Exponentially small splitting for the pendulum: a classical problem revisited

Marcel Guardia[1], Carme Olivé[2] and Tere M. Seara[1]

[1] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

[2] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avda Països Catalans 26, 43007 Tarragona, Spain

E-mail: `marcel.guardia@upc.edu`, `carme.olive@urv.cat`, `tere.m-seara@upc.edu`

Abstract. In this paper, we study the classical problem of the exponentially small splitting of separatrices of the rapidly forced pendulum. Firstly, we give an asymptotic formula for the distance between the perturbed invariant manifolds in the so-called singular case and we compare it with the prediction of Melnikov Theory. Secondly, we give exponentially small upper bounds in some cases in which the perturbation is bigger than in the singular case and we give some heuristic ideas how to obtain an asymptotic formula for these cases. Finally, we study how the splitting of separatrices behaves when the parameters are close to a codimension-2 bifurcation point.

AMS classification scheme numbers: 34C29, 34C37 ,37C29, 34E10

1. Introduction

In this paper we consider the classical problem of the splitting of separatrices for the rapidly forced pendulum whose equation is

$$\ddot{x} = \sin x + \frac{\mu}{\varepsilon^2} \sin \frac{t}{\varepsilon} \tag{1}$$

where μ is a real parameter and $\varepsilon > 0$ is a small parameter.

When $\mu\varepsilon^{-2}$ is small, this equation is a small perturbation of the classical pendulum

$$\ddot{x} = \sin x \tag{2}$$

and has been considered as a model of a two dimensional integrable system perturbed by a rapidly forcing term. Reparameterizing time $\tau = \frac{t}{\varepsilon}$ and writing $' = \frac{d}{d\tau}$, equation (1) can be considered as a nearly integrable system with slow dynamics

$$x'' = \varepsilon^2 \sin x + \mu \sin \tau. \tag{3}$$

Rewriting this equation as a first order system

$$\left\{ \begin{array}{l} x' = \varepsilon y \\ y' = \varepsilon \sin x + \mu\varepsilon^{-1} \sin \tau, \end{array} \right. \tag{4}$$

one can see that it is a Hamiltonian system of one and a half degrees of freedom, with Hamiltonian function

$$H(x, y, \tau, \mu, \varepsilon) = \varepsilon\Big(\frac{y^2}{2} + \cos x - 1 - \mu\varepsilon^{-2} x \sin\tau\Big). \tag{5}$$

Due to the $2\pi\varepsilon$-periodicity of the forcing in equation (1), its dynamical properties can be better visualized with the help of the Poincaré map P defined in a section $\Sigma_{t_0} = \{(x, y, t_0),\ (x, y) \in \mathbb{R}^2\}$.

If $\mu = 0$, the phase portrait of P is very simple. It is given by the level curves of the Hamiltonian $H_0(x, y) = (\frac{y^2}{2} + \cos x - 1)$. Our interest will be the stable and unstable manifolds of the hyperbolic fixed point $(0, 0)$ which, in this case, coincide along two separatrices given by the homoclinic orbits of the pendulum: $(x_0(t), \pm y_0(t))$ where

$$x_0(t) = 4\arctan(e^t), \quad y_0(t) = \dot{x}_0(t).$$

The phase space looks more complicated when $\mu \neq 0$. Roughly speaking, if $\mu\varepsilon^{-2}$ is small enough, there exists still a hyperbolic fixed point of P corresponding to a hyperbolic periodic orbit of (1), as well as the stable and unstable invariant curves $C^s(t_0)$ and $C^u(t_0)$, which lie near the unperturbed separatrix. Due to the symmetries of system (4), $C^s(t_0)$ and $C^u(t_0)$ intersect on a point z_h, which lies on the line $x = \pi$ in the case $t_0 = 0$. If this intersection is transversal at z_h, the curves enclose lobes whose area $\mathcal{A}$ does not depend on the homoclinic point we have chosen (see Figure 1). The measure of this area in terms of ε and μ is the main purpose of this paper. Another quantity that can be measured at the homoclinic points to check the transversality of the intersection, is the angle between the curves $C^s(t_0)$ and $C^u(t_0)$, but this quantity depends on the chosen homoclinic point. In fact, the corresponding invariant quantity is the so called *Lazutkin invariant* (see, for instance [GLT91]).

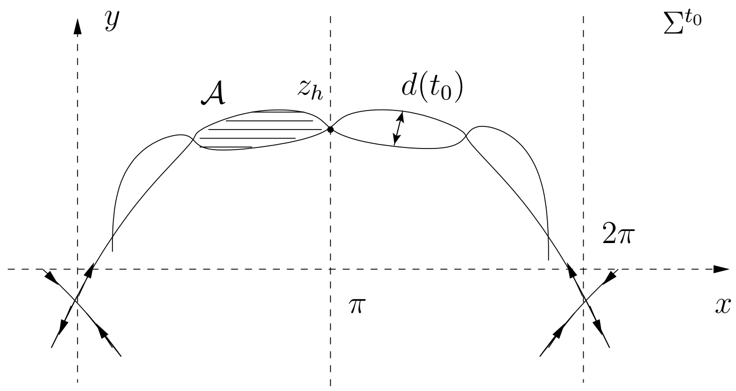

Figure 1. Splitting of separatrices.

As it is known in the Dynamical Systems community, the existence of transversal intersections between stable and unstable manifolds of one or more critical points in a dynamical system was described by Poincaré as the fundamental problem of mechanics [Poi99]. In fact, the transversal intersection of stable and unstable manifolds of fixed points of smooth planar diffeomorphisms, is the simplest setting where this phenomenon gives rise to the existence of chaotic behavior (see [Sma65]). For this reason, it has been one of the most studied problems in the last century.

An easy way to produce planar diffeomorphisms is to consider the Poincaré map of a T-periodically perturbed planar vector field defined in a Poincaré section $t = t_0$. Furthermore, in this regular perturbative context, Poincaré, and later Melnikov (see [Mel63]), developed a method which measures the distance between the invariant manifolds of hyperbolic critical points which coincide in the unperturbed integrable system. The Poincaré-Melnikov method provides a function $L(t_0)$, called Poincaré function or Melnikov potential, whose non-degenerate critical points give rise to transversal intersections between the stable and unstable perturbed manifolds (see, for instance, [Mel63, GH83]). In fact, $\frac{\partial L}{\partial t_0}$ gives the main term of the distance $d(t_0)$ between the perturbed invariant manifolds in the Poincaré section $t = t_0$. Moreover, the main term of the area $\mathcal{A}$ is given by $L(t_0^1) - L(t_0^2)$, where t_0^1 and t_0^2 are two consecutive critical points of $L(t_0)$.

The generalization of this problem to higher dimensional systems has been achieved by several authors, mainly in the Hamiltonian case. See, for instance, [Eli94, Tre94, DG00, LMS03] and references therein. A Melnikov theory for twist maps can be found in [DR97].

In the case of rapidly forced systems, a difficult problem arises due to the fact that the Poincaré function depends on the perturbed parameter and, in fact, turns out to be exponentially small with respect to it. In [San82], Sanders noticed this problem, previously stated by Poincaré, arising from the direct application of the Poincaré-Melnikov method in these cases. A similar problem occurs in families of area preserving diffeomorphisms close to the identity. In all these cases, even if the prediction of the Poincaré function for the splitting is exponentially small, it is not clear if this function gives the main term of this distance and even if the distance itself is exponentially small or not (see [SMH91, Fon95]).

Several authors have given partial answers to this problem. The first group of results is concerned about exponentially small upper bounds. Neishtadt in [Neĭ84] gave exponentially small upper bounds for the splitting in two degrees of freedom Hamiltonians. For area preserving maps close to the identity, Fontich and Simó provided upper bounds in [FS90]. For second order equations with a rapidly forced periodic term, several authors gave sharp exponentially small upper bounds in [Fon93, Fon95, FS96] and, for the higher dimensional case, the papers [Sau01, Sim94] gave (not sharp) exponentially small upper bounds.

The second group of results is concerned about the validity of the Melnikov potential in the exponentially small case. This problem needs more information about the system under consideration and, for this reason, the results existing in this direction deal mostly with specific examples. The first result was obtained by Holmes, Marsden and Scheurle in [HMS88] (followed by [Sch89, Ang93]), where they studied the rapidly perturbed pendulum (1). Taking $\mu = \mathcal{O}(\varepsilon^p)$, they confirmed the prediction of the

Melnikov potential establishing exponentially small upper and lower bounds for the area $\mathcal{A}$ provided $p \geq 10$,

$$c_2 \varepsilon^{p-1} e^{-\frac{\pi}{2\varepsilon}} \leq \mathcal{A} \leq c_1 \varepsilon^{p-1} e^{-\frac{\pi}{2\varepsilon}}.$$

Let us observe that for equation (1), the Melnikov potential is given by

$$L(t_0) = -\frac{2\pi\varepsilon}{\cosh\frac{\pi}{2\varepsilon}} \cos t_0$$

and, therefore, the prediction for the area is

$$\mathcal{A} \sim \frac{4\pi\mu}{\varepsilon\cos\frac{\pi}{2\varepsilon}} \sim 8\pi\varepsilon^{p-1} e^{-\frac{\pi}{2\varepsilon}}. \tag{6}$$

In [EKS93] the range for p was extended to $p \geq 5$ using the same approach. Following the ideas in [GLT91], Gelfreich in [Gel94] established an asymptotic expression for the splitting provided $p > 7$ and Delshams and Seara established rigourously the result in [DS92] for $p > 2$. An alternative proof, using Parametric Resurgence, was done in [Sau95] for a simplified perturbation of the pendulum equation. Later, in [DS97, Gel97a], the authors gave a proof for the validity of the Melnikov method for general rapidly periodic Hamiltonian perturbations of a class of second order equations. The case of a perturbed second order equation with a parabolic point was studied in [BF04, BF05]. Some results about the validity of the prediction given by the Poincaré function for area preserving maps were given in [DR98] and for higher dimensional Hamiltonian systems in [Gal94, CG94, DGJS97, GGM99, Sau01, DGS04]. Finally, in a non Hamiltonian setting, in [BS06] the splitting of a heteroclinic orbit for some degenerate unfoldings of the Hopf-zero singularity of vector fields in $\mathbb{R}^3$ was found.

The third group of results deal with what is called the *singular case*. In these cases the splitting of separatrices is still exponentially small but its asymptotic value is no longer given by the Melnikov potential. The first paper that dealt with this kind of problem was the paper by Lazutkin [Laz84, Laz03]. There, the author studied the splitting of separatrices in the classical Chirikov standard map, and gave the main tools to obtain an asymptotic formula for it. The problem, in this case, can be approximated by an integrable flow, but there is not a good Melnikov potential which gives the asymptotic value of the area between the stable and unstable invariant curves.

Even if [Laz84] was not complete (the complete proof was achieved by Gelfreich in [Gel99]), the ideas in this paper have inspired most of the work in this area. As it is known by the experts in the subject, the detection of an exponentially small splitting relies on suitable complex parameterizations of the invariant manifolds. These parameterizations are analytic in a complex strip, whose size is limited by the singularities of the unperturbed homoclinic orbit. In the regular case, these manifolds are well approximated by the unperturbed homoclinic orbit even for complex values of the parameter. Hence, the prediction of the Melnikov potential, which is mainly an integral over the unperturbed homoclinic orbit, gives the main term of the distance.

However, in the singular case, one has to deal with different approximations of the invariant manifolds in different zones of the complex plane. Close to the singularities of the homoclinic orbit, an equation for the leading term is obtained and it is called the *inner equation*. It is a non integrable equation without parameters, which needs a deep study itself. Once this equation is solved, matching techniques are required to match the different approximations obtained for the invariant manifolds. Roughly speaking, the difference between the solutions of the inner equation replaces the Melnikov potential in the asymptotic formula for the splitting.

Following these ideas, some authors have obtained partial results for some specific equations. The study of the inner equation of the Hénon map, using Resurgence Theory, can be found in [GS01] and the inner system associated to the Hopf-zero singularity was studied in [BS08] using functional analysis techniques. For several periodically perturbed second order equations, Gelfreich stated in [Gel97b] the corresponding inner equation which he called Reference System. The analysis of the inner equation for the Hamilton-Jacobi equation associated to a system analogous to (1) was done in [OSS03] using Resurgence Theory, and for a wider class of second order equations in [Bal06]. Using the results of the inner equation, Gelfreich in [Gel00] gave a detailed sketch of the proof for the splitting of separatrices for equation (1). Numerical results about the splitting for this problem can be found in [BO93, Gel97b]. The complete proof for the pendulum with a different perturbation was achieved in [Oli06]. Following the same approach, Lombardi proved in [Lom00] the splitting of separatrix connections for a certain class of reversible systems in $\mathbb{R}^4$. Finally, Treschev gave in [Tre97] an asymptotic formula for the splitting in the case of a pendulum with a suspension point using a different method called Continuous Averaging and a related problem about adiabatic invariants in the harmonic oscillator using matching techniques and Resurgence Theory was done in [BSSV98].

As we have said before, most of the previous works adapted the ideas in [Laz84, Laz03] to rigourously prove the asymptotic formula for the splitting in different settings. A fundamental tool in Lazutkin's work is the use of "flow box coordinates", called "straightening the flow" in [Gel00], around one of the manifolds. In this way, one obtains a periodic function whose values are related with the distance between the manifolds and whose zeros correspond to the intersections between them. Consequently, the result about exponentially small splitting is derived from some properties of analytic periodic functions bounded in complex strips (see, for instance, Proposition 2.7 in [DS97]).

A significantly different method was presented by Lochak, Marco and Sauzin in the papers [Sau01, LMS03]. There, the authors were able to measure the distance between the manifolds (and all the related quantities like the angle, etc) in the original variables of the problem without using "flow box coordinates". The authors used a very simple but clever idea: being both manifolds solutions of the same equation, their difference satisfies a linear equation and is bounded in some complex strip. Studying the properties of bounded solutions of this linear equation, where periodicity also plays a role, one obtains exponentially small results.

In the present paper, following the method of [Sau01, LMS03], we obtain mainly three results. The main part of the paper is devoted to rigorously prove the asymptotic formula of the splitting of separatrices for equation (1) in the singular case ($|\mu| \leq \mathcal{O}(1)$) following the ideas given in [Sau01, OSS03, Oli06]. The key idea is to look for the invariant manifolds as graphs of the differential of certain functions $S^{\pm}$. These functions, called generating functions, are solutions of the Hamilton-Jacobi equation associated to equation (1), which is a first order partial differential equation. In this way, the method to establish an asymptotic formula for the area of the lobes relies on computing the difference $S^+ - S^-$ in their common domain.

As we have already pointed out, the main difference of this approach from the previous ones, is the fact that we do not use "complex flow box variables" to obtain a good "splitting function" which measures the distance between both manifolds. Instead we give a formula for the distance in the original parameterizations of the manifolds, working in the original variables of the problem. The main idea, that was already used in [Sau01, OSS03, Oli06], is the fact that the difference $S^+ - S^-$ verifies a linear partial differential equation whose solutions can be characterized and bounded by exponentially small terms in $\mathbb{R}$, provided they are bounded in a complex strip. The results obtained in this singular case coincide with [Gel00].

On the other hand, studying the manifolds in the original variables, one can see that they exist even if the parameter μ in (1) is big with respect to ε or when μ is finite but approaches some values of bifurcation μ_i, which correspond to the zeros of the zero order Bessel function of first type (see (9)) . So, there is still the question of the size of the splitting of these manifolds in these cases, that we call *below the singular case* and *close to a bifurcation case*. The rest of the paper is devoted to study the splitting of separatrices in these cases, that is, when $\mu = \varepsilon^p$ with $p < 0$ or $\mu = \mu_i - \varepsilon^r$ with $r > 0$, which, as far as the authors know, had not been studied before.

As a first result, we state the existence of the periodic orbit for equation (1) for any μ, including the case $\mu = \varepsilon^p$ with $p < 0$, even if in this case it is no longer close to the hyperbolic critical point of the pendulum, but to the periodic solution of equation (3) for $\varepsilon = 0$. Later, for $\mu = \varepsilon^p$ with $p \in (-4, 0)$ or $\mu = \mu_i - \varepsilon^r$ with $r \in (0, 2)$, we state that this periodic orbit is still hyperbolic and we obtain an exponentially small upper bound for the distance between its global stable and unstable manifolds. We have not tried in this paper to give an asymptotic formula for the splitting in these cases.

2. Main results

Equation (2) has a hyperbolic fixed point at the origin. The first result in this paper is Theorem 2.1 where we prove that, if ε is small enough, equation (1) has a $2\pi\varepsilon$-periodic orbit for any value of μ, even if $\mu = \varepsilon^p$ for $p < 0$.

Theorem 2.1. *Let us consider equation (1), then there exists* $\varepsilon_0 > 0$ *such that:*

(i) For any $\mu > 0$ *and* $\varepsilon \in (0, \varepsilon_0)$*, this system has a* $2\pi\varepsilon$*-periodic orbit* $x_p(t)$ *and there*

exists a constant $C_1 > 0$ (independent of ε and μ) such that for $t \in \mathbb{R}$:

$$\left| x_p(t) + \mu \sin \frac{t}{\varepsilon} \right| \leq C_1 \varepsilon^2.$$

(ii) Moreover, for any fixed $\bar{\mu} > 0$, for $\mu \in (0, \bar{\mu})$ and $\varepsilon \in (0, \varepsilon_0)$, there exists a constant $C_2 > 0$ (independent of ε and μ) such that for $t \in \mathbb{R}$, x_p holds the sharper bound:

$$\left| x_p(t) + \mu \sin \frac{t}{\varepsilon} \right| \leq C_2 \mu \varepsilon^2.$$

This theorem is proved in Section 4.

Remark 2.2. *Once we know the existence of the periodic orbit for equation (1), the next step is to establish its hyperbolicity as a solution of the corresponding system. When it is hyperbolic, we study the existence of stable and unstable manifolds associated to it and their possible intersection.*

Classical perturbation theory applied to equation (1), gives a positive answer for these questions provided $\mu\varepsilon^{-2}$ is small enough. Nevertheless, to obtain the widest range of values of μ where the periodic orbit is still hyperbolic, one can average several times the system associated to equation (1). In fact, two steps of averaging lead to

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = \sin\left(x - \mu \sin \dfrac{t}{\varepsilon} \right), \end{array} \right. \tag{7}$$

whose averaged system is

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = J_0(\mu) \sin x \end{array} \right. \tag{8}$$

where

$$J_0(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\mu \sin \tau) d\tau \tag{9}$$

is the zero order Bessel function of first type. This function has simple zeros μ_i for $i \geq 0$ (see [AS92]). Thus, for $\mu < \mu_0$, $J_0(\mu) > 0$ and equation (1) is close to a pendulum slightly modified by this coefficient $J_0(\mu)$.

The next question would be if more steps of averaging show that when $J_0(\mu) \to 0$, system (7) is still close to pendulum (8). For instance, for $\mu = \varepsilon^p$ with $p < 0$, using that

$$J_0(\mu) \sim \sqrt{\frac{2}{\pi\mu}} \cos\left(\mu - \frac{\pi}{4} \right) \quad \text{as } \mu \to +\infty, \tag{10}$$

(see [AS92]), one can see that $J_0(\mu)$ has separated zeros $\mu_0 < \mu_1 < \mu_2 < \dots$ and tends to zero as $\mu \to +\infty$. Thus, in order to study system (7) for values of $\mu = \varepsilon^p$ with $p < 0$, we restrict ourselves to ε such that μ belongs to compact intervals $I_i \subset (\mu_{2i}, \mu_{2i+1})$, in such a way that $\cos(\mu - \pi/4)$ has a positive lower bound independent of ε. In particular, $J_0(\mu) > 0$ and $J_0(\mu) = \mathcal{O}(\varepsilon^{-p/2})$.

Taking ε with these restrictions, we can perform one step more of averaging. First, we write system (7) as

$$\left\{\begin{array}{rl} \dot{x} = & y \\ \dot{y} = & J_0(\mu)\sin x + \sin x\left(\cos\left(\mu\sin\dfrac{t}{\varepsilon}\right) - J_0(\mu)\right) - \\ & -\cos x\sin\left(\mu\sin\dfrac{t}{\varepsilon}\right), \end{array}\right. \tag{11}$$

and we consider the non-symplectic change $(\bar{x},\bar{y}) = (x, y/\sqrt{J_0(\mu)})$ and the reparameterization of time $s = t\sqrt{J_0(\mu)}$, obtaining

$$\left\{\begin{array}{rl} \dfrac{d\bar{x}}{ds} = & \bar{y} \\ \dfrac{d\bar{y}}{ds} = & \sin\bar{x} + \dfrac{1}{J_0(\mu)}\sin\bar{x}\left(\cos\left(\mu\sin\dfrac{s}{\sqrt{J_0(\mu)}\varepsilon}\right) - J_0(\mu)\right) - \\ & -\dfrac{1}{J_0(\mu)}\cos\bar{x}\sin\left(\mu\sin\dfrac{s}{\sqrt{J_0(\mu)}\varepsilon}\right). \end{array}\right. \tag{12}$$

Now, this system is a $2\pi\sqrt{J_0(\mu)}\varepsilon$-periodic big perturbation of a pendulum. To study the possible splitting of the invariant manifolds of the periodic orbit it is natural to perform one step more of averaging in this new time, obtaining:

$$\left\{\begin{array}{rl} \dfrac{d\bar{x}}{ds} = & \bar{y} + \dfrac{\varepsilon}{\sqrt{J_0(\mu)}}\left(h_1\left(\dfrac{s}{\varepsilon\sqrt{J_0(\mu)}}\right)\cos\bar{x} + h_2\left(\dfrac{s}{\varepsilon\sqrt{J_0(\mu)}}\right)\sin\bar{x}\right) \\ \dfrac{d\bar{y}}{ds} = & \sin\bar{x} - \\ & -\dfrac{\varepsilon}{\sqrt{J_0(\mu)}}\bar{y}\left(h_1\left(\dfrac{s}{\varepsilon\sqrt{J_0(\mu)}}\right)\sin\bar{x} - h_2\left(\dfrac{s}{\varepsilon\sqrt{J_0(\mu)}}\right)\cos\bar{x}\right) + \\ & +\dfrac{\varepsilon^2}{J_0(\mu)}m\left(\bar{x}, \dfrac{s}{\varepsilon\sqrt{J_0(\mu)}}\right) \end{array}\right. \tag{13}$$

where $m(\bar{x},\tau)$ is a function 2π-periodic in τ which contains the higher order terms and does not depend on y, and $h_1(\tau)$ and $h_2(\tau)$ are 2π-periodic functions with zero average which depend on $\mu = \varepsilon^p$, but are of order 1 with respect to ε for $\tau \in \mathbb{R}$ and $p < 0$.

So that, by (10), if $\varepsilon^{1+p/4} \ll 1$ and $\mu = \varepsilon^p \in I_i$ with $i \in \mathbb{N}$, this system is still a small and fast perturbation of period $2\pi\varepsilon\sqrt{J_0(\mu)} = \mathcal{O}(\varepsilon^{1-p/4})$ of the classical pendulum. Thus, considering $\mu = \varepsilon^p$ with $\mu \in I_i$ and $p \in (-4,0)$, the periodic orbit is still hyperbolic and consequently it has local stable and unstable manifolds.

The question about the global behavior of these manifolds and their spliting is more involved. In Theorems 2.6 and 2.10 we give a rigorous proof of their existence and a bound of their distance to the unperturbed homoclinic orbit. We also provide, in these theorems, an asymptotic formula for the splitting for $p \geq 0$ and an exponentially small upper bound for $p \in (-4,0)$. Nevertheless, to obtain an asymptotic expression for the splitting in this last case, other techniques, as it is explained in Remark 2.11, are needed. In any case, the authors think that the "true limit" of the splitting problem is given by

$p=-4$, in the sense that for $p\le -4$, the leading term of the averaged system will be given by $\langle m\rangle$ instead of the pendulum equation. See Remark 2.11 for a more detailed explanation of these limiting values of p.

Remark 2.3. *Another case in which $J_0(\mu)\to 0$ is when $\mu\to\mu_i$ (being μ_i any zero of $J_0(\mu)$) since $J_0(\mu)\sim\mu-\mu_i$. For this case, taking $\mu=\mu_0-\varepsilon^r$ (or close to any other zero) an analogous reasoning to Remark 2.2 applies. Now system (13) is a $2\pi\sqrt{J_0(\mu)}=\mathcal{O}(\varepsilon^{1+r/2})$-periodic perturbation of size $\mathcal{O}(\varepsilon^{1-r/2})$ of the classical pendulum. Thus, in Theorem 2.12 we will be able to provide a rigorous proof of the existence of the invariant manifolds and an exponentially small upper bound of the size of the splitting provided $r<2$, analogously to the the results obtained for the* below the singular case *in Theorem 2.10. For μ closer to μ_0, that is taking $\mu=\mu_0-\varepsilon^r$ with $r\ge 2$, the leading term of the averaged system is not the pendulum anymore and thus the hyperbolic structure, and so the splitting, changes drastically (see Remark 2.13).*

For the rest of this paper we consider $\mu\varepsilon^4\ll 1$ and $|\mu-\mu_i|\gg\varepsilon^2$. We prove that the periodic orbit is hyperbolic and we study the splitting between its stable and unstable manifolds. The results are split in three cases depending on μ. The results for $\mu<\mu_0$ are stated in Theorems 2.4 and 2.6. The asymptotic formula for the splitting of separatrices is given, in the Geometric Theorem 2.4, in terms of the area of the lobes between the invariant curves $C^s(t_0)$ and $C^u(t_0)$ in the Poincaré section (see Figure 1). A more analytic version of this result is given in Theorem 2.6.

Later, we consider the *below the singular case*, that is $\mu=\varepsilon^p$ with $p\in(-4,0)$ and, in Theorem 2.10, we provide exponentially small upper bounds for the distance between the invariant manifolds.

Finally, in Theorem 2.12, we give the result for $\mu=\mu_0-\varepsilon^r$ with $r\in(0,2)$. Even if we focus in the first zero of $J_0(\mu)$, the results are analogous for all the zeros. The asymptotic size of the distance between the invariant manifolds in these two last cases is still an open problem.

2.1. The classical singular case

Theorem 2.4 (Main Theorem: geometric version)**.** *Consider the hyperbolic fixed point of the Poincaré map of system (4) given by Theorem 2.1. Then, for any fixed $\bar\mu<\mu_0$ (where μ_0 is the first zero of the Bessel function $J_0(\mu)$), there exists $\varepsilon_0>0$ (depending on $\bar\mu$) such that for $\varepsilon\in(0,\varepsilon_0)$ and $\mu\in(0,\bar\mu)$, the area of the lobes between the invariant stable and unstable curves associated to this point is given by the asymptotic formula*

$$\mathcal{A}=\frac{\mu}{\varepsilon}e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}}\left(4|f(\mu)|+\mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right)\right)$$

where $f(\mu)=2\pi+\mathcal{O}(\mu^2)$ is a real-analytic even function.

Remark 2.5. *The result presented in this theorem is valid for μ belonging to any compact set I_i with $i\in\mathbb{N}$ in which $J_0(\mu)>0$ (see Remark 2.2). However, in order to* *simplify the notation and to make this result comparable to the previous ones (see [DS92], [Gel94]) we restrict ourselves to* $\mu \in (0, \bar{\mu}] \subset (0, \mu_0)$.

When μ *belongs to a compact set in which* $J_0(\mu) < 0$*, an analogous result can be obtained making little changes (see [Gel97b, Gel00]).*

The Geometric Theorem 2.4 will be a consequence of the next Theorem 2.6. To state it we work with the fast time $\tau = t/\varepsilon$ from now on, and therefore we deal with system (4).

In order to study the invariant manifolds of the periodic orbit found in Theorem 2.1, as a first step, we consider the symplectic time-dependent change of variables,

$$\begin{cases} q = x - x_p(\tau) \\ p = y - x_p'(\tau). \end{cases} \tag{14}$$

With this change, system (4) becomes a new Hamiltonian system with a hyperbolic periodic orbit at the origin which has stable and unstable 2-dimensional invariant manifolds. Unlike the unperturbed system, these invariant manifolds do not coincide, but since are Lagrangian (see [LMS03]), they can be written locally as the graph of the differential of certain functions as $p = \partial_q S^\pm(q, \tau)$, where $S^\pm$ are usually called generating functions and verify the so-called Hamilton-Jacobi equation (see (25)). However, since for the unperturbed system they can be written as a graph globally, it is expected that the same happens for the perturbed case.

Moreover, once we know the functions $S^\pm(q, \tau)$, we are able to compute the difference between the manifolds in the original coordinates simply by computing the difference between $\partial_q S^+(q, \tau)$ and $\partial_q S^-(q, \tau)$.

Theorem 2.6 (Main Theorem: analytic version)**.** *For fixed* $Q_0 \in (0, \pi/4)$ *and* $\bar{\mu} \in (0, \mu_0)$ *(where* μ_0 *is the first zero of* $J_0(\mu)$*), there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *and* $\mu \in (0, \bar{\mu}]$*:*

(i) There exists a hyperbolic 2π*-periodic orbit* $(x_p(\tau), y_p(\tau))$ *of system (4). Moreover, it has stable and unstable invariant manifolds which can be parameterized as graphs* $y = y_p(\tau) + \partial_q S^\pm(x - x_p(\tau), \tau)$.

(ii) Writing $x = x_p(\tau) + q$*, for* $q \in (\pi - Q_0, \pi + Q_0)$ *and* $\tau \in \mathbb{R}$*, the functions* $S^\pm$ *hold*

$$\partial_q S^\pm(q, \tau) = y_p(\tau) + \partial_q S_0(q, \tau) + \mathcal{O}(\mu\varepsilon^2)$$

where S_0 *corresponds to the separatrix of the pendulum equation.*

(iii) For $q \in (\pi - Q_0, \pi + Q_0)$ *and* $\tau \in \mathbb{R}$*,*

$$\begin{aligned} &\partial_q S^+(q, \tau) - \partial_q S^-(q, \tau) = \\ &\qquad = \frac{2\mu}{\sqrt{J_0(\mu)}\varepsilon^2} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(f(\mu) \sin h(q, \tau) + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right)\right) \end{aligned} \tag{15}$$

where

$$h(q, \tau) = \tau - \left(\varepsilon\sqrt{J_0(\mu)}\right)^{-1} \ln\left(\tan(q/4)\right)$$

and $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$ *is a real-analytic even function.*

(iv) There exists a constant $\overline{\alpha}(\mu,\varepsilon)$ such that,

$$\begin{aligned} &S^+(q,\tau) - S^-(q,\tau) - \overline{\alpha}(\mu,\varepsilon) = \\ &\qquad = \frac{2\mu}{\varepsilon} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(f(\mu)\cos h(q,\tau) + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right)\right). \end{aligned} \tag{16}$$

Proof of the Geometric Theorem 2.4. The result follows almost directly from equality (16) of Theorem 2.6. The intersections of the invariant manifolds with a transversal section $\Sigma^{\tau_0} = \{(q,p,\tau_0),\, (q,p) \in \mathbb{R}^2\}$ are two different curves $C^{s,u}(\tau_0)$, which can be parameterized by $\gamma^{\pm}(q) = (q, y_p(\tau_0) + \partial_q S^{\pm}(q,\tau_0))$. Then, the area is given by

$$\mathcal{A} = \left| \int_{q_0}^{q_1} \partial_q S^+(q,\tau_0) - \partial_q S^-(q,\tau_0)\, dq \right| = \left| S^+(q,\tau_0) - S^-(q,\tau_0) \Big|_{q_0}^{q_1} \right| \tag{17}$$

where q_0 and q_1 are two consecutive points where $C^{s,u}(\tau_0)$ intersect each other in Σ^{τ_0}.

To simplify the notation, we define

$$F_0(q,\tau) = \frac{2}{\varepsilon} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} f(\mu)\cos h(q,\tau). \tag{18}$$

Then, by (16) and (17),

$$\mathcal{A} = \mu \left| F_0(q_1,\tau_0) - F_0(q_0,\tau_0) + (q_0 - q_1)\mathcal{O}\left(\frac{1}{\varepsilon \ln(1/\varepsilon)} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \right) \right|.$$

Using (15) we have that q_0 and q_1 are given by $h(q_i,\tau_0) = r_i$, where r_i verifies

$$f(\mu)\sin r_i + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right) = 0,$$

and therefore,

$$r_1 = 0 + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right) \text{ and } r_2 = \pi + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right).$$

Their existence for ε small enough is obtained applying the Implicit Function Theorem using that $1/f(\mu)$ is bounded for $\mu \in [0,\bar{\mu}]$.

Now,

$$\begin{aligned} \mathcal{A} &= \mu \left| \frac{2}{\varepsilon} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} f(\mu)(\cos r_1 - \cos r_0) + \mathcal{O}\left(\frac{1}{\varepsilon\ln(1/\varepsilon)} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \right) \right| \\ &= \frac{\mu}{\varepsilon} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(4|f(\mu)| + \mathcal{O}\Big(\frac{1}{\ln(1/\varepsilon)}\Big)\right). \end{aligned}$$

□

Remark 2.7. *As it can be seen in the proof of the Geometric Theorem 2.4, the constant $\overline{\alpha}$, which appears in formula (16), does not play any role. Nevertheless, straightforward computations lead to $|\overline{\alpha}| \leq \mathcal{O}(\mu\varepsilon^2)$.*

Remark 2.8. *The computation of the maximal distance between the stable and unstable manifolds, the splitting angle or the Lazutkin invariant at any homoclinic point can be done in an analogous way.*

Remark 2.9. *The results of Theorems 2.4 and 2.6 look more natural if we take into account that the invariant manifolds of system (4) are also, up to the term* $\mu\sin(t/\varepsilon)$ *and the rescaling of time, the invariant manifolds of system (12). Applying Melnikov Theory to this system, it seems natural the exponent* $\pi/(2\varepsilon\sqrt{J_0(\mu)})$ *appearing in the formulas of the area in Theorem 2.4, since the perturbation in system (12) is* $2\pi\varepsilon\sqrt{J_0(\mu)}$*-periodic in time.*

Comparing the result of Theorem 2.4 with the prediction of the Melnikov formula (6) and using that $J_0(\mu) = 1 + \mathcal{O}(\mu^2)$ *for* μ *small (see [AS92]), it is clear that, since the singularities of the homoclinic are* $i\pi/2 + \mathcal{O}(\mu^2)$*, both coincide provided* $\mu = \varepsilon^p$ *with* $p > 1/2$.

For $\mu = \varepsilon^p$ *with* $p \in (0, 1/2]$*, the result of Theorem 2.4 does not coincide with the Melnikov formula applied to system (4) but to system (11).*

Finally, for the classical limit case $\mu = \mathcal{O}(1)$*, Melnikov theory fails to predict correctly the exponentially small splitting given in Theorem 2.4.*

2.2. Below the singular case

In the *below the singular case*, that is $\mu = \varepsilon^p$ with $p \in (-4, 0)$, next theorem states that the periodic orbit $(x_p(\tau), y_p(\tau))$ of the Hamiltonian system (4) given in Theorem 2.1 is still hyperbolic. Furthermore, it gives an exponentially small upper bound for the splitting of its invariant manifolds.

Theorem 2.10 (Below the singular case)**.** *For fixed* $Q_0 \in (0, \pi/4)$*,* $p \in (-4, 0)$*,* $\bar{a} > 0$ *and* $\bar{\gamma} \in [0, 1 + p/4)$*, there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *and taking* $\mu = \varepsilon^p \in I_i$ *for* $i \in \mathbb{N}$ *(see Remark 2.2):*

(i) The 2π*-periodic orbit* $(x_p(\tau), y_p(\tau))$ *of system (4) is hyperbolic and its stable and unstable invariant manifolds can be parameterized as graphs* $y = y_p(\tau) + \partial_x S^{\pm}(x - x_p(\tau), \tau)$*.*

(ii) Writing $x = x_p(\tau) + q$*, for* $q \in (\pi - Q_0, \pi + Q_0)$ *and* $\tau \in \mathbb{R}$*, the invariant manifolds hold*

$$\partial_x S^{\pm}(q, \tau) = y_p(\tau) + \partial_x S_0(q, \tau) + \mathcal{O}\left(\varepsilon^{1+\frac{p}{4}} \ln(1/\varepsilon)\right)$$

where S_0 *corresponds to the separatrix of the pendulum equation.*

(iii) There exists a constant $\bar{C} > 0$ *independent of* ε *such that, for* $q \in (\pi - Q_0, \pi + Q_0)$ *and* $\tau \in \mathbb{R}$*,*

$$\left|\partial_x S^{+}(q, \tau) - \partial_x S^{-}(q, \tau)\right| \leq \bar{C}\varepsilon^{2-4\bar{\gamma}} e^{-\frac{1}{\varepsilon\sqrt{J_0(\mu)}}\left(\frac{\pi}{2} - \bar{a}\varepsilon^{\bar{\gamma}}\right)}.$$

Remark 2.11. *Theorem 2.10 gives an exponentially small upper bound for the splitting of separatrices for system (4) when* $\mu = \varepsilon^p$ *with* $p \in (-4, 0)$*. Nevertheless, in this theorem the exponent* $\bar{\gamma}$*, appearing in the correcting term* $\bar{a}\varepsilon^{\bar{\gamma}}$*, can not be taken equal to* $1 - p/4$*, which corresponds to the size of* $\varepsilon\sqrt{J_0(\mu)} = \mathcal{O}(\varepsilon^{1-p/4})$*.*

A sharper estimate for the splitting needs to consider as the unperturbed system the full averaged system of (13), obtained after another step of averaging, instead of (8).

So, the first step is to study how the singularities of the homoclinic orbit of the averaged system have changed with respect to the pendulum ones. In fact, taking into account that $\mu = \varepsilon^p$, $-4 < p < 0$, *and that* $J_0(\mu) = O(\varepsilon^{-p/2})$, *the averaged system of system (13) reads:*

$$\left\{ \begin{array}{l} \dfrac{d\bar{x}}{ds} = \bar{y} \\ \dfrac{d\bar{y}}{ds} = \sin \bar{x} + \mathcal{O}\left(\varepsilon^{2+p/2}\right) \langle m \rangle \left(\bar{x}\right) \end{array} \right. \tag{19}$$

and we have to locate the singularities of its homoclinic orbit. Recalling that system (13) is a $2\pi\varepsilon\sqrt{J_0(\mu)} = O(\varepsilon^{1-p/4})$*-periodic perturbation of system (19), we know that an upper bound of the distance between the perturbed invariant manifolds would be of order* $\mathcal{O}\left(\varepsilon^\beta e^{-a/(\varepsilon\sqrt{J_0})}\right)$, *where* a *corresponds to the imaginary part of the singularity of the homoclinic orbit of the averaged system (19) and* $\beta \in \mathbb{R}$. *Therefore, a necessary condition to have bounds of the form* $\mathcal{O}\left(\varepsilon^\beta e^{-\pi/(2\varepsilon\sqrt{J_0})}\right)$ *is that the imaginary part of the singularity holds*

$$a = \frac{\pi}{2} + \mathcal{O}(\varepsilon^\alpha), \quad \text{with } \alpha > 1 - p/4.$$

Even if, in general, it is not an easy question to know how the singularities of system (19) behave when $\varepsilon > 0$, *one can produce trivial examples (for instance when* $\bar{m}(\bar{x}) = \sin\bar{x}$*) where the singularities move at least the same order as the size of the perturbation, that is,* $\mathcal{O}(\varepsilon^\alpha)$ *with* $\alpha = 2 + p/2$. *In these cases the condition for* p *that would make the exponential size of the splitting to change dramatically would be* $2 + p/2 \leq 1 - p/4$, *and therefore* $p \leq -4/3$.

We do not claim in this paper that $p = -4/3$ *is the limiting value in which the size of the splitting changes, but we believe that studying how this singularity moves is an essential step to obtain the asymptotic value of the splitting.*

Once we knew about the singularity of the homoclinic orbit of system (19), a deeper study of the below the singular case *could be done studying the corresponding Hamilton-Jacobi equation, and thus reparameterizing through the new homoclinic orbit, and then looking for a suitable inner equation which gave the first order approximation of the invariant manifolds in the inner domain. Nevertheless, the main purpose of Theorem 2.10 is to illustrate that the splitting problem of the pendulum still has meaning for* $\mu = \varepsilon^p$ *with* $p \in (-4, 0)$, *which is below the usually considered "limiting case"* $p = 0$.

2.3. Close to a codimension-2 bifurcation

Finally we deal with the case in which μ is close to μ_0 where μ_0 is the first zero of the Bessel function. As we have said in Remark 2.3,

$$J_0(\mu) \sim \mu - \mu_0,$$

and therefore, we can deal with this case as we have done with the *below the singular case* in Theorem 2.10. Recall that in Remark 2.3 we have noticed that, taking $\mu = \mu_0 - \varepsilon^r$, we have that $J_0(\mu) = \mathcal{O}(\varepsilon^r)$, and thus the periodic orbit is still hyperbolic provided $r < 2$ (which corresponded in the previous case to $p > -4$).

We want to point out that the study of the splitting of separatrices in system (11) has interest in itself, since it can be seen as a very simple toy model for two degrees of freedom Hamiltonian system close to a second order resonance. In fact, when one studies perturbations of a completely integrable Hamiltonian system close to a simple resonance and averages with respect to the fast frequency, at first order one obtains a pendulum-like structure of size square root of the size of the perturbation. Nevertheless, in some degenerate cases this first order vanishes in the resonance, and therefore there is no impediment to perform another step of non-resonant averaging. In that case, we have a higher order resonance in which the hyperbolic structure is smaller (at least, same order as the size of the perturbation) as is happening in this problem. Even if a more accurate model of a Hamiltonian system close to second order resonances should depend on the actions in a more complicated form, it would be interesting, as a first step, to study system (11) with μ close to μ_0.

Theorem 2.12. *Let μ_0 be the first zero of the Bessel function $J_0(\mu)$, then for fixed $Q_0 \in (0, \pi/4)$, $r \in (0, 2)$, $\bar{a} > 0$ and $\bar{\gamma} \in [0, 1 - r/2)$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and taking $\mu = \mu_0 - \varepsilon^r$:*

(i) The 2π-periodic orbit $(x_p(\tau), y_p(\tau))$ of system (4) is hyperbolic and its stable and unstable invariant manifolds can be parameterized as graphs $y = y_p(\tau) + \partial_x S^{\pm}(x - x_p(\tau), \tau)$.

(ii) Writing $x = x_p(\tau) + q$, for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$, the invariant manifolds hold

$$\partial_x S^{\pm}(q, \tau) = y_p(\tau) + \partial_x S_0(q, \tau) + \mathcal{O}\left(\varepsilon^{1-\frac{r}{2}} \ln(1/\varepsilon)\right)$$

where S_0 corresponds to the separatrix of the pendulum equation.

(iii) There exists a constant $\bar{C} > 0$ independent of ε such that, for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$,

$$\left|\partial_x S^+(q, \tau) - \partial_x S^-(q, \tau)\right| \leq \bar{C} \varepsilon^{2-4\bar{\gamma}} e^{-\frac{1}{\varepsilon\sqrt{J_0(\mu)}}\left(\frac{\pi}{2} - \bar{a}\varepsilon^{\bar{\gamma}}\right)}.$$

Remark 2.13. *The problem of considering μ closer to μ_0, namely $\mu = \mu_0 - \varepsilon^r$ with $r \geq 2$ have to be studied in a different way, since now this system is better understood considering it as being close to the bifurcating value μ_0. For $\mu = \mu_0$, system (11) has zero average and period $2\pi\varepsilon$ and thus one has to perform again another step of averaging to obtain, after rescaling $y = \varepsilon \bar{y}$ and $t = s/\varepsilon$,*

$$\left\{\begin{array}{rcl} \dfrac{d\bar{x}}{ds} &=& \bar{y} + h_1\left(\frac{s}{\varepsilon^2}\right)\cos\bar{x} + h_2\left(\frac{s}{\varepsilon^2}\right)\sin\bar{x} \\ \dfrac{d\bar{y}}{ds} &=& \langle m\rangle(\bar{x}) - \dfrac{1}{\varepsilon}\bar{y}\left(h_1\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x} - h_2\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x}\right) + \\ && + \left(m\left(\bar{x}, \dfrac{s}{\varepsilon^2}\right) - \langle m\rangle(\bar{x})\right). \end{array}\right.$$

where m is the function considered in Remark 2.2 and its average is of the form

$$\langle m\rangle(x) = a_1 \sin(2x) + a_2 \cos(2x)$$

for certain constants a_1 and a_2. Therefore, the averaged system

$$\left\{\begin{aligned} \frac{d\bar{x}}{ds} &= \bar{y} \\ \frac{d\bar{y}}{ds} &= \langle m\rangle\,(\bar{x}), \end{aligned}\right. \tag{20}$$

has a double-well potential with two hyperbolic critical points in the same level of energy which are connected by four heteroclinic orbits (see Figure 2).

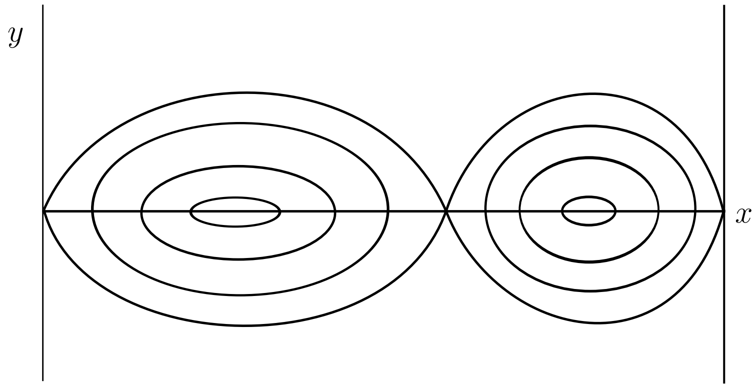

Figure 2. Phase space of the averaged system (20).

Taking now $\mu = \mu_0 - \varepsilon^r$ with $r \geq 2$, we obtain as averaged system

$$\left\{\begin{aligned} \frac{d\bar{x}}{ds} &= \bar{y} \\ \frac{d\bar{y}}{ds} &= \langle m\rangle\,(\bar{x}) + \mathcal{O}(\varepsilon^{r-2})\sin x. \end{aligned}\right. \tag{21}$$

which is a perturbation of (20). For this system, the hyperbolic critical points belong to different levels of energy, and therefore the heteroclinic connections bifurcate into four homoclinic orbits, as it can be seen in Figure 3: two of them forming an "eye" as in the classical pendulum and the other two forming a figure eight. To study the splitting of any of these homoclinic orbits, as a first step one has to study how its singularities change with respect to the ones of the case $\mu = \mu_0$.

Between this dynamics and the one studied in Theorem 2.12, which is close to the classical pendulum, there exist another codimension-1 bifurcation of the averaged system which takes place at a curve in the (μ, ε) parameter space which is given at first order by $\mu = \mu - c\varepsilon^2$ for certain constant $c > 0$. In this curve ocurrs a saddle-center bifurcation in such a way that at the bifurcation value there exists a parabolic point resulting from the collision between one of the saddles and its nearest elliptic point. Consequently, after crossing the bifurcation, one of the homoclinic orbits shrinks to the parabolic point and disappears whereas the other one becomes a periodic orbit around the other elliptic point. After crossing the bifurcation curve we obtain again the classical picture of the pendulum equation.

Remark 2.14. *Once we have studied how is the averaged integrable system (21), we add the non-autonomous perturbation to see how behaves the splitting of separatrices for*

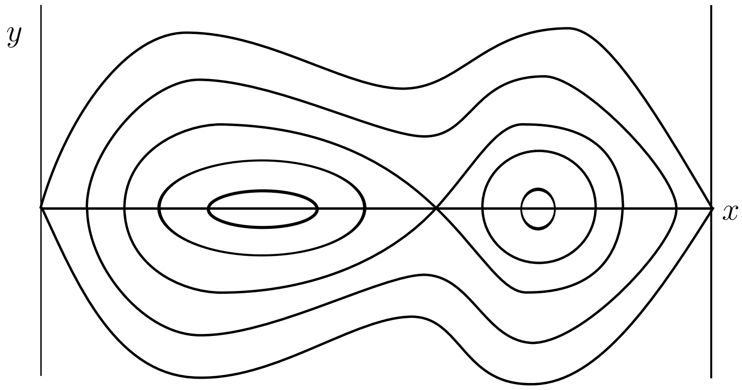

Figure 3. Phase space of the averaged system (21) for ε small enough.

$\mu = \mu_0 - \varepsilon^r$ *for* $r \geq 2$. *Recall that, after the rescaling performed to obtain system (20), the period of the perturbation is* $2\pi\varepsilon^2$. *The systems reads*

$$\left\{ \begin{array}{rcl} \dfrac{d\bar{x}}{ds} & = & \bar{y} + h_1\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x} + h_2\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x} \\ \dfrac{d\bar{y}}{ds} & = & \langle m\rangle(\bar{x}) + \dfrac{J_0(\mu)}{\varepsilon^2}\sin\bar{x} - \dfrac{1}{\varepsilon}\bar{y}\left(h_1\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x} - h_2\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x}\right) \\ & & + \left(m\left(\bar{x}, \dfrac{s}{\varepsilon^2}\right) - \langle m\rangle(\bar{x})\right). \end{array} \right. \quad (22)$$

where $J_0(\mu)/\varepsilon^2 = \mathcal{O}(\varepsilon^{r-2})$. *Note that the perturbation now is of order* $\mathcal{O}(1/\varepsilon)$ *and thus bigger than the averaged system. Therefore, to understand fully the splitting of any of these homoclinic orbits we should perform another step of averaging, obtaining a new averaged system and decreasing the size of the non-averaged part. Thus, we want to point out that in Remark 2.13 we are not claiming that system (21) is the* good *system to consider as unperturbed but that at least it contains its leading term, and therefore one has to know how are the singularities of the separatrices of system (21) before we add the smaller averaged terms. In any case, since the perturbation is* $2\pi\varepsilon^2$*-periodic, it is expected that the splitting of any of the separatrices has an upper bound of the size* $\mathcal{O}(\varepsilon^\beta e^{-d/\varepsilon^2})$, *creating two exponentially small chaotic layers in the phase space.*

Remark 2.15. *Looking at the averaged system, it is straightforward to see that the existence of heteroclinic orbits is only possible in the line* $\mu = \mu_0$. *Nevertheless, the transversal splitting of the homoclinic orbits which exist away from* $\mu = \mu_0$ *might imply the transversal intersection between the stable and the unstable invariant manifolds of the two hyperbolic points. However, since this splitting is exponentially small, it is expected that the region in the parameter space where these heteroclinic orbits can exist will be delimited by two curves emanating from* $(\varepsilon, \mu) = (0, \mu_0)$ *which are exponentially close (see [GN09] for a related problem). Away from this region, the two exponentially chaotic zones seem to be enough separated to allow the existence of some invariant cruves, as is indicated in [Tre98].*

The rest of the paper is organized as follows. In Section 3 we write the invariant manifolds as graphs of functions $S^{\pm}(q,\tau)$ which are solutions of the Hamilton-Jacobi equation (25) associated to (4). We also give the complete description of the proof of Theorem 2.6. However, to make the paper more readable we postpone the proofs of the theorems stated in Section 3 to Sections 4, 5, 6, 7 and 8.

Section 4 is devoted to the proof of the first part of Theorem 2.1. The existence of stable and unstable manifolds and their first order approximations for $|\mu| \leq \mathcal{O}(1)$ are established, in different domains, in Sections 5 and 6. Firstly, in Section 5 we present results in the *outer domain*, a region of the complex plane up to a distance $\mathcal{O}(\varepsilon^{\gamma})$, with $0 < \gamma < 1$, from $\pm i\pi/2$. These points are the singularities of the unperturbed homoclinic orbit closest to $\mathbb{R}$. In this domain, the manifolds are well approximated by the unperturbed homoclinic orbit. Secondly, in Section 6, we present the results in the *inner domain*, a region of the complex plane up to a distance $\mathcal{O}(\varepsilon\ln(1/\varepsilon))$ from the singularities. As it is shown in Section 3, in the inner region the manifolds are well approximated by the solutions of the so-called inner equation, which corresponds to the Reference System in [Gel97b]. This is done using complex matching techniques in the Hamilton-Jacobi equation.

In Sections 7 and 8, we obtain a change of variables which allows us to have the difference between manifolds as a solution of a homogeneous linear partial differential equation with constant coefficients. Proceeding as in the preceding sections, first we obtain this change of variables in the outer domain and afterwards, using complex matching techniques, we extend it to the inner one. The final result follows from straightforward properties of linear partial differential equations with constant coefficients. This idea was already used in [Sau01, OSS03, Bal06].

Finally, the results about the *below the singular case* given in Theorem 2.10 are proved in Section 9. In this case the manifolds, as well as the change of variables, are just studied in the outer domain where they are well approximated by the unperturbed homoclinic orbit. Consequently, we only obtain exponentially small upper bounds which are not sharp.

The proof of Theorem 2.12 will not be given since it is completely analogous to the one of Theorem 2.10 considering $p = -2r$ with $r < 2$.

3. Description of the proof of Theorem 2.6

3.1. The Hamilton-Jacobi equation

Even if Hamiltonian (5) is entire as a function of τ, we do not take advantage of it. Instead, we restrict ourselves to strips which make the proofs easier. We consider a complex strip around the torus: $\mathbb{T}_\sigma = \{\tau \in \mathbb{C}/2\pi\mathbb{Z} : |\Im\tau| \leq \sigma\}$ with σ independent of ε and μ. In fact, since through the proof we have to reduce slightly the strip of analyticity, we consider $\sigma_1 > \sigma_2 > \ldots > \sigma_5 > 0$. Note that σ_1 will be given by Theorem 3.2 and will be independent of ε and μ, and therefore, we will be able to make this finite number of reductions.

Notation 3.1. *From now on, in order to simplify the notation, if there is no danger of confusion, the dependence on μ or ε of all the functions will be omitted.*

First, we state the existence of the periodic orbit in the complex extension of the torus.

Theorem 3.2. *There exists $\varepsilon_0 > 0$ such that for any $\bar{\mu} > 0$ there exists $\sigma_1 > 0$, such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \bar{\mu})$, system (4) has a 2π-periodic orbit $(x_p(\tau), y_p(\tau)) : \mathbb{T}_{\sigma_0} \to \mathbb{C}^2$ which is real-analytic and verifies*

$$\sup_{\tau \in \mathbb{T}_{\sigma_0}} |x_p(\tau) + \mu \sin \tau| \leq b_0 \mu \varepsilon^2 \tag{23}$$

where b_0 is a constant depending only on σ_1, $\bar{\mu}$ and ε_0.

The proof of this theorem, which is done in [Fon93] for μ small enough, is postponed to Section 4.

Remark 3.3. *The strip of analyticity of the periodic orbit does not play any role in the results of this paper as it does in the case of a quasiperiodic perturbation (see [DGJS97]). For this reason, even though it is possible to prove that the periodic orbit is analytic in a strip of size $\sigma_\varepsilon = \mathcal{O}(\ln(C/\mu))$ for a suitable constant $C > 0$ (see [SV00] for a related problem), we content ourselves to work in a strip of fixed width $\sigma_1 > 0$.*

Once we know the existence of the hyperbolic periodic orbit by Theorem 3.2, With change (14), system (4) becomes a Hamiltonian system of Hamiltonian

$$K(q, p, \tau) = \varepsilon \left(\frac{p^2}{2} + (\cos q - 1) \cos x_p(\tau) + (q - \sin q) \sin x_p(\tau) \right) \tag{24}$$

which has a hyperbolic periodic orbit at the origin $p = q = 0$.

Since its local invariant manifolds are Lagrangian, they have a parameterization of the form $(q, \partial_q S^{\pm}(q, \tau))$, where $S^{\pm}$ are 2π-periodic in τ solutions of the non-autonomous Hamilton-Jacobi equation

$$\partial_\tau S + K(q, \partial_q S(q, \tau), \tau) = 0,$$

that is

$$\partial_\tau S + \varepsilon \left(\frac{1}{2} (\partial_q S)^2 + (\cos q - 1) \cos x_p(\tau) + (q - \sin q) \sin x_p(\tau) \right) = 0, \tag{25}$$

with asymptotic conditions

$$\begin{cases} \lim_{q \to 0} \partial_q S^-(q, \tau) = 0 & \text{(for the unstable manifold)} \\ \lim_{q \to 2\pi} \partial_q S^+(q, \tau) = 0 & \text{(for the stable manifold).} \end{cases}$$

Note that the separatrix of the unperturbed system is given by $(q, \partial_q S_0(q))$ where $S_0(q) = 4(1 - \cos(q/2))$, which is a solution of equation (25) for $\mu = 0$.

Recalling that $x_p(\tau) = -\mu \sin \tau + \mathcal{O}(\mu \varepsilon^2)$, equation (25) is approximated up to first order in ε by

$$\partial_\tau S + \varepsilon \Big(\frac{1}{2} (\partial_q S)^2 + (\cos q - 1) \cos(\mu \sin \tau) - (q - \sin q) \sin(\mu \sin \tau) \Big) = 0, \tag{26}$$

which is the Hamilton-Jacobi equation of the Hamiltonian

$$K_0(q,p,\tau) = \varepsilon\left(\frac{1}{2}p^2 + (\cos q - 1)\cos(\mu\sin\tau) - (q - \sin q)\sin(\mu\sin\tau)\right),$$

whose associated system of differential equations is

$$\begin{cases} q' = \varepsilon p \\ p' = \varepsilon\left(\sin q\cos(\mu\sin\tau) + (1-\cos q)\sin(\mu\sin\tau)\right). \end{cases}$$

Even if this system is a perturbation of the pendulum

$$\begin{cases} q' = \varepsilon p \\ p' = \varepsilon \sin q \end{cases} \tag{27}$$

for μ small enough with respect to ε, this is not true for general values of μ. On the other hand, as it is non-autonomous and periodic in time, averaging theory ensures that it is close to the averaged system

$$\begin{cases} q' = \varepsilon p \\ p' = \varepsilon J_0(\mu)\sin q, \end{cases} \tag{28}$$

where $J_0(\mu)$ is the zero order Bessel function of first type defined in (9). Recall that when μ is small, $J_0(\mu)$ has the asymptotic expansion

$$J_0(\mu) = 1 - \frac{\mu^2}{4} + \mathcal{O}(\mu^4),$$

so that (28) is an $\mathcal{O}(\varepsilon^2)$-perturbation of the pendulum (27) provided $\mu^2 = \mathcal{O}(\varepsilon)$.

The Hamilton-Jacobi equation of (28) is

$$\frac{1}{2}(\partial_q S)^2 + J_0(\mu)(\cos q - 1) = 0.$$

Due to the coefficient $J_0(\mu)$, this equation does not correspond exactly to the Hamilton-Jacobi equation of the unperturbed pendulum (27). In fact, the solution of this equation is $\sqrt{J_0(\mu)}S_0(q) = S_0(q) + \mathcal{O}(\mu^2)$.

In order to have the classical pendulum as a first order approximation of (25), we perform the change

$$S(q,\tau) = \sqrt{J_0(\mu)}\cdot\bar{S}(q,\tau) \tag{29}$$

for $\mu \in (0,\mu_0)$, where μ_0 is the first zero of the Bessel function. This change is well defined since $J_0(\mu) > 0$ for $\mu \in (0,\mu_0)$ (see Remark 2.5). From now on, we consider μ belonging to compact sets $[0,\bar{\mu}_0]$ for $\bar{\mu}_0 < \mu_0$. Therefore all the results which are stated in the rest of this section and Sections 5 to 8 are only valid provided this condition holds.

In order to simplify the notation, from now on we write J instead of $J_0(\mu)$. Moreover, we consider as a new small parameter

$$\delta = \varepsilon\sqrt{J},$$

so that the transformed Hamilton-Jacobi equation reads

$$\partial_\tau\bar{S} + \delta\Big(\frac{1}{2}(\partial_q\bar{S})^2 + \frac{1}{J}(\cos q - 1)\cos x_p(\tau) + \frac{1}{J}(q - \sin q)\sin x_p(\tau)\Big) = 0.$$

Remark 3.4. *With this new parameter δ and taking $\mu \in (0, \bar{\mu}_0]$, Theorem 3.2 is still valid if we replace ε by δ, for $\delta < \delta_0 = \varepsilon_0/J$.*

Following the idea given by Poincaré in [Poi99], we reparameterize the invariant manifolds with the time through the homoclinic orbit, namely

$$q = q_0(u) = 4\arctan(e^u). \tag{30}$$

Taking

$$T^{\pm}(u,\tau) = \bar{S}^{\pm}(q_0(u),\tau) \tag{31}$$

and defining

$$\psi(u) = q_0(u) - \sin q_0(u), \tag{32}$$

we obtain that the stable and unstable manifolds $T^{\pm}(u,\tau)$ are 2π-periodic in τ solutions of the equation

$$\begin{aligned} \partial_\tau T(u,\tau) \ &+ \\ + \ &\delta\Big(\frac{\cosh^2 u}{8}\left(\partial_u T(u,\tau)\right)^2 - \frac{2}{J\cosh^2 u}\cos x_p(\tau) + \frac{1}{J}\psi(u)\sin x_p(\tau)\Big) = 0, \end{aligned} \tag{33}$$

with asymptotic conditions

$$\begin{cases} \lim\limits_{u\to-\infty} \cosh u \cdot \partial_u T^-(u,\tau) = 0 \\ \lim\limits_{u\to+\infty} \cosh u \cdot \partial_u T^+(u,\tau) = 0. \end{cases} \tag{34}$$

In order to understand the behavior of $T^{\pm}$, we expand them formally in power series

$$T^{\pm}(u,\tau) \sim \sum_{k=0}^{\infty} \delta^k T_k^{\pm}(u,\tau), \tag{35}$$

where $T_k^{\pm}$ are 2π-periodic in τ and hold asymptotic conditions (34). Next proposition, whose proof is straightforward, gives the first terms in these asymptotic expansions.

Proposition 3.5. *Except for constant terms, the first terms in (35) are*

$$T_0(u) \quad = 4\frac{e^u}{\cosh u} \tag{36}$$

$$T_1(u,\tau) = \ -\frac{2}{\cosh^2 u} f_1(\tau) + \psi(u) f_2(\tau), \tag{37}$$

where ψ is the function defined in (32) and $f_1(\tau)$ and $f_2(\tau)$ are the primitives of $l_1(\tau) = (J - \cos(\mu\sin\tau))/J$ and $l_2(\tau) = \sin(\mu\sin\tau)/J$ which are 2π-periodic and have zero mean. Moreover, we remark that for $\tau \in \mathbb{T}_{\sigma_0}$, it holds that

$$|f_1(\tau)|, |f_2(\tau)| \le \mathcal{O}(\mu). \tag{38}$$

One can see that $T_k^+ = T_k^- = T_k$ for all $k \in \mathbb{N}$, so that the formal expansion $\tilde{T} = \sum_{k=0}^{\infty} \delta^k T_k$ is the same for both manifolds and is of Gevrey type (in suitable domains, there exist $M > 0$ and $\rho > 0$ such that $|T_k| < M\rho^k k!$, see [Bal94]). This fact, as it is well known, leads to the exponentially smallness of the difference between T^+ and T^-.

The first term of these asymptotic expansions, T_0, corresponds to the homoclinic orbit for the unperturbed problem, namely for $\mu = 0$. In other words, for the unperturbed system, the homoclinic orbit in the new variable u can be parameterized by $\left(u, \frac{\cosh u}{2}\partial_u T_0(u)\right)$. This term has polar singularities at $u = i\pi/2 + k\pi i$ for $k \in \mathbb{Z}$ which propagate to all the terms in the series (35).

On the other hand, in order to compute the exponentially small splitting of the perturbed invariant manifolds $T^{\pm}(u,\tau)$, it is a crucial step to prove their existence for u in a complex strip as wide as possible. In fact, following the idea given by Lazutkin in [Laz03], we study the invariant manifolds up to a distance of order $\mathcal{O}(\delta \ln(1/\delta))$ of the singularities $u = \pm i\pi/2$, which are the closest to the real axis.

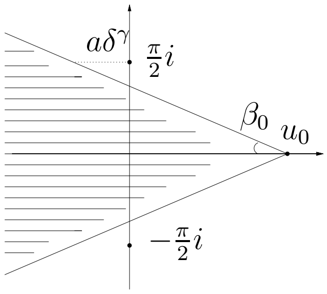

Figure 4. The outer domain D^u_γ.

3.2. Existence and approximation of the invariant manifolds in the outer domains

As a first step, we prove the existence of $T^{\pm}$ in certain domains that are called *outer domains*, which correspond to sectorial neighborhoods of $\pm\infty$ which are far from the singularities of $T_0(u)$, that is, at a distance $\mathcal{O}(\delta^\gamma)$ with $\gamma \in (0,1)$. In the Section 3.3, we study the invariant manifolds when u is closer to $\pm i\pi/2$.

The outer domains, as it can be seen in Figure 4, are given by:

$$\begin{cases} D^u_\gamma = \left\{ u \in \mathbb{C} : |\Im u| < -\dfrac{\pi}{2(u_0 + a\delta^\gamma)}(\Re u + a\delta^\gamma) + \dfrac{\pi}{2} \right\} \\ D^s_\gamma = \{ u \in \mathbb{C} : -u \in D^u_\gamma \}, \end{cases} \tag{39}$$

where $a \in (0, \pi/4)$, $\gamma \in [0,1)$ and $u_0 > 0$ are fixed. Throughout the article we will also consider the constant β_0 related to the slopes of the boundaries of the outer domains, which can be obtained from u_0, a and δ as

$$\beta_0 = \arctan\left(\frac{\pi}{2(u_0 + a\delta^\gamma)} \right) < \frac{\pi}{2}. \tag{40}$$

It is necessary to split these domains into two parts in order to study the behavior of the invariant manifolds whether $\Re u \to \pm\infty$ or near the singularities $u = \pm i\pi/2$. Thus, we fix real constants $\rho > 0$, small but independent of δ, and $U > u_0$, and we consider the domains (see Figure 5),

$$D_\gamma^{u,c} = \{u \in D_\gamma^u | \, \Re u > -U-\rho\} \quad \text{where `c' is written for close to the singularity} \tag{41}$$

$$D_\gamma^{u,f} = \{u \in D_\gamma^u | \, \Re u < -U+\rho\} \quad \text{where `f' is written for far from the singularity.} \tag{42}$$

Let us note that $D_\gamma^{u,f} \cap D_\gamma^{u,c} \neq \emptyset$ provided $\rho > 0$. We define analogously $D_\gamma^{s,c}$ and $D_\gamma^{s,f}$.

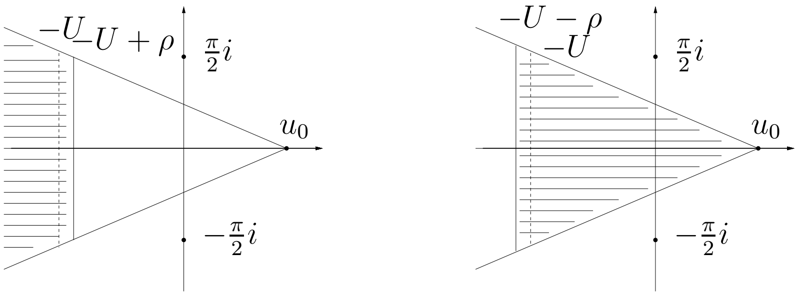

Figure 5. The domain far from the singularity and close to the singularity: $D_\gamma^{u,f}$, $D_\gamma^{u,c}$.

Moreover, for technical reasons, through the article we have to reduce slightly the domains a finite number $N = 14$ of times. These reductions, which are called $D_\gamma^{u(i)}$, $D_\gamma^{u,c(i)}$ and $D_\gamma^{u,f(i)}$ for $0 \leq i \leq N$, are done in such a way that they preserve the shape of the original ones, as it can be seen in Figure 6. We take parameters $0 = h_0 < \ldots < h_N < 1$. Then, for fixed δ and γ, we consider the modified parameters

$$\begin{aligned} a^{(i)} &= (1+h_i)a \\ u_0^{(i)} &= u_0 - h_i a\delta^\gamma, \quad u_0^{(0)} = u_0 \\ \rho^{(i)} &= (1-h_i)\rho, \end{aligned}$$

in such a way that the new domains are defined as

$$D_\gamma^{u(i)} = \left\{ u \in \mathbb{C} : |\Im u| < -\tan\beta_0(\Re u + a^{(i)}\delta^\gamma) + \pi/2 \right\} \tag{43}$$

and analogously for $D_\gamma^{*,c(i)}$ and $D_\gamma^{*,f(i)}$, where $*$ denotes either s or u.

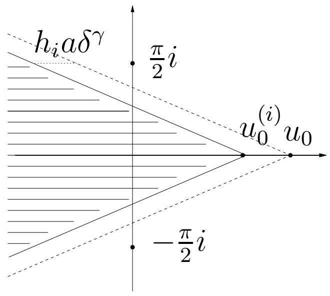

Figure 6. Reduction of the outer domain, $D_\gamma^{u(i)} \subset D_\gamma^u$.

With these reductions, we obtain that for $u \in D_\gamma^{u(i)}$,

$$\mathrm{d}\left(u, \partial D_\gamma^{u(i-1)}\right) > \mathcal{O}(\delta^\gamma), \tag{44}$$

which gives, writing $*$ for either s or u, the following inclusions:

$$D_\gamma^{*(N)} \subset \ldots \subset D_\gamma^{*(1)} \subset D_\gamma^{*(0)},$$

which also hold for $D_\gamma^{*,f(i)}$ and $D_\gamma^{*,c(i)}$.

Next theorem gives the existence and the asymptotic expressions of the invariant manifolds in the outer domains.

Theorem 3.6. *Let σ_1, $\bar\mu_0$ and γ be real numbers such that $0 < \sigma_1 < \sigma_0$, $0 < \bar\mu_0 < \mu_0$ and $\gamma \in [0,1)$. Then, there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$ and $\mu \in (0,\bar\mu_0]$, equation (33) has unique (module an additive constant) solutions in $D_\gamma^{u(1)} \times \mathbb{T}_{\sigma_1}$ and $D_\gamma^{s(1)} \times \mathbb{T}_{\sigma_1}$ of the form*

$$T^\pm(u,\tau) = T_0(u) + \delta T_1(u,\tau) + Q^\pm(u,\tau)$$

holding asymptotic conditions (34).

Moreover, there exists a real constant $b_1 > 0$ independent of δ and μ, such that

$$\left|\partial_u^{i+1} Q^-(u,\tau)\right| \le \begin{cases} \dfrac{b_1\mu\delta^{2-i\gamma}}{\left|\cosh^4 u\right|} & \text{for } (u,\tau) \in D_\gamma^{u,c(1)} \times \mathbb{T}_{\sigma_1} \\ \dfrac{b_1\mu\delta^{2-i\gamma}}{\left|\cosh^2 u\right|} & \text{for } (u,\tau) \in D_\gamma^{u,f(1)} \times \mathbb{T}_{\sigma_1} \end{cases}$$

for $i = 0,1$. Analogous bounds for Q^+ also hold.

This theorem is proved in Section 5. It shows that the invariant manifolds in the outer domains are well approximated by the unperturbed separatrix. In fact, using (37), as a consequence of this theorem we have that,

$$\left|\partial_u T^\pm(u,\tau) - \partial_u T_0(u)\right| \le \frac{1}{\left|\cosh^3 u\right|}\mathcal{O}(\mu\delta)$$

for $u \in D_\gamma^{u,c(1)}$. However, in the proofs of Theorems 3.10 and 3.12, it will be convenient to have explicitly the order δ term T_1 in the expansion of $T^\pm$.

3.3. Existence and approximation of the invariant manifolds in the inner domains

In order to prove the exponentially small difference between T^+ and T^-, it is necessary to obtain good approximations of the invariant manifolds close to the singularities $u = \pm i\pi/2$. In fact, from Theorem 3.6 and formulas (36) and (37), near these singularities the invariant stable and unstable manifolds grow considerably. In order to study the unstable manifold T^- close to the singularities $\pm i\pi/2$, following [Laz84], we extend the unstable outer domain to the inner domain defined as

$$\begin{aligned} D^u_{\delta,+} = \;& \{u \in \mathbb{C} : \;\; \Im u > -\tan\beta_0\left(\Re u + \tilde a\delta^\gamma\right) + \pi/2,\ \Im u < \pi/2 - c\delta\ln(1/\delta), \\ & \qquad\qquad\; \Im u < -\tan\beta_1 \Re u + \pi/2 - c\delta\ln(1/\delta)\} \\ D^u_{\delta,-} = \;& \{u \in \mathbb{C} : \;\; \bar u \in D^u_{\delta,+}\} \end{aligned} \tag{45}$$

where β_0 is the parameter defined in (40), c is a parameter that we choose such that $0 < c < 1$ and $\beta_1 > \beta_0$ is taken such that $\beta_1 - \beta_0 = \mathcal{O}(1)$ (see Figure 7).

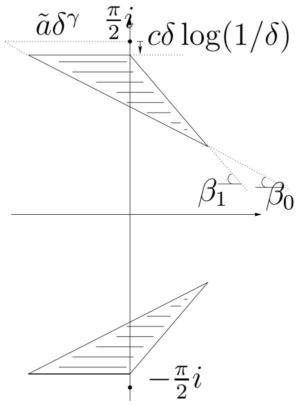

Figure 7. The inner domain for T^-, $D^u_{\delta,+}\cup D^u_{\delta,-}$.

As for the outer domain, the inner one is reduced, obtaining

$$D^{u(i)}_{\delta,+}=\left\{u\in\mathbb{C}:\Im u>-\tan\beta_0\left(\Re u+\tilde{a}^{(i)}\delta^\gamma\right)+\pi/2,\Im u<\pi/2-c^{(i)}\delta\ln(1/\delta),\right.\\ \left.\Im u<-\tan\beta_1\Re u+\pi/2-c^{(i)}\delta\ln(1/\delta)\right\}, \tag{46}$$

where

$$\begin{aligned}\tilde{a}^{(i)}&=(1-h_i)\tilde{a}\\ c^{(i)}&=(1+h_i)c\end{aligned} \tag{47}$$

and hence

$$c^{(i)}<2 \text{ for } i=1,\dots,N. \tag{48}$$

For the other domains the reduction is done analogously. Moreover, this reduction guarantees that for $u\in D^{*(i)}_{\delta,\pm}$ it holds that,

$$\mathrm{d}\left(u,\partial D^{*(i-1)}_{\delta,\pm}\right)>\mathcal{O}(\delta\ln(1/\delta)) \tag{49}$$

and

$$D^{*(N)}_{\delta,\pm}\subset\ldots\subset D^{*(1)}_{\delta,\pm}\subset D^{*(0)}_{\delta,\pm}.$$

Furthermore, since we need an overlapping domain between the inner and outer domains, we take $\tilde{a}>a$ in (39), such that for all i,j with $1\le i,j\le N$, $D^{u(i)}_{\delta,\pm}\cap D^{u(j)}_\gamma\neq\emptyset$.

Since the behavior near both singularities is analogous, we deal only with the proof for $D^{u(i)}_{\delta,+}$. In order to study the unstable manifold in this domain, we consider the change

$$z=\delta^{-1}(u-i\pi/2). \tag{50}$$

The variable z is called the *inner variable*, in contraposition to the *outer variable* u. Formulas (36) and (37) suggest the change

$$\phi^-(z,\tau)=\delta T^-(\delta z+i\pi/2,\tau). \tag{51}$$

We will prove the existence of ϕ^- in

$$\mathcal{D}^{u(i)}_{\delta,+}=\left\{z\in\mathbb{C}:\Im z>-\tan\beta_0\left(\Re z+\tilde{a}^{(i)}\delta^{\gamma-1}\right),\Im z<-c^{(i)}\ln(1/\delta),\right.\\ \left.\Im z<-\tan\beta_1\Re z-c^{(i)}\ln(1/\delta)\right\} \tag{52}$$

which corresponds to $D^{u(i)}_{\delta,+}$ expressed in the inner variable. Let us observe that for $z \in \mathcal{D}^{u(i)}_{\delta,+}$,

$$\mathrm{d}\left(z, \partial\mathcal{D}^{u(i-1)}_{\delta,+}\right) > \mathcal{O}\left(\ln(1/\delta)\right). \tag{53}$$

By (33), the unstable manifold in the inner variables, $\phi^-(z,\tau)$, has to verify

$$\partial_\tau\phi + \frac{\cosh^2\left(\frac{\pi}{2}i + \delta z\right)}{8\delta^2}\left(\partial_z\phi\right)^2 - \frac{2\delta^2}{\cosh^2\left(\frac{\pi}{2}i + \delta z\right)}\frac{\cos x_p(\tau)}{J} + \delta^2\psi\left(i\frac{\pi}{2} + \delta z\right)\frac{\sin x_p(\tau)}{J} = 0 \tag{54}$$

where ψ is the function defined in (32).

Using (36), (37) and Theorem 3.6, one expects that for $\Re z < 0$, $\Im z < 0$ with $|\delta z| \ll 1$, the following equality holds

$$\begin{aligned}\phi^-(z,\tau) &= \delta T_0\left(i\frac{\pi}{2} + \delta z\right) + \delta^2 T_1\left(i\frac{\pi}{2} + \delta z, \tau\right) + \dots \\ &= \frac{4}{z} + \mathcal{O}\left(\frac{1}{z^2}\right).\end{aligned}$$

So, we look for the solution of equation (54) which is 2π-periodic in τ and holds the asymptotic condition

$$\lim_{\Re z \to -\infty} \phi^-(z,\tau) = 0. \tag{55}$$

Analogously, for the stable manifold we define the domains $D^{s(i)}_{\delta,\pm}$ and $\mathcal{D}^{s(i)}_{\delta,\pm}$, and the function ϕ^+ defined in $\mathcal{D}^{s(i)}_{\delta,+}$ which has to hold (54) and

$$\lim_{\Re z \to +\infty} \phi^+(z,\tau) = 0. \tag{56}$$

As a first step, we have to study equation (54) for the first order terms in δ. Taking $\delta = 0$ we obtain the so-called *inner equation*

$$\partial_\tau\phi_0 - \frac{z^2}{8}\left(\partial_z\phi_0\right)^2 + \frac{2}{Jz^2}\left(\cos(\mu\sin\tau) + i\sin(\mu\sin\tau)\right) = 0 \tag{57}$$

and we look for 2π-periodic in τ solutions ϕ_0^+ and ϕ_0^- verifying asymptotic conditions (56) and (55), defined for z in domains of the form

$$\begin{aligned}\mathcal{D}^u_{\mathrm{i}} &= \{z \in \mathbb{C} : \Im z < -\tan\beta_1 \Re z - d\} \\ \mathcal{D}^s_{\mathrm{i}} &= \{z \in \mathbb{C} : \Im z > \tan\beta_1 \Re z - d\}\end{aligned}$$

where β_1 was introduced to define the domains $D^{u,s}_{\delta,\pm}$ in (45) and $d > 0$ is a real constant (see Figure 8).

Remark 3.7. *Let us observe that the inner equation (57) is the Hamilton-Jacobi equation of the Hamiltonian system with Hamiltonian*

$$\mathcal{H}(z,\omega,\tau) = -\frac{z^2}{8}w^2 + \frac{2}{Jz^2}\left(\cos(\mu\sin\tau) + i\sin(\mu\sin\tau)\right).$$

The dynamical system associated to this Hamiltonian was called Reference System *in [Gel97b].*

Theorem 3.8. *The following statements hold:*

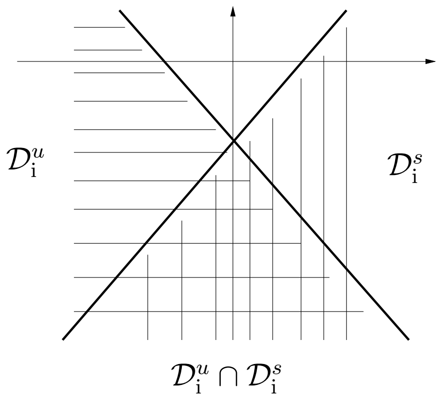

Figure 8. The inner domain for ϕ_0^+ and ϕ_0^-, $\mathcal{D}^{s,u}_{\rm i}$.

- *For each $\mu\in\mathbb{C}$, equation (57) has a unique formal solution*

$$\tilde\phi_0(z,\tau)=\sum_{n\geq 0}\frac{c_n(\tau)}{z^{n+1}} \tag{58}$$

where $c_0=4$, $c_1(\tau)=2\left(f_1(\tau)-if_2(\tau)\right)$ (see Proposition 3.5) and $c_n(\tau)$ are 2π-periodic entire functions.

- *Let us consider the constant σ_1 defined in Theorem 3.6 and any $\bar\mu>0$. Then, there exists a constant $d>0$ such that, for $\mu\in(0,\bar\mu)$, equation (57) has unique 2π-periodic in τ solutions $\phi_0^\pm(z,\tau)$ asymptotic to $\tilde\phi_0$ in the corresponding domains $(z,\tau)\in\mathcal{D}^{u,s}_{\rm i}\times\mathbb{T}_{\sigma_1}$. In particular, they hold the asymptotic conditions (56) and (55) respectively.*
Furthermore, their difference is exponentially small. There exists a constant $b_2>0$, such that, for $z\in\mathcal{D}^s_{\rm i}\cap\mathcal{D}^u_{\rm i}$, $\tau\in\mathbb{T}_{\sigma_1}$ and $\mu\in(0,\bar\mu)$:

$$\left|\phi_0^+(z,\tau)-\phi_0^-(z,\tau)-\mu f(\mu)e^{-i(z-\tau)}\right|\leq b_2\mu\left(|z|^{-1}e^{\Im z}+e^{a_1\Im z}\right), \tag{59}$$

where f is a real-analytic even function which holds $f(\mu)=2\pi+\mathcal{O}(\mu^2)$ for μ small and a_1 is any constant such that $a_1\in(1,2)$.

- *There exists a change of variables defined in $\mathcal{D}^u_{\rm i}\times\mathbb{T}_{\sigma_1}$*

$$z=x+R^-(x,\tau) \tag{60}$$

which conjugates

$$\bar{\mathcal{L}}=\partial_\tau-\frac{1}{4}z^2\partial_z\phi_0^-(z,\tau)\partial_z \tag{61}$$

with

$$\mathcal{L}=\partial_\tau+\partial_x. \tag{62}$$

Moreover, there exists $\delta_0>0$, such that for $\delta\in(0,\delta_0)$, $\mathcal{D}^{u(i)}_{\delta,+}\subset\mathcal{D}^u_{\rm i}$ for $i=0,\dots,N$ and:

(i) If $(x,\tau)\in\mathcal{D}^{u(i)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$, $z=x+R^-(x,\tau)\in\mathcal{D}^{u(i-1)}_{\delta,+}$ for $i=1,\dots,N$.

(ii) There exists a constant $b_3 > 0$, such that for $(x,\tau) \in \mathcal{D}^{u(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$ with $i = 1, \ldots, N$, it holds

$$\left|\partial_x^j R^-(x,\tau)\right| \leq \frac{b_3}{|x|^{j+1}} \text{ for } j = 0,1,2.$$

- *There exists the inverse change of variables $x = z + S^-(z,\tau)$ which holds:*
  (i) If $(z,\tau) \in \mathcal{D}^{u(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$, $x = z + S^-(z,\tau) \in \mathcal{D}^{u(i-1)}_{\delta,+}$ for $i = 1, \ldots, N$.
  (ii) There exists a constant $b_4 > 0$, such that for $(z,\tau) \in \mathcal{D}^{u(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$ with $i = 1, \ldots, N$, it holds

$$\left|\partial_z^j S^-(z,\tau)\right| \leq \frac{b_4}{|z|^{j+1}} \text{ for } j = 0,1,2.$$

Remark 3.9. *In [OSS03], the equation*

$$\partial_\tau \phi_0 - \frac{z^2}{8}(\partial_z \phi_0)^2 + \frac{2}{z^2}(1 - \mu \sin \tau) = 0, \tag{63}$$

which corresponds to the inner equation of

$$x'' = \varepsilon^2 \sin x + \mu \varepsilon^2 \sin x \sin \tau,$$

was considered and the complete and detailed proof of the results of Theorem 3.8 for this equation was given using Resurgence Theory techniques.

Let us observe that both equations (57) and (63) are of the form

$$\partial_\tau \phi_0 - \frac{z^2}{8}(\partial_z \phi_0)^2 + \frac{1}{z^2} P_\mu(\tau) = 0$$

where $P_\mu(\tau)$ is an entire 2π-periodic function.

As it is stated in Remark 1 of that paper, the proof of these results for general 2π-periodic functions $P_\mu(\tau)$ can be handled by the same method with little effort. For this reason we skip the proof and we just observe that the function $P_\mu(\tau)$ for (57) holds the same symmetry $P_\mu(\tau) = P_{-\mu}(\tau - \pi)$ as the function $P_\mu(\tau)$ of the inner equation studied in [OSS03]. Therefore, $\bar{f}(\mu) = \mu f(\mu)$, where $f(\mu)$ is the function considered in Theorem 3.8, does still have the oddness property of the function $f_0^{[i]}(\mu)$ of Theorem 2 and Corollary 1 of [OSS03], and holds $\bar{f}(\mu) = 2\pi\mu + \mathcal{O}(\mu^3)$. Therefore, $f(\mu)$ is even and holds $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$.

Another proof of the first part of Theorem 3.8 (in particular of (59)) can be found in [Bal06]. Let us remark that in that paper the result is obtained without the use of Resurgence Theory but with direct methods of functional analysis. Moreover, it applies to more general Hamiltonians which only are $\mathcal{C}^1$ in τ.

Theorem 3.8 gives the existence and behavior of $\phi_0^\pm$. Next theorem proves the existence of the invariant manifolds $\phi^\pm$ in $\mathcal{D}^{s,u(7)}_{\delta,+}$, considering them as a perturbation of $\phi_0^\pm$.

Theorem 3.10. *Let $\bar{\mu}_0$ be the constant considered in Theorem 3.6 and σ_3 and γ be real numbers such that $0 < \sigma_3 < \sigma_1$ and $\gamma \in (1/3, 1/2)$. Then, there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $\mu \in [0, \bar{\mu}_0]$, there exists functions $\phi^\pm(z,\tau)$ defined in $\mathcal{D}^{s,u(7)}_{\delta,+} \times \mathbb{T}_{\sigma_3}$ which are solutions of equation (54) and hold asymptotic conditions (56) and (55) respectively.*

The functions $\delta^{-1}\phi^{\pm}(\frac{u-i\pi/2}{\delta})$ *are the analytic continuation of the invariant manifolds* $T^{\pm}(u,\tau)$ *given by Theorem 3.6, to the corresponding inner domains* $D^{u,s(7)}_{\delta,+}$.

Moreover, there exists $b_5 > 0$ *independent of* δ *and* μ *such that for* $(z,\tau) \in \mathcal{D}^{u(7)}_{\delta,+} \times \mathbb{T}_{\sigma_3}$,

$$\begin{aligned}\left|\partial_z\phi^{\pm}(z,\tau) - \partial_z\phi_0^{\pm}(z,\tau)\right| &\le b_5\delta^2\\ \left|\partial_z^2\phi^{\pm}(z,\tau) - \partial_z^2\phi_0^{\pm}(z,\tau)\right| &\le b_5\frac{\delta^2}{\ln^2(1/\delta)}.\end{aligned}$$

This theorem is proved in Section 6 using complex matching techniques. That is, using a characteristic-like method which consists in integrating the Hamilton-Jacobi equation (54) from the overlapping zone between the inner and outer domains where the invariant manifolds are already defined by Theorem 3.6. This procedure ensures that the functions we obtain are the analytic continuation of the invariant manifolds defined in the outer domains.

In conclusion, Theorems 3.6 and 3.10 give the existence of the invariant manifolds $T^{\pm}$ in the domains $D^{u(7)}$ and $D^{s(7)}$ (see Figure 9) defined as

$$\begin{aligned}D^{u(i)} &= D^{u(i)}_{\gamma} \cup D^{u(i)}_{\delta,+} \cup D^{u(i)}_{\delta,-}\\ D^{s(i)} &= D^{s(i)}_{\gamma} \cup D^{s(i)}_{\delta,+} \cup D^{s(i)}_{\delta,-}.\end{aligned} \tag{64}$$

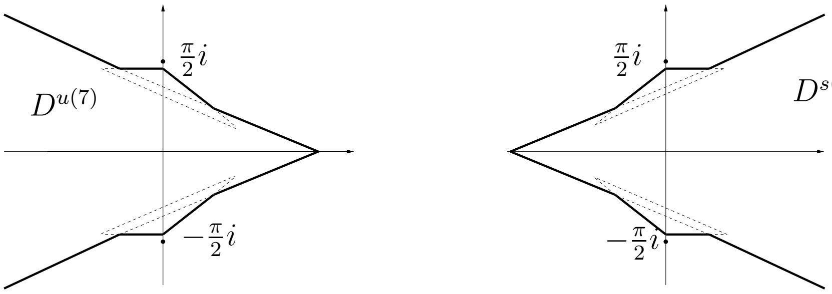

Figure 9. The domains for T^- and T^+.

3.4. Study of the difference between the invariant manifolds

Next step is to compute $\Delta T(u,\tau) = T^+(u,\tau) - T^-(u,\tau)$ in $D^{(7)} \subset D^{s(7)} \cap D^{u(7)}$ which is of the form

$$D^{(i)} = \left\{u \in \mathbb{C} : |\Im u| + \tan\beta_2|\Re u| < \frac{\pi}{2} - c^{(i)}\delta\ln(1/\delta)\right\}, \tag{65}$$

where $c^{(i)}$ are the constants defined in (47). The angle $\beta_2 > \beta_1$, holding $|\beta_2| = \mathcal{O}(1)$, has been chosen in such a way that the domains $D^{(i)}$ can be split as

$$D^{(i)} = D^{(i)}_{\gamma} \cup D^{(i)}_{\delta,+} \cup D^{(i)}_{\delta,-} \tag{66}$$

where $D^{(i)}_{\gamma} = D^{(i)} \cap D^{u(i)}_{\gamma} \cap D^{s(i)}_{\gamma}$, and the same for $D^{(i)}_{\delta,\pm}$ (see Figure 10).

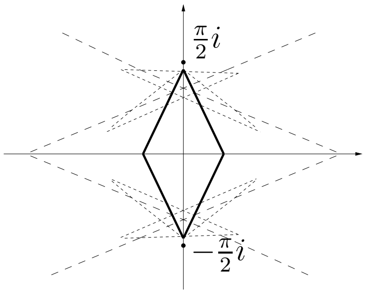

Figure 10. The domain for $T^+ - T^-$, $D^{(7)}$.

Let us consider in $D^{(7)}$ the linear differential operator

$$\tilde{\mathcal{L}}_\delta = \delta^{-1}\partial_\tau + \left(\frac{\cosh^2 u}{8}\left(\partial_u T^+(u,\tau) + \partial_u T^-(u,\tau)\right)\right)\partial_u \tag{67}$$

which holds $\tilde{\mathcal{L}}_\delta(\Delta T) = 0$. To deal with $\tilde{\mathcal{L}}_\delta$, we observe that, heuristically, as $\partial_u T^+$ and $\partial_u T^-$ are close to $\partial_u T_0$, $\tilde{\mathcal{L}}_\delta$ is close to the linear differential operator with constants coefficients

$$\mathcal{L}_\delta = \delta^{-1}\partial_\tau + \partial_u. \tag{68}$$

The following lemma, that will be crucial to prove the exponentially small asymptotic expansion of ΔT, shows that analytic solutions $\zeta(u,\tau)$ of $\mathcal{L}_\delta\zeta = 0$ defined for $(u,\tau) \in (-ir_0, ir_0) \times \mathbb{T}_{\sigma_4}$ are exponentially small for real (u,τ).

Lemma 3.11. *Let us consider a function $\zeta(u,\tau)$ analytic in $(-ir_0, ir_0) \times \mathbb{T}_{\sigma_4}$ which is solution of $\mathcal{L}_\delta\zeta = 0$. Then, ζ can be extended analytically to $\{|\Im u| < r_0\} \times \mathbb{T}_{\sigma_4}$ and its mean value*

$$\langle\zeta\rangle = \frac{1}{2\pi}\int_0^{2\pi} \zeta(u,\tau)d\tau$$

does not depend on u. Moreover, for $r \in (0, r_0)$ and $\sigma \in (0, \sigma_4)$, we define

$$M_r = \max_{(u,\tau)\in[-ir,ir]\times\overline{\mathbb{T}}_\sigma} \left|\partial_u^2\zeta(u,\tau)\right| \tag{69}$$

Then, provided δ is small enough, the following inequalities hold

$$\forall (u,\tau) \in \mathbb{R}\times\mathbb{T}_\sigma, \begin{cases} |\partial_u^2\zeta(u,\tau)| \leq 4M_r e^{-\frac{r}{\delta}} \\ |\partial_u\zeta(u,\tau)| \leq 4\delta M_r e^{-\frac{r}{\delta}} \\ |\zeta(u,\tau) - \langle\zeta\rangle| \leq 4\delta^2 M_r e^{-\frac{r}{\delta}}. \end{cases}$$

Proof. Since ζ is periodic in τ, we can write

$$\zeta(u,\tau) = \sum_{k\in\mathbb{Z}} \zeta^k(u)e^{ik\tau}.$$

On the other hand, being ζ solution of $\mathcal{L}_\delta\zeta = 0$, there exists a 2π-periodic function $\Lambda(s)$ such that

$$\zeta(u,\tau) = \Lambda\left(\tau - \delta^{-1}u\right).$$

Thus, it is straightforward to see that $\langle\zeta\rangle$ does not depend on u and it can be extended to $u \in \{|\Im u| < r_0\}$ as

$$\zeta(u,\tau) = \Lambda\left(\tau - \delta^{-1}\Re u - i\delta^{-1}\Im u\right) = \zeta\left(i\delta^{-1}\Im u, \tau - \delta^{-1}\Re u\right).$$

Considering M_r defined in (69), one has that $|\partial_u^2\zeta(u,\tau)| \leq M_r$ for $(u,\tau) \in \{|\Im u| \leq r\} \times \overline{\mathbb{T}_\sigma}$. Moreover since ζ is analytic for $(u,\tau) \in \{|\Im u| \leq r\} \times \overline{\mathbb{T}_\sigma}$,

$$\left|\partial_u^2\zeta^k(u)\right| \leq M_r e^{-|k|\sigma}.$$

On the other hand, it can be seen that for $k \in \mathbb{Z}$

$$\partial_u^2\zeta^k(u) = -\frac{k^2}{\delta^2}\Lambda^k e^{-iku/\delta} \tag{70}$$

and taking $u = \pm ir$,

$$\Lambda^k = -\frac{\delta^2}{k^2}\partial_u^2\zeta^k(\pm ir)e^{\pm kr/\delta}.$$

Hence, with this equality and (70), we obtain that for $u \in \mathbb{R}$,

$$\left|\partial_u^2\zeta^k(u)\right| \leq \left|\partial_u^2\zeta^k(-\mathrm{sgn}(k)ir)\right| e^{-|k|r/\delta} \leq M_r e^{-|k|\sigma}e^{-|k|r/\delta}.$$

With this bound and recalling that $\partial_u^2\zeta^0(u) = \partial_u^2\langle\zeta\rangle = 0$, for $u \in \mathbb{R}$ and $|\Im\tau| < \sigma$,

$$|\partial_u^2\zeta(u,\tau)| \leq \sum_{k\neq 0}\left|\partial_u^2\zeta^k(u)\right| e^{|k||\Im\tau|} \leq 2M_r\sum_{k>0}\left(e^{|\Im\tau|-\sigma-r/\delta}\right)^k \leq 4M_r e^{-r/\delta}.$$

The other two bounds are obtained proceeding analogously. □

In order to apply Lemma 3.11 to ΔT, it is natural to look for a change of variables that conjugates operators (67) and (68). The operator $\tilde{\mathcal{L}}_\delta$ can be split as

$$\tilde{\mathcal{L}}_\delta = \tilde{\mathcal{L}}_\delta^- + \left(\frac{\cosh^2 u}{8}\left(\partial_u T^+(u,\tau) - \partial_u T^-(u,\tau)\right)\right)\partial_u,$$

where

$$\tilde{\mathcal{L}}_\delta^- = \partial_\tau + \frac{\cosh^2 u}{4}\partial_u T^-(u,\tau)\partial_u. \tag{71}$$

As a first step we prove the existence of a change which straightens $\tilde{\mathcal{L}}_\delta^-$.

Theorem 3.12. *Let σ_3 and $\bar{\mu}_0$ be the constants considered in Theorem 3.10. Then, for any fixed $\gamma \in (1/3, 1/2)$ there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$ and $\mu \in (0,\bar{\mu}_0]$, there exists a real-analytic function $\mathcal{C}^-(w,\tau)$ defined in $D^{u(10)} \times \mathbb{T}_{\sigma_3}$, such that the change*

$$(u,\tau) = \left(w + \mathcal{C}^-(w,\tau), \tau\right)$$

conjugates the operators $\tilde{\mathcal{L}}_\delta^-$ and $\mathcal{L}_\delta$ defined in (71) and (68).
Moreover there exists a constant $b_6 > 0$, independent of δ and μ, such that:

- *For $(w,\tau) \in D_\gamma^{u(10)} \times \mathbb{T}_{\sigma_3}$, it holds that $u = w + \mathcal{C}^-(w,\tau) \in D_\gamma^{u(9)}$ and*

$$\begin{aligned} |\mathcal{C}^-(w,\tau)| &\leq b_6\left(\delta^2|w| + \delta^{2-2\gamma}\right) \\ |\partial_w^j\mathcal{C}^-(w,\tau)| &\leq b_6\delta^{2-(j+2)\gamma} \quad \text{for } j = 1,2, \end{aligned}$$

- *For* $(w,\tau)\in D^{u(10)}_{\delta,\pm}\times\mathbb{T}_{\sigma_3}$, *it holds that* $u=w+\mathcal{C}^-(w,\tau)\in D^{u(9)}_{\delta,\pm}$ *and*
$$|\partial_u^j\mathcal{C}^-(w,\tau)|\le b_6\delta^{1-j}\ln(1/\delta)^{-1-j}\quad \textit{for } j=0,1,2.$$

The proof of this theorem is given in Section 7. We note that we will need to proceed as in the proof of the existence of the invariant manifolds in Sections 5 and 6. First, in Section 7.1 it is proved the existence of the change of variables in the outer domain $D^{u(10)}_\gamma$ and afterwards it is extended to the inner ones $D^{u(10)}_{\delta,\pm}$ using complex matching techniques in Section 7.2.

Applying the change $\mathcal{C}^-$ to $\tilde{\mathcal{L}}_\delta$ (in (67)) as a first step, it can be found the global change which conjugates $\tilde{\mathcal{L}}_\delta$ to $\mathcal{L}_\delta$.

Theorem 3.13. *Let* $\bar\mu_0$ *be the constant considered in Theorem 3.12 and* σ_4 *and* γ *real numbers such that* $0<\sigma_4<\sigma_3$ *and* $\gamma\in(1/3,1/2)$. *Then, there exist* $\delta_0>0$ *such that for all* $\delta\in(0,\delta_0)$ *and* $\mu\in(0,\bar\mu_0]$, *there exists a real-analytic function* $\mathcal{U}(v,\tau)$ *in* $D^{(12)}\times\mathbb{T}_{\sigma_4}$, *such that the change*
$$(u,\tau)=(v+\mathcal{U}(v,\tau),\tau)$$
conjugates the operators $\tilde{\mathcal{L}}_\delta$ *and* $\mathcal{L}_\delta$ *defined in (67) and (68). Moreover, there exist constants* $b_7>0$ *and* $\nu_1>0$ *independent of* δ *and* μ *such that:*

- *For all* $(v,\tau)\in D^{(12)}_\gamma\times\mathbb{T}_{\sigma_4}$, *it holds that* $v+\mathcal{U}(v,\tau)\in D^{(11)}_\gamma$ *and*
$$\left|\partial_v^j\mathcal{U}(v,\tau)\right|\le b_7\delta^{1+\nu_1-j\gamma}\quad \textit{for } j=0,1,2.$$
- *For all* $(v,\tau)\in D^{(12)}_{\delta,\pm}\times\mathbb{T}_{\sigma_4}$, *it holds that* $v+\mathcal{U}(v,\tau)\in D^{(11)}_{\delta,\pm}$ *and*
$$\left|\partial_v^j\mathcal{U}(v,\tau)\right|\le b_7\delta^{1-j}(\ln(1/\delta))^{-1-j}\quad \textit{for } j=0,1,2.$$

It will also be necessary to consider the inverse change of variables in the outer domain $D^{(17)}_\gamma$. The existence and properties of it are stated in the following theorem whose proof is straightforward considering a fixed point argument.

Theorem 3.14. *Let* $\bar\mu_0$, σ_4 *and* ν_1 *be the constants defined in Theorem 3.13 and* γ *be any fixed constant* $\gamma\in(1/3,1/2)$. *Then, there exists* $\delta_0>0$ *such that for all* $\delta\in(0,\delta_0)$ *and* $\mu\in(0,\bar\mu_0]$, *there exists a real-analytic function* $\mathcal{V}(u,\tau)$ *in* $D^{(14)}_\gamma\times\mathbb{T}_{\sigma_4}$, *such that*
$$(v,\tau)=(u+\mathcal{V}(u,\tau),\tau)$$
is the inverse of the change given in Theorem 3.13. Moreover, there exists a constant $b_8>0$ *independent of* δ *and* μ *such that for all* $(u,\tau)\in D^{(14)}_\gamma\times\mathbb{T}_{\sigma_4}$, *it holds that* $u+\mathcal{V}(u,\tau)\in D^{(13)}_\gamma$ *and*
$$\left|\partial_u^j\mathcal{V}(u,\tau)\right|\le b_8\delta^{1+\nu_1-j\gamma}\quad \textit{for } j=0,1,2.$$

With Lemma 3.11 and the change of variables obtained in Theorem 3.13, we have all the tools which will be necessary to obtain the exponentially small asymptotic expression of the splitting of separatrices.

In fact, in the following theorem, we prove that the first asymptotic term of ΔT is given essentially by the function

$$\begin{aligned} F(u,\tau) &= 2\delta^{-1}\mu f(\mu) e^{-\dfrac{\pi}{2\delta}} \cos\left(\tau - \delta^{-1}u\right) \\ &= \delta^{-1}\mu f(\mu)\left(e^{i\tau}e^{-i\frac{u-i\pi/2}{\delta}} + e^{-i\tau}e^{i\dfrac{u+i\pi/2}{\delta}}\right) \end{aligned} \tag{72}$$

which comes from the difference between the solutions of the inner equation $\phi_0^{\pm}(z,\tau)$ (see (59)) stated in Theorem 3.8.

Theorem 3.15. *Let $\bar{\mu}_0$ be the constant defined in Theorem 3.14, then for any fixed $0 < \bar{\mu} < \bar{\mu}_0$ there exist $\delta_0 > 0$ and $b_9 > 0$ independent of δ and μ such that, for $\delta \in (0,\delta_0)$ and $\mu \in (0,\bar{\mu}]$, $u \in (-u_1,u_1) := D^{(14)} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$, the following bounds hold:*

$$\begin{aligned} |\Delta T(u,\tau) - \alpha(\mu,\delta) - F(u,\tau)| &\le b_9\mu\frac{1}{\ln(1/\delta)}\delta^{-1}e^{-\dfrac{\pi}{2\delta}} \\ |\partial_u\Delta T(u,\tau) - \partial_u F(u,\tau)| &\le b_9\mu\frac{1}{\ln(1/\delta)}\delta^{-2}e^{-\dfrac{\pi}{2\delta}} \\ \left|\partial_u^2\Delta T(u,\tau) - \partial_u^2 F(u,\tau)\right| &\le b_9\mu\frac{1}{\ln(1/\delta)}\delta^{-3}e^{-\dfrac{\pi}{2\delta}} \end{aligned}$$

where $\alpha(\mu,\delta) = \langle\Delta T(v+\mathcal{U}(v,\tau),\tau)\rangle$ (which is independent of v) and $\mathcal{U}$ is the change given in Theorem 3.13.

Proof of Theorem 2.6. After the statement of Theorem 3.15, we are ready to prove Theorem 2.6. Changes (31), (30), (29) and (14), Theorems 3.2 and 3.6, and formula (37) give the two first statements. For the last two ones we also need Theorem 3.15. □

In order to prove Theorem 3.15, we consider the change of variables found in Theorem 3.13 and we split $\Delta T(u,\tau) - F(u,\tau)$ as

$$\Delta T(v+\mathcal{U}(v,\tau),\tau) - F(v+\mathcal{U}(v,\tau),\tau) = \zeta_1(v,\tau) + \zeta_2(v,\tau)$$

where

$$\begin{aligned} \zeta_1(v,\tau) &= \Delta T(v+\mathcal{U}(v,\tau),\tau) - F(v,\tau) \\ \zeta_2(v,\tau) &= F(v,\tau) - F(v+\mathcal{U}(v,\tau),\tau). \end{aligned} \tag{73}$$

In the following two propositions, whose proofs are delayed to the end of this section, we obtain the desired bounds of ζ_1 and ζ_2 which will lead to the proof of Theorem 3.15.

Proposition 3.16. *Let $\bar{\mu}_0$ be the constant defined in Theorem 3.14, then for any fixed $0 < \bar{\mu} < \bar{\mu}_0$ there exist $\delta_0 > 0$ and $b_{10} > 0$ independent of δ and μ such that for $\delta \in (0,\delta_0)$, $\mu \in (0,\bar{\mu}]$, $v \in D^{(13)} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$, the function ζ_1 defined in (73) satisfies:*

$$\begin{aligned} |\zeta_1(v,\tau) - \langle\zeta_1\rangle| &\le b_{10}\frac{\mu}{\ln(1/\delta)}\delta^{-1}e^{-\dfrac{\pi}{2\delta}} \\ \left|\partial_v^j\zeta_1(v,\tau)\right| &\le b_{10}\frac{\mu}{\ln(1/\delta)}\delta^{-(j+1)}e^{-\dfrac{\pi}{2\delta}} \quad \textit{for } j = 1,2, \end{aligned}$$

where $\langle \zeta_1 \rangle$ is independent of u.

Proposition 3.17. *Let $\bar{\mu}_0$ and ν_1 be the constants defined in Theorem 3.13, then there exist $\delta_0 > 0$ and $b_{11} > 0$ independent of δ and μ such that for $\delta \in (0, \delta_0)$, $\mu \in (0, \bar{\mu}_0)$, $v \in D^{(12)} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$,*

$$|\partial_v^j \zeta_2(v, \tau)| \leq b_{11} \mu \delta^{\nu_1 - j - 1} e^{-\frac{\pi}{2\delta}} \quad \textit{for } j = 1, 2. \tag{74}$$

Proof of Theorem 3.15. In order to recover the bounds stated in Theorem 3.15 from the bounds obtained in Propositions 3.16 and 3.17, it is enough to perform the real-analytic change of variables obtained in Theorem 3.14. □

In the rest of the paper, we will use the following convention.

Notation 3.18. *In order to make the proofs more readable, we will use the following convention: we will say that $g_1 \leq \mathcal{O}(g_2)$ in some domain D if there exists some constant $C > 0$, that may depend on other constants that will be defined through the article, but does not depend on ε neither μ (and therefore neither on δ), such that $|g_1| \leq C|g_2|$ in D. Moreover, we will say that $g_1 = \mathcal{O}(g_2)$ if $g_1 \leq \mathcal{O}(g_2)$ and $g_2 \leq \mathcal{O}(g_1)$.*

Proof of Proposition 3.16. By Theorem 3.13 and definition of F in (72), $\mathcal{L}_\delta \zeta_1(v, \tau) = 0$. Then, Lemma 3.11 can be applied and it gives that $\langle \zeta_1 \rangle$ does not depend on v. In order to bound ζ_1, we recall that $\Delta T = T^+ - T^- = Q^+ - Q^-$. Differentiating ζ_1:

$$\begin{aligned} \partial_v^2 \zeta_1(v, \tau) = \; & \partial_u^2 \Delta T(v + \mathcal{U}(v, \tau), \tau)(1 + \partial_v \mathcal{U}(v, \tau))^2 \\ & + \partial_u \Delta T(v + \mathcal{U}(v, \tau), \tau) \partial_v^2 \mathcal{U}(v, \tau) - \partial_v^2 F(v, \tau). \end{aligned}$$

Next step is to bound $|\partial_v^2 \zeta_1(v, \tau)|$ for $(v, \tau) \in [-ir, ir] \times \overline{\mathbb{T}_{\sigma_5}}$ for any fixed $\sigma_5 < \sigma_4$ and $r = \pi/2 - c^{(13)} \delta \ln(1/\delta)$.

For $v \in \overline{D_\gamma^{(13)}}$ we apply Theorems 3.6 and 3.13, and the expression of F in (72), obtaining, since $\gamma > 1/3$,

$$|\partial_v^2 \zeta_1(v, \tau)| \leq \mathcal{O}\left(\mu \delta^{2 - 5\gamma}\right) \leq \mathcal{O}\left(\delta^{-1}\right). \tag{75}$$

For $v \in \overline{D_{\delta,+}^{(13)}} \subset \overline{D_{\delta,+}^{(12)}}$ we have that

$$z = (v + \mathcal{U}(v, \tau) - i\pi/2)/\delta \in \overline{\mathcal{D}_{\delta,+}^{(11)}}.$$

We split ζ_1 using the functions $\phi^\pm$ and $\phi_0^\pm$ whose existence has been proved in Theorems 3.8 and 3.10

$$\begin{aligned} \delta \zeta_1(v, \tau) = \; & \left(\phi^+ - \phi_0^+\right)(z, \tau) - \left(\phi^- - \phi_0^-\right)(z, \tau) + \\ & + \left(\phi_0^+ - \phi_0^-\right)(z, \tau) - \delta F(v, \tau). \end{aligned}$$

Thus, differentiating it twice we obtain

$$\delta^3 \partial_z^2 \zeta_1(v, \tau) = \partial_z^2 \left(\phi^+ - \phi_0^+\right)(z, \tau)\left(1 + \partial_v \mathcal{U}(v, \tau)\right)^2 - \tag{76}$$

$$- \partial_z^2 \left(\phi^- - \phi_0^-\right)(z, \tau)\left(1 + \partial_v \mathcal{U}(v, \tau)\right)^2 + \tag{77}$$

$$+ \delta \partial_z \left(\phi^+ - \phi_0^+\right)(z, \tau)\, \partial_v^2 \mathcal{U}(v, \tau) - \tag{78}$$

$$- \delta \partial_z \left(\phi^- - \phi_0^-\right)(z, \tau)\, \partial_v^2 \mathcal{U}(v, \tau) + \tag{79}$$

$$+ \partial_z^2(\phi_0^+ - \phi_0^-)(z,\tau)\left(1+\partial_v \mathcal{U}(v,\tau)\right)^2 - \delta^3\partial_v^2 F(v,\tau) + \quad (80)$$
$$+ \delta\partial_z(\phi_0^+ - \phi_0^-)(z,\tau)\,\partial_v^2\mathcal{U}(v,\tau). \quad (81)$$

Applying Theorems 3.10 and 3.13, we obtain that terms (76) and (77) can be bounded by $\mathcal{O}(\delta^2 \ln^{-2}(1/\delta))$ and terms (78) and (79) by $\mathcal{O}(\delta^2 \ln^{-3}(1/\delta))$.

To bound (80), we differentiate twice formula (59) for $\phi_0^+ - \phi_0^-$ and (72) for F:

$$\begin{aligned}
&\left|\partial_z^2(\phi_0^+ - \phi_0^+)\left((v+\mathcal{U}(v,\tau)-i\pi/2)/\delta,\tau\right)\left(1+\partial_v\mathcal{U}(v,\tau)\right)^2 - \delta^3\partial_v^2 F(v,\tau)\right| \\
&\quad \le \mathcal{O}\left(\left(\mu e^{\Im\left(\frac{v+\mathcal{U}(v,\tau)-i\pi/2}{\delta}\right)}\left[-1+\mathcal{O}\left(\ln^{-1}(1/\delta)\right)\right]+\mathcal{O}\left(\delta^{a_1 c^{(13)}}\right)\right)\left|1+\partial_v\mathcal{U}(v,\tau)\right|^2\right)+ \\
&\qquad +\mathcal{O}\left(\mu e^{\Im\left(\frac{v-i\pi/2}{\delta}\right)}+\mu e^{-\Im\left(\frac{v+i\pi/2}{\delta}\right)}\right) \\
&\quad \le \mathcal{O}\left(\mu e^{\Im\left(\frac{v-i\pi/2}{\delta}\right)}\left(e^{\frac{|\mathcal{U}(v,\tau)|}{\delta}}\left[-1+\mathcal{O}(\ln^{-1}(1/\delta))\right]\left(1+\partial_v\mathcal{U}(v,\tau)\right)^2+1\right)\right)+ \\
&\qquad +\mathcal{O}\left(\mu e^{-\Im\left(\frac{v+i\pi/2}{\delta}\right)}+\mathcal{O}\left(\delta^{a_1 c^{(13)}}\right)\left|1+\partial_v\mathcal{U}(v,\tau)\right|^2\right).
\end{aligned}$$

Applying the bounds obtained in Theorem 3.13 and recalling that $0 \le \Im v \le \pi/2 - c^{(13)}\delta\ln(1/\delta)$, we obtain the following statements:

$$\begin{aligned}
&e^{\frac{|\mathcal{U}(v,\tau)|}{\delta}} = 1+\mathcal{O}\left(\ln^{-1}(1/\delta)\right) \\
&e^{\Im\left(\frac{v-i\pi/2}{\delta}\right)} \le \mathcal{O}\left(\delta^{c^{(13)}}\right) \\
&e^{-\Im\left(\frac{v+i\pi/2}{\delta}\right)} \le \mathcal{O}\left(e^{-\frac{\pi}{2\delta}}\right).
\end{aligned}$$

From these bounds and applying again Theorem 3.13, it can be seen that term (80) is bounded by $\mathcal{O}(\delta^{c^{(13)}}\ln^{-1}(1/\delta))$.

The last term (81) can be bounded analogously by $\mathcal{O}(\delta^{c^{(13)}}(\ln(1/\delta))^{-3})$ differentiating formula (59) once and using Theorem 3.13.

Therefore, since $0 < c^{(13)} < 2$ (see (48)), for $(v,\tau) \in \overline{D^{(13)}_{\delta,+}} \times \overline{\mathbb{T}_{\sigma_5}}$,

$$\left|\partial_v^2\zeta_1(v,\tau)\right| \le \mathcal{O}\left(\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right).$$

Thus, joining this bound with (75), for $(v,\tau) \in \overline{D^{(13)}} \times \overline{\mathbb{T}_{\sigma_5}}$,

$$\left|\partial_v^2\zeta_1(v,\tau)\right| \le \mathcal{O}\left(\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right).$$

Furthermore, taking $m(v,\tau,\delta,\mu) = \partial_z^2\zeta_1(v,\tau)$, m is analytic in μ and holds that $m(v,\tau,\delta,0) = 0$. Thus, considering an arbitrary $\bar{\mu} \in (0,\bar{\mu}_0)$ such that $\rho = |\bar{\mu}_0 - \bar{\mu}|$ is independent of δ, and applying Cauchy integral formula, for $\mu < \bar{\mu}$

$$|\partial_\mu m(v,\tau,\delta,\mu)| = \left|\frac{1}{2\pi}\int_{|\xi-\mu|=\rho}\frac{m(v,\tau,\delta,\xi)}{(\xi-\mu)^2}d\xi\right| \le \mathcal{O}\left(\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right).$$

Therefore,

$$\left|\partial_v^2\zeta_1(v,\tau)\right| = |m(v,\tau,\delta,\mu)| = \left|\int_0^\mu \partial_\mu m(v,\tau,\delta,\xi)d\xi\right| \le \mathcal{O}\left(\mu\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right). \quad (82)$$

Finally we apply Lemma 3.11 to ζ_1 with $r = \pi/2 - c^{(13)}\delta \ln(1/\delta)$ and

$$M_r = \mathcal{O}\left(\mu\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right),$$

and we obtain the bounds stated in the proposition having into account that

$$e^{-\frac{r}{\delta}} = \delta^{-c^{(13)}} e^{-\frac{\pi}{2\delta}}.$$

□

Proof of Proposition 3.17. It is enough to apply carefully the Mean Value Theorem to the function F given in (72) using the bounds of Theorem 3.13. □

Notation 3.19. *In the forthcoming sections, we will use several Banach spaces $(\mathcal{X}, \|\cdot\|)$. We will denote the ball of center* 0 *and radius* r *as*

$$B(r) = \{x \in \mathcal{X} : \|x\| < r\}.$$

4. Proof of Theorem 2.1 and 3.2

As it was noticed in [DS92], the periodic orbit x_p of system (3) holds the symmetry $x_p(-\tau) = -x_p(\tau)$, and thus it has zero mean. On the other hand, it has to hold the functional equation

$$\mathcal{L}_0(x_p) = \varepsilon^2 \sin(x_p) + \mu \sin\tau$$

where $\mathcal{L}_0 x = x''$.

However, for a fixed value of $\mu > 0$ independent of ε, $x_p(\tau)$ is not small. Expanding formally the periodic orbit in powers of ε is obtained that

$$x_p(\tau) = -\mu\sin\tau + \mathcal{O}(\varepsilon^2).$$

Thus, in order to apply a fixed point argument, we consider $z_p(\tau) = x_p(\tau) + \mu\sin\tau$. Hence, we will look for a solution of the equation

$$\mathcal{L}_0 z_p = \varepsilon^2 \sin(z_p - \mu\sin\tau)$$

in the Banach space

$$\mathcal{X}_0 = \{z : \mathbb{T}_{\sigma_0} \to \mathbb{C} : \text{real-analytic}, 2\pi\text{-periodic}, \langle z\rangle = 0, \|z\|_{\sigma_0} < \infty\} \quad (83)$$

where $\langle z\rangle = (1/2\pi)\int_0^{2\pi} z(\tau)d\tau$ and the norm $\|\cdot\|_{\sigma_0}$ is defined using the Fourier series of z and is given by

$$\|z\|_{\sigma_0} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|z^{[k]}\right| e^{|k|\sigma_0}.$$

Moreover, we take σ depending on μ, since the strip of analyticity of the periodic orbit seems to shrink when $\mu \to \infty$. Indeed, one has to take the width of the strip in such a way that the function $b(\tau) = \sin(\mu\sin\tau)$ holds $|b(\tau)| \le \mathcal{O}(1)$. As it will be seen in the proof of Proposition 4.1, this bound holds provided $\mu \sinh \sigma \leq \mathcal{O}(1)$. Therefore one has to take

$$\sigma_0(\mu) = \ln\left(\frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} + 1}\right). \tag{84}$$

Notice that when $\mu \to 0$ it holds that $\sigma_0(\mu) \sim \ln(1/\mu)$ whereas as $\mu \to \infty$ it holds that $\sigma_0(\mu) \sim 1/\mu$.

On the other hand, for functions belonging to $\mathcal{X}_0$, the operator $\mathcal{L}_0$ has an inverse which is defined through the Fourier series

$$\mathcal{G}_0(z) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{z^{[k]}}{k^2} e^{ik\tau}.$$

and thus it can be bounded by $\|\mathcal{G}_0\|_{\sigma_0} \leq \mathcal{O}(1)$. Using this operator, first statement of Theorem 2.1 will be a straightforward consequence of the following proposition:

Proposition 4.1. *There exists ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ and any $\mu > 0$ there exists a 2π-periodic real-analytic function $z_p(\tau)$ defined in $\mathbb{T}_{\sigma_0}$, which is a fixed point of the functional*

$$\mathcal{F}_0(z) = \mathcal{G}_0\left(\varepsilon^2 \sin(z + \mu \sin \tau)\right). \tag{85}$$

Moreover, it holds $\|z_p(\tau)\|_{\sigma_0} \leq \mathcal{O}\left(\varepsilon^2\right)$.

From this proposition, we obtain the following corollary which has Theorem 3.2, and thus also the second statement of Theorem 2.1, as a consequence.

Corollary 4.2. *There exists ε_0 such that, for any fixed $\bar{\mu}$, there exists $\sigma_1 > 0$ (independent of μ and ε), such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \bar{\mu})$, system (3) has a periodic orbit, which holds the following bound,*

$$\|x_p(\tau) + \mu \sin \tau\|_{\sigma_1} \leq \mathcal{O}(\mu\varepsilon^2).$$

In order to prove the proposition, we need the following technical lemma:

Lemma 4.3. *There exists constants $c_1 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and any $\mu > 0$:*

1. $\|\mathcal{F}_0(0)\|_{\sigma_0} \leq \frac{c_1}{2}\varepsilon^2$.
2. *For all $z_1, z_2 \in B\left(c_1\mu\varepsilon^2\right)$, $\|\mathcal{F}_0(z_1) - \mathcal{F}_0(z_2)\|_{\sigma_0} \leq \mathcal{O}(\varepsilon^2)\|z_1 - z_2\|_{\sigma_0}$.*

Proof. For the first statement, we take $b(\tau) = \sin(\mu \sin \tau)$, and thus

$$\mathcal{F}_0(0) = \mathcal{G}_0\left(-\varepsilon^2 b(\tau)\right) = \varepsilon^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{b^{[k]}}{k^2} e^{ik\tau}.$$

As a first step we bound $b(\tau) = \sin(\mu \sin \tau)$. Writing τ as $\tau = \zeta + i\theta$ with $|\theta| \leq \sigma_0(\mu)$,

$$\begin{aligned} |b(\tau)| = \ & |\sin(\mu \sin \tau)| \\ \leq \ & |\sin(\mu \sin \zeta \cosh \theta)||\cosh(\mu \cos \zeta \sinh \theta)| \\ & + |\cos(\mu \sin \zeta \cosh \theta)||\sinh(\mu \cos \zeta \sinh \theta)| \\ \leq \ & c_0 \end{aligned}$$

for certain constant $c_0 > 0$ independent of ε and μ. Therefore $|b^{[k]}| \leq c_0 \mu e^{-|k|\sigma_0}$. Thus, taking $c_1 = 2c_0\pi^2/3$,

$$\|\mathcal{F}_0(0)\|_{\sigma_0} \leq \varepsilon^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|b^{[k]}|}{|k|^2} e^{|k|\sigma_0} \leq \varepsilon^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{c_0}{|k|^2} \leq \frac{c_1}{2}\varepsilon^2.$$

For the second statement,

$$\begin{aligned} \|\mathcal{F}_0(z_1) - \mathcal{F}(z_2)\|_{\sigma_0} &\leq \mathcal{O}(\varepsilon^2)\|\sin(z_1 - \mu\sin(\tau)) - \sin(z_2 - \mu\sin(\tau))\|_{\sigma_0} \\ &\leq \mathcal{O}(\varepsilon^2)\|z_1 - z_2\|_{\sigma_0}. \end{aligned}$$

□

Proof of Proposition 4.1. Considering the bounds of Lemma 4.3, $\mathcal{F}_0$ is a contraction from $B(c_1\varepsilon^2) \subset \mathcal{X}_0$ to itself. Thus, it has a fixed point $z_p(\tau)$ that gives the periodic orbit $x_p(\tau) = z_p(\tau) - \mu\sin\tau$ and holds the wanted bound. □

5. Invariant manifolds in the outer domains: proof of Theorem 3.6

5.1. Banach spaces and technical lemmas

We define several norms which will be used throughout the proof. Since we work with functions that near the singularities $\pm i\pi/2$ behave like $h(u,\tau) \sim \cosh^{-n} u$ and near infinity like $h(u,\tau) \sim e^{\mp mu}$ we will use certain weighted norms in the variable u. However, for simplicity and in order to have symmetry between $+\infty$ and $-\infty$, we will also use $\cosh^m u$ as weight at infinity. Moreover, since we work with analytic functions 2π-periodic in τ, Fourier norms are considered, as well as weighted supremum norms.

In this way, we define for analytic functions $h(u)$ the following norms in the domains defined in (41), (42) and (43).

(i) For $h : D^{u,c(j)}_\gamma \to \mathbb{C}$: $\|h\|_{n,c} = \sup_{u\in D^{u,c(j)}_\gamma} |\cosh^n u \cdot h(u)|$

(ii) For $h : D^{u,f(j)}_\gamma \to \mathbb{C}$: $\|h\|_{m,f} = \sup_{u\in D^{u,f(j)}_\gamma} |\cosh^m u \cdot h(u)|$

(iii) For $h : D^{u(j)}_\gamma \to \mathbb{C}$: $\|h\|_{n,m} = \|h\|_{n,c} + \|h\|_{m,f}$.

Given $h(u,\tau)$ an analytic function of 2 variables 2π-periodic in τ, we consider the corresponding Fourier norms

(i) For $h : D^{u,c(j)}_\gamma \times \mathbb{T}_{\sigma_i} \to \mathbb{C}$: $\|h\|_{n,c,\sigma_i} = \sum_{k\in\mathbb{Z}} \|h^{[k]}\|_{n,c} e^{|k|\sigma_i}$

(ii) For $h : D^{u,f(j)}_\gamma \times \mathbb{T}_{\sigma_i} \to \mathbb{C}$: $\|h\|_{m,f,\sigma_i} = \sum_{k\in\mathbb{Z}} \|h^{[k]}\|_{m,f} e^{|k|\sigma_i}$

(iii) For $h : D^{u(j)}_\gamma \times \mathbb{T}_{\sigma_i} \to \mathbb{C}$: $\|h\|_{n,m,\sigma_i} = \|h\|_{n,c,\sigma_i} + \|h\|_{m,f,\sigma_i}$.

On the other hand, it will be used also the Fourier supremum norm defined by $\|h\|_{\infty,\sigma_i} = \sum_{k\in\mathbb{Z}} \|h^{[k]}\|_\infty e^{|k|\sigma_i}$ where a subindex c or f will be added if we are restricted to the domains $D^{u,f(j)}_\gamma$ or $D^{u,c(j)}_\gamma$. Note that when h does not depend on u, this norm coincides with the Fourier norm $\|\cdot\|_{\sigma_i}$ introduced in Section 4. In order to clarify the notation, we will denote the classical supremum norm for functions defined in $D^{u(j)}_\gamma \times \mathbb{T}_{\sigma_i}$ as $\|\cdot\|_{\infty,i}$. All these norms are defined analogously for the domains where the stable manifold is defined. Using the Fourier expansion properties one can see the following relations between these norms:

Lemma 5.1. *The following inequalities hold:*

1. $\|h\|_{\infty,i} \leq \|h\|_{\infty,\sigma_i}$.
2. $\|\cosh^n u \cdot h\|_{\infty,*,i} \leq \|h\|_{n,*,\sigma_i}$ *for* $* = f, c$.
3. $\|h\|_{n,*,\sigma_{i+1}} \leq \mathcal{O}\left(\frac{1}{|\sigma_{i+1}-\sigma_i|}\right) \|\cosh^n u \cdot h\|_{\infty,*,i}$ *for* $* = f, c$ *and for any* $0 < \sigma_{i+1} < \sigma_i$.
4. $\|h_1 \cdot h_2\|_{n,*,\sigma_i} \leq \|h_1\|_{\infty,*,\sigma_i} \|h_2\|_{n,*,\sigma_i}$ *for* $* = f, c$.
5. $\|h_1 \cdot h_2\|_{n,m,\sigma_i} \leq \|h_1\|_{\infty,\sigma_i} \|h_2\|_{n,m,\sigma_i}$.
6. $\|\cosh^n u \cdot h_1 h_2\|_{n,*,\sigma_i} \leq \|h_1\|_{n,*,\sigma_i} \|h_2\|_{n,*,\sigma_i}$ *for* $* = f, c$.

Since the proof for both invariant manifolds is analogous, we will deal only with the unstable case and then for the proof of Theorem 3.6, we will look for a solution of (33) with a fixed point argument in the Banach space

$$\mathcal{E}_{n,m} = \{h(u,\tau) \mid h : D_\gamma^{u(0)} \times \mathbb{T}_{\sigma_1} \to \mathbb{C}, \text{ real-analytic}, \|h\|_{n,m,\sigma_1} < \infty\} \quad (86)$$

for certain naturals n and m.

For the proof of Theorem 5.9 it will be needed the Banach space:

$$\mathcal{E}_m = \{h(u,\tau) \mid h : D_\gamma^{u,f(0)} \times \mathbb{T}_{\sigma_1} \to \mathbb{C}, \text{ real-analytic}, \|h\|_{m,f,\sigma_1} < \infty\} \quad (87)$$

We state some previous technical lemmas.

Lemma 5.2. *For* $\delta > 0$ *small enough and* $u \in D_\gamma^{u(j)}$:

1. $|\cosh^{-1} u| \leq \mathcal{O}(\delta^{-\gamma})$.
2. $|\tanh u| \leq \mathcal{O}(\delta^{-\gamma})$.
3. $\forall \beta \in [0, \beta_0/2]$ *(see (40)) and* $\forall \xi \in \mathbb{R}^-$, $\left|\dfrac{\cosh u}{\cosh(u + \xi e^{\pm i\beta})}\right| \leq \mathcal{O}(1)$.
4. *For* $m \geq 1$, $\displaystyle\int_{-\infty}^0 \left|\frac{\cosh u}{\cosh(u+t)}\right|^m dt \leq \mathcal{O}(1)$.

Lemma 5.3. *The following inequalities hold:*

1. *(Cauchy inequalities) Considering the nested domains* $D_\gamma^{u(j+1)} \subset D_\gamma^{u(j)} \subset D_\gamma^{u(0)}$, *and denoting the corresponding norms* $\|\cdot\|_{n,m,\sigma_i}^{(j+1)}$ *and* $\|\cdot\|_{n,m,\sigma_i}^{(j)}$, *for* $h \in \mathcal{E}_{n,m}$,
$$\|\partial_u h\|_{n,m,\sigma_i}^{(j+1)} \leq \mathcal{O}(\delta^{-\gamma}) \|h\|_{n,m,\sigma_i}^{(j)}.$$
2. *For* $h \in \mathcal{E}_{n,m}$ *or* $h \in \mathcal{E}_m$ *with* $m > 0$, *and* $l < m$, $\displaystyle\lim_{\Re u \to -\infty} \cosh^l u \cdot h(u,\tau) = 0$ *and* $\displaystyle\int_{-\infty}^u h(v,\tau) dv$ *exists and is bounded.*

Proof. For the first statement, we consider $h(u,\tau) = \sum_{k\in\mathbb{Z}} h^{[k]}(u) e^{ik\tau}$ and we have:

$$\cosh^n u \cdot \partial_u h^{[k]}(u) = \partial_u(\cosh^n u \cdot h^{[k]}(u)) - n \cosh^{n-1} u \sinh u \cdot h^{[k]}(u)$$

and we bound each summand. For the first one, it is enough to apply Cauchy estimates to $D^{u(j+1)}$:

$$\|\partial_u(\cosh^n u \cdot h^{[k]}(u))\|_{\infty,c}^{(j+1)} \leq \mathcal{O}(\delta^{-\gamma})\|h^{[k]}\|_{n,c}^{(j)}.$$

The second summand, using the second statement of Lemma 5.2, can be bounded as:

$$\|n \cosh^{n-1} u \sinh u \cdot h^{[k]}(u)\|_{\infty,c}^{(j+1)} \leq \mathcal{O}(1)\| \tanh u\|_{\infty,c}^{(j)} \cdot \|h^{[k]}\|_{n,c}^{(j)} \leq \mathcal{O}(\delta^{-\gamma})\|h^{[k]}\|_{n,c}^{(j)}.$$

Moreover, analogous bounds hold in $D_\gamma^{u,f}$. Thus, $\|\partial_u h^{[k]}\|_{n,m}^{(j+1)} \leq \mathcal{O}(\delta^{-\gamma})\|h^{[k]}\|_{n,m}^{(j)}$, and hence $\|\partial_u h\|_{n,m,\sigma_i}^{(j+1)} \leq \mathcal{O}(\delta^{-\gamma})\|h\|_{n,m,\sigma_i}^{(j)}$.

For the proof of the second statement is enough to consider the relation between supremum and weighted norms. □

Throughout this section we are going to solve equations of the form $\mathcal{L}_\delta(h) = g$ where $\mathcal{L}_\delta = \delta^{-1}\partial_\tau + \partial_u$ is the differential operator defined in (68). Despite not being invertible, this operator has a right inverse (in fact, more than one) which will be used in several sections of this paper:

$$\mathcal{G}_\delta(h) = \int_{\mathbb{R}^-} h(u+t, \tau+\delta^{-1}t)dt. \tag{88}$$

Thus, we state some technical lemmas which will be needed in the proofs of Theorems 3.6 and 5.9.

Lemma 5.4. *For all $h \in \mathcal{E}_{n,m}$ or $h \in \mathcal{E}_m$ with $m > 0$, we have*

$$\partial_u \int_{\mathbb{R}^-} h(u+t,\tau)dt = \int_{\mathbb{R}^-} \partial_u h(u+t,\tau)dt = h(u,\tau) \tag{89}$$

$$\partial_u \int_{\mathbb{R}^-} h(u+t,\tau+t)dt = \int_{\mathbb{R}^-} \partial_u h(u+t,\tau+t)dt. \tag{90}$$

Proof. Since all the statements are proved analogously, we deal only with the first one. In order to prove that the integral and ∂_u commute for $h \in \mathcal{E}_{n,m}$, it is enough to prove the uniform convergence of the integral at ∞ since $h \in \mathcal{E}_{n,m}$ guarantees that $\partial_u h$ is differentiable in $D_\gamma^{u(i)}$. Thus, we split the integral as

$$\int_{\mathbb{R}^-} \partial_u h(u+t,\tau)dt = \int_{-\infty}^{-T} \partial_u h(u+t,\tau)dt + \int_{-T}^{0} \partial_u h(u+t,\tau)dt$$

taking $T > 0$ big enough such that $u + t \in D_\gamma^{u,f(i+1)}$ for $t \in (-\infty, -T)$ and we have to bound the first one. Using the first statement of Lemma 5.3,

$$\left| \int_{-\infty}^{-T} \partial_u h(u+t,\tau)dt \right| \leq \int_{-\infty}^{-T} \left|\cosh^{-m}(u+t)\right| \|\partial_u h\|_{n,m}^{(i+1)}dt \leq \mathcal{O}(\delta^\gamma)\|h\|_{n,m}^{(i)}.$$

□

Lemma 5.5. *The operators $\mathcal{G}_\delta$ defined in (88) and $\bar{\mathcal{G}}_\delta = \partial_u \mathcal{G}_\delta$ are well defined in $\mathcal{E}_{n,m}$ and $\mathcal{E}_m$ for $m > 0$. Moreover*

1. *$\mathcal{G}_\delta$ is linear from $\mathcal{E}_{n,m}$ to itself, commutes with ∂_u and $\mathcal{L}_\delta \circ \mathcal{G}_\delta = \mathrm{Id}$.*

2. *$\bar{\mathcal{G}}_\delta$ is linear from $\mathcal{E}_{n,m}$ to itself.*
3. *For $h \in \mathcal{E}_{n,m}$, $\|\mathcal{G}_\delta(h)\|_{n,m,\sigma_i} \leq \mathcal{O}(1)\|h\|_{n,m,\sigma_i}$. Furthermore, if*
$$\langle h\rangle = \frac{1}{2\pi}\int_0^{2\pi} h(u,\tau)d\tau = 0,$$
then
$$\|\mathcal{G}_\delta(h)\|_{n,m,\sigma_i} \leq \mathcal{O}(\delta)\|h\|_{n,m,\sigma_i}.$$
4. *For $h \in \mathcal{E}_{n,m}$, $\|\bar{\mathcal{G}}_\delta(h)\|_{n,m,\sigma_i} \leq \mathcal{O}(1)\|h\|_{n,m,\sigma_i}$.*
5. *The same properties and bounds hold for $h \in \mathcal{E}_m$.*

Proof. We take
$$g(u,\tau) = \mathcal{G}_\delta(h)(u,\tau) = \sum_{k\in\mathbb{Z}} g^{[k]}(u)e^{ik\tau}$$
where
$$g^{[k]}(u) = \int_{-\infty}^0 h^{[k]}(u+t)e^{ik\delta^{-1}t}dt$$
which is integrable for $h \in \mathcal{E}_{n,m}$. In order to prove the lemma we want bounds of $g^{[k]}$ in terms of $h^{[k]}$. These bounds are computed in different ways depending on the domain and whether $k \neq 0$ or $k = 0$.

We consider first de case $k \neq 0$. For $u \in D_\gamma^{u,f(i)}$:
$$\cosh^m u \cdot g^{[k]}(u) = \int_{-\infty}^0 h^{[k]}(u+t)\cosh^m u \cdot e^{ik\delta^{-1}t}dt.$$
Since the integrand has exponential decay for $\Re t \to -\infty$, we change the integration path to the line from 0 to $-\infty$ with angle $\mp\beta_0/2$, where the sign $+$ is taken for $k > 0$ and $-$ for $k < 0$. Moreover, using the third statement of Lemma 5.2, we obtain:
$$\begin{aligned}|\cosh^m u \cdot g^{[k]}(u)| &\leq \mathcal{O}(1)\int_{-\infty}^0 \|h^{[k]}\|_{m,f}|e^{ik\delta^{-1}\xi e^{\mp i\beta_0/2}} \cdot e^{\mp i\beta_0/2}|d\xi \\ &\leq \mathcal{O}(\delta)\frac{\|h^{[k]}\|_{m,f}}{|k|\sin(\beta_0/2)} \leq \mathcal{O}(\delta)\frac{1}{|k|}\|h^{[k]}\|_{n,m}.\end{aligned}$$

For $u \in D_\gamma^{u,c(i)}$, we have to change the path of integration in the same way. Moreover, since $u+t$ can belong either to $D_\gamma^{u,c(i)}$ or $D_\gamma^{u,f(i)}$ and we use different norms in both domains, the integral must be split in two parts $(-\infty,T)$ and $(T,0)$ where T

is chosen such that $u+t \in D_\gamma^{u,f(i)}$ for t with $\Re t < \Re T$. We recall that, in this case, $u \in D_\gamma^{u,c(i)}$ and then $\cosh u$ is bounded.

$$\begin{aligned}
|\cosh^n u \cdot g^{[k]}(u)| = \ & \Big| \int_{-\infty}^{T} \cosh^n u \cdot h^{[k]} \left(u + \xi e^{\mp i\beta_0/2}\right) \cdot e^{ik\delta^{-1}\xi e^{\mp i\beta_0/2}} e^{\mp i\beta_0/2} d\xi + \\
& + \int_T^0 \cosh^n u \cdot h^{[k]} \left(u + \xi e^{\mp i\beta_0/2}\right) \cdot e^{ik\delta^{-1}\xi e^{\mp i\beta_0/2}} e^{\mp i\beta_0/2} d\xi \Big| \\
\leq \ & \mathcal{O}(1) \|h^{[k]}\|_{m,f} \int_{-\infty}^{T} e^{|k|\sin(\beta_0/2)\delta^{-1}\xi} d\xi + \\
& + \|h^{[k]}\|_{n,c} \int_T^0 \left(\frac{\cosh u}{\cosh\left(u + \xi e^{\mp i\beta_0/2}\right)} \right)^n e^{|k|\sin(\beta_0/2)\delta^{-1}\xi} d\xi \\
\leq \ & \mathcal{O}(\delta) \frac{1}{|k|} \|h^{[k]}\|_{m,f} + \mathcal{O}(\delta) \frac{1}{|k|} \|h^{[k]}\|_{n,c} \leq \mathcal{O}(\delta) \frac{1}{|k|} \|h^{[k]}\|_{n,m}
\end{aligned}$$

where we have used the third statement of Lemma 5.2.

Thus, for $k \neq 0$ we have obtained:

$$\|g^{[k]}\|_{n,m} \leq \mathcal{O}(\delta) \frac{\|h^{[k]}\|_{n,m}}{|k|}. \tag{91}$$

For $k = 0$ and $u \in D_\gamma^{u,f(i)}$, using the fourth statement of Lemma 5.2:

$$|\cosh^m u \cdot g^{[0]}(u)| \leq \|h^{[0]}\|_{m,f} \int_{-\infty}^{0} \left| \frac{\cosh u}{\cosh(u+t)} \right|^m dt \leq \mathcal{O}(1) \|h^{[0]}\|_{n,m},$$

and splitting the integral again for $u \in D_\gamma^{u,c(i)}$:

$$\begin{aligned}
|\cosh^n u \cdot g^{[0]}(u)| & \leq \int_{-\infty}^{T} \left|\cosh^n u \cdot h^{[0]}(u+t)\right| dt + \int_T^0 \left|\cosh^n u \cdot h^{[0]}(u+t)\right| dt \\
& \leq \mathcal{O}(1) \|h^{[0]}\|_{m,f} + \mathcal{O}(1) \|h^{[0]}\|_{n,c} \leq \mathcal{O}(1) \|h^{[0]}\|_{n,m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{G}_\delta(h)\|_{n,m,\sigma_i} & = \textstyle\sum_{k\in\mathbb{Z}} \|g^{[k]}\|_{n,m} e^{|k|\sigma_i} \leq \mathcal{O}(1) \|h^{[0]}\|_{n,m} + \mathcal{O}(\delta) \sum_{k\neq 0} \|h^{[k]}\|_{n,m} \frac{1}{|k|} e^{|k|\sigma_i} \\
& \leq \mathcal{O}(1) \|h\|_{n,m,\sigma_i}.
\end{aligned}$$

Moreover, if $\langle h \rangle = h^{[0]}(u) = 0$, then

$$\|\mathcal{G}_\delta(h)\|_{n,m,\sigma_i} \leq \mathcal{O}(\delta) \|h\|_{n,m,\sigma_i}.$$

For the fourth statement, we consider the derivative of the Fourier coefficients and Lemmas 5.3 and 5.4:

$$\begin{aligned}
\frac{d}{du} g^{[k]}(u) & = \int_{-\infty}^{0} \frac{d}{du} h^{[k]}(u+t) e^{ik\delta^{-1}t} dt \\
& = \int_{-\infty}^{0} \frac{d}{dt} (h^{[k]}(u+t) e^{ik\delta^{-1}t}) dt - ik\delta^{-1} \int_{-\infty}^{0} h^{[k]}(u+t) e^{ik\delta^{-1}t} dt \\
& = h^{[k]}(u) - \lim_{t\to-\infty} (h^{[k]}(u+t) e^{ik\delta^{-1}t}) - ik\delta^{-1} g^{[k]}(u) \\
& = h^{[k]}(u) - ik\delta^{-1} g^{[k]}(u).
\end{aligned} \tag{92}$$

Hence, using (91), it is clear that for $k \neq 0$, $\left\| \frac{d}{du} g^{[k]} \right\|_{n,m} \leq \mathcal{O}(1) \|h^{[k]}\|_{n,m}$. Moreover, recalling that $\frac{d}{du} g^{[0]} = h^{[0]}$, we obtain

$$\|\partial_u \mathcal{G}_\delta(h)\|_{n,m,\sigma_i} = \|\partial_u g\|_{n,m\sigma_i} = \sum_{k\in\mathbb{Z}} \left\| \frac{d}{du} g^{[k]} \right\|_{n,m} e^{|k|\sigma_i} \leq \mathcal{O}(1) \|h\|_{n,m,\sigma_i}.$$

Therefore, we have seen that if $h \in \mathcal{E}_{n,m}$, then both g and $\partial_u g$ belong to $\mathcal{E}_{n,m}$.

In order to prove $\mathcal{L}_\delta \circ \mathcal{G}_\delta = \mathrm{Id}$ we use (92):

$$\mathcal{L}_\delta \circ \mathcal{G}_\delta(h)(u,\tau) = \sum_{k\in\mathbb{Z}} \mathcal{L}_\delta(g^{[k]}(u) e^{ik\tau}) = \sum_{k\in\mathbb{Z}} \Big(ik\delta^{-1} g^{[k]}(u) + \frac{d}{du} g^{[k]}(u) \Big) e^{ik\tau} = h(u,\tau).$$

For functions belonging to $\mathcal{E}_m$ all the statements are proved analogously. □

5.2. Proof of Theorem 3.6

In order to prove Theorem 3.6 through a fixed point argument, we work with $Q = T - T_0 - \delta T_1$. Replacing it in (33), using 32 and considering the linear operator $\mathcal{L}_\delta$ defined in (68), we obtain

$$\mathcal{L}_\delta Q = \mathcal{F}(\partial_u Q) \tag{93}$$

where

$$\begin{aligned} \mathcal{F}(h) = \; & -\frac{\cosh^2 u}{8} h^2 \\ & -\delta \Big(\tanh u f_1(\tau) + \frac{1}{\cosh u} f_2(\tau) \Big) h + \frac{2}{J \cosh^2 u} \Big(\cos x_p(\tau) - \cos(\mu \sin \tau) \Big) \\ & -\frac{1}{J} \psi(u) \Big(\sin x_p(\tau) + \sin(\mu \sin \tau) \Big) - 4\delta \Big(\frac{\sinh u}{\cosh^3 u} f_1(\tau) + \frac{1}{\cosh^3 u} f_2(\tau) \Big) \\ & -2\delta^2 \Big((f_1(\tau))^2 \frac{\sinh^2 u}{\cosh^4 u} + 2 f_1(\tau) f_2(\tau) \frac{\sinh u}{\cosh^4 u} + (f_2(\tau))^2 \frac{1}{\cosh^4 u} \Big) \end{aligned} \tag{94}$$

and f_1 and f_2 are the functions introduced in Proposition 3.5.

Using the notation introduced in Section 5.1 and considering a fixed point argument in $\partial_u Q^-$ instead of Q^-, Theorem 3.6 can be rewritten as:

Proposition 5.6. *There exists δ_0 such that for $\delta \in (0, \delta_0)$, there exists a function $Q^-(u,\tau)$ for $(u,\tau) \in D_\gamma^{u(1)} \times \mathbb{T}_{\sigma_1}$ such that $\partial_u Q^- \in \mathcal{E}_{4,2}$ is a fixed point of the functional*

$$\bar{\mathcal{F}}(h) = \bar{\mathcal{G}}_\delta \mathcal{F}(h) \tag{95}$$

where $\bar{\mathcal{G}}_\delta$ is the operator defined in Lemma 5.5. Furthermore,

$$\begin{aligned} \|\partial_u Q^-\|_{4,2,\sigma_1} &\leq \mathcal{O}(\mu\delta^2) \\ \|\partial_u^2 Q^-\|_{4,2,\sigma_1} &\leq \mathcal{O}(\mu\delta^{2-\gamma}). \end{aligned}$$

Considering the expected behavior of $\partial_u Q^-$ at infinity and near the singularities in $u = \pm i\pi/2$ of T_0, we look for it in the Banach space $\mathcal{E}_{4,2}$ defined in (86) using the corresponding norm.

Next lemma gives the main properties of the functional $\bar{\mathcal{F}}$.

Lemma 5.7. *There exists a constant $b_1 > 0$ such that the functional $\bar{\mathcal{F}}$ defined in (95) holds the following properties:*

1. *If $h \in \mathcal{E}_{4,2}$, $\bar{\mathcal{F}}(h) \in \mathcal{E}_{4,2}$.*
2. *$\|\bar{\mathcal{F}}(0)\|_{4,2,\sigma_1} \leq \frac{b_1}{2}\mu\delta^2$.*
3. *For $h_1, h_2 \in B(b_1\mu\delta^2) \subset \mathcal{E}_{4,2}$, $\|\bar{\mathcal{F}}(h_2) - \bar{\mathcal{F}}(h_1)\|_{4,2,\sigma_1} \leq \mathcal{O}(\mu\delta^{1-\gamma})\|h_2 - h_1\|_{4,2,\sigma_1}$.*

Proof. The first statement is clear recalling that all the terms of $\mathcal{F}$ are analytic in $D_\gamma^{u(0)}$, considering their behavior at infinity and applying the fourth statement of Lemma 5.5.

For the second statement, we split $\mathcal{F}(0)$ as $\mathcal{F}(0)(u,\tau) = q_1(u,\tau) + q_2(u,\tau)$ where

$$\begin{aligned} q_1(u,\tau) = \; & \frac{2}{J\cosh^2 u}(\cos x_p(\tau) - \cos(\mu\sin\tau)) - \frac{1}{J}\psi(u)(\sin x_p(\tau) + \sin(\mu\sin\tau)) \\ & +\delta^2\left(2(f_1(\tau))^2\frac{\sinh^2 u}{\cosh^4 u} + 4f_1(\tau)f_2(\tau)\frac{\sinh u}{\cosh^4 u} + 2(f_2(\tau))^2\frac{1}{\cosh^4 u}\right) \\ q_2(u,\tau) = \; & -4\delta\left(\frac{\sinh u}{\cosh^3 u}f_1(\tau) + \frac{1}{\cosh^3 u}f_2(\tau)\right). \end{aligned}$$

Using Corollary 4.2 and statement (38), one can see that $\|q_1\|_{4,2,\sigma_1} \leq \mathcal{O}(\mu\delta^2)$. Therefore, this bound and the fourth statement of Lemma 5.5, give $\|\bar{\mathcal{G}}_\delta(q_1)\|_{4,2,\sigma_1} \leq \mathcal{O}(\mu\delta^2)$.

On the other hand, q_2 has zero mean. Thus, commuting ∂_u with $\mathcal{G}_\delta$, applying statement three of Lemma 5.5 and statement (38), we obtain $\|\bar{\mathcal{G}}_\delta(q_2)\|_{4,2,\sigma_1} \leq \mathcal{O}(\delta)\|\partial_u q_2\|_{4,2,\sigma_1} \leq \mathcal{O}(\mu\delta^2)$.

Taking $b_1 = 2\|\bar{\mathcal{F}}\|_{4,2}/\mu\delta^2$ we obtain the second statement.

For the third statement, for $h_1, h_2 \in B(b_1\mu\delta^2) \subset \mathcal{E}_{4,2}$,

$$\begin{aligned} & \|\mathcal{F}(h_1) - \mathcal{F}(h_2))\|_{4,2,\sigma_1} \leq \\ & \leq \left(\frac{\mathcal{O}(1)}{|\cosh^2 u|}\|h_1 + h_2\|_{4,2,\sigma_1} + \mathcal{O}(\delta)\left\|\frac{\sinh u}{\cosh u}f_1(\tau) + \frac{1}{\cosh u}f_2(\tau)\right\|_{\infty,\sigma_1}\right)\|h_1 - h_2\|_{4,2,\sigma_1} \\ & \leq \mathcal{O}(\mu\delta^{1-\gamma})\|h_1 - h_2\|_{4,2,\sigma_1}. \end{aligned}$$

Thus, applying the fourth statement of Lemma 5.5, we obtain the result. □

With this lemma, we are ready to prove the existence of the unstable manifold in the outer domain.

Proof of Proposition 5.6. Lemma 5.7 gives that the functional $\bar{\mathcal{F}}$ is a contraction from $B(b_1\mu\delta^2) \subset \mathcal{E}_{4,2}$ to itself, and hence it has a fixed point h^-. Furthermore, calling $Q^- = \mathcal{G}_\delta\mathcal{F}(h^-)$, we have that $h^- = \partial_u Q^-$ and Q^- is solution of (93).

In order to bound $\partial_u^2 Q^-$ it is enough to apply the first statement of Lemma 5.3 to the nested domains $D_\gamma^{u(1)}$ and $D_\gamma^{u(0)}$. □

For the stable manifold, the proof can be done analogously.

Corollary 5.8. *Equation (33) has a solution of the form* $T^+ = T_0 + \delta T_1 + Q^+$ *defined in* $D_\gamma^{s(1)} \times \mathbb{T}_{\sigma_1}$ *such that* $\lim_{\Re u \to +\infty} \cosh u \cdot Q^+(u,\tau) = 0$. *Moreover, the following bounds hold*

$$\begin{aligned} \|\partial_u Q^+\|_{4,2,\sigma_1} &\leq \mathcal{O}(\mu\delta^2) \\ \|\partial_u^2 Q^+\|_{4,2,\sigma_1} &\leq \mathcal{O}(\mu\delta^{2-\gamma}). \end{aligned}$$

5.3. Behavior of the invariant unstable manifold at infinity

Even though the existence of the invariant manifolds in the outer domains has already been proved in Section 5.2, in order to prove Theorem 3.12 it will be necessary to have more information about the behavior of the unstable invariant manifold T^- as $\Re u \to -\infty$.

In fact, the use of the weighted norm in the proof of Theorem 3.6, has showed that $Q^- \sim e^{2u}$ as $\Re u \to -\infty$. Replacing $Q^-(u,\tau)$ by $Q^-(u,\tau) = 8\lambda(\tau)e^{2u} + \mathcal{O}(e^{3u})$ in (93) shows the equation (96) which $\lambda(\tau)$ has to verify. The solution of this equation is studied in Lemma 5.10 and the behavior of $Q^-(u,\tau)$ as $\Re u \to -\infty$ is given in next theorem.

Theorem 5.9. *There exists* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$, *the differential equation*

$$\lambda' = -2\delta\lambda + 2\delta^2 f_1(\tau)\lambda - \delta\lambda^2 + \frac{\delta}{J}(\cos x_p(\tau) - \cos(\mu \sin \tau)) + 2\delta^2 f_1(\tau) - \delta^3 (f_1(\tau))^2, \quad (96)$$

where f_1 *is the function defined in Proposition 3.5, has a unique real-analytic solution defined in* $\mathbb{T}_{\sigma_0}$, *which holds* $|\lambda(\tau)| \leq \mathcal{O}(\mu\delta^2)$.

Moreover, for all $(u,\tau) \in D_\gamma^{u,f(0)} \times \mathbb{T}_{\sigma_1}$,

$$\left| \frac{\cosh^2 u}{4} \partial_u Q^-(u,\tau) - \lambda(\tau) \right| \leq |\cosh u|^{-1} \mathcal{O}(\mu\delta^2).$$

The proof of this theorem is done in several steps.

Lemma 5.10. *There exists* $\delta_0 > 0$ *such that for* $\delta \in (0, \delta_0)$, *equation (96) has a unique real-analytic solution defined in* $\mathbb{T}_{\sigma_0}$. *Moreover, it holds* $|\lambda(\tau)| \leq \mathcal{O}(\mu\delta^2)$.

Proof. The equation (96) can be rewritten as $\lambda' = -2\delta\lambda - \delta F(\tau, \lambda(\tau))$, with

$$F(\tau, z) = -2\delta f_1(\tau) z + z^2 - \frac{1}{J}(\cos x_p(\tau) - \cos(\mu \sin \tau)) - 2\delta f_1(\tau) + \delta^2 (f_1(\tau))^2.$$

Thus, λ is a 2π-periodic solution of (96) provided it holds:

$$\lambda(\tau) = \mathcal{F}(\lambda)(\tau) = -\delta e^{-2\delta\tau} \left(\int_0^\tau e^{2\delta s} F(s, \lambda(s)) ds + \frac{1}{e^{4\pi\delta} - 1} \int_0^{2\pi} e^{2\delta s} F(s, \lambda(s)) ds \right). \quad (97)$$

We will prove that, for certain $C_1 > 0$, $\mathcal{F}$ is a contraction from $B(C_1\mu\delta^2) \subset \mathcal{X}_p$ to itself, where $\mathcal{X}_p$ is the Banach space

$$\mathcal{X}_p = \{ z : \mathbb{T}_{\sigma_0} \to \mathbb{C} : \text{real-analytic}, 2\pi\text{-periodic}, \|z\|_\infty < \infty \}.$$

Note that $F(\tau,0)=\tilde{F}(\tau)-2\delta f_1(\tau)$ where $\tilde{F}(\tau)=-\left(\cos x_p(\tau)-\cos(\mu\sin\tau)\right)/J+\delta^2\left(f_1(\tau)\right)^2$. We split $\mathcal{F}(0)$ as $\mathcal{F}(0)(\tau)=h_1(\tau)+h_2(\tau)$ where

$$
\begin{aligned}
h_1(\tau) &= -\delta e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s}\tilde{F}(s)ds+\frac{1}{e^{4\pi\delta}-1}\int_0^{2\pi} e^{2\delta s}\tilde{F}(s)ds\right)\\
h_2(\tau) &= 2\delta^2 e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s}f_1(s)ds+\frac{1}{e^{4\pi\delta}-1}\int_0^{2\pi} e^{2\delta s}f_1(s)ds\right)
\end{aligned}
$$

and we bound each term in different ways. For the first one, by Corollary 4.2 and statement (38), since $|\tilde{F}(\tau)|\leq\mathcal{O}(\mu\delta^2)$,

$$
|h_1(\tau)|\leq\mathcal{O}(\mu\delta^3)e^{-2\delta\tau}\left(\int_0^\tau ds+\frac{1}{e^{4\pi\delta}-1}\int_0^{2\pi} ds\right)\leq\mathcal{O}(\mu\delta^2).
$$

For the second one, we define $\hat{f}_1$ such that $\hat{f}_1'=f_1$ and $\langle\hat{f}_1\rangle=0$ and we integrate it by parts. From statement (38), it is clear that $\hat{f}_1\leq\mathcal{O}(\mu)$. Using this bound and the 2π-periodicity of $\hat{f}_1$, it is obtained:

$$
h_2(\tau)=2\delta^2\hat{f}_1(\tau)-4\delta^3 e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s}\hat{f}_1(s)ds+\frac{1}{e^{4\pi\delta}-1}\int_0^{2\pi} e^{2\delta s}\hat{f}_1(s)ds\right).
$$

Hence, reasoning as for h_1 it is clear that $|h_2(\tau)|\leq\mathcal{O}(\mu\delta^2)$. Thus, there exists C_1 such that $\|\mathcal{F}(0)\|_\infty\leq\dfrac{C_1}{2}\mu\delta^2$.

Moreover, taking $\lambda_1,\lambda_2\in B(C_1\mu\delta^2)\subset\mathcal{X}_p$,

$$
\begin{aligned}
\mathcal{F}(\lambda_1)(\tau)-\mathcal{F}(\lambda_2)(\tau)\leq{}& \delta e^{-2\delta\tau}\int_0^\tau e^{2\delta s}\left(-2\delta f_1(\tau)\left(\lambda_1(s)-\lambda_2(s)\right)+\left(\lambda_1(s)-\lambda_2(s)\right)^2\right)ds\\
&+\frac{\delta e^{-2\delta\tau}}{e^{4\pi\delta}-1}\int_0^{2\pi} e^{2\delta s}\left(-2\delta f_1(\tau)\left(\lambda_1(s)-\lambda_2(s)\right)+\left(\lambda_1(s)-\lambda_2(s)\right)^2\right)ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\mathcal{F}(\lambda_1)(\tau)-\mathcal{F}(\lambda_2)(\tau)| &\leq\mathcal{O}(\mu\delta^2)\|\lambda_1-\lambda_2\|_\infty\int_0^\tau ds+\mathcal{O}(\mu\delta)\|\lambda_1-\lambda_2\|_\infty\int_0^{2\pi} ds\\
&\leq\mathcal{O}(\mu\delta)\|\lambda_1-\lambda_2\|_\infty.
\end{aligned}
$$

Therefore reducing δ if it was necessary, $\mathcal{F}$ is a contraction from $B(C_1\mu\delta^2)\subset\mathcal{X}_p$ to itself and there exists a unique fixed point $\lambda(\tau)$ which is a solution of equation (96). $\square$

In order to see that $Q^-=8\lambda(\tau)e^{2u}+\mathcal{O}(e^{3u})$, we replace $Q^-(u,\tau)=8\lambda(\tau)e^{2u}+P^-(u,\tau)$ in equation (93). Using also equation (96) for $\lambda(\tau)$, we obtain,

$$
\mathcal{L}_\delta P^-=\mathcal{H}_2(\partial_u P^-,u,\tau) \tag{98}
$$

where

$$\begin{aligned}
\mathcal{H}_2(h,u,\tau) = & -\frac{\cosh^2 u}{8}h^2 - 4\cosh^2 u e^{2u}\lambda(\tau)h - \delta\Big(\frac{\sinh u}{\cosh u}f_1(\tau) + \frac{1}{\cosh u}f_2(\tau)\Big)h \\
& +8(\lambda(\tau))^2 e^{2u}(1-4\cosh^2 u e^{2u}) - 16\lambda(\tau)e^{2u}\delta\Big((\tanh u+1)f_1(\tau) + \frac{1}{\cosh u}f_2(\tau)\Big) \\
& +\frac{2}{J}\left(\cos x_p(\tau) - \cos(\mu\sin\tau)\right)\left(\frac{1}{\cosh^2 u} - 4e^{2u}\right) + \frac{1}{J}\psi(u)\left(\sin x_p(\tau) + \sin(\mu\sin\tau)\right) \\
& -4\delta\left(\left(\frac{\sinh u}{\cosh^3 u} + 4e^{2u}\right)f_1(\tau) + \frac{1}{\cosh^3 u}f_2(\tau)\right) \\
& -2\delta^2\left(\left(\frac{\sinh^2 u}{\cosh^4 u} - 4e^{2u}\right)(f_1(\tau))^2 + 2f_1(\tau)f_2(\tau)\frac{\sinh u}{\cosh^4 u} + (f_2(\tau))^2\frac{1}{\cosh^4 u}\right)
\end{aligned}$$

and f_1 and f_2 are the functions defined in Proposition 3.5.

In order to solve equation (98), using the operator $\bar{\mathcal{G}}_\delta$ defined in Lemma 5.5, we observe that $\partial_u P^-$ will be a fixed point of

$$\bar{\mathcal{H}}_2 = \bar{\mathcal{G}}_\delta \circ \mathcal{H}_2. \tag{99}$$

Then, we will look for it in the Banach space $\mathcal{E}_3$ defined in (87). We will need the following technical lemma.

Lemma 5.11. *There exists a constant $C_2 > 0$ such that the functional $\bar{\mathcal{H}}_2$ defined in (99) holds the following properties:*

1. *If $h \in \mathcal{E}_3$, $\bar{\mathcal{H}}_2(h,\cdot,\cdot) \in \mathcal{E}_3$.*
2. *There exists $C_2 > 0$ such that $\|\bar{\mathcal{H}}_2(0,\cdot,\cdot)\|_{3,f,\sigma_1} \leq \frac{C_2}{2}\mu\delta^2$.*
3. *For all $h_1, h_2 \in B(C_2\mu\delta^2) \subset \mathcal{E}_3$, $\|\bar{\mathcal{H}}_2(h_2,\cdot,\cdot) - \bar{\mathcal{H}}_2(h_1,\cdot,\cdot)\|_{3,f,\sigma_1} \leq \mathcal{O}(\mu\delta)\|h_2 - h_1\|_{3,f,\sigma_1}$.*

Proof. The first statement is straightforward recalling that all the terms of $\mathcal{H}_2$ are real-analytic in $D^{u,f(0)}_\gamma$, considering their behavior at infinity (taking advantage of the cancellations) and applying the Lemma 5.5. For the second statement, $\mathcal{H}_2(0,u,\tau)$ is split as $\mathcal{H}_2(0,u,\tau) = p_1(u,\tau) + p_2(u,\tau)$ where

$$\begin{aligned}
p_1(u,\tau) = &\quad 8\left(\lambda(\tau)\right)^2 e^{2u}(1-4\cosh^2 u e^{2u}) \\
& -16\lambda(\tau)e^{2u}\delta\left((\tanh u+1)f_1(\tau) + \frac{1}{\cosh u}f_2(\tau)\right) \\
& +\frac{2}{J}\left(\cos x_p(\tau) - \cos(\mu\sin\tau)\right)\left(\frac{1}{\cosh^2 u} - 4e^{2u}\right) + \frac{1}{J}\psi(u)\left(\sin x_p(\tau) + \sin(\mu\sin\tau)\right) \\
& -2\delta^2\left(\left(\frac{\sinh^2 u}{\cosh^4 u} - 4e^{2u}\right)(f_1(\tau))^2 + 2f_1(\tau)f_2(\tau)\frac{\sinh u}{\cosh^4 u} + (f_2(\tau))^2\frac{1}{\cosh^4 u}\right) \\
p_2(u,\tau) = & -4\delta\left(\left(\frac{\sinh u}{\cosh^3 u} + 4e^{2u}\right)f_1(\tau) + \frac{1}{\cosh^3 u}f_2(\tau)\right).
\end{aligned}$$

Using the properties of the norm given in Lemma 5.1, statement (38), Corollary 4.2 and the bound of λ obtained in Lemma 5.10, it is clear that $\|p_1(u,\tau)\|_{3,f,\sigma_1} \leq \mathcal{O}(\mu\delta^2)$ and thus, by Lemma 5.5,

$$\|\bar{\mathcal{G}}_\delta(p_1(u,\tau))\|_{3,f,\sigma_1} \leq \mathcal{O}(\mu\delta^2).$$

On the other hand, since $p_2(u,\tau)$ has zero mean, commuting $\mathcal{G}_\delta$ and ∂_u and using again Lemmas 5.5 and 5.1 and statement (38), we obtain:

$$\|\bar{\mathcal{G}}_\delta(p_1(u,\tau))\|_{3,f,\sigma_1} \leq \mathcal{O}(\delta)\|\partial_u p_2(u,\tau))\|_{3,f,\sigma_1} \leq \mathcal{O}(\mu\delta^2),$$

which gives the second statement of the lemma with a suitable constant C_2.

The proof of the third statement, is analogous to the proof of the third statement of Lemma 5.7. □

Proof of theorem 5.9. Using the results obtained in Lemma 5.11, it is clear that the functional $\bar{\mathcal{H}}_2$ defined in (99) is a contraction from $B(C_2\mu\delta^2) \subset \mathcal{E}_{3,f}$ to itself. Thus, it has a unique fixed point h^- which holds $h^- = \partial_u P^-$ with $P^- = \mathcal{G}_\delta \mathcal{H}_2(h^-,u,\tau)$ and then $\|\partial_u P^-\|_{3,f,\sigma_1} \leq C_2\mu\delta^2$.

Therefore, since $|\partial_u P^-(u,\tau)| \leq |\cosh^{-3} u|\mathcal{O}(\mu\delta^2)$, we obtain

$$\begin{aligned}\left|\frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau) - \lambda(\tau)\right| &\leq \left|\frac{\cosh^2 u}{4}\partial_u P^-(u,\tau)\right| + |4\cosh^2 u e^{2u} - 1||\lambda(\tau)| \\ &\leq |\cosh^{-1} u|\mathcal{O}(\mu\delta^2).\end{aligned}$$

□

6. Invariant manifolds in the inner domains: proof of Theorem 3.10

In this section we will prove the existence of the unstable invariant manifold in the inner domain (45) (the proof of the existence of the stable one is analogous). As we will work in the inner variables (z,τ) (see (50)), the manifold $\phi^-(z,\tau)$ (see (51)) is solution of equation (54). Moreover, we will have to impose that the solution has a given initial condition, which comes from the information that we have obtained from the outer domain. This will allow us to guarantee that the solution we will find in this section is the analytic continuation of the invariant manifold $T^-(u,\tau)$ obtained in Theorem 3.6.

We recall that we have constructed the inner and outer domains in such a way that, for all $0 \leq i,j \leq N$, $D_\gamma^{u(i)} \cap D_{\delta,\pm}^{u(j)} \neq \emptyset$. Therefore, it will be in this intersecting zone, usually called matching domain, where we will use the information given by the invariant manifold in the outer domain in order to continue it to the inner one. We recall that in this section we will work in the inner variables (z,τ), and hence we will consider the domains $\mathcal{D}_{\delta,\pm}^{u(i)} \times \mathbb{T}_{\sigma_1}$ defined in (52), so that we will take the initial conditions in curves of the form

$$\begin{aligned}\Gamma_{\gamma,+}^{u(i)} = \quad &\{z \in \mathbb{C} : \Im z = -\tan\beta_0\left(\Re z + \tilde{a}^{(i)}\delta^{\gamma-1}\right), \\ &\frac{\tan\beta_0}{\tan\beta_1 - \tan\beta_0}\left(c^{(i)}\ln(1/\delta) - \tan\beta_1\tilde{a}^{(i)}\delta^{\gamma-1}\right) < \Im z < -c^{(i)}\ln(1/\delta)\},\end{aligned} \tag{100}$$

which belong to the matching domain (see Figure 11).

It is expected that in the inner domains the unstable invariant manifold ϕ^- is well approximated by the solution ϕ_0^- of equation (57) given in Theorem 3.8. Thus, we

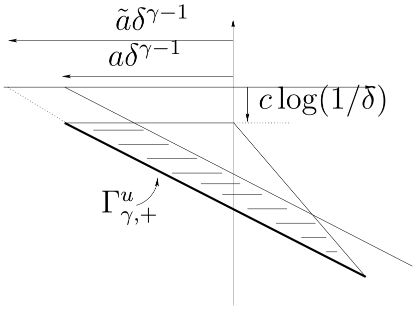

Figure 11. The matching domain and $\Gamma^u_{\gamma,+}$ in variable z.

consider the corresponding partial differential equation for $\varphi^-(z,\tau) = \phi^-(z,\tau) - \phi_0^-(z,\tau)$ for $(z,\tau) \in \mathcal{D}^{u(0)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$:

$$\partial_\tau \varphi - \frac{1}{4} z^2 \partial_z \phi_0^- \partial_z \varphi = a(z,\tau)\partial_z\varphi + b(z,\tau)(\partial_z\varphi)^2 + c(z,\tau), \tag{101}$$

where

$$a(z,\tau) = -\Big(\frac{\cosh^2(\frac{\pi}{2}i + \delta z)}{4\delta^2} + \frac{z^2}{4}\Big)\partial_z\phi_0^-(z,\tau) \tag{102}$$

$$b(z,\tau) = -\frac{\cosh^2(\frac{\pi}{2}i + \delta z)}{8\delta^2} \tag{103}$$

$$\begin{aligned} c(z,\tau) = {} & -\frac{2\delta^2}{J\cosh^2(\frac{\pi}{2}i + \delta z)}\cos x_p(\tau) - \frac{2}{Jz^2}\cos(\mu\sin\tau) + \frac{\delta^2}{J}\psi\left(\frac{\pi}{2}i + \delta z\right)\sin x_p(\tau) \\ & -\frac{2i}{Jz^2}\sin(\mu\sin\tau) + \Big(\frac{\cosh^2(\frac{\pi}{2}i + \delta z)}{8\delta^2} + \frac{z^2}{8}\Big)\left(\partial_z\phi_0^-(z,\tau)\right)^2. \end{aligned} \tag{104}$$

We know that the function $\varphi^-(z,\tau)$ is an analytic solution of (101) in a neighborhood of $\Gamma^{u(0)}_{\gamma,+}$ contained in the matching domain. So, we look for a solution of (101) with prescribed initial conditions $\varphi^-(z,\tau) = \phi^-(z,\tau) - \phi_0^-(z,\tau)$ where (z,τ) are in a suitable curve in the matching domain (close to $\Gamma^{u(0)}_{\gamma,+}$), which a posteriori will be the analytic continuation of φ^-. In fact, as we are interested in $\partial_z\varphi^-$, the initial condition we will take is $\partial_z\varphi^- = \partial_z\phi^- - \partial_z\phi_0^-$.

In order to solve (101), the first step is to perform the change of variables $z = x + R^-(x,\tau)$, given in Theorem 3.8, which conjugates the linear differential operator $\partial_\tau - \frac{1}{4}z^2\partial_z\phi_0^-\partial_z$ to $\partial_\tau + \partial_x$ and is defined from $\mathcal{D}^{u(1)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$ to $\mathcal{D}^{u(0)}_{\delta,+}$.

The second step is to extend the function $\Phi^-(x,\tau) = \varphi^-(x + R^-(x,\tau),\tau)$ already defined in a neighborhood of $\Gamma^{u(1)}_{\gamma,+}$ to the domain $\mathcal{D}^{u(1)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$, that verifies:

$$\mathcal{L}\Phi = A(x,\tau)\partial_x\Phi + B(x,\tau)(\partial_x\Phi)^2 + C(x,\tau) \tag{105}$$

where

$$A(x,\tau) = a(x + R^-(x,\tau),\tau)\frac{1}{1 + \partial_x R^-(x,\tau)} \tag{106}$$

$$B(x,\tau) = b(x + R^-(x,\tau),\tau)\frac{1}{(1 + \partial_x R^-(x,\tau))^2} \tag{107}$$

$$C(x,\tau) = c(x + R^-(x,\tau),\tau). \tag{108}$$

and $\mathcal{L}$ is the operator defined in (62).

Next proposition shows the existence of $\Phi^-(x,\tau)$ in a reduced inner domain $\mathcal{D}^{u(4)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$ as a solution of (105) with prescribed initial condition expressed in the new variables (x,τ):

$$\begin{aligned}\partial_x\Phi^-(x^*,\tau) &= \eta(x^*,\tau)\\ &= \left(\partial_z\phi^-\left(x^*+R^-(x^*,\tau),\tau\right)-\partial_z\phi_0^-\left(x^*+R^-(x^*,\tau),\tau\right)\right)\frac{1}{1+\partial_xR^-(x^*,\tau)}\end{aligned} \tag{109}$$

when $(x^*,\tau)\in\Gamma^{u(1)}_{\gamma,+}$ defined in (100).

Proposition 6.1. *Let us consider a constant $\sigma_2<\sigma_1$ and $\gamma\in(1/3,1/2)$. Then, there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$, there exists an analytic function $\Phi^-(x,\tau)$ for $(x,\tau)\in\mathcal{D}^{u(3)}_{\delta,+}\times\mathbb{T}_{\sigma_2}$, which holds the partial differential equation (105) and the initial condition $\partial_x\Phi^-(x^*,\tau)=\eta(x^*,\tau)$ for $(x^*,\tau)\in\Gamma^{u(1)}_{\gamma,+}\times\mathbb{T}_{\sigma_2}$. Moreover,*

$$\|\partial_x\Phi\|_\infty\le\mathcal{O}(\delta^2).$$

Considering $\Psi=\partial_x\Phi^-$ and differentiating expression (105) we obtain

$$\tilde{\mathcal{L}}\Psi=\partial_xA(x,\tau)\Psi+\partial_xB(x,\tau)\Psi^2+\partial_xC(x,\tau), \tag{110}$$

where

$$\tilde{\mathcal{L}}=\partial_\tau+\left(1-A(x,\tau)-2B(x,\tau)\Psi(x,\tau)\right)\partial_x. \tag{111}$$

Since this partial differential equation is quasilinear it can be solved through a characteristics-like method.

The initial condition $\eta(x^*,\tau)$ defined in (109), which is given in terms of the invariant manifold ϕ^- and ϕ_0^- is bounded in the next lemma.

Lemma 6.2. *For any fixed $\gamma\in(0,1/2)$, there exists δ_0 such that for all $\delta\in(0,\delta_0)$, the following bound holds*

$$\forall(x,\tau)\in\Gamma^{u(1)}_{\gamma,+}\times\mathbb{T}_{\sigma_1},\quad|\eta(x,\tau)|\le\mathcal{O}(\delta^2)$$

Proof. By definition

$$\Phi^-(x,\tau)=\varphi^-(x+R^-(x,\tau),\tau)=\phi^-(x+R^-(x,\tau),\tau)-\phi_0^-(x+R^-(x,\tau),\tau),$$

and since $x\in\Gamma^{u(1)}_{\gamma,+}$, $|\delta x|\le\mathcal{O}(\delta^\gamma)$. Using (51) and the definition of T_0 and T_1 given in (36) and (37), we bound first $\partial_z\varphi^-(z,\tau)$:

$$\begin{aligned}|\partial_z\varphi^-(z,\tau)| &\le\delta^2\left|\partial_uT^-(\tfrac{\pi}{2}i+\delta z,\tau)-\partial_uT_0(\tfrac{\pi}{2}i+\delta z)-\delta\partial_uT_1(\tfrac{\pi}{2}i+\delta z,\tau)\right|+\\ &+\left|\delta^2\partial_uT_0(\tfrac{\pi}{2}i+\delta z)+4z^{-2}\right|+\\ &+\left|\delta^3\partial_uT_1(\tfrac{\pi}{2}i+\delta z,\tau)+4(f_1(\tau)-if_2(\tau))z^{-3}\right|+\\ &+\left|\partial_z\phi_0^-(z,\tau)+4z^{-2}+4(f_1(\tau)-if_2(\tau))z^{-3}\right|.\end{aligned}$$

Applying the bound obtained in Theorem 3.6 and using that $\gamma>1/2$, one can see that the first summand is of order $\mathcal{O}(\mu\delta^2)$. For the second one, we have to recall that

$$\partial_uT_0\left(\frac{\pi}{2}i+\delta z\right)=\frac{-4}{\delta^2z^2}+\mathcal{O}(1),$$

and thus, $\left|\delta^2\partial_u T_0(\frac{\pi}{2}i+\delta z)+4z^{-2}\right|\leq\mathcal{O}(\delta^2)$. Reasoning in the same way, the third term can be bounded by $\mathcal{O}(\delta^2)$. Since by Theorem 3.8 the last one holds

$$\partial_z\phi_0^-(z,\tau)+4z^{-2}+4\left(f_1(\tau)-if_2(\tau)\right)z^{-3}=\mathcal{O}\left(z^{-4}\right),$$

it is of order $\mathcal{O}(\delta^{4-4\gamma})\leq\mathcal{O}(\delta^2)$. To obtain the bound of $\partial_x\Phi^-(x,\tau)$, we use, by Theorem 3.8, the function R^- holds $\partial_x R^-(x,\tau)=\mathcal{O}(\delta^{2-2\gamma})$ for $(x,\tau)\in\Gamma^{u(1)}_{\delta,+}$. □

To solve (110) we need to consider a change of variables which straightens the differential operator (111). We look for it by applying the following lemma, whose proof is straightforward.

Lemma 6.3. *Let us consider a function $W(x,\tau)$, then if the function $g(y,\tau)$ is a solution of*

$$\mathcal{L}g(y,\tau)=W(g(y,\tau),\tau),$$

where $\mathcal{L}$ is the operator defined in (62), the change of variables $x=g(y,\tau)$ conjugates the operator $\partial_\tau+W(x,\tau)\partial_x$ to $\mathcal{L}$.

If we knew a function $g(y,\tau)$ such that the change $x=g(y,\tau)$ conjugated $\tilde{\mathcal{L}}$ in (111) with $\mathcal{L}$, it would be natural to apply this change to (110) obtaining

$$\mathcal{L}\xi(y,\tau)=\partial_x A(g(y,\tau),\tau)\xi(y,\tau)+\partial_x B(g(y,\tau),\tau)(\xi(y,\tau))^2+\partial_x C(g(y,\tau),\tau). \tag{112}$$

Once we knew $\xi(y,\tau)$ verifying this equation, the solution of (110) would be given by the implicit relation

$$\Psi(g(y,\tau),\tau)=\xi(y,\tau).$$

However, by Lemma 6.3, the equation that $g(y,\tau)$ has to verify to conjugate $\tilde{\mathcal{L}}$ and $\mathcal{L}$ is

$$\mathcal{L}g(y,\tau)=1-A(g(y,\tau),\tau)-2B(g(y,\tau),\tau)\xi(y,\tau)$$

which has ξ on the right hand side. So that, we will look for the solutions of both equations at the same time.

Remark 6.4. *Let us observe that the method that we will use to construct g, ξ and g^{-1} will provide functions of class $\mathcal{C}^1$ in x (but analytic in τ). Nevertheless, since the partial differential equation (110) is quasilinear and analytic and the initial condition η in (109) is also analytic, the solution $\Psi(x,\tau)$ thus obtained will be analytic in both variables for $(x,\tau)\in\mathcal{D}^{u(4)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$.*

We look for a function of the form $g(y,\tau)=y+\hat{g}(y,\tau)$ with $|\hat{g}|\leq\mathcal{O}(\delta^s)$ for certain $s>0$ in such a way that for $(y,\tau)\in\mathcal{D}^{u(2)}_{\delta,+}$, it holds that $y+\hat{g}(y,\tau)\in\mathcal{D}^{u(1)}_{\delta,+}$. Thus, $\hat{g}$ has to be a solution of the partial differential equation

$$\mathcal{L}\hat{g}(y,\tau)=-A(y+\hat{g}(y,\tau),\tau)-2B(y+\hat{g}(y,\tau),\tau)\xi(y,\tau), \tag{113}$$

and equation (112) reads:

$$\begin{aligned}\mathcal{L}\xi(y,\tau)=\;&\partial_x A(y+\hat{g}(y,\tau),\tau)\xi(y,\tau)+\partial_x B(y+\hat{g}(y,\tau),\tau)(\xi(y,\tau))^2+\\&+\partial_x C(y+\hat{g}(y,\tau),\tau).\end{aligned} \tag{114}$$

Respect to the initial conditions for $\hat{g}$ and ξ in $\Gamma^{u(2)}_{\gamma,+}$ we choose

$$\hat{g}(y^*,\tau)=0 \quad \text{for} \quad y^*\in\Gamma^{u(2)}_{\gamma,+} \tag{115}$$

and consequently, using (109),

$$\xi(y^*,\tau)=\Psi(y^*+\hat{g}(y^*,\tau),\tau)=\Psi(y^*,\tau)=\eta(y^*,\tau) \quad \text{for} \quad y^*\in\Gamma^{u(2)}_{\gamma,+}. \tag{116}$$

The next lemma shows that both equations (113) and (114) with initial conditions (115) and (116), can be expressed as integral equations.

Lemma 6.5. *For $(y,\tau)\in\mathcal{D}^{(2)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$, it can be considered the following integral operator, which is a left inverse of $\mathcal{L}$:*

$$\mathcal{G}(h)(y,\tau)=\int_{y^*-y}^{0} h(y+t,\tau+t)dt, \tag{117}$$

where given $y\in\mathcal{D}^{u(2)}_{\delta,+}$, we consider $y^*=y^*(y)\in\Gamma^{u(2)}_{\gamma,+}$ such that $\Im y^*=\Im y$.

Then $(\hat{g},\xi)$ is a solution $\mathcal{C}^1$ in y and analytic in τ of equations (113) and (114) with initial conditions (115) and (116), provided they hold

$$\begin{cases} \hat{g}=\mathcal{G}\left(-A(y+\hat{g},\tau)-2B(y+\hat{g},\tau)\xi\right) \\ \xi=\mathcal{G}\left(\partial_x A(y+\hat{g},\tau)\xi+\partial_x B(y+\hat{g},\tau)\xi^2+\partial_x C(y+\hat{g},\tau)\right)+\eta(y^*,\tau+y^*-y), \end{cases} \tag{118}$$

and are continuous in y and analytic in τ.

Proof. As (118) is an integral equation, its solutions $(\hat{g},\xi)$ are $\mathcal{C}^1$ in y and analytic in τ. Thus, it is enough to differentiate by $\mathcal{L}$ in both sides of the equations in order to prove that they are classical solutions of (113) and (114). For the initial condition, we only have to evaluate the equations for $y=y^*$. □

To solve equation (118), it is necessary to perform the change

$$\hat{\xi}(y,\tau)=\xi(y,\tau)-\eta(y^*,\tau+y^*-y) \tag{119}$$

which lead to the following equations:

$$\begin{cases} \hat{g}= & \mathcal{G}\left(-A(y+\hat{g},\tau)-2B(y+\hat{g},\tau)\left(\hat{\xi}+\eta(y^*,\tau+y^*-y)\right)\right) \\ \hat{\xi}= & \mathcal{G}\left(\partial_x A(y+\hat{g},\tau)\left(\hat{\xi}+\eta(y^*,\tau+y^*-y)\right)+\right. \\ & \left. +\partial_x B(y+\hat{g},\tau)\left(\hat{\xi}+\eta(y^*,\tau+y^*-y)\right)^2+\partial_x C(y+\hat{g},\tau)\right), \end{cases} \tag{120}$$

whose solutions will be found in the following proposition:

Proposition 6.6. *For any fixed $\gamma\in(1/3,1/2)$, there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$, there exists a solution $(\hat{g},\hat{\xi})$ of (120) defined in $\mathcal{D}^{u(2)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$, $\mathcal{C}^1$ in y and analytic in τ. Moreover, for all $(y,\tau)\in\mathcal{D}^{u(2)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$,*

$$|\hat{g}(y,\tau)|\le\mathcal{O}(\delta^{3\gamma-1})$$
$$|y\hat{\xi}(y,\tau)|\le\mathcal{O}(\delta^2).$$

The proof of this proposition will be done in Subsection 6.2, but before it, we need some technicalities which will be explained in the following subsection.

6.1. Banach spaces and technical lemmas

Since we expect that $\hat{g} \sim \delta^{3\gamma-1}$ and $\hat{\xi} \sim y^{-1}\delta^2$, we consider the norms

1. For $\hat{g}$: $\llbracket \hat{g} \rrbracket_1 = \|\hat{g}\|_\infty$.
2. For $\hat{\xi}$: $\llbracket \hat{\xi} \rrbracket_2 = \|\delta^{3\gamma-3} y \hat{\xi}\|_\infty$.

Therefore, for the pairs $(\hat{g}, \hat{\xi})$ we will consider the norm $\llbracket (\hat{g}, \hat{\xi}) \rrbracket = \sup\{\llbracket \hat{g} \rrbracket_1, \llbracket \hat{\xi} \rrbracket_2\}$ and the corresponding Banach space

$$\hat{\mathcal{X}} = \{(\hat{g}, \hat{\xi}) : \mathcal{D}^{u(2)}_{\delta,+} \times \mathbb{T}_{\sigma_1} \to \mathbb{C}^2, \text{continuous in } y \text{ and analytic in } \tau, \llbracket (\hat{g}, \hat{\xi}) \rrbracket < \infty\}. \quad (121)$$

Lemma 6.7. *The following bounds hold for all* $(x,\tau) \in \mathcal{D}^{u(1)}_{\delta,+}$*:*

$$\begin{array}{ll} |A(x,\tau)| \leq \mathcal{O}(\delta^{2\gamma}) & |B(x,\tau)| \leq \mathcal{O}(1)|x|^2 \\ |\partial_x A(x,\tau)| \leq \mathcal{O}(\delta^{1+\gamma}) & |\partial_x B(x,\tau)| \leq \mathcal{O}(1)|x| \\ |\partial_x^2 A(x,\tau)| \leq \mathcal{O}(\delta^2) & |\partial_x^2 B(x,\tau)| \leq \mathcal{O}(1). \end{array}$$

Moreover,

$$\partial_x C(x,\tau) = -\mathcal{L}\Big(\frac{i\delta^2 f_2(\tau)}{2(x + R^-(x,\tau))}\Big) + \mathcal{O}(\delta^2)|x|^{-2}$$

where f_2 is the function defined in Proposition 3.5.

Proof. For the bounds of function A and its derivatives, one can check that $a(z,\tau) = \mathcal{O}(\varepsilon z)^2$ (see (102)). Differentiating, one can obtain $\partial_z a(z,\tau) = \varepsilon\mathcal{O}(\varepsilon z)$ and $\partial_z^2 a(z,\tau) = \varepsilon^2\mathcal{O}(1)$, and considering the properties of the change R^- stated in Theorem 3.8 it is straightforward to obtain the wanted bounds.

Since $b(z,\tau) = \frac{1}{4}z^2\left(1 + \mathcal{O}(\varepsilon z)^2\right)$ (see (103)), proceeding in the same way one can obtain the bounds of B and its derivatives.

For the last expression, we consider first $c(z,\tau)$ (see (104)) and we split it in several terms. We will call these terms $c_i(z,\tau)$, and we will also deal with the corresponding functions in the new variable $C_i(x,\tau)$. We recall that we are not interested in bounding $C(x,\tau)$ but its derivative $\partial_x C(x,\tau)$. We consider $c(z,\tau) = c_1(z,\tau) + c_2(z,\tau) + c_3(z,\tau)$, where

$$\begin{aligned} c_1(z,\tau) = \;& -\frac{2\delta^2}{J\cosh^2(\frac{\pi}{2}i + \delta z)}(\cos x_p(\tau) - \cos(\mu \sin\tau)) \\ & + \frac{\delta^2}{J}\psi\left(\frac{\pi}{2}i + \delta z\right)(\sin x_p(\tau) - \sin(-\mu\sin\tau)) \\ & + \Big(\frac{\cosh^2(\frac{\pi}{2}i + \delta z)}{8\delta^2} + \frac{z^2}{8}\Big)(\partial_z\phi_0)^2 \\ c_2(z,\tau) = \;& -\Big(\frac{2\delta^2}{J\cosh^2(\frac{\pi}{2}i + \delta z)} + \frac{2}{Jz^2}\Big)\cos(\mu\sin\tau) \\ c_3(z,\tau) = \;& -\Big(\frac{\delta^2}{J}\psi\left(\frac{\pi}{2}i + \delta z\right) + \frac{2i}{Jz^2}\Big)\sin(\mu\sin\tau). \end{aligned}$$

In order to bound the first and the third terms of c_1 we need to know the behavior of $\partial_u\psi(u) = \frac{4}{\cosh^3 u}$ (see (32)) in the inner domain:

$$\delta^2\partial_z\left[\psi\left(\frac{\pi}{2}i + \delta z\right)\right] = \frac{4i}{z^3} + \frac{i\delta^2}{2z} + \delta^3\mathcal{O}(\delta z). \quad (122)$$

With this fact and Corollary 4.2, the first term holds $|\partial_z c_1(z,\tau)| \le |z|^{-2}\mathcal{O}(\delta^2)$.

For c_2, since $|z| \le \mathcal{O}(\delta^{\gamma-1})$ and $\gamma < 1/3$, $|\partial_z c_2(z,\tau)| = \delta^3\mathcal{O}(\delta z) \le |z|^{-2}\mathcal{O}(\delta^2)$.

The third one, c_3, has to be treated more carefully. Using (122), $\partial_z c_3$ can be split as:

$$\partial_z c_3(z,\tau) = \partial_z c_{31}(z,\tau) + \partial_z c_{32}(z,\tau)$$

where $\partial_z c_{31}(z,\tau) = -\dfrac{i\delta^2}{2Jz}\sin(\mu\sin\tau)$ and $\partial_z c_{32}(z,\tau) = \delta^3\mathcal{O}(\delta z)$. For this last term, since holds $|c_{32}(z,\tau)| \le \mathcal{O}(\delta^{3+\gamma}) \le |z|^{-2}\mathcal{O}(\delta^2)$, we can proceed as for the previous ones.

For c_{31}, we consider the operator $\bar{\mathcal{L}}$ defined in Theorem 3.8, and we recall that $\bar{\mathcal{L}} = \partial_\tau + \mathcal{O}(1)\partial_z$,

$$\partial_z c_{31}(z,\tau) = -\partial_\tau\left(\frac{i\delta^2 f_2(\tau)}{2z}\right) = -\bar{\mathcal{L}}\left(\frac{i\delta^2 f_2(\tau)}{2z}\right) + |z|^{-2}\mathcal{O}(\delta^2).$$

Performing the change of variables defined in (60) and using the properties stated in Theorem 3.8, it is obtained:

$$\begin{aligned}\partial_x C_{31}(x,\tau) =\ & \partial_z c_{31}(x+R^-(x,\tau),\tau)(1+\partial_x R^-(x,\tau))\\ =\ & -\mathcal{L}\Big(\frac{i\delta^2 f_2(\tau)}{2(x+R^-(x,\tau))}\Big) + |x|^{-2}\mathcal{O}(\delta^2),\end{aligned}$$

obtaining the desired bound for $\partial_x C$. ∎

Lemma 6.8. *Let us consider $\gamma \in (0,1)$. Then, there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$, for $y \in \mathcal{D}^{u(i)}_{\delta,+}$ and $y^* \in \Gamma^{u(i)}_{\delta,+}$ with $\Im y = \Im y^*$, it holds that*

$$\int_{y^*-y}^0 |y+t|^n dt \le \begin{cases} \mathcal{O}(1)|y|^{n+1} & \text{if } n \le -2\\ \mathcal{O}(\ln(1/\delta)) & \text{if } n = -1\\ \mathcal{O}(\delta^{(n+1)(\gamma-1)}) & \text{if } n \ge 0\end{cases}$$

and for $n \ge -1$,

$$\int_{y^*-y}^0 |y+t|^n dt \le \begin{cases} \mathcal{O}\left(\ln(1/\delta)\delta^{\gamma-1}\right)|y|^{-1} & \text{if } n = -1\\ \mathcal{O}\left(\delta^{(n+2)(\gamma-1)}\right)|y|^{-1} & \text{if } n \ge 0.\end{cases}$$

Proof. It is straightforward. ∎

6.2. Proof of Propositions 6.6 and 6.1 and Theorem 3.10

First we prove Proposition 6.6. We define the functional $\mathcal{K} = (\mathcal{K}_1,\mathcal{K}_2)$ as

$$\begin{aligned}\mathcal{K}_1(\hat g,\hat\xi) =\ & -A(y+\hat g,\tau) - 2B(y+\hat g,\tau)\left(\hat\xi + \eta(y^*,\tau+y^*-y)\right)\\ \mathcal{K}_2(\hat g,\hat\xi) =\ & \partial_x A(y+\hat g,\tau)\left(\hat\xi + \eta(y^*,\tau+y^*-y)\right)\\ & +\partial_x B(y+\hat g,\tau)\left(\hat\xi + \eta(y^*,\tau+y^*-y)\right)^2 + \partial_x C(y+\hat g,\tau))\end{aligned}$$

and $\bar{\mathcal{K}}_i = \mathcal{G}\circ\mathcal{K}_i$ for $i = 1,2$ (where $\mathcal{G}$ is the operator defined in (117)).

Lemma 6.9. *For any fixed $\gamma \in (1/3,1/2)$, there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$, the functional $\bar{\mathcal{K}}$ holds the following properties*

- *If* $(\hat{g},\hat{\xi})\in\hat{\mathcal{X}}$, $\bar{\mathcal{K}}(\hat{g},\hat{\xi})\in\hat{\mathcal{X}}$, *where* $\hat{\mathcal{X}}$ *is the Banach space in (121).*
- *There exists* $C_3>0$ *such that* $\llbracket\bar{\mathcal{K}}(0,0)\rrbracket\leq\frac{C_3}{2}\delta^{3\gamma-1}$.
- *For all* $(\hat{g}_1,\hat{\xi}_1),(\hat{g}_2,\hat{\xi}_2)\in B(C_3\delta^{3\gamma-1})\subset\hat{\mathcal{X}}$,
$$\llbracket\bar{\mathcal{K}}(\hat{g}_1,\hat{\xi}_1)-\bar{\mathcal{K}}(\hat{g}_2,\hat{\xi}_2)\rrbracket\leq\mathcal{O}(\delta^{3\gamma-1})\llbracket(\hat{g}_1,\hat{\xi}_1)-(\hat{g}_2,\hat{\xi}_2)\rrbracket.$$

Proof. The first statement is straightforward. For the second statement, since $\hat{g}=0$, we can apply Lemmas 6.2 for η and 6.7. Recalling that $\mathcal{O}(\ln(1/\delta))\leq|y|\leq\mathcal{O}(\delta^{\gamma-1})$ and $\gamma>1/3$,

$$|\mathcal{K}_1(0,0)|=|A(y,\tau)|+2|B(y,\tau)||\eta(y^*,\tau+y^*-y)|\leq\mathcal{O}(\delta^{2\gamma})+\mathcal{O}(\delta^2)|y|^2\leq\mathcal{O}(\delta^{2\gamma})$$

and

$$\begin{aligned}\left|\mathcal{K}_2(0,0)-\mathcal{L}\left(\frac{i\delta^2f_2(\tau)}{2(y+R^-(y,\tau))}\right)\right| = {} & |\partial_xA(y,\tau)||\eta(y^*,\tau+y^*-y)|+\\ & +|\partial_xB(y,\tau)||(\eta(y^*,\tau+y^*-y))|^2+\\ & +\left|\partial_xC(y,\tau)-\mathcal{L}\left(\frac{i\delta^2f_2(\tau)}{2(y+R^-(y,\tau))}\right)\right|\\ \leq {} & \mathcal{O}(\delta^{3+\gamma})+\mathcal{O}(\delta^4)|y|+\mathcal{O}(\delta^2)|y|^{-2}\leq\mathcal{O}(\delta^2)|y|^{-2}.\end{aligned}$$

Therefore, using that $\mathcal{G}\circ\mathcal{L}=\mathrm{Id}$ and Lemma 6.8:

$$\begin{aligned}|\bar{\mathcal{K}}_1(0,0)|&=|\mathcal{G}\circ\mathcal{K}_1(0,0)|\leq\mathcal{O}(\delta^{3\gamma-1})\\ |\bar{\mathcal{K}}_2(0,0)|&=|\mathcal{G}\circ\mathcal{K}_2(0,0)|\leq\mathcal{O}(\delta^2)\int_{y^*-y}^0|y+t|^{-2}dt+\left|\frac{i\delta^2f_2(\tau)}{2(y+R^-(y,\tau))}\right|\leq\mathcal{O}(\delta^2)|y|^{-1}.\end{aligned}$$

Thus, $\llbracket\bar{\mathcal{K}}(0,0)\rrbracket\leq\mathcal{O}(\delta^{3\gamma-1})$, and then there exists a constant $C_3>0$ such that the second statement holds.

Finally, for the third statement, we consider $(\hat{g}_1,\hat{\xi}_1),(\hat{g}_2,\hat{\xi}_2)\in B(C_3\delta^{3\gamma-1})\subset\hat{\mathcal{X}}$ and we use the mean value theorem. In order to apply it, we consider $\hat{g}_\rho=\rho\hat{g}_1+(1-\rho)\hat{g}_2$ for some $\rho\in(0,1)$, obtaining:

$$\begin{aligned}&|\mathcal{K}_1(\hat{g}_1,\hat{\xi}_1,y,\tau)-\mathcal{K}_1(\hat{g}_2,\hat{\xi}_2,y,\tau)|\leq\\ &\quad\leq|A(y+\hat{g}_1,\tau)-A(y+\hat{g}_2,\tau)|+2|B(y+\hat{g}_1,\tau)|\cdot|\hat{\xi}_1-\hat{\xi}_2|+\\ &\qquad+2|B(y+\hat{g}_1,\tau)-B(y+\hat{g}_2,\tau)|\cdot|\hat{\xi}_2+\eta(y^*,\tau+y^*-y)|+\\ &\quad\leq|\partial_xA(y+\hat{g}_\rho,\tau)|\cdot|\hat{g}_1-\hat{g}_2|+2|B(y+\hat{g}_1,\tau)||\hat{\xi}_1-\hat{\xi}_2|+\\ &\qquad+2|\partial_xB(y+\hat{g}_\rho,\tau)|\cdot|\hat{g}_1-\hat{g}_2|\cdot|\hat{\xi}_2+\eta(y^*,\tau+y^*-y)|\end{aligned}$$

and

$$\begin{aligned}&|\mathcal{K}_2(\hat{g}_1,\hat{\xi}_1,y,\tau)-\mathcal{K}_2(\hat{g}_2,\hat{\xi}_2,y,\tau)|\leq\\ &\quad\leq|\partial_xA(y+\hat{g}_1,\tau)|\cdot|\hat{\xi}_1-\hat{\xi}_2|+|\hat{\xi}_2+\eta(y^*,\tau+y^*-y)|\cdot|\partial_xA(y+\hat{g}_1,\tau)-\partial_xA(y+\hat{g}_2,\tau)|\\ &\qquad+|\partial_xB(y+\hat{g}_1,\tau)|\cdot|\hat{\xi}_1+\hat{\xi}_2+2\eta(y^*,\tau+y^*-y)|\cdot|\hat{\xi}_1-\hat{\xi}_2|\\ &\qquad+|\hat{\xi}_2+\eta(y^*,\tau+y^*-y)|^2\cdot|\partial_xB(y+\hat{g}_1,\tau)-\partial_xB(y+\hat{g}_2,\tau)|\\ &\qquad+|\partial_xC(y+\hat{g}_1,\tau)-\partial_xC(y+\hat{g}_2,\tau)|\end{aligned}$$

Now, using the bounds of Lemma 6.7 and recalling that for $i = 1, 2$, $|\hat{\xi}_i| = |y|^{-1}\mathcal{O}(\delta^2) \leq \mathcal{O}(\delta^2)$, $|y + \hat{g}_\rho| \leq \mathcal{O}(1)|y|$ and the definition of the norms:

$$
\begin{aligned}
&|\mathcal{K}_1(\hat{g}_1, \hat{\xi}_1, y, \tau) - \mathcal{K}_1(\hat{g}_2, \hat{\xi}_2, y, \tau)| \leq \\
&\qquad \leq \mathcal{O}(\delta^{1+\gamma})\llbracket \hat{g}_1 - \hat{g}_2 \rrbracket_1 + \mathcal{O}(\delta^{3-3\gamma})|y|\llbracket \hat{\xi}_1 - \hat{\xi}_2 \rrbracket_2 + \mathcal{O}(\delta^2)|y|\llbracket \hat{g}_1 - \hat{g}_2 \rrbracket_1 \\
&|\mathcal{K}_2(\hat{g}_1, \hat{\xi}_1, y, \tau) - \mathcal{K}_2(\hat{g}_2, \hat{\xi}_2, y, \tau)| \leq \\
&\qquad \leq \mathcal{O}(\delta^{1+\gamma})|\hat{\xi}_1 - \hat{\xi}_2| + \mathcal{O}(\delta^4)|\hat{g}_1 - \hat{g}_2| + \mathcal{O}(\delta^2)|y||\hat{\xi}_1 - \hat{\xi}_2| + \\
&\qquad\quad + \mathcal{O}(\delta^4)|\hat{g}_1 - \hat{g}_2| + \mathcal{O}(\delta^2)|y|^{-2}|\hat{g}_1 - \hat{g}_2| \\
&\qquad \leq \mathcal{O}(\delta^{4-2\gamma})|y|^{-1}\llbracket \hat{\xi}_1 - \hat{\xi}_2 \rrbracket_2 + \mathcal{O}(\delta^4)\llbracket \hat{g}_1 - \hat{g}_2 \rrbracket_1 + \\
&\qquad\quad + \mathcal{O}(\delta^{5-3\gamma})\llbracket \hat{\xi}_1 - \hat{\xi}_2 \rrbracket_2 + \mathcal{O}(\delta^2)|y|^{-2}\llbracket \hat{g}_1 - \hat{g}_2 \rrbracket_1.
\end{aligned}
$$

Moreover, taking into account that $\gamma \in (1/2, 1/3)$ and $|y| \leq \mathcal{O}(\delta^{\gamma-1})$, we obtain that

$$
\begin{aligned}
|\mathcal{K}_1(\hat{g}_1, \hat{\xi}_1, y, \tau) - \mathcal{K}_1(\hat{g}_2, \hat{\xi}_2, y, \tau)| &\leq \mathcal{O}(\delta^{2-2\gamma})\llbracket (\hat{g}_1, \hat{\xi}_1) - (\hat{g}_2, \hat{\xi}_2) \rrbracket \\
|\mathcal{K}_2(\hat{g}_1, \hat{\xi}_1, y, \tau) - \mathcal{K}_2(\hat{g}_2, \hat{\xi}_2, y, \tau)| &\leq \mathcal{O}(\delta^2)|y|^{-2}\llbracket (\hat{g}_1, \hat{\xi}_1) - (\hat{g}_2, \hat{\xi}_2) \rrbracket.
\end{aligned}
$$

Using these bounds, Lemma 6.8 and recalling the definition of the norms, we obtain the third statement. □

With the bounds of the previous lemma we can prove Proposition 6.6.

Proof of Proposition 6.6. Since $\bar{\mathcal{K}}$ is a contraction from $B(C_3\varepsilon^{3\gamma-1}) \subset \hat{\mathcal{X}}$ to itself, there exists a unique fixed point of this functional which is indeed a solution of (120) with the desired bounds. □

In order to prove Proposition 6.1, the original variables have to be recovered. So, we have to find the inverse change of variables of $x = y + \hat{g}(y, \tau)$.

Lemma 6.10. *Let us consider any constants $\sigma_2 < \sigma_1$ and $\gamma \in (1/3, 1/2)$. Then, for $(y, \tau) \in \mathcal{D}^{u(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$, the function $g(y, \tau) = y + \hat{g}(y, \tau)$ is invertible and its inverse can be written as $y = x + \hat{h}(x, \tau)$ and it is defined for $(x, \tau) \in \mathcal{D}^{u(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$. Moreover, it is $\mathcal{C}^1$ and holds $y = x + \hat{h}(x, \tau) \in \mathcal{D}^{u(2)}_{\delta,+}$.*

Proof. It can be seen that $\hat{h}$ is solution of the functional equation

$$\hat{h}(x, \tau) = \mathcal{F}(\hat{h})(x, \tau) = -\hat{g}(x + \hat{h}(x, \tau), \tau).$$

We prove its existence with a fixed point argument in the Banach Space:

$$\mathcal{Y} = \{\hat{h} : \mathcal{D}^{u(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2} \to \mathbb{C}, \text{continuous in } y \text{ and analytic in } \tau, \|\hat{h}\|_\infty < +\infty\}.$$

A bound of $\partial_y \hat{g}$ in $\mathcal{D}^{u(2)}_{\delta,+}$ will be required in the proof. Since we will obtain it from equation (113) (recall definition of $\mathcal{L}$ in (68)), we have to bound $\partial_\tau \hat{g}$. For this purpose, we reduce slightly the domain in τ taking any constant $\sigma_2 \in (0, \sigma_1)$ and we apply Cauchy estimates. Hence, we obtain that for $(y, \tau) \in \mathcal{D}^{u(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$,

$$|\partial_\tau \hat{g}(y, \tau)| \leq \mathcal{O}(1)|\hat{g}(y, \tau)| \leq \mathcal{O}(\delta^{3\gamma-1}).$$

Using (113) and the bounds obtained in Lemma 6.9, for $(y, \tau) \in \mathcal{D}^{u(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$,

$$|\partial_y \hat{g}(y, \tau)| \leq |\partial_\tau \hat{g}(y, \tau)| + |A(y + \hat{g}(y, \tau), \tau)| + 2|B(y + \hat{g}(y, \tau), \tau)||\xi(y, \tau)| \leq \mathcal{O}\left(\delta^{3\gamma-1}\right). \quad (123)$$

The bound obtained in Proposition 6.6 leads to $\|\mathcal{F}(0)\|_\infty \leq C_3\delta^{3\gamma-1}$. Moreover, considering $\hat{h}_1, \hat{h}_2 \in B(2C_3\delta^{3\gamma-1}) \subset \mathcal{Y}$, we take $\hat{h}_\rho = \rho\hat{h}_1 + (1-\rho)\hat{h}_2$ for $\rho \in (0,1)$, and thus

$$\|\mathcal{F}(\hat{h}_1) - \mathcal{F}(\hat{h}_2)\|_\infty \leq \|\partial_y \hat{g}(x + \hat{h}_\rho(x,\tau),\tau)\|_\infty \|\hat{h}_1 - \hat{h}_2\|_\infty \leq \mathcal{O}\left(\delta^{3\gamma-1}\right)\|\hat{h}_1 - \hat{h}_2\|_\infty.$$

Hence, this functional is a contraction from $B(2C_3\delta^{3\gamma-1}) \subset \mathcal{Y}$ to itself and thus, there exists a function $\hat{h}$ continuous in x such that $y = x + \hat{h}(x,\tau)$ is the inverse change of $x = g(y,\tau)$. In order to prove it is $\mathcal{C}^1$, it is enough to recall that considering (123), for $(y,\tau) \in \mathcal{D}^{u(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$,

$$\partial_y g(y,\tau) = 1 + \mathcal{O}\left(\delta^{3\gamma-1}\right) \neq 0$$

and therefore g is a diffeomorphism. □

Proof of proposition 6.1. From Lemma 6.10, Proposition 6.6 and (119), we can recover

$$\begin{aligned}\partial_x\Phi^-(x,\tau) &= \Psi(x,\tau) = \xi(x + \hat{h}(x,\tau),\tau) = \\ &= \hat{\xi}(x + \hat{h}(x,\tau),\tau) + \eta(x^*, \tau + x^* - x - \hat{h}(x,\tau)),\end{aligned}$$

for $(x,\tau) \in \mathcal{D}^{u(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$ and $\|\partial_x\Phi(x,\tau)\|_\infty \leq \mathcal{O}(\delta^2)$.

As we have observed in Remark 6.4, this function is the analytic continuation of $\partial_x\Phi^-$ to the inner domain.

Moreover, we can recover Φ taking

$$\forall (x,\tau) \in \mathcal{D}^{u(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2},\ \Phi(x,\tau) = \Phi(x^*,\tau) + \int_{x^*}^{x} \Psi(r,\tau)dr.$$

□

One can see that Cauchy estimates to $\partial_x\Phi^-$ in $\mathcal{D}^{u(4)}_{\delta,+}$ lead to

$$|\partial_x^2\Phi^-(x,\tau)| \leq \mathcal{O}(\delta^2 \ln^{-1}(1/\delta)). \tag{124}$$

However a better estimate will be required. To this end, an analogous fixed point procedure is needed. In fact, differentiating both sides of (110) and taking $\Theta(x,\tau) = \partial_x^2\Phi(x,\tau)$:

$$\begin{aligned}\tilde{\mathcal{L}}_2\Theta = \ & (2\partial_x A(x,\tau) + 4\partial_x B(x,\tau)\partial_x\Phi(x,\tau))\Theta + 2B(x,\tau)\Theta^2 + \\ & + \partial_x^2 A(x,\tau)\partial_x\Phi(x,\tau) + \partial_x^2 B(x,\tau)(\partial_x\Phi(x,\tau))^2 + \partial_x^2 C(x,\tau).\end{aligned} \tag{125}$$

where

$$\tilde{\mathcal{L}}_2 = \partial_\tau + (1 - A(x,\tau) - 2B(x,\tau)\partial_x\Phi(x,\tau))\partial_x. \tag{126}$$

Hence, we want to solve this partial differential equation with initial condition

$$\eta_2(x^*,\tau) = \partial_x^2\Phi^-(x^*,\tau) = \partial_x\left[\phi^-\left(x + R^-(x,\tau),\tau\right) - \phi_0^-\left(x + R^-(x,\tau),\tau\right)\right]\Big|_{x=x^*} \tag{127}$$

for $(x^*,\tau) \in \Gamma^{u(5)}_{\gamma,+} \times \mathbb{T}_{\sigma_2}$ which can be bounded analogously to Lemma 6.2 as

$$\eta_2(x^*,\tau) \leq \mathcal{O}\left(\delta^3 + \delta^{5-5\gamma}\right).$$

Proposition 6.11. *Let us fix any constants $\sigma_3 < \sigma_2$ and $\gamma \in (1/3, 1/2)$. Then, there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists an analytic function $\Theta(x, \tau)$ for $(x, \tau) \in \mathcal{D}^{u(6)}_{\delta,+} \times \mathbb{T}_{\sigma_3}$, which holds the partial differential equation (125) and the initial condition $\Theta(x^*, \tau) = \eta_2(x^*, \tau)$ for $(x^*, \tau) \in \Gamma^{u(5)}_{\gamma,+} \times \mathbb{T}_{\sigma_3}$. Moreover $\Theta = \partial_x^2 \Phi$ and $\|\Theta\|_\infty \leq |x|^{-1}\mathcal{O}(\delta^2 \ln^{-1}(1/\delta))$.*

Proof. We proceed as in Proposition 6.1. However, since the differential operator of the left hand side of the equation does not depend on Θ, it will be easier.

In fact, as a first step, we solve equation

$$\mathcal{L}\tilde{g} = -A(x, \tau) - 2B(x, \tau)\partial_x \Phi(x, \tau)|_{x=y+\tilde{g}(y,\tau)}$$

using Lemma 6.7, Proposition 6.1 and bound (124). Then, there exists a change of variables $x = y + \tilde{g}(y, \tau)$ defined in $(y, \tau) \in \mathcal{D}^{u(5)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$, which holds $x = y + \tilde{g}(y, \tau) \in \mathcal{D}^{u(4)}_{\delta,+}$ and conjugates the operator $\tilde{\mathcal{L}}_2$ defined in (126) to $\mathcal{L}$, and moreover holds $|\tilde{g}(y, \tau)| \leq \mathcal{O}(\delta^{2\gamma})$ and for $y^* \in \Gamma^{u(5)}_{\gamma,+}$, $\tilde{g}(y^*, \tau) = 0$.

Hence, performing this change to (125) and considering $\bar{\Theta}(y, \tau) = \Theta(y + \tilde{g}(y, \tau), \tau)$, we obtain an equation of the form

$$\mathcal{L}\bar{\Theta} = L(y, \tau)\bar{\Theta} + M(y, \tau)\bar{\Theta}^2 + N(y, \tau).$$

Thus, considering again the operator $\mathcal{G}$ defined in Lemma 6.5 and the initial condition η_2 in (127), a fixed point argument can be used in certain Banach space with norm $\llbracket \bar{\Theta} \rrbracket_1 = \|y\bar{\Theta}\|_\infty$ obtaining the wanted $\bar{\Theta}$ with bound $|\bar{\Theta}| \leq |y|^{-1}\mathcal{O}(\delta^2 \ln(1/\delta))$.
Proceeding as in Lemma 6.10 (and thus taking from now on $\tau \in \mathbb{T}_{\sigma_3}$ for any fixed $\sigma_3 < \sigma_2$), we can find the inverse change of $x = y + \tilde{g}(y, \tau)$ so that we recover function $\Theta(x, \tau)$ defined in $\mathcal{D}^{u(6)}_{\delta,+} \times \mathbb{T}_{\sigma_3}$ with the desired bound. Analogously to Remark 6.4, the solution thus obtained is analytic in its domain. □

With Propositions 6.1 and 6.11 is straightforward to prove Theorem 3.10:

Proof. We consider $x = z + S^-(z, \tau)$ the inverse change of variables of $z = x + R^-(x, \tau)$ defined in Theorem 3.8 which holds that for $(z, \tau) \in \mathcal{D}^{u(7)}_{\delta,+} \times \mathbb{T}_{\sigma_3}$, $x = z + S^-(z, \tau) \in \mathcal{D}^{u(6)}_{\delta,+}$. Hence:

$$\phi^-(z, \tau) = \phi_0^-(z, \tau) + \Phi^-\left(z + S^-(z, \tau), \tau\right)$$

and

$$\begin{aligned}
\partial_z(\phi - \phi_0)(z, \tau) = &\ \partial_x \Phi(z + S^-(z, \tau), \tau)\left(1 + \partial_z S^-(z, \tau)\right) \\
\partial_z^2(\phi - \phi_0)(z, \tau) = &\ \partial_x^2 \Phi(z + S^-(z, \tau), \tau)\left(1 + \partial_z S^-(z, \tau)\right)^2 \\
&+ \partial_x \Phi(z + S^-(z, \tau), \tau)\partial_z^2 S^-(z, \tau).
\end{aligned}$$

To end the proof of Theorem 3.10 it is enough to use the bounds of Propositions 6.1 and 6.11 and the bound of $\partial_z^i S$ in Theorem 3.8. □

7. First approximation of the change of the variables: proof of Theorem 3.12

The proof of theorem 3.12 is divided into two parts. First, in Proposition 7.1, we find the change of variables $u = w + \mathcal{C}^-(w,\tau)$ in the outer domain $D_\gamma^{u(9)}$. Later, in Proposition 7.2, we find its analytic continuation to the inner domains $D_{\delta,\pm}^{u(11)}$, using matching techniques as we did for the proof of the existence of the invariant manifolds.

During this section, γ will be any fixed constant $\gamma \in (1/3, 1/2)$.

Proposition 7.1. *There exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists a change of variables $u = w + \mathcal{C}^-(w,\tau)$ which conjugates $\tilde{\mathcal{L}}_\delta^-$ (in 71) to $\mathcal{L}_\delta = \delta^{-1}\partial_\tau + \partial_w$, such that for $(w,\tau) \in D_\gamma^{u(9)} \times \mathbb{T}_{\sigma_3}$, it holds that $u = w + \mathcal{C}^-(w,\tau) \in D_\gamma^{u(8)}$ and*

$$\begin{aligned}
&|\mathcal{C}^-(w,\tau)| \le \mathcal{O}(\delta^2)|w| + \mathcal{O}(\delta^{2-2\gamma})\\
&|\partial_w \mathcal{C}^-(w,\tau)| \le \mathcal{O}(\delta^{2-3\gamma})\\
&|\partial_w^2 \mathcal{C}^-(w,\tau)| \le \mathcal{O}(\delta^{2-4\gamma}).
\end{aligned}$$

Proposition 7.2. *There exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists an analytic continuation of the change of variables found in Proposition 7.1 to the inner domains $D_{\delta,\pm}^{u(10)}$, such that for $(w,\tau) \in D_{\delta,\pm}^{u(10)} \times \mathbb{T}_{\sigma_3}$, it holds that $u = w + \mathcal{C}^-(w,\tau) \in D_{\delta,\pm}^{u(9)}$ and for $j = 0, 1, 2$,*

$$|\partial_w^j \mathcal{C}^-(w,\tau)| \le \mathcal{O}(\delta^{1-j}\ln(1/\delta)^{-1-j}).$$

7.1. Change of variables in the outer domain: proof of Proposition 7.1

Due to Lemma 6.3, if we call $u = g(w,\tau)$ to the change that conjugates $\tilde{\mathcal{L}}_\delta^-$ and $\mathcal{L}_\delta$, the function g has to hold the following equation:

$$\mathcal{L}_\delta g(w,\tau) = \frac{\cosh^2 u}{4} \partial_u T^-(u,\tau)\Big|_{u=g(w,\tau)}. \tag{128}$$

Let us observe that

$$\frac{\cosh^2 u}{4}\partial_u T^-(u,\tau) = 1 + \delta \frac{\cosh^2 u}{4}\partial_u T_1(u,\tau) + \frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau)$$

where T_1 is the function defined in (37). Thus, by Theorem 3.6,

$$\frac{\cosh^2 u}{4}\partial_u T^-(u,\tau)$$

does not tend to 1 when $\Re u \to -\infty$. In fact, by Theorem 5.9 its limit as $\Re u \to -\infty$ is $1 - \delta f_1(\tau) + \lambda(\tau)$, where f_1 and λ are the functions defined in Proposition 3.5 and Theorem 5.9 respectively.

Hence, if we look for a change of the form $u = g(w,\tau) = g_0 w + g_1(\tau) + g_2(w,\tau)$ we obtain that

$$g_0 = 1 + \langle\lambda\rangle \tag{129}$$

and g_1 satisfies

$$\delta^{-1} g_1' = \lambda - \langle\lambda\rangle - \delta f_1(\tau) \text{ and } \langle g_1\rangle = 0. \tag{130}$$

Moreover, by the definitions of f_1 and λ stated in Proposition 3.5 and Theorem 5.9 respectively, one has that:

$$|g_0 - 1| \leq \mathcal{O}(\mu\delta^2) \tag{131}$$

$$|g_1(\tau)| \leq \mathcal{O}(\mu\delta^2). \tag{132}$$

Replacing these terms in (128), we look for a solution of

$$\mathcal{L}_\delta g_2 = \mathcal{H}(g_2, w, \tau) \tag{133}$$

where

$$\begin{aligned} &\mathcal{H}(g_2, w, \tau) = \\ &= \left(\frac{\cosh^2 u}{4}\partial_u Q^- - \lambda(\tau) + \delta(\tanh u + 1)f_1(\tau) + \frac{\delta}{\cosh u} f_2(\tau)\right)\Big|_{u=g_0 w + g_1(\tau) + g_2(w,\tau)}. \end{aligned} \tag{134}$$

In fact, we look for a fixed point of equation $g_2(w,\tau) = \bar{\mathcal{H}}(g_2) = \mathcal{G}_\delta(\mathcal{H}(g_2, w, \tau))$ where $\mathcal{G}_\delta$ is the operator defined in (88).

Although the change of variables g is not close to the identity, in the following lemma we will see that $u = g(w,\tau) \in D_\gamma^{(7)}$ provided $(w,\tau) \in D_\gamma^{(8)} \times \mathbb{T}_{\sigma_3}$.

Lemma 7.3. *Let us consider the change of variables* $u = g(w,\tau) = g_0 w + g_1(\tau) + g_2(w,\tau)$ *where* g_0 *and* g_1 *are the functions defined in (129) and (130) and* g_2 *is any function defined in* $(w,\tau) \in D_\gamma^{u(8)} \times \mathbb{T}_{\sigma_3}$ *verifying* $|g_2(w,\tau)| \leq \mathcal{O}(\delta^{2-2\gamma})$*. Then, there exists* $\delta_0 > 0$ *such that for* $\delta < \delta_0$, $u = g(w,\tau) \in D_\gamma^{u(7)}$ *provided* $(w,\tau) \in D_\gamma^{u(8)} \times \mathbb{T}_{\sigma_3}$.

Proof. Recalling that $g_0 \in \mathbb{R}$ and $\gamma < 2 - 2\gamma$, for $(w,\tau) \in D_\gamma^{u(8)} \times \mathbb{T}_{\sigma_3}$:

$$\begin{aligned} |\Im u| < |\Im g(w,\tau)| < \ & g_0|\Im w| + \mathcal{O}(\delta^{2-2\gamma}) < -\tan\beta_0\left(\Re(g_0 w) + a^{(8)}\delta^\gamma\right) + \frac{\pi}{2} + \mathcal{O}(\delta^{2-2\gamma}) \\ & < -\tan\beta_0\left(\Re u + a^{(7)}\delta^\gamma\right) + \frac{\pi}{2} \end{aligned}$$

and therefore $u = g(w,\tau) \in D_\gamma^{(7)}$. □

In order to prove the existence of the function g_2, we will prove that the operator $\bar{\mathcal{H}}$ is a contraction in a certain subset of the Banach space

$$\mathcal{Y}_1 = \{g_2(w,\tau) \,|\, g_2 : D_\gamma^{u(8)} \times \mathbb{T}_{\sigma_3} \to \mathbb{C}, \ \text{real-analytic}, \ \|g_2\|_1 < \infty\}$$

where $\|h\|_1 = \|\cosh(g_0 w) \cdot h\|_\infty$ and g_0 is the constant defined in (129).

Proposition 7.4. *There exists a constant* $C_4 > 0$ *such that* $\bar{\mathcal{H}}$ *is a contraction from* $B(C_4\delta^{2-\gamma}) \subset \mathcal{Y}_1$ *to itself. Therefore, there exists a function* $g_2 \in B(C_4\delta^{2-\gamma}) \subset \mathcal{Y}_1$ *which is a solution of (133).*

In order to prove this proposition some technical lemmas are required.

Lemma 7.5. *The operator* $\mathcal{G}_\delta$ *defined in (88) is linear and is well defined from* $\mathcal{Y}_1$ *to itself. Moreover, if* $h \in \mathcal{Y}_1$*, then*

$$\|\mathcal{G}_\delta(h)\|_1 \leq \mathcal{O}(1)\|h\|_1$$

Proof. Applying the third statement of Lemma 5.2 adapted to the norm $\|\cdot\|_1$ we are dealing with, we obtain that

$$\begin{aligned} |\cosh(g_0 w)\mathcal{G}_\delta(h)(w,\tau)| &\leq \left|\cosh(g_0 w)\int_{-\infty}^{0} h(w+t,\tau+\delta^{-1}t)dt\right| \\ &\leq \|h\|_1 \int_{-\infty}^{0}\left|\frac{\cosh(g_0 w)}{\cosh(g_0(w+t))}\right|dt \leq \mathcal{O}(1)\|h\|_1. \end{aligned}$$

As a consequence, if $h \in \mathcal{Y}_1$, then $\mathcal{G}_\delta(h) \in \mathcal{Y}_1$. □

Lemma 7.6. *The following inequality holds:* $\|\bar{\mathcal{H}}(0)\|_1 = \|\mathcal{G}_\delta(\mathcal{H}(0,\cdot,\cdot))\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma})$.

Proof. We split $\mathcal{H}$ defined in (134) in several terms in order to simplify the proof

- $\mathcal{H}_1(g_2,w,\tau) = \dfrac{\cosh^2 u}{4}\partial_u Q^-(u,\tau) - \lambda(\tau)\Big|_{u=g_0w+g_1(\tau)+g_2(w,\tau)}$
- $\mathcal{H}_2(g_2,w,\tau) = \delta(\tanh(g_0w+g_1(\tau)+g_2(w,\tau))+1)f_1(\tau)$
- $\mathcal{H}_3(g_2,w,\tau) = \delta f_2(\tau)(\cosh(g_0w+g_1(\tau))+g_2(w,\tau))^{-1}$.

First, using (132), it is straightforward to see that

$$\frac{\cosh(g_0w)}{\cosh(g_0w+g_1(\tau))} = \mathcal{O}(1) \text{ for } (w,\tau) \in D_\gamma^{u(8)} \times \mathbb{T}_{\sigma_3}.$$

Since we want to use the bounds obtained in Proposition 5.6 and Theorem 5.9, we split the domain $D_\gamma^{u(8)} = D_\gamma^{u,c(8)} \cup D_\gamma^{u,f(8)}$ (see (41) and (42)). For these domains we define two auxiliar norms

$$\|h\|_{1,c} = \|\cosh(g_0w)\cdot h\|_{\infty,c} \text{ and } \|h\|_{1,f} = \|\cosh(g_0w)\cdot h\|_{\infty,f},$$

which are the weighted supremum norm in the corresponding domains, in such a way that $\|h\|_1 = \sup\{\|h\|_{1,c}, \|h\|_{1,f}\}$.

Since $g_0w \in D_\gamma^{u(7)}$, then, by Proposition 5.6,

$$\begin{aligned} \|\mathcal{H}_1(0,w,\tau)\|_{1,c} \leq\ & \frac{1}{4}\left|\frac{1}{\cosh(g_0w)}\right|\left|\frac{\cosh(g_0w)}{\cosh(g_0w+g_1(\tau))}\right|^2\|\partial_u Q^-\|_{4,2,\sigma_1} + \\ & + \|\cosh(g_0w)\lambda(\tau)\|_{\infty,c} \leq \mathcal{O}(\mu\delta^{2-\gamma} \end{aligned}$$

and using Theorem 5.9

$$\|\mathcal{H}_1(0,w,\tau)\|_{1,f} \leq \left\|\cosh(g_0w)\left(\frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau) - \lambda(\tau)\Big|_{u=g_0w+g_1(\tau)}\right)\right\|_{\infty,f} \leq \mathcal{O}(\mu\delta^2)$$

and, applying Lemma 7.5, we obtain $\|\mathcal{G}_\delta(\mathcal{H}_1(0,w\tau))\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma})$.

For the other two terms integrating by parts, considering 2π-periodic functions $\hat{f}_i(\tau)$ such that $\hat{f}_i' = f_i$ and $\langle \hat{f}_i \rangle = 0$, and using the properties of g_0 and g_1:

$$\begin{aligned}
\mathcal{G}_\delta \mathcal{H}_2(0,w,\tau) = {} & \delta \int_{-\infty}^{0} (\tanh(g_0(w+t) + g_1(\tau+\delta^{-1}t)) + 1) f_1(\tau + \delta^{-1}t) dt \\
= {} & \delta^2 (\tanh(g_0 w + g_1(\tau)) + 1)\hat{f}_1(\tau) - \delta^2 \mathcal{G}_\delta \left(\frac{g_0 + \delta^{-1} g_1'(\tau)}{\cosh^2(g_0 w + g_1(\tau))} \hat{f}_1(\tau) \right) \\
\mathcal{G}_\delta \mathcal{H}_3(0,w,\tau) = {} & \delta \int_{-\infty}^{0} \frac{1}{\cosh(g_0(w+t) + g_1(\tau + \delta^{-1}t))} f_2(t + \delta^{-1}t) dt \\
= {} & \delta^2 \frac{1}{\cosh(g_0 w + g_1(\tau))} \hat{f}_2(\tau) + \\
& + \delta^2 \mathcal{G}_\delta \left(\frac{(g_0 + \delta^{-1} g_1'(\tau)) \sinh(g_0 w + g_1(\tau))}{\cosh^2(g_0 w + g_1(\tau))} \hat{f}_2(\tau) \right).
\end{aligned}$$

Hence, using that $|\hat{f}_i(\tau)| \leq \mathcal{O}(\mu)$, statement (132) and Lemma 7.5,

$$\begin{aligned}
& \|\mathcal{G}_\delta \mathcal{H}_2(0,w,\tau)\|_1 \leq \mathcal{O}(\mu\delta^2) + \mathcal{O}(\delta^2) \left\| \frac{g_0 + \delta^{-1} g_1'(\tau)}{\cosh^2(g_0 w + g_1(\tau))} \hat{f}_1(\tau) \right\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma}) \\
& \|\mathcal{G}_\delta \mathcal{H}_3(0,w,\tau)\|_1 \leq \mathcal{O}(\mu\delta^2) + \\
& \qquad + \mathcal{O}(\delta^2) \left\| \frac{(g_0 + \delta^{-1} g_1'(\tau)) \sinh(g_0 w + g_1(\tau))}{\cosh^2(g_0 w + g_1(\tau))} \hat{f}_2(\tau) \right\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma}).
\end{aligned}$$

□

Lemma 7.7. *For all $g_2, \tilde{g}_2$ such that $\|g_2\|_1, \|\tilde{g}_2\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma})$, it holds that*

$$\|\mathcal{G}_\delta \mathcal{H}(g_2, w, \tau) - \mathcal{G}_\delta \mathcal{H}(\tilde{g}_2, w, \tau)\|_1 \leq \mathcal{O}(\mu\delta^{1-2\gamma}) \|g_2 - \tilde{g}_2\|_1.$$

Proof. In order to apply the Mean Value Theorem, we split $\mathcal{H}$ again and we define $g_2^\rho = \rho g_2 + (1-\rho)\tilde{g}_2$ and $u^\rho = g_0 w + g_1(\tau) + g_2^\rho(w,\tau)$ for $\rho \in (0,1)$. In fact, it is clear that there exists $\rho \in (0,1)$ such that,

$$\begin{aligned}
\mathcal{H}_1(g_2, w, \tau) - \mathcal{H}_1(\tilde{g}_2, w, \tau) &= \partial_u \Big(\frac{\cosh^2 u}{4} \partial_u Q^- \Big) \Big|_{u=u^\rho} (g_2(w,\tau) - \tilde{g}_2(w,\tau)) \\
\mathcal{H}_2(g_2, w, \tau) - \mathcal{H}_2(\tilde{g}_2, w, \tau) &= \frac{\delta f_1(\tau)}{\cosh^2 u^\rho} (g_2(w,\tau) - \tilde{g}_2(w,\tau)) \\
\mathcal{H}_3(g_2, w, \tau) - \mathcal{H}_3(\tilde{g}_2, w, \tau) &= \frac{\delta f_2(\tau) \sinh u^\rho}{\cosh^2 u^\rho} (g_2(w,\tau) - \tilde{g}_2(w,\tau)),
\end{aligned}$$

where we have used $u^\rho \in D_\gamma^{u(7)}$ for $(w,\tau) \in D_\gamma^{u(8)} \times \mathbb{T}_{\sigma_3}$. Applying Proposition 5.6, we obtain:

$$\begin{aligned}
& \|\mathcal{H}_1(g_2,\cdot,\cdot) - \mathcal{H}_1(\tilde{g}_2,\cdot,\cdot)\|_1 \leq \\
& \quad \leq \left(\left\| \frac{\cosh^2 u^\rho}{4} \partial_u^2 Q^-(u^\rho,\tau) \right\|_\infty + \left\| \frac{\cosh u^\rho \sinh u^\rho}{2} \partial_u Q^-(u^\rho,\tau) \right\|_\infty \right) \|g_2 - \tilde{g}_2\|_1 \\
& \quad \leq \mathcal{O}(\mu\delta^{2-3\gamma}) \|g_2 - \tilde{g}_2\|_1 \\
& \|\mathcal{H}_2(g_2,\cdot,\cdot) - \mathcal{H}_2(\tilde{g}_2,\cdot,\cdot)\|_1 \leq \left\| \frac{\delta f_1(\tau)}{\cosh^2 u^\rho} \right\|_\infty \|g_2 - \tilde{g}_2\|_1 \leq \mathcal{O}(\mu\delta^{1-2\gamma}) \|g_2 - \tilde{g}_2\|_1 \\
& \|\mathcal{H}_3(g_2,\cdot,\cdot) - \mathcal{H}_3(\tilde{g}_2,\cdot,\cdot)\|_1 \leq \left\| \frac{\delta f_2(\tau) \sinh u^\rho}{\cosh^2 u^\rho} \right\|_\infty \|g_2 - \tilde{g}_2\|_1 \leq \mathcal{O}(\mu\delta^{1-2\gamma}) \|g_2 - \tilde{g}_2\|_1.
\end{aligned}$$

Thus, to prove the lemma it is enough to join all the bounds and apply Lemma 7.5. □

Thus, we are ready to prove Proposition 7.4.

Proof of Proposition 7.4. Joining the results of Lemmas 7.6 and 7.7, the conclusion is that there exists a constant $C_4 > 0$ such that $\|\bar{\mathcal{H}}(0)\|_1 \leq C_4\mu\delta^{2-\gamma}/2$ and for all $g_2, \tilde{g}_2 \in B(C_4\mu\delta^{2-\gamma})$, $\|\bar{\mathcal{H}}(g_2) - \bar{\mathcal{H}}(\tilde{g}_2)\|_1 \leq \mathcal{O}(\mu\delta^{1-2\gamma})\|g_2 - \tilde{g}_2\|_1$. Thus, reducing δ if it is necessary, $\bar{\mathcal{H}}$ is a contraction from $B(C_4\mu\delta^{2-\gamma}) \subset \mathcal{Y}_1$ to itself, and therefore it has a unique fixed point. □

Hence, for the proof of Proposition 7.1 it is sufficient to consider $\mathcal{C}^-(w,\tau) = (g_0 - 1)w + g_1(\tau) + g_2(w,\tau)$. Moreover, the first bound holds and we only need to reduce slightly the domain to $\mathcal{D}_\gamma^{u(9)}$ and apply Cauchy estimates to obtain the bounds of the derivatives.

7.2. Change of variables in the inner domain: proof of Proposition 7.2

In the last subsection we obtained a change of variables $\mathcal{C}^-$ such that conjugates $\tilde{\mathcal{L}}_\delta^-$ to $\mathcal{L}_\delta$ in the outer domain $D_\gamma^{u(9)}$. We will now see that there exists an analytical continuation of it to the inner domains $D_{\delta,\pm}^{u(9)}$. Since the procedure is analogous for both regions, we will only deal with $D_{\delta,+}^{u(9)}$.

Proceeding as in the outer domain, we know that the change $u = w + g(w,\tau)$ has to hold (128) for $(w,\tau) \in D_{\delta,\pm}^{u(9)} \times \mathbb{T}_{\sigma_3}$. This equation will be solved using complex matching techniques, like we did in Section 6. For this purpose we consider inner variables and as initial condition in a curve $\Gamma_{\gamma,+}^{u(9)}$ (see (100)) we take the change of variables $u = w + \mathcal{C}^-(w,\tau)$ already obtained in the outer domain. Then, performing a characteristics-like method, we will obtain the continuation of this change to the inner domains which will be analytic *a posteriori*.

In order to translate equation (128) to inner variables, we consider

$$z = \delta^{-1}\left(u - i\frac{\pi}{2}\right) \quad x = \delta^{-1}\left(w - i\frac{\pi}{2}\right) \quad g_{\mathrm{i}}(x,\tau) = \delta^{-1}g\left(i\frac{\pi}{2} + \delta x, \tau\right). \tag{135}$$

Thus, the change in inner variables is given by $z = x + g_{\mathrm{i}}(x,\tau)$, which is a solution of

$$\mathcal{L}g_{\mathrm{i}}(x,\tau) = \frac{\cosh^2(i\frac{\pi}{2} + \delta z)}{4\delta^2}\partial_z\phi^-(z,\tau) - 1\Big|_{z=x+g_{\mathrm{i}}(x,\tau)}, \tag{136}$$

where $\mathcal{L} = \partial_\tau + \partial_x$.

We will consider as a first approximation of g_{i} the change of variables $z = x + R^-(x,\tau)$ given by Theorem 3.8. Since this change conjugates the operators $\bar{\mathcal{L}}$ defined in (61) and $\mathcal{L}$, using Lemma 6.3, the function R^- holds the partial differential equation

$$\partial_\tau R^-(x,\tau) + \partial_x R^-(x,\tau) + 1 = -\frac{1}{4}z^2\partial_z\phi_0^-(z,\tau)\Big|_{z=x+R^-(x,\tau)}.$$

We will look for a function g_{i} of the form $g_{\mathrm{i}}(x,\tau) = R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau)$ with $|\hat{g}_{\mathrm{i}}| \leq \mathcal{O}(\delta^{\nu_0})$ for certain $\nu_0 > 0$. As a consequence, using (53), $z = x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau) \in \mathcal{D}_{\delta,+}^{(8)}$ provided $(x,\tau) \in \mathcal{D}_{\delta,+}^{u(9)} \times \mathbb{T}_{\sigma_3}$.

Hence, we will look for a solution of the following equation:

$$\mathcal{L}\hat{g}_{\rm i}(x,\tau) = F_1(\hat{g}_{\rm i}(x,\tau),x,\tau) + F_2(\hat{g}_{\rm i}(x,\tau),x,\tau) \tag{137}$$

where

$$\begin{aligned} F_1(\hat{g}_{\rm i}(x,\tau),x,\tau) = &\ \frac{\cosh^2(i\frac{\pi}{2}+\delta z)}{4\delta^2}(\partial_z\phi^-(z,\tau)-\partial_z\phi_0^-(z,\tau))+ \\ &+\Big(\frac{\cosh^2(i\frac{\pi}{2}+\delta z)}{4\delta^2}+\frac{z^2}{4}\Big)\partial_z\phi_0^-(z,\tau)\Big|_{z=x+R^-(x,\tau)+\hat{g}_{\rm i}(x,\tau)} \\ = &\ -2b(z,\tau)(\partial_z\phi^-(z,\tau)-\partial_z\phi_0^-(z,\tau))-a(z,\tau)\Big|_{z=x+R^-(x,\tau)+\hat{g}_{\rm i}(x,\tau)} \\ F_2(\hat{g}_{\rm i}(x,\tau),x,\tau) = &\ -\frac{z^2}{4}\partial_z\phi_0^-(z,\tau)\Big|_{z=x+R^-(x,\tau)+\hat{g}_{\rm i}(x,\tau)}+\frac{z^2}{4}\partial_z\phi_0^-(z,\tau)\Big|_{z=x+R^-(x,\tau)} \\ = &\ d(x+R^-(x,\tau),\tau)-d(x+R^-(x,\tau)+\hat{g}_{\rm i}(x,\tau),\tau) \end{aligned}$$

and a and b are the functions defined in (102) and (103) and $d(z,\tau) = \dfrac{z^2}{4}\partial_z\phi_0^-(z,\tau)$.

As in Section 6, in order to solve equation (137) we will use we know the initial condition in a curve $\Gamma^{u(9)}_{\gamma,+}$ defined in (100).

Lemma 7.8. *For all $(x^*,\tau)\in\Gamma^{u(9)}_{\gamma,+}\times\mathbb{T}_{\sigma_3}$, $|\hat{g}_{\rm i}(x^*,\tau)|\le\mathcal{O}(\delta^{1-2\gamma})$.*

Proof. We consider the change obtained in Proposition 7.1 and we express it in inner variables using (135):

$$z = x+\delta^{-1}\mathcal{C}^-(i\pi/2+\delta x,\tau).$$

Thus for $(x^*,\tau)\in\Gamma^{u(9)}_{\gamma,+}\times\mathbb{T}_{\sigma_3}$, since these points belong to the intersection between the inner and the outer domains, $R^-+\hat{g}_{\rm i}$ must coincide with the function $\delta^{-1}\mathcal{C}^-$ and so $\hat{g}_{\rm i}(x^*,\tau)=\delta^{-1}\mathcal{C}^-(i\frac{\pi}{2}+\delta x^*,\tau)-R^-(x^*,\tau)$. Finally, to obtain the statement of the lemma it is enough to use the bounds obtained in Proposition 7.1 and Theorem 3.8. □

With this initial condition, we write equation (137) as an integral equation using the operator $\mathcal{G}$ defined in Lemma 6.5 and defining a new operator $\mathcal{F}_{\rm i}$:

$$\hat{g}_{\rm i}(x,\tau) = \mathcal{F}_{\rm i}(\hat{g}_{\rm i}) = \mathcal{G}\left(F_1(\hat{g}_{\rm i}(x,\tau),x,\tau)+F_2(\hat{g}_{\rm i}(x,\tau),x,\tau)\right)+\hat{g}_{\rm i}(x^*,\tau+x^*-x), \tag{138}$$

where we have used $\mathcal{L}\hat{g}_{\rm i}(x^*,\tau+x^*-x)=0$. We will solve this equation with a fixed point argument in the Banach space:

$$\mathcal{Y}_\infty = \{\hat{g}_{\rm i} : \mathcal{D}^{u(9)}_{\delta,+}\times\mathbb{T}_{\sigma_3}\to\mathbb{C},\text{continuous},\|\hat{g}_{\rm i}\|_\infty\le+\infty\}.$$

Lemma 7.9. *The operator $\mathcal{G}$ defined in Lemma 6.5 is linear from $\mathcal{Y}_\infty$ to itself, and*

$$\|\mathcal{G}(h)\|_\infty\le\mathcal{O}(\delta^{\gamma-1})\|h\|_\infty.$$

Proof. It is straightforward recalling that $y^*-y=\mathcal{O}(\delta^{\gamma-1})$. □

Lemma 7.10. *There exists $C_5>0$ and $\nu_0>0$ such that the following inequalities hold*

1. $\|\mathcal{F}_{\rm i}(0)\|_\infty\le\frac{C_5}{2}\delta^{\nu_0}$.

2. *For all* $\hat{g}_{\rm i}^1, \hat{g}_{\rm i}^2 \in B(C_5\delta^{\nu_0}) \subset \mathcal{Y}_\infty$, $\|\mathcal{F}_{\rm i}(\hat{g}_{\rm i}^1) - \mathcal{F}_{\rm i}(\hat{g}_{\rm i}^2)\|_\infty \leq \mathcal{O}(\ln^{-1}(1/\delta))\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty$.

Proof. Since $F_2(0,x,\tau) = 0$, $\mathcal{F}_{\rm i}(0) = \mathcal{G}(F_1(0,x,\tau)) + \hat{g}_{\rm i}(x^*, \tau + x^* - x)$. Moreover, using the functions $A(x,\tau)$ and $B(x,\tau)$ defined in (106) and (107):

$$F_1(0,x,\tau) = -2B(x,\tau)(\partial_z\phi^- - \partial_z\phi_0^-)\Big|_{z=x+R^-(x,\tau)} - A(x,\tau).$$

Thus, using the bounds obtained in Lemma 6.7 and Theorem 3.10, it holds that $\|F_1(0,x,\tau)\|_\infty \leq \mathcal{O}(\delta^{2\gamma})$. Therefore, applying Lemmas 7.8 and 7.9, and taking $\nu_0 = \min\{1-2\gamma, 3\gamma - 1\}$,

$$\|\mathcal{F}_{\rm i}(0)\|_\infty \leq \mathcal{O}(\delta^{\gamma-1})\|F_1(0,x,\tau)\|_\infty + |\hat{g}_{\rm i}(x^*, \tau + x^* - x)| \leq \mathcal{O}(\delta^{\nu_0}).$$

For the second statement,

$$\|\mathcal{F}_{\rm i}(\hat{g}_{\rm i}^1) - \mathcal{F}_{\rm i}(\hat{g}_{\rm i}^2)\|_\infty \leq \|\mathcal{G}(F_1(\hat{g}_{\rm i}^1,x,\tau) - F_1(\hat{g}_{\rm i}^2,x,\tau))\|_\infty + \|\mathcal{G}(F_2(\hat{g}_{\rm i}^1,x,\tau) - F_2(\hat{g}_{\rm i}^1,x,\tau))\|_\infty.$$

Thus, we study each term separately. For the first one, we consider $\hat{g}_{\rm i}^\rho = \rho\hat{g}_{\rm i}^1 + (1-\rho)\hat{g}_{\rm i}^2$ in order to apply the mean value theorem:

$$\begin{aligned} F_1(\hat{g}_{\rm i}^1,x,\tau) - F_1(\hat{g}_{\rm i}^2,x,\tau) = \left(\hat{g}_{\rm i}^1(x,\tau) - \hat{g}_{\rm i}^2(x,\tau)\right)\Big(&-2\partial_z b(z,\tau)\left(\partial_z\phi^-(z,\tau) - \partial_z\phi_0^-(z,\tau)\right) \\ &-2b(z,\tau)\left(\partial_z^2\phi^-(z,\tau) - \partial_z^2\phi_0^-(z,\tau)\right) - \partial_z a(z,\tau)\Big)\Big|_{z=x+R^-(x,\tau)+\hat{g}_{\rm i}^\rho(x,\tau)}. \end{aligned}$$

Therefore, applying again the bounds of Lemma 6.7 and Theorem 3.10, since $\gamma < 1/2$,

$$\|F_1(\hat{g}_{\rm i}^1,x,\tau) - F_1(\hat{g}_{\rm i}^2,x,\tau)\|_\infty \leq \mathcal{O}(\delta^{2\gamma}\ln^{-2}(1/\delta))\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty,$$

and applying Lemma 7.9,

$$\|\mathcal{G}(F_1(\hat{g}_{\rm i}^1,x,\tau) - F_1(\hat{g}_{\rm i}^2,x,\tau))\|_\infty \leq \mathcal{O}(\delta^{3\gamma-1}\ln^{-2}(1/\delta))\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty.$$

For the second term, using Theorem 3.10, we have $\partial_z d(z,\tau) = \mathcal{O}(z^{-2})$. Considering again $\hat{g}_{\rm i}^\rho$:

$$|\partial_z d(x + R^-(x,\tau) + \hat{g}_{\rm i}^\rho(x,\tau)| \leq \mathcal{O}(1)\frac{1}{|x|^2}.$$

Therefore, applying mean value theorem

$$\|F_2(\hat{g}_{\rm i}^1,x,\tau) - F_2(\hat{g}_{\rm i}^2,x,\tau)\|_\infty \leq \mathcal{O}(1)|x|^{-2}\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty.$$

Hence, considering Lemma 6.8 we obtain,

$$\begin{aligned} \|\mathcal{G}(F_2(\hat{g}_{\rm i}^1,x,\tau) - F_2(\hat{g}_{\rm i}^2,x,\tau))\|_\infty &\leq \mathcal{O}(1)\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty \int_{x^*-x}^0 |x+t|^{-2}dt \\ &\leq \mathcal{O}(\ln^{-1}(1/\delta))\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty, \end{aligned}$$

and considering the bound of each term

$$\|\mathcal{F}_{\rm i}(\hat{g}_{\rm i}^1) - \mathcal{F}_{\rm i}(\hat{g}_{\rm i}^2)\|_\infty \leq \mathcal{O}(\ln^{-1}(1/\delta))\|\hat{g}_{\rm i}^1 - \hat{g}_{\rm i}^2\|_\infty$$

that is the second statement of the lemma. □

Thus, now we are ready to prove Proposition 7.2, *Proof of Proposition 7.2.* The bounds obtained in Lemma 7.10 give us that $\mathcal{F}_{\rm i}$ is a contraction from $B(C_5\delta^{\nu_0}) \subset \mathcal{Y}_\infty$ to itself. Hence, there exists a unique fixed point $\hat{g}_{\rm i}$ which holds $|\hat{g}_{\rm i}(x,\tau)| \leq \mathcal{O}(\delta^{\nu_0})$.

Therefore, we have obtained the desired change of variables, which is given in the inner variables by

$$g_{\rm i}(x,\tau) = R^-(x,\tau) + \hat{g}_{\rm i}(x,\tau)$$

and then $\|g_{\rm i}\|_\infty \leq \mathcal{O}(\ln^{-1}(1/\delta))$. Moreover, reasoning as in Section 6, it can be guaranteed that it is analytic. Translating it to outer variables, for $w \in D^{u(9)}_{\delta,+}$

$$\mathcal{C}^-(w,\tau) = \delta g_{\rm i}\left(\delta^{-1}\left(w - i\frac{\pi}{2}\right), \tau\right)$$

and then $|\mathcal{C}^-(w,\tau)| \leq \mathcal{O}(\delta \ln^{-1}(1/\delta))$. For the bounds of the derivatives is enough to reduce the domain to $D^{u(10)}_{\delta,+}$ and apply Cauchy estimates. □

8. The global change of variables

Considering the change of variables $u = w + \mathcal{C}^-(w,\tau)$ defined in Theorem 3.12, we will find in this section the change of variables which conjugates $\tilde{\mathcal{L}}_\delta$ and $\mathcal{L}_\delta$.

Recalling that the operator $\tilde{\mathcal{L}}_\delta$ is defined by

$$\tilde{\mathcal{L}}_\delta = \tilde{\mathcal{L}}_\delta^- + \Big(\frac{\cosh^2 u}{8}(\partial_u T^+(u,\tau) - \partial_u T^-(u,\tau))\Big)\partial_u$$

and applying the change of variables $u = w + \mathcal{C}^-(w,\tau)$ defined in Theorem 3.12 we see that this change conjugates $\tilde{\mathcal{L}}_\delta$ with

$$\delta^{-1}\partial_\tau + \partial_w + H(w,\tau)\partial_w, \tag{139}$$

where

$$H(w,\tau) = \frac{1}{1+\partial_w \mathcal{C}^-(w,\tau)}\Big(\frac{\cosh^2 u}{8}\left(\partial_u T^+(u,\tau) - \partial_u T^-(u,\tau)\right)\Big)\Big|_{u=w+\mathcal{C}^-(w,\tau)}. \tag{140}$$

Thus, we look for a new change of variables $w = v + \mathcal{C}(v,\tau)$ which conjugates (139) to $\mathcal{L}_\delta$ for $(w,\tau) \in D^{(10)} \times \mathbb{T}_{\sigma_4}$ where $D^{(10)}$ is the domain defined in (65) where both manifolds are defined. Moreover, we will need $|\mathcal{C}| \leq \mathcal{O}(\delta^{1+\nu_1})$ for certain $\nu_1 > 0$, since, from (44) and (49), this will guarantee that this change of variables is well defined from $D^{(11)}$ to $D^{(10)}$, and also from $D^{(11)}_\gamma$ to $D^{(10)}_\gamma$ and from $D^{(11)}_{\delta,\pm}$ to $D^{(10)}_{\delta,\pm}$.

Applying Lemma 6.3, we will look for a function $\mathcal{C}$ solution of the partial differential equation

$$\mathcal{L}_\delta \mathcal{C} = H(v + \mathcal{C}(v,\tau),\tau). \tag{141}$$

In this way, the change which will conjugate $\tilde{\mathcal{L}}_\delta$ with $\mathcal{L}_\delta$ will be the composition of both changes, namely,

$$u = v + \mathcal{C}(v,\tau) + \mathcal{C}^-(v + \mathcal{C}(v,\tau),\tau) = v + \mathcal{U}(v,\tau).$$

8.1. Banach spaces and technical lemmas

In order to prove the existence of this change of variables we will work with the following Banach spaces

$$\mathcal{Z}_\infty = \{\mathcal{C}(v,\tau) \mid \mathcal{C} : D^{(11)} \times \mathbb{T}_{\sigma_4} \to \mathbb{C} \text{ real-analytic, } \|\mathcal{C}\|_{\infty,\sigma_4} < \infty\} \quad (142)$$

$$\mathcal{Z}_1 = \{\mathcal{C}(v,\tau) \mid \mathcal{C} : D^{(11)} \times \mathbb{T}_{\sigma_4} \to \mathbb{C} \text{ real-analytic, } \|\mathcal{C}\|_{1,\sigma_4} < \infty\} \quad (143)$$

where $D^{(11)}$ is the domain defined in (65) and $\|\cdot\|_{1,\sigma_4}$ is a Fourier norm defined analogously to the norm $\|\cdot\|_{1,c,\sigma_4}$ in Section 5.1 but in this new domain.

In the following lemmas we will consider also the norms

$$\|\cdot\|_{1,\sigma_4,\mathrm{o}} \text{ and } \|\cdot\|_{1,\sigma_4,\mathrm{i}}, \quad (144)$$

which are the norm $\|\cdot\|_{1,\sigma_4}$ restricted to the inner and outer parts of the domain $D^{(11)}$ defined in (66).

We note that σ_4 is any real number such that $0 < \sigma_4 < \sigma_3$. This reduction of domain has been considered in order to apply the following lemma.

Lemma 8.1. *Let σ_4 be any number such that $0 < \sigma_4 < \sigma_3$, the following bound holds*

$$\begin{array}{ll} \|\partial_w \mathcal{C}^-(w,\tau)\|_{\infty,\mathrm{o},\sigma_4} \le \mathcal{O}(\delta^{2-3\gamma}) & \|\partial_w^2 \mathcal{C}^-(w,\tau)\|_{\infty,\mathrm{o},\sigma_4} \le \mathcal{O}(\delta^{2-4\gamma}) \\ \|\partial_w \mathcal{C}^-(w,\tau)\|_{\infty,\mathrm{i},\sigma_4} \le \mathcal{O}(\ln^{-2}(1/\delta)) & \|\partial_w \mathcal{C}^-(w,\tau)\|_{\infty,\mathrm{i},\sigma_4} \le \mathcal{O}(\delta^{-1}\ln^{-3}(1/\delta)). \end{array} \quad (145)$$

Proof. It is enough to apply the third statement of Lemma 5.1 (adapted to the domain $D^{(11)}$) to the bounds obtained in Theorem 3.12. □

Corollary 8.2. *Let $\nu_1 > 0$ be any positive constant and $\mathcal{C}$ a function defined in $D^{(11)}$ holding $|\mathcal{C}| \le \mathcal{O}(\delta^{1+\nu_1})$. Then, for $u = v + \mathcal{C}(v,\tau) + \mathcal{C}^-(v + \mathcal{C}(v,\tau),\tau)$ where $\mathcal{C}^-$ is the function found in Theorem 3.12, the following bound holds:*

$$\left|\frac{\cosh v}{\cosh u}\right| \le \mathcal{O}(1). \quad (146)$$

In order to find a solution of (141) we need to consider a right inverse of the operator $\mathcal{L}_\delta$.

Lemma 8.3. *Let us consider $\rho^* = \pi/2 - c^{(11)}\delta\ln(1/\delta)$. For all $f \in \mathcal{Z}_1$ we consider the operator $\mathcal{G}_\delta^{\rho^*}$, defined acting on the Fourier coefficients of f:*

$$\mathcal{G}_\delta^{\rho^*}(f)^{[k]}(v) = \begin{cases} \displaystyle\int_{-i\rho^*}^{v} e^{ik\delta^{-1}(r-v)} f^{[k]}(r)\,dr & \text{if } k < 0 \\ \displaystyle\int_0^v f^{[k]}(r)\,dr & \text{if } k = 0 \\ \displaystyle -\int_v^{i\rho^*} e^{ik\delta^{-1}(r-v)} f^{[k]}(r)\,dr & \text{if } k > 0 \end{cases}$$

Then, $\mathcal{G}_\delta^{\rho^*}$ is well defined from $\mathcal{Z}_1$ to $\mathcal{Z}_\infty$ and has the following properties,

1. *$\mathcal{G}_\delta^{\rho^*}$ is linear.*
2. *$\mathcal{L}_\delta \circ \mathcal{G}_\delta^{\rho^*} = \mathrm{Id}$.*
3. *If δ is small enough, $\|\mathcal{G}_\delta^{\rho^*}(f)\|_{\infty,\sigma_4} \le \mathcal{O}(\ln(1/\delta))\|f\|_{1,\sigma_4}$.*

Proof. The first two statements are straightforward. For the third one, we consider as a path of integration the line between points v and $\pm i\rho^*$, so that $-k\Im(r-v) \leq 0$ and therefore, for all $k \in \mathbb{Z}$:

$$\left| e^{ik\delta^{-1}(r-v)} \right| \leq 1.$$

We deal only with the case $k < 0$ since the other ones are analogous. Bounding the integral:

$$\left| \mathcal{G}_\delta^{\rho^*}(f)^{[k]}(v) \right| \leq \| f^{[k]}(v) \|_1 \left| \int_{-i\rho^*}^{v} \frac{1}{|\cosh r|} dr \right|.$$

Since $|\cosh t|^{-1}$ has poles at $\pm i\pi/2$, we split the integral whether $\Im r > 0$ or $\Im r < 0$. In fact, for the case $\Im v > 0$ we consider r_0 as the value in the path of integration such that $\Im r_0 = 0$, we split the path of integration as $[-i\rho^*, r_0]$ and $[r_0, v]$ and we bound each integral.

$$\begin{aligned} \left| \int_{-i\rho^*}^{v} \frac{1}{|\cosh r|} dr \right| &\leq \mathcal{O}(1) \left| \int_{-i\rho^*}^{r_0} \frac{1}{|r + i\pi/2|} dr + \int_{r_0}^{v} \frac{1}{|r - i\pi/2|} dr \right| \\ &\leq \mathcal{O}(1) \sup_{r \in [-i\rho^*, r_0]} |\ln |r + i\pi/2|| + \mathcal{O}(1) \sup_{r \in [r_0, v]} |\ln |r - i\pi/2|| \\ &\leq \mathcal{O}(\ln(1/\delta)). \end{aligned}$$

For the case $\Im v < 0$ the integral can be bounded directly as the first integral in the previous case. Both bounds give the third statement.

The third statement guarantees that $\mathcal{G}_\delta^{\rho^*}(f)$ is analytic. Thus, in order to check that $\mathcal{G}_\delta^{\rho^*}(f) \in \mathcal{Z}_\infty$ it only remains to check that it is real-analytic. To prove that, it is enough to use the reflection principle. □

8.2. Proof of Theorem 3.13

If we want a solution of equation (141), it is enough to find a solution of

$$\mathcal{C}(v, \tau) = \mathcal{G}_\delta^{\rho^*}(F_2(\mathcal{C}, v, \tau)) \tag{147}$$

where $\mathcal{G}_\delta^{\rho^*}$ is the operator given in Lemma 8.3,

$$F_2(\mathcal{C}, v, \tau) = H(v + \mathcal{C}(v, \tau), \tau) \tag{148}$$

and H is the function defined in (140). We will find this function $\mathcal{C}$ through a fixed point argument.

Proposition 8.4. *There exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists a function $\mathcal{C}(v, \tau)$ defined in $D^{(11)} \times \mathbb{T}_{\sigma_4}$, which is a fixed point of the functional*

$$\bar{F}_2 = \mathcal{G}_\delta^{\rho^*} \circ F_2. \tag{149}$$

The next lemma gives some bounds which will be needed.

Lemma 8.5. *There exists δ_0, $\nu_1 > 0$, $\nu_2 > 0$ such that for all $\delta \in (0, \delta_0)$, the functional F_2 has the following properties*

1. *There exists $C_6 > 0$ such that $\|F_2(0, \cdot, \cdot)\|_{1,\sigma_4} \leq \dfrac{C_6}{2} \delta^{1+\nu_1}$.*

2. For all $\mathcal{C}_1, \mathcal{C}_2 \in B(C_6\delta^{1+\nu_1}) \subset \mathcal{Z}_\infty$,

$$\|F_2(\mathcal{C}_1,\cdot,\cdot) - F_2(\mathcal{C}_2,\cdot,\cdot)\|_{1,\sigma_4} \le \mathcal{O}(\delta^{\nu_2})\|\mathcal{C}_1 - \mathcal{C}_2\|_{\infty,\sigma_4}.$$

Proof. For the first statement, we recall that,

$$F_2(0,v,\tau) = (1+\partial_w\mathcal{C}^-(w,\tau))^{-1}\left(\frac{\cosh^2 u}{8}(\partial_u T^+ - \partial_u T^-)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}\right).$$

Using bound (145), it is clear that $\|\left(1+\partial_w\mathcal{C}^-(w,\tau)\right)^{-1}\|_{\infty,\sigma_4} \le \mathcal{O}(1)$. In order to bound the second factor, we consider the auxiliar norms stated in (144). Recalling that $T^+ - T^- = Q^+ - Q^-$ and applying the bounds obtained in Theorem 3.6:

$$\|F_2(0,\cdot,\cdot)\|_{1,\sigma_4,\mathrm{o}} \le \mathcal{O}(1)\|\partial_u Q^+ - \partial_u Q^-\|_{3,c,\sigma_4} \le \mathcal{O}(\mu\delta^{2-\gamma}).$$

In the inner domain,

$$\begin{aligned}(\partial_u T^+ - \partial_u T^-)(u,\tau) = \; & \delta^{-2}(\partial_z\phi^+ - \partial_z\phi_0^+)(\delta^{-1}(u-i\pi/2)) \\ & -\delta^{-2}(\partial_z\phi^- - \partial_z\phi_0^-)(\delta^{-1}(u-i\pi/2)) \\ & +\delta^{-2}(\partial_z\phi_0^+ - \partial_z\phi_0^-)(\delta^{-1}(u-i\pi/2)).\end{aligned} \tag{150}$$

Applying Theorem 3.8,

$$\|\cosh^2 u\delta^{-2}(\partial_z\phi_0^+ - \partial_z\phi_0^-)(\delta^{-1}(u-i\pi/2))\|_{1,\sigma_4,\mathrm{i}} \le \mathcal{O}(\delta)|z|^3 e^{\Im z} \le \mathcal{O}(\delta^{1+c^{(12)}}\ln^3(1/\delta)) \tag{151}$$

since $|z|^3 e^{\Im z}$ reaches its maximum for the biggest imaginary part of the inner domain. Thus, applying this bound and Theorem 3.10,

$$\|F_2(0,\cdot,\cdot)\|_{1,\sigma_4,\mathrm{i}} \le \mathcal{O}(\delta^{3\gamma} + \delta^{1+c^{(11)}}\ln^3(1/\delta)) \tag{152}$$

and thus, taking ν_1 such that $0 < \nu_1 < \min\{3\gamma - 1, c^{(11)}\}$

$$\|F_2(0,\cdot,\cdot)\|_{1,\sigma_4} \le \mathcal{O}(\delta^{1+\nu_1}).$$

For the second statement, consider $\mathcal{C}^\rho = \rho\mathcal{C}_1 + (1-\rho)\mathcal{C}_2$ for $\rho \in (0,1)$, we recall that

$$F_2(\mathcal{C}_1,v,\tau) - F_2(\mathcal{C}_2,v,\tau) = \partial_w H(v+\mathcal{C}^\rho(v,\tau))(\mathcal{C}_1(v,\tau) - \mathcal{C}_2(v,\tau))$$

where

$$\begin{aligned}\partial_w H(w,\tau) = \; & \frac{-\partial_w^2\mathcal{C}^-(w,\tau)}{(1+\partial_w C^-(w,\tau))^2}\left(\frac{\cosh^2 u}{8}(\partial_u T^+ - \partial_u T^-)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}\right) \\ & +\frac{\cosh u\sinh u}{4}(\partial_u T^+ - \partial_u T^-)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)} \\ & +\frac{\cosh^2 u}{8}(\partial_u^2 T^+ - \partial_u^2 T^-)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}.\end{aligned}$$

We bound it in different ways in the outer and inner domains. For the outer one, recalling again that $\partial_u^j T^+ - \partial_u^j T^- = \partial_u^j Q^+ - \partial_u^j Q^-$ and applying the bounds obtained in Lemma 8.1, Corollary 8.2 and Theorem 3.6,

$$\|\partial_w H(v+\mathcal{C}^\rho(v,\tau),\tau)\|_{1,\mathrm{o},\sigma_4} \le \mathcal{O}(\delta^{2-2\gamma}).$$

For the inner domain, applying Lemma 8.1, using (150) and

$$\begin{aligned}(\partial_u^2 T^+ - \partial_u^2 T^-)(u,\tau) = \;& \delta^{-3}(\partial_z^2\phi^+ - \partial_z^2\phi_0^+)(\delta^{-1}(u-i\pi/2))\\ & -\delta^{-3}(\partial_z^2\phi^- - \partial_z^2\phi_0^-)(\delta^{-1}(u-i\pi/2))\\ & +\delta^{-3}(\partial_z^2\phi_0^+ - \partial_z^2\phi_0^-)(\delta^{-1}(u-i\pi/2))\end{aligned}$$

and proceeding analogously to (151),

$$\|\partial_w H(v+\mathcal{C}^\rho(v,\tau),\tau)\|_{1,\mathrm{i},\sigma_4} \le \mathcal{O}(\delta^{3\gamma-1}\ln^{-2}(1/\delta) + \delta^{c^{(11)}}\ln^3(1/\delta)).$$

Therefore, taking ν_2 such that $0<\nu_2<\min\{2-2\gamma, 3\gamma-1, c^{(11)}\}$,

$$\|F_2(\mathcal{C}_1,v,\tau) - F_2(\mathcal{C}_2,v,\tau)\|_{1,\sigma_4} \le \mathcal{O}(\delta^{\nu_2})\|\mathcal{C}_1-\mathcal{C}_2\|_{\infty,\sigma_4}.$$

□

With this lemma we are ready to prove Proposition 8.4.

Proof of Proposition 8.4. The results obtained in Lemmas 8.3 and 8.5 give that the operator $\bar{F}_2$ defined in (149) is a contraction from $B(C_6\delta^{\nu_1})\subset\mathcal{Z}_\infty$ to itself. Hence, it has a fixed point $\mathcal{C}$ which holds that for $(w,\tau)\in D^{(11)}\times\mathbb{T}_{\sigma_4}$

$$|\mathcal{C}(w,\tau)| \le \mathcal{O}(\delta^{1+\nu_1}).$$

□

Having proved the proposition, we can now prove Theorem 3.13.

Proof of Theorem 3.13. It is sufficient to define $\mathcal{U}(v,\tau)=v+\mathcal{C}(v,\tau)+\mathcal{C}^-(v+\mathcal{C}(v,\tau),\tau)$ and consider the bounds of Theorem 3.12 and Proposition 8.4.

For the bounds of the derivatives it is enough to reduce the domain to $D^{(12)}$ and apply Cauchy estimates to the function $\mathcal{C}$ in the outer and inner domains of $D^{(12)}$, and then use again the bounds obtained in Theorem 3.12. □

9. Proof of Theorem 2.10

In this section we give the complete proof of Theorem 2.10 following the same scheme that was presented in Section 3 to prove Theorem 2.6. However, since in this case we are only looking for an upper bound of the splitting of separatrices, the proof will be easier. In fact, we will only need to study the invariant manifolds $T^\pm(u,\tau)$ defined in (31) in the outer domains (see (43)). The existence of these manifolds is obtained in Theorem 9.2 and its proof is analogous to the proof of Theorem 3.6 in Section 5. Later, we consider the equation $\tilde{\mathcal{L}}_\delta\Delta T=0$ verified by the difference between the manifolds (see (67)) and in Theorem 9.3 we look for a change of variables $\mathcal{U}$ that straightens $\tilde{\mathcal{L}}_\delta$ in the intersection of the outer domain. This procedure is analogous to the one presented in Section 8. In this case, we are not going to consider the inner equation nor the matching procedure.

In this Section we restrict ourselves to $\mu=\varepsilon^p$ with $p\in(-4,0)$ and $\mu\in I_i$ (see Remark 2.2) and thus μ tends to infinity as ε tends to 0. On the other hand, we recall that the existence of the periodic orbit has been proved in Proposition 4.1, which has the following corollary.

Corollary 9.1. *For any $p < 0$, there exists $\varepsilon_0 > 0$ such that for any s_1 holding $s_1\varepsilon^{-p} < \sigma_0(\varepsilon^p)$ (see (84)), system (4) with $\mu = \varepsilon^p$ has a periodic orbit $x_p(\tau)$ defined in $\mathbb{T}_{\sigma_1}$ with $\sigma_1 = s_1\varepsilon^{-p}$, which holds*

$$\|x_p(\tau) + \mu \sin \tau\|_{\sigma_1} \leq \mathcal{O}(\varepsilon^2) \tag{153}$$

where $\|\cdot\|_{\sigma_1}$ is the Fourier norm defined in Section 4.

From now on, we restrict ourselves to $p \in (-4, 0)$. On the other hand, we want to point out that through this section we will have to reduce once again the analyticity strip in τ, and therefore we will take $\sigma_2 = s_2\varepsilon^{-p}$ for any $s_2 > 0$ such that $\sigma_2 < \sigma_1$.

As we did in Section 1, we perform the change of variables (14) in order to translate the periodic orbit to the origin, and we obtain the Hamilton-Jacobi equation (25). Considering changes (29), (30) and (31) as in Section 3, we obtain again equation (33) with the corresponding asymptotic conditions (34).

We recall that in order to perform these changes, it is needed $J_0(\mu) > 0$. As it has been explained in Remark 2.2, since for μ big,

$$J_0(\mu) \sim \sqrt{\frac{2}{\pi\mu}} \cos\left(\mu - \frac{\pi}{4}\right)$$

(see [AS92]), the zeros $\mu_0 < \mu_1 < \dots$ of $J_0(\mu)$ are separated when $\mu \to +\infty$. Thus, from now on, for any fixed $p < 0$, we will consider ε arbitrarily small but defined in compact intervals such that $\mu = \varepsilon^p \in I_i$ for $i \in \mathbb{N}$, in such a way that $\cos(\mu - \pi/4)$ has a positive lower bound independent of ε and, therefore, $J_0(\mu) > 0$ holds. On the other hand, in these intervals I_i we have the following bounds

$$|J_0(\varepsilon^p)| \leq \mathcal{O}\left(\varepsilon^{-\frac{p}{2}}\right) \quad \text{and} \quad |J_0(\varepsilon^p)|^{-1} \leq \mathcal{O}\left(\varepsilon^{\frac{p}{2}}\right). \tag{154}$$

Therefore, in this case, the parameter $\delta = \varepsilon\sqrt{J}$ is not of the same order as ε but smaller: $\delta = \mathcal{O}(\varepsilon^{1-\frac{p}{4}})$. Hence, in terms of δ, bounds (154) read

$$|J_0(\varepsilon^p)| \leq \mathcal{O}\left(\delta^{-\frac{2p}{4-p}}\right) \quad \text{and} \quad |J_0(\varepsilon^p)|^{-1} \leq \mathcal{O}\left(\delta^{\frac{2p}{4-p}}\right). \tag{155}$$

On the other hand, δ will have to be chosen such that $\mu = (\delta/\sqrt{J})^p \in I_i$ for $i \in \mathbb{N}$. Thus, all the results stated from now on will hold provided this condition holds.

Next step is to prove the existence of the stable and unstable invariant manifolds in the outer domains (see (43)). We proceed as in Section 5 looking for solutions of the Hamilton-Jacobi equation (33) with the corresponding asymptotic conditions (34).

Theorem 9.2. *Let p and γ be real numbers such that $p \in (-4, 0)$ and $\gamma \in (0, \nu(p))$ where*

$$\nu(p) = \frac{4+p}{4-p} > 0. \tag{156}$$

Then, there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ holding $\mu = (\delta/\sqrt{J})^p \in I_i$, in such a way that $J_0(\mu) > 0$, equation (33) has unique (module an additive constant) solutions in $D_\gamma^{u(1)} \times \mathbb{T}_{\sigma_1}$ and $D_\gamma^{s(1)} \times \mathbb{T}_{\sigma_1}$, with $\sigma_1 = s_1\varepsilon^{-p}$, of the form

$$T^\pm(u,\tau) = T_0(u) + W^\pm(u,\tau)$$

holding asymptotic conditions (34).

Moreover, there exists a real constant $d_1 > 0$ independent of δ such that, for $i = 0, 1$,

$$\|\partial_u^{i+1} W^-\|_{3,2,\sigma_1} \leq d_1 \delta^{\nu(p)-i\gamma} \ln(1/\delta)$$

where $\|\cdot\|_{3,2,\sigma_1}$ is the Fourier norm defined in Section 5.1. Analogous bounds for W^+ also hold.

Proof of Theorem 9.2. We deal only with the case of the unstable manifold. Analogously to the proof of Theorem 3.6 in Section 5, we prove the existence of the invariant manifold using a fixed point argument, but in this case for $W = T - T_0$ in the Banach space $\mathcal{E}_{3,2}$ defined in (86) with $\sigma_1 = s_1\varepsilon^{-p}$.

Thus, we replace $W = T - T_0$ in equation (33) and, using the operator $\bar{\mathcal{G}}_\delta$ defined in Lemma 5.5, we obtain the following equation

$$\partial_u W = \bar{\mathcal{F}}(\partial_u W) = \bar{\mathcal{G}}_\delta \left(\mathcal{F}(\partial_u W)\right)$$

where

$$\mathcal{F}(h) = -\frac{\cosh^2 u}{8} h^2 + \frac{2}{J\cosh^2 u}\left(\cos x_p(\tau) - J\right) + \frac{1}{J}\psi(u)\sin x_p(\tau).$$

Although the width σ_1 of the strip of analyticity in τ depends on ε, the properties of operator $\bar{\mathcal{G}}_\delta$ stated in Lemma 5.5 still hold. On the other hand, it is straightforward to see that the operator $\bar{\mathcal{F}}$ is well defined from $\mathcal{E}_{3,2}$ to itself. Thus, in order to prove the existence of a fixed point of $\bar{\mathcal{F}}$ we have to obtain bounds analogous to the ones in Lemma 5.7.

First, we express bound (153) in terms of δ,

$$\left\|x_p(\tau) + \varepsilon^p \sin\tau\right\|_{\sigma_1} \leq \mathcal{O}\left(\delta^{\frac{8}{4-p}}\right), \tag{157}$$

with $\sigma_1 = s_1\varepsilon^{-p} = s_1(\delta/\sqrt{J})^{-p}$ for any s_1 such that $s_1\varepsilon^{-p} < \sigma(\mu)$.

We split $\mathcal{F}(0)(u,\tau) = p_1(u,\tau) + p_2(u,\tau)$ with

$$p_1(u,\tau) = \frac{2}{J\cosh^2 u}\left(\cos x_p(\tau) - \cos(\varepsilon^p \sin\tau)\right) + \frac{1}{J}\psi(u)\left(\sin x_p(\tau) + \sin(\varepsilon^p\sin\tau)\right)$$

and

$$p_2(u,\tau) = \frac{2}{J\cosh^2 u}\left(\cos(\varepsilon^p\sin\tau) - J\right) - \frac{1}{J}\psi(u)\sin(\varepsilon^p\sin\tau).$$

Using bounds (155) and (157), we obtain

$$\|p_1(u,\tau)\|_{3,2,\sigma_1} \leq \mathcal{O}\left(\delta^{\frac{2(p+4)}{4-p}}\right)$$

and applying Lemma 5.5, we have that $\|\bar{\mathcal{G}}(p_1)\|_{3,2,\sigma_1}$ has the same bound.

The term $\bar{\mathcal{G}}(p_2) = \partial_u \mathcal{G}(p_2)$ has to be bounded more carefully. First, we recall that p_2 has zero average, thus proceeding as in Lemma 5.7, we use that $\bar{\mathcal{G}}(p_2) = \mathcal{G}(\partial_u p_2)$. On the other hand, using bound (155) and that for $\tau \in \mathbb{T}_{\sigma(\mu)}$ (see (84)) $|\sin(\mu \sin\tau)| \leq \mathcal{O}(1)$ and $|\cos(\mu\sin\tau)| \leq \mathcal{O}(1)$ (see Section 4), we have that

$$\|\partial_u p_2^{[k]}(u)\|_{3,2} \leq e^{-|k|\sigma(\mu)} \mathcal{O}\left(\delta^{\frac{2p}{4-p}}\right)$$

Therefore using this bound and (91), we have that

$$\begin{aligned}\|\mathcal{G}\left(\partial_u p_2\right)\|_{3,2,\sigma_1} &\leq \mathcal{O}(\delta) \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathcal{O}\left(\delta^{\frac{2p}{4-p}}\right) \frac{1}{|k|} e^{|k|(\sigma_1-\sigma(\mu))} \leq \mathcal{O}\left(\delta^{\frac{4+p}{4-p}}\right) \ln\left(1-e^{|k|(\sigma_1-\sigma(\mu))}\right) \\ &\leq \mathcal{O}\left(\delta^{\nu(p)} \ln(1/\delta)\right)\end{aligned}$$

Therefore, one can see that there exists a constant $d_1 > 0$ such that

$$\|\bar{\mathcal{F}}(0)\|_{3,2,\sigma_1} \leq d_1 \delta^{\nu(p)} \ln(1/\delta)/2.$$

Taking now $h_1, h_2 \in B(d_1\delta^{\nu(p)} \ln(1/\delta)) \subset \mathcal{E}_{3,2}$,

$$\|\bar{\mathcal{F}}(h_1) - \bar{\mathcal{F}}(h_2)\|_{3,2,\sigma_1} \leq \mathcal{O}(\delta^{\nu(p)-\gamma}) \|h_1 - h_2\|_{3,2,\sigma_1}.$$

Since $\gamma < \nu(p)$ and reducing δ if necessary, $\bar{\mathcal{F}}$ is contractive from $B(d_1\delta^{\nu(p)}) \subset \mathcal{E}_{3,2}$ to itself. Thus, reasoning as in the proof of Proposition 5.6, we obtain the existence of the invariant manifold $T^-(u,\tau)$ in $D_\gamma^{u(0)}$. Finally, reducing the domain to $D_\gamma^{u(1)}$, we get the wanted bounds for the second derivative. □

The last step is to bound the difference between the invariant manifolds. We consider $\Delta T(u,\tau) = T^+(u,\tau) - T^-(u,\tau)$ which holds $\tilde{\mathcal{L}}_\delta \Delta T = 0$ where $\tilde{\mathcal{L}}_\delta$ is the operator defined in (67). We recall that ΔT is defined in $D_\gamma^{(1)} = D_\gamma^{u(1)} \cap D_\gamma^{s(1)}$ which is of the form

$$D_\gamma^{(i)} = \left\{ u \in \mathbb{C} : |\Im u| + \tan\beta_0 |\Re u| < \frac{\pi}{2} - \tan\beta_0 a^{(i)} \delta^\gamma \right\}.$$

As we have done in Section 8 we look for a change of variables that conjugates the operator $\tilde{\mathcal{L}}_\delta$ to $\mathcal{L}_\delta$ (see (68)) in order to apply Lemma 3.11. As we are working in the outer domain, where the invariant manifolds $T^\pm$ are well approximated by T_0, the change $\mathcal{U}$ we look for is close to the identity.

Theorem 9.3. *Let s_1 be the constant defined in Theorem 9.2, and p and γ be real numbers such that $p \in (-4, 0)$ and $\gamma \in (0, \nu(p))$ where $\nu(p)$ has been defined in (156). Then, there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists a real-analytic function $\mathcal{U}(v,\tau)$ in $D_\gamma^{(3)} \times \mathbb{T}_{\sigma_1}$, with $\sigma_1 = s_1\varepsilon^{-p}$, such that the change*

$$(u,\tau) = (v + \mathcal{U}(v,\tau), \tau)$$

conjugates the operators $\tilde{\mathcal{L}}_\delta$ and $\mathcal{L}_\delta$ defined in (67) and (68). Moreover, for $(v,\tau) \in D_\gamma^{(3)} \times \mathbb{T}_{\sigma_1}$,

(i) $v + \mathcal{U}(v,\tau) \in D_\gamma^{(2)}$.

(ii) $|\partial_v^j \mathcal{U}(v,\tau)| \leq \mathcal{O}(\delta^{\nu(p)-j\gamma} \ln^2(1/\delta))$ *for* $j = 0, 1, 2$.

There exists the inverse change of variables $(v,\tau)=(u+\mathcal{V}(u,\tau),\tau)$ which is real-analytic and for $(u,\tau)\in D_\gamma^{(4)}\times\mathbb{T}_{\sigma_1}$ holds:

(i) $u+\mathcal{V}(u,\tau)\in D_\gamma^{(3)}$.

(ii) $|\partial_u^j\mathcal{V}(u,\tau)|\leq\mathcal{O}(\delta^{\nu(p)-j\gamma}\ln^2(1/\delta))$ *for* $j=0,1,2$.

Proof. Applying Lemma 6.3, function $\mathcal{U}$ holds

$$\mathcal{L}_\delta\mathcal{U}=\frac{\cosh^2 u}{8}\left(\partial_u W^+(u,\tau)+\partial_u W^-(u,\tau)\right)\Big|_{u=v+\mathcal{U}(v,\tau)}. \tag{158}$$

We find $\mathcal{U}$ as in Section 8. We consider the Banach spaces $\mathcal{Z}_\infty$ and $\mathcal{Z}_1$ (see (142) and (143)) for functions defined in $D_\gamma^{(2)}\times\mathbb{T}_{\sigma_1}$ with $\sigma_1=s_1\varepsilon^{-p}$.

Recalling the operator $\mathcal{G}_\delta^{\rho^*}$ defined in Lemma 8.3 with $\rho^*=\pi/2-a^{(2)}\delta^\gamma$, one can see that Lemma 8.3 still holds. Thus, in order to prove the theorem, it is enough to look for a fixed point of the functional

$$\tilde{\mathcal{H}}=\mathcal{G}_\delta^{\rho^*}\circ\mathcal{H}$$

where

$$\mathcal{H}(\mathcal{U})=\frac{\cosh^2 u}{8}\left(\partial_u W^+(u,\tau)+\partial_u W^-(u,\tau)\right)\Big|_{u=v+\mathcal{U}(v,\tau)}.$$

We will need that $|\mathcal{U}|\leq\mathcal{O}(\delta^s)$ with $s>\gamma$ in order to guarantee that this change is well defined from $D_\gamma^{(2)}$ to $D_\gamma^{(1)}$, in such a way that $\mathcal{H}(\mathcal{U})$ can be defined.

Considering the bounds obtained in Theorem 9.2 and Lemma 8.3, it is straightforward to see that there exists $d_2>0$ such that

$$\left\|\tilde{\mathcal{H}}(0)\right\|_{\infty,\sigma_1}\leq\mathcal{O}\left(\ln(1/\delta)\right)\|\mathcal{H}(0)\|_{1,\sigma_1}\leq\frac{d_2}{2}\delta^{\nu(p)}\ln^2(1/\delta).$$

Thus, taking $h_1,h_2\in B(d_2\delta^{\nu(p)}\ln^2(1/\delta))\subset\mathcal{Z}_\infty$ we have that

$$\begin{aligned}\left\|\tilde{\mathcal{H}}(h_2)-\tilde{\mathcal{H}}(h_1)\right\|_{\infty,\sigma_1} &\leq \mathcal{O}\left(\ln(1/\delta)\right)\|\mathcal{H}(h_2)-\mathcal{H}(h_1)\|_{1,\sigma_1}\\ &\leq \mathcal{O}\left(\delta^{\nu(p)-\gamma}\ln^2(1/\delta)\right)\|h_1-h_2\|_{\infty,\sigma_1}.\end{aligned}$$

Recalling that $\nu(p)-\gamma>0$ and reducing δ if necessary, $\tilde{\mathcal{H}}$ is a contraction from $B(d_2\delta^{\nu(p)}\ln^2(1/\delta))\subset\mathcal{Z}_\infty$ to itself and therefore there exists a unique fixed point $\mathcal{U}$ of equation (158) which gives the change of variables. In order to obtain the bounds of the derivatives it is enough to reduce the domain to $D_\gamma^{(3)}$ and apply Cauchy estimates.

To obtain the existence and bounds of the inverse change, one has to use again a fixed point argument. □

With the existence of this change of variables, we are ready to prove Theorem 2.10.

Proof of Theorem 2.10. The last statement of this theorem follows applying Lemma 3.11 to the function

$$\zeta(v,\tau)=\Delta T(v+\mathcal{U}(v,\tau),\tau)$$

which verifies $\mathcal{L}_\delta\zeta=0$.

We observe that,

$$\partial_v^2\zeta(v,\tau)=\partial_u^2\Delta T(v+\mathcal{U}(v,\tau),\tau)\partial_v\left(1+\mathcal{U}(v,\tau)\right)^2+\partial_u\Delta T(v+\mathcal{U}(v,\tau),\tau)\partial_v^2\mathcal{U}(v,\tau).$$

Recalling that $\Delta T=W^+-W^-$ and using the bounds obtained in Theorems 9.2 and 9.3, it is straightforward to see that for $(v,\tau)\in\overline{D_\gamma^{(4)}}\times\overline{\mathbb{T}_{\sigma_2}}$, with $\sigma_2=s_2\varepsilon^{-p}$ for any $s_2<s_1$,

$$\left|\partial_v^2\zeta(v,\tau)\right|\leq\mathcal{O}(\delta^{\nu(p)-4\gamma}).$$

Thus applying Lemma 3.11 with $r=\pi/2-a^{(4)}\delta^\gamma$ and $M_r=\mathcal{O}(\delta^{\nu(p)-4\gamma})$, we obtain that for $v\in D_\gamma^{(4)}\cap\mathbb{R}$ and $\tau\in\mathbb{T}$

$$\left|\partial_v^2\zeta(v,\tau)\right|\leq\mathcal{O}\left(\delta^{\nu(p)-4\gamma}e^{-\frac{1}{\delta}\left(\frac{\pi}{2}-a^{(4)}\delta^\gamma\right)}\right)$$

and analogous bounds for $|\partial_v\zeta(v,\tau)|$ and $|\zeta(v,\tau)-\langle\zeta\rangle|$. Therefore, using the inverse change of variables $v=u+\mathcal{V}(u,\tau)$ obtained in Theorem 9.3, recalling that $\delta=\varepsilon\sqrt{J}$,

$$\delta^{\nu(p)}\leq\mathcal{O}\left(\varepsilon^{1+\frac{p}{4}}\right)$$

(where $\kappa(p)$ is the function defined in Theorem 2.10) and taking $\bar\gamma=\gamma(4-p)/4>0$, we obtain the bounds of Theorem 2.10, after performing the changes (31), (30), (29) and (14). □

10. Conclusions

In this paper we present some results and some open questions related to a classical problem: to obtain rigorously the measure of the splitting of separatrices of the rapidly forced pendulum

$$\ddot{x}=\sin x+\frac{\mu}{\varepsilon^2}\sin\frac{t}{\varepsilon}$$

where $\varepsilon>0$ is a small parameter.

This problem was considered firstly by Holmes, Marsden and Scheurle in [HMS88] and solved in the case that $\mu=\varepsilon^p$ with $p>10$. Basically, they showed that the prediction given by the Melnikov formula in this problem is exponentially small due to two facts: the homoclinic orbit of the pendulum has complex singularities (poles, in this case) and the perturbation is periodic with a small period $2\pi\varepsilon$. Then, they showed that this prediction is valid for $p>10$.

Since then, several authors have tried (and achieved) to weaken the hypothesis of the size of the perturbation, that is, to decrease the values of p, and give a rigorous proof of the splitting of the invariant manifolds arising from the periodic orbit of the system.

In this paper we present three results about this problem.

First, accordingly to [Gel00], we rigorously show that the prediction of the Melnikov formula is valid if $\mu=\varepsilon^p$ with $p>1/2$ and we give an alternative formula for the splitting when $0\leq p\leq 1/2$. It is important to stress that, if $0<p\leq 1/2$ the alternative formula can be obtained if we apply the Melnikov method to the system obtained after

two steps of averaging. Of course, after this averaging, the "unperturbed system" is changed having a new homoclinic orbit with different singularities. This makes that the splitting has a different exponential size.

An important feature of the proof presented here relies on the fact that, as noticed in [Sau01, LMS03] being the perturbed pendulum a Hamiltonian system, the invariant manifolds can be written as graphs of the differential of certain functions $S^{\pm}$. Moreover, these functions verify the so called Hamilton-Jacobi equation, which is a partial differential equation of first order. To obtain these functions, matching techniques in the complex plane are used to obtain different approximations for them. In particular, we use some results about the so called "inner equation" in [OSS03].

It is important to stress that the methods used in this paper can be extended to any trigonometric polynomial Hamiltonian perturbation of the pendulum using the results about the corresponding inner equation in [Bal06].

Second, we show that the problem still has sense below the so called "singular case" $\mu = O(1)$. We see that the system under consideration has a periodic orbit even if $\mu = \varepsilon^p$, for $p < 0$ and that it is also hyperbolic for $-4 < p < 0$. Moreover, we obtain in that case an exponentially small upper bound for the splitting. We also give some ideas about the possible size of the splitting in this case which mainly rely, again, on doing more steps of averaging to the system and obtain the singularities of the (new) homoclinic orbit.

Even if this "below the singular case" can seem quite "artificial" (the size of μ being too big), it is important to note that this case appears in lots of systems even if the perturbation is small. For instance, considering the perturbed pendulum:

$$\ddot{x} = \sin x + \mu \sin(nx) \sin \frac{t}{\varepsilon}$$

the singular case where the methods we present here can work would be $\mu = \varepsilon^p$ for $p = 2n - 2$. In the cases $p < 2n - 2$, even if the perturbation of the pendulum is small, the methods used up to now would not give any answer to the question of the splitting and other approximations to the problem, like the averaging procedure, should be applied to obtain the correct exponential size of the splitting.

Finally, the authors study the problem of splitting of separatrices when μ is close to the zeros of the Bessel function $J_0(\mu)$, namely $\mu = \mu_i - \varepsilon^r$ where μ_i is one of the zeros of this function. In that case, the averaged system obtained after two steps of averaging change drastically the hyperbolic behavior, and then the size of the splitting too. In this setting the authors prove that the periodic orbit is still hyperbolic provided $r \in (0, 2)$ and give a non-sharp exponentially small upper bound of the size of the splitting. For $r \geq 2$, we give some insight on the appearance of some bifurcation phenomena which change substantially the topology of the hyperbolic structure, even for the averaged system (which is integrable). In fact, $\mu(\varepsilon) = \mu_i \pm c_i^{\pm} \varepsilon^2$ for certain $c_i^{\pm} > 0$, is the first order of codimension-1 curves $\gamma_i^{\pm}$ in which ocurrs a saddle-center bifurcation: a parabolic critical point appears and then bifurcates to an elliptic point and a hyperbolic, giving birth to a figure eight made of two homoclinic connections. In the curve $\mu = \mu_i$ this figure eight merges with the separatrix of the former pendulum. The study of the splitting in the domains in the parameter space delimited by the couple of curves $\gamma_i^{\pm}$ in which both hyperbolic structures coexist remains open.

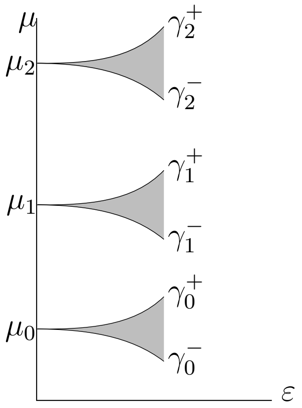

Figure 12. Parameter space (μ, ε). In white we show the domain in which we obtain exponentially small results and in grey the domain in which the problem remains open.

In Figure 12 we illustrate the set of values of the parameters (μ, ε) for which we can provide results about the exponentially small splitting.

The authors think that this paper shows that the problem of the exponentially small splitting of separatrices is still a problem that is far from being completely understood and bring some new ideas that open new points of view to deal with it.

Acknowledgments

The authors have been partially supported by the Spanish MCyT/FEDER grant MTM2006-00478. Moreover, the research of M. G. has been supported by the Spanish PhD grant FPU AP2005-1314.

References

[Ang93] S. Angenent. A variational interpretation of Mel′nikov's function and exponentially small separatrix splitting. In *Symplectic geometry*, volume 192 of *London Math. Soc. Lecture Note Ser.*, pages 5–35. Cambridge Univ. Press, Cambridge, 1993.

[AS92] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover Publications Inc., New York, 1992. Reprint of the 1972 edition.

[Bal94] W. Balser. *From divergent power series to analytic functions*, volume 1582 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994. Theory and application of multisummable power series.

[Bal06] I. Baldomá. The inner equation for one and a half degrees of freedom rapidly forced Hamiltonian systems. *Nonlinearity*, 19(6):1415–1445, 2006.

[BF04] I. Baldomá and E. Fontich. Exponentially small splitting of invariant manifolds of parabolic points. *Mem. Amer. Math. Soc.*, 167(792):x–83, 2004.

[BF05] I. Baldomá and E. Fontich. Exponentially small splitting of separatrices in a weakly hyperbolic case. *J. Differential Equations*, 210(1):106–134, 2005.

[BO93] A. Benseny and C. Olivé. High precision angles between invariant manifolds for radpidly forced hamiltonian systems. *Proceedings Equadiff91*, pages 315–319, 1993.

[BS06] I. Baldomá and T. M. Seara. Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity. *J. Nonlinear Sci.*, 16(6):543–582, 2006.

[BS08] I. Baldomá and T. M. Seara. The inner equation for generic analytic unfoldings of the Hopf-zero singularity. *Discrete Contin. Dyn. Syst. Ser. B*, 10(2-3):323–347, 2008.

[BSSV98] Carles Bonet, David Sauzin, Tere Seara, and Marta València. Adiabatic invariant of the harmonic oscillator, complex matching and resurgence. *SIAM J. Math. Anal.*, 29(6):1335–1360 (electronic), 1998.

[CG94] L. Chierchia and G. Gallavotti. Drift and diffusion in phase space. *Ann. Inst. H. Poincaré Phys. Théor.*, 60(1):144, 1994.

[DG00] A. Delshams and P. Gutiérrez. Splitting potential and the Poincaré-Melnikov method for whiskered tori in Hamiltonian systems. *J. Nonlinear Sci.*, 10(4):433–476, 2000.

[DGJS97] A. Delshams, V. Gelfreich, À. Jorba, and T.M. Seara. Exponentially small splitting of separatrices under fast quasiperiodic forcing. *Comm. Math. Phys.*, 189(1):35–71, 1997.

[DGS04] A. Delshams, P. Gutiérrez, and T.M. Seara. Exponentially small splitting for whiskered tori in Hamiltonian sysems: flow-box coordinates and upper bounds. *Discrete Contin. Dyn. Syst.*, 11(4):785–826, 2004.

[DR97] A. Delshams and R. Ramírez-Ros. Melnikov potential for exact symplectic maps. *Comm. Math. Phys.*, 190(1):213–245, 1997.

[DR98] A. Delshams and R. Ramírez-Ros. Exponentially small splitting of separatrices for perturbed integrable standard-like maps. *J. Nonlinear Sci.*, 8(3):317–352, 1998.

[DS92] A. Delshams and T. M. Seara. An asymptotic expression for the splitting of separatrices of the rapidly forced pendulum. *Comm. Math. Phys.*, 150(3):433–463, 1992.

[DS97] A. Delshams and T.M. Seara. Splitting of separatrices in Hamiltonian systems with one and a half degrees of freedom. *Math. Phys. Electron. J.*, 3:Paper 4, 40 pp. (electronic), 1997.

[EKS93] James A. Ellison, Martin Kummer, and A. W. Sáenz. Transcendentally small transversality in the rapidly forced pendulum. *J. Dynam. Differential Equations*, 5(2):241–277, 1993.

[Eli94] L. H. Eliasson. Biasymptotic solutions of perturbed integrable Hamiltonian systems. *Bol. Soc. Brasil. Mat. (N.S.)*, 25(1):57–76, 1994.

[Fon93] E. Fontich. Exponentially small upper bounds for the splitting of separatrices for high frequency periodic perturbations. *Nonlinear Anal.*, 20(6):733–744, 1993.

[Fon95] E. Fontich. Rapidly forced planar vector fields and splitting of separatrices. *J. Differential Equations*, 119(2):310–335, 1995.

[FS90] E. Fontich and C. Simó. The splitting of separatrices for analytic diffeomorphisms. *Ergodic Theory Dynam. Systems*, 10(2):295–318, 1990.

[FS96] B. Fiedler and J. Scheurle. Discretization of homoclinic orbits, rapid forcing and "invisible" chaos. *Mem. Amer. Math. Soc.*, 119(570):viii+79, 1996.

[Gal94] G. Gallavotti. Twistless KAM tori, quasi flat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review. *Rev. Math. Phys.*, 6(3):343–411, 1994.

[Gel94] V. G. Gelfreich. Separatrices splitting for the rapidly forced pendulum. In *Seminar on Dynamical Systems (St. Petersburg, 1991)*, volume 12 of *Progr. Nonlinear Differential Equations Appl.*, pages 47–67. Birkhäuser, Basel, 1994.

[Gel97a] V. G. Gelfreich. Melnikov method and exponentially small splitting of separatrices. *Phys. D*, 101(3-4):227–248, 1997.

[Gel97b] V. G. Gelfreich. Reference systems for splittings of separatrices. *Nonlinearity*, 10(1):175–193, 1997.

[Gel99] V. G. Gelfreich. A proof of the exponentially small transversality of the separatrices for the standard map. *Comm. Math. Phys.*, 201(1):155–216, 1999.

[Gel00] V. G. Gelfreich. Separatrix splitting for a high-frequency perturbation of the pendulum. *Russ. J. Math. Phys.*, 7(1):48–71, 2000.

[GGM99] G. Gallavotti, G. Gentile, and V. Mastropietro. Separatrix splitting for systems with three time scales. *Comm. Math. Phys.*, 202(1):197–236, 1999.

[GH83] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983.

[GLT91] V. G. Gelfreich, V. F. Lazutkin, and M. B. Tabanov. Exponentially small splittings in Hamiltonian systems. *Chaos*, 1(2):137–142, 1991.

[GN09] V. G. Gelfreich and V. Naudot. Width of homolinic zone in the parameter space for quadratic maps. *Experimental Mathematics*, 2009. In print.

[GS01] V. Gelfreich and D. Sauzin. Borel summation and splitting of separatrices for the Hénon map. *Ann. Inst. Fourier (Grenoble)*, 51(2):513–567, 2001.

[HMS88] P. Holmes, J. Marsden, and J. Scheurle. Exponentially small splittings of separatrices with applications to KAM theory and degenerate bifurcations. In *Hamiltonian dynamical systems*, volume 81 of *Contemp. Math.* 1988.

[Laz84] V. F. Lazutkin. Splitting of separatrices for the Chirikov standard map. VINITI 6372/82, 1984. Preprint (Russian).

[Laz03] V. F. Lazutkin. Splitting of separatrices for the Chirikov standard map. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 300(Teor. Predst. Din. Sist. Spets. Vyp. 8):25–55, 285, 2003.

[LMS03] P. Lochak, J.-P. Marco, and D. Sauzin. On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems. *Mem. Amer. Math. Soc.*, 163(775):viii+145, 2003.

[Lom00] E. Lombardi. *Oscillatory integrals and phenomena beyond all algebraic orders*, volume 1741 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000. With applications to homoclinic orbits in reversible systems.

[Mel63] V. K. Melnikov. On the stability of the center for time periodic perturbations. *Trans. Moscow Math. Soc.*, 12:1–57, 1963.

[Neĭ84] A. I. Neĭshtadt. The separation of motions in systems with rapidly rotating phase. *Prikl. Mat. Mekh.*, 48(2):197–204, 1984.

[Oli06] C. Olivé. *Càlcul de l'escissió de separatrius usant tècniques de matching complex i ressurgència aplicades a l'equació de Hamilton-Jacobi.* `http://www.tdx.cat/TDX-0917107-125950`, 2006.

[OSS03] C. Olivé, D. Sauzin, and T. M. Seara. Resurgence in a Hamilton-Jacobi equation. In *Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002)*, volume 53(4), pages 1185–1235, 2003.

[Poi99] H. Poincaré. *Les méthodes nouvelles de la mécanique céleste*, volume 1, 2, 3. Gauthier-Villars, Paris, 1892–1899.

[San82] J. A. Sanders. Melnikov's method and averaging. *Celestial Mech.*, 28(1-2):171–181, 1982.

[Sau95] D. Sauzin. Résurgence paramétrique et exponentielle petitesse de l'écart des séparatrices du pendule rapidement forcé. *Ann.Ins.Fourier*, 45(2):453–511, 1995.

[Sau01] D. Sauzin. A new method for measuring the splitting of invariant manifolds. *Ann. Sci. École Norm. Sup. (4)*, 34, 2001.

[Sch89] J. Scheurle. Chaos in a rapidly forced pendulum equation. *Contemp. Math. AMS*, 97, 1989.

[Sim94] C. Simó. Averaging under fast quasiperiodic forcing. In *Hamiltonian mechanics (Toruń, 1993)*, volume 331 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 13–34. Plenum, New York, 1994.

[Sma65] S. Smale. Diffeomorphisms with many periodic points. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 63–80. Princeton Univ. Press, Princeton, N.J., 1965.

[SMH91] Jürgen Scheurle, Jerrold E. Marsden, and Philip Holmes. Exponentially small estimates for separatrix splittings. In *Asymptotics beyond all orders (La Jolla, CA, 1991)*, volume 284 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 187–195. Plenum, New York, 1991.

[SV00] T.M. Seara and J. Villanueva. Asymptotic behaviour of the domain of analyticity of invariant curves of the standard map. *Nonlinearity*, 13(5):1699–1744, 2000.

[Tre94] D. Treschev. Hyperbolic tori and asymptotic surfaces in Hamiltonian systems. *Russian J. Math. Phys.*, 2(1):93–110, 1994.

[Tre97] D. Treschev. Separatrix splitting for a pendulum with rapidly oscillating suspension point. *Russ. J. Math. Phys.*, 5(1):63–98, 1997.

[Tre98] D. Treschev. Width of stochastic layers in near-integrable two-dimensional symplectic maps. *Phys. D*, 116(1-2):21–43, 1998.