Differentiability of Mather's average action and integrability on closed surfaces

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In this article we study the differentiability of Mather's β-function on closed surfaces and its relation to the integrability of the system.

Dans cet article nous étudions la différentiabilité de la fonction β de Mather sur surfaces et ses conséquences sur l'intégrabilité du système.

1 Introduction

In the study of Tonelli Lagrangian and Hamiltonian systems, a central role in understanding the dynamical and topological properties of the action-minimizing sets (also called *Mather* and *Aubry sets*), is played by the so-called *Mather's average action* (sometimes referred to as *β-function* or *effective Lagrangian*), with particular attention to its differentiability and non-differentiability properties. Roughly speaking, this is a convex superlinear function on the first homology group of the base manifold, which represents the minimal action of invariant probability measures within a prescribed *homology class*, or *rotation vector* (see Equation (1) for a more precise definition). Understanding whether or not this function is differentiable, or even smoother, and what are the implications of its regularity to the dynamics of the system is a formidable problem, which is still far from being completely understood. Examples of Lagrangians admitting a smooth β-function are easy to construct. Trivially, if the base manifold M is such that $\dim \mathrm{H}_1(M;\mathbb{R}) = 0$ then β is a function defined on a single point and it is therefore smooth. Furthermore, if $\dim \mathrm{H}_1(M;\mathbb{R}) = 1$ then a result by M. Dias Carneiro [8] allows one to conclude that β is differentiable everywhere, except possibly at the origin. As soon as $\dim \mathrm{H}_1(M;\mathbb{R}) \geq 2$ the situation becomes definitely less clear and the smoothness of β becomes a more "untypical" phenomenon. Nevertheless, it is still possible to find some interesting examples in which it is smooth. For instance, let $H : \mathrm{T}^*\mathbb{T}^n \longrightarrow \mathbb{R}$ be a completely integrable (in the sense of Liouville) Tonelli Hamiltonian system, given by $H(x,p) = h(p)$, and consider the associated Lagrangian $L(x,v) = \ell(v)$ on $\mathrm{T}\mathbb{T}^n$. It is easy to check that in this case, up to identifying $\mathrm{H}_1(\mathbb{T}^n;\mathbb{R})$ with $\mathbb{R}^n$, one has $\beta(h) = \ell(h)$ and therefore β is as smooth as the Lagrangian and the Hamiltonian are. One can weaken the assumption on the completely integrability of the system and consider *C^0-integrable systems, i.e.,* Hamiltonian systems that admit a foliation of the phase space by disjoint invariant continuous Lagrangian graphs, one for each possible cohomology class (see Definition 1). It is then possible to prove that in this case the associated β function is C^1 (see Lemma 2). These observations arise the following question: with the exception of the mentioned trivial cases, does the regularity of β imply the integrability of the system?

In this article we address the above problem in the case of Tonelli Lagrangians on closed surfaces, not necessarily orientable. We prove the following.

Main Results. *Let M be a closed surface and $L : \mathrm{T}M \longrightarrow \mathbb{R}$ a Tonelli Lagrangian.*

(i) *If M is not the sphere, the projective plane, the Klein bottle or the torus, then β cannot be C^1 on $H_1(M,\mathbb{R})$* [Proposition 3].

(ii) *If M is the sphere, the projective plane or the Klein bottle, then the Lagrangian cannot be C^0-integrable* [Proposition 4].

(iii) *If M is the torus, then β is C^1 if and only if the system is C^0-integrable* [Theorem 1].

Moreover in Corollary 1 we shall deduce several information on the dynamics of C^0-integrable systems and in section 3.1 we discuss the case of mechanical Lagrangians on the two-torus and show that in this case: *β is C^1 if and only if the system is the geodesic flow associated to a flat metric on $\mathbb{T}^2$* [Proposition 6].

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2 Mather's average action and Aubry-Mather theory

Let us start by recalling some basic facts about Mather's theory for Tonelli Lagrangians. Let M be a compact and connected smooth n-dimensional manifold without boundary. Denote by $\mathrm{T}M$ its tangent bundle and T^*M the cotangent one and denote points in $\mathrm{T}M$ and T^*M respectively by (x, v) and (x, p). We shall also assume that the cotangent bundle T^*M is equipped with the canonical symplectic structure, which we shall denote ω. A *Tonelli Lagrangian* is a C^2 function $L : \mathrm{T}M \to \mathbb{R}$, which is strictly convex and uniformly superlinear in the fibers; in particular this Lagrangian defines a flow on $\mathrm{T}M$, known as *Euler-Lagrange flow* Φ_t^L, whose integral curves are solutions of $\frac{d}{dt}\frac{\partial L}{\partial v}(x, v) = \frac{\partial L}{\partial x}(x, v)$. Let $\mathfrak{M}(L)$ be the space of compactly supported probability measures on $\mathrm{T}M$ invariant under the Euler-Lagrange flow of L. To every $\mu \in \mathfrak{M}(L)$, we may associate its *average action*

$$A_L(\mu) = \int_{\mathrm{T}M} L \, d\mu \, .$$

It is quite easy to check that since μ is invariant under the Euler-Lagrange flow, then for each $f \in C^1(M)$ we have $\int df(x) \cdot v \, d\mu = 0$. Therefore we can define a linear functional

$$\begin{array}{rcl} \mathrm{H}^1(M;\mathbb{R}) & \longrightarrow & \mathbb{R} \\ c & \longmapsto & \displaystyle\int_{\mathrm{T}M} \eta_c(x) \cdot v \, d\mu \, , \end{array}$$

where η_c is any closed 1-form on M whose cohomology class is c. By duality, there exists $\rho(\mu) \in \mathrm{H}_1(M;\mathbb{R})$ such that

$$\int_{\mathrm{T}M} \eta_c(x) \cdot v \, d\mu = \langle c, \rho(\mu) \rangle \qquad \forall\, c \in \mathrm{H}^1(M;\mathbb{R})$$

(the bracket on the right–hand side denotes the canonical pairing between cohomology and homology). We call $\rho(\mu)$ the *rotation vector* of μ. It is possible to show [19] that the map $\rho : \mathfrak{M}(L) \longrightarrow \mathrm{H}_1(M;\mathbb{R})$ is surjective and hence there exist invariant probability measures with any given rotation vector. Let us consider the minimal value of the average action A_L over the set of probability measures with rotation vector h (this minimum exists because of the lower semicontinuity of the action functional):

$$\begin{array}{rcl} \beta : \mathrm{H}_1(M;\mathbb{R}) & \longrightarrow & \mathbb{R} \\ h & \longmapsto & \displaystyle\min_{\mu \in \mathfrak{M}(L):\, \rho(\mu)=h} A_L(\mu)\,. \end{array} \tag{1}$$

This function β is what is generally known as *Mather's β-function.* A measure $\mu \in \mathfrak{M}(L)$ realizing such a minimum amongst all invariant probability measures with the same rotation vector, *i.e.,* $A_L(\mu) = \beta(\rho(\mu))$, is called an *action minimizing measure with rotation vector* $\rho(\mu)$. The β-function is convex, and therefore one can consider its *conjugate* function (given by Fenchel's duality) $\alpha : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \alpha(c) &:= \max_{h \in \mathrm{H}_1(M;\mathbb{R})} \left(\langle c, h\rangle - \beta(h)\right) = -\min_{h \in \mathrm{H}_1(M;\mathbb{R})} \left(\beta(h) - \langle c, h\rangle\right) = \\ &= -\min_{\mu \in \mathfrak{M}(L)} \left(A_L(\mu) - \langle c, \rho(\mu)\rangle\right) = -\min_{\mu \in \mathfrak{M}(L)} A_{L-\eta_c}(\mu), \end{aligned}$$

where η_c is any smooth closed 1-form on M with cohomology class c. Observe that the modified Lagrangian $L - \eta_c$ (η_c can be also seen as a function on $\mathrm{T}M$: $\eta_c(x,v) := \eta_c(x) \cdot v$) is still of Tonelli type and has the same Euler-Lagrange flow as L (it follows easily from the closedness of η_c). Nevertheless, invariant probability measures that minimize the action functional $A_{L-\eta_c}$ change according to the chosen cohomology class. A measure $\mu \in \mathfrak{M}(L)$ realizing the minimum of $A_{L-\eta_c}$ (withouth any constraint on the rotation vector) is called *c-action minimizing.* This leads to the definition of a first important family of invariant subsets of $\mathrm{T}M$:

- for a homology class $h \in \mathrm{H}_1(M;\mathbb{R})$, we define the *Mather set of rotation vector h* as:

$$\widetilde{\mathcal{M}}^h := \bigcup \{\mathrm{supp}\, \mu : \ \mu \text{ is action minimizing with rotation vector } \ h\}\,;$$

- for a cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we define the *Mather set of cohomology class c* as:

$$\widetilde{\mathcal{M}}_c := \bigcup \{\mathrm{supp}\, \mu : \ \mu \text{ is } c-\text{action minimizing}\}\,.$$

The relation between these sets is described in the following lemma. To state it, let us recall that, like any convex function on a finite-dimensional space, Mather's β function admits a subderivative at any point $h \in \mathrm{H}_1(M;\mathbb{R})$, *i.e.,* we can find $c \in \mathrm{H}^1(M;\mathbb{R})$ such that

$$\forall h' \in \mathrm{H}_1(M;\mathbb{R}), \quad \beta(h') - \beta(h) \geq \langle c, h' - h\rangle.$$

As it is usually done, we denote by $\partial\beta(h)$ the set of $c \in \mathrm{H}^1(M;\mathbb{R})$ which are subderivatives of β at h, that is, the set of c's which satisfy the above inequality (this set is also called the *Legendre transform of* h). By Fenchel's duality, we have

$$c \in \partial\beta(h) \Longleftrightarrow \langle c, h\rangle = \alpha(c) + \beta(h). \tag{2}$$

Lemma 1. *Let h, c be respectively an arbitrary homology class in $\mathrm{H}_1(M;\mathbb{R})$ and an arbitrary cohomology class in $\mathrm{H}^1(M;\mathbb{R})$. We have*

$$(1)\ \widetilde{\mathcal{M}}^h \cap \widetilde{\mathcal{M}}_c \neq \emptyset \quad \Longleftrightarrow \quad (2)\ \widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c \quad \Longleftrightarrow \quad (3)\ c \in \partial\beta(h)\,.$$

Proof. The implication (2) $\Longrightarrow$ (1) is trivial. Let us prove that (1) $\Longrightarrow$ (3). If $\widetilde{\mathcal{M}}^h \cap \widetilde{\mathcal{M}}_c \neq \emptyset$, then there exists a c-minimizing invariant measure μ with rotation vector h. Let η_c be a closed 1-form with $[\eta_c] = c$. From the definitions of α, β and rotation vector:

$$-\alpha(c) = \int_{\mathrm{T}M} (L - \eta_c)\, d\mu = \int_{\mathrm{T}M} L\, d\mu - \langle c, h\rangle = \beta(h) - \langle c, h\rangle\,;$$

since β and α are convex conjugates, then c is a subderivative of β at h.
Finally, in order to show (3) $\Longrightarrow$ (2), let us prove that any action minimizing measure with rotation vector h is c-minimizing. In fact, if $c \in \partial\beta(h)$ then $\beta(h) = \langle c, h\rangle - \alpha(c)$; therefore for any action minimizing measure μ with $\rho(\mu) = h$ and for any η_c as above:

$$-\alpha(c) = \beta(h) - \langle c, h\rangle = \int_{\mathrm{T}M} (L - \eta_c)\, d\mu.$$

This proves that μ is c-action minimizing and it concludes the proof. ☐

Remark 1. One can also find a relation between the Mather sets corresponding to different cohomology classes, in terms of the strictly convexity of the α-function or, better, the lack thereof. The regions where α is not strictly convex are called *flats*: for instance the Legendre transform of α at c, denoted $\partial\alpha(c)$, which is the set of homology classes for which equality (2) holds, is an example of flat. It is possible to check that if two cohomology classes are in the relative interior of the same flat F of α, then their Mather sets coincide (see [19, 15]). We denote by $\widetilde{\mathcal{M}}(F)$ the common Mather set to all the cohomologies in the relative interior of F.

In addition to the Mather sets, one can also construct another interesting family of compact invariant sets. We define the *Aubry set* $\widetilde{\mathcal{A}}_c$ *of cohomology class* c by looking at a special kind of *global minimizers*: we say $(x, v) \in TM$ lies in $\widetilde{\mathcal{A}}_c$ if there exists a sequence of C^1 curves $\gamma_n : [0, T_n] \longrightarrow M$, such that

- $\gamma_n(0) = \gamma_n(T_n) = x$ for all n
- $\dot\gamma_n(0)$ and $\dot\gamma_n(T_n)$ tend to v as n tends to infinity
- T_n goes to infinity with n
- $\int_0^{T_n} (L - \eta_c + \alpha(c))(\gamma_n(t), \dot\gamma_n(t))\, dt$ tends to 0 as n tends to infinity.

One can prove [20, 11] that if $(x,v) \in \widetilde{\mathcal{A}}_c$, then the curve $\gamma(t) := \pi \circ \Phi_t^L$, where $\pi : \mathrm{T}M \longrightarrow M$ denotes the canonical projection, is a *c-global minimizer*, that is, it minimizes on any compact time interval the action of $L - \eta_c$, among all curves with the same endpoints and time length (see [19]); observe, however, that not all c-global minimizers can be obtained in this way.
These action-minimizing sets that we have just defined, are such that $\widetilde{\mathcal{M}}_c \subseteq \widetilde{\mathcal{A}}_c$ for all $c \in \mathrm{H}^1(M;\mathbb{R})$ and moreover one of their most important features is that they are graphs over M (*Mather's graph theorem* [19, 20]), *i.e.,* the projection along the fibers $\pi|\widetilde{\mathcal{A}}_c(L)$ is injective and its inverse $\left(\pi|\widetilde{\mathcal{A}}_c(L)\right)^{-1}\colon \pi\big(\widetilde{\mathcal{A}}_c(L)\big) \longrightarrow \widetilde{\mathcal{A}}_c(L)$ is Lipschitz. Hereafter we shall denote by $\mathcal{M}^h$, $\mathcal{M}_c$ and $\mathcal{A}_c$ the corresponding projected sets.

Another interesting characterization of the Aubry set is provided by *weak KAM theory* [11], in terms of *critical subsolutions* of Hamilton-Jacobi equation or, in a more geometric way, in terms of some special *Lipschitz Lagrangian graphs.* Let us consider the Hamiltonian system associated to our Tonelli Lagrangian. In fact, if one considers the Legendre transform associated to L, *i.e.,* the diffeomorphism $\mathcal{L}_L : \mathrm{T}M \longrightarrow \mathrm{T}^*M$ defined by $\mathcal{L}_L(x,v) = (x, \frac{\partial L}{\partial v}(x,v))$, then it is possible to introduce a Hamiltonian system $H : \mathrm{T}^*M \to \mathbb{R}$, where $H(x,p) = \sup_{v \in \mathrm{T}_xM}(\langle p, v\rangle - L(x,v))$. It is easy to check that H is also C^2, strictly convex and uniformly superlinear in each fiber: H is also said to be *Tonelli* (or sometimes *optical*). Recall that the flow Φ_t^H on T^*M associated to this Hamiltonian, known as the *Hamiltonian flow* of H, is conjugated - via the Legendre transform - to the Euler-Lagrange flow of L. Therefore one can define the analogue of the Mather and Aubry sets in the cotangent space, simply considering $\mathcal{M}_c^* := \mathcal{L}_L\big(\widetilde{\mathcal{M}}_c\big)$ and $\mathcal{A}_c^* := \mathcal{L}_L\big(\widetilde{\mathcal{A}}_c\big)$. These sets still satisfy all the properties mentioned above, including the graph theorem. Moreover, they will be contained in the energy level $\{H(x,p) = \alpha(c)\}$, where $\alpha : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is exactly *Mather's α-function* introduced before.
Using the results in [11] it is possible to obtain a nice characterization of the Aubry set. As above, let η_c be a closed 1-form with cohomology class c; we shall say that $u \in C^{1,1}(M)$ is an η_c-critical subsolution if it satisfies $H(x, \eta_c + d_x u) \leq \alpha(c)$ for all $x \in M$. The existence of such functions has been showed by P. Bernard [4]. If one denotes by $\mathcal{S}_{\eta_c}$ the set of η_c-critical subsolutions, then:

$$\mathcal{A}_c^* = \bigcap_{u \in \mathcal{S}_{\eta_c}} \{(x, \eta_c(x) + d_x u) : \ x \in M\}\,. \tag{3}$$

This set does not depend on the particular choice of η_c, but only on its cohomology class. Observe that since in (T^*M, ω) there is a 1-1 correspondence between Lagrangian graphs and closed 1-forms, then we can interpret the graphs of these critical subsolutions as Lipschitz Lagrangian graphs in T^*M and the Aubry set may be seen as their *non-removable* intersection.

3 Main results

In this section we prove the main results stated in the introduction. Let M be a closed surface, not necessarily orientable, $L : \mathrm{T}M \longrightarrow \mathbb{R}$ a Tonelli Lagrangian on M and $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ the associated Hamiltonian.

Let us recall that a homology class h is said to be k-irrational, if k is the dimension of the smallest subspace of $\mathrm{H}_1(M;\mathbb{R})$ generated by integer classes and containing h. In particular, 1-*irrational* means "on a line with rational slope", while *completely irrational* stands for

$\dim \mathrm{H}_1(M;\mathbb{R})$-irrational. Moreover, a homology h is said to be *singular* if its Legendre transform $\partial\beta(h)$ is a *singular flat, i.e.,* its Mather set $\widetilde{\mathcal{M}}(\partial\beta(h))$ - see Remark 1 - contains fixed points. Observe that the set of singular classes, unless it is empty, contains the zero class and is compact.

For $h \in \mathrm{H}_1(M;\mathbb{R})\setminus\{0\}$, we define the maximal radial flat R_h of β containing h as the largest subset of the half-line $\{th\ : t \in [0,+\infty[\}$ containing h (not necessarily in its relative interior) in restriction to which β is affine. The possibility of radial flats is the most obvious difference between the β functions of Riemannian metrics [14, 3] and those of general Lagrangians. An instance of radial flat is found for instance in [9]. We define the Mather set $\widetilde{\mathcal{M}}(R_h)$ as the union of the supports of all action minimizing measures with rotation vector th, for $th \in R_h$.

Let h be a homology class. Assume h is 1-irrational. Then for all t such that $th \in R_h$, th is also 1-irrational and $\partial\beta(th) = \partial\beta(h)$ (see [16, Lemma 17]). In particular, R_h is contained in a flat of β. So Mather's Graph Theorem and [16, Lemma C.3] combine to say that $\widetilde{\mathcal{M}}(R_h)$ is a union of pairwise disjoint periodic orbits, or fixed points, γ_i, $i \in I$ where I is some set, not necessarily finite.

We shall start with the following proposition, which generalizes a result by Bangert [2] and Mather [18] for twist maps and geodesic flows on the two-torus.

Proposition 1. *Let L be a Tonelli Lagrangian on a closed surface M and let $h \in \mathrm{H}_1(M;\mathbb{R})$ be a non-singular 1-irrational homology class. Then, β is differentiable at h if and only if $\mathcal{M}(R_h) = M$. In that case, $\mathcal{M}(R_h) = M$ is foliated by closed extremals.*

Proof. [$\Longleftarrow$] Since $\mathcal{M}(R_h) = M$, then β is differentiable at h. In fact, suppose that $c, c' \in \partial\beta(h)$; then $\widetilde{\mathcal{M}}(R_h) \subseteq \widetilde{\mathcal{M}}_c$ and $\widetilde{\mathcal{M}}(R_h) \subseteq \widetilde{\mathcal{M}}_{c'}$ (Lemma 1). From the graph property of the Mather and Aubry sets it follows that $\widetilde{\mathcal{M}}_c = \widetilde{\mathcal{M}}_{c'}$ and $\widetilde{\mathcal{A}}_c = \widetilde{\mathcal{A}}_{c'}$. In particular, if $\mathcal{L}_L : \mathrm{T}M \longrightarrow \mathrm{T}^*M$ denotes the Legendre transform associated to L, then $\mathcal{A}^*_c = \mathcal{L}_L(\widetilde{\mathcal{A}}_c)$ and $\mathcal{A}^*_{c'} = \mathcal{L}_L(\widetilde{\mathcal{A}}_{c'})$ are invariant Lipschitz Lagrangian graphs and since they coincide, they must have the same cohomology class, *i.e.,* $c = c'$.
[$\Longrightarrow$] Since β is differentiable at h, let us denote $c_h := \partial\beta(h)$. From [16, Theorem 3] it follows that $\widetilde{\mathcal{M}}(R_h) = \widetilde{\mathcal{M}}_{c_h} = \widetilde{\mathcal{A}}_{c_h}$ and it is a union of periodic orbits. From this, one can deduce that:

- for all $c' \neq c_h$, $\widetilde{\mathcal{M}}_{c_h} \cap \widetilde{\mathcal{M}}_{c'} = \emptyset$; otherwise, $\widetilde{\mathcal{M}}(R_h) \cap \widetilde{\mathcal{M}}_{c'} \neq \emptyset$ and this would imply that $c' \in \partial\beta(h)$ (Lemma 1), contradicting the differentiability of β at h.

- α is strictly convex at c_h: it is an easy consequence of the disjointness of $\widetilde{\mathcal{M}}_{c_h}$ from the other Mather sets and Remark 1.

In order to complete the proof, we need to show that $\mathcal{A}_{c_h} = M$. If this were not true, then, there would exist an open region Γ in the complement of the $\mathcal{A}_{c_h}$, whose boundary consists of two (minimizing) periodic orbits. Γ would be homologically non trivial, therefore one could find a closed non-exact 1-form on M whose support is disjoint from $\mathcal{A}_{c_h}$. From [15, Theorem 1] (see [17, Theorem 2] for the non-orientable case), this would imply that c_h is contained in the interior of a non-trivial flat of α, contradicting the strict convexity of α at c_h. $\square$

Remark 2. Observe that the above result is not true if h is singular. Indeed, take a vector field X on a closed surface Σ_2 of genus 2, which consists of periodic orbits, except for a singular graph with two fixed points and four hetero/homoclinic orbits between the two fixed points (see figure 1). One can embed it into the Euler-Lagrange flow of a Tonelli Lagrangian, given by $L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2$. It is possible to show (see for instance [13]) that Graph(X) is invariant and that in this case $\widetilde{\mathcal{A}}_0 = \mathrm{Graph}(X)$; therefore, the projected Aubry set $\mathcal{A}_0$ is the whole surface Σ_2, so β is differentiable at all homology classes in $\partial\alpha(0)$ (see [15, Theorem 1]). However, $\mathcal{A}_0 = \Sigma_2$ is not foliated by closed extremals.

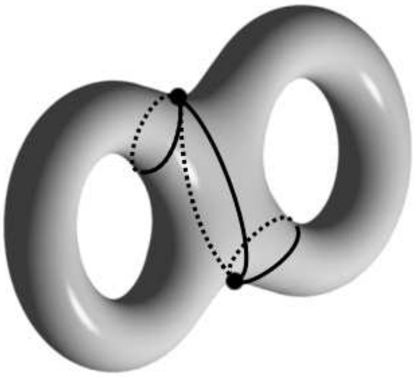

Figure 1: An example of singular Aubry set.

For the sake of completeness, let us also recall that:

Proposition 2. *β is always differentiable at completely irrational homology classes.*

It follows from [15, Corollary 3] (see [17, Theorem 2] for the non-orientable case).

Let us now recall the definition of C^0-integrability (see also [1]).

Definition 1. *A Tonelli Hamiltonian $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ is said to be C^0-integrable, if there exists a foliation of T^*M made of invariant Lipschitz Lagrangian graphs, one for each cohomology class.*

Lemma 2. *Let M be a compact manifold of any dimension, $L : \mathrm{T}M \longrightarrow \mathbb{R}$ a Tonelli Lagrangian and $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ the associated Hamiltonian. If H is C^0-integrable, then β is C^1.*

Proof. Suppose that H is C^0-integrable. This means that the cotangent space T^*M is foliated by disjoint invariant Lipschitz Lagrangian graphs. Let us denote by Λ_c the invariant Lagrangian graph of such a foliation, corresponding to the cohomology class c. It is easy to check that each Λ_c is the graph of a solution of Hamilton-Jacobi equation $H(x, \eta_c + du) = \alpha(c)$, where η_c is a closed 1-form on M with cohomology class c, and therefore from weak KAM theory [11] it follows that for each $c \in \mathrm{H}^1(M;\mathbb{R})$ the Aubry set $\mathcal{A}_c^* \subseteq \Lambda_c$. If for some $h \in \mathrm{H}_1(M;\mathbb{R})$ there exist $c \neq c'$ such that $c, c' \in \partial\beta(h)$, then $\widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c \cap \widetilde{\mathcal{M}}_{c'}$ (Lemma 1). But this implies that $\widetilde{\mathcal{M}}_c \cap \widetilde{\mathcal{M}}_{c'} \neq 0$ and therefore $\Lambda_c \cap \Lambda_{c'} \supseteq \mathcal{A}_c^* \cap \mathcal{A}_{c'}^* \neq 0$, which contradicts the disjointness of the Lagrangian graphs forming the foliation. Hence, β must be differentiable everywhere.

$\square$

We can now prove the main results stated in the Section 1.

Lemma 3. *Let M be a closed surface other than the sphere, the projective plane, the Klein bottle or the torus, and let $L : \mathrm{T}M \longrightarrow \mathbb{R}$ be a Tonelli Lagrangian. Then, β is not differentiable at any 1-irrational non-singular homology class.*

Proof. If β is differentiable at some non-singular 1-irrational homology class, from Proposition 1 it follows that the Mather set corresponding to such a class projects over the whole manifold and consists of periodic orbits. But this leads to a contradiction, since it implies the existence of a fixed-point-free vector field on M. □

When β is C^1 everywhere, we can improve on Proposition 1, by ruling out radial flats of β.

Proposition 3. *Let L be an autonomous Tonelli Lagrangian on the two-torus such that β is differentiable at every point of $\mathrm{H}_1(\mathbb{T}^2;\mathbb{R})$. Then for all $h \in \mathrm{H}_1(\mathbb{T}^2;\mathbb{R}) \setminus \{0\}$, we have $R_h = \{h\}$. In particular, β is strictly convex.*

Proof. First case: h is 1-irrational.

Replacing, if we have to, h with an extremal point of R_h, which is also 1-irrational, we may assume that $R_h = [\lambda h, h]$ for some $0 \le \lambda \le 1$. We want to prove that $\lambda = 1$. Pick:

- $(x, v) \in \mathrm{T}\mathbb{T}^2$ such that the probability measure carried by the orbit $\phi_t(x, v)$ has homology λh,
- a sequence of real numbers $t_n \ge 1$ such that t_n converge to 1
- for each $n \in \mathbb{N}$, $h_n := t_n h$ and $c_n \in \mathrm{H}^1(\mathbb{T}^2;\mathbb{R})$ such that $\partial\beta(h_n) = c_n$ (recall that β is differentiable at h_n).

Then by Proposition 1, since β is differentiable at $t_n h$, which is 1-irrational, we have $\mathcal{M}_{c_n} = M$; thus for every n there exists a $v_n \in T_x M$ such that $(x, v_n) \in \widetilde{\mathcal{M}}_{c_n}$, and moreover the orbit $\phi_t(x, v_n)$ is periodic. By semicontinuity of the Aubry set in dimension two [5] and by the Graph Theorem, v_n converges to v when n goes to infinity (in fact c_n converges to $c := \partial\beta(R_h)$). Let T and T_n be the minimal periods of $\phi_t(x, v)$ and $\phi_t(x, v_n)$, respectively. If (x, v) is a fixed point we just set $T := +\infty$. We now prove that $\liminf T_n \ge T$. Indeed, if some subsequence $T_{n_k} := T_k$ converged to $T' < T$, we would have $\phi_{T'}(x, v) = (x, v)$, contradicting the minimality of T. If (x, v) is a fixed point, we have $v = 0$, so v_n converges to zero, so T_n tends to infinity.

Let $h_0 \in \mathrm{H}_1(\mathbb{T}^2;\mathbb{Z})$ be a generator of $\mathbb{R}h \cap \mathrm{H}_1(\mathbb{T}^2;\mathbb{Z})$ such that the probability measures carried by the orbits $\phi_t(x, v)$ and $\phi_t(x, v_n)$ have homologies $T^{-1}h_0 = \lambda h$ and $T_n^{-1}h_0$, respectively.

Now if $\lambda < 1$, since $\liminf T_n \ge T$, for n large enough there would exist a c_n-minimizing measure with homology in $[\lambda h, h)$. This means that the radial flats R_h and $R_{t_n h}$ overlap, in other words, $t_n h \in R_h$. This contradicts the fact that $R_h = [\lambda h, h]$, hence $\lambda = 1$.

Second case: h is 2-irrational, that is, completely irrational.

Then any action minimizing measure with rotation vector h is supported on a lamination of the torus without closed leaves. Any such lamination is uniquely ergodic, in particular h is not contained in any non-trivial flat of β, radial or not, regardless of the Lagrangian.

The statement about the differentiability of β implying its strict convexity is now just a consequence of the fact, observed by M. Dias Carneiro in [8], that for an autonomous Lagrangian on a closed, orientable surface M, the flats of β are contained in isotropic subspaces of $\mathrm{H}_1(M;\mathbb{R})$ with respect to the intersection symplectic form on $\mathrm{H}_1(M;\mathbb{R})$. In particular, when $M=\mathbb{T}^2$, all flats are radial. □

Lemma 4. *Let $L:\mathrm{T}\mathbb{T}^2\longrightarrow\mathbb{R}$ be a Tonelli Lagrangian. Assume that β is C^1 everywhere. Then zero is (possibly) the only singular class, and for every 1-irrational, nonzero homology class h, $\mathcal{M}^h=\mathbb{T}^2$, and $\mathcal{M}^h$ is foliated by periodic orbits with the same homology class and minimal period.*

Proof. First note that if a non-zero class h is singular, then the Mather set of R_h contains a fixed point, so R_h contains the homology of the Dirac measure on the fixed point, which is zero. This contradicts Proposition 3, which says that $R_h=\{h\}$.

Now take a non-zero, 1-irrational homology class h; so h is non-singular and by Proposition 1, $\mathcal{M}(R_h)=\mathbb{T}^2$ and $\mathcal{M}(R_h)$ is foliated by periodic orbits. Since $R_h=\{h\}$ by Proposition 3, we have $\mathcal{M}^h=\mathbb{T}^2$. It is well known that all leaves of a foliation of the two-torus by homologically non trivial closed curves are homologous. So there exists $h_0\in\mathrm{H}_1(\mathbb{T}^2;\mathbb{Z})$ such that the projection to $\mathbb{T}^2$ of any orbit contained in $\mathcal{M}^h$ is homologous to h_0. Besides, for each $x\in\mathbb{T}^2$, let $T(x)$ be the minimal period of the periodic orbit γ_x in $\mathcal{M}^h$ through x. The homology class of the probability measure carried by γ_x is $T^{-1}h_0$. Now the fact that there are no non-trivial radial flats of β implies that $T(x)$ is independent of x. □

Proposition 4. *The torus is the only closed surface which admits a C^0-integrable Hamiltonian.*

Proof. First, no Hamiltonian on the sphere can be C^0-integrable. Indeed, any Lagrangian graph is exact since the sphere is simply connected, and any two exact Lagrangian graphs intersect, because any C^1 function on the sphere has critical points.

Likewise, no Hamiltonian on the projective plane can be C^0-integrable, for its lift to the sphere would be C^0-integrable.

For the Klein bottle $\mathbb{K}$, we need to work a little bit more. For each $x\in\mathbb{K}$, let us define

$$\begin{aligned} F_x:\mathrm{H}^1(\mathbb{K};\mathbb{R})\simeq\mathbb{R} &\longrightarrow \mathrm{T}^*_x\mathbb{K}\simeq\mathbb{R}^2\\ c &\longmapsto \Lambda_c(x), \end{aligned}$$

where Λ_c, for $c\in\mathrm{H}^1(\mathbb{K};\mathbb{R})$, are the Lagrangian graphs foliating $\mathrm{T}^*\mathbb{K}$. It is easy to check that F_x is continuous and injective (as it follows from the disjointness of the Λ_c's). Moreover, if the Hamiltonian is C^0-integrable, the map F_x is surjective. Now there is no such thing as a continuous bijection from $\mathbb{R}$ to $\mathbb{R}^2$, so there is no C^0-integrable Hamiltonian on the Klein bottle.

The same argument can be used for any surface with first Betti number >2.

Finally, no Hamiltonian on the connected sum of three projective planes (first Betti number equal to 2) can be C^0-integrable, for it would lift to a C^0-integrable Hamiltonian on a surface of genus two.

□

Theorem 1. *Let $L:\mathrm{T}\mathbb{T}^2\longrightarrow\mathbb{R}$ be a Tonelli Lagrangian on the two-torus. Then, β is C^1 if and only if the system is C^0-integrable.*

Proof. [$\Longleftarrow$] It follows from Lemma 2. [$\Longrightarrow$] For each homology class h, let us denote $c_h:=\partial\beta(h)$. If h is non-singular and 1-irrational, then $\Lambda_{c_h}:=\mathcal{A}^*_{c_h}$ is an invariant Lipschitz Lagrangian graph of cohomology class c_h, which is foliated by periodic orbits of homology h.

Observe that such cohomology classes c_h are dense in $\mathrm{H}^1(M;\mathbb{R})$. Indeed, 1-irrational non-singular homology classes are dense in $\mathrm{H}_1(M;\mathbb{R})$ because 1-irrational homology classes are dense in $\mathrm{H}_1(M;\mathbb{R})$ and, by Lemma 4, zero is the only possibly singular one. On the other hand, since β is differentiable, the Legendre transform is a homeomorphism from $\mathrm{H}_1(M;\mathbb{R})$ to $\mathrm{H}^1(M;\mathbb{R})$.

Using the semicontinuity of the Aubry set in dimension 2 (see [5]), we can deduce that for each $c \in \mathrm{H}^1(M;\mathbb{R})$ the Aubry set $\mathcal{A}^*_c$ projects over the whole manifold and therefore it is still an invariant Lipschitz Lagrangian graph, which we shall denote Λ_c. Observe that all these Λ_c's are disjoint. This is a straightforward consequence of the differentiability of β. In fact if for some $c \neq c'$ we had that $\Lambda_c \cap \Lambda_{c'} \neq 0$, then $\widetilde{\mathcal{M}}_c \cap \widetilde{\mathcal{M}}_{c'} \neq 0$; but this would contradict the differentiability of β at some homology class. It only remains to prove that the union is the whole T^*M. Observe that for each $c \in \mathrm{H}^1(M;\mathbb{R})$, Λ_c is the graph of the unique solution of the Hamilton-Jacobi equation $H(x, \eta_c + du_c) = \alpha(c)$, where $u_c \in C^{1,1}(M)$ and η_c is a closed 1-form on M with cohomology class c. For each $x_0 \in M$, let us define

$$\begin{array}{rcl} F_{x_0} : \mathrm{H}^1(M;\mathbb{R}) \simeq \mathbb{R}^2 & \longrightarrow & \mathrm{T}^*_{x_0}M \simeq \mathbb{R}^2 \\ c & \longmapsto & \eta_c + d_{x_0}u_c. \end{array}$$

It is easy to check that F_{x_0} is continuous (actually locally Lipschitz); moreover, it is injective (as it follows from the disjointness of the Λ_c's). Therefore, $F_{x_0}(\mathbb{R}^2)$ is open [6]. Using Ascoli-Arzelà's theorem, one can show that this image is also closed. In fact, let $y_k = F_{x_0}(c_k)$ be a sequence in $\mathrm{T}^*_x M$ converging to some y_0. Since each family of classical solutions of Hamilton-Jacobi on which the α-function (*i.e.,* the energy) is bounded gives rise to a family of functions with uniformly bounded Lipschitz constants, then there exist $\tilde{u} \in C^{1,1}(M)$ and $\tilde{c} \in \mathrm{H}^1(M;\mathbb{R})$, such that $c_k + du_{c_k} \to \tilde{c} + d\tilde{u}$ uniformly. From the continuity of the α-function, it follows that $H(x, \tilde{c} + d_x\tilde{u}) = \alpha(\tilde{c})$. Since $\Lambda_{\tilde{c}}$ is the unique invariant Lagrangian graph with cohomology $\tilde{c}$ (essentially because it coincides with the Aubry set, see [11]), then $\Lambda_{\tilde{c}} = \{(x, \tilde{c} + d_x\tilde{u} :\ x \in M\}$ and therefore $y_0 = \tilde{c} + d_{x_0}\tilde{u} = F_{x_0}(\tilde{c}) \in F_{x_0}(\mathbb{R}^2)$. This shows that $F_{x_0}(\mathbb{R}^2)$ is also closed and therefore it is all of $\mathbb{R}^2$. Since this holds for all $x_0 \in M$, it follows that $\cup_c \Lambda_c = \mathrm{T}^*M$. ☐

We can deduce more information about the dynamics on a C^0-integrable system.

Corollary 1. *Let $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ be a C^0-integrable Hamiltonian on a two-dimensional closed manifold M. Then, M is diffeomorphic to $\mathbb{T}^2$. Moreover:*

- *(i) for each 1-irrational homology class h, there exists an invariant Lagrangian graph foliated by periodic orbits with the same homology and minimal period;*
- *(ii) for each completely irrational homology class, there exist an invariant Lagrangian graph on which the motion is conjugated to an irrational rotation on the torus or to a Denjoy type homeomorphism;*
- *(iii) there exists a dense G_δ set of (co)homology classes, for which the motion on the corresponding invariant torus is conjugated to a rotation;*
- *(iv) as for the 0-homology class, there exists a C^1 invariant torus $\Lambda_{c(0)} = \{(x, \frac{\partial L}{\partial v}(x,0) :\ x \in \mathbb{T}^2\}$ consisting of fixed points.*

Proof. If H is C^0 integrable, then β is C^1 (Lemma 2). By Proposition 4, M is diffeomorphic to the 2-torus.

(i) It is a consequence of Lemma 4. (ii) Let h be a completely irrational homology class and let $c_h = \partial\beta(h)$. We shall follow the discussions in [8, p. 1084] (see also [21, p. 66 et seqq.] for more technical details). The Hamiltonian flow on Λ_{c_h} induces a flow on $\mathbb{T}^2$ without fixed points or closed trajectories. Thus, one can construct a non-contractible closed curve Γ, which is transverse to the flow. Using an argument *à la* Poincaré-Bendixon, it follows that all trajectories starting on Γ must return to it. Hence, one can define a continuous map from a compact subset of Γ to itself, which is order preserving and with irrational rotation number. Therefore, it is either conjugate to an irrational rotation or a Denjoy type homeomorphism.
(iii) From ii), it is sufficient to show that the set of homology classes for which the Mather set projects over the whole torus is a dense G_δ set; in fact if this is the case, the flow cannot be conjugated to a Denjoy type homeomorphism. Let us start by observing that this set is clearly dense since it contains all 1-irrational homology classes. The fact that it is also G_δ follows from [10, Corollaire 4.5], in which it is proven that the set of strictly ergodic flows on a compact set is G_δ in the C^0 topology. One could also observe that since our Lagrangian is C^0-integrable, by [1, Théorème 3], there is a dense G_δ of cohomology classes with a C^2 weak KAM solution; and by a recent result of Fathi [12], when the flow is of Denjoy type, there cannot be a C^2 weak KAM solution.
(iv) Since the union of the Aubry sets foliates all the phase space, then any point $(x, 0)$ will be contained in some Aubry set and therefore, using [13, Proposition 3.2], it follows that it is a fixed point of the Euler-Lagrange flow. Hence, the Dirac-measure $\delta_{(x,0)}$ is invariant and action minimizing (since its support is contained in some Aubry set); clearly, such a measure has rotation vector equal to 0. Therefore, $\widetilde{\mathcal{M}}^0 \supseteq \mathbb{T}^2 \times \{0\}$ and from the graph property it follows that they coincide. Then, $\Lambda_{c(0)} := \mathcal{A}^*_{c(0)} = \left\{(x, \frac{\partial L}{\partial v}(x, 0) : \ x \in \mathbb{T}^2\right\}$, which is at least C^1.

□

Remark 3. Although we believe that the motions on all invariant Lagrangian tori must be conjugated to rotations, we have not been able to show more than iii). It is not clear to us whether it is possible or not that a Denjoy type homeomorphism is embedded into a C^0-integrable Hamiltonian system.

3.1 The case of mechanical Lagrangians

Recall that a mechanical Lagrangian is of the form $L(x, v) = 1/2\, g_x(v, v) + f(x)$, where g is a Riemannian metric and f is a C^2 function on $\mathbb{T}^2$. In this case we can bridge the gap between C^0-integrability and complete integrability (in the sense of Liouville), using Burago and Ivanov's theorem on metrics without conjugate points [7].

Proposition 5. *Let L be a C^0-integrable mechanical Lagrangian on an n-dimensional torus. Then the potential f is identically zero and the metric g is flat. In particular, L is completely integrable.*

Proposition 6. *Let L be a mechanical Lagrangian on a 2-dimensional torus, whose β-function is C^1. Then the potential f is identically zero and the metric g is flat. In particular, L is completely integrable.*

First we need the following lemma (see also [22, Lemma 1]).

Lemma 5. *Let*

- *M be a closed manifold of any dimension*
- *L be an autonomous Tonelli Lagrangian on M, such that $L(x,v) = L(x,-v)$ for all (x,v) in TM.*

Then, the Aubry set $\widetilde{\mathcal{A}}_0$ consists of fixed points of the Euler-Lagrange flow.

Proof. Take $(x,v) \in \widetilde{\mathcal{A}}_0$. Then, by [11] there exists a sequence of C^1 curves $\gamma_n : [0,T_n] \longrightarrow M$, such that

- $\gamma_n(0) = \gamma_n(T_n) = x$ for all $n \in \mathbb{N}$
- $T_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$
- $\dot{\gamma}_n(0) \longrightarrow v$ and $\dot{\gamma}_n(T_n) \longrightarrow v$ as $n \longrightarrow +\infty$
- $\int_0^{T_n} (L(\gamma_n, \dot{\gamma}_n) + \alpha(0))\, dt \longrightarrow 0$ as $n \longrightarrow +\infty$.

Consider the sequence of curves $\delta_n : [0,T_n] \longrightarrow M$, such that $\delta_n(t) := \gamma_n(T_n - t)$ for all $n \in \mathbb{N}$, $t \in [0,T_n]$. Then, since L is symmetrical, $\int_0^{T_n}(L(\delta_n, \dot{\delta}_n) + \alpha(0))dt \longrightarrow 0$ as $n \longrightarrow +\infty$, which proves that $(x,-v) \in \widetilde{\mathcal{A}}_0(L)$. Therefore, by Mather's Graph Theorem, $v = 0$, so (x,v) lies in the zero section of TM. Now, by [13, Proposition 3.2], for any Lagrangian, the intersection of $\widetilde{\mathcal{A}}_0$ with the zero section of TM consists of fixed points of the Euler-Lagrange flow. This proves the lemma. ☐

Now let us assume that the Lagrangian is mechanical. Then, the only fixed points of the Euler-Lagrange flow are the critical points of the potential f, and the only minimizing fixed points are the minima of f. So if f is not constant, the Lagrangian cannot be C^0-integrable. Furthermore, since the Lagrangian is C^0-integrable, every orbit is minimizing, in particular, there are no conjugate points. So by Burago and Ivanov's proof of the Hopf Conjecture [7], the metric g is flat. This proves Proposition 5. Proposition 6 is now just a consequence of Theorem 1.

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