UNIQUE CONTINUATION FOR DISCRETE NONLINEAR WAVE EQUATIONS

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Abstract. We establish unique continuation for various discrete nonlinear wave equations. For example, we show that if two solutions of the Toda lattice coincide for one lattice point in some arbitrarily small time interval, then they coincide everywhere. Moreover, we establish analogous results for the Toda, Kac–van Moerbeke, and Ablowitz–Ladik hierarchies. Although all these equations are integrable, the proof does not use integrability and can be adapted to other equations as well.

1. Introduction

Unique continuation results for wave equations have a long tradition and seem to originate in control theory. One of the first result seems to be the one by Zhang [20], where he proves that if a short range solution of the Korteweg–de Vries (KdV) equation vanishes on an open subset in the xt-plane, then it must vanish everywhere. Since then, this result has been extended in various directions and for different equations (see for example [2], the introduction in [10] for the case of the nonlinear Schrödinger equation, [5], [11], [12] for the generalized KdV equation, [14] for the Camassa–Holm equation).

However, all the results so far seem to only deal with wave equations which are continuous in the spatial direction and this clearly raises the question for such unique continuation results for wave equations which are discrete in the spatial variable. In particular, to the best of our knowledge there are no results for example for the Toda equation, one of the most prominent discrete system. While in principle the strategy from Zhang [20] would be applicable to the Toda lattice, it is the purpose of this note to advocate a much simpler direct approach in the discrete case. We will start with the Toda lattice as our prototypical example and then show how the entire Toda hierarchy as well as the Kac–van Moerbeke and Ablowitz–Ladik hierarchies can be treated. It is important to stress that our approach does not use integrability of these equations and hence can be adapted to more general systems. On the other hand, our approach seems to need one dimensionality in the spatial variable and thus does not apply to the discrete Schrödinger equation on $\mathbb{Z}^d$. Due to the connections with localization for discrete Anderson–Bernoulli models, unique continuation for this model is an important open problem; see [3], [4].

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2. The Toda lattice

In this section we want to treat the Toda lattice as the prototypical example. To this end, recall the Toda lattice [19] (in Flaschka's variables [7])

$$\begin{aligned}\dot{a}(n,t) &= a(n,t)\Big(b(n+1,t) - b(n,t)\Big),\\ \dot{b}(n,t) &= 2\Big(a(n,t)^2 - a(n-1,t)^2\Big), \qquad n \in \mathbb{Z},\end{aligned} \tag{2.1}$$

where the dot denotes a derivative with respect to t. It is a well studied physical model and one of the prototypical discrete integrable wave equations. We refer to the monographs [6], [16], [19] or the review articles [13], [17] for further information.

Theorem 2.1. *Assume that $a_0(n,t) \neq 0, b_0(n,t)$ and $a(n,t), b(n,t)$ are complex-valued solutions of (2.1) such that there is one $n_0 \in \mathbb{Z}$ and two times $t_0 < t_1$ such that*

$$a_0(n_0,t)^2 = a(n_0,t)^2, \quad b_0(n_0,t) = b(n_0,t), \tag{2.2}$$

for $t \in (t_0, t_1)$. Then

$$a_0(n,t)^2 = a(n,t)^2, \quad b_0(n,t) = b(n,t) \tag{2.3}$$

for all $(n,t) \in \mathbb{Z} \times \mathbb{R}$.

Proof. It suffices to prove that (2.2) for n_0 implies (2.2) for $n_0 - 1$ and $n_0 + 1$. We start with $N_0 - 1$ and first observe that (2.1) implies

$$\begin{aligned}0 = \dot{b}(n_0,t) - \dot{b}_0(n_0,t) &= 2\big(a(n_0,t)^2 - a_0(n_0,t)^2 - a(n_0-1,t)^2 + a_0(n_0-1,t)^2\big)\\ &= -2\big(a(n_0-1,t)^2 - a_0(n_0-1,t)^2\big)\end{aligned}$$

and thus $a(n_0-1,t)^2 = a_0(n_0-1,t)^2$. Using this we compute

$$\begin{aligned}0 &= \frac{\dot{a}(n_0-1,t)}{a(n_0-1,t)} - \frac{\dot{a}_0(n_0-1,t)}{a_0(n_0-1,t)}\\ &= b(n_0,t) - b_0(n_0,t) - b(n_0-1,t) + b_0(n_0-1,t)\\ &= -b(n_0-1,t) + b_0(n_0-1,t),\end{aligned}$$

so $b(n_0-1,t) = b_0(n_0-1,t)$. Now for n_0+1, we begin with

$$\begin{aligned}0 &= \frac{\dot{a}(n_0,t)}{a(n_0,t)} - \frac{\dot{a}_0(n_0,t)}{a_0(n_0,t)} = b(n_0+1,t) - b_0(n_0+1,t) - b(n_0,t) + b_0(n_0,t)\\ &= b(n_0+1,t) - b_0(n_0+1,t),\end{aligned}$$

so $b(n_0+1,t) = b_0(n_0+1,t)$. Now, use that

$$\begin{aligned}0 &= \dot{b}(n_0+1,t) - \dot{b}_0(n_0+1,t)\\ &= 2\big(a(n_0+1,t)^2 - a_0(n_0+1,t)^2 - a(n_0,t)^2 + a_0(n_0,t)^2\big)\\ &= 2\big(a(n_0+1,t)^2 - a_0(n_0+1,t)^2\big)\end{aligned}$$

to conclude $a(n_0+1,t)^2 = a_0(n_0+1,t)^2$. This finishes the proof. □

It is worthwhile to note that the assumption $a(n,t)^2 \neq 0$ is crucial. In fact, if $a(n_0,t) = 0$ for one (and hence for all) $t \in \mathbb{R}$, then the Toda lattice decouples into two independent parts to the left and right of n_0, and the above result is clearly wrong.

As a simple consequence, this also porves finite propagation speed for the Toda lattice.

Corollary 2.2. *Let $a(n,t)$, $b(n,t)$ be a solution of the Toda lattice for which $a(n,t_0) - \frac{1}{2}$, $b(n,t_0)$ is compactly supported. Then this does not remain true for $t \in (t_0, t_1)$ unless $a(n,t) = \frac{1}{2}$, $b(n,t) = 0$.*

In fact, in the case of real-valued soultions, one can even show the somewhat stronger result, that $a(n,t_0) - \frac{1}{2}$, $b(n,t_0)$ can be compactly supported for at most one time [18]. However, on the other hand the Toda lattice does preserve certain asymptotic properties of the initial conditions, see again [18].

3. Extension to the Toda and Kac–van Moerbeke hierarchy

In this section we show that our main result extends to the entire Toda hierarchy (which will cover the Kac–van Moerbeke hierarchy as well). To this end, we introduce the Toda hierarchy using the standard Lax formalism following [1] (see also [9], [16]).

Associated with two sequences $a(t)^2 \neq 0, b(t)$ is a Jacobi operator

$$H(t) = a(t)S^+ + a^-(t)S^- + b(t) \tag{3.1}$$

acting on sequences over $\mathbb{Z}$, where $S^\pm f(n) = f^\pm(n) = f(n \pm 1)$ are the usual shift operators. Moreover, choose constants $c_0 = 1$, c_j, $1 \leq j \leq r$, $c_{r+1} = 0$, and set

$$P_{2r+2}(t) = \sum_{j=0}^{r} c_{r-j} \tilde{P}_{2j+2}(t), \qquad \tilde{P}_{2j+2}(t) = [H(t)^{j+1}]_+ - [H(t)^{j+1}]_-, \tag{3.2}$$

where $[A]_\pm$ denote the upper and lower triangular parts of an operator with respect to the standard basis $\delta_m(n) = \delta_{m,n}$ (with $\delta_{m,n}$ the usual Kronecker delta). Then the Toda hierarchy is equivalent to the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{3.3}$$

where $[A,B] = AB - BA$ is the usual commutator. Abbreviating

$$g_j(n,t) = \sum_{\ell=0}^{j} c_{j-\ell} \tilde{g}_\ell(n,t), \quad \tilde{g}_\ell(n,t) = \langle \delta_n, H(t)^\ell \delta_n \rangle,$$

$$h_j(n,t) = 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \tilde{h}_\ell(n,t) + c_{j+1}, \quad \tilde{h}_\ell(n,t) = \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle. \tag{3.4}$$

one explicitly obtains

$$\mathrm{TL}_r(a(t), b(t)) = \begin{pmatrix} \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) \\ \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) \end{pmatrix} = 0, \qquad r \in \mathbb{N}_0, \tag{3.5}$$

for the r'th equation $\mathrm{TL}_r(a,b) = 0$ in the Toda hierarchy (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Our main point in this section is the following generalization of Theorem 2.1 to the entire Toda hierarchy:

Theorem 3.1. *Assume that $a_0(n,t)^2 \neq 0, b_0(n,t)$ and $a(n,t)^2, b(n,t)$ are complex-valued solutions of some equation in the Toda hierarchy TL_r such that there is one $n_0 \in \mathbb{Z}$ and two times $t_0 < t_1$ such that*

$$a_0(n_0+j,t)^2 = a(n_0+j,t)^2, \quad b_0(n_0+j,t) = b(n_0+j,t), \qquad j = 0,\dots,r, \tag{3.6}$$

for $t \in (t_0, t_1)$. Then

$$a_0(n,t)^2 = a(n,t)^2, \quad b_0(n,t) = b(n,t) \tag{3.7}$$

for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$.

Proof. Let us drop the dependence on t for notational simplicity during this proof. The key observation is the following structure for the homogenous quantities $\tilde{g}_j$, $\tilde{h}_j$:

$$\tilde{g}_j(n) = \begin{cases} \left(\prod\limits_{\ell=0}^{k-1} a(n+\ell)^2\right) b(n+k) + R(n+k-1, n-k+1) + \\ + \left(\prod\limits_{\ell=1}^{k} a(n-\ell)^2\right)\left(b(n-k) + 2\sum\limits_{\ell=0}^{k-1} b(n-\ell)\right), & j = 2k+1, \\ \left(\prod\limits_{\ell=0}^{k-2} a(n+\ell)^2\right)\Big(a(n+k-1)^2 + b(n+k-1)^2 \\ +2b(n+k-1)\sum\limits_{\ell=0}^{k-2} b(n+\ell)\Big) + \\ +R(n+k-2, n-k+1) + \prod\limits_{\ell=1}^{k} a(n-\ell)^2, & j = 2k, \end{cases}$$

and

$$\tilde{h}_j(n) = \begin{cases} 2\left(\prod\limits_{\ell=0}^{k-1} a(n+\ell)^2\right)\Big(a(n+k)^2 + b(n+k)^2 \\ +2b(n+k)\sum\limits_{\ell=0}^{k-1} b(n+\ell)\Big) + \\ +R(n+k-1, n-k+1) + 2\prod\limits_{\ell=0}^{k} a(n-\ell)^2, & j = 2k+1, \\ 2\left(\prod\limits_{\ell=0}^{k-1} a(n+\ell)^2\right) b(n+k) + R(n+k-1, n-k+2) \\ +2\prod\limits_{\ell=0}^{k-1} a(n-\ell)^2\left(b(n+1) + b(n-k+1) + 2\sum\limits_{\ell=0}^{k-2} b(n-\ell)\right), & j = 2k, \end{cases}$$

for $j > 1$. Here $R(n,m)$ denotes terms which involve only $a(\ell)$ and $b(\ell)$ with $m \le \ell \le n$ and we set $R(n,m) = 0$ if $n < m$. In fact, this can be verified using $\tilde{g}_0 = 1$, $\tilde{h}_0 = 0$ together with the recursions ([16, Chap. 6])

$$\tilde{g}_{j+1} = \frac{\tilde{h}_j + \tilde{h}_j^-}{2} + b\tilde{g}_j, \tag{3.8}$$

$$\tilde{h}_{j+1} = 2a^2\sum_{\ell=0}^{j}\tilde{g}_{j-\ell}\tilde{g}_\ell^+ - \frac{1}{2}\sum_{\ell=0}^{j}\tilde{h}_{j-\ell}\tilde{h}_\ell, \quad j \in \mathbb{N}_0. \tag{3.9}$$

Now we are ready for the main part of the proof. It suffices to show that (3.7) hold for $n = n_0 - 1$ and $n = n_0 + r + 1$.

We first look at the case $r+1=2k+1$. Then

$$\begin{aligned} 0 &= \frac{\dot a(n_0+k)}{a(n_0+k)} - \frac{\dot a_0(n_0+k)}{a_0(n_0+k)} \\ &= g_{r+1}(n_0+k+1) - g_{r+1}(n_0+k) - g_{0,r+1}(n_0+k+1) + g_{0,r+1}(n_0+k) \\ &= \left(\prod_{\ell=k+1}^{r} a_0(n_0+\ell)^2\right)\big(b(n_0+r+1) - b_0(n_0+r+1)\big) \end{aligned}$$

shows $b(n_0+r+1)=b_0(n_0+r+1)$. Similarly,

$$\begin{aligned} 0 &= \dot b(n_0+k) - \dot b_0(n_0+k) \\ &= h_{r+1}(n_0+k) - h_{r+1}(n_0+k-1) - h_{0,r+1}(n_0+k) + h_{0,r+1}(n_0+k-1) \\ &= 2\left(\prod_{\ell=0}^{k-1} a_0(n_0+\ell)^2\right)\big(a(n_0-1)^2 - a_0(n_0-1)^2\big) \end{aligned}$$

shows $a(n_0-1)^2=a_0(n_0-1)^2$. Proceeding like this, and using the result found in the previous steps,

$$\begin{aligned} 0 &= \dot b(n_0+k+1) - \dot b_0(n_0+k+1) \\ &= h_{r+1}(n_0+k+1) - h_{r+1}(n_0+k) - h_{0,r+1}(n_0+k+1) + h_{0,r+1}(n_0+k) \\ &= \left(\prod_{\ell=k+1}^{r} a_0(n_0+\ell)^2\right)\big(a(n_0+r+1)^2 - a_0(n_0+r+1)^2\big) \end{aligned}$$

shows $a(n_0+r+1)^2=a_0(n_0+r+1)^2$ and

$$\begin{aligned} 0 &= \frac{\dot a(n_0+k-1)}{a(n_0+k-1)} - \frac{\dot a_0(n_0+k-1)}{a_0(n_0+k-1)} \\ &= g_{r+1}(n_0+k) - g_{r+1}(n_0+k-1) - g_{0,r+1}(n_0+k) + g_{0,r+1}(n_0+k-1) \\ &= \left(\prod_{\ell=-1}^{k} a_0(n_0+\ell)^2\right)\big(b(n_0-1) - b_0(n_0-1)\big) \end{aligned}$$

shows $b(n_0-1)=b_0(n_0-1)$, which finishes the case $r+1=2k+1$. The case $r+1=2k$ is analogous. $\square$

Finally since the Kac–van Moerbeke hierarchy can be obtained by setting $b=0$ in the odd equations of the Toda hierarchy, $\mathrm{KM}_r(a)=\mathrm{TL}_{2r+1}(a,0)$ (see [15]), this last result also covers the Kac–van Moerbeke hierarchy. In particular,

Corollary 3.2. *Assume that $\rho_0(n,t)>0$ and $\rho(n,t)>0$ are solutions of the Kac–van Moerbeke equation*

$$\dot\rho(n,t) = \rho(n,t)\big(\rho(n+1,t)-\rho(n-1,t)\big) \tag{3.10}$$

such that there is one $n_0\in\mathbb{Z}$ and two times $t_0<t_1$ such that

$$\rho_0(n_0,t)=\rho(n_0,t), \quad \rho_0(n_0+1,t)=\rho(n_0+1,t), \tag{3.11}$$

for $t\in(t_0,t_1)$. Then

$$\rho_0(n,t)=\rho(n,t) \tag{3.12}$$

for all $(n,t)\in\mathbb{Z}\times\mathbb{R}$.

4. The Ablowitz–Ladik hierarchy

In this section we show that our main result extends to the Ablowitz–Ladik (AL) hierarchy [9]. We first state the result for the simplest case, whose proof follows as the one of Theorem 2.1.

Theorem 4.1. *Let $C_{0,\pm}, c_1 \in \mathbb{C}\backslash\{0\}$. Assume that $\alpha_0(n,t), \beta_0(n,t)$, with $\rho_0(n,t) \neq 0$, and $\alpha(n,t), \beta(n,t)$ are solutions of the Ablowitz–Ladik equation*

$$\begin{aligned} \mathrm{i}\dot{\alpha}(n,t) &= -\rho(n,t)^2\big(c_{0,-}\alpha(n-1,t) + c_{0,+}\alpha(n+1,t)\big) - c_1\alpha(n,t), \\ \mathrm{i}\dot{\beta}(n,t) &= \rho(n,t)^2\big(c_{0,+}\beta(n-1,t) + c_{0,-}\beta(n+1,t)\big) + c_1\beta(n,t), \end{aligned} \tag{4.1}$$

where

$$\rho(n,t) = (1-\alpha(n,t)\beta(n,t))^{1/2}, \tag{4.2}$$

such that there is one $n_0 \in \mathbb{Z}$ and two times $t_0 < t_1$ such that

$$\alpha_0(n_0+j,t) = \alpha(n_0+j,t), \quad \beta_0(n_0+j,t) = \beta(n_0+j,t), \qquad j = 0,1, \tag{4.3}$$

for $t \in (t_0, t_1)$. Then

$$\alpha_0(n,t) = \alpha(n,t), \qquad \beta_0(n,t) = \beta(n,t) \tag{4.4}$$

for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$.

The special choices $c_{0,\pm} = 1$, $c_1 = -2$, and $\beta = \pm\overline{\alpha}$ yield the focusing, defocusing discrete nonlinear Schrödinger equations, respectively. The alternative choice $c_{0,\pm} = \pm\mathrm{i}$, $c_1 = 0$, and $\beta = \overline{\alpha}$ yield the Schur flow.

We next turn to the AL hierarchy following [8], [9]. Associated with two sequences $\alpha(t), \beta(t)$ is a CMV operator

$$\begin{aligned} L(t) = {} & \rho^-(t)\rho(t)\delta_{\mathrm{even}}S^{--} + (\beta^-(t)\rho(t)\delta_{\mathrm{even}} - \alpha^+(t)\rho(t)\delta_{\mathrm{odd}})S^- - \beta(t)\alpha^+(t) \\ & + (\beta(t)\rho^+(t)\delta_{\mathrm{even}} - \alpha^{++}(t)\rho^+(t)\delta_{\mathrm{odd}})S^+ + \rho^+(t)\rho^{++}(t)\delta_{\mathrm{odd}}S^{++}, \end{aligned} \tag{4.5}$$

acting on sequences over $\mathbb{Z}$, where δ_{even} and δ_{odd} denote the characteristic functions of the even, odd integers,

$$\delta_{\mathrm{even}} = \chi_{2\mathbb{Z}}, \quad \delta_{\mathrm{odd}} = 1 - \delta_{\mathrm{even}} = \chi_{2\mathbb{Z}+1}, \tag{4.6}$$

respectively. Next, consider

$$\begin{aligned} P_{\underline{p}}(t) = {} & \frac{\mathrm{i}}{2}\sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\big([L^{\ell}(t)]_+ - [L^{\ell}(t)]_-\big) \\ & - \frac{\mathrm{i}}{2}\sum_{\ell=1}^{p_-} c_{p_--\ell,-}\big([L^{-\ell}(t)]_+ - [L^{-\ell}(t)]_-\big) - \frac{\mathrm{i}}{2}c_{\underline{p}}\, Q_d, \qquad \underline{p} \in \mathbb{N}_0^2, \end{aligned} \tag{4.7}$$

with Q_d denoting the doubly infinite diagonal matrix

$$Q_d = \big((-1)^k\delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}. \tag{4.8}$$

Then the AL hierarchy is equivalent to the Lax equation

$$\frac{d}{dt}L(t) - [P_{\underline{p}}(t), L(t)] = 0, \qquad t \in \mathbb{R}. \tag{4.9}$$

To find an explicit expression we introduce

$$f_{\ell,\pm}(t) = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat{f}_{k,\pm}(t), \quad g_{\ell,\pm}(t) = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat{g}_{k,\pm}(t),$$
$$h_{\ell,\pm}(t) = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat{h}_{k,\pm}(t), \tag{4.10}$$

where

$$\hat{f}_{\ell,+}(n,t) = \alpha(n,t)\langle \delta_n, L^{\ell+1}\delta_n\rangle + \rho(n,t) \begin{cases} \langle \delta_{n-1}, L^{\ell+1}(t)\delta_n\rangle, & n \text{ even},\\ \langle \delta_n, L^{\ell+1}(t)\delta_{n-1}\rangle, & n \text{ odd}, \end{cases}$$
$$\hat{f}_{\ell,-}(n,t) = \alpha(n,t)(\delta_n, L^{-\ell}\delta_n) + \rho(n,t) \begin{cases} \langle \delta_{n-1}, L^{-\ell}(t)\delta_n\rangle, & n \text{ even},\\ \langle \delta_n, L^{-\ell}(t)\delta_{n-1}\rangle, & n \text{ odd}, \end{cases}$$
$$\hat{g}_{0,\pm} = 1/2, \quad \hat{g}_{\ell,\pm}(n,t) = \langle \delta_n, L^{\pm\ell}(t)\delta_n\rangle, \tag{4.11}$$
$$\hat{h}_{\ell,+}(n,t) = \beta(n,t)\langle \delta_n, L^{\ell}(t)\delta_n\rangle + \rho(n,t) \begin{cases} \langle \delta_n, L^{\ell}(t)\delta_{n-1}\rangle, & n \text{ even},\\ \langle \delta_{n-1}, L^{\ell}(t)\delta_n\rangle, & n \text{ odd}, \end{cases}$$
$$\hat{h}_{\ell,-}(n,t) = \beta(n,t)\langle \delta_n, L^{-\ell-1}\delta_n\rangle + \rho(n,t) \begin{cases} \langle \delta_n, L^{-\ell-1}(t)\delta_{n-1}\rangle, & n \text{ even},\\ \langle \delta_{n-1}, L^{-\ell-1}(t)\delta_n\rangle, & n \text{ odd}. \end{cases}$$

Then the $\underline{p}$th equation, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2$, in the AL hierarchy is given by

$$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -\mathrm{i}\dot{\alpha}(t) - \alpha(g_{p_+,+}(t) + g^-_{p_-,-}(t)) + f_{p_+-1,+}(t) - f^-_{p_--1,-}(t) \\ -\mathrm{i}\dot{\beta}(t) + \beta(g^-_{p_+,+}(t) + g_{p_-,-}(t)) - h_{p_--1,-}(t) + h^-_{p_+-1,+}(t) \end{pmatrix} = 0, \tag{4.12}$$
$$\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2.$$

Theorem 4.2. *Fix some $\underline{p} = (p_+, p_-) \in \mathbb{N}_0^2$ such that $p_- = p_+ > 0$ and set $p = p_+ + p_- - 1$. Assume that $\alpha_0(n,t), \beta_0(n,t)$, with $\rho_0(n,t) \neq 0$, and $\alpha(n,t), \beta(n,t)$ are solutions of some equation in the Toda hierarchy $\mathrm{AL}_{\underline{p}}$ such that there is one $n_0 \in \mathbb{Z}$ and two times $t_0 < t_1$ such that*

$$\alpha_0(n_0 + j, t) = \alpha(n_0 + j, t), \quad \beta_0(n_0 + j, t) = \beta(n_0 + j, t), \qquad 0 \le j \le p, \tag{4.13}$$

for $t \in (t_0, t_1)$. Then

$$\alpha_0(n,t) = \alpha(n,t), \quad \beta_0(n,t) = \beta(n,t) \tag{4.14}$$

for all $(n,t) \in \mathbb{Z} \times \mathbb{R}$.

Proof. Again drop the dependence on t for notational simplicity during this proof and we will use the same conventions as in the proof of Theorem 3.1.

The homogeneous quantities $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, $\hat{h}_{\ell,\pm}$ are uniquely defined by the following recursion relations [9, Lem. C.5]:

$$
\begin{aligned}
\hat{g}_{0,+} &= \frac{1}{2}, \quad \hat{f}_{0,+} = -\alpha^+, \quad \hat{h}_{0,+} = \beta, \\
\hat{g}_{l+1,+} &= \sum_{k=0}^{l} \hat{f}_{l-k,+}\hat{h}_{k,+} - \sum_{k=1}^{l} \hat{g}_{l+1-k,+}\hat{g}_{k,+}, \\
\hat{f}^-_{l+1,+} &= \hat{f}_{l,+} - \alpha(\hat{g}_{l+1,+} + \hat{g}^-_{l+1,+}), \\
\hat{h}_{l+1,+} &= \hat{h}^-_{l,+} + \beta(\hat{g}_{l+1,+} + \hat{g}^-_{l+1,+}),
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_{0,-} &= \frac{1}{2}, \quad \hat{f}_{0,-} = \alpha, \quad \hat{h}_{0,-} = -\beta^+, \\
\hat{g}_{l+1,-} &= \sum_{k=0}^{l} \hat{f}_{l-k,-}\hat{h}_{k,-} - \sum_{k=1}^{l} \hat{g}_{l+1-k,-}\hat{g}_{k,-}, \\
\hat{f}_{l+1,-} &= \hat{f}^-_{l,-} + \alpha(\hat{g}_{l+1,-} + \hat{g}^-_{l+1,-}), \\
\hat{h}^-_{l+1,-} &= \hat{h}_{l,-} - \beta(\hat{g}_{l+1,-} + \hat{g}^-_{l+1,-}).
\end{aligned}
$$

From them we obtain

$$
\begin{aligned}
\hat{f}_{j,+}(n) = &- \left(\prod_{l=1}^{j} \rho(n+l)^2\right) \alpha(n+j+1) + R(n+j, n-j+2) \\
&+ \left(\prod_{l=0}^{j-2} \rho(n-l)^2\right) \alpha(n+1)^2\beta(n-j+1), \qquad (4.15)\\
\hat{f}_{j,-}(n) = &- \left(\prod_{l=1}^{j-1} \rho(n+l)^2\right) \alpha(n)^2\beta(n+j) + R(n+j-1, n-j+1) \\
&+ \left(\prod_{l=0}^{j-1} \rho(n-l)^2\right) \alpha(n-j), \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
\hat{g}_{j,+}(n) = &- \left(\prod_{l=1}^{j-1} \rho(n+l)^2\right) \beta(n)\alpha(n+j) + R(n+j-1, n-j+2) \\
&- \left(\prod_{l=0}^{j-2} \rho(n-l)^2\right) \alpha(n+1)\beta(n-j+1) \qquad (4.17)\\
\hat{g}_{j,-}(n) = &- \left(\prod_{l=1}^{j-1} \rho(n+l)^2\right) \alpha(n)\beta(n+j) + R(n+j-1, n-j+2) \\
&- \left(\prod_{l=0}^{j-2} \rho(n-l)^2\right) \beta(n+1)\alpha(n-j+1) \qquad (4.18)
\end{aligned}
$$

$$\hat{h}_{j,+}(n) = -\left(\prod_{l=1}^{j-1}\rho(n+l)^2\right)\beta(n)^2\alpha(n+j) + R(n+j-1,n-j+1)$$
$$+\left(\prod_{l=0}^{j-1}\rho(n-l)^2\right)\beta(n-j), \tag{4.19}$$
$$\hat{h}_{j,-}(n) = -\left(\prod_{l=1}^{j}\rho(n+l)^2\right)\beta(n+j+1) + R(n+j,n-j+2)$$
$$+\left(\prod_{l=0}^{j-2}\rho(n-l)^2\right)\beta(n+1)^2\alpha(n-j+1) \tag{4.20}$$

for $j\in\mathbb{N}$. Note that it suffices to verify the $+$ case since the $-$ case follows from $\hat{f}_{j,\pm}(\alpha,\beta) = \hat{h}_{j,\pm}(\alpha,\beta)$ and $\hat{g}_{j,+}(\alpha,\beta) = \hat{g}_{j,}(\alpha,\beta)$ ([9, Lem. 3.7]).

Now we can proceed as in the case of the Toda hierarchy. For example

$$0 = \mathrm{i}(\dot{\alpha}(n+p_-) - \dot{\alpha}_0(n+p_-) = -c_{0,+}\left(\prod_{l=p_-}^{p}\rho(n-l)^2\right)\big(\alpha(n+p+1)-\alpha_0(n+p+1)\big)$$

implies $\alpha(n+p+1) = \alpha_0(n+p+1)$ etc. □

Interestingly, the above approach does not seem to work for $p_- \neq p_+$ in general. In any case, the above result covers the discrete nonlinear Schrödinger and Schur hierarchies via the above mentioned special choices $c_{0,\pm}=1$, $\beta=\pm\overline{\alpha}$ and $c_{0,\pm}=\pm\mathrm{i}$, $\beta=\overline{\alpha}$.

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