Time ordering and counting statistics

S. Bachmann [(a)], G.M. Graf [(a)], G.B. Lesovik [(a,b)]

[(a)] *Theoretische Physik, ETH-Hönggerberg, 8093 Zürich, Switzerland*

[(b)] *L.D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia*

July 22, 2009

Abstract

The basic quantum mechanical relation between fluctuations of transported charge and current correlators is discussed. It is found that, as a rule, the correlators are to be time-ordered in an unusual way. Instances where the difference with the conventional ordering matters are illustrated by means of a simple scattering model. We apply the results to resolve a discrepancy concerning the third cumulant of charge transport across a quantum point contact.

1 Introduction

Transport in mesoscopic systems has been discussed using different approaches, based on *current* correlators [1, 2, 3], either time- or in/out-ordered, or on the statistics of the transferred *charge* [4] (and, related to the latter, on the precession of a spin coupled to current [5]). In this work we intend to address two main points. First, we describe the relation between the counting statistics and the time ordering of correlators. It turns out that the correct time ordering differs as a rule from the conventional one, T, and is given by the Matthews' T^*-ordering [6]. Second, we will present a model of energy independent scattering, where that difference matters. Though the model is implicit in previous works [2, 5, 7], its formalization allows to establish the equivalence between the in/out- and the T^*-ordering of currents and hence between the two approaches mentioned at the beginning.

An application of these findings is the clarification of a discrepancy between [8] and [7] concerning the third cumulant of charge transfer through a quantum point contact. In [7] the discrepancy was claimed to be entirely due to the difference existing between unordered and T-ordered correlators [9, 4]. Our explanation is different, as we spell out momentarily.

An often chosen framework rests on states $|t, \alpha\rangle$ forming a basis and labelled by their time t of passage across a fiducial point and w.r.t. a reference dynamics, as well as by further quantum numbers α. To be precise, in such a scheme t is an observable which entails as its canonically conjugate operator a Hamiltonian whose spectrum similarly covers the real line.

The description is therefore an effective one valid near the Fermi energy; indeed, from that perspective the energy spectrum appears unbounded. This remark being made, the scattering amplitude from $|t_1, \alpha_1\rangle$ to $|t_2, \alpha_2\rangle$ is denoted by $S_{\alpha_2\alpha_1}(t_2 - t_1)$. Equivalence between in/out- and T-ordering has been shown [10, 7] under the assumption that the scattering matrix satisfies

$$S(t) = 0 \,, \quad (t \le 0) \,. \tag{1.1}$$

The assumption, there called causality, should be called strict causality, as it e.g. rules out the limiting, but simple case of instantaneous, $S(t) \propto \delta(t)$, or equivalently, energy independent scattering. Actually, in the latter case, disagreement between the two orderings was found [8, 11]. Our observation that T^* rather than T matters explains this difference, in that only for strictly causal scattering the two orderings agree. Generally the difference takes the guise of a Schwinger term related to the infinite depth of the Fermi sea, at least for the lowest order cumulant where the difference matters, i.e. the third. That cumulant has been the object of experimental work [12], which may be read as a confirmation of the T^* ordering in the above mentioned model.

The plan of the paper is as follows. In Section 2 we discuss the general relation between the generating function of the moments of the tranferred charge and the time ordering of current correlators. It is unrelated to scattering and not necessarily placed within a scheme of second quantization. In Section 3 we introduce a model of energy independent scattering at the level of first quantization. We also discuss the equivalence between in/out- and T^*-ordering. In Section 4 we will promote the model to second quantization. The quantities of Section 2, as well as others, can then be computed explicitly and the relation found there illustrated on this example.

2 The T^*-ordering of current correlators

2.1 The result and its context

We consider a current carrying device as symbolically illustrated in Fig. 1. The generating function of counting statistics [9] is

$$\chi(\lambda, t) = \langle \mathrm{e}^{\mathrm{i}\lambda\Delta Q} \rangle = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\lambda)^k}{k!} \langle (\Delta Q)^k \rangle \,, \tag{2.1}$$

where $\langle (\Delta Q)^k \rangle$ is the k-th moment of the charge transported during time t. Similarly, $\log \chi$ generates the cumulants $\langle\!\langle (\Delta Q)^k \rangle\!\rangle$. Several quantum mechanical expression for χ have been proposed, based in part on different measurement protocols. The first proposal [9] is

$$\chi(\lambda, t) = \langle \exp\,(\mathrm{i}\lambda \int_0^t I(t')dt') \rangle \tag{2.2}$$

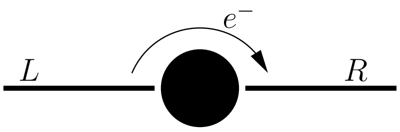

Figure 1: A device connecting two leads.

where $I(t)$ is the current operator at the junction and at time t, and $\langle\cdot\rangle$ has become the expectation in the initial quantum state of the system. The second one [4] reads, as recast by [13],

$$\chi(\lambda,t)=\langle \mathrm{e}^{\mathrm{i}\lambda Q(t)}\mathrm{e}^{-\mathrm{i}\lambda Q}\rangle=\langle \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{\mathrm{i}\lambda Q}\mathrm{e}^{-\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}\lambda Q}\rangle\,, \tag{2.3}$$

where $Q(t)=\exp(\mathrm{i}Ht)Q\exp(-\mathrm{i}Ht)$ is the charge operator on the right of the junction and at time t, and H is the Hamiltonian. The definition is appropriate to the situation where the initial state is an eigenstate of Q and the charge is measured after time t. In fact, if the initial eigenvalue is q, then $\chi(\lambda,t)=\langle \mathrm{e}^{\mathrm{i}\lambda Q(t)}\rangle\mathrm{e}^{-\mathrm{i}\lambda q}$ just describes the statistics of $q'-q$, where q' is the (random) outcome of the measurement of $Q(t)$. The quantity $q'-q$ is identified with the transported charge ΔQ. We mention in passing that there is a generalization [14] of this definition to the case where the assumption on the initial state does not apply and q is the outcome of the measurement of $Q(0)=Q$.

A further proposal [15, 5] is based on observing a spin coupled to current

$$\chi(\lambda,t)=\langle \mathrm{e}^{\mathrm{i}H(-\lambda/2)t}\mathrm{e}^{-\mathrm{i}H(\lambda/2)t}\rangle=\langle \mathrm{e}^{-\mathrm{i}\frac{\lambda}{2}Q}\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{\mathrm{i}\lambda Q}\mathrm{e}^{-\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}\frac{\lambda}{2}Q}\rangle\,, \tag{2.4}$$

where $H(\lambda)=\mathrm{e}^{\mathrm{i}\lambda Q}H\mathrm{e}^{-\mathrm{i}\lambda Q}$. This approach does not require the initial state to be an eigenstate of Q, but if it does, then it agrees with (2.3). A related proposal was put forward in [8] (with similar expressions found in [16, 10]). It is given by the Keldysh time-ordered expression

$$\chi(\lambda,t)=\langle \vec{\mathrm{T}}\exp\bigl(\mathrm{i}(\lambda/2)\int_0^t I(t')dt'\bigr)\mathrm{T}\,\exp\bigl(\mathrm{i}(\lambda/2)\int_0^t I(t')dt'\bigr)\rangle\,, \tag{2.5}$$

where T denotes the usual time-ordering and $\vec{\mathrm{T}}$ the one in the opposite direction. For later purposes we recall that the time ordering is supposed to occur inside the integrals once the exponential is expanded in powers of λ.

Eqs. (2.4) and (2.5) may differ in some applications, as it is the case in the very simple situation of independent fermions passing with fixed transparency T across a scatterer biased by V. In fact it was found in [5] and [8] that the third cumulants are, respectively,

$$\langle\!\langle(\Delta Q)^3\rangle\!\rangle=T(1-T)(1-2T)\frac{Vt}{2\pi}\,,\qquad \langle\!\langle(\Delta Q)^3\rangle\!\rangle=-2T^2(1-T)\frac{Vt}{2\pi}\,. \tag{2.6}$$

In [7] the first answer has been found for the second approach, i.e. for (2.5), by trading the Keldysh ordering for in/out-ordering. This step in turn relies on the strict causality of the scattering matrix, which however is not consistent with an energy-independent transparency.

The results of this article explain the discrepancies. We will consider (2.3) for simplicity, though similar considerations could be made for (2.4). As mentioned before, the two definitions agree in some cases, including the model underlying (2.6). We want to point out that some care is to be exercised when expressing the moments in terms of time-ordered current correlation functions. The result is

$$\langle(\Delta Q)^k\rangle = \int_0^t d^k t \, \langle \mathrm{T}^* \left(I(t_1) \cdots I(t_k) \right) \rangle \,, \tag{2.7}$$

where $d^k t = dt_1 \cdots dt_k$, $I(t) = dQ(t)/dt$, $Q(t) = \exp(\mathrm{i}Ht) Q \exp(-\mathrm{i}Ht)$, and T^* means that the derivative has to be taken after the time ordering:

$$\mathrm{T}^* \left(I(t_1) \cdots I(t_n) \right) := \frac{\partial}{\partial t_n} \cdots \frac{\partial}{\partial t_1} \mathrm{T} \left(Q(t_1) \cdots Q(t_n) \right) . \tag{2.8}$$

Hence, eq. (2.3) may be summarized as $\chi(\lambda, t) = \langle \mathrm{T}^* \exp \left(\mathrm{i}(\lambda/2) \int_0^t I(t')dt' \right) \rangle$; by (2.8), this amounts to keeping the time ordering outside of the integrals of the exponential series. Similarly, (2.4) is equivalent to (2.5) once stars are added to the time orderings.

It is known [6, 17] and will be shown below that, as a rule, (2.8) differs from $\mathrm{T} \left(I(t_1) \cdots I(t_n) \right)$ by contact terms supported at coinciding times but agrees with it if $[Q, I] = 0$.

2.2 The derivation

In this section we show (2.7) and obtain an expansion of (2.8) in contact terms. Any arising operators will be assumed to be properly defined and that will be implicitely verified in later sections in the context of applications.

Either side of

$$\langle \left(\mathrm{e}^{\mathrm{i}Ht} \mathrm{e}^{\mathrm{i}\lambda Q} \mathrm{e}^{-\mathrm{i}Ht} \right) \mathrm{e}^{-\mathrm{i}\lambda Q} \rangle = \langle \mathrm{e}^{\mathrm{i}Ht} \left(\mathrm{e}^{\mathrm{i}\lambda Q} \mathrm{e}^{-\mathrm{i}Ht} \mathrm{e}^{-\mathrm{i}\lambda Q} \right) \rangle \tag{2.9}$$

is an expression for $\chi(\lambda, t)$ and may serve as a starting point for the derivation of (2.7). We will follow both avenues, because the l.h.s. leads to a simple proof, while the r.h.s. has often been considered in the literature [5, 18].

We begin with the l.h.s., which yields

$$\chi(\lambda, t) = \langle \mathrm{e}^{\mathrm{i}\lambda Q(t)} \mathrm{e}^{-\mathrm{i}\lambda Q} \rangle = \langle \mathrm{T} \, \mathrm{e}^{\mathrm{i}\lambda (Q(t) - Q)} \rangle \tag{2.10}$$

and hence

$$\langle(\Delta Q)^k\rangle = \langle \mathrm{T} \, (Q(t) - Q)^k \rangle = \langle \mathrm{T} \left((Q(t_1) - Q) \cdots (Q(t_k) - Q) \right) \rangle \big|_{t_1 = \ldots = t_k = t} \,. \tag{2.11}$$

We note that the expression in brackets vanishes if $t_i = 0$ for some $i = 1, \ldots, n$. By repeated use of the fundamental theorem of calculus we obtain

$$\begin{aligned} \langle(\Delta Q)^k\rangle &= \int_0^t dt_1 \, \frac{\partial}{\partial t_1} \langle \mathrm{T} \left((Q(t_1) - Q) \cdots (Q(t_k) - Q) \right) \rangle \\ &= \int_0^t d^k t \, \frac{\partial}{\partial t_k} \cdots \frac{\partial}{\partial t_1} \langle \mathrm{T} \left((Q(t_1) - Q) \cdots (Q(t_k) - Q) \right) \rangle \,. \end{aligned}$$

Expanding the correlator in $Q(t_i)$ and $-Q$, the resulting second term is independent of t_i and does not contribute to the derivative. This proves (2.7).

Let us now turn to the r.h.s. of (2.9):

$$\chi(\lambda,t)=\langle \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H(\lambda)t}\rangle\,, \tag{2.12}$$

where

$$H(\lambda)=\mathrm{e}^{\mathrm{i}\lambda Q}H\mathrm{e}^{-\mathrm{i}\lambda Q}=\sum_{j=0}^{\infty}\frac{(\mathrm{i}\lambda)^j}{j!}\mathrm{ad}_Q^j(H) \tag{2.13}$$

with multiple commutators defined by $\mathrm{ad}_A^0(B)=B$, $\mathrm{ad}_A^j(B)=[A,\mathrm{ad}_A^{j-1}(B)]$, $(j\geq 1)$. Therefore, $\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H(\lambda)t}$ is the propagator in the interaction picture for H with interaction

$$W=H(\lambda)-H=\mathrm{i}\sum_{j=1}^{\infty}\frac{(\mathrm{i}\lambda)^j}{j!}\mathrm{ad}_Q^{j-1}(I)\,. \tag{2.14}$$

It is expressed by the Dyson expansion

$$\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H(\lambda)t}=\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int_0^t d^nt\,\mathrm{T}\left(W(t_1)\cdots W(t_n)\right) \tag{2.15}$$

with $W(t)=\exp(\mathrm{i}Ht)W\exp(-\mathrm{i}Ht)$. The term of order λ^k in the expansion is obtained by picking the power λ^{j_i} from $W(t_i)$ through (2.14), in such a way that $\sum_{i=1}^n j_i=k$:

$$\langle(\Delta Q)^k\rangle=\sum_{\substack{n,(j_1,\dots j_n)\\ \sum_i j_i=k}}\frac{k!}{n!j_1!\cdots j_n!}\int_0^t d^nt\,\langle\mathrm{T}\left(\mathrm{ad}_Q^{(j_1-1)}(I)(t_1)\cdots\mathrm{ad}_Q^{(j_n-1)}(I)(t_n)\right)\rangle\,. \tag{2.16}$$

The domain of integration is invariant under permutations of the times and the integrand under those of its factors. Thus, n-tuples $(j_1,\dots,j_n)$ differing only by order may be binned together. To this end, let

$$n_j=\sharp\{l\mid j_l=j\} \tag{2.17}$$

be the number of times the power λ^j has been picked. Each bin then consists of

$$\frac{n!}{n_1!\cdots n_k!} \tag{2.18}$$

tuples, since permutations among equal j_i's do not generate new tuples. Thus,

$$\langle(\Delta Q)^k\rangle=\sum_{\substack{(n_1,n_2,\dots)\\ \sum_j jn_j=k}}\frac{k!}{\left(\prod_{i=1}^{\infty}n_i!\right)\left(\prod_{j=1}^{\infty}j!^{n_j}\right)}\int_0^t d^nt\,\langle\mathrm{T}\left(\mathrm{ad}_Q^{(j_1-1)}(I)(t_1)\cdots\mathrm{ad}_Q^{(j_n-1)}(I)(t_n)\right)\rangle\,, \tag{2.19}$$

where $n=\sum_j n_j$ and the j_l's satisfy (2.17). The products are finite because of the condition $\sum_j jn_j=k$.

The derivation just given establishes the equality between (2.19) and (2.7). In the following we give an independent one by expanding (2.7) in contact terms.

Proposition 1.

$$\begin{aligned}\mathrm{T}^*\big(I(t_1)\cdots I(t_k)\big) &= \frac{\partial}{\partial t_k}\cdots\frac{\partial}{\partial t_1}\mathrm{T}\left(Q(t_1)\cdots Q(t_k)\right)\\ &= \sum_{P\in\mathcal{P}_k}\mathrm{T}\Big(\prod_{C\in P}\mathrm{ad}_Q^{n(C)-1}(I)(t_C)\delta_C\Big)\,,\end{aligned}\tag{2.20}$$

where the sum runs over all partitions P of $\{1,\ldots,k\}$ into nonempty disjoint subsets C. By δ_C we understand a product of δ-functions collapsing the times t_i, $(i\in C)$ to a single time $t_C = t_i$. More precisely, for C consisting of $n(C)$ elements, $C=\{i_1,\ldots,i_{n(C)}\}$, we set $\delta_C = \prod_{j=1}^{n(C)-1}\delta(t_{i_j}-t_{i_{j+1}})$; in particular $\delta_C = 1$ for $n(C)=1$.

Remark. The partition into single-element clusters contributes $\mathrm{T}\big(I(t_1)\cdots I(t_k)\big)$.

Proof. We claim the slightly more general statement for $0\le l\le k$:

$$\frac{\partial}{\partial t_l}\cdots\frac{\partial}{\partial t_1}\mathrm{T}\left(Q(t_1)\cdots Q(t_k)\right) = \sum_{P\in\mathcal{P}_l}\mathrm{T}\Big(\big(\prod_{C\in P}\mathrm{ad}_Q^{n(C)-1}(I)(t_C)\delta_C\big)\cdot Q(t_{l+1})\cdots Q(t_k)\Big)\,,\tag{2.21}$$

where the sum now runs over $\mathcal{P}_l$ instead of $\mathcal{P}_k$. This is trivially true for $l=0$ and identical to (2.20) for $l=k$. To run the induction in l, we note that for operators A_i associated with times t_i,

$$\frac{\partial}{\partial t_j}\mathrm{T}\left(A_1\cdots A_r\right) = \mathrm{T}\big(A_1\cdots\dot{A}_j\cdots A_r\big) + \sum_{i=1,i\neq j}^{r}\delta(t_j-t_i)\mathrm{T}\left([A_j,A_i]A_1\cdots\check{A}_i\cdots\check{A}_j\cdots A_r\right),\tag{2.22}$$

where $\check{}$ denotes omission. Using this with $j=l+1$ on (2.21) we observe that $\delta(t_{l+1}-t_i)[Q(t_{l+1}),Q(t_i)]=0$. The other commutators so generated are

$$\delta(t_{l+1}-t_C)[Q(t_{l+1}),\mathrm{ad}_Q^{n(C)-1}(I)(t_C)] = \delta(t_{l+1}-t_C)\mathrm{ad}_Q^{n(C)}(I)(t_C)\,,\tag{2.23}$$

so that

$$\begin{aligned}&\frac{\partial}{\partial t_{l+1}}\cdots\frac{\partial}{\partial t_1}\mathrm{T}\left(Q(t_1)\cdots Q(t_k)\right)\\ &\quad= \sum_{P\in\mathcal{P}_l}\mathrm{T}\Big(\big(\prod_{C\in P}\mathrm{ad}_Q^{n(C)-1}(I)(t_C)\delta_C\big)\cdot I(t_{l+1})Q(t_{l+2})\cdots Q(t_k)\Big)\\ &\qquad+\sum_{P\in\mathcal{P}_l}\sum_{C'\in P}\mathrm{T}\Big(\mathrm{ad}_Q^{n(C')}(I)(t_{C'})\delta_{C'}\delta(t_{l+1}-t_C)\cdot\big(\prod_{\substack{C\in P\\ C\neq C'}}\mathrm{ad}_Q^{n(C)-1}(I)(t_C)\delta_C\big)\\ &\qquad\qquad\cdot Q(t_{l+2})\cdots Q(t_k)\Big)\,.\end{aligned}$$

This agrees with (2.21) for $l{+}1$ in place of l. In fact, partitions of $\{1,\ldots,l{+}1\}$ are distinguished by whether the cluster $\widetilde{C}$ containing $l+1$ is $\widetilde{C}=\{l+1\}$ (first line) or $\widetilde{C}=C'\cup\{l+1\}$ with $C'\in P\in\mathcal{P}_l$ (second line); in this case, $n(\widetilde{C})-1=n(C')$. $\square$

We can now verify the stated equality. After inserting (2.20) in (2.7), partitions $P \in \mathcal{P}_k$ can be binned according to the numbers n_j, ($j = 1, 2, \ldots$) of their clusters of size j (hence $\sum_j j n_j = k$). Each bin consists of

$$\frac{k!}{\left(\prod_{i=1}^{\infty} n_i!\right)\left(\prod_{j=1}^{\infty} j!^{n_j}\right)} \tag{2.24}$$

partitions. In fact, permutations of k elements permuting or preserving clusters do not generate new partitions. Since in (2.20), $\sum_C n(C) = k$ and $\sum_j n_j = n$, the equality between (2.19) and (2.7) is established.

3 The instantaneous scattering model

3.1 The model

Transport of charge across a scatterer at zero temperature and at a small bias ought to be determined fully by the Fermi velocity (equal to 1 in suitable units) and the scattering matrix at the Fermi energy. The Fermi sea is, effectively, infinitely deep. The model we are about to define is minimal in the sense that it is fully determined by that matrix. The left and right reservoirs are represented by two infinite one-dimensional leads, which are however chiral. The scattering between leads occurs instantaneously at a single point. The model may be seen as describing a quantum point contact.

Let $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \cong L^2(\mathbb{R}; \mathbb{C}^2)$ be the single-particle Hilbert space of the model. The charge operator Q is the projection onto the right lead,

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

the second quantization of which will later represent the number of particles there. In absence of interaction, states $\psi \in \mathcal{H}$ evolve in time according to $\psi(x - t)$. The corresponding Hamiltonian H_0 is linear in the momentum:

$$H_0 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \tag{3.2}$$

where $p = -\mathrm{i}d/dx$ and the derivative is taken in the sense of distributions. Scattering between leads is determined by the matrix

$$S = \begin{pmatrix} \mathfrak{r} & \mathfrak{t}' \\ \mathfrak{t} & \mathfrak{r}' \end{pmatrix}. \tag{3.3}$$

We will specify the resulting Hamiltonian [19] at the end of this section. First however, we define the unitary group generated by it, because it is simpler and it is all which is needed in the rest of the paper. It is

$$(U(t)\psi)(x) = \begin{cases} \psi(x-t) + (S-1)\theta(0 < x < t)\psi(x-t)\,, & (t > 0)\,, \\ \psi(x-t) + (S^*-1)\theta(t < x < 0)\psi(x-t)\,, & (t < 0)\,, \end{cases} \tag{3.4}$$

and is motivated by the idea that the part of the freely evolved wave function $\psi(x-t)$ which, say for times $t>0$, has crossed the scatterer at $x=0$ between times 0 and t gets replaced by $S\psi(x-t)$. One readily verifies that $U(t)$ is a strongly continuous 1-parameter group.

Letting $U(t)$ act on the charge operator, we get

$$Q(t)=U(t)^*QU(t)=\begin{cases} Q(\theta(-x-t)+\theta(x))+S^*QS\theta(x+t)\theta(-x)\,, & (t>0)\,,\\ Q(\theta(-x)+\theta(x+t))+SQS^*\theta(-x-t)\theta(x)\,, & (t<0)\,,\end{cases} \tag{3.5}$$

or equivalently

$$Q(t)=\big(Q\theta(-x-t)+S^*QS\theta(x+t)\big)\theta(-x)+\big(SQS^*\theta(-x-t)+Q\theta(x+t)\big)\theta(x)\,. \tag{3.6}$$

It yields the current

$$I(t)=\frac{dQ(t)}{dt}=\big((S^*QS-Q)\theta(-x)+(Q-SQS^*)\theta(x)\big)\delta(x+t)\,, \tag{3.7}$$

which is well-defined as a distribution in t.

We now come to the description of the generator H, a self-adjoint operator by Stone's theorem. To this end, let

$$\mathcal{D}_+=\{\psi_+\in L^2(\mathbb{R}_+;\mathbb{C}^2)\mid \psi_+'\in L^2(\mathbb{R}_+;\mathbb{C}^2)\}\,, \tag{3.8}$$

where the derivative ψ_+' is that of a distribution on $\mathbb{R}_+=(0,\infty)$, i.e. away from the origin. Any function $\psi\in\mathcal{D}_+$ is continuous up to the boundary $x=0$, and let $\psi_+(0)$ be its boundary value. Any function in $L^2(\mathbb{R}_+;\mathbb{C}^2)$ may be seen as a distribution on all of $\mathbb{R}$. In this sense, the derivative of $\psi_+\in\mathcal{D}_+$ is

$$\frac{d}{dx}\psi_+=\psi_+'(x)+\psi_+(0)\delta(x)\,. \tag{3.9}$$

We may similarly define $\psi_-\in\mathcal{D}_-$ based on $\mathbb{R}_-=(-\infty,0)$. Then,

$$\frac{d}{dx}\psi_-=\psi_-'(x)-\psi_-(0)\delta(x)\,. \tag{3.10}$$

Given $\psi\in L^2(\mathbb{R};\mathbb{C}^2)$, let $\psi_\pm$ be their restrictions to the half-lines $\mathbb{R}_\pm$.

Proposition 2. *The generator H of $U(t)$ has domain*

$$D(H)=\{\psi\in\mathcal{H}\mid \psi_\pm\in\mathcal{D}_\pm,\ \psi_+(0)=S\psi_-(0)\} \tag{3.11}$$

and is given by

$$(H\psi)(x)=-\mathrm{i}(\psi_+'(x)+\psi_-'(x))\,. \tag{3.12}$$

We recall that $D(H)$ consists of those states $\psi\in\mathcal{H}$ for which the limit $\lim_{t\to0}t^{-1}(U(t)-1)\psi=-\mathrm{i}H\psi$ exists, thereby defining $H\psi$.

Proof. Let first $t > 0$. Then

$$(U(t)\psi)(x) = \psi_+(x-t) + \psi_-(x-t) + (S-1)\theta(x)\psi_-(x-t)\,. \tag{3.13}$$

Using (3.9, 3.10), we find

$$\begin{aligned} \frac{U(t)-1}{t}\psi &\xrightarrow{t\downarrow 0} -\frac{d}{dx}\psi_+ - \frac{d}{dx}\psi_- + (S-1)\psi_-(0)\delta \qquad (3.14)\\ &= -(\psi'_+ + \psi'_-) + (S\psi_-(0) - \psi_+(0))\delta \qquad (3.15) \end{aligned}$$

in the sense of distributions if $\psi_\pm \in \mathcal{D}_\pm$. Actually, this last condition is implied by the requirement $\psi \in D(H)$. In fact, by using $t^{-1}(\phi, (U(t)-1)\psi) = t^{-1}((U(t)-1)\phi, \psi)$ on a test function ϕ supported away from $x = 0$, we obtain in the limit $-\mathrm{i}(\phi, H\psi) = (\phi'_+ + \phi'_-, \psi)$ and hence $\psi'_\pm \in L^2(\mathbb{R}_\pm, \mathbb{C}^2)$. At that point $\psi \in D(H)$ further implies $S\psi_-(0) = \psi_+(0)$. Moreover, the convergence (3.15) is attained in $\mathcal{H}$. The same conclusion in reached for $t < 0$, whence (3.15) is $-\mathrm{i}H\psi$. $\square$

More casually, the Hamiltonian can also be written as

$$H = H_0 + \mathrm{i}(S-1)\cdot\delta_- = H_0 - \mathrm{i}(S^*-1)\cdot\delta_+ = H_0 + 2\mathrm{i}\frac{S-1}{S+1}\cdot\delta\,, \tag{3.16}$$

with quadratic forms $\delta_\pm$ on $\mathcal{D}_+ \cap \mathcal{D}_-$ defined as $(\delta_\pm\psi)(x) := \delta(x)\psi(0\pm)$ and $\delta = (\delta_+ + \delta_-)/2$. The first expression in (3.16) corresponds to (3.14) and the remaining ones follow from the boundary condition. Eq. (3.16) is a variant of one found in [20].

We note in passing that (3.7) may alternatively be understood as a quadratic form on $D(H)$ for fixed t. The expression is unambiguous even at $t = 0$, because $(S^*QS - Q)\delta_-$ and $(Q - SQS^*)\delta_+$ agree as a result of the boundary condition.

3.2 Comparison of time orderings

The T^*-ordering of currents is identical to the in/out-ordering, $\widetilde{T}$, whose definition [10] we shall recall momentarily. For the time being, we establish this fact at the level of first quantization, but we shall show in Section 4 that it persists under second quantization. The current (3.7) may be split, $I(t) = I_+(t) + I_-(t)$, into outgoing and incoming parts,

$$\begin{aligned} I_+(t) &= \big(S^*QS\theta(-x) + Q\theta(x)\big)\delta(x+t)\,,\\ I_-(t) &= -\big(Q\theta(-x) + SQS^*\theta(x)\big)\delta(x+t)\,. \end{aligned} \tag{3.17}$$

In particular, $[I_\pm(t), I_\pm(s)] = 0$.

Proposition 3. *We have*

$$\mathrm{T}^*\left(I(t_1)\cdots I(t_k)\right) = \widetilde{\mathrm{T}}\left(I(t_1)\cdots I(t_k)\right), \tag{3.18}$$

where the ordering $\widetilde{\mathrm{T}}$ places any $I_-(t)$ to the right of any $I_+(t')$, regardless of $t \gtrless t'$ (the order of currents of the same type is irrelevant), and extends by linearity to $I(t) = I_+(t) + I_-(t)$.

Before giving the proof, let us make an observation. The proof of (2.20) was based on

$$\frac{\partial}{\partial s}\mathrm{T}\left(I(t)Q(s)\right) = \mathrm{T}\left(I(t)I(s)\right) + \delta(t-s)[Q(s), I(t)]\,, \tag{3.19}$$

which is a sensible equation in the context of our model. The formal analogue

$$\frac{\partial}{\partial s}\mathrm{T}\left(I_+(t)Q(s)\right) = \mathrm{T}\left(I_+(t)I(s)\right) + \delta(t-s)[Q(s), I_+(t)] \tag{3.20}$$

is meaningless on its r.h.s., and should not be used. These claims are based on the commutators

$$\begin{aligned} [Q(s), I_+(t)] &= \theta(t-s)\delta(x+t)\big([Q, S^*QS]\theta(-x) + [SQS^*, Q]\theta(x)\big)\,, \\ [Q(s), I_-(t)] &= \theta(s-t)\delta(x+t)\big([Q, S^*QS]\theta(-x) + [SQS^*, Q]\theta(x)\big)\,, \\ [I(s), I_+(t)] &= -\delta(t-s)\delta(x+t)\big([Q, S^*QS]\theta(-x) + [SQS^*, Q]\theta(x)\big)\,, \end{aligned} \tag{3.21}$$

which can be obtained from (3.6–3.17). The first commutator is discontinuous at $t-s=0$, hence its multiplication with $\delta(t-s)$ in (3.20) is ambiguous. Also the first term there is seen to exhibit an ambiguity proportional to $\delta(t-s)$. By contrast, the sum of the first two commutators,

$$[Q(s), I(t)] = \delta(x+t)\big([Q, S^*QS]\theta(-x) + [SQS^*, Q]\theta(x)\big)\,, \tag{3.22}$$

is independent of s, whence the r.h.s. of (3.19) is well-defined.

Proof. The inductively stable generalization of (3.18) is

$$\frac{\partial}{\partial t_l}\cdots\frac{\partial}{\partial t_1}\mathrm{T}\left(Q(t_1)\cdots Q(t_k)\right) = \widetilde{\mathrm{T}}\left(I(t_1)\cdots I(t_l); Q(t_{l+1})\cdots Q(t_k)\right)\,, \qquad (0 \le l \le k)\,, \tag{3.23}$$

where, on the r.h.s., the Q's are time-ordered and the I_- (resp. I_+) are placed to their right (resp. left). Upon differentiating the r.h.s. w.r.t. t_{l+1}, eq. (2.22) generates equal time commutators only among Q's, which vanish. Thus,

$$\frac{\partial}{\partial t_{l+1}}\widetilde{\mathrm{T}}\left(I(t_1)\cdots I(t_l); Q(t_{l+1})Q(t_{l+2})\cdots Q(t_k)\right) = \widetilde{\mathrm{T}}\left(I(t_1)\cdots I(t_l); I(t_{l+1})Q(t_{l+2})\cdots Q(t_k)\right)\,, \tag{3.24}$$

indicating that $I(t_{l+1})$ is still subject to time ordering relatively to $Q(t_{l+2})\cdots Q(t_k)$. After splitting it, $I(t_{l+1}) = I_+(t_{l+1}) + I_-(t_{l+1})$, the incoming part $I_-(t_{l+1})$ commutes with the earlier Q's on its right by (3.21); similarly, $I_+(t_{l+1})$ with those on its left. The result is (3.23) with $l+1$ instead of l. $\square$

4 Second quantization

4.1 Binomial statistics

For fermionic many-body systems consisting of independent particles, the generating function (2.3) has been computed by Levitov and Lesovik [4] as

$$\chi(\lambda, t) = \big((1-T) + \mathrm{e}^{\mathrm{i}\lambda}T\big)^{Vt/2\pi} \cdot (1 + o(t))\,, \qquad (t \to \infty)\,, \tag{4.1}$$

where $T = |\mathfrak{t}|^2$ is the transmission probability and V is the bias across the scatterer. The result describes a binomial distribution with $N = tV/2\pi$ attempts. In particular, it yields the third cumulant

$$\langle\!\langle \Delta Q^3 \rangle\!\rangle_\rho = \frac{Vt}{2\pi} T(1-T)(1-2T)\,. \tag{4.2}$$

Before presenting a derivation of (4.1) among others [4, 21], let us make a digression on second quantization. In vague terms, the second quantization $\widehat{A}$ of a single-particle operator A is the sum of its contributions from all particles i, $\widehat{A} = \sum_i (A_i - \langle A_i \rangle_\rho)$, though with expectation value subtracted. The notion can be formalized for infinitely many particles in the frame of the GNS space of the multi-particle state ρ, i.e. a Hilbert space containing ρ and its local perturbations as vectors. We shall do for ρ being a quasi-free fermionic state, determined by a single-particle density matrix of the same name, $1 \geq \rho = \rho^* \geq 0$. In the special case of a pure many-particle state, i.e. for $\rho = \rho^2$, A admits a second quantization $\widehat{A}$ acting on the GNS space iff

$$[A, \rho] \in \mathcal{I}_2\,, \tag{4.3}$$

where $\mathcal{I}_2$ denotes the Hilbert-Schmidt operators. The exponentiated version of this criterion is: A unitary operator U admits an implementation $\widehat{U}$ on the GNS space iff $[U, \rho] \in \mathcal{I}_2$. The notation is slightly abusive as the construction differs from $\widehat{A}$: The propagator $U = \exp(-\mathrm{i}Ht)$ is promoted to $\widehat{U} = \exp(-\mathrm{i}\widehat{H}t)$. With these preliminaries, the generating function (2.3) reads

$$\chi(\lambda) = \left\langle \widehat{U}^* \mathrm{e}^{\mathrm{i}\lambda\widehat{Q}} \widehat{U} \mathrm{e}^{-\mathrm{i}\lambda\widehat{Q}} \right\rangle_\rho\,, \tag{4.4}$$

where, in line with the hypothesis made there, we assume

$$[\rho, Q] = 0\,. \tag{4.5}$$

The generating function (4.4) can be expressed in terms of the first quantized operators through the Levitov-Lesovik (infinite) determinant [4]. In [22] a particle-hole symmetric variant of the determinant was given a mathematical foundation on the basis of the GNS space. We recall the result, in the case of a pure state. Set $A_U := U^* A U$.

Proposition 4. *Let*

$$[U, \rho] \in \mathcal{I}_1\,, \tag{4.6}$$

where $\mathcal{I}_1$ $(\subset \mathcal{I}_2)$ denotes the trace class operators. Then

$$\chi(\lambda) = \det\left(1 + (\mathrm{e}^{-\mathrm{i}\lambda} - 1) Q_U \rho_U \rho' + (\mathrm{e}^{\mathrm{i}\lambda} - 1) Q_U \rho'_U \rho\right)\,, \tag{4.7}$$

where $\rho' = 1 - \rho$ is the density matrix of holes. The determinant is Fredholm.

In the context of (4.1) and of the instantaneous scattering model (3.16), the initial density matrix is the projection

$$\rho = \begin{pmatrix} \rho_L & 0 \\ 0 & \rho_R \end{pmatrix} \tag{4.8}$$

with $\rho_i = \theta(\mu_i - p)$, $(i = L, R)$ representing two Fermi seas biased by $V = \mu_L - \mu_R$. While assumption (4.5) holds true, (4.6) unfortunately does not. The failure is however of minor importance. It can be traced back to the discontinuities of $(U(t)\psi)(x)$ at $x = 0$ and $x = t$ introduced by U, as seen in (3.4). Technically, the hypothesis could be met by making the scattering matrix time-dependent with $S(\tau) = S$ for most of the relevant time interval $\tau \in (0, t)$, but $= 1$ near its ends (see [22] for a similar model). The particular choice of rounding would affect the determinant only to sub-leading order in t (cf. a $\log t$ contribution to $\chi(\lambda)$ in [5]). We neglect such contributions from the result, which in view of $\theta A_U \theta = \theta S^* A S \theta$ for $\theta = \theta(-t < x < 0)$ becomes

$$\chi(\lambda, t) = \det{}_{\mathcal{H}(t)} \left(1 + (\mathrm{e}^{-\mathrm{i}\lambda} - 1) S^* Q \rho S \rho' + (\mathrm{e}^{\mathrm{i}\lambda} - 1) S^* Q \rho' S \rho\right) , \tag{4.9}$$

where $\mathcal{H}(t) = L^2([-t, 0]; \mathbb{C}^2)$. The determinant is now of a form considered by Kac [23] and Akhiezer [24], extended to the matrix case [25]: Consider a translation invariant operator A on $L^2(\mathbb{R}; \mathbb{C}^n)$ with Fourier multiplier $A(k)$. For its truncation to $[-t, 0]$, one has

$$\log\det{}_{\mathcal{H}(t)}(1 + A) \longrightarrow \frac{t}{2\pi} \int_{-\infty}^{\infty} dk \, \log\det(1 + A(k)) + o(t) , \qquad (t \to \infty) , \tag{4.10}$$

where the determinant on the r.h.s is that of an $n \times n$ matrix. We apply the result to (4.9) with $n = 2$ and

$$S^* Q \rho S \rho' = \rho_R \rho'_L \begin{pmatrix} T & 0 \\ \bar{\mathfrak{r}}' \mathfrak{t} & 0 \end{pmatrix} , \tag{4.11}$$

as well as with ρ and ρ' interchanged. For $\mu_L > \mu_R$ we have $\rho_R \rho'_L = 0$ and $\rho'_R \rho_L = \theta(\mu_R < p < \mu_L)$. We so obtain

$$\log \chi(\lambda, t) = \frac{Vt}{2\pi} (1 + (\mathrm{e}^{\mathrm{i}\lambda} - 1) T) + o(t) , \qquad (t \to \infty) , \tag{4.12}$$

as announced.

4.2 The third cumulant

In this section we apply the expansion in contact terms, eq. (2.20), in order to compute the third cumulant of the transported charge. The currents will have to be understood as second quantized operators, which we emphasize by adding a hat. Eq. (2.7) reads

$$\langle\!\langle \Delta Q^3 \rangle\!\rangle_\rho = \int_0^t d^3t \, \langle\!\langle \mathrm{T}^* \left(\widehat{I}(t_1) \widehat{I}(t_2) \widehat{I}(t_3)\right) \rangle\!\rangle_\rho \tag{4.13}$$

and the result to be re-derived is (4.2). It agrees with the binomial statistics as derived again in the previous section and the result [7] based on the $\widetilde{\mathrm{T}}$ time ordering; on the other hand, it should be contrasted with

$$\int_0^t d^3t \, \langle\!\langle \mathrm{T} \left(\widehat{I}(t_1) \widehat{I}(t_2) \widehat{I}(t_3)\right) \rangle\!\rangle_\rho = -\frac{Vt}{2\pi} 2T^2(1 - T) \tag{4.14}$$

which is the result obtained for $\langle\!\langle \Delta Q^3 \rangle\!\rangle_\rho$ in other schemes [9, 8]. The difference between (4.13) and (4.14) is accounted for by contact terms. Indeed, in eq. (2.20) with $k = 3$, the partitions P are (1)(2)(3), (12)(3), (23)(1), (31)(2), (123), which yields

$$\langle\!\langle \Delta Q^3 \rangle\!\rangle_\rho = \int_0^t d^3t \, \langle\!\langle \mathrm{T}\, \widehat{I}_1 \widehat{I}_2 \widehat{I}_3 \rangle\!\rangle_\rho + 3 \int_0^t d^2t \, \langle\!\langle \mathrm{T}\, \widehat{I}_1 [\widehat{Q}_2, \widehat{I}_2] \rangle\!\rangle_\rho + \int_0^t dt_1 \, \langle\!\langle [\widehat{Q}_1, [\widehat{Q}_1, \widehat{I}_1]] \rangle\!\rangle_\rho \,, \quad (4.15)$$

where $\widehat{Q}_i = \widehat{Q}(t_i)$, $\widehat{I}_i = \widehat{I}(t_i)$. Besides of (4.14), we will show that the two further integrals are 0, resp. $(Vt/2\pi) \cdot T(1-T)$, resulting in

$$\langle\!\langle \Delta Q^3 \rangle\!\rangle_\rho = \frac{Vt}{2\pi}(-2T^2(1-T) + 0 + T(1-T)) = \frac{Vt}{2\pi} T(1-T)(1-2T) \,. \quad (4.16)$$

Before going into the proper proof of these claims, let us recall the following elementary rules [26], which reduce computations of correlators and commutators of second quantized operators to the level of first quantization:

$$\begin{aligned} \langle \widehat{A} \rangle_\rho &= 0 \,, \\ \langle\!\langle \widehat{A}\widehat{B} \rangle\!\rangle_\rho &= \langle \widehat{A}\widehat{B} \rangle_\rho = \mathrm{tr}(\rho A \rho' B \rho) \,, \\ \langle\!\langle \widehat{A}\widehat{B}\widehat{C} \rangle\!\rangle_\rho &= \mathrm{tr}(\rho A \rho' B \rho' C \rho) - \mathrm{tr}(\rho A \rho' C \rho B \rho) \,, \end{aligned} \quad (4.17)$$

with $\rho' = 1 - \rho$. The r.h.s. of (4.17) may be cast as $\mathrm{tr}(\rho A \rho' B \rho) = \mathrm{tr}([\rho, A]\rho'[B, \rho])$, which is finite for A, B enjoying (4.3). Eq. (4.17) implies

$$[\widehat{A}, \widehat{B}] = \widehat{[A, B]} + \big(\mathrm{tr}(\rho A \rho' B \rho) - \mathrm{tr}(\rho' A \rho B \rho')\big) \cdot 1 \,, \quad (4.18)$$

where the last term is known as a Schwinger term. Since $[H, \rho] = 0$ and $\rho\rho' = 0$ we have $[\widehat{H}, \widehat{A}] = \widehat{[H, A]}$ and hence $\widehat{A}(t) = \widehat{A(t)}$.

In our case, the current $\widehat{I}(t) = d\widehat{Q}(t)/dt$ is a distribution in t. In order to give rise to an operator on which the above may be applied, a test function is required, which will however remain implicit. It should also be remarked that $[Q, \rho] = 0$, whence the hypothesis about the initial state, discussed after eq. (2.3) and underlying eq. (4.13), is satisfied. Together with (4.18) that also implies that $\widehat{I} = \mathrm{i}\widehat{[H, Q]}$ satisfies $[\widehat{Q}, \widehat{I}] = \widehat{[Q, I]}$, where

$$[Q(t), I(t)] = \big([Q, S^* Q S]\theta(-x) + [S Q S^*, Q]\theta(x)\big)\delta(x + t) \quad (4.19)$$

by (3.22). It indicates the importance of contact terms.

The computation of the integrals (4.15) will make repeated use of the following expressions. The translation invariant density matrices ρ and ρ' have integral kernels

$$\begin{aligned} \rho(x, y) &= \frac{1}{2\pi\mathrm{i}} \frac{1}{x - y - \mathrm{i}0} \cdot D(x - y) \,, \\ \rho'(x, y) &= \frac{1}{2\pi\mathrm{i}} \frac{1}{y - x - \mathrm{i}0} \cdot D(x - y) \end{aligned}$$

with

$$D(z) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\mu_L z} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\mu_R z} \end{pmatrix} . \tag{4.20}$$

Moreover,

$$S^*QS = \begin{pmatrix} T & \mathfrak{r}'\bar{\mathfrak{t}} \\ \bar{\mathfrak{r}}'\mathfrak{t} & 1-T \end{pmatrix} , \qquad [Q, S^*QS] = \begin{pmatrix} 0 & -\mathfrak{r}'\bar{\mathfrak{t}} \\ \bar{\mathfrak{r}}'\mathfrak{t} & 0 \end{pmatrix} . \tag{4.21}$$

We compute the first integrand (4.15) by temporarily dropping the time ordering.

$$\langle\!\langle \widehat{I}_1 \widehat{I}_2 \widehat{I}_3 \rangle\!\rangle_\rho = \mathrm{tr}(I_1\rho' I_2 \rho' I_3 \rho) - \mathrm{tr}(I_1 \rho' I_3 \rho I_2 \rho) \tag{4.22}$$

$$= \frac{1}{(2\pi\mathrm{i})^3} \frac{\mathrm{tr}\big(A(t_2-t_1)A(t_3-t_2)A(t_1-t_3)\big) - \mathrm{tr}\big(A(t_3-t_1)A(t_2-t_3)A(t_1-t_2)\big)}{(t_1-t_2-\mathrm{i}0)(t_2-t_3-\mathrm{i}0)(t_1-t_3-\mathrm{i}0)} \tag{4.23}$$

with

$$A(t_i - t_j) = (S^*QS - Q)D(t_i - t_j) = \begin{pmatrix} T & \mathfrak{r}'\bar{\mathfrak{t}} \\ \bar{\mathfrak{r}}'\mathfrak{t} & -T \end{pmatrix} D(t_i - t_j) , \tag{4.24}$$

where we used (3.7). That results in

$$\langle\!\langle \widehat{I}_1 \widehat{I}_2 \widehat{I}_3 \rangle\!\rangle_\rho = \frac{-4T^2(1-T)}{(2\pi)^3} \cdot \frac{\sin V(t_1-t_2) + \sin V(t_2-t_3) + \sin V(t_3-t_1)}{(t_1-t_2-\mathrm{i}0)(t_2-t_3-\mathrm{i}0)(t_1-t_3-\mathrm{i}0)} , \tag{4.25}$$

where $V = \mu_L - \mu_R$. In fact, the first trace equals

$$2\mathrm{i}T^2(1-T)\sin V(t_3-t_1) + (\text{cyclic}) \tag{4.26}$$

by the following argument. It may be expressed as $\sum_{i,j,k=L,R} A_{ij}A_{jk}A_{ki}$. The word $ijk = LLL$ involves a single diagonal entry of (4.24) with overall compensating phases. Its contribution, T^3, cancels an opposite contribution from the similar term RRR. For LRR, we find

$$\mathfrak{r}'\bar{\mathfrak{t}}\,\mathrm{e}^{\mathrm{i}\mu_R(t_2-t_1)} \cdot (-T)\mathrm{e}^{\mathrm{i}\mu_R(t_3-t_2)} \cdot \bar{\mathfrak{r}}'\mathfrak{t}\,\mathrm{e}^{\mathrm{i}\mu_L(t_1-t_3)} = -T^2(1-T)\mathrm{e}^{-\mathrm{i}V(t_3-t_1)} , \tag{4.27}$$

which can be combined with $T^2(1-T)\mathrm{e}^{\mathrm{i}V(t_3-t_1)}$ from RLL. The remaining words provide the cyclic permutations. To conclude, it suffices to note that the second trace is obtained by exchanging $2 \leftrightarrow 3$, which induces a change of sign in all the exponents.

We then note that (4.25) is locally integrable and we drop the regularizations. Eq. (4.14) then follows from

$$\int_0^t d^3t \, \frac{\sin V(t_1-t_2) + \sin V(t_2-t_3) + \sin V(t_3-t_1)}{(t_1-t_2)(t_2-t_3)(t_1-t_3)} = 2\pi^2 V t + o(t) , \quad (t \to \infty) . \tag{4.28}$$

Actually, the time ordering should have been reinstated, but since the expression is permutation symmetric that is superfluous. Eq. (4.28) can be derived by reinstating the regularizations $-\mathrm{i}0$; then the integral can be broken into three terms, the first two of which vanish in the limit of large t. In fact, the first term has poles at $t_3 = t_2 - \mathrm{i}0$, $t_1 - \mathrm{i}0$ which do not pinch the real axis. The same applies to the second and to the t_1-integration. Using $\int dt_2(t_1-t_2-\mathrm{i}0)^{-1}\cdot(t_2-t_3-\mathrm{i}0)^{-1}=2\pi\mathrm{i}(t_1-t_3-\mathrm{i}0)^{-1}$, one is left with

$$2\pi\mathrm{i}\int_0^t dt_1 dt_3\,\frac{\sin V(t_3-t_1)}{(t_1-t_3-\mathrm{i}0)^2}=t\cdot 2\pi\mathrm{i}\int dx\,\frac{\sin Vx}{(x+\mathrm{i}0)^2}+o(t)=2\pi^2 Vt+o(t)\,; \tag{4.29}$$

in fact, the odd part of $(x+\mathrm{i}0)^{-2}$ is $\bigl((x+\mathrm{i}0)^{-2}-(x-\mathrm{i}0)^{-2}\bigr)/2=\pi\mathrm{i}\delta'(x)$.

We next consider the middle integral in (4.15). We find

$$\begin{aligned}\langle\!\langle\widehat{I}_1[\widehat{Q}_2,\widehat{I}_2]\rangle\!\rangle_\rho=\operatorname{tr}(I_1\rho'[Q_2,I_2]\rho)&=\frac{1}{(2\pi\mathrm{i})^2}\frac{1}{(t_1-t_2-\mathrm{i}0)^2}\cdot\operatorname{tr}\bigl(A(t_2-t_1)[Q,S^*QS]D(t_1-t_2)\bigr)\\&=\frac{1}{(2\pi\mathrm{i})^2}\frac{1}{(t_1-t_2-\mathrm{i}0)^2}\cdot\sin V(t_1-t_2)\,,\end{aligned}$$

and the same result for $\langle\!\langle[\widehat{Q}_2,\widehat{I}_2]\widehat{I}_1\rangle\!\rangle_\rho$ except for $+\mathrm{i}0$ instead of $-\mathrm{i}0$. Thus, $\langle\!\langle \mathrm{T}\,\widehat{I}_1[\widehat{Q}_2,\widehat{I}_2]\rangle\!\rangle_\rho$ is odd in $t_1\leftrightarrow t_2$, and the integral vanishes.

Finally the last integrand in (4.15), being the expectation value of a commutator, reduces to its Schwinger term

$$\langle\!\langle[\widehat{Q}_1,[\widehat{Q}_1,\widehat{I}_1]]\rangle\!\rangle_\rho=\operatorname{tr}(\rho Q_1\rho'[Q_1,I_1]\rho)-\operatorname{tr}(\rho' Q_1\rho[Q_1,I_1]\rho')\,. \tag{4.30}$$

Using (4.19, 3.5) it equals

$$\begin{aligned}&\frac{1}{(2\pi\mathrm{i})^2}\int_{-t_1}^0 dy\,\left(\frac{1}{(y+t_1+\mathrm{i}0)^2}-\frac{1}{(y+t_1-\mathrm{i}0)^2}\right)\operatorname{tr}(S^*QSD(t_1+y)[Q,S^*QS]D(-t_1-y))\\&\qquad=\frac{-2\mathrm{i}}{(2\pi\mathrm{i})^2}T(1-T)\int_{-t_1}^0 dy\,\sin V(y+t_1)\left(\frac{1}{(y+t_1+\mathrm{i}0)^2}-\frac{1}{(y+t_1-\mathrm{i}0)^2}\right)\,,\end{aligned}$$

where we noted that additional contributions from $y<-t_1$ and $y>0$ vanish because of $\operatorname{tr}(QD(t_1+y)[Q,S^*QS]D(-t_1-y))=0$. After changing the integration variable to $x=y+t_1$, both factors are odd, whence the integral equals

$$\frac{1}{2}\int_{-t_1}^{t_1} dx\,\sin Vx\left(\frac{1}{(x+\mathrm{i}0)^2}-\frac{1}{(x-\mathrm{i}0)^2}\right)=-\pi\mathrm{i}V+o(1)\,, \tag{4.31}$$

as in (4.29). We conclude that

$$\int_0^t dt_1\,\langle\!\langle[\widehat{Q}_1,[\widehat{Q}_1,\widehat{I}_1]]\rangle\!\rangle_\rho=T(1-T)\frac{Vt}{2\pi}\,, \tag{4.32}$$

as claimed.

4.3 A strictly causal scattering process

We may trade the instantaneous scattering process in use in the previous section with a strictly causal one, all while remaining within the model of Section 3. That is achieved by simply forfeiting a piece of length $l > 0$ of the leads in favor of the scatterer. So reinterpreted, the scattering process lasts $2l$ and the contact terms should disappear. Nevertheless the same result for the third cumulant will be obtained, though only the first terms on the r.h.s. of (4.15) will contribute.

In physical terms we place the detector a distance $l > 0$ away from the scatterer; in mathematical terms we replace Q by its regularization $Q_l = Q\theta(|x| \geq l)$ and hence I by $I_l = \mathrm{i}[H, Q_l] = Q[\delta(x-l) - \delta(x+l)]$. It ought to be noted that $[Q_l, \rho] \neq 0$, so that the physical appropriateness of (2.3), and hence of (4.13), could be questioned. Nevertheless, we may just view $l > 0$ as a regulator before taking the limit $l \to 0$. However, even this is troublesome, at least in this form, because $[Q_l, \rho] \notin \mathcal{I}_2$ makes $\widehat{Q}_l$ undefined. That can be remedied by smoothing the step function $\theta(|x| \geq l)$ on a length $< l$. With these preliminaries taken, the contact terms in (4.15) vanish as expected. This follows from $[\widehat{Q}_l, \widehat{I}_l] \propto 1$ and in turn from $[Q_l, I_l] = 0$, while the corresponding Schwinger term no longer vanishes. On the other hand, the current correlator $\langle\langle \mathrm{T}\, \widehat{I}_{l1} \widehat{I}_{l2} \widehat{I}_{l3} \rangle\rangle$ in (4.15) has a well-defined limit when the smoothing is removed. Omitting the time ordering, it is for times $t_i > l$ [11]

$$\begin{aligned}\langle\langle \widehat{I}_{l1} \widehat{I}_{l2} \widehat{I}_{l3} \rangle\rangle_\rho = {} & T(1-T)(1-2T) \cdot \frac{2}{(2\pi)^3} \cdot \frac{\sin V(t_1-t_2) + \sin V(t_2-t_3) + \sin V(t_3-t_1)}{(t_1-t_2)(t_2-t_3)(t_1-t_3)} \\ & - T(1-T) \cdot \frac{2}{(2\pi)^3} \biggl(\frac{\sin V(t_1-t_2)}{(t_1-t_2)(t_2-t_3-2l-\mathrm{i}0)(t_1-t_3-2l-\mathrm{i}0)} + \\ & \qquad \frac{\sin V(t_2-t_3)}{(t_1-t_2+2l-\mathrm{i}0)(t_2-t_3)(t_1-t_3+2l-\mathrm{i}0)} + \\ & \qquad \frac{\sin V(t_3-t_1)}{(t_1-t_2-2l-\mathrm{i}0)(t_2-t_3+2l-\mathrm{i}0)(t_1-t_3)} \biggr) \\ = {} & \mathrm{I} + \mathrm{II}\,. \end{aligned} \tag{4.33}$$

We remark that the expression reduces to (4.25) in the limit $l \to 0$; the integral of the latter is (4.14) and should be distinguished from the limit of the integral. We add that time-unordered correlators, suitably symmetrized, can be measured by means of detectors discussed in [27].

The derivation of (4.33) begins as in (4.22), but with currents

$$I(t) = S^* Q S \delta(x-l+t) - Q\delta(x+l+t) \equiv I_+(t) + I_-(t) \tag{4.34}$$

split into outgoing and incoming parts. The terms involving only outgoing parts yield I. Indeed, their contribution has the structure (4.23, 4.24), but with $S^*QS - Q$ replaced by S^*QS. In the description used there in reference to the first trace (4.23), the words LLL and RRR contribute T^3 and $(1-T)^3$, respectively; LRR and RLL contribute $T(1-T)^2 \mathrm{e}^{-\mathrm{i}V(t_3-t_1)}$ and $T^2(1-T)\mathrm{e}^{\mathrm{i}V(t_3-t_1)}$, with the remaining words providing cyclic permutations thereof. Again, the second trace is obtained by flipping signs in the exponents, and the difference of the two is

$$-2\mathrm{i}T(1-T)^2 \sin V(t_3-t_1) + 2\mathrm{i}T^2(1-T)\sin V(t_3-t_1) = -2\mathrm{i}T(1-T)(1-2T)\sin V(t_3-t_1)\,, \tag{4.35}$$

plus cyclic permutations. Terms involving precisely one incoming currents yield II. Finally, terms containing more than one incoming current vanish.

Each term of II exhibits two singularities not cancelled by the numerator. The terms differ however in regard as to whether the singularities may be attained within the region $t_1 \geq t_2 \geq t_3$: both in the first term, neither in the second, and $t_1 = t_2 + 2l$ but not $t_2 = t_3 - 2l$ in the last one. Within said region the regularization of the unaccessible singularities may be changed from $-\mathrm{i}0$ to $+\mathrm{i}0$. The so modified expression II$'$ is permutation symmetric, like I. This proves

$$\langle\!\langle \mathrm{T}\, \widehat{I}_{l1} \widehat{I}_{l2} \widehat{I}_{l3} \rangle\!\rangle = \mathrm{I} + \mathrm{II}' \tag{4.36}$$

for all $t_1, t_2, t_3 \geq l$. We can now compute its integral over $0 \leq t_i \leq t$, $(i = 1, 2, 3)$, for large t. The contribution from I is inferred from (4.28). The three terms of II$'$ contain singularities in the variables t_3, t_1, respectively t_2, which do not pinch the real axis. Hence II$'$ does not contribute and the result is

$$\langle\!\langle (\Delta Q_l)^3 \rangle\!\rangle = \int_0^t d^3t\, \langle\!\langle \mathrm{T}\, \widehat{I}_{l1} \widehat{I}_{l2} \widehat{I}_{l3} \rangle\!\rangle = \frac{Vt}{2\pi} T(1-T)(1-2T) + o(t)\,. \tag{4.37}$$

4.4 Comparison of time orderings (continued)

The purpose of this subsection is to show that eq. (3.18) persists upon replacing the currents by their second quantized counterparts:

$$\mathrm{T}^* \left(\widehat{I}(t_1) \cdots \widehat{I}(t_k) \right) = \widetilde{\mathrm{T}} \left(\widehat{I}(t_1) \cdots \widehat{I}(t_k) \right). \tag{4.38}$$

Inspection of the proof of Proposition 3 shows that we need to establish $[\widehat{Q}(s), \widehat{I}_+(t)] = 0$, $(t < s)$ and a similar property for $\widehat{I}_-(t)$. By time covariance it suffices to prove the claim for $s = 0$, where $Q(0) = Q$. Since by (3.21) we have

$$[Q(s), I_+(t)] = 0, \quad (t < s)\,, \tag{4.39}$$

at the level of first quantization, we have to make sure that the property is not destroyed by Schwinger terms:

$$\mathrm{tr}\big(\rho Q(s) \rho' I_+(t) \rho\big) - \mathrm{tr}\big(\rho' Q(s) \rho I_+(t) \rho'\big) = 0\,, \tag{4.40}$$

which for $s = 0$ indeed follows from $\rho Q \rho' = 0$. Alternatively, one can verify that the Schwinger term does not change under a common translation of t and s, i.e. under conjugation of $Q(s)$ and $I_+(t)$ by the propagator $U(\tau)$. Denoting by $S(A, B)$ the last term in (4.18), it follows from the Hochschild identity $S(AB, C) - S(BC, A) + S(CA, B) = 0$ that

$$S([A, B], C) - S([B, C], A) + S([C, A], B) = 0\,. \tag{4.41}$$

As a result of (4.39) we have

$$-\mathrm{i}\frac{d}{d\tau} S(Q(s+\tau), I_+(t+\tau)) = S([H, Q(s+\tau)], I_+(t+\tau)) + S(Q(s+\tau), [H, I_+(t+\tau)]) = 0\,. \tag{4.42}$$

Acknowledgements. We thank A. Lebedev, M. Reznikov, M. Suslov for valuable discussions.

References

[1] G. B. Lesovik. Excess quantum noise in 2D ballistic point contacts. *JETP Lett.*, 49:592–594, 1989.

[2] M. Büttiker. Scattering theory of thermal and excess noise in open conductors. *Phys. Rev. Lett.*, 65(23):2901, 1990.

[3] Ya. M. Blanter and M. Büttiker. Shot noise in mesoscopic conductors. *Phys. Rep.*, 336:1, 2000.

[4] L. S. Levitov and G. B. Lesovik. Charge distribution in quantum shot noise. *JETP Lett.*, 58(3):230–235, 1993.

[5] L. S. Levitov, H. W. Lee, and G. B. Lesovik. Electron counting statistics and coherent states of electric current. *J. Math. Phys.*, 37(10):4845–4866, 1996.

[6] P. T. Matthews. The application of Dyson's method to meson interactions. *Phys. Rev.*, 76(5):684–685, 1949.

[7] J. Salo, F. W. J. Hekking, and J. P. Pekola. Frequency-dependent current correlation functions from scattering theory. *Phys. Rev. B*, 74:125427, 2006.

[8] G. B. Lesovik and N. M. Chtchelkatchev. Quantum and classical binomial distributions for the charge transmitted through coherent conductor. *JETP Lett.*, 77(7):393–396, 2003.

[9] L. S. Levitov and G. B. Lesovik. Charge-transport statistics in quantum conductors. *JETP Lett.*, 55(9):555–559, 1992.

[10] C. W. J Beenakker and H. Schomerus. Counting statistics of photons produced by electronic shot noise. *Phys. Rev. Lett.*, 86(4):700–703, 2001.

[11] K. V. Bayandin, A. V. Lebedev, and G. B. Lesovik. Calculation of three-current correlators for QPC. Unpublished.

[12] G. Gershon, Yu. Bomze, E. V. Sukhorukov, and M. Reznikov. Detection of non-Gaussian fluctuations in quantum point contacts. *Phys. Rev. Lett.*, 101:016803, 2008.

[13] B. A. Muzykanskii and Y. Adamov. Scattering approach to counting statistics in quantum pumps. *Phys. Rev. B*, 68(15):155304–155313, 2003.

[14] A. Shelankov and J. Rammer. Charge transfer counting statistics revisited. *Europhys. Lett.*, 63:485–491, 2003.

[15] L. S. Levitov and G. B. Lesovik. Quantum measurement in electric circuit. `cond-mat/9401004`.

[16] M. Fleischhauer. Quantum theory of photodetection without the rotating wave approximation. *J. Phys. A*, 31:453–463, 1998.

[17] H. Epstein and V. Glaser. The role of locality in perturbation theory. *Ann. Inst. H. Poincaré*, 19(3):211–295, 1973.

[18] M. Kindermann and Yu. V. Nazarov. Full counting statistics in electric circuits. In *Quantum noise*. Kluwer, 2003.

[19] P. Falkensteiner and H. Grosse. Quantization of fermions interacting with point-like external fields. *Lett. Math. Phys.*, 14:139–148, 1987.

[20] S. Albeverio and P. Kurasov. Rank one perturbations of not semibounded operators. *Integr. Equ. Oper. Th.*, 27:379–400, 1997.

[21] F. Hassler, M. V. Suslov, G. M. Graf, M. V. Lebedev, G. B. Lesovik, and G. Blatter. Wave-packet formalism of full counting statistics. *Phys. Rev. B*, 78:165330, 2008.

[22] J. E. Avron, S. Bachmann, G. M. Graf, and I. Klich. Fredholm determinants and the statistics of charge transport. *Comm. Math. Phys.*, 280:807–829, 2008.

[23] M. Kac. Toeplitz matrices, translation kernels and a related problem in probability theory. *Duke Math. J.*, 21(3):501–509, 1954.

[24] N. I. Akhiezer. A functional analogue of some theorems on Toeplitz matrices. *Amer. Math. Soc. Transl. II*, 50:295–316, 1966.

[25] A. Böttcher and B. Silbermann. *Analysis of Toeplitz Operators*. Springer Monographs in Mathematics. Springer-Verlag, second edition, 2006.

[26] L.-E. Lundberg. Quasi-free 'Second quantization'. *Comm. Math. Phys.*, 50:103–112, 1976.

[27] K. V. Bayandin, A. V. Lebedev and G. B. Lesovik. Measurement of high order current correlators. *JETP*, 106: 117-129, 2008.