Asymptotics near $\pm m$ of the spectral shift function for Dirac operators with non-constant magnetic fields

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Abstract

We consider a 3-dimensional Dirac operator H_0 with non-constant magnetic field of constant direction, perturbed by a sign-definite matrix-valued potential V decaying fast enough at infinity. Then we determine asymptotics, as the energy goes to $+m$ and $-m$, of the spectral shift function for the pair $(H_0, H_0 + V)$. We obtain, as a by-product, a generalised version of Levinson's Theorem relating the eigenvalues asymptotics of $H_0 + V$ near $+m$ and $-m$ to the scattering phase shift for the pair $(H_0, H_0 + V)$.

Contents

1 Introduction

It is known [38] that the free Dirac Hamiltonian H_m acting in the Hilbert space $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^3;\mathbb{C}^4)$ is unitarily equivalent to the operator $h(P) \oplus -h(P)$, where $P := -i\nabla$ and $\mathbb{R}^3 \ni \xi \mapsto h(\xi) := (\xi^2 + m^2)^{1/2}$. For this reason, the set $\{\pm m\} = h\big[(\nabla h)^{-1}(\{0\})\big]$ of critical values of h plays an important role in spectral analysis and scattering theory for Dirac operators. For instance, one cannot prove at $\pm m$ the usual limiting absorption principle for operators $H_m + V$, even with V a regular perturbation of H_m, by using standard commutator methods. Both the statements and the proofs have to be modified (see *e.g.* [4, 19]).

In this paper, we provide a new account on the spectral analysis of Dirac operators at the critical values by discussing the behaviour at $\pm m$ of the spectral shift function associated to sign-definite perturbations of Dirac operators with non-constant magnetic fields. Our work is closely related to [27] where G. D. Raikov treats a similar issue in the case of magnetic Pauli operators. It can also be considered as a complement of [33], where general properties of the spectrum of Dirac operators with variable magnetic fields of constant direction and matrix perturbations are determined. Other related results on the spectrum of 3-dimensional magnetic Dirac operators can be found in [2, 3, 5, 7, 11, 13, 15, 16, 17, 20, 23, 34, 36, 37].

Let us describe the content of this paper. We consider a relativistic spin-$\frac{1}{2}$ particle evolving in $\mathbb{R}^3$ in presence of a variable magnetic field of constant direction. By virtue of the Maxwell equations, we may assume with no loss of generality that the magnetic field has the form

$$\vec{B}(x_1, x_2, x_3) = \big(0, 0, b(x_1, x_2)\big).$$

The system is described in $\mathcal{H}$ by the Dirac operator

$$H_0 := \alpha_1\Pi_1 + \alpha_2\Pi_2 + \alpha_3 P_3 + \beta m,$$

where $\beta \equiv \alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the usual Dirac-Pauli matrices, $m > 0$ is the mass of the particle and $\Pi_j := -i\partial_j - a_j$ are the generators of the magnetic translations with a vector potential

$$\vec{a}(x_1, x_2, x_3) = \big(a_1(x_1, x_2), a_2(x_1, x_2), 0\big)$$

that satisfies $B = \partial_1 a_2 - \partial_2 a_1$. Since $a_3 = 0$, we write $P_3 = -i\partial_3$ instead of Π_3. We assume that the function $b : \mathbb{R}^2 \to \mathbb{R}$ is continuous (see Section 2 for details), so that H_0, defined on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, can be extended uniquely to a selfadjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H_0)$.

Then we consider a bounded positive multiplication operator $V \in C\big(\mathbb{R}^3; \mathscr{B}_{\mathsf{h}}(\mathbb{C}^4)\big)$, where $\mathscr{B}_{\mathsf{h}}(\mathbb{C}^4)$ is the set of 4×4 hermitian matrices, and define the perturbed Hamiltonian $H_\pm := H_0 \pm V$. Since V is bounded and symmetric, the operator $H_\pm$ is selfadjoint in $\mathcal{H}$ and has domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. We also assume that $|V(x)|$ decays more rapidly than $|x|^{-3}$ as $|x| \to \infty$ and that

$$(H_\pm - z)^{-3} - (H_0 - z)^{-3} \in S_1(\mathcal{H}) \quad \text{for each} \quad z \in \mathbb{R} \setminus \{\sigma(H_0) \cup \sigma(H_\pm)\}, \tag{1.1}$$

where $S_1(\mathcal{H})$ denotes the set of trace class operators in $\mathcal{H}$.

Under these assumptions, there exists a unique function $\xi(\,\cdot\,; H_\pm, H_0) \in \mathsf{L}^1\big(\mathbb{R}; (1 + |\lambda|)^{-4}\mathrm{d}\lambda\big)$ such that the Lifshits-Krein trace formula

$$\mathsf{Tr}\left[f(H_\pm) - f(H_0)\right] = \int_{\mathbb{R}} \mathrm{d}\lambda\, f'(\lambda)\xi(\lambda; H_\pm, H_0) \tag{1.2}$$

holds for each $f \in C_0^\infty(\mathbb{R})$ (see [39, Sec. 8.11]). The function $\xi(\,\cdot\,; H_\pm, H_0)$ is called the spectral shift function for the pair $(H_\pm, H_0)$. It vanishes identically on $\mathbb{R} \setminus \{\sigma(H_0) \cup \sigma(H_\pm)\}$, and can be related to the number of eigenvalues of $H_\pm$ in $(-m, m)$ (see Remark 4.5). Morever, for almost every $\lambda \in \sigma_{\mathrm{ac}}(H_0)$ the spectral shift function is related to the scattering matrix $S(\lambda; H_\pm, H_0)$ for the pair $(H_\pm, H_0)$ by the Birman-Krein formula

$$\det S(\lambda; H_\pm, H_0) = \mathrm{e}^{-2\pi i \xi(\lambda; H_\pm, H_0)}.$$

After identification of $\xi(\,\cdot\,; H_\pm, H_0)$ with some representative of its equivalence class, our results are the following. In Proposition 4.4, we show that there exists a constant $\zeta > 0$ defined in terms of b (*cf.* Proposition

2.1) such that $\xi(\,\cdot\,;H_\pm,H_0)$ is bounded on each compact subset of $(-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\{\pm m\}$ and is continuous on $(-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\big(\{\pm m\}\cup\sigma_{\rm p}(H_\pm)\big)$. In Theorem 6.5, we determine the asymptotic behaviour of $\xi(\lambda;H_\pm,H_0)$ as $\lambda\to\pm m$, $|\lambda|<m$, and in Theorem 6.14, we determine the asymptotic behaviour of $\xi(\lambda;H_\pm,H_0)$ as $\lambda\to\pm m$, $|\lambda|>m$. In both cases, one has $\xi(\lambda;H_\pm,H_0)\to\pm\infty$ as $\lambda\to\mp m$. The divergence of $\xi(\lambda;H_\pm,H_0)$ near $\lambda=\pm m$ scales as the number of eigenvalues near 0 of certain Berezin-Toeplitz type operators. When V admits a power-like or exponential decay at infinity, or when it has a compact support, we give the first term of the asymptotic expansion of $\xi(\lambda,;H_\pm,H_0)$ near $\lambda=\pm m$ (see Proposition 6.10 and Corollary 6.17). In these cases, we show that the limits

$$\lim_{\varepsilon\searrow 0}\frac{\xi\big(m+\varepsilon;H_-,H_0\big)}{\xi\big(m-\varepsilon;H_-,H_0\big)}\qquad\text{and}\qquad\lim_{\varepsilon\searrow 0}\frac{\xi\big(-m-\varepsilon;H_+,H_0\big)}{\xi\big(-m+\varepsilon;H_+,H_0\big)}$$

exist and are equal to positive constants depending on the decay rate of V at infinity (see Corollary 6.18 for a precise statement). This can be interpreted as a generalised version of Levinson's Theorem for the pair $(H_\pm,H_0)$ (see [21, 22] for usual versions of Levinson's Theorem for Dirac operators). The relation between the behaviour of the spectral shift function near $\lambda=+m$ and near $\lambda=-m$ is explained in Remark 6.15 by using the charge conjugation symmetry.

These results are similar to the results of [27] (where Pauli operators with non-constant magnetic fields are considered) and [12] (where Schrödinger operators with constant magnetic field are considered). Part of the interest of this work relies on the fact that we were able to exhibit a non-trivial class of matrix potentials V satisfying (1.1) even though H_0 is not a bounded perturbation of the free Dirac operator. We refer to Remark 3.3 and Section 7 for a discussion of this issue.

Let us fix the notations that are used in the paper. The norm and scalar product of $\mathcal H\equiv\mathsf L^2(\mathbb R^3;\mathbb C^4)$ are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. The symbol $\otimes$ stands for the closed tensor product of Hilbert spaces and $S_p(\mathcal H)$, $p\in[1,\infty]$, denotes the p-th Schatten-von Neumann class of operators in $\mathcal H$ ($S_\infty(\mathcal H)$ is the set of compact operators in $\mathcal H$). We denote by $\|\cdot\|_p$ the corresponding operator norm. The variable $x\in\mathbb R^3$ is often written as $x\equiv(x_\perp,x_3)$, with $x_\perp\in\mathbb R^2$ and $x_3\in\mathbb R$. The symbol Q_j, $j=1,2,3$, denotes the multiplication operator by x_j in $\mathcal H$, $Q:=(Q_1,Q_2,Q_3)$, and $Q_\perp:=(Q_1,Q_2)$. Sometimes, when the context is unambiguous, we consider the operators Q_j and P_j as operators in $\mathsf L^2(\mathbb R)$ instead of $\mathcal H$ without changing the notations. Given a selfadjoint operator A in a Hilbert space $\mathcal G$, the symbol $E^A(\cdot)$ stands for the spectral measure of A.

2 Unperturbed operator

Throughout this paper we assume that the component $b:\mathbb R^2\to\mathbb R$ of the magnetic field $\vec B\equiv(0,0,b)$ belongs to the class of "admissible" magnetic fields defined in [27, Sec. 2.1]. Namely, we assume that $b=b_0+\widetilde b$, where $b_0>0$ is a constant while the function $\widetilde b:\mathbb R^2\to\mathbb R$ is such that the Poisson equation

$$\Delta\widetilde\varphi=\widetilde b$$

admits a solution $\widetilde\varphi:\mathbb R^2\to\mathbb R$, continuous and bounded together with its derivatives of order up to two. We also define $\varphi_0(x_\perp):=\frac14 b_0|x_\perp|^2$ for each $x_\perp\in\mathbb R^2$ and set $\varphi:=\varphi_0+\widetilde\varphi$. Then we obtain a vector potential $\vec a\equiv(a_1,a_2,a_3)\in C^1(\mathbb R^2;\mathbb R^3)$ for the magnetic field $\vec B$ by putting

$$a_1:=\partial_1\varphi,\qquad a_2:=\partial_2\varphi\qquad\text{and}\qquad a_3:=0.$$

(changing, if necessary, the gauge, we shall always assume that the vector potential $\vec a$ is of this form). We refer to [27] for further properties and examples of admissible magnetic fields.

Since the vector potential $\vec a$ belongs to $\mathsf L^\infty_{\rm loc}(\mathbb R^2;\mathbb R^3)$, the magnetic Dirac operator

$$H_0=\alpha_1\Pi_1+\alpha_2\Pi_2+\alpha_3P_3+\beta m$$

satisfies all the properties of [33, Sec. 2.1]. The operator H_0 is essentially selfadjoint on $C_0^\infty(\mathbb R^3;\mathbb C^4)$, with domain $\mathcal D(H_0)\subset\mathcal H^{1/2}_{\rm loc}(\mathbb R^3;\mathbb C^4)$, the spectrum of H_0 satisfies

$$\sigma(H_0)=\sigma_{\rm ac}(H_0)=(-\infty,-m]\cup[m,\infty),\tag{2.1}$$

and we have the identity

$$H_0^2 = \begin{pmatrix} H_\perp^- \otimes 1 + 1 \otimes (P_3^2 + m^2) & 0 & 0 & 0 \\ 0 & H_\perp^+ \otimes 1 + 1 \otimes (P_3^2 + m^2) & 0 & 0 \\ 0 & 0 & H_\perp^- \otimes 1 + 1 \otimes (P_3^2 + m^2) & 0 \\ 0 & 0 & 0 & H_\perp^+ \otimes 1 + 1 \otimes (P_3^2 + m^2) \end{pmatrix} \tag{2.2}$$

with respect to the tensorial decomposition $\mathsf{L}^2(\mathbb{R}^2) \otimes \mathsf{L}^2(\mathbb{R})$ of $\mathsf{L}^2(\mathbb{R}^3)$. Here the operators $H_\perp^\pm$ are the components of the Pauli operator $H_\perp := H_\perp^- \oplus H_\perp^+$ in $\mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^2)$ associated with the vector potential (a_1, a_2).

We recall from [27, Sec. 2.2] that $\dim \ker(H_\perp^-) = \infty$, that $\dim \ker(H_\perp^+) = 0$ and that we have the following result.

Proposition 2.1. *Let b be an admissible magnetic field with $b_0 > 0$. Then $0 = \inf \sigma(H_\perp)$ is an isolated eigenvalue of infinite multiplicity. More precisely, we have*

$$\dim \ker(H_\perp) = \infty \qquad \textit{and} \qquad (0, \zeta) \subset \mathbb{R} \setminus \sigma(H_\perp),$$

where

$$\zeta := 2b_0 \, \mathrm{e}^{-2\operatorname{osc}(\widetilde{\varphi})} \qquad \textit{and} \qquad \operatorname{osc}(\widetilde{\varphi}) := \sup_{x_\perp \in \mathbb{R}^2} \widetilde{\varphi}(x_\perp) - \inf_{x_\perp \in \mathbb{R}^2} \widetilde{\varphi}(x_\perp).$$

Finally, since $(0, \zeta) \subset \mathbb{R} \setminus \sigma(H_\perp)$, we know from [33, Thm. 1.2.(d)] that the limits

$$\lim_{\varepsilon \searrow 0} \langle Q_3 \rangle^{-\nu_3/2} (H_0 - \lambda \mp i\varepsilon)^{-1} \langle Q_3 \rangle^{-\nu_3/2}, \qquad \nu_3 > 1, \tag{2.3}$$

exist for each $\lambda \in (-\sqrt{m^2 + \zeta}, \sqrt{m^2 + \zeta}) \setminus \{\pm m\}$ (note that we use the usual notation $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$).

3 Perturbed operator

We consider now the perturbed operators $H_\pm = H_0 \pm V$, where $V \equiv \{V_{jk}\}$ is the multiplication operator associated to the following matrix-valued function V.

Assumption 3.1. *The function $V \in C\big(\mathbb{R}^3; \mathscr{B}_{\mathsf{h}}(\mathbb{C}^4)\big)$ satisfies for each $x \equiv (x_\perp, x_3) \in \mathbb{R}^3$ and each $j, k \in \{1, \ldots, 4\}$*

$$V(x) \geq 0 \qquad \textit{and} \qquad |V_{jk}(x)| \leq \mathrm{Const.} \langle x_\perp \rangle^{-\nu_\perp} \langle x_3 \rangle^{-\nu_3} \quad \textit{for some } \nu_\perp > 2 \textit{ and } \nu_3 > 1. \tag{3.1}$$

The potential V in Assumption 3.1 is short-range along x_3. So we know from [33, Thm. 1.2] that

(i) $\sigma_{\mathrm{ess}}(H_\pm) = \sigma_{\mathrm{ess}}(H_0) = (-\infty, -m] \cup [m, \infty)$.

(ii) The point spectrum of $H_\pm$ in $\big(-\sqrt{m^2 + \zeta}, \sqrt{m^2 + \zeta}\big) \setminus \{\pm m\}$ is composed of eigenvalues of finite multiplicity and with no accumulation point.

(iii) $H_\pm$ has no singular continuous spectrum in $\big(-\sqrt{m^2 + \zeta}, \sqrt{m^2 + \zeta}\big)$. In particular, H_0 and $H_\pm$ have a common spectral gap in $(-m, m)$.

Using the formula

$$(A + \lambda)^{-\gamma} = \Gamma(\gamma)^{-1} \int_0^\infty \mathrm{d}t \, t^{\gamma - 1} \, \mathrm{e}^{-t(A + \lambda)}, \qquad A : \mathcal{D}(A) \to \mathcal{H}, \ A \geq 0, \ \lambda, \gamma > 0,$$

the diamagnetic inequality [1, Thm. 2.3], and the compactness criterion [9, Thm. 5.7.1], we find that

$$|V_{jk}|^{1/2} \big(\textstyle\sum_{\ell \leq 3} \Pi_\ell^* \Pi_\ell + m^2 \big)^{-1/4} \in S_\infty[\mathsf{L}^2(\mathbb{R}^3)].$$

Since b is bounded this implies that

$$|H_0|^{-1/2}V|H_0|^{-1/2} \le |H_0|^{-1/2}\big(\textstyle\sum_{j,k\le 4}|V_{jk}|\big)|H_0|^{-1/2} \in S_\infty(\mathcal{H}).$$

So $|H_0|^{-1/2}V|H_0|^{-1/2}$ also belongs to $S_\infty(\mathcal{H})$, since $S_\infty(\mathcal{H})$ is an hereditary C^*-subalgebra of $\mathscr{B}(\mathcal{H})$ [24, Cor. 3.2.3]. One has in particular

$$V^{1/2}(|H_0|+1)^{-1/2} \in S_\infty(\mathcal{H}). \tag{3.2}$$

The standard criterion [31, Thm. XI.20] shows that

$$|V_{jk}|^{1/2}\big(-\Delta+m^2\big)^{-\gamma} \in S_q[\mathsf{L}^2(\mathbb{R}^3)] \quad \text{if } q\in[2,\infty) \text{ and } \gamma q > 3/2.$$

This together with arguments as above implies that

$$V^{1/2}|H_0|^{-\gamma} \in S_q(\mathcal{H}) \quad \text{if } q\ge 2 \text{ is even and } \gamma q > 3. \tag{3.3}$$

So we have in particular that

$$V^{1/2}E^{H_0}(B) \in S_2(\mathcal{H}) \quad \text{for any bounded borel set } B\subset\mathbb{R}. \tag{3.4}$$

In the sequel we shall need a more restrictive assumption on V. For this, we recall that there exists numbers $z\in\mathbb{R}\setminus\{\sigma(H_0)\cup\sigma(H_\pm)\}$ since H_0 and $H_\pm$ have a common spectral gap in $(-m,m)$. We also set $R_0(z) := (H_0-z)^{-1}$ and $R_\pm(z) := (H_\pm - z)^{-1}$ for $z\in\mathbb{C}\setminus\sigma(H_0)$ and $z\in\mathbb{C}\setminus\sigma(H_\pm)$, respectively.

Assumption 3.2. *The function $V\in C\big(\mathbb{R}^3;\mathscr{B}_{\mathsf{h}}(\mathbb{C}^4)\big)$ satisfies for each $x\in\mathbb{R}^3$ and each $j,k\in\{1,\dots,4\}$*

$$V(x)\ge 0 \qquad \textit{and} \qquad |V_{jk}(x)| \le \mathrm{Const.}\,\langle x\rangle^{-\nu} \quad \textit{for some constant } \nu > 3. \tag{3.5}$$

Furthermore, V is chosen such that

$$R_\pm^3(z) - R_0^3(z) \in S_1(\mathcal{H}) \quad \text{for each } z\in\mathbb{R}\setminus\{\sigma(H_0)\cup\sigma(H_\pm)\}. \tag{3.6}$$

Note that (3.5) implies (3.1) if one takes $\nu_3\in(1,\nu-2)$ and $\nu_\perp := \nu-\nu_3$. Note also that the choice of function $\lambda\mapsto(\lambda-z)^{-3}$ in the trace class condition (3.6) has been made for convenience. Many other choices would also guarantee the existence of the spectral shift function for the pair $(H_\pm, H_0)$ (see *e.g.* [39, Sec. 8.11]).

Remark 3.3. *Since the operator H_0 is not a bounded perturbation of the free Dirac operator, we cannot apply the results of [40, Sec. 4] to prove the inclusion* (3.6) *under the condition* (3.5)*. In general, one has to impose additional assumptions on V to get the result. For instance, if V verifies* (3.5)*, and*

(i) $[V,\alpha_1] = [V,\alpha_2] = 0$*,*

(ii) for each $x\in\mathbb{R}^3$ and each $j,k,\ell\in\{1,\dots,4\}$, one has $|(\partial_\ell V_{jk})(x)| \le \mathrm{Const.}\,\langle x\rangle^{-\varsigma}$ for some $\varsigma > 3$,

(iii) for each $j,k,\ell\in\{1,\dots,4\}$, one has $(\partial_\ell\partial_3 V_{jk})\in\mathsf{L}^\infty(\mathbb{R}^3)$,

then (3.6) *is satisfied. Furthermore, if V is scalar, then the same is true without assuming (iii) (and (i) is trivially satisfied). The proof of these statements can be found in the appendix. Here, we only note that a matrix $\mathsf{V}\in\mathscr{B}_{\mathsf{h}}(\mathbb{C}^4)$ satisfying (i) is necessarily of the form*

$$\mathsf{V} = \begin{pmatrix} \mathsf{v}_1 & 0 & \mathsf{v}_3 & 0 \\ 0 & \mathsf{v}_2 & 0 & \overline{\mathsf{v}_3} \\ \overline{\mathsf{v}_3} & 0 & \mathsf{v}_2 & 0 \\ 0 & \mathsf{v}_3 & 0 & \mathsf{v}_1 \end{pmatrix},$$

with $\mathsf{v}_1,\mathsf{v}_2\in\mathbb{R}$ and $\mathsf{v}_3\in\mathbb{C}$.

4 Spectral shift function

In this section we recall some results due to A. Pushnitski on the representation of the spectral shift function for a pair of not semibounded selfadjoint operators.

Given a a Lebesgue measurable set $B \subset \mathbb{R}$, we set $\mu(B) := \frac{1}{\pi}\int_B \frac{\mathrm{d}t}{1+t^2}$, and note that $\mu(\mathbb{R}) = 1$. Furthermore, if $T = T^*$ is a compact operator in a separable Hilbert space $\mathcal{G}$, we set

$$n_\pm(s;T) := \operatorname{rank} E^{\pm T}\big((s,\infty)\big) \quad \text{for } s > 0.$$

Then we have the following estimates.

Lemma 4.1 (Lemma 2.1 of [26]). *Let $T_1 = T_1^* \in S_\infty(\mathcal{H})$ and $T_2 = T_2^* \in S_1(\mathcal{H})$. Then one as for each $s_1, s_2 > 0$*

$$\int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm(s_1+s_2;T_1+tT_2) \le n_\pm(s_1;T_1) + \frac{1}{\pi s_2}\|T_2\|_1.$$

For $z \in \mathbb{C}\setminus\sigma(H_0)$, we define the usual weighted resolvent

$$T(z) := V^{1/2}(H_0-z)^{-1}V^{1/2}$$

and the corresponding real and imaginary parts

$$A(z) := \mathsf{Re}\, T(z) \qquad \text{and} \qquad B(z) := \mathsf{Im}\, T(z).$$

Then the next lemma is direct consequence of the inclusions (3.2)-(3.4) and [25, Prop. 4.4.(i)].

Lemma 4.2. *Let V satisfy Assumption 3.1. Then, for almost every $\lambda \in \mathbb{R}$, the limits $A(\lambda+i0) := \lim_{\varepsilon\searrow 0} A(\lambda+i\varepsilon)$ and $B(\lambda+i0) := \lim_{\varepsilon\searrow 0} B(\lambda+i\varepsilon) \ge 0$ exist in $S_4(\mathcal{H})$.*

Next theorem follows from the inclusions (3.2), (3.4), (3.6), from the equations (1.9), (8.1), (8.2) of [25], and from Theorem 8.1 of [25].

Theorem 4.3. *Let V satisfy Assumption 3.2. Then, for almost every $\lambda \in \mathbb{R}$, $\xi(\lambda;H_\pm,H_0)$ exists and is given by*

$$\xi(\lambda;H_\pm,H_0) = \pm\int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\mp\big(1;A(\lambda+i0)+tB(\lambda+i0)\big). \tag{4.1}$$

We know from (2.3) that $A(\lambda+i0)$ and $B(\lambda+i0)$ exist in $\mathscr{B}(\mathcal{H})$ for each $\lambda \in (-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\{\pm m\}$. In Propositions 5.2-5.3 and Corollary 5.5 below we show that in fact $A(\lambda+i0) \in S_4(\mathcal{H})$ and $B(\lambda+i0) \in S_1(\mathcal{H})$ for each $\lambda \in (-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\{\pm m\}$. Hence, by Lemma 4.1, the r.h.s. of (4.1) will turn out to be well-defined for every $\lambda \in (-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\{\pm m\}$. In the next proposition we state some regularity properties of the function

$$(-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\{\pm m\} \ni \lambda \mapsto \widetilde{\xi}(\lambda;H_\pm,H_0) := \pm\int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\mp\big(1;A(\lambda+i0)+tB(\lambda+i0)\big).$$

The proof (which relies on Propositions 5.2-5.3, Lemma 5.4, Corollary 5.5 and the stability result [14, Thm. 3.12]) is similar to the one of [6, Sec. 4.2.1].

Proposition 4.4. *Let V satisfy Assumption 3.1. Then $\widetilde{\xi}(\,\cdot\,;H_\pm,H_0)$ is bounded on each compact subset of $(-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\{\pm m\}$ and is continuous on $(-\sqrt{m^2+\zeta},\sqrt{m^2+\zeta})\setminus\big(\{\pm m\}\cup\sigma_{\rm p}(H_\pm)\big)$.*

In the sequel, we identify the functions $\widetilde{\xi}(\,\cdot\,;H_\pm,H_0)$ and $\xi(\,\cdot\,;H_\pm,H_0)$ since they are equal for almost every $\lambda \in \mathbb{R}$ due to Theorem 4.3 (see [35] for a study where the r.h.s. of (4.1) is directly treated as a definition of $\xi(\lambda;H_\pm,H_0)$).

Remark 4.5. *In the interval $(-m,m)$, H_0 has no spectrum and the spectrum of $H_\pm$ is purely discrete. Thus the spectral shift function $\xi(\,\cdot\,;H_\pm,H_0)$ can be related to the number of eigenvalues of $H_\pm$ as follows: for $\lambda_1,\lambda_2 \in (-m,m)\setminus\sigma(H_\pm)$ with $\lambda_1<\lambda_2$, we have (see [25, Thm. 9.1])*

$$\xi(\lambda_1;H_\pm,H_0) - \xi(\lambda_2;H_\pm,H_0) = \operatorname{rank} E^{H_\pm}\big([\lambda_1,\lambda_2)\big).$$

5 Decomposition of the weighted resolvent

In this section we decompose the weighted resolvent

$$T(z) = V^{1/2}(H_0 - z)V^{1/2}, \quad z \in \mathbb{C} \setminus \sigma(H_0),$$

into a sum $T(z) = T_{\mathsf{div}}(z) + T_{\mathsf{bound}}(z)$, where $T_{\mathsf{div}}(z)$ (respectively $T_{\mathsf{bound}}(z)$) corresponds to the diverging (respectively non-diverging) part of $T(z)$ as $z \to \pm m$. Then we estimate the behaviour, in suitable Schatten norms, of each term as $z \to \pm m$. We refer to [12, Sec. 4] and [27, Sec. 4.2] for similar approaches in the case of the Schrödinger and Pauli operators.

Let a and a^* be the closures in $\mathsf{L}^2(\mathbb{R}^2)$ of the operators given by

$$\mathsf{a}\varphi := (\Pi_1 - i\Pi_2)\varphi \qquad \text{and} \qquad \mathsf{a}^*\varphi := (\Pi_1 + i\Pi_2)\varphi,$$

for $\varphi \in C_0^\infty(\mathbb{R}^2)$. Then one has (see [38, Sec. 5.5.2] and [28, Sec. 5])

$$H_0 = \begin{pmatrix} m & 0 & 1\otimes P_3 & \mathsf{a}\otimes 1 \\ 0 & m & \mathsf{a}^*\otimes 1 & -1\otimes P_3 \\ 1\otimes P_3 & \mathsf{a}\otimes 1 & -m & 0 \\ \mathsf{a}^*\otimes 1 & -1\otimes P_3 & 0 & -m \end{pmatrix}, \tag{5.1}$$

with

$$\ker(\mathsf{a}^*) = \ker(\mathsf{a}\mathsf{a}^*) = \ker(H_\perp^-) \subset \mathsf{L}^2(\mathbb{R}^2). \tag{5.2}$$

Now, let

$$\mathsf{P} := \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be the orthogonal projection onto the union of the eigenspaces of H_0 corresponding to the values $\lambda = \pm m$. Since $P \equiv p \otimes 1$ is the orthogonal projection onto $\ker(H_\perp^-) \otimes \mathsf{L}^2(\mathbb{R})$, the equations (5.1) and (5.2) imply that H_0 and P commute:

$$H_0^{-1}\mathsf{P} = \mathsf{P}H_0^{-1}. \tag{5.3}$$

In fact, by using (2.2) and (5.1), one gets for each $z \in \mathbb{C} \setminus \sigma(H_0)$ the equalities

$$\begin{aligned} &(H_0 - z)^{-1}\mathsf{P} \\ &= (H_0 + z)\big(H_0^2 - z^2\big)^{-1}\mathsf{P} \\ &= \big[p \otimes R(z^2 - m^2)\big] \begin{pmatrix} (z+m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z-m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \big[p \otimes P_3 R(z^2 - m^2)\big] \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $R(z) := \big(P_3^2 - z\big)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$, is the resolvent of P_3^2 in $\mathsf{L}^2(\mathbb{R})$. This allows us to decompose $T(z)$ as $T(z) = T_{\mathsf{div}}(z) + T_{\mathsf{bound}}(z)$, with

$$\begin{aligned} T_{\mathsf{div}}(z) &:= V^{1/2}\big[p \otimes R(z^2 - m^2)\big] \begin{pmatrix} (z+m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z-m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^{1/2}, \\ T_{\mathsf{bound}}(z) &:= V^{1/2}\big[p \otimes P_3 R(z^2 - m^2)\big] \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^{1/2} + V^{1/2}(H_0 - z)^{-1}\mathsf{P}^\perp V^{1/2} \qquad (\mathsf{P}^\perp := 1 - \mathsf{P}). \end{aligned}$$

One may note that this decomposition of $T(z)$ differs slightly from the simpler decomposition

$$T(z) = V^{1/2}(H_0 - z)\mathsf{P}V^{1/2} + V^{1/2}(H_0 - z)\mathsf{P}^\perp V^{1/2},$$

since the first term in $T_{\mathsf{bound}}(z)$ is associated to the projection P and not the projection $\mathsf{P}^\perp$. This choice is motivated by the will of distinguishing clearly the contribution $T_{\mathsf{div}}(z)$, that diverge as $z \to \pm m$, from the contribution $T_{\mathsf{bound}}(z)$, that stays bounded as $z \to \pm m$.

For $\lambda \in \mathbb{R} \setminus \{0\}$, we can define the boundary value $R(\lambda)$ of the resolvent $R(z)$ as the operator with convolution kernel $r_\lambda(\,\cdot\,)$, where

$$r_\lambda(x_3) := \begin{cases} \frac{\mathrm{e}^{-\sqrt{-\lambda}|x_3|}}{2\sqrt{-\lambda}} & \text{if } \lambda < 0, \\ \frac{i\,\mathrm{e}^{\sqrt{\lambda}|x_3|}}{2\sqrt{\lambda}} & \text{if } \lambda > 0, \end{cases}$$

for each $x_3 \in \mathbb{R}$. So, we can extend the definition of $T_{\mathsf{div}}(\,\cdot\,)$ to the values $\lambda \in \mathbb{R} \setminus \{\pm m\}$:

$$T_{\mathsf{div}}(\lambda) := V^{1/2}\big[p \otimes R(\lambda^2 - m^2)\big] \begin{pmatrix} (\lambda+m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (\lambda-m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^{1/2}.$$

In the following proposition, we show that the trace norm of $T_{\mathsf{div}}(z)$ is continuous in $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \mathsf{Im}(z) \geq 0\}$ outside the points $z = \pm m$, where it may diverge as $|z \mp m|^{-1/2}$. The proof of the proposition relies on a technical result that we now recall.

Lemma 5.1 (Lemma 2.4 of [27]). *Let $U \in \mathsf{L}^q(\mathbb{R}^2)$, $q \in [1, \infty)$, and assume that b is an admissible magnetic field. Then $pUp \in S_q[\mathsf{L}^2(\mathbb{R}^2)]$, and*

$$\|pUp\|^q_{S_q[\mathsf{L}^2(\mathbb{R}^2)]} \leq \frac{b_0}{2\pi}\,\mathrm{e}^{2\mathrm{osc}(\widetilde{\varphi})}\,\|U\|^q_{\mathsf{L}^q(\mathbb{R}^2)}\,.$$

The symbol y_+ denotes the postive part of $y \in \mathbb{R}$.

Proposition 5.2. *Let V satisfy Assumption 3.1. Then the operator-valued function*

$$\mathbb{C}_+ \setminus \{\pm m\} \ni z \mapsto T_{\mathsf{div}}(z) \in S_1(\mathcal{H})$$

is well-defined and continuous. Moreover, we have for each $\lambda \in \mathbb{R} \setminus \{\pm m\}$ the bound

$$\|T_{\mathsf{div}}(\lambda)\|_1 \leq \mathrm{Const.}\left(\left|\tfrac{\lambda+m}{\lambda-m}\right|^{1/2} + \left|\tfrac{\lambda-m}{\lambda+m}\right|^{1/2}\right)\big(1 + (\lambda^2 - m^2)_+^{1/4}\big).$$

Proof. We have for each $z \in \mathbb{C} \setminus \sigma(H_0)$ the identity

$$T_{\mathsf{div}}(z) = M\big(G \otimes J_{z^2-m^2}\big) \begin{pmatrix} (z+m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z-m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} M,$$

where

$$M := V^{1/2}\langle Q_\perp\rangle^{\nu_\perp/2}\langle Q_3\rangle^{\nu_3/2}, \tag{5.4}$$

$$G := \langle Q_\perp\rangle^{-\nu_\perp/2} p \langle Q_\perp\rangle^{-\nu_\perp/2}, \tag{5.5}$$

$$J_z := \langle Q_3\rangle^{-\nu_3/2} R(z) \langle Q_3\rangle^{-\nu_3/2}.$$

The operator M is bounded due to Assumption 3.1. So

$$\|T_{\mathsf{div}}(z)\|_1 \leq \mathrm{Const.}\,(|z+m| + |z-m|)\,\|G\|_1 \|J_{z^2-m^2}\|_1.$$

But we know from Lemma 5.1 that $\|G\|_1 \leq \mathrm{Const.}$, and from [6, Sec. 4.1] that the operator-valued function $\mathbb{C}_+ \setminus \{0\} \ni z \mapsto J_z$ is continuous in the trace norm and admits the bound

$$\|J_\lambda\|_1 \leq \mathrm{Const.}\big(1 + \lambda_+^{1/4}\big)|\lambda|^{-1/2}, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

It follows that

$$\|T_{\mathsf{div}}(z)\|_1 \leq \mathrm{Const.}\left(\left|\tfrac{\lambda+m}{\lambda-m}\right|^{1/2} + \left|\tfrac{\lambda-m}{\lambda+m}\right|^{1/2}\right)\big(1 + (\lambda^2 - m^2)_+^{1/4}\big)$$

for each $\lambda \in \mathbb{R} \setminus \{\pm m\}$. $\square$

In the following proposition, we show that the function $z \mapsto T_{\mathsf{bound}}(z) \in S_4(\mathcal{H})$ is continuous in $\mathbb{C} \setminus \{(-\infty, -\sqrt{m^2+\zeta}] \cup [\sqrt{m^2+\zeta}, \infty)\}$. The symbols $H^\pm$ stand for the operators $H^\pm := H_\perp^\pm \otimes 1 + 1 \otimes P_3^2$ acting in $\mathsf{L}^2(\mathbb{R}^3)$.

Proposition 5.3. *Let V satisfy Assumption 3.1. Then the operator-valued function*

$$\mathbb{C} \setminus \{(-\infty, -\sqrt{m^2+\zeta}] \cup [\sqrt{m^2+\zeta}, \infty)\} \ni z \mapsto T_{\mathsf{bound}}(z) \in S_4(\mathcal{H})$$

is well-defined and continuous. Moreover, we have for each $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta})$ the bound

$$\|T_{\mathsf{bound}}(\lambda)\|_4 \le \text{Const.}\left(|\lambda| + \lambda^2\right)\left(1 + \tfrac{(\lambda^2 - m^2 + 1)_+}{\zeta + m^2 - \lambda^2}\right) + \text{Const.} \tag{5.6}$$

Proof. One has the identity

$$(H_0 - z)^{-1} = H_0^{-1} + z\left(1 + zH_0^{-1}\right)\left(H_0^2 - z^2\right)^{-1}$$

for each $z \in \mathbb{C} \setminus \sigma(H_0)$. Thus the operator $T_{\mathsf{bound}}(z)$ can be written as

$$\begin{aligned} T_{\mathsf{bound}}(z) &= M\left(G \otimes S_z\right)\left(\begin{smallmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{smallmatrix}\right) M + V^{1/2} H_0^{-1} \mathsf{P}^\perp V^{1/2} + z V^{1/2}\left(1 + zH_0^{-1}\right)\left(H_0^2 - z^2\right)^{-1} \mathsf{P}^\perp V^{1/2} \\ &\equiv T_1(z) + T_2 + T_3(z), \end{aligned} \tag{5.7}$$

with M and G given by (5.4)-(5.5), and

$$S_z := \langle Q_3 \rangle^{-\nu_3/2} P_3 R(z^2 - m^2) \langle Q_3 \rangle^{-\nu_3/2}.$$

The integral kernel of S_z is

$$\tfrac{i}{2} \langle x_3 \rangle^{-\nu_3/2} \tfrac{(x_3 - x_3')}{|x_3 - x_3'|} \, \mathrm{e}^{i\sqrt{z^2 - m^2}|x_3 - x_3'|} \langle x_3' \rangle^{-\nu_3/2}, \tag{5.8}$$

with the branch of $\sqrt{z^2 - m^2}$ chosen so that $\mathsf{Im}\sqrt{z^2 - m^2} > 0$. So S_z extends to an element of $S_2[\mathsf{L}^2(\mathbb{R})]$ for each $z \in \mathbb{C}$, with $\|S_z\|_2 \le$ Const. Since M is bounded and $\|G\|_1 \le$ Const., this implies that

$$\|T_1(z)\|_2 \le \text{Const.}\, \|M\|^2 \|G\|_1 \|S_z\|_2 \le \text{Const.} \tag{5.9}$$

for each $z \in \mathbb{C}$. One also has

$$\|T_2\|_4 \le \text{Const.} \tag{5.10}$$

due to (3.3). So, it only remains to bound the term $T_3(z)$.

Let $z \in \mathbb{C} \setminus \{(-\infty, -\sqrt{m^2+\zeta}] \cup [\sqrt{m^2+\zeta}, \infty)\}$ and $P^\perp := 1 - P$. Then $\left(H^- + m^2 - z^2\right)^{-1} P^\perp$ and $\left(H^+ + m^2 - z^2\right)^{-1}$ belong to $\mathscr{B}[\mathsf{L}^2(\mathbb{R}^3)]$, and we have

$$\left(H^- + m^2 - z^2\right)^{-1} P^\perp = P^\perp \left(H^- + m^2 - z^2\right)^{-1}.$$

Thus

$$\begin{aligned} \left(H_0^2 - z^2\right)^{-1} \mathsf{P}^\perp V^{1/2} &= \left(H_0^2 - z^2\right)^{-1} \left(\begin{smallmatrix} P^\perp & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & P^\perp & 0\\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) V^{1/2} \\ &= \begin{pmatrix} P^\perp (H^- + m^2 - z^2)^{-1} & 0 & 0 & 0\\ 0 & (H^+ + m^2 - z^2)^{-1} & 0 & 0\\ 0 & 0 & P^\perp (H^- + m^2 - z^2)^{-1} & 0\\ 0 & 0 & 0 & (H^+ + m^2 - z^2)^{-1} \end{pmatrix} V^{1/2}, \end{aligned}$$

and

$$\left\|\left(H_0^2-z^2\right)^{-1}\mathsf{P}^{\perp}V^{1/2}\right\|_2^2 \le 2\,\|M\|^2\Big\{\left\|P^{\perp}\left(H^-+m^2-z^2\right)^{-1}M_2\right\|_2^2+\left\|\left(H^++m^2-z^2\right)^{-1}M_2\right\|_2^2\Big\},$$

where $M_2 := \langle Q_\perp\rangle^{-\nu_\perp/2}\langle Q_3\rangle^{-\nu_3/2}$. But, we know from the proof of [27, Prop. 4.4] that

$$\left\|P^{\perp}(H^-+m^2-z^2)^{-1}M_2\right\|_2 \le \mathrm{Const.}\,C(z) \qquad \text{and} \qquad \left\|(H^++m^2-z^2)^{-1}M_2\right\|_2 \le \mathrm{Const.}\,C(z),$$

where

$$C(z) := \sup_{y\in[\zeta,\infty)} \frac{y+1}{|y+m^2-z^2|}\,.$$

It follows that

$$\|T_3(z)\|_2 \le \mathrm{Const.}\left\|zV^{1/2}\left(1+zH_0^{-1}\right)\right\|\,\|M\|\,C(z) \le \mathrm{Const.}\left(|z|+|z|^2\right)C(z). \tag{5.11}$$

The claim follows then by putting together (5.9), (5.10), and (5.11). □

In the next lemma we give some results on the imaginary part of the operator S_z in $\mathsf{L}^2(\mathbb{R})$ appearing in the proof of Proposition 5.3

$$S_z = \langle Q_3\rangle^{-\nu_3/2}P_3R(z^2-m^2)\langle Q_3\rangle^{-\nu_3/2}, \qquad z\in\mathbb{C}\setminus\sigma(H_0),\ \nu_3>1.$$

Lemma 5.4. *(a) One has $\operatorname{Im} S_\lambda = 0$ for each $\lambda\in(-m,m)$.*

(b) Let $p\ge 1$ be an integer. Then one has for each $\lambda\in\mathbb{R}$ with $|\lambda|>m$

$$\|\operatorname{Im} S_\lambda\|_p \le \mathrm{c}_p\,,$$

where c_p is a constant independent of λ. Furthermore

$$\lim_{\lambda\to\pm m,\,|\lambda|>m}\|\operatorname{Im} S_\lambda\|_p = 0.$$

Proof. (a) This is a direct consequence of the spectral theorem.

(b) Let $\lambda\in\mathbb{R}$, $|\lambda|>m$. Then one shows by using (5.8) that $\operatorname{Im} S_\lambda$ is equal to the rank two operator

$$\operatorname{Im} S_\lambda = \langle v_\lambda,\cdot\rangle\,u_\lambda + \langle u_\lambda,\cdot\rangle\,v_\lambda,$$

with

$$u_\lambda(x_3) := \langle x_3\rangle^{-\nu_3/2}\sin\left(x_3\sqrt{\lambda^2-m^2}\right) \qquad \text{and} \qquad v_\lambda(x_3) := -\tfrac{i}{2}\langle x_3\rangle^{-\nu_3/2}\cos\left(x_3\sqrt{\lambda^2-m^2}\right).$$

Since $\langle v_\lambda,u_\lambda\rangle = 0$, this implies that

$$|\operatorname{Im} S_\lambda|^p = \|u_\lambda\|^p\langle v_\lambda,\cdot\rangle\,v_\lambda + \|v_\lambda\|^p\langle u_\lambda,\cdot\rangle\,u_\lambda.$$

Thus

$$\|\operatorname{Im} S_\lambda\|_p^p = \operatorname{Tr}\left(|\operatorname{Im} S_\lambda|^p\right) = \|u_\lambda\|^p\,\|v_\lambda\|^2 + \|v_\lambda\|^p\,\|u_\lambda\|^2.$$

This, together with the equality

$$\lim_{\lambda\to\pm m,\,|\lambda|>m}\|u_\lambda\| = 0,$$

implies the claim. □

In the next corollary we combine some of the results of Propositions 5.2, 5.3 and Lemma 5.4.

Corollary 5.5. *Let V satisfy Assumption 3.1. Then the identity*

$$T(\lambda + i0) = T_{\mathsf{div}}(\lambda) + T_{\mathsf{bound}}(\lambda) \tag{5.12}$$

holds for each $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$, and the estimate

$$\big\| \operatorname{\mathsf{Im}} T_{\mathsf{bound}}(\lambda)\big\|_p \le \mathrm{Const.} \| \operatorname{\mathsf{Im}} S_\lambda \|_p \tag{5.13}$$

holds for each integer $p \ge 1$ an each $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta})$. In particular, we have

$$\lim_{\lambda\to\pm m} \big\| \operatorname{\mathsf{Im}} T_{\mathsf{bound}}(\lambda)\big\|_p = 0, \tag{5.14}$$

due to Lemma 5.4.

Proof. The first identity follows from Propositions 5.2 and 5.3. Let $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta})$. Using (5.7) and the commutation rule (5.3) one obtains that

$$\operatorname{\mathsf{Im}} T_{\mathsf{bound}}(\lambda) = M\big(G \otimes \operatorname{\mathsf{Im}} S_\lambda\big) \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix} M,$$

with M and G defined by (5.4)-(5.5). Since M is bounded and $\|G\|_1 \le \mathrm{Const.}$, this implies (5.13). □

6 Proof of the main results

We begin this section by showing that the value of $\xi(\lambda; H, H_\pm)$ as $\lambda \to \pm m$ is bounded from below and from above by expressions involving only the term $T_{\mathsf{div}}(\lambda)$ of the decomposition $T(\lambda+i0) = T_{\mathsf{div}}(\lambda) + T_{\mathsf{bound}}(\lambda)$. Then we consider separately the limits $\lambda \to \pm m$ with $|\lambda| < m$ and the limits $\lambda \to \pm m$ with $|\lambda| > m$.

We start by recalling two standard properties of the counting functions $n_\pm$. Given two compact operators $T_1 = T_1^*$ and $T_2 = T_2^*$ in a separable Hilbert space $\mathcal{G}$, we have the Weyl inequalities

$$n_\pm(s_1 + s_2; T_1 + T_2) \le n_\pm(s_1; T_1) + n_\pm(s_2; T_2) \quad \text{for each } s_1, s_2 > 0. \tag{6.1}$$

Moreover, if $T = T^*$ belongs to $S_p(\mathcal{G})$ for some $p \in [1, \infty)$, then

$$n_\pm(s; T) \le s^{-p} \|T\|_p^p \quad \text{for each } s > 0. \tag{6.2}$$

Proposition 6.1. *Let V satisfy Assumption 3.2. Then the estimates*

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm\big(1+\varepsilon; \operatorname{\mathsf{Re}} T_{\mathsf{div}}(\lambda) + t \operatorname{\mathsf{Im}} T_{\mathsf{div}}(\lambda)\big) + \mathcal{O}(1)\\
&\le \mp\xi(\lambda; H_\mp, H_0)\\
&\le \int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm\big(1-\varepsilon; \operatorname{\mathsf{Re}} T_{\mathsf{div}}(\lambda) + t \operatorname{\mathsf{Im}} T_{\mathsf{div}}(\lambda)\big) + \mathcal{O}(1)
\end{aligned}$$

hold as $\lambda \to \pm m$ for each $\varepsilon \in (0,1)$.

Proof. Using (5.12), the Weyl inequalities (6.1), and Lemma 4.1 we get

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm\big(1+\varepsilon; \operatorname{\mathsf{Re}} T_{\mathsf{div}}(\lambda) + t \operatorname{\mathsf{Im}} T_{\mathsf{div}}(\lambda)\big) - n_\mp\big(\varepsilon/2; \operatorname{\mathsf{Re}} T_{\mathsf{bound}}(\lambda)\big) - \frac{2}{\pi\varepsilon}\big\|\operatorname{\mathsf{Im}} T_{\mathsf{bound}}(\lambda)\big\|_1\\
&\le \int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm\big(1; A(\lambda+i0) + tB(\lambda+i0)\big)\\
&\le \int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm\big(1-\varepsilon; \operatorname{\mathsf{Re}} T_{\mathsf{div}}(\lambda) + t \operatorname{\mathsf{Im}} T_{\mathsf{div}}(\lambda)\big) + n_\pm\big(\varepsilon/2; \operatorname{\mathsf{Re}} T_{\mathsf{bound}}(\lambda)\big) + \frac{2}{\pi\varepsilon}\big\|\operatorname{\mathsf{Im}} T_{\mathsf{bound}}(\lambda)\big\|_1.
\end{aligned} \tag{6.3}$$

Due to (6.2), we have

$$n_\pm\big(\varepsilon/2; \operatorname{Re} T_{\mathsf{bound}}(\lambda)\big) \le 16\,\varepsilon^{-4}\|T_{\mathsf{bound}}(\lambda)\|_4^4,$$

which combined with (5.6) gives

$$n_\pm\big(\varepsilon/2; \operatorname{Re} T_{\mathsf{bound}}(\lambda)\big) = \mathcal{O}(1) \quad \text{as} \quad \lambda \to \pm m.$$

Moreover, we know from (5.14) that

$$\lim_{\lambda\to\pm m} \big\| \operatorname{Im} T_{\mathsf{bound}}(\lambda)\big\|_1 = 0.$$

So the claim follows from the estimates (6.3) and Formula (4.1) □

6.1 The case $|\lambda| < m$

In this section we prove asymptotic estimates for $\xi(\lambda; H, H_\pm)$ as $\lambda \to \pm m$ with $|\lambda| < m$. We start with a corollary of Proposition 6.1, which follows from the fact that $\operatorname{Im} T_{\mathsf{div}}(\lambda) = 0$ and $\operatorname{Re} T_{\mathsf{div}}(\lambda) = T_{\mathsf{div}}(\lambda)$ for $\lambda \in (-m, m)$.

Corollary 6.2. *Let V satisfy Assumption 3.2. Then the estimates*

$$n_\pm\big(1+\varepsilon; T_{\mathsf{div}}(\lambda)\big) + \mathcal{O}(1) \le \mp\xi(\lambda; H_\mp, H_0) \le n_\pm\big(1-\varepsilon; T_{\mathsf{div}}(\lambda)\big) + \mathcal{O}(1)$$

hold as $\lambda \to \pm m$, $|\lambda| < m$, for each $\varepsilon \in (0,1)$.

Define the bounded operators $K_\pm : \mathcal{H} \to \mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^4)$ by

$$(K_+\varphi)(x_\perp) := \int_{\mathbb{R}^3} \mathrm{d}x'_\perp \mathrm{d}x'_3\, p(x_\perp, x'_\perp) \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} V^{1/2}(x'_\perp, x'_3)\varphi(x'_\perp, x'_3),$$

$$(K_-\varphi)(x_\perp) := \int_{\mathbb{R}^3} \mathrm{d}x'_\perp \mathrm{d}x'_3\, p(x_\perp, x'_\perp) \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix} V^{1/2}(x'_\perp, x'_3)\varphi(x'_\perp, x'_3),$$

where $p(\,\cdot\,,\,\cdot\,)$ is the integral kernel of the projection p. One shows easily that $K_\pm^* : \mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^4) \to \mathcal{H}$ are given by

$$(K_+^*\psi)(x_\perp, x_3) = V^{1/2}(x_\perp, x_3) \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} (p\psi)(x_\perp),$$

$$(K_-^*\psi)(x_\perp, x_3) = V^{1/2}(x_\perp, x_3) \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix} (p\psi)(x_\perp),$$

and that

$$O_+(\lambda) := \tfrac{1}{2}\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2} K_+^* K_+ \qquad \text{and} \qquad O_-(\lambda) := -\tfrac{1}{2}\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2} K_-^* K_-$$

belong to $S_2(\mathcal{H})$ for each $\lambda \in (-m, m)$.

In the next proposition we show that the functions $n_\pm\big(\,\cdot\,; T_{\mathsf{div}}(\lambda)\big)$ as $\lambda \to \pm m$, $|\lambda| < m$, can be bounded, up to $\mathcal{O}(1)$ terms, from below and from above by expressions involving $O_\pm(\lambda)$.

Proposition 6.3. *Let V satisfy Assumption 3.2. Then the estimates*

$$n_+\big((1+\varepsilon)s; O_+(\lambda)\big) + \mathcal{O}(1) \le n_+\big(s; T_{\mathsf{div}}(\lambda)\big) \le n_+\big((1-\varepsilon)s; O_+(\lambda)\big) + \mathcal{O}(1), \tag{6.4}$$

$$\mathcal{O}(1) \le n_-\big(s; T_{\mathsf{div}}(\lambda)\big) \le \mathcal{O}(1), \tag{6.5}$$

hold as $\lambda \nearrow m$, for each $\varepsilon \in (0,1)$ and $s > 0$, and the estimates

$$\mathcal{O}(1) \le n_+\big(s; T_{\mathsf{div}}(\lambda)\big) \le \mathcal{O}(1), \tag{6.6}$$

$$n_-\big((1+\varepsilon)s; O_-(\lambda)\big) + \mathcal{O}(1) \le n_-\big(s; T_{\mathsf{div}}(\lambda)\big) \le n_-\big((1-\varepsilon)s; O_-(\lambda)\big) + \mathcal{O}(1), \tag{6.7}$$

hold as $\lambda \searrow -m$, for each $\varepsilon \in (0,1)$ and $s > 0$.

Proof. We only give the proof of (6.4)-(6.5), since the proof of (6.6)-(6.7) is similar. In point (i) below we show that the difference $T_{\mathsf{div}}(\lambda) - O_+(\lambda)$ can be approximated in norm, as $\lambda \nearrow m$, by a compact operator independent of λ. Then we prove (6.4)-(6.5) in point (ii) by using this result.

(i) Let $\lambda \in (-m, m)$ and take $\nu' \in (3, \nu)$. A direct calculation shows that

$$T_{\mathsf{div}}(\lambda) - O_+(\lambda) = \widetilde{M}\big(G_{\nu-\nu'} \otimes J_{\nu'}^{(\lambda)}\big) \begin{pmatrix} (\lambda+m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (\lambda-m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \widetilde{M} + O_-(\lambda), \tag{6.8}$$

where $J_{\nu'}^{(\lambda)} : \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R})$ is given by

$$\big(J_{\nu'}^{(\lambda)}\psi\big)(x_3) := -\langle x_3\rangle^{-\nu'/2} \int_{\mathbb{R}} \mathrm{d}x_3' \frac{\mathrm{e}^{-\frac{1}{2}\sqrt{m^2-\lambda^2}|x_3-x_3'|}}{\sqrt{m^2-\lambda^2}} \sinh\Big(\frac{\sqrt{m^2-\lambda^2}|x_3-x_3'|}{2}\Big)\langle x_3'\rangle^{-\nu'/2}\psi(x_3'),$$

and

$$\widetilde{M} := V^{1/2}\langle Q_\perp\rangle^{(\nu-\nu')/2}\langle Q_3\rangle^{\nu'/2}, \tag{6.9}$$

$$G_{\nu-\nu'} := \langle Q_\perp\rangle^{-(\nu-\nu')/2} p \langle Q_\perp\rangle^{-(\nu-\nu')/2}. \tag{6.10}$$

The operator $\widetilde{M}$ is bounded due to Assumption 3.2, $G_{\nu-\nu'}$ is compact in $\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^4)$ due to Lemma 5.1, and $O_-(\lambda)$ satisfies

$$\lim_{\lambda\to m,\, |\lambda|<m} \big\|O_-(\lambda)\big\|_2 = 0. \tag{6.11}$$

Define

$$T_\pm := \widetilde{M}\big(G_{\nu-\nu'} \otimes J_{\nu'}^{(m)}\big) \begin{pmatrix} (m\pm m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(m\mp m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \widetilde{M}, \tag{6.12}$$

with $J_{\nu'}^{(m)} : \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R})$ given by

$$\big(J_{\nu'}^{(m)}\psi\big)(x_3) := -\tfrac{1}{2}\langle x_3\rangle^{-\nu'/2} \int_{\mathbb{R}} \mathrm{d}x_3'\, |x_3 - x_3'| \langle x_3'\rangle^{-\nu'/2}\psi(x_3').$$

Since $\nu' > 3$, $J_{\nu'}^{(m)}$ belongs to $S_2[\mathsf{L}^2(\mathbb{R})]$, and $T_\pm$ is compact in $\mathcal{H}$. Moreover, by using Lebesgue's dominated convergence theorem, one shows that

$$\lim_{\lambda\to\pm m,\, |\lambda|<m} \big\|J_{\nu'}^{(m)} - J_{\nu'}^{(\lambda)}\big\|_2^2 = 0.$$

This, together with (6.8), (6.11) and (6.12), implies that

$$\lim_{\lambda\nearrow m} \big\|T_{\mathsf{div}}(\lambda) - O_+(\lambda) - T_+\big\| = 0. \tag{6.13}$$

(ii) Take $\lambda \in (-m, m)$, $\varepsilon \in (0,1)$, and $s > 0$. Using the Weyl inequalities (6.1) we get

$$\begin{aligned} n_\pm\big((1+\varepsilon)s; O_+(\lambda)\big) - n_\mp\big(\varepsilon s; T_{\mathsf{div}}(\lambda) - O_+(\lambda)\big) &\le n_\pm\big(s; T_{\mathsf{div}}(\lambda)\big) \\ &\le n_\pm\big((1-\varepsilon)s; O_+(\lambda)\big) + n_\pm\big(\varepsilon s; T_{\mathsf{div}}(\lambda) - O_+(\lambda)\big). \end{aligned}$$

Now we have $n_-\big(t; O_+(\lambda)\big) = 0$ for each $t > 0$ and $\lambda \in (-m, m)$, since $O_+(\lambda)$ is a positive operator. So, to prove (6.4)-(6.5), it is sufficient to show that $n_\pm\big(\varepsilon s; T_{\mathsf{div}}(\lambda) - O_+(\lambda)\big) = \mathcal{O}(1)$ as $\lambda \nearrow m$, for each $\varepsilon \in (0,1)$ and $s > 0$. Let $t > 0$ be fixed. Then we know from (6.13) that we can chose $\lambda_+ \in (-m, m)$, close enough to m, so that $\big\|T_{\mathsf{div}}(\lambda_+) - O_+(\lambda_+) - T_+\big\| < t/2$. Thus, using again the Weyl inequalities, we get

$$n_\pm\big(t; T_{\mathsf{div}}(\lambda_+) - O_+(\lambda_+)\big) \le n_\pm\big(t/2; T_{\mathsf{div}}(\lambda_+) - O_+(\lambda_+) - T_+\big) + n_\pm\big(t/2; T_+\big) = n_\pm\big(t/2; T_+\big).$$

Since the r.h.s. is independent of λ_+ we have shown that $n_\pm\big(t; T_{\mathsf{div}}(\lambda) - O_+(\lambda)\big) = \mathcal{O}(1)$ as $\lambda \nearrow m$. This concludes the proof of (6.4)-(6.5). $\square$

We show now that the counting functions $n_\pm\big(\,\cdot\,; O_\pm(\lambda)\big)$ in Proposition 6.3 can be rewritten in terms of Berezin-Toeplitz type operators. Define for each $\lambda \in (-m, m)$

$$\omega_+(\lambda) := \tfrac{1}{2}\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2} pW_+p \qquad \text{and} \qquad \omega_-(\lambda) := -\tfrac{1}{2}\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2} pW_-p,$$

where the functions $W_\pm : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$W_+(x_\perp) := \int_{\mathbb{R}} \mathrm{d}x_3\, V_{11}(x_\perp, x_3) \qquad \text{and} \qquad W_-(x_\perp) := \int_{\mathbb{R}} \mathrm{d}x_3\, V_{33}(x_\perp, x_3). \tag{6.14}$$

Under the condition (3.5) one has

$$0 \le W_\pm(x_\perp) \le \mathrm{Const.}\, \langle x_\perp \rangle^{-\nu+1} \quad \text{for all } x_\perp \in \mathbb{R}^2,$$

and $\omega_\pm(\lambda) \in S_1[\mathsf{L}^2(\mathbb{R}^2)]$ if V satisfies Assumption 3.1 (see Lemma 5.1). Moreover, one has the following.

Proposition 6.4. *Let V satisfy Assumption 3.1. Then we have for each $\lambda \in (-m, m)$ and $s > 0$*

$$n_\pm\big(s; O_\pm(\lambda)\big) = n_\pm\big(s; \omega_\pm(\lambda)\big). \tag{6.15}$$

Proof. Given $s > 0$ and two separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, one has

$$n_\pm\big(s; B^*B\big) = n_\pm\big(s; BB^*\big) \tag{6.16}$$

for any $B \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $B^*B \in S_\infty(\mathcal{H}_1)$. Moreover, one can easily check that

$$K_+K_+^* = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} pW_+p \qquad \text{and} \qquad K_-K_-^* = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix} pW_-p.$$

Thus

$$n_+\big(s; O_+(\lambda)\big) = n_+\left(s; \tfrac{1}{2}\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2} \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} pW_+p\right) = n_+\Big(s; \tfrac{1}{2}\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2} pW_+p\Big) = n_+\big(s; \omega_+(\lambda)\big).$$

The proof of the second equality in (6.15) is similar. $\square$

The next theorem is direct consequence of Corollary 6.2 and Propositions 6.3-6.4.

Theorem 6.5. *Let V satisfy Assumption 3.2. Then one has for each $\varepsilon \in (0, 1)$*

$$\mathcal{O}(1) \le \xi(\lambda; H_+, H_0) \le \mathcal{O}(1) \tag{6.17}$$

$$-n_+\big(1-\varepsilon; \omega_+(\lambda)\big) + \mathcal{O}(1) \le \xi(\lambda; H_-, H_0) \le -n_+\big(1+\varepsilon; \omega_+(\lambda)\big) + \mathcal{O}(1) \tag{6.18}$$

as $\lambda \nearrow m$, and

$$n_-\big(1+\varepsilon; \omega_-(\lambda)\big) + \mathcal{O}(1) \le \xi(\lambda; H_+, H_0) \le n_-\big(1-\varepsilon; \omega_-(\lambda)\big) + \mathcal{O}(1) \tag{6.19}$$

$$\mathcal{O}(1) \le \xi(\lambda; H_-, H_0) \le \mathcal{O}(1) \tag{6.20}$$

as $\lambda \searrow -m$.

Remark 6.6. *The inequalities* (6.17) *together with Remark* (4.5) *imply that the eigenvalues of $H_0 + V$ in $(-m, m)$ near $+m$ (if any) do not accumulate at $+m$. On the other hand the inequalities* (6.18) *tell us that the number of eigenvalues of $H_0 - V$ in $(-m, m)$ near $\lambda = +m$ scales, up to $\mathcal{O}(1)$ terms, as*

$$n_+\big(s; \omega_+(\lambda)\big) \equiv \operatorname{rank} E^{pW_+p}\Big(\Big(s\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2}, \infty\Big)\Big)$$

with $s \approx 2$. Accordingly, the problem of counting the number of eigenvalues of $H_0 - V$ in $(-m, m)$ near $+m$ reduces to the problem of counting the number of eigenvalues of the positive Berezin-Toeplitz type operator

pW_+p near 0. The inequalities (6.19)-(6.20) *lead to similar conclusions on the number of eigenvalues of $H_0 \pm V$ in $(-m,m)$ near $-m$.*

One can compare these results with the results of [8] and [19] on the finiteness in $(-m,m)$ of the discrete spectrum of the Dirac operator perturbed by a matrix potential $Q \equiv \{Q_{jk}(x)\}_{j,k=1}^4$. In Corollary 2.2 of [8], the author shows that the spectrum in $(-m,m)$ of the Dirac operator perturbed by Q is finite if the 2×2 diagonal blocks of Q are of order $\mathcal{O}\big(|x|^{-2-\delta}\big)$ and the anti-diagonal blocks are of order $\mathcal{O}\big(|x|^{-1-\delta}\big)$, for some $\delta > 0$ as $|x| \to \infty$. In Corollary 2.1 of [19], the authors show that the Dirac operator perturbed by γQ, with $|\gamma|$ small enough and

$$\big|Q_{jk}(x)\big| \le \langle x\rangle^{-2}, \qquad j,k \in \{1,2,3,4\},$$

does not have any point spectrum. Therefore, in our case where $Q = -\alpha_1 a_1 - \alpha_2 a_2 + V$, we would not have had any accumulation of eigenvalues in $(-m,m)$ if we would have imposed such decay assumptions on the magnetic part $-\alpha_1 a_1 - \alpha_2 a_2$ of the perturbation.

As seen in Theorem 6.5 the behaviour of the function $\xi(\,\cdot\,;H_\pm,H_0)$ in $(-m,m)$ depends on the distribution of eigenvalues of the trace class operator $pW_\mp p$. In our next proposition we shall exhibit different types of behaviours depending on the choice of the functions V_{11} and V_{33} appearing in $W_\pm$. For that purpose, we first have to recall some technical results taken from [27], [29] and [30].

In the first lemma, an integrated density of states (IDS) for the operator $H_\perp^-$ in $\mathsf{L}^2(\mathbb{R}^2)$ is defined as follows (see *e.g.* [10, 18]): Let $\chi_{T,x_\perp}$ be the characteristic function of the square $x_\perp + \big(-\frac{T}{2},\frac{T}{2}\big)^2$, with $x_\perp \in \mathbb{R}^2$ and $T > 0$. Then a non-increasing function $\varrho : [0,\infty) \to \mathbb{R}$ is called IDS for the operator $H_\perp^-$ if for each $x_\perp \in \mathbb{R}^2$ it satisfies

$$\varrho(\lambda) = \lim_{T\to\infty} T^{-2}\,\mathsf{Tr}\left[\chi_{T,x_\perp}(Q_\perp)E^{H_\perp^-}\big((-\infty,\lambda)\big)\chi_{T,x_\perp}(Q_\perp)\right]$$

for each point $\lambda \in \mathbb{R}$ of continuity of ϱ.

Lemma 6.7 (Lemma 3.3 of [27])**.** *Let $U \in C^1(\mathbb{R}^2)$ satisfy*

$$0 \le U(x_\perp) \le \mathrm{Const.}\,\langle x_\perp\rangle^{-\alpha} \qquad \textit{and} \qquad \big|(\nabla U)(x_\perp)\big| \le \mathrm{Const.}\,\langle x_\perp\rangle^{-\alpha-1}$$

for all $x \in \mathbb{R}^2$ and some $\alpha > 0$. Assume moreover that

- *$U(x_\perp) = u\big(\frac{x_\perp}{|x_\perp|}\big)\big(1+o(1)\big)$ as $|x_\perp| \to \infty$, where u is a continuous function on $\mathbb{S}^1$ which does not vanish identically,*
- *b is an admissible magnetic field,*
- *there exists an IDS ϱ_b for the operator $H_\perp^-$.*

Then we have

$$n_+\big(s;pUp\big) = \frac{b_0}{2\pi}\big|\{x_\perp \in \mathbb{R}^2 \mid U(x_\perp) > s\}\big|\big(1+o(1)\big) = \Psi_\alpha(s;u,b_0)\big(1+o(1)\big) \quad \textit{as} \quad s \searrow 0,$$

where $|\cdot|$ denotes the Lebesgue measure, and

$$\Psi_\alpha(s;u,b_0) := \frac{s^{-2/\alpha}b_0}{4\pi}\int_{\mathbb{S}^1} \mathrm{d}\vartheta\, u(\vartheta)^{2/\alpha}, \qquad s > 0. \tag{6.21}$$

Lemma 6.8 (Lemma 3.4 of [27])**.** *Let $0 \le U \in \mathsf{L}^\infty(\mathbb{R}^2)$. Assume that*

$$\ln\big(U(x_\perp)\big) = -\eta|x_\perp|^{2\beta}\big(1+o(1)\big) \quad \textit{as} \quad |x_\perp| \to \infty,$$

for some $\eta,\beta > 0$. Let b be an admissible magnetic field. Then we have

$$n_+\big(s;pUp\big) = \Phi_\beta(s,\eta,b_0)\big(1+o(1)\big) \quad \textit{as} \quad s \searrow 0,$$

where

$$\Phi_\beta(s,\eta,b_0) := \begin{cases} \frac{b_0}{2\eta^{1/\beta}}\,|\ln(s)|^{1/\beta} & \text{if } \beta\in(0,1),\\ \frac{1}{\ln(1+2\eta/b_0)}\,|\ln(s)| & \text{if } \beta=1,\\ \frac{\beta}{\beta-1}\big(\ln|\ln(s)|\big)^{-1}|\ln(s)| & \text{if } \beta>1, \end{cases} \qquad s\in(0,\mathrm{e}^{-1}). \tag{6.22}$$

Lemma 6.9 (Lemma 3.5 of [27])**.** *Let $0\le U\in \mathsf{L}^\infty(\mathbb{R}^2)$. Assume that the support of U is compact, and that there exists a constant $\mathsf{c}>0$ such that $U\ge \mathsf{c}$ on an open non-empty subset of $\mathbb{R}^2$. Let b be an admissible magnetic field. Then we have*

$$n_+\big(s;pUp\big)=\Phi_\infty(s)\big(1+o(1)\big) \quad \textit{as} \quad s\searrow 0,$$

where

$$\Phi_\infty(s) := \big(\ln|\ln(s)|\big)^{-1}|\ln(s)|, \qquad s\in(0,\mathrm{e}^{-1}). \tag{6.23}$$

Combining Theorem 6.5 with Lemmas 6.7-6.9 we obtain the behaviour of $\xi(\lambda;H_\pm,H_0)$ as $|\lambda|\to m$, $|\lambda|<m$, when the functions $W_\pm$ admit a power-like or exponential decay at infinity, or when they have a compact support.

Proposition 6.10. *Let V satisfy Assumption 3.2.*

(a) Assume that the hypotheses of Lemma 6.7 hold with $U_\pm=W_\pm$ and $\alpha=\nu-1$. Then we have

$$\xi(\lambda;H_-,H_0)=-\Psi_{\nu-1}\Big(2\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2};u_+,b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda\nearrow m,$$

and

$$\xi(\lambda;H_+,H_0)=\Psi_{\nu-1}\Big(2\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2};u_-,b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda\searrow -m,$$

with $\Psi_{\nu-1}$ given by Equation (6.21)*.*

(b) Assume that the hypotheses of Lemma 6.8 hold with $U_\pm=W_\pm$. Then we have

$$\xi(\lambda;H_-,H_0)=-\Phi_{\beta_+}\Big(2\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2};\eta_+,b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda\nearrow m,$$

and

$$\xi(\lambda;H_+,H_0)=\Phi_{\beta_-}\Big(2\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2};\eta_-,b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda\searrow -m,$$

with $\beta_\pm\in(0,\infty)$ and $\Phi_{\beta_\pm}$ given by Equation (6.22)*.*

(c) Assume that the hypotheses of Lemma 6.9 hold with $U_\pm=W_\pm$. Then we have

$$\xi(\lambda;H_-,H_0)=-\Phi_\infty\Big(2\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2}\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda\nearrow m,$$

and

$$\xi(\lambda;H_+,H_0)=\Phi_\infty\Big(2\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2}\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda\searrow -m,$$

with Φ_∞ given by Equation (6.23)*.*

The estimates of Proposition 6.10 are similar to the ones of [27, Cor. 3.6], where the corresponding situation for magnetic Pauli operators is considered.

6.2 The case $|\lambda| > m$

In this section we prove asymptotic estimates for $\xi(\lambda; H, H_\pm)$ as $\lambda \to \pm m$, when $|\lambda| > m$. We start by showing an estimate for $n_\pm\big(s; \operatorname{Re} T_{\mathsf{div}}(\lambda)\big)$.

Proposition 6.11. *Let V satisfy Assumption 3.2. Then the estimates*

$$n_\pm\big(s; \operatorname{Re} T_{\mathsf{div}}(\lambda)\big) = \mathcal{O}(1) \quad \text{as} \quad \lambda \to \pm m,\ |\lambda| > m,$$

hold for each $s > 0$.

Proof. Take $\lambda \in \mathbb{R}$ with $|\lambda| > m$, and let $\nu' \in (3, \nu)$. Then we have

$$\operatorname{Re} T_{\mathsf{div}}(\lambda) = \widetilde{M}\big(G_{\nu-\nu'} \otimes R^{(\lambda)}_{\nu'}\big) \begin{pmatrix} (\lambda+m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (\lambda-m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \widetilde{M},$$

with $\widetilde{M}$ and $G_{\nu-\nu'}$ as in (6.9)-(6.10), and

$$R^{(\lambda)}_{\nu'} := \langle Q_3\rangle^{-\nu'/2} \operatorname{Re} R(\lambda^2 - m^2)\langle Q_3\rangle^{-\nu'/2}.$$

By using Lebesgue's dominated convergence theorem, one shows that

$$\lim_{\lambda\to\pm m,\,|\lambda|>m} \big\| \operatorname{Re} T_{\mathsf{div}}(\lambda) - T_\pm \big\| = 0,$$

with $T_\pm$ as in (6.12). So the claim can be proved as in point (ii) of the proof of Proposition 6.3. ☐

The next result follows from applying Propositions 6.1 and 6.11, the Weyl inequalities (6.1) and the identities [12, Sec. 5.4]

$$\int_{\mathbb{R}} \mathrm{d}\mu(t)\, n_\pm\big(s; tT\big) = \pi^{-1} \operatorname{Tr} \arctan(s^{-1}T), \qquad s > 0, \tag{6.24}$$

where $T \in S_1(\mathcal{H})$, $T = T^* \geq 0$. We also use the fact that $\operatorname{sgn}(\lambda) \operatorname{Im} T_{\mathsf{div}}(\lambda)$ is a positive operator if $|\lambda| > m$.

Corollary 6.12. *Let V satisfy Assumption 3.2. Then the estimates*

$$\begin{aligned} &\pi^{-1} \operatorname{Tr} \arctan\big[(1+\varepsilon)^{-1} \operatorname{sgn}(\lambda) \operatorname{Im} T_{\mathsf{div}}(\lambda)\big] + \mathcal{O}(1) \\ &\leq \mp\xi(\lambda; H_\mp, H_0) \\ &\leq \pi^{-1} \operatorname{Tr} \arctan\big[(1-\varepsilon)^{-1} \operatorname{sgn}(\lambda) \operatorname{Im} T_{\mathsf{div}}(\lambda)\big] + \mathcal{O}(1) \end{aligned}$$

hold as $\lambda \to \pm m$, $|\lambda| > m$, for each $\varepsilon \in (0, 1)$.

As in the case $|\lambda| < m$, we introduce auxiliary operators in order to express the lower and upper bounds for $\mp\xi(\lambda; H_\mp, H_0)$ in terms of Berezin-Toeplitz type operators. For $\lambda \in \mathbb{R}$ with $|\lambda| > m$, we define the operators $K_{1,\lambda}, K_{2,\lambda} : \mathcal{H} \to \mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^4)$ by

$$(K_{1,\lambda}\varphi)(x_\perp) := \int_{\mathbb{R}^3} \mathrm{d}x'_\perp \mathrm{d}x'_3\, p(x_\perp, x'_\perp) \cos\big(x'_3\sqrt{\lambda^2 - m^2}\big) \begin{pmatrix} \sqrt{|\lambda+m|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{|\lambda-m|} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^{1/2}(x'_\perp, x'_3)\varphi(x'_\perp, x'_3),$$

$$(K_{2,\lambda}\varphi)(x_\perp) := \int_{\mathbb{R}^3} \mathrm{d}x'_\perp \mathrm{d}x'_3\, p(x_\perp, x'_\perp) \sin\big(x'_3\sqrt{\lambda^2 - m^2}\big) \begin{pmatrix} \sqrt{|\lambda+m|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{|\lambda-m|} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^{1/2}(x'_\perp, x'_3)\varphi(x'_\perp, x'_3).$$

Direct calculations show that the adjoint operators $K^*_{1,\lambda}, K^*_{2,\lambda} : \mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^4) \to \mathcal{H}$ are given by

$$(K^*_{1,\lambda}\psi)(x_\perp, x_3) = \cos\big(x_3\sqrt{\lambda^2 - m^2}\big) V^{1/2}(x_\perp, x_3) \begin{pmatrix} \sqrt{|\lambda+m|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{|\lambda-m|} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (p\psi)(x_\perp),$$

$$(K^*_{2,\lambda}\psi)(x_\perp, x_3) = \sin\big(x_3\sqrt{\lambda^2 - m^2}\big) V^{1/2}(x_\perp, x_3) \begin{pmatrix} \sqrt{|\lambda+m|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{|\lambda-m|} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (p\psi)(x_\perp),$$

and that

$$\operatorname{sgn}(\lambda)\,\mathsf{Im}\,T_{\mathsf{div}}(\lambda)=\frac{1}{2\sqrt{\lambda^2-m^2}}\big(K_{1,\lambda}^*K_{1,\lambda}+K_{2,\lambda}^*K_{2,\lambda}\big).$$

This last equation can be written more compactly as

$$\operatorname{sgn}(\lambda)\,\mathsf{Im}\,T_{\mathsf{div}}(\lambda)=\frac{1}{2\sqrt{\lambda^2-m^2}}\,K_\lambda^*K_\lambda \tag{6.25}$$

if we use the operator

$$K_\lambda:\mathcal{H}\to\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^8),\qquad K_\lambda\varphi:=\begin{pmatrix}K_{1,\lambda}\varphi\\ K_{2,\lambda}\varphi\end{pmatrix},$$

with adjoint

$$K_\lambda^*:\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^8)\to\mathcal{H},\qquad K_\lambda^*\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=K_{1,\lambda}^*\psi_1+K_{2,\lambda}^*\psi_2.$$

For the next proposition we also need to introduce for each $\lambda\in\mathbb{R}$ with $|\lambda|>m$ the positive operator $\Omega(\lambda):\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^8)\to\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^8)$ defined by

$$\Omega(\lambda):=\frac{1}{2\sqrt{\lambda^2-m^2}}\,K_\lambda K_\lambda^*.$$

A direct calculation shows that

$$K_\lambda K_\lambda^*=p\begin{pmatrix}M_{1,\lambda}&M_{2,\lambda}\\ M_{2,\lambda}&M_{3,\lambda}\end{pmatrix}p,$$

where

$$M_{1,\lambda}(x_\perp):=\int_{\mathbb{R}}\mathrm{d}x_3\,\cos^2\big(x_3\sqrt{\lambda^2-m^2}\big)\begin{pmatrix}|\lambda+m|V_{11}(x_\perp,x_3)&0&\sqrt{\lambda^2-m^2}V_{13}(x_\perp,x_3)&0\\0&0&0&0\\ \sqrt{\lambda^2-m^2}V_{31}(x_\perp,x_3)&0&|\lambda-m|V_{33}(x_\perp,x_3)&0\\0&0&0&0\end{pmatrix},$$

$$M_{2,\lambda}(x_\perp)=\int_{\mathbb{R}}\mathrm{d}x_3\,\sin\big(x_3\sqrt{\lambda^2-m^2}\big)\cos\big(x_3\sqrt{\lambda^2-m^2}\big)\begin{pmatrix}|\lambda+m|V_{11}(x_\perp,x_3)&0&\sqrt{\lambda^2-m^2}V_{13}(x_\perp,x_3)&0\\0&0&0&0\\ \sqrt{\lambda^2-m^2}V_{31}(x_\perp,x_3)&0&|\lambda-m|V_{33}(x_\perp,x_3)&0\\0&0&0&0\end{pmatrix},$$

$$M_{3,\lambda}(x_\perp)=\int_{\mathbb{R}}\mathrm{d}x_3\,\sin^2\big(x_3\sqrt{\lambda^2-m^2}\big)\begin{pmatrix}|\lambda+m|V_{11}(x_\perp,x_3)&0&\sqrt{\lambda^2-m^2}V_{13}(x_\perp,x_3)&0\\0&0&0&0\\ \sqrt{\lambda^2-m^2}V_{31}(x_\perp,x_3)&0&|\lambda-m|V_{33}(x_\perp,x_3)&0\\0&0&0&0\end{pmatrix}.$$

This implies that

$$\|\Omega(\lambda)\|_1\le\big(\tfrac{\lambda+m}{\lambda-m}\big)^{1/2}\big\|pW_+p\big\|_1+\big(\tfrac{\lambda-m}{\lambda+m}\big)^{1/2}\big\|pW_-p\big\|_1,$$

and thus $\Omega(\lambda)\in S_1[\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^8)]$ if V satisfies Assumption 3.1.

Next Proposition is a direct consequence of Equations (6.16) and (6.25).

Proposition 6.13. *Let V satisfy Assumption 3.1. Then we have for each $\lambda\in\mathbb{R}$ with $|\lambda|>m$ and each $s>0$*

$$n_\pm\big(s;\operatorname{sgn}(\lambda)\,\mathsf{Im}\,T_{\mathsf{div}}(\lambda)\big)=n_\pm\big(s;\Omega(\lambda)\big).$$

In particular, it follows by Equation (6.24) *that*

$$\mathsf{Tr}\arctan\big(s^{-1}\operatorname{sgn}(\lambda)\,\mathsf{Im}\,T_{\mathsf{div}}(\lambda)\big)=\mathsf{Tr}\arctan\big(s^{-1}\Omega(\lambda)\big). \tag{6.26}$$

The combination of Corollary 6.12 and Equation (6.26) gives the following.

Theorem 6.14. *Let V satisfy Assumption 3.2. Then one has for each $\varepsilon\in(0,1)$*

$$\pm\pi^{-1}\,\mathsf{Tr}\arctan\big[(1\pm\varepsilon)^{-1}\Omega(\lambda)\big]+\mathcal{O}(1)\le\xi(\lambda;H_\pm,H_0)\le\pm\pi^{-1}\,\mathsf{Tr}\arctan\big[(1\mp\varepsilon)^{-1}\Omega(\lambda)\big]+\mathcal{O}(1)$$

as $\lambda\to\pm m$, $|\lambda|>m$.

Remark 6.15. *The fact that the operators $\omega_\pm(\lambda)$ and $\Omega(\lambda)$ in Theorems 6.5 and 6.14 depend in a distinguished way on the components V_{11} and V_{33} of V is due to our initial assumption $b_0 > 0$. Indeed, this choice implies that $\ker(H_\perp^-)$ is non trivial, whereas $\ker(H_\perp^+) = \{0\}$. This lead us to introduce in Section 5 the projection $\mathsf{P} \equiv \operatorname{diag}(P, 0, P, 0)$, which put into light the priviledged role of the components V_{11} and V_{33} of V.*

The variation of $\xi(\lambda; H_\pm, H_0)$ under the change $\lambda \mapsto -\lambda$ can be explained using the antinunitary transformation of charge conjugation [38, Sec. 1.4.6]

$$C : \mathcal{H} \to \mathcal{H}, \qquad \varphi \mapsto U_C \overline{\varphi},$$

where $U_C := i\beta\alpha_2$. Indeed, if we write $H(\vec{a}, \pm V)$ and $H_0(\vec{a})$ for $H_\pm$ and H_0, then a direct calculation using the Lifshits-Krein trace formula (1.2) *shows that*

$$CH(\vec{a}, \pm V)C^{-1} = -H(-\vec{a}, \mp U_C \overline{V} U_C^*),$$

which entails

$$\xi\big(\lambda; H(\vec{a}, \pm V), H_0(\vec{a})\big) = -\xi\big(-\lambda; H(-\vec{a}, \mp U_C \overline{V} U_C^*), H_0(-\vec{a})\big).$$

This obviously explains why the overall sign of the spectral shift function is reversed under the change $\lambda \mapsto -\lambda$. But it also explains why the roles of V_{11} and V_{33} are interchanged in the estimates. Indeed, the natural projection corresponding to the vector potential $\vec{a}$ is $\mathsf{P} = \operatorname{diag}(P, 0, P, 0)$ since we have $b_0 > 0$ for $\vec{a}$, whereas $\mathsf{P}' := \operatorname{diag}(0, P, 0, P)$ is the natural choice for the vector potential $-\vec{a}$ since we have $b_0 < 0$ for $-\vec{a}$. Now, one has

$$\mp U_C \overline{V} U_C^* = \mp \begin{pmatrix} V_{44} & -\overline{V_{43}} & -\overline{V_{42}} & \overline{V_{41}} \\ -\overline{V_{34}} & V_{33} & \overline{V_{32}} & -\overline{V_{31}} \\ -\overline{V_{24}} & \overline{V_{23}} & V_{22} & -\overline{V_{21}} \\ \overline{V_{14}} & -\overline{V_{13}} & -\overline{V_{12}} & V_{11} \end{pmatrix}.$$

So, the projection P which selects the components $\pm(V_{11}, V_{33})$ of the potential $\pm V$ is replaced, after the change $\lambda \mapsto -\lambda$, by the projection P' which selects the components $\mp(V_{33}, V_{11})$ of the transformed potential $\mp U_C \overline{V} U_C^*$.

For the next proposition we define for each $\lambda \in \mathbb{R}$ with $|\lambda| > m$ the positive operator $\Omega^{(1)}(\lambda)$ in $\mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^8)$ given by

$$\Omega^{(1)}(\lambda) := \frac{1}{2\sqrt{\lambda^2 - m^2}} \begin{pmatrix} pM_\lambda p & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{where} \qquad M_\lambda := \begin{pmatrix} |\lambda+m|W_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & |\lambda-m|W_- & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 6.16. *(a) Let V satisfy Assumption 3.2 with $\nu \in (3, 4]$. Then one has for each $s > 0$ and each $\delta \in \big(\frac{4-\nu}{2}, \frac{1}{2}\big)$*

$$\mathsf{Tr}\big\{\arctan\big[s^{-1}\Omega(\lambda)\big] - \arctan\big[s^{-1}\Omega^{(1)}(\lambda)\big]\big\} = O\big(|\lambda \mp m|^{-\delta}\big) \quad \textit{as} \quad \lambda \to \pm m, \ |\lambda| > m.$$

(b) Let V satisfy Assumption 3.1 with $\nu_\perp > 2$ and $\nu_3 > 2$. Then one has for each $s > 0$

$$\mathsf{Tr}\big\{\arctan\big[s^{-1}\Omega(\lambda)\big] - \arctan\big[s^{-1}\Omega^{(1)}(\lambda)\big]\big\} = \mathcal{O}(1) \quad \textit{as} \quad \lambda \to \pm m, \ |\lambda| > m. \tag{6.27}$$

Proof. Points (a) and (b) are proved by using the Lifshits-Krein trace formula (1.2) with $f(\lambda) = \arctan(\lambda)$, $\lambda \in \mathbb{R}$. We do not give the details, since the argument is analogous to the one of [12, Cor. 2.2]. □

Note that if V satisfy Assumption 3.2 with $\nu \in (3, 4]$, we can choose $\delta \in \big(\frac{4-\nu}{2}, \frac{1}{\nu-1}\big)$, and so Proposition 6.16.(a) entails

$$\mathsf{Tr}\big\{\arctan\big[s^{-1}\Omega(\lambda)\big] - \arctan\big[s^{-1}\Omega^{(1)}(\lambda)\big]\big\} = o\big(|\lambda \mp m|^{-\frac{1}{\nu-1}}\big) \quad \text{as} \quad \lambda \to \pm m, \ |\lambda| > m. \tag{6.28}$$

Moreover, if V satisfy Assumption 3.2 with $\nu > 4$, then it satisfies Assumption 3.1 with $\nu_\perp > 2$ and $\nu_3 > 2$, and, hence (6.27) is valid. Finally, we have for $s > 0$ and $|\lambda| > m$

$$\mathsf{Tr}\arctan\big[s^{-1}\Omega^{(1)}(\lambda)\big] = \int_0^\infty \frac{\mathrm{d}t}{1+t^2}\, n_+\Big(2st\big(\tfrac{\lambda-m}{\lambda+m}\big)^{1/2}; pW_+p\Big) + \int_0^\infty \frac{\mathrm{d}t}{1+t^2}\, n_+\Big(2st\big(\tfrac{\lambda+m}{\lambda-m}\big)^{1/2}; pW_-p\Big). \tag{6.29}$$

Combining Equations (6.27)-(6.29), Theorem 6.14 and Lemmas 6.7-6.9, we get the following.

Corollary 6.17. *Let V satisfy Assumption 3.2.*

(a) Assume that the hypotheses of Lemma 6.7 hold with $U_\pm = W_\pm$ and $\alpha = \nu - 1$. Then we have

$$\xi(\lambda; H_-, H_0) = -\frac{1}{2\cos\big(\pi/(\nu-1)\big)}\, \Psi_{\nu-1}\Big(2\big(\tfrac{\lambda-m}{\lambda+m}\big)^{1/2}; u_+, b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda \searrow m,$$

and

$$\xi(\lambda; H_+, H_0) = \frac{1}{2\cos\big(\pi/(\nu-1)\big)}\, \Psi_{\nu-1}\Big(2\big(\tfrac{\lambda+m}{\lambda-m}\big)^{1/2}; u_-, b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda \nearrow -m,$$

with $\Psi_{\nu-1}$ given by Equation (6.21)*.*

(b) Suppose that V also satisfies (3.1) *with $\nu_\perp > 2$ and $\nu_3 > 2$, and assume that the hypotheses of Lemma 6.8 hold with $U_\pm = W_\pm$. Then we have*

$$\xi(\lambda; H_-, H_0) = -\frac{1}{2}\, \Phi_{\beta_+}\Big(2\big(\tfrac{\lambda-m}{\lambda+m}\big)^{1/2}; \eta_+, b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda \searrow m,$$

and

$$\xi(\lambda; H_-, H_0) = \frac{1}{2}\, \Phi_{\beta_-}\Big(2\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2}; \eta_-, b_0\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda \nearrow -m,$$

with $\beta_\pm \in (0,\infty)$ and $\Phi_{\beta_\pm}$ given by Equation (6.22)*.*

(c) Suppose that V also satisfies (3.1) *with $\nu_\perp > 2$ and $\nu_3 > 2$, and assume that the hypotheses of Lemma 6.9 hold with $U_\pm = W_\pm$. Then we have*

$$\xi(\lambda; H_-, H_0) = -\frac{1}{2}\, \Phi_\infty\Big(2\big(\tfrac{m-\lambda}{m+\lambda}\big)^{1/2}\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda \searrow m,$$

and

$$\xi(\lambda; H_+, H_0) = \frac{1}{2}\, \Phi_\infty\Big(2\big(\tfrac{m+\lambda}{m-\lambda}\big)^{1/2}\Big)\big(1+o(1)\big) \quad \textit{as} \quad \lambda \nearrow -m,$$

with Φ_∞ given by Equation (6.23)*.*

Putting together the results of Proposition 6.10 and Corollary 6.17, we obtain the following.

Corollary 6.18. *Under the assumptions of Corollary 6.17.(a), we have*

$$\lim_{\varepsilon\searrow 0} \frac{\xi\big(m(1-\varepsilon)^{-1}; H_-, H_0\big)}{\xi\big(m(1-\varepsilon); H_-, H_0\big)} = \frac{1}{2\cos\big(\pi/(\nu-1)\big)} = \lim_{\varepsilon\searrow 0} \frac{\xi\big(-m(1-\varepsilon)^{-1}; H_+, H_0\big)}{\xi\big(-m(1-\varepsilon); H_+, H_0\big)}\,,$$

and under the assumptions of Corollary 6.17.(b)-(c), we have

$$\lim_{\varepsilon\searrow 0} \frac{\xi\big(m(1-\varepsilon)^{-1}; H_-, H_0\big)}{\xi\big(m(1-\varepsilon); H_-, H_0\big)} = \frac{1}{2} = \lim_{\varepsilon\searrow 0} \frac{\xi\big(-m(1-\varepsilon)^{-1}; H_+, H_0\big)}{\xi\big(-m(1-\varepsilon); H_+, H_0\big)}\,.$$

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7 Appendix

We give in this appendix the proof of the inclusion (3.6) for the class of potentials V given in Remark 3.3. We start with a technical lemma. We use the notations $\alpha := (\alpha_1, \alpha_2, \alpha_3)^{\mathsf{T}}$ and

$$(\partial_\ell V) := \{(\partial_\ell V_{jk})\}, \qquad \nabla V := (\partial_1 V, \partial_2 V, \partial_3 V)^{\mathsf{T}}, \qquad (\partial_\ell \partial_m V) := \{(\partial_\ell \partial_m V_{jk})\}.$$

Lemma 7.1. *Let V be as in Remark 3.3. Then*

(a) One has in $\mathscr{B}\big(\mathcal{D}(H_0), \mathcal{D}(H_0)^*\big)$ the equalities

$$[H_0, H] = -i\alpha \cdot (\nabla V) + [\alpha_3, V]P_3 + m[\beta, V] \tag{7.1}$$

$$= -i\alpha \cdot (\nabla V) + P_3[\alpha_3, V] + i[\alpha_3, (\partial_3 V)] + m[\beta, V]. \tag{7.2}$$

(b) Let $z \in \mathbb{R} \setminus \{\sigma(H_0) \cup \sigma(H_\pm)\}$. Then there exist operators $B_\pm \in \mathscr{B}(\mathcal{H})$ such that

$$R_\pm^2(z) = B_\pm H_0^{-2} \qquad \textit{and} \qquad R_\pm^2(z) = H_0^{-2} B_\pm^*. \tag{7.3}$$

Proof. (a) We know from Lemma 2.2(b) of [32] that $\mathcal{D}(H_0) \subset \mathcal{D}(P_3)$. So each member of Equations (7.1)-(7.2) belongs to $\mathscr{B}\big(\mathcal{D}(H_0), \mathcal{D}(H_0)^*\big)$.

Let $\varphi \in \mathcal{D}(H_0)$, take a sequence $\{\varphi_n\} \subset C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$ such that $\lim_n \|\varphi_n - \varphi\|_{\mathcal{D}(H_0)} = 0$, and denote by $\langle \cdot, \cdot \rangle_{1,-1}$ the anti-duality map between $\mathcal{D}(H_0)$ and $\mathcal{D}(H_0)^*$. Then

$$\begin{aligned}\langle \varphi, [H_0, H]\varphi \rangle_{1,-1} &\equiv \langle H_0\varphi, V\varphi \rangle - \langle V\varphi, H_0\varphi \rangle \\ &= \lim_n \big\langle \varphi_n, [\alpha_1(P_1 - a_1) + \alpha_2(P_2 - a_2) + \alpha_3 P_3 + \beta m, V]\varphi_n \big\rangle \\ &= \lim_n \big\langle \varphi_n, \big\{ -i\alpha \cdot (\nabla V) + [\alpha_3, V]P_3 + m[\beta, V] \big\} \varphi_n \big\rangle. \end{aligned} \tag{7.4}$$

Since $\mathcal{D}(H_0) \subset \mathcal{D}(P_3)$, we also have $\lim_n \|\varphi_n - \varphi\|_{\mathcal{D}(P_3)} = 0$, and thus

$$\langle \varphi, [H_0, H]\varphi \rangle_{1,-1} = \big\langle \varphi, \big\{ -i\alpha \cdot (\nabla V) + [\alpha_3, V]P_3 + m[\beta, V] \big\} \varphi \big\rangle$$

This proves (7.1). Using (7.4), one also gets the equality (7.2).

(b) In what follows, we omit the indices "$\pm$" to simplify the notations and we write $B_1, B_2, \dots$ for elements of $\mathscr{B}(\mathcal{H})$. Since $\mathcal{D}(H) = \mathcal{D}(H_0)$, we have

$$R^2(z) = B_1 H_0^{-1} R(z) = B_1 R(z) H_0^{-1} + B_1 \big[H_0^{-1}, R(z)\big] = B_2 H_0^{-2} + B_1 H_0^{-1} R(z) \big[H_0, H\big] R(z) H_0^{-1}.$$

Now, one has

$$R(z)\big[H_0, H\big] R(z) H_0^{-1} = R(z) \big\{ -i\alpha \cdot (\nabla V) + P_3[\alpha_3, V] + i[\alpha_3, (\partial_3 V)] + m[\beta, V] \big\} R(z) H_0^{-1} = B_3 H_0^{-2}$$

due to Equation (7.2), the equality $\mathcal{D}(H) = \mathcal{D}(H_0)$, and the inclusion $\mathcal{D}(H_0) \subset \mathcal{D}(P_3)$. This, together with the preceding equation, implies the first identity in (7.3). The second identity follows from the first one by adjunction. $\square$

Proposition 7.2. *Take $z \in \mathbb{R} \setminus \{\sigma(H_0) \cup \sigma(H_\pm)\}$ and let V be as in Remark 3.3. Then we have*

$$R_\pm^3(z) - R_0^3(z) \in S_1(\mathcal{H}).$$

Proof. In what follows, we omit the indices "$\pm$" to simplify the notations and we write $B_1, B_2, \dots$ for elements of $\mathscr{B}(\mathcal{H})$. Differentiating twice the resolvent identity

$$R(z) - R_0(z) = -R(z) V R_0(z)$$

we find that

$$R^3(z) - R_0^3(z) = -R(z)VR_0^3(z) - R^2(z)VR_0^2(z) - R^3(z)VR_0(z).$$

So it is sufficient to show that each term on the r.h.s. belongs to $S_1(\mathcal{H})$. This is done in points (i), (ii) and (iii) below.

(i) For the term $R(z)VR_0^3(z)$, one has

$$R(z)VR_0^3(z) = R(z)R_0(z)VR_0^2(z) + R(z)[V, R_0(z)]R_0^2(z). \tag{7.5}$$

Since $\mathcal{D}(H) = \mathcal{D}(H_0)$, one has

$$R(z)R_0(z)VR_0^2(z) = R(z)(H_0 - z)R_0^2(z)VR_0^2(z) = \big(B_1 H_0^{-2} V^{1/2}\big)\big(V^{1/2} H_0^{-2} B_2\big).$$

So, by (3.3), $R(z)R_0(z)VR_0^2(z)$ is the product of two Hilbert-Schmidt operators, and thus belongs to $S_1(\mathcal{H})$.

For the second term of (7.5), we have by (7.1)

$$R(z)[V, R_0(z)]R_0^2(z) = R(z)R_0(z)[H_0, H]R_0^3(z) = B_1 H_0^{-2}\big\{ - i\alpha \cdot (\nabla V) + [\alpha_3, V]P_3 + m[\beta, V]\big\} H_0^{-3} B_3.$$

Due to the hypotheses on V and $(\partial_j V)$, one can use (3.3) to write the first and third term as a product of two Hilbert-Schmidt operators. So it only remains to show that $H_0^{-2}[\alpha_3, V]P_3 H_0^{-3}$ belongs to $S_1(\mathcal{H})$. For this, we use the inclusion $\mathcal{D}(H_0) \subset \mathcal{D}(P_3)$ and the commutation of P_3 and H_0^{-1} on $\mathcal{D}(P_3)$ [32, Lemma 2.2(b)] to get

$$H_0^{-2}[\alpha_3, V]P_3 H_0^{-3} = H_0^{-2}[\alpha_3, V]H_0^{-2} P_3 H_0^{-1} = H_0^{-2}[\alpha_3, V]H_0^{-2} B_4.$$

This, together with (3.3), implies that $H_0^{-2}[\alpha_3, V]P_3 H_0^{-3}$ belongs to $S_1(\mathcal{H})$.

(ii) One can write $R^2(z)VR_0^2(z)$ as the product of two Hilbert-Schmidt operators by using (7.3) and (3.3):

$$R^2(z)VR_0^2(z) = B_5 H_0^{-2} V H_0^{-2} B_2 = \big(B_5 H_0^{-2} V^{1/2}\big)\big(V^{1/2} H_0^{-2} B_2\big).$$

Thus $R^2(z)VR_0^2(z)$ belongs to $S_1(\mathcal{H})$.

(iii) For the term $R^3(z)VR_0(z)$ we have

$$R^3(z)VR_0(z) = R^2(z)VR(z)R_0(z) + R^2(z)[R(z), V]R_0(z).$$

One shows that $R^2(z)VR(z)R_0(z) \in S_1(\mathcal{H})$ as in point (ii). For the second term, we have by (7.2) and (7.3)

$$\begin{aligned} R^2(z)[R(z), V]R_0(z) &= R^3(z)[H, H_0]R(z)R_0(z) \\ &= B_5 H_0^{-2} R(z)\big\{i\alpha \cdot (\nabla V) - P_3[\alpha_3, V] - i[\alpha_3, (\partial_3 V)] - m[\beta, V]\big\} H_0^{-2} B_6. \end{aligned}$$

Due to the hypotheses on V and $(\partial_j V)$, one can use (7.3) and (3.3) to write the first, third, and fourth term as a product of two Hilbert-Schmidt operators. So it only remains to show that $H_0^{-2} R(z) P_3 [\alpha_3, V] H_0^{-2}$ belongs to $S_1(\mathcal{H})$. Using [32, Lemma 2.2(b)] and (7.3), one gets

$$\begin{aligned} H_0^{-2} R(z) P_3[\alpha_3, V] H_0^{-2} &= H_0^{-2} P_3 R(z)[\alpha_3, V] H_0^{-2} + H_0^{-2}[R(z), P_3][\alpha_3, V] H_0^{-2} \\ &= P_3 H_0^{-2} R(z)[\alpha_3, V] H_0^{-2} - i H_0^{-2} R(z)(\partial_3 V) R(z)[\alpha_3, V] H_0^{-2} \\ &= B_7 H_0^{-2}[\alpha_3, V] H_0^{-2} + B_8 R(z)(\partial_3 V) R(z)[\alpha_3, V] H_0^{-2}. \end{aligned}$$

The first term on the r.h.s. belongs to $S_1(\mathcal{H})$, and for the second term we have by (7.2) and (7.3)

$$\begin{aligned} & R(z)(\partial_3 V) R(z)[\alpha_3, V] H_0^{-2} \\ &= (\partial_3 V) R^2(z)[\alpha_3, V] H_0^{-2} + R(z)[(\partial_3 V), H] R^2(z)[\alpha_3, V] H_0^{-2} \\ &= B_9 H_0^{-2}[\alpha_3, V] H_0^{-2} + R(z)\big\{i\alpha \cdot [\nabla(\partial_3 V)] - P_3[\alpha_3, (\partial_3 V)] \\ &\qquad - i[\alpha_3, (\partial_3^2 V)] - m[\beta, (\partial_3 V)] + [(\partial_3 V), V]\big\} B_5 H_0^{-2}[\alpha_3, V] H_0^{-2}. \end{aligned}$$

Due to the hypotheses on V, $(\partial_j V)$, and $(\partial_{j3} V)$, one can use (3.3) to show that the first, the second, the fourth, the fifth, and the sixth term are trace class. For the third term we have to use (3.3) and the fact that $R(z)P_3$ extends to a bounded operator. □

Remark 7.3. *When the potential V is scalar, the equations* (7.1)-(7.2) *reduce to the single equality*

$$[H_0, H] = -i\alpha \cdot (\nabla V)$$

in $\mathscr{B}\big(\mathcal{D}(H_0), \mathcal{D}(H_0)^*\big)$. So the calculations in points (i) and (iii) of the proof of Proposition 7.2 simplify accordingly, and we obtain the inclusion

$$R^3_\pm(z) - R^3_0(z) \in S_1(\mathcal{H})$$

without assuming anything on the derivatives of V of order 2.

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