

# Functional integral representations for self-avoiding walk\*

David C. Brydges<sup>†</sup>

*Department of Mathematics  
University of British Columbia, Vancouver, BC V6T 1Z2, Canada  
e-mail: [db5d@math.ubc.ca](mailto:db5d@math.ubc.ca)*

John Z. Imbrie

*Department of Mathematics  
University of Virginia, Charlottesville, VA 22904-4137, U.S.A.  
e-mail: [ji2k@virginia.edu](mailto:ji2k@virginia.edu)*

and

Gordon Slade<sup>†</sup>

*Department of Mathematics  
University of British Columbia, Vancouver, BC V6T 1Z2, Canada  
e-mail: [slade@math.ubc.ca](mailto:slade@math.ubc.ca)*

**Abstract:** We give a survey and unified treatment of functional integral representations for both simple random walk and some self-avoiding walk models, including models with strict self-avoidance, with weak self-avoidance, and a model of walks and loops. Our representation for the strictly self-avoiding walk is new. The representations have recently been used as the point of departure for rigorous renormalization group analyses of self-avoiding walk models in dimension 4. For the models without loops, the integral representations involve fermions, and we also provide an introduction to fermionic integrals. The fermionic integrals are in terms of anticommuting Grassmann variables, which can be conveniently interpreted as differential forms.

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## 1. Introduction

The use of random walk representations for functional integrals in mathematical physics has a long history going back to Symanzik [25], who showed how such representations can be used to study quantum field theories. Representations of this type were exploited systematically in [1, 4, 5, 11, 12]. It is also possible to use such representations in reverse, namely to rewrite a random walk problem in terms of an equivalent problem for a functional integral.

Our goal in this paper is to provide an introductory survey of functional integral representations for some problems connected with self-avoiding walks, with both strict and weak self-avoidance. In particular, we derive a new representation for the strictly self-avoiding walk. These representations have proved useful recently in the analysis of various problems concerning 4-dimensional self-avoiding walks, by providing a setting in which renormalization group methods can be applied. This has allowed for a proof of  $|x|^{-2}$  decay of the critical Green function and existence of a logarithmic correction to the end-to-end distance for weakly self-avoiding walk on a 4-dimensional hierarchical lattice [3, 6, 7]. It is also the basis for work in progress on the critical Green function for weakly self-avoiding walk on  $\mathbb{Z}^4$  and a particular (spread-out) model of strictly self-avoiding walk on  $\mathbb{Z}^4$  [10]. In addition, the renormalization group trajectory for a specific model of weakly self-avoiding walk on  $\mathbb{Z}^3$  (one with upper critical dimension  $3 + \epsilon$ ) has been constructed in [20], in this context. In this paper, we explain and derive the representations, but we make no attempt to analyze the representations here, leaving those details to [3, 6, 7, 10, 20].

The representations we will discuss can be divided into two classes: purely bosonic, and mixed bosonic-fermionic. The bosonic representations will be the

most familiar to probabilists, as they are in terms of ordinary Gaussian integrals. They represent simple random walks, and also systems of self-avoiding and mutually-avoiding walks and loops.

The mixed bosonic-fermionic representations eliminate the loops, leaving only the self-avoiding walk. They involve Gaussian integrals with anticommuting Grassmann variables. A classic reference for Grassmann integrals is the text by Berezin [2], and there is a short introduction in [23, Appendix B]. Such integrals, although familiar in physics, are less so in probability theory. It turns out, however, that these more exotic integrals share many features in common with ordinary Gaussian integrals. One of our goals is to provide a minimal introduction to these integrals, for probabilists.

Representations for self-avoiding walks go back to an observation of de Gennes [13]. The  $N$ -vector model has a random walk representation given by a self-avoiding walk in a background of mutually-avoiding self-avoiding loops, with every loop contributing a factor  $N$ . This led de Gennes to consider the limit  $N \rightarrow 0$ , in which closed loops no longer contribute, leading to a representation for the self-avoiding walk model as the  $N = 0$  limit of the  $N$ -vector model (see also [18, Section 2.3]). Although this idea has been very useful in physics, it has been less productive within mathematics, because  $N$  is a natural number and so it is unclear how to understand a limit  $N \rightarrow 0$  in a rigorous manner.

On the other hand, the notion was developed in [19, 21] that while an  $N$ -component boson field  $\phi$  contributes a factor  $N$  to each closed loop, an  $N$ -component fermion field  $\psi$  contributes a complementary factor  $-N$ . The net effect is to associate zero to each closed loop. We give a concrete demonstration of this effect in Section 5.2.1 below. This provides a way to realize de Gennes' idea, without any nonrigorous limit.

Moreover, it was pointed out by Le Jan [16, 17] that the anticommuting variables can be represented by differential forms: the fermion field can be regarded as nothing more than the differential of the boson field. This observation was further developed in [8, 6], and we will follow the approach based on differential forms in this paper. In this approach, the anticommuting nature of fermions is represented by the anticommuting wedge product for differential forms. Thus the world of Grassmann variables, initially mysterious, can be replaced by differential forms, objects which are fundamental in differential geometry in the way that random variables are fundamental in probability.

We have attempted to keep this paper self-contained. In particular, our discussion of differential forms for the representations involving fermions is intended to be introductory.

The rest of the paper is organized as follows. In Section 2, we derive integral representations for simple random walk, and for a model of a self-avoiding walk and self-avoiding loops all of which are mutually avoiding. These are purely bosonic representations, without anticommuting fermionic variables. In Section 3, we define the self-avoiding walk models (without loops). Their representations are derived in Section 5, using the fermionic integration introduced in Section 4. The mixed bosonic-fermionic integrals are examples of *supersymmetric* field theories. Although an appreciation of this fact is not necessary to

understand the representations, in Section 6 we briefly discuss this important connection.

## 2. Bosonic representations

### 2.1. Gaussian integrals

By “bosonic representations” we mean representations for random walk models in terms of ordinary Gaussian integrals. For our purposes, these integrals are in terms of a two-component field  $(u_x, v_x)_{x \in \{1, \dots, M\}}$ , which is most conveniently represented by the complex pair  $(\phi_x, \bar{\phi}_x)$ , where

$$\phi_x = u_x + iv_x, \quad \bar{\phi}_x = u_x - iv_x. \quad (2.1)$$

The differentials  $d\phi_x, d\bar{\phi}_x$  are given by

$$d\phi_x = du_x + idv_x, \quad d\bar{\phi}_x = du_x - idv_x, \quad (2.2)$$

and their product  $d\bar{\phi}_x d\phi_x$  is given by

$$d\bar{\phi}_x d\phi_x = 2idu_x dv_x, \quad (2.3)$$

where we adopt the convention that differentials are multiplied together with the anticommutative wedge product; in particular  $du_x du_x$  and  $dv_x dv_x$  vanish and do not appear in the above product. This anticommutative product will play a central role when we come to fermions in Section 4, but until then plays no role beyond the formula (2.3). We are using the letter “ $x$ ” as index for the field in anticipation of the fact that in our representations the field will be indexed by the space in which our random walks take steps.

We now briefly review some elementary properties of Gaussian measures. Let  $C$  be an  $M \times M$  complex matrix. We assume that  $C$  has positive Hermitian part, i.e.,  $\sum_{x,y=1}^M \phi_x (C_{x,y} + \bar{C}_{y,x}) \bar{\phi}_y > 0$  for all nonzero  $\phi \in \mathbb{C}^M$ . Let  $A = C^{-1}$ . We write  $d\mu_C$  for the Gaussian measure on  $\mathbb{R}^{2M}$  with covariance  $C$ , namely

$$d\mu_C(\phi, \bar{\phi}) = \frac{1}{Z_C} e^{-\phi A \bar{\phi}} d\bar{\phi}_1 d\phi_1 \cdots d\bar{\phi}_M d\phi_M, \quad (2.4)$$

where  $\phi A \bar{\phi} = \sum_{x,y=1}^M \phi_x A_{x,y} \bar{\phi}_y$ , and where  $Z_C$  is the normalization constant

$$Z_C = \int_{\mathbb{R}^{2M}} e^{-\phi A \bar{\phi}} d\bar{\phi}_1 d\phi_1 \cdots d\bar{\phi}_M d\phi_M. \quad (2.5)$$

We will need the value of  $Z_C$  given in the following lemma.

**Lemma 2.1.** *For  $C$  with positive Hermitian part and inverse  $A = C^{-1}$ ,*

$$Z_C = \int e^{-\phi A \bar{\phi}} d\bar{\phi}_1 d\phi_1 \cdots d\bar{\phi}_M d\phi_M = \frac{(2\pi i)^M}{\det A}. \quad (2.6)$$

*Proof.* Consider first the case where  $C$ , and hence  $A$ , is Hermitian. In this case, there is a unitary matrix  $U$  and a diagonal matrix  $D$  such that  $A = U^{-1}DU$ . Then  $\phi A \bar{\phi} = w D \bar{w}$ , where  $w = U\phi$ , so

$$\frac{1}{(2\pi i)^M} Z_C = \prod_{x=1}^M \left( \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-d_x(u_x^2 + v_x^2)} du_x dv_x \right) = \prod_{x=1}^M \frac{1}{d_x} = \frac{1}{\det A}. \quad (2.7)$$

For the general case, we write  $A(z) = G + izH$  with  $G = \frac{1}{2}(A + A^\dagger)$ ,  $H = \frac{1}{2i}(A - A^\dagger)$  and  $z = 1$ . Since  $\phi(iH)\bar{\phi}$  is imaginary, when  $G$  is positive definite the integral in (2.6) converges and defines an analytic function of  $z$  in a neighborhood of the real axis. Furthermore, for  $z$  small and purely imaginary,  $A(z)$  is Hermitian and positive definite, and hence (2.6) holds in this case. Since  $(\det A(z))^{-1}$  is a meromorphic function of  $z$ , (2.6) follows from the uniqueness of analytic extension.  $\square$

A basic tool is the integration by parts formula given in the following lemma. The derivative appearing in its statement is defined by

$$\frac{\partial}{\partial \phi_x} = \frac{1}{2} \left( \frac{\partial}{\partial u_x} - i \frac{\partial}{\partial v_x} \right). \quad (2.8)$$

With  $\partial/\partial \bar{\phi}_x$  defined to be its conjugate, this leads to the equations

$$\frac{\partial \phi_y}{\partial \phi_x} = \frac{\partial \bar{\phi}_y}{\partial \bar{\phi}_x} = \delta_{x,y}, \quad \frac{\partial \bar{\phi}_y}{\partial \phi_x} = \frac{\partial \phi_y}{\partial \bar{\phi}_x} = 0. \quad (2.9)$$

**Lemma 2.2.** *Let  $C$  have positive Hermitian part. Then*

$$\int_{\mathbb{R}^{2M}} \bar{\phi}_a F d\mu_C(\phi, \bar{\phi}) = \sum_{x \in \Lambda} C_{a,x} \int_{\mathbb{R}^{2M}} \frac{\partial F}{\partial \phi_x} d\mu_C(\phi, \bar{\phi}), \quad (2.10)$$

where  $F$  is any  $C^1$  function such that both sides are integrable.

*Proof.* Let  $A = C^{-1}$ . We begin with the integral on the right-hand side, and make the abbreviation  $d\bar{\phi}d\phi = d\bar{\phi}_1 d\phi_1 \cdots d\bar{\phi}_M d\phi_M$ . By (2.8), we can use standard integration by parts to move the derivative from one factor to the other, and with (2.9) this gives

$$\int \frac{\partial F}{\partial \phi_x} e^{-\phi A \bar{\phi}} d\bar{\phi}d\phi = - \int \frac{\partial e^{-\phi A \bar{\phi}}}{\partial \phi_x} F d\bar{\phi}d\phi = \int \sum_y A_{x,y} \bar{\phi}_y F e^{-\phi A \bar{\phi}} d\bar{\phi}d\phi. \quad (2.11)$$

Now we multiply by  $C_{a,x}$ , sum over  $x$ , and use  $C = A^{-1}$ , to complete the proof.  $\square$

The equations

$$\begin{aligned} \int_{\mathbb{R}^{2M}} \phi_a \phi_b d\mu_C(\phi, \bar{\phi}) &= \int_{\mathbb{R}^{2M}} \bar{\phi}_a \bar{\phi}_b d\mu_C(\phi, \bar{\phi}) = 0, \\ \int_{\mathbb{R}^{2M}} \bar{\phi}_a \phi_b d\mu_C(\phi, \bar{\phi}) &= C_{a,b}. \end{aligned} \quad (2.12)$$

are simple consequences of Lemma 2.2. The last equality is a special case of Wick's theorem, which provides a formula for the calculation of arbitrary moments of the Gaussian measure. We will only need the following special case of Wick's theorem, in which a particular Gaussian expectation is evaluated as the permanent of a submatrix of  $C$ .

**Lemma 2.3.** *Let  $\{x_1, \dots, x_k\}$  and  $\{y_1, \dots, y_k\}$  each be sets of  $k$  distinct points in  $\Lambda$ , and let  $S_k$  denote the set of permutations of  $\{1, \dots, k\}$ . Then*

$$\int_{\mathbb{R}^{2M}} \left( \prod_{l=1}^k \bar{\phi}_{x_l} \phi_{y_l} \right) d\mu_C(\phi, \bar{\phi}) = \sum_{\sigma \in S_k} \prod_{l=1}^k C_{x_l, \sigma(y_l)}. \quad (2.13)$$

*Proof.* This follows by repeated use of integration by parts.  $\square$

## 2.2. Simple random walk

Our setting throughout the paper is a fixed finite set  $\Lambda = \{1, 2, \dots, M\}$  of cardinality  $M \geq 1$ . Given points  $a, b \in \Lambda$ , a *walk*  $\omega$  from  $a$  to  $b$  is a sequence of points  $x_0 = a, x_1, x_2, \dots, x_n = b$ , for some  $n \geq 0$ . We write  $|\omega|$  for the length  $n$  of  $\omega$ . Sometimes it is useful to regard  $\omega$  as consisting of the directed edges  $(x_{i-1}, x_i)$ ,  $1 \leq i \leq n$ , rather than vertices. Let  $\mathcal{W}_{a,b}$  denote the set of all walks from  $a$  to  $b$ , of any length.

Let  $J$  be a  $\Lambda \times \Lambda$  complex matrix with zero diagonal part (i.e.,  $J_{x,x} = 0$  for all  $x \in \Lambda$ ). Let  $D$  be a diagonal matrix with nonzero entries  $D_{x,x} = d_x \in \mathbb{C}$ . We assume that  $D - J$  is *diagonally dominant*; this means that

$$\max_{x \in \Lambda} \sum_{y \in \Lambda} \left| \frac{J_{x,y}}{d_x} \right| < 1. \quad (2.14)$$

Given  $\omega \in \mathcal{W}_{a,b}$ , let

$$J^\omega = \prod_{e \in \omega} J_e. \quad (2.15)$$

Here we regard  $\omega$  as a set of labeled edges  $e = (\omega(i-1), \omega(i))$  (the empty product is 1 if  $|\omega| = 0$ ). The simple random walk two-point function is defined by

$$G_{a,b}^{\text{srw}} = \sum_{\omega \in \mathcal{W}_{a,b}} J^\omega \prod_{i=0}^{|\omega|} d_{\omega(i)}^{-1}. \quad (2.16)$$

The assumption that  $D - J$  is diagonally dominant ensures that the sum in (2.16) converges absolutely. The following theorem was proved in [5].

**Theorem 2.4.** *Suppose that  $D - J$  is diagonally dominant. Then  $C = (D - J)^{-1}$  exists and  $G_{a,b}^{\text{srw}} = (D - J)_{a,b}^{-1}$ . In addition, if  $D - J$  has positive Hermitian part then*

$$G_{a,b}^{\text{srw}} = (D - J)_{a,b}^{-1} = \int_{\mathbb{R}^{2M}} \bar{\phi}_a \phi_b d\mu_C(\phi, \bar{\phi}). \quad (2.17)$$

*Proof.* The sum in (2.16) can be evaluated explicitly as

$$G_{a,b}^{\text{srw}} = \sum_{\omega \in \mathcal{W}_{a,b}} J^\omega \prod_{i=0}^{|\omega|} d_{\omega(i)}^{-1} = \sum_{n=0}^{\infty} (D^{-1}(JD^{-1})^n)_{a,b}. \quad (2.18)$$

It is easily verified that  $D - J$  applied to the right-hand side gives the identity, and hence

$$G_{a,b}^{\text{srw}} = (D - J)_{a,b}^{-1}. \quad (2.19)$$

When  $D - J$  has positive Hermitian part, we may use (2.12) to complete the proof.  $\square$

Next, we suppose that  $d_x > 0$ ,  $J_{x,y} \geq 0$ , and give two alternate representations for  $G_{a,b}^{\text{srw}}$  in terms of continuous-time Markov chains. For the first, which appeared in [11], we consider the continuous-time Markov chain  $X$  defined as follows. The state space of  $X$  is  $\Lambda \cup \{\partial\}$ , where  $\partial$  is an absorbing state called the cemetery. When  $X$  arrives at state  $x$  it waits for an  $\text{Exp}(d_x)$  holding time and then jumps to  $y$  with probability  $\pi_{x,y} = d_x^{-1} J_{x,y}$  and jumps to the cemetery with probability  $\pi_{x,\partial} = 1 - \sum_{y \in \Lambda} d_x^{-1} J_{x,y}$ . The holding times are independent of each other and of the jumps. Let  $\zeta$  denote the time at which the process arrives in the cemetery. Note that if  $D - J$  is diagonally dominant then  $\zeta < \infty$  with probability 1, and by right-continuity of the sample paths the last state visited by  $X$  before arriving in the cemetery is  $X(\zeta^-)$ . For  $x \in \Lambda$ , let  $L_x$  denote the total (continuous) time spent by  $X$  at  $x$ . We denote the expectation for  $X$ , started from  $a \in \Lambda$ , by  $\mathbb{E}_a$ .

**Theorem 2.5.** *Suppose that  $D - J$  is diagonally dominant, with  $d_x > 0$ ,  $J_{x,y} \geq 0$ , and let  $\bar{d}_x = \sum_{y \in \Lambda} J_{x,y}$ . Let  $V$  be a diagonal matrix with entries  $V_{x,x} = v_x$ , and suppose that  $0 < \bar{d}_x < d_x + \text{Re } v_x$  for all  $x \in \Lambda$ . Let  $G_{a,b}^{\text{srw}}(v)$  denote the two-point function (2.16), with matrix  $D + V - J$  in place of  $D - J$ . Then*

$$G_{a,b}^{\text{srw}}(v) = \frac{1}{d_b \pi_{b,\partial}} \mathbb{E}_a \left( e^{-\sum_{x \in \Lambda} v_x L_x} \mathbb{I}_{X(\zeta^-)=b} \right). \quad (2.20)$$

*Proof.* The Markov chain  $X$  is equivalent to a discrete-time Markov chain  $Y$  which jumps with the above transition probabilities, together with a sequence  $\sigma_0, \sigma_1, \dots$  of exponential holding times. Let  $\eta$  denote the discrete random time after which the process  $Y$  jumps to  $\partial$ . By partitioning on the events  $\{\eta = n\}$ , noting that  $\eta$  is almost surely finite, we see that the right-hand side of (2.20) is equal to

$$\frac{1}{d_b} \sum_{n=0}^{\infty} \mathbb{E}_a \left( e^{-\sum_{i=0}^n v_{Y_i} \sigma_i} \mathbb{I}_{Y_n=b} \right). \quad (2.21)$$

Given the sequence  $Y_0, Y_1, \dots, Y_n$ , the  $\sigma_i$  are independent  $\text{Exp}(d_{Y_i})$  random variables and hence

$$\frac{1}{d_b} \mathbb{E}_a \left( e^{-\sum_{i=0}^n v_{Y_i} \sigma_i} \mathbb{I}_{Y_n=b} | Y_0, Y_1, \dots \right) = \frac{1}{d_b + v_b} \prod_{i=0}^{n-1} \frac{d_{Y_i}}{d_{Y_i} + v_{Y_i}}. \quad (2.22)$$

If we then take the expectation with respect to the Markov chain  $Y$ , we find that (2.21) is equal to

$$\sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_{a,b}; |\omega|=n} \pi^\omega \frac{1}{d_b + v_b} \prod_{i=0}^{n-1} \frac{d_{\omega(i)}}{d_{\omega(i)} + v_{\omega(i)}} = \sum_{\omega \in \mathcal{W}_{a,b}} J^\omega \prod_{i=0}^{|\omega|} \frac{1}{d_{\omega(i)} + v_{\omega(i)}}, \quad (2.23)$$

which is the desired result.  $\square$

Next, we derive a third representation for  $G_{a,b}^{\text{srw}}(v)$ , which is more general than Theorem 2.5 as it does not require diagonal dominance of  $D - J$  (it does require  $\text{Re } v_x > 0$  when  $d_x = \bar{d}_x$ ). This representation was obtained in [3] using the Feynman–Kac formula, but we give a different proof based on Theorem 2.5. The representation involves a second continuous-time Markov process, with generator  $\bar{D} - J$  where we set  $\bar{d}_x = \sum_{y \in \Lambda} J_{x,y}$  and assume  $\bar{d}_x > 0$  for each  $x \in \Lambda$ . This process is like the one described above, but has no cemetery site and continues for all time. Let  $\bar{\mathbb{E}}_a$  denote the expectation for this process started at  $a \in \Lambda$ . Let

$$L_{x,T} = \int_0^T \mathbb{I}_{X(s)=x} ds. \quad (2.24)$$

denote the time spent by  $X$  at  $x$  during the time interval  $[0, T]$ .

**Theorem 2.6.** *Suppose that  $d_x > 0$ ,  $J_{x,y} \geq 0$ , and let  $\bar{d}_x = \sum_{y \in \Lambda} J_{x,y}$ . Let  $V$  be a diagonal matrix with entries  $V_{x,x} = v_x$ , and suppose that  $0 < \bar{d}_x < d_x + \text{Re } v_x$  for all  $x \in \Lambda$ . Then*

$$G_{a,b}^{\text{srw}}(v) = \int_0^\infty \bar{\mathbb{E}}_a \left( e^{-\sum_{x \in \Lambda} (v_x + d_x - \bar{d}_x) L_{x,T}} \mathbb{I}_{X(T)=b} \right) dT. \quad (2.25)$$

*Proof.* Let  $\mu = \min_{x \in \Lambda} (\text{Re } v_x + d_x - \bar{d}_x)$  and let  $0 < \epsilon < \mu$ . We write

$$D + V - J = D^{(\epsilon)} + V^{(\epsilon)} - J \quad (2.26)$$

with

$$D_{x,x}^{(\epsilon)} = d_x^{(\epsilon)} = \bar{d}_x + \epsilon, \quad V_{x,x}^{(\epsilon)} = v_x^{(\epsilon)} = v_x + d_x - \bar{d}_x - \epsilon. \quad (2.27)$$

Let  $\mathbb{E}_a^{(\epsilon)}$  denote the expectation for the Markov process defined in terms of  $D^{(\epsilon)} - J$ . Since  $D^{(\epsilon)} - J$  is diagonally dominant and  $\text{Re } v_x^{(\epsilon)} \geq \mu - \epsilon$ , by Theorems 2.4 and 2.5 we have

$$\begin{aligned} G_{a,b}^{\text{srw}}(v) &= (D + V - J)_{a,b}^{-1} = (D^{(\epsilon)} + V^{(\epsilon)} - J)_{a,b}^{-1} \\ &= \frac{1}{\epsilon} \mathbb{E}_a^{(\epsilon)} \left( e^{-\sum_{x \in \Lambda} v_x^{(\epsilon)} L_x} \mathbb{I}_{X(\zeta^-)=b} \right), \end{aligned} \quad (2.28)$$

where the  $\epsilon$  in the denominator is equal to the product of  $d_b^{(\epsilon)}$  and  $\pi_{b,\partial}^{(\epsilon)} = \epsilon/d_b^{(\epsilon)}$ .

We partition on the values of  $\zeta$ , the time of transition to  $\partial$ . For  $\delta > 0$ , let

$$I(\delta) = \{j\delta : j = 0, 1, 2, \dots\}. \quad (2.29)$$



Then

$$G_{a,b}^{\text{srw}}(v) = \sum_{T \in I(\delta)} \frac{1}{\epsilon} \mathbb{E}_a^{(\epsilon)} \left( e^{-\sum_{x \in \Lambda} v_x^{(\epsilon)} L_x} \mathbb{I}_{Y_\eta = b} \mathbb{I}_{T < \zeta \leq T + \delta} \right). \quad (2.30)$$

The probability of the symmetric difference

$$\{Y_\eta = b, T < \zeta \leq T + \delta\} \Delta \{X(T) = b, X(T + \delta) = \partial\} \quad (2.31)$$

is  $O(\delta^2)$  because this event requires two jumps in time  $\delta$ . Also,  $L_{x,T} \leq L_x \leq L_{x,T} + \delta$  on the event  $\{T < \zeta \leq T + \delta\}$ , so

$$G_{a,b}^{\text{srw}}(v) = \lim_{\delta \rightarrow 0} \sum_{T \in I(\delta)} \frac{1}{\epsilon} \mathbb{E}_a^{(\epsilon)} \left( e^{-\sum_{x \in \Lambda} v_x^{(\epsilon)} L_{x,T}} \mathbb{I}_{X(T)=b, X(T+\delta)=\partial} \right). \quad (2.32)$$

By the Markov property and the fact that

$$\bar{\mathbb{P}}(X(T + \delta) = \partial | X(T) = b) = d_b^{(\epsilon)} \delta \pi_{b,\partial}^{(\epsilon)} + O(\delta^2) = \epsilon \delta + O(\delta^2), \quad (2.33)$$

we obtain

$$\begin{aligned} G_{a,b}^{\text{srw}}(v) &= \lim_{\delta \rightarrow 0} \sum_{T \in I(\delta)} \mathbb{E}_a^{(\epsilon)} \left( e^{-\sum_{x \in \Lambda} v_x^{(\epsilon)} L_{x,T}} \mathbb{I}_{X(T)=b} \right) \delta \\ &= \int_0^\infty \mathbb{E}_a^{(\epsilon)} \left( e^{-\sum_{x \in \Lambda} v_x^{(\epsilon)} L_{x,T}} \mathbb{I}_{X(T)=b} \right) dT. \end{aligned} \quad (2.34)$$

Now taking the limit  $\epsilon \rightarrow 0$ ,  $\mathbb{E}_a^{(\epsilon)}$  converges to  $\bar{\mathbb{E}}_a$  on bounded functions of  $\{X(t) : 0 \leq t \leq T\}$  since the transition probabilities and the densities of the holding times  $\sigma_i$  converge to their analogues in  $\bar{\mathbb{E}}_a$ . Noting that

$$\left| e^{-\sum_{x \in \Lambda} v_x^{(\epsilon)} L_{x,T}} \right| \leq e^{-(\mu - \epsilon)T}, \quad (2.35)$$

we obtain (2.25) by dominated convergence.  $\square$

The two representations for  $G_{a,b}^{\text{srw}}$  in Theorems 2.5–2.6 show that the right-hand sides of (2.20) and (2.25) are equal. The following proposition generalizes this equality.

**Proposition 2.7.** *Suppose that  $D - J$  is diagonally dominant, with  $d_x > 0$ ,  $J_{x,y} \geq 0$ . Fix  $0 < \epsilon < \min_{x \in \Lambda} (d_x - \bar{d}_x)$ . Let  $F : [0, \infty)^M \rightarrow \mathbb{C}$  be a Borel function such that there is a constant  $C$  for which  $|F(t)| \leq C \exp(\epsilon \sum_x t_x)$ . Let  $L = (L_x)_{x \in \Lambda}$  and similarly for  $L_T$ . Then*

$$\frac{1}{d_b \pi_{b,\partial}} \mathbb{E}_a (F(L) \mathbb{I}_{X(\zeta^-) = b}) = \int_0^\infty \bar{\mathbb{E}}_a \left( F(L_T) e^{-\sum_{x \in \Lambda} (d_x - \bar{d}_x) L_{x,T}} \mathbb{I}_{X(T) = b} \right) dT. \quad (2.36)$$

*Proof.* Let  $S$  be a Borel subset of  $[0, \infty)^M$ , and let  $\chi_S$  denote the characteristic function of  $S$ . We define  $\mu(S)$  and  $\nu(S)$  by evaluating the left- and right-hand sides of (2.36) on  $F = \chi_S$ , respectively. With these definitions,  $\mu$  and  $\nu$  are finite Borel measures. Together, Theorems 2.5–2.6 establish (2.36) for the special case  $F(t) = e^{-\sum_{x \in \Lambda} v_x t_x}$  when  $\text{Re } v_x \geq 0$ . Therefore, for this choice of  $F$ ,

$$\int_{[0, \infty)^M} F d\mu = \int_{[0, \infty)^M} F d\nu. \quad (2.37)$$

This proves (2.36) in the general case, since finite measures are characterized by their Laplace transforms. The hypothesis on the growth of  $F$  assures its integrability.  $\square$

### 2.3. Self-avoiding walk with loops

Next, we derive a representation for a model of a self-avoiding walk in a background of loops. This requires the introduction of some terminology and notation.

Given not necessarily distinct points  $a, b \in \Lambda$ , a *self-avoiding walk*  $\omega$  from  $a$  to  $b$  is a sequence  $x_0 = a, x_1, x_2, \dots, x_n = b$ , for some  $n \geq 1$ , where  $x_1, x_2, \dots, x_{n-1}$  are distinct points in  $\Lambda \setminus \{a, b\}$ . In other words, for  $a \neq b$ ,  $\omega$  is a non-intersecting path from  $a$  to  $b$  on the complete graph on  $M$  vertices and for  $a = b$  it is non-intersecting except at  $a = b$ . We again write  $|\omega|$  for the length  $n$  of  $\omega$ , and sometimes regard  $\omega$  as consisting of directed edges rather than vertices. Let  $\mathcal{S}_{a,b}$  denote the set of all self-avoiding walks from  $a$  to  $b$ . For  $X \subset \Lambda$ , we write  $\mathcal{S}_{a,b}(X)$  for the subset of  $\mathcal{S}_{a,b}$  consisting of walks with  $x_0 = a, x_n = b$  and  $x_1, x_2, \dots, x_{n-1} \in X$ . A *loop*  $\gamma$  is an unrooted directed cycle (consisting of distinct vertices) in the complete graph, regarded sometimes as a cyclic list of vertices and sometimes as directed edges. We include the *self-loop* which joins a vertex to itself by a single edge, as a possible loop (see Remark 2.9 below). We write  $\mathcal{L}$  for the set of all loops. We write  $\Gamma$  for a subgraph of  $\Lambda$  consisting of mutually-avoiding loops, i.e.,  $\Gamma = \{\gamma_1, \dots, \gamma_m\}$  with each  $\gamma_i \in \mathcal{L}$  and  $\gamma_i \cap \gamma_j = \emptyset$  (as sets of vertices) for  $i \neq j$ . We write  $\mathcal{G}$  for the set of all such  $\Gamma$  (including  $\Gamma = \emptyset$ ), and  $\mathcal{G}(X)$  for the subset of  $\mathcal{G}$  which uses only vertices in  $X \subset \Lambda$ . We write  $|\gamma|$  for the length of  $\gamma$ , and  $|\Gamma| = \sum_{i=1}^m |\gamma_i|$  for the total length of loops in  $\Gamma$ .

Given a  $\Lambda \times \Lambda$  real matrix  $C$ ,  $\omega \in \mathcal{W}_{a,b}$  and  $\Gamma \in \mathcal{G}$ , let

$$C^\Gamma = \prod_{e \in \Gamma} C_e, \quad C^{\omega \cup \Gamma} = C^\omega C^\Gamma, \quad (2.38)$$

where here we regard self-avoiding walks and loops as collections of directed edges and write, e.g.,  $e = (\omega(i-1), \omega(i))$ . An empty product is equal to 1. We define the two-point function

$$G_{a,b}^{\text{loop}} = \sum_{\omega \in \mathcal{S}_{a,b}} \sum_{\Gamma \in \mathcal{G}(\Lambda \setminus \omega)} C^{\omega \cup \Gamma}. \quad (2.39)$$

The representation for  $G_{a,b}^{\text{loop}}$  is elementary and we derive it now.

**Theorem 2.8.** *Let  $C$  have positive Hermitian part. Let  $a, b \in \Lambda$  (not necessarily distinct) and let  $X \subset \Lambda \setminus \{a, b\}$ . Then*

$$\int_{\mathbb{R}^{2M}} d\mu_C \bar{\phi}_a \phi_b \prod_{x \in X} (1 + \phi_x \bar{\phi}_x) = \sum_{\omega \in \mathcal{S}_{a,b}(X)} C^\omega \int_{\mathbb{R}^{2M}} d\mu_C \prod_{x \in X \setminus \omega} (1 + \phi_x \bar{\phi}_x), \quad (2.40)$$

$$\int_{\mathbb{R}^{2M}} d\mu_C \prod_{x \in X} (1 + \phi_x \bar{\phi}_x) = \sum_{\Gamma \in \mathcal{G}(X)} C^\Gamma, \quad (2.41)$$

and, finally,

$$G_{a,b}^{\text{loop}} = \int_{\mathbb{R}^{2M}} d\mu_C \bar{\phi}_a \phi_b \prod_{x \in \Lambda: x \neq a, b} (1 + \phi_x \bar{\phi}_x). \quad (2.42)$$

*Proof.* To prove (2.40), we write  $F = \phi_b \prod_{x \in X} (1 + \phi_x \bar{\phi}_x)$  and apply the integration by parts formula (2.10), which replaces  $\bar{\phi}_a F$  by  $\sum_{v \in \Lambda} C_{a,v} \partial F / \partial \phi_v$ . The first step in the walk  $\omega$  is  $(a, v)$ . If the derivative acts on a factor in the product over  $x$ , then it replaces that factor by  $\bar{\phi}_v$ , and the procedure can be iterated until the derivative acts on  $\phi_b$ , in which case  $\omega$  terminates. The result is (2.40).

For (2.41), we expand the product to obtain

$$\prod_{x \in X} (1 + \phi_x \bar{\phi}_x) = \sum_{Y \subset X} \prod_{y \in Y} \phi_y \bar{\phi}_y. \quad (2.43)$$

and hence

$$\int_{\mathbb{R}^{2M}} d\mu_C \prod_{x \in X} (1 + \phi_x \bar{\phi}_x) = \sum_{Y \subset X} \int_{\mathbb{R}^{2M}} d\mu_C(u) \prod_{y \in Y} \phi_y \bar{\phi}_y. \quad (2.44)$$

We then evaluate the integral on the right-hand side using Lemma 2.3, and this gives (2.41).

The representation (2.42) follows from the combination of (2.40)–(2.41).  $\square$

**Remark 2.9.** *Self-loops can be eliminated in the representation by replacing the right-hand side of (2.42) by*

$$\int_{\mathbb{R}^{2M}} d\mu_C \bar{\phi}_a \phi_b \prod_{w \in \Lambda: w \neq a, b} (1 + : \phi_w \bar{\phi}_w :), \quad (2.45)$$

where

$$: \phi_x \bar{\phi}_x : = \phi_x \bar{\phi}_x - C_{x,x}, \quad (2.46)$$

using a modification of the above proof.

### 3. Self-avoiding walk models

#### 3.1. Self-avoiding walk

We define the two-point function:

$$G_{a,b}^{\text{saw}} = \sum_{\omega \in \mathcal{S}_{a,b}} C^\omega. \quad (3.1)$$

When  $a = b$ , the walks are self-avoiding except for the fact that the walk begins and ends at the same site. In this case, there is, in particular, a contribution due to the one-step walk that steps from  $a$  to  $a$ , which has weight  $C_{a,a} \neq 0$ . The only new result in this paper is the integral representation for  $G_{a,b}^{\text{SAW}}$ . The representation for the loop model (2.39) is easier than for (3.1), as (2.39) is in terms of a bosonic (ordinary) Gaussian integral. To eliminate the loops and obtain a representation for the walk model (3.1), we will need fermionic (Grassmann) integrals involving anticommuting variables. The necessary mathematical background for this is developed in Section 4, and the representation is stated and derived in Section 5.2. This representation is the point of departure for the analysis of the 4-dimensional self-avoiding walk in [10], for a convenient particular choice of  $C$ .

### 3.2. Weakly self-avoiding walk

The two-point functions (2.39) and (3.1) are for strictly self-avoiding walks and loops. We also consider the continuous-time weakly self-avoiding walk, which is defined as follows.

Let  $D$  have diagonal entries  $d_x > 0$ ,  $J$  have zero diagonal entries and  $J_{x,y} \geq 0$ , and suppose that  $D - J$  is diagonally dominant. Let  $X$  and  $\mathbb{E}_a$  be the continuous-time Markov process and corresponding expectation, as in Theorem 2.5. In particular, the process dies at the random time  $\zeta$  at which it makes a transition to the cemetery state. The local time at  $x$  is given by  $L_x = \int_0^\infty \mathbb{1}_{X(s)=x} ds$  (note that the integral effectively terminates at  $\zeta < \infty$ ). By definition,

$$\sum_{x \in \Lambda} L_x^2 = \int_0^\infty ds_1 \int_0^\infty ds_2 \sum_{x \in \Lambda} \mathbb{1}_{X(s_1)=x} \mathbb{1}_{X(s_2)=x} \quad (3.2)$$

$$= \int_0^\infty ds_1 \int_0^\infty ds_2 \mathbb{1}_{X(s_1)=X(s_2) \neq \partial}, \quad (3.3)$$

so  $\sum_{x \in \Lambda} L_x^2$  is a measure of the amount of self-intersection of  $X$  up to time  $\zeta$ . The continuous-time weakly self-avoiding walk two-point function is defined by

$$G_{a,b}^{\text{wsaw}} = \frac{1}{d_b \pi_{b,\partial}} \mathbb{E}_a \left( e^{-g \sum_{x \in \Lambda} L_x^2} e^{-\lambda \zeta} \mathbb{1}_{X(\zeta^-)=b} \right), \quad (3.4)$$

where  $g > 0$ , and  $\lambda$  is a parameter (possibly negative) which is chosen in such a way that the integral converges. In (3.4), self-intersections are suppressed by the factor  $\exp[-g \sum_{x \in \Lambda} L_x^2]$ . We will derive a representation for (3.4) in Section 5.1.

It follows from Proposition 2.7 that there is also the alternate representation:

$$G_{a,b}^{\text{wsaw}} = \int_0^\infty \overline{\mathbb{E}}_a \left( e^{-g \sum_{x \in \Lambda} L_{x,T}^2} e^{-\sum_x (\lambda + d_x - \bar{d}_x) L_{x,T}} \mathbb{1}_{X(T)=b} \right) dT. \quad (3.5)$$

In the homogeneous case, in which  $d_x - \bar{d}_x = a$  is independent of  $x$ , the second exponential can be written as  $e^{-\lambda' T}$  where  $\lambda' = \lambda + a$ . This representation is the starting point for the analysis of the weakly self-avoiding walk on a 4-dimensional hierarchical lattice in [3, 6, 7], on  $\mathbb{Z}^4$  in [10], and for a model on  $\mathbb{Z}^3$  in [20].

#### 4. Gaussian integrals with fermions

In this section, we review some standard material about Gaussian integrals which incorporate anticommuting Grassmann variables. We realize these Grassmann variables as differential forms.

##### 4.1. Differential forms

We recall and extend the formalism introduced in Section 2. Let  $\Lambda = \{1, \dots, M\}$  be a finite set of cardinality  $M$ . Let  $u_1, v_1, \dots, u_M, v_M$  be standard coordinates on  $\mathbb{R}^{2M}$ , so that  $du_1 \wedge dv_1 \wedge \dots \wedge du_M \wedge dv_M$  is the standard volume form on  $\mathbb{R}^{2M}$ , where  $\wedge$  denotes the usual anticommuting wedge product (see [22, Chapter 10] for an introduction). We will drop the wedge from the notation and write simply  $du_i dv_j$  in place of  $du_i \wedge dv_j$ . The one-forms  $du_i, dv_j$  generate the Grassmann algebra of differential forms on  $\mathbb{R}^{2M}$ . A form which is a function of  $u, v$  times a product of  $p$  differentials is said to have *degree*  $p$ , for  $p \geq 0$ .

The integral of a differential form over  $\mathbb{R}^{2M}$  is defined to be zero unless the form has degree  $2M$ . A form  $K$  of degree  $2M$  can be written as  $K = f(u, v) du_1 dv_1 \dots du_M dv_M$ , and we define

$$\int K = \int_{\mathbb{R}^{2M}} f(u, v) du_1 dv_1 \dots du_M dv_M, \quad (4.1)$$

where the right-hand side is the usual Lebesgue integral of  $f$  over  $\mathbb{R}^{2M}$ .

We again complexify by setting  $\phi_x = u_x + iv_x, \bar{\phi}_x = u_x - iv_x$  and  $d\phi_x = du_x + idv_x, d\bar{\phi}_x = du_x - idv_x$ , for  $x \in \Lambda$ . Since the wedge product is anticommutative, the following pairs all anticommute for every  $x, y \in \Lambda$ :  $d\bar{\phi}_x$  and  $d\phi_y, d\bar{\phi}_x$  and  $d\phi_y, d\bar{\phi}_x$  and  $d\bar{\phi}_y$ . Given an  $M \times M$  matrix  $A$ , we write  $\phi A \bar{\phi} = \sum_{x, y \in \Lambda} \phi_x A_{x, y} \bar{\phi}_y$ . As in (2.3),

$$d\bar{\phi}_x d\phi_x = 2i du_x dv_x. \quad (4.2)$$

The integral of a function  $f(\phi, \bar{\phi})$  (a *zero form*) with respect to  $\prod_{x \in \Lambda} d\bar{\phi}_x d\phi_x$  is thus given by  $(2i)^M$  times the integral of  $f(u + iv, u - iv)$  over  $\mathbb{R}^{2M}$ . Note that the product over  $x$  can be taken in any order, since each factor  $d\bar{\phi}_x d\phi_x$  has even degree (namely degree two). To simplify notation, it is convenient to introduce

$$\psi_x = \frac{1}{(2\pi i)^{1/2}} d\phi_x, \quad \bar{\psi}_x = \frac{1}{(2\pi i)^{1/2}} d\bar{\phi}_x, \quad (4.3)$$

where we fix a choice of the square root and use this choice henceforth. Then

$$\bar{\psi}_x \psi_x = \frac{1}{2\pi i} d\bar{\phi}_x d\phi_x = \frac{1}{\pi} du_x dv_x. \quad (4.4)$$

Given any matrix  $A$ , the *action* is the even form defined by

$$S_A = \phi A \bar{\phi} + \psi A \bar{\psi}. \quad (4.5)$$

In the special case  $A_{u,v} = \delta_{u,x}\delta_{x,v}$ ,  $S_A$  becomes the form  $\tau_x$  defined by

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x. \quad (4.6)$$

Let  $K = (K_j)_{j \in J}$  be a collection of forms. When each  $K_j$  is a sum of forms of even degree, we say that  $K$  is *even*. Let  $K_j^{(0)}$  denote the degree-zero part of  $K_j$ . Given a  $C^\infty$  function  $F : \mathbb{R}^J \rightarrow \mathbb{C}$  we define  $F(K)$  by its power series about the degree-zero part of  $K$ , i.e.,

$$F(K) = \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^{(0)})(K - K^{(0)})^{\alpha}. \quad (4.7)$$

Here  $\alpha$  is a multi-index, with  $\alpha! = \prod_{j \in J} \alpha_j!$ , and  $(K - K^{(0)})^{\alpha} = \prod_{j \in J} (K_j - K_j^{(0)})^{\alpha_j}$ . Note that the summation terminates as soon as  $\sum_{j \in J} \alpha_j = M$  since higher order forms vanish, and that the order of the product on the right-hand side is irrelevant when  $K$  is even. For example,

$$e^{-S_A} = e^{-\phi A \bar{\phi}} \sum_{n=0}^M \frac{(-1)^n}{n!} (\psi A \bar{\psi})^n. \quad (4.8)$$

Because the formal power series of a composition of two functions is the same as the composition of the two formal power series, we may regard  $e^{-S_A}$  either as a function of the single form  $S_A$  or of the  $M^2$  forms  $\phi_x \bar{\phi}_y + \frac{1}{2\pi i} d\phi_x d\bar{\phi}_y$ . The same result is obtained for  $e^{-S_A}$  in either case.

## 4.2. Gaussian integrals

We refer to the integral  $\int e^{-S_A} K$  as the mixed bosonic-fermionic Gaussian expectation of  $K$ , or, more briefly, as a mixed expectation. The following proposition shows that if  $K$  is a product of a zero form and factors of  $\psi$  and  $\bar{\psi}$  then the mixed expectation factorizes. Moreover, if  $K$  is a zero form then the mixed expectation is just the usual Gaussian expectation of  $K$ , and if  $K$  is a product of factors of  $\psi$  and  $\bar{\psi}$  then its expectation is a determinant. It also shows that  $\int e^{-S_A}$  is self-normalizing in the sense that it is equal to 1 without any normalization required. The determinant in (4.9) appears also e.g. in [23, Lemma B.7], in a related purely fermionic context and with a different proof.

**Proposition 4.1.** *Let  $A$  have positive Hermitian part, with inverse  $C = A^{-1}$ . Suppose that  $f$  is a zero form. Let  $F = \prod_{r=1}^p \bar{\psi}_{i_r} \prod_{s=1}^q \psi_{j_s}$ . If  $p \neq q$  then  $\int e^{-S_A} f F = 0$ . When  $p = q$ , up to sign we can take  $F = \psi_{i_1} \psi_{j_1} \cdots \psi_{i_p} \psi_{j_p}$  and in this case*

$$\int e^{-S_A} f F = \left( \int e^{-S_A} f \right) \left( \int e^{-S_A} F \right) = I_f \det C_{i_1, \dots, i_p; j_1, \dots, j_p} \quad (4.9)$$

where  $I_f = \int f d\mu_C(\phi, \bar{\phi})$ , and where  $C_{i_1, \dots, i_p; j_1, \dots, j_p}$  is the  $p \times p$  matrix whose  $r, s$  element is  $C_{i_r, j_s}$  when  $p \neq 0$ , and the determinant is replaced by 1 when

$p = 0$ . In particular,

$$\int e^{-S_A} = 1. \quad (4.10)$$

*Proof.* We first note that if  $p \neq q$  then no form of degree  $2M$  can be obtained by expanding  $e^{-\psi A \bar{\psi}} F$  and the integral vanishes. Thus we assume  $p = q$ .

Let  $i = i_1, \dots, i_p$ ,  $j = j_1, \dots, j_p$ , and

$$B_{i,j} = \int e^{-S_A} f \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_p} \psi_{j_p}. \quad (4.11)$$

For  $k \in \Lambda$ , let

$$\tilde{\psi}_k = \sum_{l \in \Lambda} A_{k,l} \bar{\psi}_l. \quad (4.12)$$

The tensor product  $A^{\otimes p}$  is a linear operator on  $V^{\otimes p}$  defined by the matrix elements

$$(A^{\otimes p})_{i,j} = A_{i_1,j_1} A_{i_2,j_2} \cdots A_{i_p,j_p}. \quad (4.13)$$

By definition, (4.8), and the anticommutation relation  $\psi_{k_l} \tilde{\psi}_{k_l} = -\tilde{\psi}_{k_l} \psi_{k_l}$ ,

$$\begin{aligned} (A^{\otimes p} B)_{i,j} &= \int e^{-S_A} f \tilde{\psi}_{i_1} \psi_{j_1} \cdots \tilde{\psi}_{i_p} \psi_{j_p} \\ &= \frac{1}{(M-p)!} \sum_{k_1, \dots, k_{M-p}} \int e^{-\phi A \bar{\phi}} f \tilde{\psi}_{k_1} \psi_{k_1} \cdots \tilde{\psi}_{k_{M-p}} \psi_{k_{M-p}} \tilde{\psi}_{i_1} \psi_{j_1} \cdots \tilde{\psi}_{i_p} \psi_{j_p}. \end{aligned} \quad (4.14)$$

By antisymmetry, for a nonzero contribution,  $k_1, \dots, k_{M-p}, i_1, \dots, i_p$  must be a permutation of  $\Lambda$ , as must be  $k_1, \dots, k_{M-p}, j_1, \dots, j_p$ . In particular,  $j_1, \dots, j_p$  must be a permutation of  $i_1, \dots, i_p$ ; let  $\epsilon_{i,j}$  be the sign of this permutation (and equal zero if it is not a permutation). Then we can rearrange the above to obtain

$$(A^{\otimes p} B)_{i,j} = \epsilon_{i,j} \int e^{-\phi A \bar{\phi}} f \tilde{\psi}_1 \psi_1 \cdots \tilde{\psi}_M \psi_M. \quad (4.15)$$

We insert (4.12) on the right-hand side and again use antisymmetry and then Lemma 2.1 to obtain

$$(A^{\otimes p} B)_{i,j} = \epsilon_{i,j} \det A \int e^{-\phi A \bar{\phi}} f \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_M \psi_M = I_f \epsilon_{i,j}. \quad (4.16)$$

When  $p = 0$  the above calculations give  $B = I_f$ , as required.

For  $p \neq 0$ , we use the fact that  $C^{\otimes p}$  is the inverse of  $A^{\otimes p}$  to obtain

$$B_{k,j} = \sum_l C_{k,l}^{\otimes p} (A^{\otimes p} B)_{l,j} = I_f \sum_l \epsilon_{l,j} C_{k,l}^{\otimes p}. \quad (4.17)$$

The sum on the right-hand side is the determinant  $\det C_{k_1, \dots, k_p; j_1, \dots, j_p}$ , as required.  $\square$

In the Gaussian integral in the above proposition, the fermionic part  $d\phi A d\bar{\phi}$  of the action gives rise to a factor  $\det A$  while the bosonic part  $\phi A \bar{\phi}$  gives rise to the reciprocal of this determinant, providing the cancellation that produces the self-normalization property (4.10).

We will use the following corollary in Section 5.2.1.

**Corollary 4.2.** *Let  $x_1, \dots, x_k$  be distinct elements of  $\Lambda$ . Then*

$$\int e^{-S_A} \psi_{x_1} \bar{\psi}_{x_1} \cdots \psi_{x_k} \bar{\psi}_{x_k} = \sum_{\sigma \in S_k} (-1)^{N(\sigma)} \prod_{l=1}^k C_{x_l, \sigma(x_l)}, \quad (4.18)$$

where  $N(\sigma)$  is the number of cycles in the permutation  $\sigma$ .

*Proof.* It follows from (4.9) and anticommutativity that

$$\int e^{-S_A} \psi_{x_1} \bar{\psi}_{x_1} \cdots \psi_{x_k} \bar{\psi}_{x_k} = (-1)^k \sum_{\sigma \in S_k} \epsilon_\sigma \prod_{l=1}^k C_{x_l, \sigma(x_l)}, \quad (4.19)$$

where  $\epsilon_\sigma$  is the sign of the permutation  $\sigma$ . Then (4.18) follows from the identity

$$\epsilon_\sigma = (-1)^k (-1)^{N(\sigma)}, \quad (4.20)$$

which itself follows from the fact that for a permutation  $\sigma \in S_k$  consisting of cycles  $c$  of length  $|c|$ ,

$$\epsilon_\sigma = \prod_{c \in \sigma} \epsilon_c = \prod_{c \in \sigma} (-1)^{|c|+1} = (-1)^k (-1)^{N(\sigma)}. \quad (4.21)$$

□

**Remark 4.3.** *The omission of the operation  $A^{\otimes p}$  in (4.14)–(4.16) leads to the alternate formula*

$$B_{i,j} = \int e^{-S_A} f \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_p} \psi_{j_p} = I_f \frac{1}{\det A} \det \hat{A}_{i_1, \dots, i_p; j_1, \dots, j_p} \epsilon_{\sigma_i} \epsilon_{\sigma_j}, \quad (4.22)$$

where  $\sigma_i \in S_M$  is the permutation that moves  $i_1, \dots, i_p$  to  $1, \dots, p$  and preserves the order of the other indices and  $\epsilon_{\sigma_i}$  is its sign (and similarly for  $\sigma_j$ ), and where  $\hat{A}_{i_1, \dots, i_p; j_1, \dots, j_p}$  is the  $(M-p) \times (M-p)$  matrix obtained from  $A$  by deleting rows  $j_1, \dots, j_p$  and columns  $i_1, \dots, i_p$ . The identity (4.22) is essentially [9, Lemma 4]. This proves the fact from linear algebra that

$$\det C_{i_1, \dots, i_p; j_1, \dots, j_p} = \frac{1}{\det A} \det \hat{A}_{i_1, \dots, i_p; j_1, \dots, j_p} \epsilon_{\sigma_i} \epsilon_{\sigma_j}. \quad (4.23)$$

The case  $p = 1$  of (4.23) states that

$$C_{i_1; j_1} = A_{i_1; j_1}^{-1} = \frac{1}{\det A} \det \hat{A}_{i_1; j_1} (-1)^{i_1 + j_1}, \quad (4.24)$$

which is Cramer's rule. Thus (4.23) is a generalization of Cramer's rule.



### 4.3. Integrals of functions of $\tau$

The identity (4.25) below provides an extension of (4.10), and will be used in Section 5.2. The identity (4.26) is sometimes called the  $\tau$ -isomorphism; it will lead to a representation for the weakly self-avoiding walk two-point function (3.4). Our method of proof follows the method of [3, 15]. Alternate approaches to (4.25) are given in Sections 5.2.1 and 6.

Recall the definitions of  $\tau_x$  in (4.6) and  $L_x$  above Theorem 2.5. We write  $\tau$  for the entire collection  $(\tau_x)_{x \in \Lambda}$ , and similarly for  $L$ .

**Proposition 4.4.** *Suppose that  $A$  has positive Hermitian part. Let  $F$  be a  $C^\infty$  function on  $[0, \infty)^M$  ( $C^\infty$  also on the boundary), and assume that for each  $\epsilon > 0$  and multi-index  $\alpha$  there is a constant  $C = C_{\epsilon, \alpha}$  such that  $F$  and its derivatives obey  $|F^{(\alpha)}(t)| \leq C \exp(\epsilon \sum_{x \in \Lambda} t_x)$  for all  $t \in [0, \infty)^M$ . Then*

$$\int e^{-S_A} F(\tau) = F(0). \quad (4.25)$$

Suppose further that  $A = D - J$  is diagonally dominant and real. Then

$$\int e^{-S_A} F(\tau) \bar{\phi}_a \phi_b = \frac{1}{d_b \pi_{b, \partial}} \mathbb{E}_a (F(L) \mathbb{1}_{X(\zeta^-) = b}). \quad (4.26)$$

*Proof.* It is straightforward to adapt the result of [24] to extend  $F$  to a  $C^\infty$  function on  $\mathbb{R}^M$ , which we also call  $F$ . By multiplying  $F$  by a suitable  $C^\infty$  function, we can further assume that  $F$  is equal to zero on the complement of  $[-1, \infty)^M$ . Fix  $\epsilon > 0$  such that  $A - \epsilon I$  has positive Hermitian part, and let  $H(t) = F(t) \exp(-\epsilon \sum_x t_x)$ . Then  $H$  is a Schwartz class function. Its Fourier transform is defined by

$$\widehat{H}(v) = \int_{\mathbb{R}^M} H(t) e^{iv \cdot t} dt_1 \dots dt_M, \quad (4.27)$$

where  $v \cdot t = \sum_{x \in \Lambda} v_x t_x$ . The function  $H$  can be recovered via the inverse Fourier transform as

$$H(t) = (2\pi)^{-M} \int_{\mathbb{R}^M} \widehat{H}(v) e^{-iv \cdot t} dv_1 \dots dv_M. \quad (4.28)$$

Since  $H$  is of Schwartz class, the above integral is absolutely convergent. Also,

$$F(t) = (2\pi)^{-M} \int_{\mathbb{R}^M} \widehat{H}(v) e^{\sum_x (-iv_x + \epsilon) t_x} dv_1 \dots dv_M. \quad (4.29)$$

We may replace  $t$  by  $\tau$  in (4.29); this amounts to a statement about differentiating under the integral since functions of  $\tau$  are defined by their power series as in (4.7). Let  $V$  be the real diagonal matrix with  $V_{x,x} = v_x$ . Since  $A - \epsilon I + iV$  has positive Hermitian part, (4.10) gives

$$\int e^{-S_A} e^{\sum_x (-iv_x + \epsilon) \tau_x} = \int e^{-S_{A - \epsilon I + iV}} = 1. \quad (4.30)$$

Assuming that it is possible to interchange the integrals, we obtain

$$\int e^{-S_A} F(\tau) = (2\pi)^{-M} \int_{\mathbb{R}^M} \widehat{H}(v) dv_1 \dots dv_M = H(0) = F(0), \quad (4.31)$$

which is (4.25).

To complete the proof of (4.25), it remains only to justify the interchange of integrals; this can be done as follows. By definition, the iterated integral

$$\int e^{-S_A} \int_{\mathbb{R}^M} dv_1 \dots dv_M \widehat{H}(v) e^{\sum_x (-iv_x + \epsilon)\tau_x} \quad (4.32)$$

is equal to

$$\begin{aligned} \sum_{n,N} \frac{(-1)^N}{n!N!} \int e^{-\phi A \bar{\phi}} (\psi A \bar{\psi})^N \left( \sum_x (-iv_x + \epsilon) \psi_x \bar{\psi}_x \right)^n \\ \times \int_{\mathbb{R}^M} dv_1 \dots dv_M \widehat{H}(v) e^{\sum_x (-iv_x + \epsilon)\phi_x \bar{\phi}_x}. \end{aligned} \quad (4.33)$$

According to our definition of integration, the outer integral is evaluated as a usual Lebesgue integral by keeping the (finitely many) terms that produce the standard volume form on  $\mathbb{R}^{2M}$ . Since  $\widehat{H}$  is Schwartz class and  $A - \epsilon I$  has positive Hermitian part, the resulting iterated Lebesgue integral is absolutely convergent and its order can be interchanged by Fubini's theorem. Once the integrals have been interchanged, the sums over  $n$  and  $N$  can be resummed to see that (4.32) has the same value when its two integrals are interchanged, and the proof of (4.25) is complete.

To prove (4.26), we fix  $\epsilon > 0$  such that  $A - \epsilon I$  is diagonally dominant. Then

$$\begin{aligned} \int e^{-S_A} e^{\sum_x (-iv_x + \epsilon)\tau_x} \bar{\phi}_a \phi_b &= \int e^{-S_{A - \epsilon I + iV}} \bar{\phi}_a \phi_b = G_{a,b}^{\text{srw}}(-\epsilon + iv) \\ &= \frac{1}{d_b \pi_{b,\partial}} \mathbb{E}_a \left( e^{(\epsilon - iv) \cdot L} \mathbb{I}_{X(\zeta^-) = b} \right), \end{aligned} \quad (4.34)$$

where we have used (4.9) and Theorem 2.4 in the second equality, and Theorem 2.5 in the third. With further application of Fubini's theorem, we obtain

$$\begin{aligned} \int e^{-S_A} \bar{\phi}_a \phi_b F(\tau) &= \frac{1}{d_b \pi_{b,\partial}} \mathbb{E}_a \left( e^{\epsilon \cdot L} (2\pi)^{-M} \int_{\mathbb{R}^M} \widehat{H}(v) e^{-iv \cdot L} dv \mathbb{I}_{X(\zeta^-) = b} \right) \\ &= \frac{1}{d_b \pi_{b,\partial}} \mathbb{E}_a (F(L) \mathbb{I}_{X(\zeta^-) = b}), \end{aligned} \quad (4.35)$$

which is (4.26).  $\square$

## 5. Self-avoiding walk representations

### 5.1. Weakly self-avoiding walk

#### 5.1.1. The representation

**Theorem 5.1.** *The weakly self-avoiding walk two-point function  $G_{a,b}^{\text{wsaw}}$  has the representation*

$$G_{a,b}^{\text{wsaw}} = \int e^{-S_A} \bar{\phi}_a \phi_b e^{-g \sum_{x \in \Lambda} \tau_x^2 - \lambda \sum_{x \in \Lambda} \tau_x}. \quad (5.1)$$

*Proof.* This is immediate when we take  $F(\tau) = e^{-g \sum_{x \in \Lambda} \tau_x^2 - \lambda \sum_{x \in \Lambda} \tau_x}$  in (4.26), and compare with (3.4).  $\square$

#### 5.1.2. The $N \rightarrow 0$ limit

If we omit the fermions from the right-hand side of (5.1) and normalize the integral then we obtain instead the two-point function of the  $|\phi|^4$  field theory, namely

$$\langle \bar{\phi}_a \phi_b \rangle = \frac{\int d\mu_C \bar{\phi}_a \phi_b e^{-g \sum_{x \in \Lambda} |\phi_x|^4 - \lambda \sum_{x \in \Lambda} |\phi_x|^2}}{\int d\mu_C e^{-g \sum_{x \in \Lambda} |\phi_x|^4 - \lambda \sum_{x \in \Lambda} |\phi_x|^2}}. \quad (5.2)$$

This is known to have a representation as the two-point function of a system of a weakly self-avoiding walk and weakly self-avoiding loops, all weakly mutually-avoiding [5, 25], as we now briefly sketch.

Let  $n_x(\omega)$  denote the number of visits to  $x$  by a walk  $\omega$ . Let

$$d\nu_n(s) = \begin{cases} \delta(s) ds & n = 0 \\ \frac{s^{n-1}}{(n-1)!} \mathbb{1}_{s \geq 0} ds & n \geq 1 \end{cases}, \quad d\nu_\omega(t) = \prod_{x \in \Lambda} d\nu_{n_x(\omega)}(t_x). \quad (5.3)$$

It follows from [5, Theorem 2.1] (see also [4, p.137] and [12, p.197]) that for a real  $N$ -component field  $\phi$ , for any component  $i$  we have

$$\begin{aligned} \langle \phi_a^{(i)} \phi_b^{(i)} \rangle &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{N}{2} \right)^n \sum_{\omega \in \mathcal{W}_{a,b}} \sum_{x_1, \dots, x_n \in \Lambda} \sum_{\omega_1 \in \mathcal{W}_{x_1, x_1}} \cdots \sum_{\omega_n \in \mathcal{W}_{x_n, x_n}} \\ &\times \frac{J^{\omega \cup \omega_1 \cup \dots \cup \omega_n}}{\|\omega_1\| \cdots \|\omega_n\|} \int d\nu_{\omega \cup \omega_1 \cup \dots \cup \omega_n}(t) e^{-4g \sum_{x \in \Lambda} t_x^2 - 2\lambda \sum_{x \in \Lambda} t_x}, \end{aligned} \quad (5.4)$$

where  $\|\omega\| = |\omega| + 1$  denotes the number of vertices in  $\omega$ ,

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{N}{2} \right)^n \sum_{x_1, \dots, x_n \in \Lambda} \sum_{\omega_1 \in \mathcal{W}_{x_1, x_1}} \cdots \sum_{\omega_n \in \mathcal{W}_{x_n, x_n}} \\ &\times \frac{J^{\omega_1 \cup \dots \cup \omega_n}}{\|\omega_1\| \cdots \|\omega_n\|} \int d\nu_{\omega_1 \cup \dots \cup \omega_n}(t) e^{-4g \sum_{x \in \Lambda} t_x^2 - 2\lambda \sum_{x \in \Lambda} t_x} \end{aligned} \quad (5.5)$$

is a normalization constant, and

$$d\nu_{\omega \cup \omega_1 \cup \dots \cup \omega_n}(t) = \prod_{x \in \Lambda} d\nu_{n_x(\omega) + n_x(\omega_1) + \dots + n_x(\omega_n) + N/2}(t_x). \quad (5.6)$$

Note the factor  $N/2$  associated to each loop. If we simply set  $N = 0$  in these formulas, then only the  $n = 0$  term survives, and we obtain the formal limit (formal, because the left-hand side is defined only for  $N = 1, 2, 3, \dots$ )

$$\lim_{N \rightarrow 0} \langle \phi_a^{(1)} \phi_b^{(1)} \rangle = \sum_{\omega \in \mathcal{W}_{a,b}} J^\omega \int d\nu_\omega(t) e^{-4g \sum_{x \in \Lambda} t_x^2 - 2\lambda \sum_{x \in \Lambda} t_x}. \quad (5.7)$$

As we argue next, the right-hand side of (5.7) is equal to the weakly self-avoiding walk two-point function  $G_{a,b}^{\text{wsaw}}$  (with modified parameters  $g, \lambda$ ). This recovers de Gennes' idea, in the context of the weakly self-avoiding walk [1].

We now show that the right-hand side of (5.7) is equal to the right-hand side in the representation (3.4) of  $G_{a,b}^{\text{wsaw}}$ , with constant  $d_x \equiv d$ . As in the proof of Theorem 2.5, we condition on the events  $\{\eta = n\}$  and also on  $Y = (Y_0, Y_1, \dots, Y_n) \in \mathcal{W}_{a,b}$ . Given both of these, the random variable  $L_x$  has a  $\Gamma(n_x(Y), d)$  distribution, since it is the sum of independent  $\text{Exp}(d)$  random variables. Thus we obtain

$$\begin{aligned} G_{a,b}^{\text{wsaw}} &= \frac{1}{d\pi_{b,\partial}} \mathbb{E}_a \left( e^{-g \sum_x L_x^2 - \lambda \sum_x L_x} \mathbb{1}_{X(\zeta^-) = b} \right) \\ &= \frac{1}{d} \sum_{n=0}^{\infty} \mathbb{E}_a \left[ \mathbb{E}_a \left( e^{-g \sum_x L_x^2 - \lambda \sum_x L_x} | Y_0, \dots, Y_n \right) \right]. \end{aligned} \quad (5.8)$$

Since

$$\mathbb{E}_a \left( e^{-g \sum_x L_x^2 - \lambda \sum_x L_x} | Y_0, \dots, Y_n \right) = \int d\Gamma_Y(t) e^{-g \sum_x t_x^2 - \lambda \sum_x t_x} \quad (5.9)$$

with

$$d\Gamma_Y(t) = \prod_{x \in \Lambda} d\nu_{n_x(Y)}(t_x) d^{n_x(Y)} e^{-dt_x} = d\nu_Y(t) d^{n+1} e^{-d \sum_x t_x}, \quad (5.10)$$

this gives

$$\begin{aligned} G_{a,b}^{\text{wsaw}} &= \frac{1}{d} \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_{a,b}: |\omega| = n} \left( \frac{J}{d} \right)^\omega \int d\nu_\omega(t) d^{n+1} e^{-d \sum_x t_x} e^{-g \sum_x t_x^2 - \lambda \sum_x t_x} \\ &= \sum_{\omega \in \mathcal{W}_{a,b}} J^\omega \int d\nu_\omega(t) e^{-g \sum_x t_x^2 - (\lambda + d) \sum_x t_x}, \end{aligned} \quad (5.11)$$

which is the right-hand side of (5.7) with a modified choice of constants in the exponent.

Theorem 5.1 provides an alternative to the above formal  $N \rightarrow 0$  limit. The inclusion of fermions in Theorem 5.1 has eliminated all the loops, leaving only the weakly self-avoiding walk. In Section 5.2.1, we will make explicit the mechanism

by which this occurs in the strictly self-avoiding walk representation: fermionic loops cancel the bosonic ones.

### 5.2. Strictly self-avoiding walk

Here we obtain the representation for (3.1). We give two proofs based on two different ideas.

#### 5.2.1. Proof by expansion and resummation

**Theorem 5.2.** *Let  $A$  have positive Hermitian part, and let  $C = A^{-1}$  denote its inverse. For all  $a, b \in \Lambda$ ,*

$$G_{a,b}^{\text{saw}} = \int e^{-S_A} \bar{\phi}_a \phi_b \prod_{x \in \Lambda \setminus \{a,b\}} (1 + \tau_x). \quad (5.12)$$

*Proof.* We write  $X = \Lambda \setminus \{a, b\}$ . By expanding the product of  $1 + \tau_x = (1 + \phi_x \bar{\phi}_x) + \psi_x \bar{\psi}_x$ , we obtain

$$\prod_{x \in X} (1 + \tau_x) = \sum_{Y \subset X} \left( \prod_{y \in Y} \psi_y \bar{\psi}_y \right) \left( \prod_{z \in X \setminus Y} (1 + \phi_z \bar{\phi}_z) \right). \quad (5.13)$$

Thus, by Proposition 4.1,

$$\begin{aligned} & \int e^{-S_A} \bar{\phi}_a \phi_b \prod_{x \in X} (1 + \tau_x) \\ &= \sum_{Y \subset X} \left( \int e^{-S_A} \prod_{y \in Y} \psi_y \bar{\psi}_y \right) \left( \int e^{-S_A} \bar{\phi}_a \phi_b \prod_{z \in X \setminus Y} (1 + \phi_z \bar{\phi}_z) \right). \end{aligned} \quad (5.14)$$

By (2.40),

$$\int e^{-S_A} \bar{\phi}_a \phi_b \prod_{z \in X \setminus Y} (1 + \phi_z \bar{\phi}_z) = \sum_{\omega \in \mathcal{S}_{a,b}(X \setminus Y)} C^\omega \int e^{-S_A} \prod_{z \in X \setminus (Y \cup \omega)} (1 + \phi_z \bar{\phi}_z), \quad (5.15)$$

where we have also used (4.9) twice to equate bosonic and mixed bosonic-fermionic integrals. Another application of Proposition 4.1 then gives

$$\begin{aligned} & \int e^{-S_A} \bar{\phi}_a \phi_b \prod_{x \in X} (1 + \tau_x) \\ &= \sum_{Y \subset X} \sum_{\omega \in \mathcal{S}_{a,b}(X \setminus Y)} C^\omega \int e^{-S_A} \prod_{y \in Y} \psi_y \bar{\psi}_y \prod_{z \in X \setminus (Y \cup \omega)} (1 + \phi_z \bar{\phi}_z). \end{aligned} \quad (5.16)$$

We now interchange the sums over  $Y$  and  $\omega$ , and then resum to obtain

$$\begin{aligned} & \int e^{-S_A} \bar{\phi}_a \phi_b \prod_{x \in X} (1 + \tau_x) \\ &= \sum_{\omega \in \mathcal{S}_{a,b}} C^\omega \sum_{Y \subset X \setminus \omega} \int e^{-S_A} \prod_{y \in Y} \psi_y \bar{\psi}_y \prod_{z \in (X \setminus \omega) \setminus Y} (1 + \phi_z \bar{\phi}_z) \\ &= \sum_{\omega \in \mathcal{S}_{a,b}} C^\omega \int e^{-S_A} \prod_{x \in X \setminus \omega} (1 + \tau_x). \end{aligned} \quad (5.17)$$

By (4.25), the integral in the last line is 1, and we obtain (5.12).  $\square$

The above proof ultimately relies on the identity

$$\int e^{-S_A} \prod_{x \in X} (1 + \tau_x) = 1, \quad (5.18)$$

for a subset  $X \subset \Lambda$ . This identity follows immediately from (4.25). We now give an alternate, more direct proof of (5.18), which demonstrates that (5.18) results from the explicit cancellation of bosonic loops carrying a factor  $+1$  with fermionic loops carrying a factor  $-1$ . The net effect of a loop is  $(+1) + (-1) = 0$ , which provides a realization of the self-avoiding walk as corresponding to an  $N = 0$  model, without the need of a mysterious  $N \rightarrow 0$  limit.

*Alternate proof of (5.18).* We expand the last product in (5.13) and apply Proposition 4.1 to obtain

$$\int e^{-S_A} \prod_{x \in X} (1 + \tau_x) = \sum_{\text{disjoint } X_1, X_2 \subset X} \int e^{-S_A} \prod_{u \in X_1} \phi_u \bar{\phi}_u \int e^{-S_A} \prod_{v \in X_2} \psi_v \bar{\psi}_v. \quad (5.19)$$

The term  $X_1 = X_2 = \emptyset$  is special, and contributes 1 to the above right-hand side. We write  $S(X_i)$  for the set of permutations of  $X_i$ ,  $c_i$  for a cycle of  $\sigma_i \in S(X_i)$ , and  $W_{c_i} = \prod_{e \in c_i} C_e$  for the weight of the loop corresponding to the cycle  $c_i$ . With this notation, we can evaluate the integrals using Lemma 2.3 and (4.18) to find that the contribution to the right-hand side of (5.19) due to all terms other than  $X_1 = X_2 = \emptyset$  is equal to

$$\sum_{Y \subset X: Y \neq \emptyset} \sum_{\substack{\text{disjoint } X_1, X_2 \\ X_1 \cup X_2 = Y}} \sum_{\substack{\sigma_1 \in S(X_1) \\ \sigma_2 \in S(X_2)}} \prod_{c_1 \in \sigma_1} W_{c_1} \prod_{c_2 \in \sigma_2} (-W_{c_2}). \quad (5.20)$$

We claim that this equals

$$\sum_{Y \subset X: Y \neq \emptyset} \sum_{\sigma \in S(Y)} \prod_{c \in \sigma} (W_c + (-W_c)) = 0. \quad (5.21)$$

This is a consequence of the fact that, for fixed  $Y$ ,

$$\sum_{\sigma \in S(Y)} \prod_{c \in \sigma} (P_c + Q_c) = \sum_{\substack{\text{disjoint } X_1, X_2 \\ X_1 \cup X_2 = Y}} \sum_{\substack{\sigma_1 \in S(X_1) \\ \sigma_2 \in S(X_2)}} \prod_{c_1 \in \sigma_1} P_{c_1} \prod_{c_2 \in \sigma_2} Q_{c_2}, \quad (5.22)$$

which follows by expanding the product on the left-hand side.  $\square$

## 5.2.2. Proof by integration by parts

The integration by parts formula (2.10) extends easily to the mixed bosonic-fermionic case, to give

$$\int e^{-SA} \bar{\phi}_x F = \sum_{v \in \Lambda} C_{x,v} \int e^{-SA} \frac{\partial F}{\partial \phi_v}, \quad (5.23)$$

where  $A$  has positive Hermitian part,  $C = A^{-1}$ , and  $F$  is any  $C^\infty$  form such that both sides are integrable. To see this, we first note that by linearity it suffices to consider the case  $F = fK$  where  $f$  is a zero form and  $K$  is a product of factors of  $\psi$  and  $\bar{\psi}$ . By Proposition 4.1 and (2.10),

$$\begin{aligned} \int e^{-SA} \bar{\phi}_x f K &= \int e^{-SA} \bar{\phi}_x f \int e^{-SA} K \\ &= \sum_{v \in \Lambda} C_{x,v} \int e^{-SA} \frac{\partial f}{\partial \phi_v} \int e^{-SA} K \\ &= \sum_{v \in \Lambda} C_{x,v} \int e^{-SA} \frac{\partial f K}{\partial \phi_v}, \end{aligned} \quad (5.24)$$

and this proves (5.23).

The special case  $F = \phi_y$  in (5.23) gives  $\int e^{-SA} \bar{\phi}_a \phi_b = C_{a,b}$ . More interestingly, the choice  $F = \phi_b(1 + \tau_x)$  gives  $\int e^{-SA} \bar{\phi}_a \phi_b(1 + \tau_x) = C_{a,b} + C_{a,x} C_{x,b}$ . In the Gaussian integral, the effect of  $\bar{\phi}_a$  is to start a walk step at  $a$ , whereas  $\phi_b$  has the effect of terminating a walk step at  $b$ . Each step receives the appropriate matrix element of the covariance  $C$  as its weight. This leads to the following alternate proof of Theorem 5.2.

*Second proof of Theorem 5.2.* The right-hand side of (5.12) is equal to

$$\int e^{-SA} \bar{\phi}_a F \quad (5.25)$$

with

$$F = \phi_b \prod_{x \neq a,b} (1 + \tau_x), \quad (5.26)$$

and hence

$$\frac{\partial F}{\partial \phi_v} = \delta_{b,v} \prod_{x \neq a,b} (1 + \tau_x) + \mathbb{I}_{v \neq a,b} \phi_b \bar{\phi}_v \prod_{x \neq a,b,v} (1 + \tau_x). \quad (5.27)$$

Substitution of (5.27) into (5.23), using (4.25), gives

$$\int e^{-SA} \bar{\phi}_a F = C_{a,b} + \sum_{v \neq a,b} C_{a,v} \int e^{-SA} \bar{\phi}_v \phi_b \prod_{x \neq a,b,v} (1 + \tau_x). \quad (5.28)$$

After iteration, the right-hand side gives  $G_{a,b}^{\text{SAW}}$ .  $\square$

### 5.3. Comparison of two self-avoiding walk representations

The representations (5.1) and (5.12) state that

$$G_{a,b}^{\text{wsaw}} = \int e^{-S_A \bar{\phi}_a \phi_b} e^{-g \sum_{x \in \Lambda} \tau_x^2 - \lambda \sum_{x \in \Lambda} \tau_x}. \quad (5.29)$$

$$G_{a,b}^{\text{saw}} = \int e^{-S_A \bar{\phi}_a \phi_b} \prod_{x \in \Lambda \setminus \{a,b\}} (1 + \tau_x). \quad (5.30)$$

These are heuristically related as follows. We insert the missing factors for  $x = a, b$  in the product in (5.30), and make the (uncontrolled) approximation

$$\prod_{x \in \Lambda} (1 + \tau_x) = e^{\sum_{x \in \Lambda} \tau_x} \prod_{x \in \Lambda} (1 + \tau_x) e^{-\tau_x} \approx e^{\sum_{x \in \Lambda} \tau_x} \prod_{x \in \Lambda} e^{-\frac{1}{2} \tau_x^2}. \quad (5.31)$$

The approximation amounts to matching terms up to order  $\tau_x^2$  in a Taylor expansion. With this approximation, (5.30) corresponds to (5.29) with  $g = \frac{1}{2}$  and  $\lambda = -1$ . A careful comparison of the two models is given in [10].

## 6. Supersymmetry

Integrals such as  $\int e^{-S_A} F(\tau)$  are unchanged if we formally interchange the pairs  $\phi, \bar{\phi}$  and  $\psi, \bar{\psi}$ . By (4.25), it is also true that  $\int e^{-S_A} F(\tau) \bar{\phi}_a \phi_b = \int e^{-S_A} F(\tau) \bar{\psi}_a \psi_b$  (the difference is  $\int e^{-S_A} \tau F(\tau) = 0$ ). This suggests the existence of a symmetry between bosons and fermions. Such a symmetry is called a *supersymmetry*.

In this section, as a brief illustration, we use methods of supersymmetry to provide an alternate proof of (4.25), following [7]. The supersymmetry generator  $Q$  is a map on the space of forms which maps bosons to fermions and vice versa. It can be defined in terms of standard operations in differential geometry, namely the exterior derivative and interior product, as follows.

An antiderivation  $F$  is a linear map on forms which obeys  $F(\omega_1 \wedge \omega_2) = F\omega_1 \wedge \omega_2 + (-1)^{p_1} \omega_1 \wedge F\omega_2$ , when  $\omega_1$  is a form of degree  $p_1$ . The *exterior derivative*  $d$  is the linear antiderivation that maps a form of degree  $p$  to a form of degree  $p + 1$ , defined by  $d^2 = 0$  and, for a zero form  $f$ ,

$$df = \sum_{x \in \Lambda} \left( \frac{\partial f}{\partial \phi_x} d\phi_x + \frac{\partial f}{\partial \bar{\phi}_x} d\bar{\phi}_x \right). \quad (6.1)$$

Consider the flow acting on  $\mathbb{C}^M$  defined by  $\phi_x \mapsto e^{-2\pi i \theta} \phi_x$ . This flow is generated by the vector field  $X$  defined by  $X(\phi_x) = -2\pi i \phi_x$ , and  $X(\bar{\phi}_x) = 2\pi i \bar{\phi}_x$ . The action by pullback of the flow on forms is

$$d\phi_x \mapsto d(e^{-2\pi i \theta} \phi_x) = e^{-2\pi i \theta} d\phi_x, \quad d\bar{\phi}_x \mapsto e^{2\pi i \theta} d\bar{\phi}_x. \quad (6.2)$$

The *interior product*  $\underline{i} = \underline{i}_X$  with the vector field  $X$  is the linear antiderivation that maps forms of degree  $p$  to forms of degree  $p - 1$  (and maps forms of degree zero to zero), given by

$$\underline{i}d\phi_x = -2\pi i \phi_x, \quad \underline{i}d\bar{\phi}_x = 2\pi i \bar{\phi}_x. \quad (6.3)$$

The interior product obeys  $\underline{i}^2 = 0$ .



The *supersymmetry generator*  $Q$  is defined by

$$Q = d + \underline{i}. \quad (6.4)$$

A form  $\omega$  that satisfies  $Q\omega = 0$  is called *supersymmetric* or *Q-closed*. A form  $\omega$  that is in the image of  $Q$  is called *Q-exact*. Note that the integral of any *Q-exact* form is zero (assuming that the form decays appropriately at infinity), since integration acts only on forms of top degree  $2N$  and the degree of  $\underline{i}\omega$  is at most  $2N - 1$ , while  $\int d\omega = 0$  by Stokes' theorem. We will use the fact that  $Q$  obeys the chain rule for even forms, in the sense that if  $K = (K_1, \dots, K_t)$  with each  $K_i$  an even form, and if  $F : \mathbb{C}^t \rightarrow \mathbb{C}$  is  $C^\infty$ , then

$$QF(K) = \sum_{i=1}^t F_i(K) QK_i, \quad (6.5)$$

where  $F_i$  denotes the partial derivative. A proof is given below.

The *Lie derivative*  $\mathcal{L} = \mathcal{L}_X$  is the infinitesimal flow obtained by differentiating with respect to the flow at  $\theta = 0$ . Thus, for example,

$$\mathcal{L} d\phi_x = \frac{d}{d\theta} e^{-2\pi i \theta} d\phi_x \Big|_{\theta=0} = -2\pi i d\phi_x. \quad (6.6)$$

A form  $\omega$  is defined to be *invariant* if  $\mathcal{L}\omega = 0$ . For example, the form

$$u_{x,y} = \phi_x d\bar{\phi}_y \quad (6.7)$$

is invariant since it is constant under the flow of  $X$ . Cartan's formula asserts that  $\mathcal{L} = d\underline{i} + \underline{i}d$  (see, e.g., [14, p. 146]). Since  $d^2 = 0$  and  $\underline{i}^2 = 0$ , we have that  $\mathcal{L} = Q^2$ , so  $Q$  is the square root of  $\mathcal{L}$ .

*Alternate proof of (4.25).* We will show that  $\int e^{-S_A} F(\lambda\tau)$  is independent of  $\lambda \in \mathbb{R}$ . Comparing the value of this integral for  $\lambda = 0$  and  $\lambda = 1$ , the identity (4.25) then follows from (4.10). Computation of the derivative gives

$$\frac{d}{d\lambda} \int e^{-S_A} F(\lambda\tau) = \int e^{-S_A} \sum_{x \in \Lambda} F_x(\lambda\tau) \tau_x, \quad (6.8)$$

where  $F_x$  denotes the partial derivative of  $F$  with respect to coordinate  $x$ . To show that the integral on the right-hand side vanishes, it suffices to show that the integrand is *Q-exact*. Let  $v_{x,y} = \frac{1}{2\pi i} u_{x,y}$ , where  $u_{x,y}$  is given by (6.7). Then  $v_{x,y}$  is invariant, and since  $Qv_{x,x} = \tau_x$ ,  $\tau_x$  is both *Q-exact* and *Q-closed*. Since  $Q(\sum_{x,y} A_{x,y} v_{x,y}) = S_A$  and  $\sum_{x,y} A_{x,y} v_{x,y}$  is invariant, the form  $S_A$  is also *Q-exact* and *Q-closed*. By (6.5),  $e^{-S_A}$  and  $F_x(\lambda\tau)$  are both *Q-closed*. Therefore, since  $Q$  is an antiderivation,

$$e^{-S_A} F_x(\lambda\tau) \tau_x = Q(e^{-S_A} F_x(\lambda\tau) v_{x,x}), \quad (6.9)$$

as required.  $\square$

*Proof of the chain rule (6.5) for  $Q$ .* Suppose first that  $K$  is a zero form. Then

$$QF(K) = dF(K) = \sum_{i=1}^t \left[ \frac{\partial F(K)}{\partial \phi_i} d\phi_i + \frac{\partial F(K)}{\partial \bar{\phi}_i} d\bar{\phi}_i \right]. \quad (6.10)$$

By the chain rule, this is  $\sum_i F_i(K) dK_i = \sum_i F_i(K) QK_i$ . This proves (6.5) for zero forms, so we may assume now that  $K$  is higher degree.

Let  $\epsilon_i$  be the multi-index that has  $i^{\text{th}}$  component 1 and all other components 0. Let  $K^{(0)}$  denote the degree zero part of  $K$ . By (4.7), the fact that  $Q$  is an antiderivation, and the chain rule applied to zero forms,

$$\begin{aligned} QF(K) &= \sum_{\alpha} \frac{1}{\alpha!} QF^{(\alpha)}(K^{(0)})(K - K^{(0)})^{\alpha} + \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^{(0)})Q(K - K^{(0)})^{\alpha} \\ &= \sum_{\alpha} \frac{1}{\alpha!} \sum_{i=1}^t F^{(\alpha+\epsilon_i)}(K^{(0)})[QK_i^{(0)}](K - K^{(0)})^{\alpha} \\ &\quad + \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^{(0)})Q(K - K^{(0)})^{\alpha}. \end{aligned} \quad (6.11)$$

Since  $Q$  is an antiderivation,

$$Q(K - K^{(0)})^{\alpha} = \sum_{i=1}^t \alpha_i (K - K^{(0)})^{\alpha - \epsilon_i} [QK_i - QK_i^{(0)}]. \quad (6.12)$$

The first term on the right-hand side of (6.11) is canceled by the contribution to the second term of (6.11) due to the second term of (6.12). And the contribution to the second term of (6.11) due to the first term of (6.12) is  $\sum_i F_i(K)QK_i$ , as required.  $\square$

## 7. Conclusion

We have given a unified treatment of three representations for simple random walk in Theorems 2.4, 2.5 and 2.6. These representations had appeared previously in [5, 11, 3]. In Theorem 2.8, we have represented a model of a self-avoiding walk in a background of self-avoiding loops, all mutually avoiding, in terms of a (bosonic) Gaussian integral.

Mixed bosonic-fermionic Gaussian integrals were introduced in Section 4, and some elements of the theory of these integrals were derived. Using these integrals, and particularly using Proposition 4.4, representations for the weakly self-avoiding walk and strictly self-avoiding walk were obtained in Theorems 5.1 and 5.2, respectively. Our representation in Theorem 5.2 is new. These representations provide the point of departure for rigorous renormalization group analyses of various self-avoiding walk problems [3, 6, 7, 10, 20]. For the strictly self-avoiding walk, two different proofs of the representation were given, in Sections 5.2.1 and 5.2.2. The role of the fermionic part of the representation in

eliminating loops was detailed in Section 5.2.1. This contrasts with the formal  $N \rightarrow 0$  limit discussed in Section 5.1.2.

The mixed bosonic-fermionic representations are examples of supersymmetric field theories. A brief discussion of some elements of supersymmetry was given in Section 6.

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