_{z_0} - L_2\widehat{\gamma}_D\big)\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\phi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle\big(\tau^N_{z_0} - L_2\widehat{\gamma}_D\big)\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\phi_1, \widehat{\gamma}_D\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)^{-1}\phi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\
&\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle\big[\tau^N_{z_0}\psi_1 - L_2\widehat{\gamma}_D\psi_1\big], \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*}}.
\end{aligned} \tag{16.93}$$

Thus, matters have been reduced to proving that

$$\begin{aligned}
&\big(\psi_2, \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)\psi_1\big)_{L^2(\Omega;d^nx)} - \big(\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)\psi_2, \psi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle\big[\tau^N_{z_0}\psi_1 - L_2\widehat{\gamma}_D\psi_1\big], \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*}}.
\end{aligned} \tag{16.94}$$

Using (12.9) for the left-hand side of (16.94) one obtains

$$\begin{aligned}
&\big(\psi_2, \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)\psi_1\big)_{L^2(\Omega;d^nx)} - \big(\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)\psi_2, \psi_1\big)_{L^2(\Omega;d^nx)} \\
&\quad = (\psi_2, (-\Delta - (\overline{z} + z_0))\psi_1)_{L^2(\Omega;d^nx)} - ((-\Delta - (z + z_0))\psi_2, \psi_1)_{L^2(\Omega;d^nx)} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle\tau^N_{z_0}\psi_1, \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*} - \overline{{}_{N^{1/2}(\partial\Omega)}\big\langle\tau^N_{z_0}\psi_2, \widehat{\gamma}_D\psi_1\big\rangle_{(N^{1/2}(\partial\Omega))^*}} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle\tau^N_{z_0}\psi_1, \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*} - {}_{X_2}\langle\widehat{\gamma}_D\psi_1, L_2\widehat{\gamma}_D\psi_2\rangle_{X_2^*} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle\tau^N_{z_0}\psi_1, \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*} - \overline{{}_{X_2}\langle\widehat{\gamma}_D\psi_2, L_2\widehat{\gamma}_D\psi_1\rangle_{X_2^*}} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle\tau^N_{z_0}\psi_1, \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*} - {}_{N^{1/2}(\partial\Omega)}\langle L_2\widehat{\gamma}_D\psi_1, \widehat{\gamma}_D\psi_2\rangle_{(N^{1/2}(\partial\Omega))^*} \\
&\quad = {}_{N^{1/2}(\partial\Omega)}\big\langle\big[\tau^N_{z_0}\psi_1 - L_2\widehat{\gamma}_D\psi_1\big], \widehat{\gamma}_D\psi_2\big\rangle_{(N^{1/2}(\partial\Omega))^*},
\end{aligned} \tag{16.95}$$

since $\psi_2 \in \operatorname{dom}(-\Delta^D_{X_2,L_2,z_0})$ and

$$\begin{aligned}
\psi_1 \in \operatorname{dom}(-\Delta^D_{X_1,L_1,z_0}) \text{ implies } \widehat{\gamma}_D(\psi_1) \in &\operatorname{dom}(L_1) \hookrightarrow \operatorname{dom}(L_2) \\
&\hookrightarrow X_2 = \big(N^{1/2}(\partial\Omega)\big)^*.
\end{aligned} \tag{16.96}$$

This justifies (16.94), completing the proof of the lemma. $\square$

We are now prepared to state and prove the Krein-type formula alluded to earlier, expressing the difference of the resolvents of two arbitrary self-adjoint extensions of $-\Delta\big|_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^nx)$.

Theorem 16.12. *Assume Hypothesis* 8.10 *and suppose that* $z_0 \in \mathbb{R}\backslash\sigma(-\Delta_{D,\Omega})$. *In addition, consider two bounded, self-adjoint operators*

$$L_j : \big(N^{1/2}(\partial\Omega)\big)^* \to N^{1/2}(\partial\Omega), \quad j = 1, 2. \tag{16.97}$$

Associated with L_j, $j = 1, 2$, *define the self-adjoint operators* $-\Delta^D_{X_j,L_j,z_0}$, $j = 1, 2$, *as in* (14.13) *corresponding to* $z = z_0$ *and* $X_1 = X_2 = \big(N^{1/2}(\partial\Omega)\big)^*$. *In addition, let* $z \in \mathbb{C}\backslash\big(\sigma(-\Delta^D_{X_1,L_1,z_0}) \cup \sigma(-\Delta^D_{X_2,L_2,z_0})\big)$. *Then* *the following resolvent relation holds on $L^2(\Omega; d^n x)$,*

$$\begin{aligned}\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1} &= \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} \\ &\quad + \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big]^* M^D_{L_1,L_2,z_0}(z)\big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big].\end{aligned} \tag{16.98}$$

Here

$$\begin{aligned}&(L_2 - L_1)\big] \\ &= (L_2 - L_1)\big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z+z_0) - L_2\big]^{-1} \\ &\times \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z+z_0) - L_1\big]\end{aligned}$$

has the property that

$$\mathbb{C}_+ \ni z \mapsto M^D_{L_1,L_2,z_0}(z) \in \mathcal{B}\big((N^{1/2}(\partial\Omega))^*, N^{1/2}(\partial\Omega)\big) \tag{16.99}$$

is an operator-valued Herglotz function which satisfies

$$\big[M^D_{L_1,L_2,z_0}(z)\big]^* = M^D_{L_1,L_2,z_0}(\overline{z}). \tag{16.100}$$

Proof. Applying $\tau^N_{z_0} - L_1\widehat{\gamma}_D$ to both sides of (16.86) and using the fact that

$$\big(\tau^N_{z_0} - L_j\widehat{\gamma}_D\big)\big(-\Delta^D_{X_j,L_j,z_0} - zI_\Omega\big)^{-1} = 0, \quad j = 1, 2, \tag{16.101}$$

one obtains

$$\begin{aligned}&(L_2 - L_1)\big[\widehat{\gamma}_D\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1}\big] \\ &\quad = \big(\tau^N_{z_0} - L_1\widehat{\gamma}_D\big)\big[\widehat{\gamma}_D\big(-\Delta^D_{X_2,L_2,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^*(L_1 - L_2)\big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big] \\ &\quad = \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z+z_0) - L_1\big]M^D_{X_2,L_2,z_0}(z)(L_1 - L_2)\big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big],\end{aligned} \tag{16.102}$$

where in the last step we have also made use of (16.39) and (16.31). By taking adjoints of both sides in (16.102) one arrives at

$$\begin{aligned}&= \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big]^*(L_1 - L_2)M^D_{X_2,L_2,z_0}(\overline{z}) \\ &\times \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(\overline{z}+z_0) - L_1\big],\end{aligned}$$

using (16.30) and (11.10). Inserting this identity (written with $\overline{z}$ in place of z) into (16.86) then yields

$$\begin{aligned}\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1} &= \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} \\ &\quad + \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^*(L_2 - L_1)M^D_{X_2,L_2,z_0}(z) \\ &\qquad \times \big[M^{(0)}_{D,N,\Omega}(z_0) - M^{(0)}_{D,N,\Omega}(z+z_0) - L_1\big]\big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big].\end{aligned} \tag{16.103}$$

Making reference to (16.75), we may then write

$$\begin{aligned}\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1} &= \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} \\ &\quad + \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^*(L_2 - L_1)M^D_{X_2,L_2,z_0}(z) \\ &\qquad \times \big[M^D_{X_2,L_2,z_0}(z)^{-1} + (L_2 - L_1)\big]\big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big],\end{aligned} \tag{16.104}$$

or equivalently,

$$\begin{aligned}\big(-\Delta^D_{X_2,L_2,z_0} - zI_\Omega\big)^{-1} &= \big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1} \\ &\quad + \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - \overline{z}I_\Omega\big)^{-1}\big]^*(L_2 - L_1)[I + M^D_{X_2,L_2,z_0}(z)(L_2 - L_1)] \\ &\qquad \times \big[\widehat{\gamma}_D\big(-\Delta^D_{X_1,L_1,z_0} - zI_\Omega\big)^{-1}\big].\end{aligned} \tag{16.105}$$

Now (16.98) follows from this and (16.75).

Finally, the advertised properties for $M^D_{L_1,L_2,z_0}(z)$ are consequences of the formula

$$M^D_{L_1,L_2,z_0}(z) = (L_2 - L_1)\big[I + M^D_{X_2,L_2,z_0}(z)(L_2 - L_1)\big] \tag{16.106}$$

and the corresponding properties of $M^D_{X_2,L_2,z_0}(z)$. □

Remark 16.13. An alternative expression of $M^D_{L_1,L_2,z_0}(z)$ is given by

$$M^D_{L_1,L_2,z_0}(z) = M^D_{X_1,L_1,z_0}(z)^{-1}\big[M^D_{X_2,L_2,z_0}(z) - M^D_{X_1,L_1,z_0}(z)\big]M^D_{X_1,L_1,z_0}(z)^{-1}. \tag{16.107}$$

This follows from the second equality in (**??**) and (16.75).

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