TRACE FORMULAS FOR SCHRÖDINGER OPERATORS IN CONNECTION WITH SCATTERING THEORY FOR FINITE-GAP BACKGROUNDS

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Abstract. We investigate trace formulas for one-dimensional Schrödinger operators which are trace class perturbations of quasi-periodic finite-gap operators using Krein's spectral shift theory. In particular, we establish the conserved quantities for the solutions of the Korteweg–de Vries hierarchy in this class and relate them to the reflection coefficients via Abelian integrals on the underlying hyperelliptic Riemann surface.

1. Introduction

Trace formulas for one-dimensional (discrete and continuous) Schrödinger operators have attracted an enormous amount of interest recently (see e.g. [3], [14], [15], [20], [23], [26], [31]). However, most results are in connection with scattering theory for a constant background. On the other hand, scattering theory for one-dimensional Schrödinger operators with periodic background is a much older topic first investigated by Firsova in a series of papers [7]–[9]. Nevertheless, many questions which have long been answered in the constant background case are still open in this more general setting.

The aim of the present paper is to help filling some of these gaps. To this end, we want to find the analog of the classical trace formulas in scattering theory for the case of a finite-gap background. In the case of zero background it is well-known that the transmission coefficient is the perturbation determinant in the sense of Krein [17] (see e.g., [30]) and our first aim is to establish this result for the case considered here; thereby establishing the connection with Krein's spectral shift theory. Our second aim is to find a representation of the transmission coefficient in terms of the scattering data — the analog of the classical Poisson–Jensen formula.

Moreover, scattering theory for one-dimensional Schrödinger operators is not only interesting in its own right, it also constitutes the main ingredient of the inverse scattering transform for the Korteweg–de Vries (KdV) hierarchy (see, e.g., [5], [21]). Again the case of decaying solutions is classical and trace formulas for this case were studied exhaustively in the past (cf. [12] and the references therein). Here we want to investigate the case of Schwartz type perturbations of a given finite-gap solution. The Cauchy problem for the KdV equation with initial conditions in this class was only solved recently by Egorova, Grunert, and Teschl [6]. Since the transmission coefficient is invariant when our Schrödinger operator evolves in

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time with respect to some equation of the KdV hierarchy, the corresponding trace formulas provide the conserved quantities for the KdV hierarchy in this setting.

Our work extends previous results for Jacobi operators by Michor and Teschl [22], [28]. For trace formulas in the pure finite-gap case see Gesztesy, Ratnaseelan, and Teschl [13] and Gesztesy and Holden [11].

2. Notation

We assume that the reader is familiar with finite-gap Schrödinger operators. Hence we only briefly recall some notation and refer to the monographs [11], [21] for further information.

Let

$$H_q = -\frac{d^2}{dx^2} + V_q(x) \tag{2.1}$$

be a finite-gap Schrödinger operator in $L^2(\mathbb{R})$ whose spectrum consists of $g+1$ bands:

$$\sigma(H_q) = \bigcup_{j=0}^{g-1} [E_{2j}, E_{2j+1}] \cup [E_{2g}, \infty). \tag{2.2}$$

It is well-known that H_q is associated with the Riemann surface $\mathbb{M}$ of the function

$$R_{2g+1}^{1/2}(z), \qquad R_{2g+1}(z) = \prod_{j=0}^{2g} (z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g}, \tag{2.3}$$

$g \in \mathbb{N}_0$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. Here $R_{2g+1}^{1/2}(z)$ is chosen to have branch cuts along the spectrum with the sign fixed by the asymptotic behavior $R_{2g+1}^{1/2}(z) = \sqrt{z} z^g + \dots$ as $z \to \infty$.

A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+1}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$. The point at infinity is denoted by $p_\infty = (\infty, \infty)$. We use $\pi(p) = z$ for the projection onto the extended complex plane $\mathbb{C} \cup \{\infty\}$. The points $\{(E_j, 0), 0 \le j \le 2g\} \cup \{p_\infty\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \{(z, \pm R_{2g+1}^{1/2}(z)) \mid z \in \mathbb{C} \backslash \Sigma\} \subset \mathbb{M}, \qquad \Sigma = \sigma(H_q), \tag{2.4}$$

are called upper and lower sheet, respectively. Note that the boundary of $\Pi_\pm$ consists of two copies of Σ corresponding to the two limits from the upper and lower half plane.

For every $z \in \mathbb{C}$ the Baker–Akhiezer functions $\psi_{q,\pm}(z,x)$ are two (weak) solutions of $H_q \psi = z\psi$. They are linearly independent away from the band-edges $\{E_j\}_{j=0}^{2g}$ since their Wronskian is given by

$$W(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{2\mathrm{i} R_{2g+1}^{1/2}(z)}{\prod_{j=1}^{g} (z - \mu_j)}. \tag{2.5}$$

Here $W_x(f,g) = f(x)g'(x) - f'(x)g(x)$ denotes the usual Wronskian and μ_j are the Dirichlet eigenvalues at base point $x_0 = 0$. We recall that $\psi_{q,\pm}(z,x)$ have the form

$$\psi_{q,\pm}(z,x) = \theta_{q,\pm}(z,x) \exp(\pm \mathrm{i} x k(z)), \tag{2.6}$$

where $\theta_{q,\pm}(z,x)$ is quasi-periodic with respect to x and

$$k(z) = -\int_{E_0}^{p} \omega_{p_\infty,0}, \qquad p=(z,+), \tag{2.7}$$

denotes the quasimomentum map. Here $\omega_{p_\infty,k}$ is a normalized Abelian differential of the second kind with a single pole at $p_\infty=(\infty,\infty)$ and principal part $\zeta^{-k-2}d\zeta$ where $\zeta = z^{-1/2}$. It is explicitly given by

$$\omega_{p_\infty,0} = -\frac{\prod_{j=1}^{g}(\pi-\lambda_j)d\pi}{2R_{2g+1}^{1/2}}, \tag{2.8}$$

where $\lambda_j \in (E_{2j-1},E_{2j})$, $1\le j\le g$. In particular, $\left|\mathrm{e}^{\mathrm{i}k(z)}\right| < 1$ for $z\in\mathbb{C}\backslash\sigma(H_q)$ and $|\mathrm{e}^{\mathrm{i}k(z)}|=1$ for $z\in\sigma(H_q)$.

3. Asymptotics of Jost solutions

After we have these preparations out of our way, we come to the study of short-range perturbations H of H_q associated with a potential V satisfying $V(x)\to V_q(x)$ as $|x|\to\infty$. More precisely, we will make the following assumption throughout this paper:

Let

$$H = -\frac{d^2}{dx^2} + V(x) \tag{3.1}$$

be a perturbation of H_q such that

$$\int_{-\infty}^{+\infty} \big|V(x)-V_q(x)\big| dx < \infty. \tag{3.2}$$

We first establish existence of Jost solutions, that is, solutions of the perturbed operator which asymptotically look like the Baker–Akhiezer solutions.

Theorem 3.1. *Assume* (3.2). *For every $z\in\mathbb{C}\backslash\{E_j\}_{j=0}^{2g}$ there exist (weak) solutions $\psi_\pm(z,.)$ of $H\psi=z\psi$ satisfying*

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mp\mathrm{i}xk(z)}\big(\psi_\pm(z,x)-\psi_{q,\pm}(z,x)\big)=0, \tag{3.3}$$

where $\psi_{q,\pm}(z,.)$ are the Baker–Akhiezer functions. Moreover, $\psi_\pm(z,.)$ are continuous (resp. holomorphic) with respect to z whenever $\psi_{q,\pm}(z,.)$ are and

$$\big|\mathrm{e}^{\mp\mathrm{i}xk(z)}\big(\psi_\pm(z,x)-\psi_{q,\pm}(z,x)\big)\big| \le C(z), \tag{3.4}$$

where $C(z)$ denotes some constant depending only on z.

Proof. Since $H\psi=z\psi$ is equivalent to $(H_q-z)\psi = -\widehat{V}\psi$, where $\widehat{V}=V-V_q$, we can use the variation of constants formula to obtain the usual Volterra integral equations for the Jost functions,

$$\begin{aligned}\psi_\pm(z,x) = \psi_{q,\pm}(z,x) - &\frac{1}{W(\psi_{q,+},\psi_{q,-})}\int_x^{\pm\infty}\big(\psi_{q,-}(z,x)\psi_{q,+}(z,y)-\\ &-\psi_{q,-}(z,y)\psi_{q,+}(z,x)\big)\widehat{V}(y)\psi_\pm(z,y)dy.\end{aligned} \tag{3.5}$$

Moreover, introducing $\tilde{\psi}_\pm(z,x)=\mathrm{e}^{\pm\mathrm{i}xk(z)}\psi_\pm(z,x)$ the resulting integral equation can be solved using the method of successive iterations in the usual way. This proves the claims. $\square$

Theorem 3.2. *Assume* (3.2). *The Jost functions have the following asymptotic behavior*

$$\psi_\pm(z,x) = \psi_{q,\pm}(z,x)\Big(1 \mp \frac{1}{2\mathrm{i}\sqrt{z}} \int_x^{\pm\infty} \big(V(y) - V_q(y)\big)dy + o(z^{-1/2})\Big), \tag{3.6}$$

as $z \to \infty$, *with the error being uniformly in* x.

Proof. Invoking (3.5) we have

$$\begin{aligned}\frac{\psi_\pm(z,x)}{\psi_{q,\pm}(z,x)} &= 1 - \frac{1}{W(\psi_{q,+},\psi_{q,-})} \int_x^{\pm\infty} \Big(\psi_{q,-}(z,x)\psi_{q,+}(z,y)\frac{\psi_{q,\pm}(z,y)}{\psi_{q,\pm}(z,x)} \\ &\quad - \int_x^{\pm\infty} \psi_{q,-}(z,y)\psi_{q,+}(z,x)\frac{\psi_{q,\pm}(z,y)}{\psi_{q,\pm}(z,x)}\Big)\widehat{V}(y)\frac{\psi_\pm(z,y)}{\psi_{q,\pm}(z,y)}dy \\ &= 1 \mp \int_x^{\pm\infty} \Big(G_q(z,x,x)\frac{\psi_{q,\pm}(z,y)^2}{\psi_{q,\pm}(z,x)^2} - G_q(z,y,y)\Big)\widehat{V}(y)\frac{\psi_\pm(z,y)}{\psi_{q,\pm}(z,y)}dy, \end{aligned} \tag{3.7}$$

where

$$G_q(z,x,y) = \frac{1}{W(\psi_{q,+},\psi_{q,-})} \begin{cases} \psi_{q,+}(z,x)\psi_{q,-}(z,y), & x \geq y, \\ \psi_{q,+}(z,y)\psi_{q,-}(z,x), & x \leq y, \end{cases} \tag{3.8}$$

is the Green function of H_q. We have

$$G_q(z,x,x) = \frac{\psi_{q,+}(z,x)\psi_{q,-}(z,x)}{W(\psi_{q,+},\psi_{q,-})} = \frac{\mathrm{i}\prod_{j=1}^g \big(z - \mu_j(x)\big)}{2R_{2g+1}^{1/2}(z)}. \tag{3.9}$$

Hence for z near ∞ one infers

$$G_q(z,x,x) = \frac{\mathrm{i}}{2\sqrt{z}}\Big(1 + \frac{1}{2}V_q(x)\frac{1}{z} + O\big(\frac{1}{z^2}\big)\Big), \tag{3.10}$$

where we made use of the fact that the quasi-periodic potential V_q can be written as

$$V_q(x) = E_0 + \sum_{j=1}^g \big(E_{2j-1} + E_{2j} - 2\mu_j(x)\big). \tag{3.11}$$

Next we insert (3.10) into (3.7) such that iteration implies

$$\frac{\psi_\pm(z,x)}{\psi_{q,\pm}(z,x)} = 1 \mp \frac{\mathrm{i}}{2\sqrt{z}}\Big(\int_x^{\pm\infty} \frac{\psi_{q,\pm}(z,y)^2}{\psi_{q,\pm}(z,x)^2}\widehat{V}(y)dy - \int_x^{\pm\infty} \widehat{V}(y)dy\Big) + O\big(\frac{1}{z}\big).$$

Next we will show that the first integral vanishes as $\sqrt{z} \to \infty$. We begin with the case $\mathrm{Im}(\sqrt{z}) \to \infty$. For that purpose note that

$$k(z) = \sqrt{z} + c + O(z^{-1/2}), \quad \text{as } z \to \infty,$$

for some constant $c \in \mathbb{C}$. Thus we compute

$$\begin{aligned} \Big|\int_x^{\pm\infty} \frac{\psi_{q,\pm}(z,y)^2}{\psi_{q,\pm}(z,x)^2}\widehat{V}(y)dy\Big| &\leq C\int_x^{\pm\infty} \exp\big(\mp 2\mathrm{Im}(\sqrt{z})(y-x)\big)\big|\widehat{V}(y)\big|dy \\ \leq C\int_x^{x+\varepsilon} \big|\widehat{V}(y)\big|dy &+ C\cdot\exp\big(\mp 2\mathrm{Im}(\sqrt{z})\varepsilon\big)\int_{x+\varepsilon}^{\pm\infty} \big|\widehat{V}(y)\big|dy, \end{aligned}$$

such that the first integral can be made arbitrary small if $\varepsilon > 0$ is small and the second integral vanishes as $\mathrm{Im}(\sqrt{z}) \to \infty$.

Otherwise, if $\mathrm{Re}(\sqrt{z}) \to \infty$, we use (2.6) to rewrite the integral as

$$\int_x^{\pm\infty} \Big(\frac{\theta_{q,\pm}(z,y)^2}{\theta_{q,\pm}(z,x)^2} \widehat{V}(y) \exp\big(\mp 2\mathrm{Im}(\sqrt{z})(y-x)\big)\Big) \exp\big(\pm 2\mathrm{i}\mathrm{Re}(\sqrt{z})(y-x)\big) dy$$

Since

$$\Big| \frac{\theta_{q,\pm}(z,y)^2}{\theta_{q,\pm}(z,x)^2} \widehat{V}(y) \exp\big(\mp 2\mathrm{Im}(\sqrt{z})(y-x)\big)\Big| \le |\widehat{V}(y)|$$

the integral vanishes as $\mathrm{Re}(\sqrt{z}) \to \infty$ by a slight variation of the Riemann–Lebesgue lemma.

Hence we finally have

$$\frac{\psi_\pm(z,x)}{\psi_{q,\pm}(z,x)} = 1 \pm \frac{\mathrm{i}}{2\sqrt{z}} \int_x^{\pm\infty} \widehat{V}(y) dy + o\big(\frac{1}{\sqrt{z}}\big) \tag{3.12}$$

as $z \to \infty$. □

For later use we note the following immediate consequence

Corollary 3.3. *Under the assumptions of the previous theorem we have*

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mp\mathrm{i}xk(z)} \Big(\psi'_\pm(z,x) \mp \mathrm{i}xk'(z)\psi_\pm(z,x) - \psi'_{q,\pm}(z,x) \pm \mathrm{i}xk'(z)\psi_{q,\pm}(z,x)\Big) = 0,$$

where the prime denotes differentiation with respect to z.

Proof. Just differentiate (3.3) with respect to z, which is permissible by uniform convergence on compact subsets of $\mathbb{C}\backslash\{E_j\}_{j=0}^{2g}$. □

We remark that if we require our perturbation to satisfy the usual short range assumption as in [1], [7, 8, 9] (i.e., the first moment is integrable, see (5.1)), then we even have $\mathrm{e}^{\mp\mathrm{i}xk(z)}(\psi'_\pm(z,x) - \psi'_{q,\pm}(z,x)) \to 0$.

From Theorem 3.2 we obtain a complete characterization of the spectrum of H.

Theorem 3.4. *Assume* (3.2). *Then $(H-z)^{-1} - (H_q - z)^{-1}$ is trace class. In particular, we have $\sigma_{ess}(H) = \sigma(H_q)$ and the point spectrum of H is confined to $\overline{\mathbb{R}\backslash\sigma(H_q)}$. Furthermore, the essential spectrum of H is purely absolutely continuous except for possible eigenvalues at the band edges.*

Proof. That $(H-z)^{-1} - (H_q-z)^{-1}$ is trace class follows as in [29, Lem. 9.34] (cf. also [19, Sect. 4]). The fact that the essential spectrum is purely absolutely continuous follows from subordinacy theory ([29, Sect. 9.5]) since the asymptotics of the Jost solutions imply that no solution is subordinate inside the essential spectrum. □

Note that (3.2) does neither exclude eigenvalues at the boundary of the essential spectrum nor an infinite number of eigenvalues inside essential spectral gaps (see [24] or [1] for conditions excluding these cases).

Our next result concerns the asymptotics of the Jost solutions at the *other side*.

Lemma 3.5. *Assume* (3.2). *Then the Jost solutions $\psi_\pm(z,.)$, $z \in \mathbb{C}\backslash\sigma(H)$, satisfy*

$$\lim_{x\to\mp\infty} \big|\mathrm{e}^{\mp\mathrm{i}xk(z)}\big(\psi_\pm(z,x) - \alpha(z)\psi_{q,\pm}(z,x)\big)\big| = 0, \tag{3.13}$$

where

$$\alpha(z) = \frac{W(\psi_-(z),\psi_+(z))}{W(\psi_{q,-}(z),\psi_{q,+}(z))} = \frac{\prod_{j=1}^g (z-\mu_j)}{2\mathrm{i}R_{2g+1}^{1/2}(z)} W(\psi_-(z),\psi_+(z)). \tag{3.14}$$

Proof. Since H and H_q have the same form domain, the second resolvent equation ([29, Lem. 6.30]) for form perturbations implies

$$G(z,x,x) - G_q(z,x,x) = \int_{-\infty}^{\infty} G(z,x,y)\widehat{V}(y)G_q(z,y,x)dy,$$

where $\widehat{V} = V - V_q$. By (3.8) and

$$G(z,x,y) = \frac{1}{W(\psi_+,\psi_-)}\begin{cases} \psi_+(z,x)\psi_-(z,y), & x \geq y, \\ \psi_+(z,y)\psi_-(z,x), & x \leq y, \end{cases}$$

we obtain

$$\begin{aligned} G(z,x,x) - G_q(z,x,x) =& \frac{\psi_{q,+}(z,x)\psi_+(z,x)}{W(z)W_q(z)}\int_{-\infty}^{x}\widehat{V}(y)\psi_{q,-}(z,y)\psi_-(z,y)dy \\ &+ \frac{\psi_{q,-}(z,x)\psi_-(z,x)}{W(z)W_q(z)}\int_{x}^{\infty}\widehat{V}(y)\psi_{q,+}(z,y)\psi_+(z,y)dy, \end{aligned} \tag{3.15}$$

where $W(z) = W(\psi_+,\psi_-)$ and $W_q(z) = W(\psi_{q,+},\psi_{q,-})$. Next, by (3.4), note that

$$\big|\psi_{q,\pm}(z,x)\big| \leq c_1 \mathrm{e}^{\mp\varepsilon x}, \quad \big|\psi_\pm(z,x)\big| \leq c_2\mathrm{e}^{\mp\varepsilon x}, \tag{3.16}$$

as $x \to +\infty$, where c_1, c_2 denote some constants and $\varepsilon > 0$ does only depend on z. Now one can show that the first term in (3.15) tends to 0 when $x \to +\infty$ using the same kind of argument as in the proof of Theorem 3.2. Similarly one then checks that the second term in (3.15) tends to 0 when $x \to -\infty$. Thus

$$\lim_{x\to\pm\infty} G(z,x,x) - G_q(z,x,x) = 0$$

and using

$$G_q(z,x,x) = \frac{\psi_{q,-}(z,x)\psi_{q,+}(z,x)}{W(\psi_{q,-}(z),\psi_{q,+}(z))}, \qquad G(z,x,x) = \frac{\psi_-(z,x)\psi_+(z,x)}{W(\psi_-(z),\psi_+(z))}$$

implies

$$\lim_{x\to\pm\infty}\big(\psi_-(z,x)\psi_+(z,x) - \alpha(z)\psi_{q,-}(z,x)\psi_{q,+}(z,x)\big) = 0,$$

respectively,

$$\lim_{x\to-\infty}\psi_{q,-}(z,x)\big(\psi_+(z,x) - \alpha(z)\psi_{q,+}(z,x)\big) = 0,$$

which is the claimed result. □

To see the connection with scattering theory (see, e.g., [1]), we introduce the scattering relations

$$T_\mp(\lambda)\psi_\pm(\lambda,x) = \overline{\psi_\mp(\lambda,x)} + R_\mp(\lambda)\psi_\mp(\lambda,x), \quad \lambda\in\sigma(H_q), \tag{3.17}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda) = \frac{W(\overline{\psi_\pm(\lambda)},\psi_\pm(\lambda))}{W(\psi_\mp(\lambda),\psi_\pm(\lambda))}, \qquad R_\pm(\lambda) := -\frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\mp(\lambda),\psi_\pm(\lambda))}, \quad \lambda\in\sigma(H_q). \tag{3.18}$$

In particular, $\alpha(z)$ is just the inverse of the transmission coefficient $T(z)$. It is holomorphic in $\mathbb{C}\backslash\sigma(H_q)$ with simple zeros at the discrete eigenvalues of H.

Corollary 3.6. *Assume* (3.2). *Then we have*

$$T(z) = \exp\Big(-\int_{-\infty}^{+\infty} \big(m_\pm(z,x) - m_{q,\pm}(z,x)\big)dx\Big), \tag{3.19}$$

where

$$m_\pm(z,x) = \pm\frac{\psi'_\pm(z,x)}{\psi_\pm(z,x)}, \quad m_{q,\pm}(z,x) = \pm\frac{\psi'_{q,\pm}(z,x)}{\psi_{q,\pm}(z,x)} \tag{3.20}$$

are the Weyl–Titchmarsh functions. Here the prime denotes differentiation with respect to x.

Proof. From the definition (3.20) we get the following representations of the Jost and Baker–Akhiezer functions

$$\begin{aligned}\psi_\pm(z,x) =&\psi_\pm(z,x_0)\exp\big(\pm\int_{x_0}^{x} m_\pm(z,y)dy\big),\\ \psi_{q,\pm}(z,x) =&\psi_{q,\pm}(z,x_0)\exp\big(\pm\int_{x_0}^{x} m_{q,\pm}(z,y)dy\big),\end{aligned}$$

and thus

$$\begin{aligned}\frac{\psi_\pm(z,x)}{\psi_{q,\pm}(z,x)} =&\frac{\psi_\pm(z,x_0)}{\psi_{q,\pm}(z,x_0)}\exp\big(\pm\int_{x_0}^{x}(m_\pm(z,y) - m_{q,\pm}(z,y))dy\big)\\ =&\exp\big(\pm\int_{\pm\infty}^{x}(m_\pm(z,y) - m_{q,\pm}(z,y))dy\big).\end{aligned}$$

Making use of that and (3.13) we get

$$\alpha(z) = \lim_{x\to\mp\infty}\frac{\psi_\pm(z,x)}{\psi_{q,\pm}(z,x)} = \exp\big(\pm\int_{\pm\infty}^{\mp\infty}(m_\pm(z,y) - m_{q,\pm}(z,y))dy\big),$$

which finishes the proof. □

Corollary 3.7. *Assume* (3.2). *Then $T(z)$ has the following asymptotic behavior*

$$T(z) = 1 + \frac{1}{2\mathrm{i}\sqrt{z}}\int_{-\infty}^{\infty}\big(V(y) - V_q(y)\big)dy + o(z^{-1/2}), \tag{3.21}$$

as $z \to \infty$.

Proof. Use (3.13) and (3.6). □

4. Connections with Krein's spectral shift theory and trace formulas

To establish the connection with Krein's spectral shift theory we next show:

Lemma 4.1. *We have*

$$\frac{d}{dz}\alpha(z) = -\alpha(z)\int_{-\infty}^{+\infty}\big(G(z,x,x) - G_q(z,x,x)\big)dx, \qquad z \in \mathbb{C}\backslash\sigma(H), \tag{4.1}$$

where $G(z,x,y)$ and $G_q(z,x,y)$ are the Green's functions of H and H_q, respectively.

Proof. The Lagrange identity ([29], eq. (9.4)) implies

$$W_x(\psi_+(z), \psi'_-(z)) - W_y(\psi_+(z), \psi'_-(z)) = \int_y^x \psi_+(z,r)\psi_-(z,r)dr, \tag{4.2}$$

hence the derivative of the Wronskian can be written as

$$\begin{aligned}\frac{d}{dz}W(\psi_-(z), \psi_+(z)) &= W_x(\psi'_-(z), \psi_+(z)) + W_x(\psi_-(z), \psi'_+(z))\\ &= W_y(\psi'_-(z), \psi_+(z)) + W_x(\psi_-(z), \psi'_+(z)) - \int_y^x \psi_+(z,r)\psi_-(z,r)dr.\end{aligned}$$

Using Corollary 3.3 and Lemma 3.5 we have

$$\begin{aligned}W_y(\psi'_-(z), \psi_+(z)) &= W_y(\psi'_- + \mathrm{i}k'y\psi_-, \psi_+) -\\ &\quad \mathrm{i}k'\big(y\, W(\psi_-, \psi_+) - \psi_-(z,y)\psi_+(z,y)\big)\\ &\to \alpha W_y(\psi'_{q,-} + \mathrm{i}k'y\psi_{q,-}, \psi_{q,+}) -\\ &\quad \alpha\mathrm{i}k'\big(y\, W(\psi_{q,-}, \psi_{q,+}) - \psi_{q,-}(z,y)\psi_{q,+}(z,y)\big)\\ &= \alpha(z)W_y(\psi'_{q,-}(z), \psi_{q,+}(z))\end{aligned}$$

as $y \to -\infty$. Similarly we obtain

$$W_x(\psi_-(z), \psi'_+(z)) \to \alpha(z)W_x(\psi_{q,-}(z), \psi'_{q,+}(z))$$

as $x \to +\infty$ and again using (4.2) we have

$$W_y(\psi'_{q,-}(z), \psi_{q,+}(z)) = W_x(\psi'_{q,-}(z), \psi_{q,+}(z)) + \int_y^x \psi_{q,+}(z,r)\psi_{q,-}(z,r)dr.$$

Collecting terms we arrive at

$$\begin{aligned}W'(\psi_-(z), \psi_+(z)) = &-\int_{-\infty}^{+\infty} \Big(\psi_+(z,r)\psi_-(z,r) - \alpha(z)\psi_{q,+}(z,r)\psi_{q,-}(z,r)\Big)dr\\ &+ \alpha(z)W'(\psi_{q,-}(z)\psi_{q,+}(z)).\end{aligned}$$

Abbreviating $W_q = W(\psi_{q,-}, \psi_{q,+})$ we now compute

$$\begin{aligned}\frac{d}{dz}\alpha(z) &= \frac{d}{dz}\Big(\frac{W}{W_q}\Big) = \Big(\frac{1}{W_q}\Big)'W + \frac{1}{W_q}W'\\ &= -\frac{W'_q}{W_q^2}W + \frac{1}{W_q}\Big(-\int_{-\infty}^{+\infty}(\psi_+\psi_- - \alpha\psi_{q,+}\psi_{q,-})dr + \alpha W'_q\Big)\\ &= -\frac{1}{W_q}\int_{-\infty}^{+\infty}\Big(\psi_+(z,r)\psi_-(z,r) - \alpha(z)\psi_{q,+}(z,r)\psi_{q,-}(z,r)\Big)dr,\end{aligned}$$

which finishes the proof. □

Since $(H-z)^{-1}-(H_q-z)^{-1}$ is trace class with continuous integral kernel $G(z,x,x)-G_q(z,x,x)$, we have ([2])

$$\mathrm{tr}\big((H-z)^{-1} - (H_q-z)^{-1}\big) = \int_{-\infty}^{+\infty}\big(G(z,x,x) - G_q(z,x,x)\big)dx, \qquad z \in \mathbb{C}\backslash\sigma(H), \tag{4.3}$$

and the last result can be rephrased as

$$\frac{d}{dz}T(z) = T(z)\mathrm{tr}\big((H-z)^{-1} - (H_q-z)^{-1}\big), \qquad z \in \mathbb{C}\backslash\sigma(H), \tag{4.4}$$

As an immediate consequence we can establish the connection with Krein's spectral shift function ([17]). We refer to [30] for Krein's spectral shift theory in the case when only the resolvent difference is trace class; which is the case needed here.

Theorem 4.2. *The function $\alpha(z)$ has the representation*

$$T(z) = \exp\Big(\int_{\mathbb{R}} \frac{\xi(\lambda) d\lambda}{\lambda - z}\Big), \tag{4.5}$$

where

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \arg T(\lambda + \mathrm{i}\epsilon) \tag{4.6}$$

is the spectral shift function of the pair H, H_q.

Proof. The function $\operatorname{Im}\log(T(z))$ is a bounded harmonic function in the upper half plane and hence has a Poisson representation (cf. [16])

$$\operatorname{Im}\log(T(z)) = \int_{\mathbb{R}} \frac{y}{(x-\lambda)^2 + y^2} \xi(\lambda) d\lambda. \quad z = x + \mathrm{i}y,$$

Moreover, by $\xi(\lambda) = 0$ for λ below the spectrum of H and $\xi(\lambda) = O(\lambda^{-1/2})$ as $\lambda \to +\infty$ (by Corollary 3.7) we obtain equality in (4.5) up to a real constant. The missing constants follows since both sides tend to 1 as $z \to \infty$. Moreover, combining (4.5) with (4.4) we see

$$\operatorname{tr}\big((H-z)^{-1} - (H_q - z)^{-1}\big) = \int_{\mathbb{R}} \frac{\xi(\lambda) d\lambda}{(\lambda - z)^2},$$

which shows that $\xi(\lambda)$ is the spectral shift function. □

Remark 4.3. *We can interpret $T(z)$ as perturbation determinant of the pair H and H_q:*

$$T(z) = \det\big(\mathbb{1} + (V - V_q)(H_q - z)^{-1}\big). \tag{4.7}$$

However, since $(V - V_q)(H_q - z)^{-1}$ is not trace class, the right-hand side has to be interpreted in a generalized sense. In fact, using [25, Prop 4.7] *it follows that $(V - V_q)(H_q - z)^{-1}$ will not be trace class even if $V_q = 0$ and V is smooth with compact support.*

5. The transmission coefficient

Throughout this section we make the somewhat stronger assumption that

$$\int_{-\infty}^{+\infty} (1 + |x|)\big|V(x) - V_q(x)\big| dx < \infty \tag{5.1}$$

in order to ensure that there is only a finite number of eigenvalues in each gap [24]. Our aim is to reconstruct the transmission coefficient $T(z)$ from its boundary values and its poles. To this end, recall that $T(z)$ is meromorphic in $\mathbb{C}\backslash\sigma(H_q)$ with simple poles at the eigenvalues ρ_j of H. Moreover, for $z \in \sigma(H_q)$ the boundary values from the upper, respectively, lower, half plane exist and satisfy $|T(z)|^2 = 1 - |R_\pm(z)|^2$, where $R_\pm(z)$ are the reflection coefficients defined in the previous section.

In the case where $V_q = 0$, this can be done via the classical Poisson–Jensen formula. In the more general setting here, the reconstruction needs to be done on the the underlying Riemann surface. We essentially follow [28], where the analog problem for Jacobi operators was solved.

First we introduce the Blaschke factor

$$B(p,\rho) = \exp\Big(g(p,\rho)\Big) = \exp\Big(\int_{E_0}^{p} \omega_{\rho\,\rho^*}\Big) = \exp\Big(\int_{E(\rho)}^{\rho} \omega_{p\,p^*}\Big), \quad \pi(\rho) \in \mathbb{R}, \tag{5.2}$$

where $E(\rho)$ is E_0 if $\rho < E_0$ and either E_{2j-1} or E_{2j} if $\rho \in (E_{2j-1}, E_{2j})$, $1 \le j \le g$. It is a multivalued function with a simple zero at ρ and simple pole at ρ^* satisfying $|B(p,\rho)| = 1$, $p \in \partial\Pi_+$. It is real-valued for $\pi(p) \in (-\infty, E_0)$ and satisfies

$$B(E_0,\rho) = 1 \quad \text{and} \quad B(p^*,\rho) = B(p,\rho^*) = B(p,\rho)^{-1}. \tag{5.3}$$

Then we have

Theorem 5.1. *The transmission coefficient is given by*

$$T(z,x) = \Big(\prod_{j=1}^{g} B(p,\rho_j)^{-1}\Big) \exp\Big(\frac{1}{2\pi\mathrm{i}} \int_{\partial\Pi_+} \log(1 - |R_\pm|^2)\omega_{p\,p_\infty}\Big), \quad p = (z,+), \tag{5.4}$$

where we set $R_\pm(p) = R_\pm(z)$ for $p = (z,+)$ and $R_\pm(p) = \overline{R_\pm(z)}$ for $p = (z,-)$.

Proof. Just literally follow the argument in [28, Sect. 3]. □

Remark 5.2. *A few remarks are in order:*

(i) *Using symmetry, $|R_\pm(p^*)| = |R_\pm(p)|$ for $p \in \partial\Pi_+$, of the integrand we can rewrite (5.4) as*

$$T(p,x) = \Big(\prod_{j=1}^{g} \exp\Big(-\int_{E(\rho_j)}^{\rho_j} \omega_{p\,p^*}\Big)\Big) \exp\Big(\frac{1}{2\pi\mathrm{i}} \int_{\Sigma} \log(1 - |R_\pm|^2)\omega_{p\,p^*}\Big), \tag{5.5}$$

where the integral over Σ is taken on the upper sheet.

(ii) *There exist explicit formulas for Abelian differentials of the third kind:*

$$\omega_{pq} = \left(\frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p)}{2(\pi - \pi(p))} - \frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(q)}{2(\pi - \pi(q))} + P_{pq}(\pi)\right) \frac{d\pi}{R_{2g+1}^{1/2}},$$

$$\omega_{pp_\infty} = \left(-\frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p)}{2(\pi - \pi(p))} + P_{pp_\infty}(\pi)\right) \frac{d\pi}{R_{2g+1}^{1/2}},$$

where $P_{pq}(z)$, $P_{pp_\infty}(z)$ are polynomials of degree $g-1$ which have to be determined from the normalization condition to have vanishing a-periods. In particular,

$$\omega_{pp^*} = \left(\frac{R_{2g+1}^{1/2}(p)}{\pi - \pi(p)} + P_{pp^*}(\pi)\right) \frac{d\pi}{R_{2g+1}^{1/2}}.$$

(iii) *The function*

$$T(p) = \begin{cases} T(z), & p = (z,+), \\ T(z)^{-1}, & p = (z,-), \end{cases}$$

solves the following scalar meromorphic Riemann–Hilbert factorization problem:

$$\begin{aligned} & T_+(p,x) = T_-(p,x)(1 - |R(p)|^2), \quad p \in \partial\Pi_+, \\ & (T(p,x)) = \mathcal{D}_\rho^* - \mathcal{D}_\rho \\ & T(p_\infty,x) = 1. \end{aligned} \tag{5.6}$$

Here the subscripts in $T_\pm(p)$ denote the limits from $\Pi_\pm$, respectively. Compare [18].

As was pointed out in [28], this implies the following algebraic constraint on the scattering data.

Theorem 5.3. *The transmission coefficient T defined via* (5.4) *is single-valued if and only if the eigenvalues ρ_j and the reflection coefficients $R_\pm$ satisfy*

$$\sum_j \int_{\rho_j^*}^{\rho_j} \zeta_\ell - \frac{1}{2\pi\mathrm{i}} \int_{\partial\Pi_+} \log(1-|R_\pm|^2)\zeta_\ell \in \mathbb{Z}. \tag{5.7}$$

6. Conserved quantities of the KdV hierarchy

Finally we turn to solutions of the KdV hierarchy (see [11]). Let $V_q(x,t)$ be a finite-gap solution of some equation in the KdV hierarchy, $\mathrm{KdV}_r(V_q(x,t)) = 0$, and let $V(x,t)$ be another solution, $\mathrm{KdV}_r(V(x,t)) = 0$, such that $V(.,t) - V_q(.,t)$ is Schwartz class for all $t \in \mathbb{R}$. Existence of such solutions has been established only recently in [6].

Since the transmission coefficient $T(z,t) = T(z,0) \equiv T(z)$ is conserved (see [6] – formally this follows from unitary invariance of the determinant), conserved quantities of the KdV hierarchy can be obtained by computing the asymptotic expansion at ∞.

To this end, we begin by recalling the following well-known asymptotics for the Weyl m-functions in case of smooth potentials:

Lemma 6.1. *Suppose $V(x) \in C^\infty(\mathbb{R})$ is smooth. The Weyl m-functions have the following asymptotic expansion for large z*

$$m_\pm(z,x) \asymp \mathrm{i}\sqrt{z} \pm \sum_{n=1}^{\infty} \frac{\chi_n(x)}{(\pm 2\mathrm{i}\sqrt{z})^n}, \tag{6.1}$$

with coefficients defined recursively via

$$\chi_1(x) = V(x), \quad \chi_{n+1}(x) = -\frac{\partial}{\partial x}\chi_n(x) - \sum_{m=1}^{n-1} \chi_{n-m}(x)\chi_m(x). \tag{6.2}$$

The corresponding expansion coefficients associated with V_q will be denoted by $\chi_{q,m}(x)$. It is also known that the even coefficients are complete differentials [4] and the first few are explicitly given by

$$\begin{aligned}
\chi_1(x) &= V(x),\\
\chi_2(x) &= -V'(x),\\
\chi_3(x) &= V''(x) - V(x)^2,\\
\chi_4(x) &= -V'''(x) + 4V(x)V'(x),\\
\chi_5(x) &= V''''(x) - 6V''(x)V(x) - 5V'(x)^2 + 2V(x)^3.
\end{aligned}$$

Theorem 6.2. *Suppose $V(x) - V_q(x) \in \mathcal{S}(\mathbb{R})$ is Schwartz. Then $\log T(z)$ has an asymptotic expansion around $z=\infty$:*

$$\log T(z) \asymp \mathrm{i}\sqrt{z} \sum_{k=1}^{\infty} \frac{\tau_k}{z^k}.$$

The quantities τ_k are given by

$$\tau_k = \int_{-\infty}^{\infty} \frac{\chi_{2k-1}(x) - \chi_{q,2k-1}(x)}{(-1)^k 2^{2k-1}} dx \tag{6.3}$$

and are conserved quantities for the KdV hierarchy. Explicitly,

$$\begin{aligned}
\tau_1 &= -\frac{1}{2} \int_{-\infty}^{\infty} \big(V(x) - V_q(x)\big) dx,\\
\tau_2 &= -\frac{1}{8} \int_{-\infty}^{\infty} \big(V^2(x) - V_q^2(x)\big) dx,\\
\tau_3 &= -\frac{1}{32} \int_{-\infty}^{\infty} \big(2V^3(x) - 5V_x^2(x) - 6V_{xx}(x)V(x)\\
&\qquad - 2V_q^3(x) + 5V_{q,x}^2(x) + 6V_{q,xx}(x)V_q(x)\big) dx,
\end{aligned}$$

etc.

Proof. Represent the Jost solutions in the form

$$\psi_\pm(z,x) = \psi_{q,\pm}(z,x) \exp\left(\mp \int_x^{\pm\infty} \big(m_\pm(z,y) - m_{q,\pm}(z,y)\big) dy\right). \tag{6.4}$$

Then iterating the Volterra integral equations (3.7) one sees that $\psi_\pm(z,x)$ have an asymptotic expansion uniformly with respect to x and given by

$$\log \frac{\psi_\pm(z,x)}{\psi_{q,\pm}(z,x)} \asymp -\sum_{n=1}^{\infty} \frac{1}{(\pm 2\mathrm{i}\sqrt{z})^n} \int_x^{\pm\infty} \big(\chi_n(y) - \chi_{q,n}(y)\big) dy. \tag{6.5}$$

Then, letting $x \to \mp\infty$ using (3.13) yields (6.3). In particular, equality of the plus and minus cases shows that all even expansion coefficients must vanish (which alternatively also follows from the fact that the even expansion coefficients are complete differentials). □

Theorem 6.3. *Consider the expansion coefficients τ_k of $\log T(z)$ defined in* (6.3). *Then the following trace formulas are valid:*

$$\tau_k = 2\mathrm{i} \sum_{j=1}^{g} \int_{E(\rho_j)}^{\rho_j} \omega_{p_\infty, 2k-2} - \frac{1}{\pi} \int_\Sigma \log|T|^2 \omega_{p_\infty,2k-2}, \tag{6.6}$$

where $\omega_{p_\infty,k}$ is the Abelian differential of the second kind with a pole of order $k+2$ at p_∞.

Proof. From $\frac{d^k}{dz^k}\omega_{p\,E_0} = k!\omega_{p_\infty,k-1}$ we get that

$$\begin{aligned}
\omega_{p\,E_0} &= \omega_{p_\infty\,E_0} + \sum_{k=1}^{\infty} \zeta^k \omega_{p_\infty,k-1}, \qquad \zeta = z^{-1/2},\\
\omega_{p^*\,E_0} &= \omega_{p_\infty\,E_0} + \sum_{k=1}^{\infty} \zeta^k \omega_{p_\infty,k-1}, \qquad \zeta = -z^{-1/2}.
\end{aligned}$$

Using this it follows

$$\omega_{p\,p^*} = \omega_{p\,E_0} - \omega_{p^*\,E_0} = 2\sum_{k=1}^{\infty} \omega_{p_\infty,2k-2}\zeta^{2k-1}, \quad \zeta = z^{-1/2}.$$

Hence we have

$$
\begin{aligned}
&-\sum_{j=1}^{g}\int_{E(\rho_j)}^{\rho_j}\omega_{p\,p^*}+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\log|T|^2\omega_{p\,p^*}=\\
&-\sum_{j=1}^{g}\int_{E(\rho_j)}^{\rho_j}2\sum_{k=1}^{\infty}\zeta^{2k-1}\omega_{p_\infty,2k-2}+\frac{1}{\pi\mathrm{i}}\int_{\Sigma}\log|T|^2\sum_{k=1}^{\infty}\zeta^{2k-1}\omega_{p_\infty,2k-2}.
\end{aligned}
$$

Thus, since $|T|^2 = 1 - |R_\pm|^2$ and $R_\pm(\lambda)$ decays faster than any polynomial as $\lambda \to \infty$ [6], one obtains

$$
\log T(z) \asymp -\sum_{k=1}^{\infty}\big(2\sum_{j=1}^{\infty}\int_{E(\rho_j)}^{\rho_j}\omega_{p_\infty,2k-2}-\frac{1}{\pi\mathrm{i}}\int_{\Sigma}\log|T|^2\omega_{p_\infty,2k-2}\big)\zeta^{2k-1},
$$

where $\zeta = z^{-1/2}$ denotes the local coordinate at $z = \infty$. □

Remark 6.4. *The differentials $\omega_{p_\infty,2k-2}$, $k = 1, 2, \dots$, are explicitly given by*

$$
\omega_{p_\infty,2k-2} = \left(\frac{\pi^{g+k-1}}{R_{2g+1}^{1/2}} + P_k(\pi)\right) d\pi. \tag{6.7}
$$

Here $P_k(\pi)$ is a polynomial of degree $g+k-2$ which has to be chosen such that all a-periods vanish.

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