Stopping dynamics of sliding and spinning bodies on a flat surface with dry friction.

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We analyze dynamics of sliding and spinning rigid bodies on a flat surface with arbitrary axisymmetric distribution of normal pressures on the contact surface and masses within the body. We show that the motion can not finish with arbitrary spin and slide velocities. We establish a specific condition on the two velocities during the final stage of the motion of the body and derive exact and closed form expressions both for the total time of motion and the distance traveled.

Dry friction is widespread both in Nature and technology, various issues involving a dynamics of a rigid body on a surface with friction play very important role in numerous engineering and transport applications. Although the problem has been under intensive consideration since ancient times, there are still many unresolved and practically important questions. Dry friction as a physical phenomenon has long time became a textbook issue, see e.g. [1].

Modern analysis of a rigid body dynamics with combined friction has been pioneered in [2]. Equations of motion for an axisymmetric body on a surface with friction include equations accounting both the total momentum and the total angular momentum change of the moving body depending on the total friction force, F, and the friction torque, M. In a general case, complete analytical expressions for the quantities can not be found. Nevertheless both F and M have been identified for a number of special cases, corresponding to a number of important models of normal pressures distributions, such as a ring [3], such as Hertz contact [4] and Galin distribution [5]. Complete solutions of the equations of motion are available either in Pade approximation [6] or for Galin distribution [7]. The limiting case of a disk with uniform distribution of normal pressures was originally studied in [3] and later on in [8] using different kinds of analytical approximations. Various aspects of equilibrium and slip were recently addressed in [9].

We start from the equations of motion for a sliding and spinning disk on a flat surface with dry friction and provide the complete and exact solution for arbitrary initial conditions. As a result of quantitative analysis we find, that the motion of the body can not complete with arbitrary spin and sliding velocities. Depending on the mass and normal pressures distribution, or in other words on asymptotic behavior of the functions M и F, for most realistic distributions of normal pressures on the contact surface the motion ends with nearly the same values spin and slide velocities. Hertz contact case is found to stay aside though: a disk modeled by Hertz distribution of normal pressures stops with spin velocity greatly exceeding the spin. In all the cases we are able to calculate both the time and the distance traveled.

Consider a movement of an axisymmetric body with slip velocity $v(t)$ and spin $u(t) = R\omega(t)$, where R is the size (radius) of the body, and $\omega(t)$ is the angular velocity corresponding to the spin. The equations for the slide velocity can be written in the form of total momentum change:

$$m\dot{v}(t) = \int dF(u, v, r, \phi) = -F(u, v),$$

where m is the mass of the body and $dF(u, v, r, \phi)$ is the elementary friction force, corresponding to an element of the contact surface characterized by certain given values of the polar coordinates r and ϕ:

$$dF(u, v, r, \phi) = -f_0\sigma(r)\frac{v - \omega r \sin\phi}{|V|}dS,$$

where, according to the Coulomb law, f_0 is the friction coefficient. Here we use the short notation $dS \equiv rdrd\phi$ for the surface element, $V = |v + \omega \times r|$ is the velocity of a point on the contact surface, and σ is the normal pressures distribution function over the contact spot.

Similarly the equation for the spin is nothing else but the change of the angular momentum due to the friction forces:

$$R^{-1}J\dot{u}(t) = \int dM(u, v, r, \phi) = -M(u, v).$$

Here J is the moment of inertia and the elementary torque, dM, is given by

$$dM(u, v, r, \phi) = -\frac{f_0}{|V|}\sigma(r)(r^2\omega - vr\sin\phi)dS.$$

For the sake of convenience let us use dimensionless variables for the radial distance, i.e. introduce the new variable ρ: $r = R\rho$, $0 < \rho < 1$. Then, after introducing the new dimensionless function $\bar{\sigma}(\rho) = \sigma(\rho)/\sigma(0)$ ($\sigma_0 = \sigma(0)$), we find that

$$F(u, v) = f_0R^2\sigma_0 \int \frac{\bar{\sigma}(\rho)(k - \rho\sin\phi)\rho d\rho d\phi}{\sqrt{\rho^2 - 2k\rho\sin\phi + k^2}}, \quad (1)$$

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and

$$M(u,v) = f_0 R^3 \sigma_0 \int \frac{\bar{\sigma}(\rho)(\rho - k\sin\phi)\rho^2 d\rho d\phi}{\sqrt{\rho^2 - 2k\rho\sin\phi + k^2}}, \qquad (2)$$

where $k(t) = v(t)/u(t)$ is the ratio of the slide and spin velocities. The latter expressions should be added by the standard definitions for the mass

$$m = 2\pi R^2 \rho_m(0) \int_0^1 \bar{\rho}_m(\rho)\rho d\rho,$$

and the moment of inertia:

$$J = 2\pi R^4 \rho_m(0) \int_0^1 \bar{\rho}_m(\rho)\rho^3 d\rho.$$

Here $\rho_m(\rho)$ is the mass distribution within the body, and $\bar{\rho}_m = \rho_m(\rho)/\rho_m(0)$.

Let us transform the equations of motion to a fully dimensionless form. First we introduce the dimensionless forms for the friction torque, $\bar{M}$, and the force, $\bar{F}$, with the help of the transformations $M(u,v) = 2\pi f_0 \sigma_0 R^3 \bar{J}\bar{M}(k)$ and $F(u,v) = 2\pi f_0 \sigma_0 \bar{m} R^2 \bar{F}(k)$. Then, we switch to dimensionless forms of mass, $\bar{m}$, and momentum of inertia, $\bar{J}$, according to $m = 2\pi R^2 \rho_m(0)\bar{m}$ and $J = 2\pi R^4 \rho_m(0)\bar{J}$. Finally, transforming to a new time variable $t = \tau \times T$ with $T = f_0\sigma_0/\rho_m(0)$ we obtain fully dimensionless form of the equations of motion:

$$\dot{v}(\tau) = -\bar{F}(k), \quad \dot{u}(\tau) = -\bar{M}(k).$$

The equations also imply, that the distance traveled is measured in the units of R.

There are different types of approximate distributions of normal pressures in the contact surface described in the literature. The simplest case is definitely a ring distribution: $\sigma(\rho) = \rho_m g = \sigma_0 \delta(\rho - 1)$, where g is the acceleration of a freely falling body, $\sigma_0 = N/2\pi R^2$, N is the total normal pressure. The integrals for the mass and the moment of inertia are obvious too: $\bar{m} = \sigma_0/g$, $\bar{J} = \sigma_0/g$. Integration over the radial coordinate in Eqs. (1) и (2) helps rewrite the friction force and torque:

$$\bar{F}_{ring}(k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(k - \sin\phi)d\phi}{\sqrt{1 - 2k\sin\phi + k^2}}. \qquad (3)$$

$$\bar{M}_{ring}(k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - k\sin\phi)d\phi}{\sqrt{1 - 2k\sin\phi + k^2}}, \qquad (4)$$

Analytical expressions for both $\bar{F}_{ring}$ and $\bar{M}_{ring}$ have been obtained in terms of Elliptic Integrals in [3]. Let us note that is such a simple case we have $\bar{F}_{ring}(k) = \bar{M}_{ring}(1/k)$. The asymptotic forms of these integrals for small and large values of k are summarized in Table I.

The integrals (4) and (3) let us calculate the friction force and torque for an arbitrary form of normal pressures distribution. Consider, for example, the case of uniform normal pressures and mass distribution within a disk characterized by $\sigma = \sigma_0 = mg/\pi R^2$. Then, $\bar{m} = m/2\pi R^2 = \sigma_0/2$, $J = mR^2/2$ ($\bar{J} = m/4\pi R^2 = \sigma_0/4$) and

$$\bar{M}_{disk}(k) = \frac{2}{\pi}\int_0^1 \rho^2 d\rho \int_0^{2\pi} \frac{(1 - k/\rho\sin\phi)d\phi}{\sqrt{1 - 2k/\rho\sin\phi + (k/\rho)^2}}.$$

Comparing the latter expression with Eq. (4) we find that:

$$\bar{M}_{disk}(k) = \frac{2}{\pi}\int_0^1 \rho^2 d\rho \bar{M}_{ring}(k/\rho).$$

Accordingly,

$$\bar{F}_{disk}(k) = \frac{1}{\pi}\int_0^1 \rho d\rho \int_0^{2\pi} \bar{F}_{ring}(k/\rho).$$

Analytical expressions for this case are hard to be found. Asymptotic forms for the force and the torque are also given in Table I.

Galin distribution of the normal pressures for a disk with uniform mass distribution is analyzed in [5]. In this case $\sigma(\rho) = \sigma_0/\sqrt{1-\rho^2}$, where $\sigma_0 = N/2\pi R^2$ (N is the total normal pressure). Then, the integral for the friction torque is

$$\bar{M}_G(k) = \frac{1}{\pi}\int \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho \bar{M}_{ring}(k/\rho).$$

In the same way

$$\bar{F}_G(k) = \frac{1}{2\pi}\int_0^1 \frac{\rho}{\sqrt{1-\rho^2}} d\rho \bar{F}_{ring}(k/\rho).$$

The integrals $\bar{F}$ and $\bar{M}$ are calculated analytically for arbitrary value of k in [5]. The asymptotic values are summarized in Table I.

Hertz distribution of normal pressures for a disk with uniform distribution of masses is given by $\sigma(\rho) = \sigma_0\sqrt{1-\rho^2}$, where $\sigma_0 = N/2\pi R^2$. The integrals for the friction force and torque are, once again, are related to those for a ring:

$$\bar{M}_H(k) = \frac{2}{\pi}\int_0^1 \rho^2 \sqrt{1-\rho^2} d\rho \bar{M}_{ring}(k/\rho),$$

and

$$\bar{F}_H(k) = \frac{1}{\pi}\int_0^1 \rho\sqrt{1-\rho^2} d\rho \bar{F}_{ring}(k/\rho).$$

The integrals $\bar{F}$ and $\bar{M}$ for the Hertz contact are found analytically in [4] (see Table I for the asymptotic values).

The goal of the further study is to determine when and how a body with arbitrary initial conditions on the sliding velocity and spin stops. In order to do that let us the integral of motion found in [3]:

$$\log u(t) - \int^{v(t)/u(t)} \frac{M(k)}{F(k) - kM(k)} dk = \text{const}. \qquad (5)$$

Normal Pressures Distribution Form	σ	$k \to 0$	$k \to \infty$	k_*
Ring	$\sigma(\rho) = \sigma_0 \delta(\rho - 1)$	$F = k/2 + O(k^3)$ $M = 1 + O(k^2)$	$F = 1 + O(k^{-2})$ $M = 1/2k + O(k^{-3})$	1
Uniform Disk	$\sigma(\rho) = \sigma_0$	$F = k + O(k^3)$ $M = 4/3 + O(k^2)$	$F = 1 + O(k^{-2})$ $M = 1/2k + O(k^{-3})$	0.65
Galin Disk	$\sigma(\rho) = \frac{\sigma_0}{\sqrt{1-\rho^2}}$	$F = k\pi/4 + O(k^3)$ $M = \pi/2 + O(k^2)$	$F = 1 + O(k^{-2})$ $M = 2/3k + O(k^{-3})$	1
Hertz Contact	$\sigma(\rho) = \sigma_0 \sqrt{1 - \rho^2}$	$F = k\pi/4 + O(k^3)$ $M = \pi/4 + O(k^2)$	$F = 2/3 + O(k^{-2})$ $M = 4/15k + O(k^{-3})$	0

TABLE I: Basic definitions and asymptotic forms of friction force and torque for various forms of normal pressures distributions.

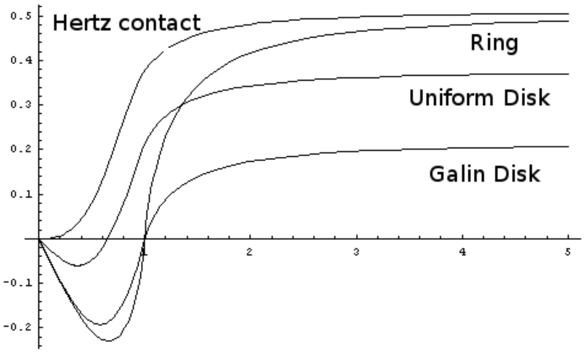

FIG. 1: Function $\Phi(k)$ for various cases of the normal pressures distributions.

Hereinafter we do not make distinctions between the original forms, M and F, and dimensionless expressions, $\bar{M}$ and $\bar{F}$, respectively. When the body stops we have $u(t) \to 0$ and therefore the logarithm in the l.h.s. becomes infinite. Since the initial conditions do not include infinite velocities (i.e. the constant in the r.h.s. is finite), the singularity in the l.h.s. can be only compensated by the singularity in the integral over k. The integral itself can be infinite only if the expression under the integral sign becomes infinite, i.e. when the upper limit of the integration k_*coincides with singularity of the denominator:

$$\Phi(k_*) = F(k_*) - k_* M(k_*) = 0.$$

In this case, close to its zero, the denominator can be approximated by its Tailor expansion, $\Phi(k) \approx \Phi'(k_*)(k - k_*)$, which helps to analyze singularities in the l.h.s. of (5) according to

$$\log u - \frac{M(k_*)}{\Phi'(k_*)} \log\left(\frac{v}{u} - k_*\right) = \text{const.}$$

Obviously the disk can stop only if $\Phi'(k_*) > 0$, since in this case both logarithms in (5) have the same sign and are infinite at the same time.

The analysis above suggests that the disk can not stop having arbitrary values of velocities of u and v. The way(s) the disk may stop depends on the number and properties of zeros of the function $\Phi(k)$. The function usually has no explicit analytical form and for all of the cases considered in our Letter is depicted on Fig. 1. In any way $\Phi(k) = 0$ if $k = 0$. Apart from the Hertz contact case, which turns out to be a special one, $\Phi'(0) < 0$ and hence the disk can not stop at $k = 0$. Normally there is another zero $k = k_* \approx 1$. Since in this case $\Phi'(k_*) > 0$ the disk indeed stops at $v/u \to k_*$, i.e. both sliding and spin ceases at roughly the same speed. Both ring and Galin disk stops at $k_* = 1$. Pade approximations to M and F developed in [6] give $k_* = 1$ as well. Our numerical research shows that a disk with uniform normal pressure distribution stops at $k \approx 0.65$, which is slightly different from the value of $k \approx 0.71$, found in the original work [3]. The difference is neither large or important and arises, in our view, from numerical errors.

Neither of the normal pressure distributions analyzed here have more than one zero k_* of the function $\Phi(k)$ with $\Phi'(k_*) > 0$. Although we can not prove it for general functions M and F, the unique value of k_* simplifies the discussion a lot in each of our model cases. For example, we are able to provide a simple expression for the time and distance traveled. Indeed, Eq. (5) represents a complete solution of the equations of motion. For example spin velocity at any moment of time is given by:

$$u(t) = u_0 \exp\left(\int_{k_0}^{k(t)} \frac{M(k)}{F(k) - kM(k)} dk\right), \tag{6}$$

where $k(t) = v(t)/u(t)$ and $k_0 = k(0)$. The time of travel t is related to the parameters of the trajectory $u(t)$ and $k(t)$ by the equation

$$t = -T \int_{k_0}^{k(t)} \frac{u}{F(k) - kM(k)} dk.$$

Since the body stops at $t = t_*$: $k(t_*) = k_*$, the total time of travel is

$$t_* = -u_0 T \int_{k_0}^{k_*} \frac{\exp\left(\int_{k_0}^{k} \frac{M(k')}{F(k') - k'M(k')} dk'\right)}{F(k) - kM(k)} dk. \tag{7}$$

Similarly the total distance traveled can be found from

$$x_* = -R \int_{k_0}^{k_*} \frac{u^2 k}{F(k) - kM(k)} dk. \tag{8}$$

The latter expressions are the central results of our research, are general and do not depend on any kind of approximation beyond those used to derive the original equation of motion. Given specific expressions for F and M both t_* and x_* can be found exactly, for example, numerically. Below we will find the time and distance traveled in a few important limiting cases.

Let us consider first the simplest case: the body starts moving with such a ration of spin and sliding velocities, that the difference between $k(0)$ and k_* is small from the very beginning: $k_0 = k_* + \varepsilon$, $|\varepsilon| \ll 1$. Consider first the case $\varepsilon > 0$ (the opposite case is straightforward):

$$\frac{M(k)}{F(k) - kM(k)} \approx \frac{M(k_*)}{\Phi'(k_*)(k - k_*)},$$

$$t_* = \frac{u_0 T}{M(k_*)},\ x_* = \frac{u_0^2 k_* R}{2M(k_*)}.$$

Therefore, if the motion starts so that the sliding and spin velocities are "locked" such that $k_0 \approx k_*$, then both the time and distance traveled are given by universal expressions depending only on the value $M_* = M(k_*)$.

Consider a more interesting case $k_0 \gg k_* \sim 1$ (i.e. the case when the sliding velocity greatly exceeds the spin velocity). The calculation above for $k \approx k_*$ shows, that although some expressions under the integral are singular at $k = k_*$, the singularity in exponential factor cancels the singularity arising from the singularity of the denominator. This means that both integrals in t_* and x_* converge and can be calculated using the asymptotic forms of the integrand at large k: $F(k) \to F_\infty$, $M(k) \to m_\infty/k$ (the specific values of the constants F_∞ and m_∞ can be extracted from Table I). Then

$$\frac{M(k)}{F(k) - kM(k)} \approx \frac{m_\infty}{k(F_\infty - m_\infty)},$$

$$t_* = \frac{u_0 k_0 T}{F_\infty},\ x_* = \frac{k_0^2 u_0^2 R}{F_\infty}.$$

Consider, at last, the opposite case, $k_0 \ll k_* \sim 1$ (now the spin velocity greatly exceeds the sliding velocity). In order to calculate approximations to the integrals we need to study the asymptotic values for the functions $M(k)$ and $F(k)$ for $k \to 0$: $F(k) - kM(k) \to -\beta k$, where $\beta > 0$ (otherwise the body would stop at $k = 0$, see Hertz contact analysis below). The torque does not lead to an additional singularity, since $M(k) \to M_0 \neq 0$, $F(k) - kM(k) \approx -\beta k$ ($\beta > 0$) and

$$t_* = \frac{u_0 T}{M_0},\ x_* = \frac{u_0^2 k_* R}{2M_0}.$$

Note that neither the time no the distance traveled depend on the value of the constant β.

Hertz contact case turns out to be a special one. The body stops at $k_* = 0$. In other words although both u and v vanish at the same time, right before the body stops we have $v \ll u$, i.e. spin velocity greatly exceeds slide velocity [4]. The situation apparently occurs in a billiard game [8], when a ball may nearly stop ($v \approx 0$) while spinning around its vertical axis. The calculations are nevertheless straightforward and give $t_* = u_0 k_0 T/F_\infty$ and $t_* = u_0 T/M_0$ for $k_0 \gg 1$ and $k_0 \ll 1$, respectively.

Combining the results obtained above we find that if $k_0 \lesssim k_*$ then $t_* \approx u_{0T}/M_0$, $M_0 = M(k_0)$, If, on the other hand $k_0 \gg k_*$, then $t_* = u_0 k_0 T/F_\infty$. A simple interpolating formula bringing the two results together may be proposed as follows

$$t_*(k_0) \approx u_0 T(\frac{1}{M(k_0)} + \frac{k_0}{F_\infty}).$$

An interpolating formula for the distance traveled may be obtained in the similar way: $x_*(k_0) = k_0 u_0 t_*(k_0)$.

Although the analysis above is somewhat involved, the conclusion is very simple: the time of travel depends on u_0, if $v_0 \ll u_0$ and v_0, if $v_0 \gg u_0$. In other words the time of movement depends on the ability of the friction force to extinguish the maximum of the slide or spin velocities. Interestingly enough, that if the ratio $k(0)$ of the velocities at the beginning is either large (or small), then it states large (or small) practically all the time, reaching the value $k(t_*) = k_*$ at the very last moment before the motion ceases.

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