On geometric perturbations of critical Schrödinger operators with a surface interaction

Pavel Exner[1,2] and Martin Fraas[3]

[1]Nuclear Physics Institute, Czech Academy of Sciences, 25068 Řež near Prague
[2]Doppler Institute, Břehová 7, 11519 Prague, Czechia
[3]Physics Department, Technion, Haifa 32000, Israel
exner@ujf.cas.cz, fraas@ujf.cas.cz

Abstract

We study singular Schrödinger operators with an attractive interaction supported by a closed smooth surface $\mathcal{A} \subset \mathbb{R}^3$ and analyze their behavior in the vicinity of the critical situation where such an operator has empty discrete spectrum and a threshold resonance. In particular, we show that if $\mathcal{A}$ is a sphere and the critical coupling is constant over it, any sufficiently small smooth area preserving radial deformation gives rise to isolated eigenvalues. On the other hand, the discrete spectrum may be empty for general deformations. We also derive a related inequality for capacities associated with such surfaces.

1 Introduction

Relations between geometrical and spectral properties belong to traditional questions in mathematical physics. Recently an isoperimetric inequality was derived for two-dimensional Schrödinger operators with a singular attractive interaction supported by a closed loop [1, 2]. It claims that if the coupling is constant along the loop the ground state eigenvalue is maximized in the class of all loops of a fixed length by a circle. The result has interested connections to both the classical electrodynamics [1, 3] and a class of isoperimetric inequalities of a purely geometric nature [3, 4, 5].

One can ask naturally whether the result has a higher-dimensional counterpart, that is, for Schrödinger operators with an attractive interaction supported by a closed hypersurface. Our aim in this paper is to address this question in dimension three. It is clear from the outset that such a problem is more involved. One difference comes from the fact that the two-dimensional operator with an attractive interaction has always a non-empty discrete spectrum while its three-dimensional counterpart may be positive if the coupling is sufficiently weak. This opens a possibility, on the other hand, that such operator may or may not have bound states depending on geometric perturbations of the interaction support.

To be specific, choose a a closed smooth surface $\mathcal{A} \subset \mathbb{R}^3$ and consider the operator formally given as $-\Delta - \alpha\delta(x - \mathcal{A})$ with an attractive coupling constant over $\mathcal{A}$. If we choose $\alpha > 0$ such that for a spherical $\mathcal{A}$ the operator is critical, i.e. any larger α will produce a nontrivial discrete spectrum, one might expect in analogy with the two dimensional case that any deformation of $\mathcal{A}$ preserving its area would lead to occurrence of negative eigenvalues. In reality the situation is more complicated. We are going to show that the above claim is valid for small enough smooth radial deformations of $\mathcal{A}$, however, it fails generally: there are "large" area-preserving deformations which leave the operator positive.

On the other hand, one can prove another global result for deformations of $\mathcal{A}$. It exhibits again a relation to electrostatics, although it is now different from the one mentioned above: the preserved quantity to replace the loop length of the two-dimensional situation is not the area of $\mathcal{A}$ but the *capacity* of the capacitor represented by the surface.

Let us briefly described the contents of the paper. After presenting in the Sections 2 and 3 the necessary preliminaries, we shall state in Section 4 the main results of this paper. Their proofs and discussion will follow. In particular, the proof of Theorem 4.1 is given in the section 6, and the section 7 deals with local deformations of the sphere which are the contents of Theorem 4.3. Finally, in Section 8 we will show that the claim of Theorem 4.3 cannot be extended to general deformations.

2 Singular interactions on a surface

Let $\mathcal{A}$ be a closed smooth surface in $\mathbb{R}^3$ and ν the "natural" measure on $\mathcal{A}$ induced by its embedding into the Euclidean space, in other words, Lebesgue measure in the appropriate local charts of $\mathcal{A}$. Consider next a bounded Borel measurable function $\alpha(x) : \mathcal{A} \to \mathbb{R}$ and $\phi,\, \psi \in \mathcal{W}^{2,1}(\mathbb{R}^3)$ and define the quadratic form

$$\mathsf{Q}_\alpha(\phi,\, \psi) := \int_{\mathbb{R}^3} \nabla\phi(x) \cdot \overline{\nabla\psi(x)}\, \mathsf{d}x - \int_{\mathcal{A}} \alpha(x)\phi(x)\overline{\psi(x)}\, \mathsf{d}\nu(x)\,;$$

by applying the Green formula one has

$$\begin{aligned}\mathsf{Q}_\alpha(\phi,\, \psi) = \int_{\mathbb{R}^3} -(\triangle\phi)(x)\overline{\psi(x)}\, \mathsf{d}x - \int_{\mathcal{A}} \left(\frac{\partial\phi(x)}{\partial n_e} + \frac{\partial\phi(x)}{\partial n_i}\right) \overline{\psi(x)}\, \mathsf{d}\nu(x) \\ - \int_{\mathcal{A}} \alpha(x)\phi(x)\overline{\psi(x)}\, \mathsf{d}\nu(x)\,,\end{aligned}$$

where n_e, n_i are exterior and interior normal, respectively. This suggest that the operator

$$\mathsf{H}_\alpha := -\triangle\,,$$

defined on functions which are locally $\mathcal{W}^{2,1}$ away from $\mathcal{A}$ and satisfy

$$\frac{\partial\phi(x)}{\partial n_e} + \frac{\partial\phi(x)}{\partial n_i} = -\alpha(x)\phi(x)\,, \tag{2.1}$$

for every $x \in \mathcal{A}$ is self-adjoint and corresponds to the form Q_α; for a proper justification of this claim and related results see [6, 7]. The operator H_α is naturally interpreted as a singular Schrödinger operator with the interaction supported by the surface $\mathcal{A}$ and the coupling "constant" $\alpha(x)$. It motivates us to define the global strength of the interaction,

$$[\alpha] := \int_{\mathcal{A}} \alpha(x)\, \mathsf{d}\nu(x) \tag{2.2}$$

and the relative density of the interaction,

$$\hat{\alpha}(x) := \frac{\alpha(x)}{[\alpha]}\,. \tag{2.3}$$

The last definition makes sense, of course, only if $[\alpha] \neq 0$, which will be true in our case, since we are going to consider only singular Schrödinger operator with attractive interactions, $\alpha(x) > 0$ for all $x \in \mathcal{A}$.

A situation of particular interest is the case of a function α constant over the surface $\mathcal{A}$ taking a value $\alpha_0 > 0$ there; for such a function we obviously have $[\alpha] = \alpha_0 S$ and $\hat{\alpha}(x) = S^{-1}$, where $S := \nu(\mathcal{A})$ is the surface area.

3 Criticality

Let us first recall the standard criticality notions. If $\mathsf{H} := -\triangle + V(x)$ is a Schrödinger operator and $W(x) \geq 0$ an arbitrary nonzero and compactly supported function, we say that H is *subcritical* if for $\epsilon > 0$ small enough the operator $-\triangle + V(x) - \epsilon W(x)$ remains to be positive. Correspondingly, H is *critical* if its positivity depends on the sign of the perturbation, and *supercritical* if $-\triangle + V(x) + \epsilon W(x)$ is negative for sufficiently small $\epsilon > 0$. Clearly, the interaction is supercritical if and only if the operator H has a negative bound state.

These notions can be carried over directly to the singular case where one has to consider the operators $\mathsf{H} := -\triangle \pm \epsilon W(x)$ with the boundary conditions (2.1) as perturbations of H_α. Let us collect without proofs several simple facts about criticality. Recall that the solution u of $\mathsf{H}u = 0$ is said to have minimal growth at infinity if for any other solution v there is a constant C such that $Cv(x) > u(x)$ holds for all $|x|$ sufficiently large [8]. With this notion, we can state the following result.

Proposition 3.1 *Let H_α be the singular Schrödinger operator H_α with the interaction supported by $\mathcal{A}$ as described above. Then the following claims are equivalent:*

- *(i)* H_α *is critical*
- *(ii) For any bounded function $\beta > 0$, $\mathsf{H}_{\alpha\pm\beta}$ is supercritical or subcritical respectively.*
- *(iii) The equation $\mathsf{H}_\alpha u = 0$ has a positive solution with a minimal growth at infinity.*

In the following we will always assume that the relative interaction density of the operator H_α is fixed and discuss how the properties of the operator depend on $[\alpha]$. Since $\mathsf{H}_0 = -\triangle$ is subcritical in dimension three it is clear that H_α is subcritical for $[\alpha]$ small enough and supercritical for $[\alpha]$ large. Our main concern is the value at which H_α is critical; we will denote as $[\hat{\alpha}]_c$.

Relations between $[\hat{\alpha}]_c$ and the geometric properties of the interaction are of a natural interest. In particular, one can ask about the critical strength for a given surface and a fixed interaction density, and about the dependence of the critical strength on the shape of the surface. To get some insight into these questions, we are going to compare the critical strength $[\hat{\alpha}]_c$ with the surface area S and its capacity C.

A comparison requires to select appropriate quantities. As an inspiration, note that in the particular situation when the surface A is a sphere of radius R and α is constant over it, $\hat{\alpha}(x) = (4\pi R^2)^{-1}$ for any $x \in \mathcal{A}$, one can compute the above named quantities explicitly,

$$[\hat{\alpha}]_c = 4\pi R\,, \quad S = 4\pi R^2\,, \quad C = R\,.$$

With this example on mind we define the *interaction radius* $\overline{[\hat{\alpha}]}_c := [\hat{\alpha}]_c/(4\pi)$ and the classical *surface radius* $\overline{S} := \sqrt{S/(4\pi)}$. Note that while α has, physically speaking, dimension length^{-1}, the integral quantities $[\alpha]$ and $[\hat{\alpha}]_c$ has the dimension of length; it will be of importance in the following that the quantities $\overline{[\hat{\alpha}]}_c$ and $\overline{S}$ scales in the same manner.

It is well known [9] that $\overline{S}$ and C are *not* comparable, in particular, that $\overline{S} \geq C$ need not hold even in the vicinity of sphere. Our aim is to try to compare these two quantities to the interaction radius for a fixed relative interaction density. Recall that our original motivation to study this problem mentioned in the introduction was to find out whether any surface-preserving deformation of a critical sphere with the interaction $\alpha(x) = \alpha_0$ of the constant relative density S^{-1} will produce a bound state. In terms of the notions introduced above we can say that it will be true if $\overline{S} > \overline{[\hat{\alpha}]}_c$ holds for the deformed surface where $\overline{S}$ is the radius of the critical sphere.

4 Main results

Let us formulate now the claims we are going to demonstrate. First of all, the above mentioned property will be valid if the quantity preserved at the deformation is not the surface area but the capacity. To this aim we can think of the surface $\mathcal{A}$ as of a capacitor charged with an unit charge denoting by $\sigma(x)$ the corresponding charge density,

$$\int_{\mathcal{A}} \sigma(x)\mathsf{d}\nu(x) = 1. \tag{4.1}$$

For the definition of capacity, charge density and their properties, see for instance [9, Chap. II] and also a brief recapitulation in the next section.

Theorem 4.1 *Let $\mathcal{A}$ be a surface with the capacity C, then*

(a) The operator H_α is critical if $\alpha = 4\pi C\sigma$, i.e. $\overline{[\hat{\sigma}]}_c = C$.

(b) Let H_α correspond to a constant interaction, $\alpha(x) = \alpha_0$ for all $x \in \mathcal{A}$, then $\overline{[\hat{\alpha}]_c} \le C$.

Remark 4.2 If $\mathcal{A}$ is a sphere, then the equality $C = \overline{[\hat{\sigma}]_c}$ holds in the part (b) of the above theorem, while the proof of the converse statement is an open question. Note that it is equivalent to the claim that the sphere is the only closed capacitor with a constant surface charge density. It is a common knowledge that the charge tends to concentrate at the points of high curvature, but we have not found a rigorous elaboration of this assertion.

In the surface-preserving situation we have a weaker result valid for a class of gentle deformations. Let A be a surface defined by the equation

$$r \equiv r(\theta,\, \phi) = r_0\,(1 + \epsilon\rho(\theta,\, \phi))\,,$$

where r, θ, ϕ are the spherical coordinates, ρ is a fixed smooth and nonzero function on the unit sphere, and $\epsilon \in (0, \|\rho\|_\infty^{-1})$. We will speak about smooth *radial* deformations of the sphere.

Theorem 4.3 *Let A be a surface described above and let S denote its area. For the corresponding operator H_α with a constant interaction, $\alpha(x) = \alpha_0$ for all $x \in \mathcal{A}$, we have*

$$\overline{S} > \overline{[\hat{\alpha}]_c} \tag{4.2}$$

provided ϵ is small enough.

In contrast to the previous result this claim cannot be extended to general deformations. In Sec. 8 we will provide an example of an area-preserving deformation for which the inequality (4.2) is violated.

Notice also that the previous result does not help us here. The part (b) of Theorem 4.1 would imply Theorem 4.3 if $\overline{S} \ge C$ held in the vicinity of the sphere, however, we have mentioned already that this is not the case.

5 Capacity and Gauss variational principle

Let $\mathcal{A}$ be a closed smooth surface, which is regard here as a capacitor. It is well know that up to multiplicative constant there is a unique solution of the Laplace equation

$$\triangle u = 0\,,$$

in the exterior of $\mathcal{A}$, denoted by $\mathcal{A}^{\rm ext}$, such that u is constant on the surface and zero at infinity. This solution describes the potential and its negative normal derivative at the surface is the charge density,

$$\sigma(x) := -\frac{1}{4\pi}\frac{\partial u(x)}{\partial n_e}\,, \quad x \in \mathcal{A}\,.$$

If we normalize the potential to the unit charge on the capacitor $\mathcal{A}$ as expressed by the relation (4.1), the capacity C is related to the value on the surface by

$$u(x) = C^{-1}\,, \quad x \in \mathcal{A}\,. \tag{5.1}$$

If we expand the potential u into spherical harmonics,

$$u(x) = \frac{S_0}{r} + \sum_{n=1}^{\infty} S_n(\theta,\,\phi)\,r^{-n-1}\,,$$

where $S_n(\theta,\,\phi) = \sum_{m=-n}^{n} c_m Y_{mn}(\theta,\,\phi)$ with some coefficients $\{c_m\}$ is the contribution of order n, then by Green's theorem we infer that $S_0 = C^{-1}$. Moreover, by the well-known formula for the Green function one gets

$$u(x) = \int_{\mathcal{A}} \frac{\sigma(y)}{|x-y|}\,\mathsf{d}\nu(y)$$

and combining with the (5.1) we get that for $x \in \mathcal{A}$

$$\int_{\mathcal{A}} \frac{\sigma(y)}{|x-y|}\,\mathsf{d}\nu(y) = \frac{1}{C}. \tag{5.2}$$

We will also need one more characterization of the capacity, called Gauss variational principle [9, Chap. II.2.8]: let $\mathcal{A}$ be a capacitor and $\mu(x)$ a positive measurable function on A satisfying

$$\int_{\mathcal{A}} \mu(x)\mathsf{d}\nu(x) = 1\,,$$

then it holds

$$\int_{\mathcal{A}\times\mathcal{A}} \frac{\mu(x)\mu(y)}{|x-y|}\,\mathsf{d}\nu(x)\mathsf{d}\nu(y) \geq \frac{1}{C} \tag{5.3}$$

and the equality holds in this relation if and only if $\mu(x) = \sigma(x)$ – cf. (5.2).

6 Krein-like resolvent formula

As usual the best way to analyze spectral properties is to employ the resolvent. For singular Schrödinger operators it is made possible due to the existence of an explicit resolvent formula of Krein (or Birman-Schwinger) type. A detailed discussion can be found in [6, 7], here we limit ourselves to quoting a few simple facts. Given $z \in \mathbb{C} \setminus (0, \infty)$ we use the free resolvent with the kernel

$$G(z)(x,y) := \frac{1}{4\pi} \frac{\exp(i\sqrt{z}|x-y|)}{|x-y|}$$

to define a pair of operators,

$$\mathsf{G}(z) : L^2(\mathcal{A}, \mathsf{d}\nu) \to L^2(\mathbb{R}^3)\,, \quad \mathsf{G}(z)u(x) := \int_{\mathcal{A}} G(z)(x,\, y)u(y)\, \mathsf{d}\nu(y)$$

and

$$\mathsf{\Gamma}(z) : L^2(\mathcal{A}, \mathsf{d}\nu) \to L^2(\mathcal{A}, \mathsf{d}\nu)\,, \quad \mathsf{\Gamma}(z)u(x) := \int_{\mathcal{A}} G(z)(x,\, y)u(y)\, \mathsf{d}\nu(y)\,.$$

Then the indicated Krein-type resolvent formula reads

$$(\mathsf{H}_\alpha(x) - z)^{-1} = (\mathsf{H}_0 - z)^{-1} + \mathsf{G}(z)(I - \alpha\mathsf{\Gamma}(z))^{-1}\alpha\mathsf{G}(z)^*,$$

where α and I are the multiplication operator by $\alpha(\cdot)$ and the unit operator on $L^2(\mathcal{A}, \mathsf{d}\nu)$, respectively. Moreover, it yields a simple characterization of the point spectrum which generalizes to the singular case the Birman-Schwinger principle: $-\kappa^2$ is an eigenvalue if and only if the operator

$$I - \alpha\mathsf{\Gamma}(i\kappa)$$

has a nontrivial kernel, and its dimension coincides with the eigenvalue multiplicity. This suggests that H_α is critical if $I - \alpha\mathsf{\Gamma}(0)$ has a nontrivial kernel. We make this claim more precise in the following proposition.

Proposition 6.1 *Let $\hat{\alpha}(\cdot)$ be a relative interaction density on the surface $\mathcal{A}$ and $\Gamma \equiv \Gamma(0)$ the above defined operator with the kernel*

$$\Gamma(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}\,, \quad x, y \in \mathcal{A}\,,$$

then the critical strength is given by $[\hat{\alpha}]_c = \|\hat{\alpha}\Gamma\|^{-1}$.

Proof: First we check that for any measurable, bounded and positive $\hat{\alpha}(\cdot)$, the number $\|\hat{\alpha}\Gamma\|$ is an eigenvalue (naturally, the largest one) of the operator

$$\hat{\alpha}\Gamma \ : \quad L^2(\mathcal{A}, \mathsf{d}\nu) \to L^2(\mathcal{A}, \mathsf{d}\nu)$$

and the corresponding eigenfunction is positive. To see the first part of the statement, observe that the operator $\mathsf{G} \equiv \mathsf{G}(0)$ maps $L^2(\mathcal{A}, \mathsf{d}\nu)$ to the set of solutions of the equation $-\triangle u = 0$ away from $\mathcal{A}$. This means, in particular, that $\operatorname{Ran}\mathsf{G} \subset W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \mathcal{A})$ and by the well known properties of trace operator [10, Section VI.4] we have $\operatorname{Ran}\Gamma \subset W^{1,2}(\mathcal{A}, \mathsf{d}\nu)$ which is compactly embedded into $L^2(\mathcal{A}, \mathsf{d}\nu)$. This shows that Γ is a compact operator, thus $\hat{\alpha}\Gamma$ is compact as well and the upper edge of its spectrum, $\|\hat{\alpha}\Gamma\|$, is an eigenvalue. Furthermore, the operator $\hat{\alpha}\Gamma$ is positivity improving, hence the positivity of the corresponding eigenfunction follows in the standard way – cf. [11, Thm XII.44].

Consequently, the operator $\alpha_c\Gamma := \frac{1}{\|\hat{\alpha}\Gamma\|}\hat{\alpha}\Gamma$ corresponding to a given relative density $\hat{\alpha}(x)$ has the number one as its largest eigenvalue, and a positive eigenfunction ϕ is associated with it. We put

$$u(x) := (\mathsf{G}\phi)(x)\,,$$

then the explicit form of the kernel yields for any $x \in \mathcal{A}$

$$\begin{aligned} \frac{\partial u(x)}{\partial n_+} + \frac{\partial u(x)}{\partial n_-} &= -\phi(x) \\ u(x) = (\tau\mathsf{G}\phi)(x) = (\Gamma\phi)(x) &= \alpha_c(x)^{-1}\phi(x)\,, \end{aligned}$$

so the above function u belongs to $D(\mathsf{H}_{\alpha_c})$ being a positive solution to $\mathsf{H}_{\alpha_c(x)}u = 0$. Moreover, the asymptotic of $\mathsf{G}(x, y)$ shows that $u(x) \sim \mathit{const}\,|x|^{-1}$ at large distances being thus of the minimal growth at infinity. Proposition 3.1 then implies that H_{α_c} is critical and $[\hat{\alpha}]_c = \|\hat{\alpha}\Gamma\|^{-1}$ what we have set out to prove. ■

With this preliminary, the *proof of theorem 4.1* follows easily:
(a) Using the relation (5.2) we get

$$\frac{1}{4\pi}\,\sigma(x)\int \frac{\sigma(y)}{|x-y|}\,\mathsf{d}\nu(y) = \frac{1}{4\pi C}\,\sigma(x)$$

for $x \in \mathcal{A}$, and since $\sigma(x)$ is positive everywhere we conclude that $1/(4\pi C)$ is the maximal eigenvalue of $\sigma\Gamma$, and hence its norm. Consequently, we have

$$\overline{[\sigma]}_c := C\,.$$

(b) By a simple variational estimate we have that for any unit vector ψ

$$\|\hat{\alpha}\Gamma\| \geq \int_{\mathcal{A}} \overline{\psi(x)}\hat{\alpha}(x)(\Gamma\psi)(x)\,\mathsf{d}\nu(x)$$

and the equality holds if and only if ψ corresponds to the maximal eigenvalue. For the constant relative density $\hat{\alpha} := S^{-1}$ and $\psi = S^{-1/2}$ we can estimate the right-hand side using the Gauss variational principle,

$$\frac{\hat{\alpha}}{4\pi}\int_{\mathcal{A}\times\mathcal{A}} \frac{\overline{\psi(x)}\psi(y)}{|x-y|}\,\mathsf{d}\nu(x)\mathsf{d}\nu(y) = \frac{1}{4\pi}\int_{\mathcal{A}\times\mathcal{A}} \frac{S^{-2}}{|x-y|}\,\mathsf{d}\nu(x)\mathsf{d}\nu(y) \geq \frac{1}{4\pi}\frac{1}{C},$$

which gives $\|\hat{\alpha}\Gamma\|^{-1} \leq 4\pi C$, or equivalently, $\overline{[\hat{\alpha}]}_c \leq C$.

7 Local deformations of a sphere

In this section we will work in the spherical coordinates $(r,\,\theta,\,\phi)$. We start from a sphere; due to the natural scaling properties it is sufficient to consider the case of unit radius. It is straightforward to check that for such a sphere the critical interaction constant over the surface is $\alpha(x) = 1$ for all $x \in \mathcal{A}$, and the corresponding positive solution with minimal growth at infinity is $u = r^{-1}$ in the exterior of the sphere (denoted as $\mathcal{A}^{\mathrm{ext}}$) and $u = 1$ inside.

We consider a radially deformed surface $\mathcal{A}_\varepsilon$ defined by the equation

$$r(\theta,\,\phi) = 1 + \epsilon\rho(\theta,\,\phi),$$

where ρ is a fixed smooth and nonzero function on the unit sphere, and $\epsilon \in (0, \|\rho\|_\infty^{-1})$, in particular, $\mathcal{A}_0$ is the sphere mentioned above. Our aim is to find a perturbed solution u_ϵ and the constant interaction, $\alpha(x) = \alpha_\epsilon$ for all $x \in \mathcal{A}$, such that the solution will remain positive and bounded, in other words, α_ϵ would be the critical strength for the said constant singular interaction on the surface $\mathcal{A}_\epsilon$. We are going to show that for a nontrivial $\rho(\theta,\,\phi)$ the inequality $\overline{S}_\epsilon\alpha_\epsilon < 1$ is valid for small and nonzero ϵ, where $\overline{S}_\epsilon$ is the corresponding surface radius. This will prove Theorem 4.3, because $[\hat{\alpha}]_c = \alpha_\epsilon S_\epsilon$ would then yield $\overline{[\hat{\alpha}]}_c = \overline{S}_\epsilon^2\alpha_\epsilon < \overline{S}_\epsilon$ for such an ϵ.

As usual in such situations the method is to employ asymptotic expansion in powers of ϵ. We put

$$\rho(\theta,\,\phi) = \sum_n X_n(\theta,\,\phi),$$

where X_n is a spherical harmonic of order n and we will look for a solution u_ϵ of $-\triangle u = 0$ away from the surface, such that it is continuous on $\mathcal{A}_\epsilon$ and has the corresponding jump in normal derivative there, i.e.

$$\frac{\partial u_\epsilon(x)}{\partial n_e} + \frac{\partial u_\epsilon(x)}{\partial n_i} = -\alpha_\epsilon u_\epsilon(x) \tag{7.1}$$

holds for $x \in \mathcal{A}_\epsilon$. We will seek it in the form

$$\begin{aligned}
u_\epsilon^{\mathrm{ext}}(r,\theta,\,\phi) &= \frac{1}{r} + \epsilon \sum_{n=1}^{\infty} S_n^{(1)}(\theta,\,\phi) r^{-n-1} + \epsilon^2 \sum_{n=1}^{\infty} S_n^{(2)}(\theta,\,\phi) r^{-n-1} + \mathcal{O}(\epsilon^3)\,,\\
u_\epsilon^{\mathrm{in}}(r,\theta,\,\phi) &= 1 + \epsilon \sum_{n=0}^{\infty} R_n^{(1)}(\theta,\,\phi) r^{n} + \epsilon^2 \sum_{n=0}^{\infty} R_n^{(2)}(\theta,\,\phi) r^{n} + \mathcal{O}(\epsilon^3)\,,\\
\alpha_\epsilon &= 1 + \epsilon\alpha^{(1)} + \epsilon^2\alpha^{(2)} + \mathcal{O}(\epsilon^3)\,,
\end{aligned}$$

where $S_n^{(i)}$, $R_n^{(i)}$ are spherical harmonics of order n. Such an Ansatz will guarantee that $u_\epsilon^{\mathrm{ext}}$ (u_ϵ^{in}) is bounded solution of $-\triangle u = 0$ in the exterior (respectively, interior) of the surface, and furthermore, that $u_\epsilon^{\mathrm{ext}}$ has the $1/r$ asymptotics at infinity. It is convenient to allow the summation in the definition of $u_\epsilon^{\mathrm{ext}}$ to run also from zero by putting $S_0^{(i)}(\theta,\,\phi) = 0$; this will allow us to write some formulæ below in a more compact form.

Due to the nature of the deformation each element of $\mathcal{A}_\epsilon$ can be uniquely characterized by $x = (\theta,\,\phi)$. The corresponding surface element equals

$$\begin{aligned}
\mathsf{d}\nu_\epsilon(\theta,\,\phi) = \Bigg\{ & 1 + 2\epsilon\rho(\theta,\,\phi) + \epsilon^2\rho(\theta,\,\phi)^2\\
& + \frac{1}{2}\epsilon^2 \left(\left(\frac{\partial\rho(\theta,\,\phi)}{\partial\phi}\right)^2 + \frac{1}{\sin^2\phi}\left(\frac{\partial\rho(\theta,\,\phi)}{\partial\theta}\right)^2 \right) \Bigg\} \sin\phi\,\mathsf{d}\theta\,\mathsf{d}\phi + \mathcal{O}(\epsilon^3)\,,
\end{aligned} \tag{7.2}$$

and for the exterior normal vector n_ϵ to $\mathcal{A}_\epsilon$ we find

$$\begin{aligned}
\mathsf{d}\nu_\epsilon(\theta,\,\phi) n_\epsilon(\theta,\,\phi) &= (1 + \epsilon^2\rho(\theta,\,\phi)^2)\,\hat{r}\,\cos\phi\\
&\quad - \epsilon(1 + \epsilon\rho(\theta,\,\phi))\frac{\partial\rho(\theta,\,\phi)}{\partial\phi}\,\hat{\phi}\,\sin\phi - (1 + \epsilon\rho(\theta,\,\phi))\frac{\partial\rho(\theta,\,\phi)}{\partial\theta}\,\hat{\theta}\,,
\end{aligned}$$

where we have introduced the standard unit vector triple $(\hat{r},\, \hat{\theta},\, \hat{\phi})$ at the surface point characterized by x. Expanding $1/r$ we find

$$\begin{aligned} u_\epsilon^{\mathrm{ext}}(x) &= 1-\epsilon\sum_{n=0}^{\infty} X_n(\theta,\, \phi)+\epsilon\sum_{n=0}^{\infty} S_n^{(1)}(\theta,\, \phi)+\mathcal{O}(\epsilon^2)\,, \\ u_\epsilon^{\mathrm{in}}(x) &= 1+\epsilon\sum_{n=0}^{\infty} R_n^{(1)}(\theta,\, \phi)+\mathcal{O}(\epsilon^2)\,, \end{aligned}$$

hence the continuity condition at the surface gives

$$S_n^{(1)} = R_n^{(1)} + X_n\,. \tag{7.3}$$

As for the normal derivatives, in the first order we may consider only the derivative in the radial direction, which gives

$$\begin{aligned} \nabla_r u_\epsilon^{\mathrm{ext}}(x) &= -1+2\epsilon\sum_{n=0}^{\infty} X_n(\theta,\, \phi)-\epsilon\sum_{n=0}^{\infty}(n+1)S_n^{(1)}(\theta,\, \phi)+\mathcal{O}(\epsilon^2)\,, \\ \nabla_r u_\epsilon^{\mathrm{in}}(x) &= \epsilon\sum_{n=0}^{\infty} nR_n^{(1)}(\theta,\, \phi)+\mathcal{O}(\epsilon^2)\,, \end{aligned}$$

leading by comparison of the first-order terms to the condition

$$2X_n-(n+1)S_n^{(1)}-nR_n^{(1)} = -R_n^{(1)}-\alpha^{(1)}\,. \tag{7.4}$$

Now the equations (7.3) and (7.4) yield $\alpha^{(1)}=-X_0$ and $R_0^{(1)}=-X_0$; recall that by convention we have $S_0^{(1)}=0$. Furthermore, for $n\geq 1$ we get

$$\begin{aligned} S_n^{(1)} &= \frac{1+n}{2n}X_n\,, \\ R_n^{(1)} &= \frac{1-n}{2n}X_n\,. \end{aligned} \tag{7.5}$$

Next we employ Green's theorem by which we have

$$\begin{aligned} \int_{\mathcal{A}_\epsilon^{\mathrm{ext}}}(\triangle u_\epsilon)(x)\,\mathsf{d}^3x &= -\int_{\mathcal{A}_\epsilon}\frac{\partial u_\epsilon(x)}{\partial n_e}\,\mathsf{d}\nu_\epsilon(x)-4\pi\,, \\ \int_{\mathcal{A}_\epsilon^{\mathrm{in}}}(\triangle u_\epsilon)(x)\,\mathsf{d}^3x &= -\int_{\mathcal{A}_\epsilon}\frac{\partial u_\epsilon(x)}{\partial n_i}\,\mathsf{d}\nu_\epsilon(x)\,, \end{aligned}$$

where the left-hand sides vanish by assumption. Summing then the equations and using the boundary condition (2.1) we get

$$\alpha_\epsilon \int_{\mathcal{A}_\epsilon} u_\epsilon(x)\,\mathsf{d}\nu_\epsilon(x) = 4\pi\,.$$

Substituting now the expansion (7.2), the above Ansatz up to the second order and the coefficients (7.5) we arrive at the condition

$$\alpha^{(2)} = X_0^2 - \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{2}\left(n^2+\frac{1}{n}\right)I_n\,, \tag{7.6}$$

where we have denoted $I_n := \int_{\mathcal{A}_0} X_n^2(\theta,\,\phi)\,\mathsf{d}\nu_0(\theta,\,\phi)$ and used the orthogonality of X_n together with the known angular-momentum formula

$$\int_{\mathcal{A}_0}\left(\left(\frac{\partial X_n(\theta,\,\phi)}{\partial\phi}\right)^2+\frac{1}{\sin^2\phi}\left(\frac{\partial X_n(\theta,\,\phi)}{\partial\theta}\right)^2\right)\sin\phi\,\mathsf{d}\phi\mathsf{d}\theta = n(n+1)I_n.$$

Using the derived coefficients, we get an explicit asymptotic formula for α_ϵ,

$$\alpha_\epsilon = 1-\epsilon X_0+\epsilon^2X_0^2-\epsilon^2\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{2}\left(n^2+\frac{1}{n}\right)I_n+\mathcal{O}(\epsilon^3)\,.$$

On the other hand, there is a well known formula [9, Sec. 1.33] for the surface radius $\overline{S}_\epsilon$, namely

$$\overline{S}_\epsilon = 1+\epsilon X_0+\frac{1}{4\pi}\epsilon^2\sum_{n=1}^{\infty}\left(\frac{n^2+n+4}{4}\right)I_n+\mathcal{O}(\epsilon^3)\,;$$

combining these two expressions we get

$$\alpha_\epsilon\overline{S}_\epsilon = 1-\frac{1}{4\pi}\epsilon^2\sum_{n=1}^{\infty}\left(\frac{1}{2}n^2+\frac{1}{2n}-\frac{n^2+n+2}{4}\right)I_n+\mathcal{O}(\epsilon^3)\,.$$

Since the I_n's are non-negative, it is easy to see that $\alpha_\epsilon\overline{S}_\epsilon\le 1$, and moreover, that the inequality is strict unless I_1 is the only nontrivial term, i.e. $\rho(\theta,\,\phi) = X_1(\theta,\,\phi)$. To prove that even in this case a small nontrivial deformation leads to diminishing of the product $\alpha_\epsilon\overline{S}_\epsilon$ we need to compute the next term in the asymptotic expansion, which means here the fourth one because the third is zero. This can be done explicitly by putting

$$X_1(\theta,\,\phi) := AY_0(\theta,\,\phi) + BY_{-1}(\theta,\,\phi) + CY_1(\theta,\,\phi)\,,$$

where Y_i is the standard basis of the first-order spherical harmonics, explicitly

$$\begin{aligned} Y_0(\theta,\,\phi) &= \sqrt{\frac{3}{4\pi}}\,\cos\phi\,, \\ Y_1(\theta,\,\phi) &= \sqrt{\frac{3}{4\pi}},\sin\phi\cos\theta\,, \\ Y_{-1}(\theta,\,\phi) &= \sqrt{\frac{3}{4\pi}}\,\sin\phi\sin\theta\,. \end{aligned}$$

After a lengthy but tractable computation we arrive at the expression

$$\alpha_\epsilon\overline{S}_\epsilon = 1 - \epsilon^4\frac{3(A^2+B^2+C^2)^2}{20\pi} + \mathcal{O}(\epsilon^5)\,,$$

which proves the desired claim.

8 A large deformation example

The aim of this section is to show that the local result of Theorem 4.3 does not extend to general surface-preserving deformations: we are going to show that for any fixed interaction strength α_0 there is a surface $\mathcal{A}$ of unit area such that a constant singular interaction, $\alpha(x) := \alpha_0$, does not induce existence of bound states. The example idea is to construct a surface with a large diameter, that is, to examine the situation when the capacity is much greater than the surface radius.

The way to achieve this goal is to show that there are surfaces for which the strict inequality

$$\|(\alpha\Gamma)^2\| < \|\alpha\Gamma\|$$

holds, where Γ is the Birman-Schwinger-type operator from Proposition 6.1. Since $\alpha\Gamma$ is strictly positive by assumption, this will yield the inequality $\|\alpha\Gamma\| < 1$ implying by the said proposition that the corresponding operator H_α has no bound states.

We have demonstrated that $\|\alpha\Gamma\|$ is an eigenvalue of the operator $\alpha\Gamma$ corresponding to a positive eigenfunction which, in particular, implies

$$\|\alpha\Gamma\| = \sup_{f\in L^2(\mathcal{A},\mathsf{d}\nu),\, f>0} \frac{\|\alpha\Gamma f\|}{\|f\|}\,;$$

note that the supremum is taken over positive functions only. Next we employ a simple geometric inequality

$$\begin{aligned}\frac{1}{|x-y||y-z|} &= \left(\frac{1}{|x-y|}+\frac{1}{|y-z|}\right)\frac{1}{|x-y|+|y-z|}\\ &\le \left(\frac{1}{|x-y|}+\frac{1}{|y-z|}\right)\frac{1}{|x-z|}\end{aligned}$$

which allows us to estimate

$$\begin{aligned}((\alpha\Gamma)^2 f)(x) &= \alpha_0^2\frac{1}{(4\pi)^2}\int_{A\times A}\frac{1}{|x-y|}\frac{1}{|y-z|}\,f(z)\,\mathsf{d}\nu(y)\mathsf{d}\nu(z)\\ &\le \frac{\alpha_0^2}{(4\pi)^2}\int_{A\times A}\left(\frac{1}{|x-y|}+\frac{1}{|y-z|}\right)\frac{1}{|x-z|}\,f(z)\,\mathsf{d}\nu(y)\mathsf{d}\nu(z)\\ &\le 2\frac{\alpha_0^2}{(4\pi)}\,\|\Gamma\|_\infty\int_A\frac{1}{|x-z|}\,f(z)\,\mathsf{d}\nu(z) = 2\alpha_0\|\Gamma\|_\infty(\alpha\Gamma f)(x),\end{aligned}$$

for any f nonnegative, where we have put

$$\|\Gamma\|_\infty := \frac{1}{4\pi}\sup_{x\in\mathcal{A}}\int_{\mathcal{A}}\frac{1}{|x-y|}\,\mathsf{d}\nu(y)\,;$$

the integral obviously converges for any $x\in\mathcal{A}$ and is continuous w.r.t. this variable, so $\|\Gamma\|_\infty$ is finite. By taking the supremum over all positive functions $f\in L^2(\mathcal{A},\mathsf{d}\nu)$ we then get

$$\|(\alpha\Gamma)^2\| \le 2\alpha_0\|\Gamma\|_\infty\|\alpha\Gamma\|\,. \tag{8.1}$$

Now we will show that the quantity $\|\Gamma\|_\infty$ tends to zero with the increasing diameter of the surface $\mathcal{A}$. To do this we choose a smooth *positive* function

$$f:(-1,\,1)\to\mathbb{R}\,,\quad 2\pi\int_{-1}^{1} f(v)\mathsf{d}v = 1\,,$$

such that $f(\pm 1) = 0$ and $f'(\pm 1) = \pm\infty$ and define a family of surfaces $\mathcal{A}_\epsilon$ by the equations

$$\begin{aligned} x &= \epsilon f(v)\cos u\,, \\ y &= \epsilon f(v)\sin u\,, \\ z &= \epsilon^{-1} v\,, \end{aligned}$$

where $v \in (-1,\, 1)$, $u \in (0,\, 2\pi)$. Thus we have

$$\mathsf{d}\nu_\epsilon(u,\, v) = f(v)\sqrt{1+\epsilon^4 f'(v)^2}\,;$$

as a consequence, the surface area satisfies $S_\epsilon > 1$ for all $\epsilon > 0$ and approaches one in the limit $\epsilon \to 0$. On the other hand for the quantity $\|\Gamma_\epsilon\|_\infty$ we have

$$\begin{aligned} 4\pi\|\Gamma_\epsilon\|_\infty = \sup_{x,y}\int_{-1}^{1}\mathsf{d}v\int_0^{2\pi}\mathsf{d}u \\ \frac{f(v)\sqrt{1+\epsilon^4 f'(v)^2}}{\sqrt{\epsilon^2(f(v)\cos u - f(x)\cos y)^2 + \epsilon^2(f(v)\sin u - f(x)\sin y)^2 + \epsilon^{-2}(v-x)^2}} \\ \le \sup_x \int_{-1}^{1}\mathsf{d}v \frac{M}{\sqrt{N\epsilon^2+\epsilon^{-2}(v-x)^2}}\,, \end{aligned}$$

for suitable, sufficiently larger constants M, N which depend on f only. It is easy to see that the last integral is maximized for $x=0$ giving

$$\|\Gamma_\epsilon\|_\infty \le 2M\epsilon\,\mathrm{arcsinh}(\sqrt{N}\epsilon^2)^{-1}$$

and since the right-hand side behaves as $-4M\epsilon\ln\epsilon + \mathcal{O}(\epsilon)$, we conclude that $\|\Gamma_\epsilon\|_\infty \to 0$ as $\epsilon \to 0$. It is therefore possible to choose ϵ in such a way that $2\alpha_0\|\Gamma_\epsilon\|_\infty < 1$ which by means of (8.1) implies $\|(\alpha\Gamma_\epsilon)^2\| < \|\alpha\Gamma_\epsilon\|$, and consequently, the operator H_α corresponding to elongated enough surface $\mathcal{A}_\epsilon$ has empty discrete spectrum.

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