EIGENFUNCTIONS AT THE THRESHOLD ENERGIES OF MAGNETIC DIRAC OPERATORS WITH POSITIVE MASS

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Abstract. We propose a framework for supersymmetric Dirac operators to have threshold energies. We apply the obtained results to magnetic Dirac operators with positive mass in order to investigate the asymptotic behaviors at infinity of the threshold eigenfunctions. It turns out that the results of most of the existing works on the zero modes of the Weyl-Dirac operators remain to hold for the threshold eigenfunctions of the magnetic Dirac operators with positive mass.

1. Introduction

This paper is concerned with eigenfunctions at the threshold energies of Dirac operators with vector potentials

$$H_A = \alpha \cdot (D - A(x)) + m\beta, \quad D = \frac{1}{i}\nabla_x, \ \ x \in \mathbb{R}^3. \tag{1.1}$$

Here $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \qquad (j = 1,\, 2,\, 3)$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}.$$

The constant m is assumed to be positive.

Throughout the present paper we assume that each component of the vector potential $A(x) = (A_1(x),\, A_2(x),\, A_3(x))$ is a real-valued measurable function. In addition to this, we shall later impose several different sets of assumptions on $A(x)$ under which the operator $-\alpha \cdot A(x)$ is relatively compact with respect to the free Dirac operator $H_0 = \alpha \cdot D + m\beta$. Therefore,

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under any set of assumptions to be made, the essential spectrum of the Dirac operator H_A is given by the union of the intervals $(-\infty, -m]$ and $[m, +\infty)$:

$$\sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, +\infty).$$

We shall later see that the discrete spectrum of H_A is empty. In other words, there are no discrete eigenvalues with finite multiplicity in the spectral gap $(-m, m)$. Under each set of the assumptions in the present paper, we thus have

$$\sigma(H_A) = \sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, +\infty).$$

By the threshold energies of H_A, we mean the values $\pm m$, the edges of the essential spectrum $\sigma_{\text{ess}}(H_A)$. It is a common practice in quantum scattering theory that the edge(s) of the essential spectrum of a quantum Hamiltonian is normally excluded. On the other hand, these values of Dirac operators are of particular importance and of interest from the physics point of view. See Pickl and Dürr [17] and Pickl [16], where they investigate generalized eigenfunctions near the edges $\pm m$ of the essential spectrum of H_A with the emphasis on the famous relativistic effect of pair creation.

The aim of the present paper is twofold. First, we consider a class of supersymmetric Dirac operators (a class of abstract Dirac operators; see (1.2) below) and investigate the eigenspaces corresponding to the energies $\pm m$. The eigenspaces corresponding to the energies $\pm m$ of supersymmetric Dirac operators seem to have not been treated in the literature (see Remark 2.1). In addition, we propose a framework in which the supersymmetric Dirac operators possess the threshold energies $\pm m$ at the edges of the essential spectrum. Second, we apply the established results to the Dirac operators (1.1) in order to describe the asymptotic behaviors at infinity of eigenfunctions of the operators (1.1) corresponding to the threshold eigenvalues $\pm m$, and investigate the set of vector potentials which give rise to eigenfunctions of the operators (1.1) corresponding to the threshold eigenvalues $\pm m$. Actually, it turns out that the asymptotic behaviors of eigenfunctions of the operators (1.1) at the threshold eigenvalues are exactly the same as those of eigenfunctions of the Weyl-Dirac operator

$$T_A = \sigma \cdot (D - A(x))$$

at the zero eigenvalue. In particular, the asymptotic behaviors of eigenfunctions of the operators (1.1) at the threshold eigenvalues at $\pm m$ are independent of m.

Explanations should be in order. The supersymmetric Dirac operator H which we shall consider in the present paper is defined as follows:

$$H := \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{on } \mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-, \tag{1.2}$$

where T is a densely defined operator from a Hilbert space $\mathcal{H}_+$ to another Hilbert $\mathcal{H}_-$, and the identity operators in $\mathcal{H}_+$ and $\mathcal{H}_-$ are both denoted by I with an abuse of notation. We should like to mention that T does not need to be a closed operator, and that T^* does not need to be densely defined; see Theorems 2.1, 2.2 and Corollary 2.1 in section 2. Also, we would like to mention that we do not need the Foldy-Wouthuysen transformation, which is a major tool in the standard theory of the supersymmetric Dirac operator. With these respects, our approach is different from the standard theory of the supersymmetric Dirac operators; see Chapter 5 of the book by Thaller [24] for the standard theory of the supersymmetric Dirac operator.

By the supercharge Q and the involution τ, we mean the first and the second term, respectively, on the right hand side of (1.2):

$$Q = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.3}$$

Note that the inner product of the Hilbert space $\mathcal{K}$ is defined as follows:

$$(f,\, g)_{\mathcal{K}} := (\varphi^+,\, \psi^+)_{\mathcal{H}_+} + (\varphi^-,\, \psi^-)_{\mathcal{H}_-} \tag{1.4}$$

for

$$f = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}, \quad g = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \in \mathcal{K}. \tag{1.5}$$

Our main result on the supersymmetric Dirac operators asserts that $\mathrm{Ker}(H-m)$ (resp. $\mathrm{Ker}(H+m)$), *i.e.* the eigenspace of H corresponding to the eigenvalue m (resp. $-m$), is given as the direct sum of $\mathrm{Ker}(T)$ (resp. $\mathrm{Ker}(T^*)$) and the zero space $\{0\}$; see Corollary 2.1. It straightforward from our results that $\mathrm{Ker}(H-m)$ (resp. $\mathrm{Ker}(H+m)$) is linearly isomorphic to $\mathrm{Ker}(T)$ (resp. $\mathrm{Ker}(T^*)$). In particular, it follows that $\mathrm{Ker}(H \pm m)$ are independent of m.

It is of special interest to consider the case where the Hilbert space $\mathcal{H}_+$ coincides with $\mathcal{H}_-$ and T is self-adjoint ($T^* = T$). In this case, our result says, in addition to symmetry between $(-\infty,\, -m)$ and $(m,\, \infty)$, that the spectrum of H at $\pm m$ is symmetric with respect to 0 in the following sense: (1) m is an eigenvalue H if and only if $-m$ is an eigenvalue of H; (2) the multiplicity of the eigenvalue m is the same as that of the eigenvalue $-m$. As mentioned above, we propose a framework under which $\sigma(H)$ is given by the union $(-\infty,\, -m] \cup [m,\, \infty)$ while $\sigma(T) \supset (0,\, \infty)$. Thus $\pm m$ are always threshold energies of the supersymmetric Dirac operator H under the assumption $\sigma(T) \supset (0,\, \infty)$. It will become clear in section 2 that the involution τ causes the gap $(-m,\, m)$.

Note that in the standard theory of supersymmetric Dirac operators, symmetry with respect to 0 of the spectrum of the operator of the form (1.2) is only assured between the intervals $(-\infty,\, -m)$ and (m, ∞), excluding $\pm m$; see Thaller [24, p. 155, Corollary 5.14]. We shall recall the discussion of

[24, §5.6.1] in our context: A supersymmetric Dirac operator H of the form (1.2) is unitarily equivalent to the operator

$$\begin{pmatrix} \sqrt{T^*T+m^2} & 0 \\ 0 & -\sqrt{TT^*+m^2} \end{pmatrix}.$$

It follows that $TT^* + m^2 = S(T^*T + m^2)S^*$, where S is a partial isometry with initial subspace $\mathcal{H} \ominus \mathrm{Ker}(T)$. (This is the reason why symmetry mentioned above has to exclude $\pm m$.)

We should like to emphasize the broad applicability of our results on the supersymmetric Dirac operator. Namely, thanks to the generality of our results on the supersymmetric Dirac operator, we are able to utilize most of the existing works on the zero modes of the Weyl-Dirac operator T_A (cf. [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15]) for the purpose of investigating the threshold eigenfunctions of the Dirac operator H_A. Among these works, we shall utilize ones by Loss and Yau [15], by Adami, Muratori and Nash [1], by Balinsky and Evans [5], and by Elton [10]. Accordingly, we shall get deep and multifaceted understandings of the threshold eigenfunctions of the magnetic Dirac operator H_A defined by (1.1) .

2. Supersymmetric Dirac operators

This section is devoted to a discussion about spectral properties of a class of supersymmetric Dirac operators. We should like to emphasize that our approach appears to be in the reverse direction in the sense that we start with two Hilbert spaces, and introduce a supersymmetric Dirac operator on the direct sum of the two Hilbert spaces. We find this approach convenient for our purpose.

Let $\mathcal{H}_\pm$ be Hilbert spaces, and let $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Note that the inner product of $\mathcal{K}$ is defined as in (1.4), (1.5). The main object in this section is a supersymmetric Dirac operator H in $\mathcal{K}$:

$$H := \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.1}$$

where T is a densely defined operator from $\mathcal{H}_+$ to $\mathcal{H}_-$, m is a positive constant, and the identity operators in $\mathcal{H}_+$ and $\mathcal{H}_-$ are both denoted by I.

We now state the main results in this section, which reveal the nature of eigenvectors of the supersymmetric Dirac operator (2.1) at the eigenvalues $\pm m$.

Theorem 2.1. *Suppose that T is a densely defined operator from $\mathcal{H}_+$ to $\mathcal{H}_-$. Let H be a supersymmetric Dirac operator defined by* (2.1).

(i) *If $f = {}^t(\varphi^+, \varphi^-) \in \mathrm{Ker}(H - m)$, then $\varphi^+ \in \mathrm{Ker}(T)$ and $\varphi^- = 0$.*

(ii) *Conversely, if* $\varphi^+ \in \mathrm{Ker}(T)$*, then* $f = {}^t(\varphi^+,\, 0) \in \mathrm{Ker}(H - m)$*.*

Theorem 2.2. *Assume that* T *and* H *are the same as in Theorem 2.1.*

(i) *If* $f = {}^t(\varphi^+,\, \varphi^-) \in \mathrm{Ker}(H + m)$*, then* $\varphi^+ = 0$ *and* $\varphi^- \in \mathrm{Ker}(T^*)$*.*

(ii) *Conversely, if* $\varphi^- \in \mathrm{Ker}(T^*)$*, then* $f = {}^t(0,\, \varphi^-) \in \mathrm{Ker}(H + m)$*.*

As immediately consequences, we have

Corollary 2.1. *Assume that* T *and* H *are the same as in Theorem 2.1. Then*

(i) $\mathrm{Ker}(H - m) = \mathrm{Ker}(T) \oplus \{0\}$, $\dim(\mathrm{Ker}(H - m)) = \dim(\mathrm{Ker}(T))$.

(ii) $\mathrm{Ker}(H + m) = \{0\} \oplus \mathrm{Ker}(T^*)$, $\dim(\mathrm{Ker}(H + m)) = \dim(\mathrm{Ker}(T^*))$.

We should like to emphasize that comparison between Corollary 2.1 above and [24, p. 144, (5.23)] (i.e. $\mathrm{Ker}(Q) = \mathrm{Ker}(T) \oplus \mathrm{Ker}(T^*)$) indicates the importance of the presence of the involution

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.2}$$

Also, we should like to emphasize that Corollary 2.1 has a significant implication: the eigenspaces of H corresponding to the eigenvalue $\mp m$ are independent of m.

Remark 2.1. To our knowledge, Thaller [24, p. 195, Theorem 7.1] is the first to mention the properties (i), (ii) in Theorem 2.1, but only for the magnetic Dirac operator (1.1). It should be noted that in his theorem mentioned above, $\pm m$ are not necessarily the thresholds energies of the magnetic Dirac operator; compare this with Theorem 2.3 below. Also, we would like to emphasize that abstractness is essential in Theorems 2.1, 2.2 and Corollary 2.1, which enable us to apply these results to the magnetic Dirac operator under various assumptions on the vector potential in the later sections: see sections 3–6. In fact, we shall even deal with vector potentials with local singularities in section 5.

Proof of Theorem 2.1 . We first prove assertion (i). Let $f = {}^t(\varphi^+,\, \varphi^-) \in \mathrm{Ker}(H - m)$. We then have

$$\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi_- \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi_- \end{pmatrix} = m \begin{pmatrix} \varphi^+ \\ \varphi_- \end{pmatrix}, \tag{2.3}$$

hence

$$\begin{cases} T^*\varphi^- + m\varphi^+ & = m\varphi^+ \\ T\varphi^+ - m\varphi^- & = m\varphi^-, \end{cases} \tag{2.4}$$

which immediately implies that $T^*\varphi^- = 0$ and $T\varphi^+ = 2m\varphi^-$. It follows that

$$\begin{aligned} \|T\varphi^+\|^2_{\mathcal{H}_-} &= (T\varphi^+, T\varphi^+)_{\mathcal{H}_-} \\ &= (T\varphi^+, 2m\varphi^-)_{\mathcal{H}_-} \\ &= (\varphi^+, 2m\, T^*\varphi^-)_{\mathcal{H}_+} \\ &= 0. \end{aligned} \tag{2.5}$$

Thus we see that $\varphi^+ \in \mathrm{Ker}(T)$, and that $\varphi^- = (2m)^{-1}\, T\varphi^+ = 0$.

We next prove assertion (ii). Let $\varphi^+ \in \mathrm{Ker}(T)$ and put $f := {}^t(\varphi^+, 0)$. Then it follows that $Hf = {}^t(m\varphi^+, T\varphi^+) = m\, {}^t(\varphi^+, 0) = mf$. □

We omit the proof of Theorem 2.2, which is quite similar to that of Theorem 2.1.

In the rest of this paper, we only deal with the case where $\mathcal{H}_+ = \mathcal{H}_- := \mathcal{H}$ and T is a self-adjoint operator in $\mathcal{H}$. In this case, the supersymmetric operator H becomes of the form

$$H = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.6}$$

in the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, and it follows from Theorems 2.1 and 2.2 that the operator H of the form (2.6) possesses of an important equivalence:

$$\begin{aligned} T \text{ has a zero mode} &\iff m \text{ is an eigenvalue of } H \\ &\iff -m \text{ is an eigenvalue of } H. \end{aligned} \tag{2.7}$$

Here we say that T has a *zero mode* if T has an eigenvector corresponding to the eigenvalue 0. In other words, the fact that T has a zero mode is equivalent to the fact that 0 is an eigenvalue of T. Furthermore, Theorems 2.1 and 2.2 imply the following equivalence for a zero mode φ of T:

$$T\varphi = 0 \iff H\begin{pmatrix} \varphi \\ 0 \end{pmatrix} = m\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \iff H\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = -m\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \tag{2.8}$$

The equivalences (2.7) and (2.8) are particularly interesting in the context of the following theorem, where $\pm m$ are the thresholds.

Theorem 2.3. *Let T be a self-adjoint operator in the Hilbert space $\mathcal{H}$. Suppose that $\sigma(T) \supset [0, +\infty)$. Then*

$$\sigma(H) = (-\infty, -m] \cup [m, +\infty).$$

In particular, $\sigma_d(H) = \emptyset$, i.e., the set of discrete eigenvalues of H with finite multiplicity is empty.

Proof. It follows from (2.6) that $\mathcal{D}(H^2) = \mathcal{D}(T^2) \oplus \mathcal{D}(T^2)$ and that

$$H^2 = \begin{pmatrix} T^2 + m^2 I & 0 \\ 0 & T^2 + m^2 I \end{pmatrix} \geq m^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.9}$$

This inequality implies that $\sigma(H) \subset (-\infty, -m] \cup [m, +\infty)$.

To complete the proof, we shall prove the fact that $\sigma(H) \supset (-\infty, -m] \cup [m, +\infty)$. To this end, suppose $\lambda_0 \in (-\infty, -m] \cup [m, +\infty)$ be given. Since $\sqrt{\lambda_0^2 - m^2} \geq 0$, we see, by the assumption of the theorem, that $\sqrt{\lambda_0^2 - m^2} \in \sigma(T)$. Therefore, we can find a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that

$$\|\psi_n\|_{\mathcal{H}} = 1, \quad \psi_n \in \operatorname{Ran}\big(E_T(\nu - \frac{1}{n}, \nu + \frac{1}{n})\big), \quad \nu_0 := \sqrt{\lambda_0^2 - m^2} \tag{2.10}$$

for each n, where $E_T(\cdot)$ is the spectral measure associated with T:

$$T = \int_{-\infty}^{\infty} \lambda \, dE_T(\lambda). \tag{2.11}$$

Here we have used a basic property of the spectral measure: see, for example, Reed and Simon [20, p. 236, Proposition]. It is straightforward to see that

$$\|(T^2 - \lambda_0^2 + m^2)\psi_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty \tag{2.12}$$

and that

$$\begin{aligned} &\|(T + \lambda_0 + m)\psi_n\|_{\mathcal{H}}^2 \to (\nu_0 + \lambda_0 + m)^2 \text{ as } n \to \infty, \\ &\|(T + \lambda_0 - m)\psi_n\|_{\mathcal{H}}^2 \to (\nu_0 + \lambda_0 - m)^2 \text{ as } n \to \infty. \end{aligned} \tag{2.13}$$

We shall construct a sequence $\{f_n\} \subset \mathcal{D}(H) = \mathcal{D}(T) \oplus \mathcal{D}(T)$ satisfying $\|f_n\|_{\mathcal{K}} = 1$ and $\|(H - \lambda_0)f_n\|_{\mathcal{K}} \to 0$ as $n \to \infty$. To this end, we first put

$$g_n := \begin{pmatrix} \psi_n \\ \psi_n \end{pmatrix}. \tag{2.14}$$

We next see that

$$\|(H + \lambda_0)g_n\|_{\mathcal{K}} \to \sqrt{2(\nu_0 + \lambda_0)^2 + 2m^2} \text{ as } n \to \infty \tag{2.15}$$

by using (2.13) and

$$(H + \lambda_0)g_n = \begin{pmatrix} (T + \lambda_0 + m)\psi_n \\ (T + \lambda_0 - m)\psi_n \end{pmatrix}.$$

It is clear that (2.15) enables us to define

$$f_n := \frac{1}{\|(H + \lambda_0)g_n\|_{\mathcal{K}}}(H + \lambda_0)g_n. \tag{2.16}$$

Since we have

$$(H - \lambda_0)(H + \lambda_0)g_n = \begin{pmatrix} (T^2 - \lambda_0^2 + m^2)\psi_n \\ (T^2 - \lambda_0^2 + m^2)\psi_n \end{pmatrix}, \tag{2.17}$$

we can conclude from (2.16), (2.17) and (2.12) that $\|(H - \lambda_0)f_n\|_{\mathcal{K}} \to 0$ as $n \to \infty$. Thus we have shown that $\lambda_0 \in \sigma(H)$. □

In all the later sections, we shall apply the obtained results on the supersymmetric Dirac operator to the Dirac operator H_A of the form (1.1) in the Hilbert space $\mathcal{K} = \left[L^2(\mathbb{R}^3)\right]^4$, where we take T to be the Weyl-Dirac operator

$$T_A = \sigma \cdot (D - A(x)) \tag{2.18}$$

acting in the Hilbert space $\mathcal{H} = \big[L^2(\mathbb{R}^3)\big]^2$.

3. Vector potentials yielding threshold eigenfunctions

This section is devoted to examples of vector potentials $A(x)$ which yield eigenfunctions of Dirac operators $H_A = \alpha\cdot(D - A(x)) + m\beta$ at the threshold energies $\pm m$. The basic idea in this section is to exploit the equivalences (2.7) and (2.8). Thus we shall apply the results in section 2 in the way described in the last paragraph of section 2. It turns out that beautiful spectral properties are in common to all the examples of Dirac operators in this section. See properties (i) – (iii) of Example 3.1.

Example 3.1. *Let*

$$A_{LY}(x) = 3\langle x\rangle^{-4}\big\{(1-|x|^2)w_0 + 2(w_0\cdot x)x + 2w_0\times x\big\} \tag{3.1}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $\phi_0 = {}^t(1,\,0)$ (ϕ_0 *can be any unit vector in* $\mathbb{C}^2$), *and*

$$w_0 = \phi_0\cdot(\sigma\phi_0) := \big((\phi_0,\,\sigma_1\phi_0)_{\mathbb{C}^2},\,(\phi_0,\,\sigma_2\phi_0)_{\mathbb{C}^2},\,(\phi_0,\,\sigma_3\phi_0)_{\mathbb{C}^2}\big). \tag{3.2}$$

Here $w_0\cdot x$ *and* $w_0\times x$ *denotes the inner product and the exterior product of* $\mathbb{R}^3$ *respectively, and* $(\phi_0,\,\sigma_1\phi_0)_{\mathbb{C}^2}$ *etc. denotes the inner product of* $\mathbb{C}^2$. *Then the Dirac operator*

$$H_{LY} := H_{A_{LY}} = \alpha\cdot(D - A_{LY}(x)) + m\beta$$

has the following properties:

(i) $\sigma(H_{LY}) = \sigma_{\mathrm{ess}}(H_{LY}) = (-\infty,\,-m]\cup[m,\,\infty)$;
(ii) *The point spectrum of* H_{LY} *consists only of* $\pm m$, *i.e.* $\sigma_p(H_{LY}) = \{-m,\,m\}$;
(iii) H_{LY} *is absolutely continuous on* $(-\infty,\,-m)\cup(m,\,\infty)$.

We shall show these properties. It is easy to see that $-\sigma\cdot A_{LY}(x)$ *is relatively compact perturbation of* $\sigma\cdot D$, *hence the Weyl-Dirac operator*

$$T_{LY} := T_{A_{LY}} = \sigma\cdot(D - A_{LY}(x))$$

is a self-adjoint operator in the Hilbert space $\mathcal{H} = \big[L^2(\mathbb{R}^3)\big]^2$ *with the domain* $\big[H^1(\mathbb{R}^3)\big]^2$. *By* $H^s(\mathbb{R}^3)$, *we mean the Sobolev space of order* s:

$$H^s(\mathbb{R}^3) = \big\{\,u \mid \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^3)} < +\infty\,\big\},$$

where $\langle D\rangle = \sqrt{1-\triangle}$. *Since the spectrum of the operator* $\sigma\cdot D$ *equals the whole real line, we see that* $\sigma(T_{LY}) = (-\infty,\,\infty)$. *Property* (i) *immediately follows from Theorem 2.3.*

We shall show property (ii). *According to Loss and Yau* [15, section II], *the Weyl-Dirac operator* T_{LY} *has a zero mode* φ_{LY} *defined by*

$$\varphi_{LY}(x) = \langle x\rangle^{-3}\big(I_2 + i\sigma\cdot x\big)\phi_0. \tag{3.3}$$

It follows from (2.7) *and* (2.8) *that* ${}^t(\varphi_{LY}, 0)$ (*resp.* ${}^t(0, \varphi_{LY})$) *is an eigenfunction of the Dirac operator* H_{LY} *corresponding to the threshold eigenvalue* m (*resp.* $-m$). *Hence* $\sigma_p(H_{LY}) \supset \{-m, m\}$. *On the other hand, it follows from Yamada* [25] *that* H_{LY} *has no eigenvalue in* $(-\infty, -m) \cup (m, \infty)$. (*Note that the vector potential* A_{LY} *satisfies the assumption of* [25, Proposition 2.5].) *This fact, together with property* (i), *implies that* $\sigma_p(H_{LY}) \subset \{-m, m\}$. *Summing up, we get property* (ii).

Property (iii) *is a direct consequence of Yamada* [25, Corollary 4.2]. *As for absolutely continuity and limiting absorption principle for Dirac operators, see also Yamada* [26], *Balslev and Helffer* [8], *and Pladdy, Saitō and Umeda* [19].

Example 3.2. *In the same spirit as in Example 3.1, we can show the existence of countably infinite number of vector potentials with which the Dirac operators have the properties* (i) – (iii) *in Example 3.1.*

In fact, we shall exploit a result on the Weyl-Dirac operator by Adam, Muratori and Nash [1], *where they construct a series of vector potentials* $A^{(\ell)}$ $(\ell = 0, 1, 2, \cdots)$, *each of which gives rise a zero mode* $\psi^{(\ell)}$ *of the Weyl-Dirac operator* $T^{(\ell)} := \sigma \cdot (D - A^{(\ell)}(x))$. *The idea of* [1] *is an extension of that of Loss and Yau* [15, section II]; *Indeed* $A^{(0)}$ *and* $\psi^{(0)}$ *give the same vector potential and zero mode as in* (3.1) *and* (3.3). *For* $\ell \geq 1$, *the construction of the zero mode* $\psi^{(\ell)}(x)$ *is based on an anzatz* (*see* (7) *in section II of* [1]) *and the definition of* $A^{(\ell)}$ *is given by*

$$A^{(\ell)}(x) = \frac{h^{(\ell)}(x)}{|\psi^{(\ell)}(x)|^2}\{\psi^{(\ell)}(x) \cdot (\sigma\psi^{(\ell)}(x))\}, \tag{3.4}$$

where $h^{(\ell)}(x)$ *is a real valued function and* $\psi^{(\ell)}(x) \cdot (\sigma\psi^{(\ell)}(x))$ *is defined in the same way as in* (3.2). *For* $\ell = 1$, *the zero mode is given by*

$$\psi^{(1)}(x) = \langle x\rangle^{-5}\{(1 - \frac{5}{3}|x|^2)I_2 + (\frac{5}{3} - |x|^2)i\sigma \cdot x\}\phi_0. \tag{3.5}$$

By the same arguments as in Example 3.1, we can deduce that the Dirac operator $H^{(\ell)} := \alpha \cdot (D - A^{(\ell)}(x)) + m\beta,\ \ell = 0, 1, 2, \cdots$, *has the properties* (i) – (iii) *of Example 3.1.*

4. Asymptotic limits of threshold eigenfunctions

In the previous section, we have seen there exists infinitely many A's such that the corresponding magnetic Dirac operators H_A have the threshold eigenvalues $\pm m$. In this section, we consider a class of magnetic Dirac operators H_A under Assumption(SU) below, and will focus on the asymptotic behaviors at infinity of eigenfunctions of H_A corresponding to $\pm m$ provided that these two energies are eigenvalues.

Assumption(SU).
Each element $A_j(x)$ $(j = 1, 2, 3)$ of $A(x)$ is a measurable function satisfying

$$|A_j(x)| \le C\langle x\rangle^{-\rho} \quad (\rho > 1), \tag{4.1}$$

where C is a positive constant.

It is easy to see that under Assumption(SU) the Dirac operator H_A is a self-adjoint operator in the Hilbert space $\mathcal{K} = \big[L^2(\mathbb{R}^3)\big]^4$ with $\mathrm{Dom}(H_A) = [H^1(\mathbb{R}^3)]^4$. Also it is easy to see that under Assumption(SU) the Weyl-Dirac operator T_A is a self-adjoint operator in the Hilbert space $\mathcal{H} = \big[L^2(\mathbb{R}^3)\big]^2$ with $\mathrm{Dom}(T_A) = \big[H^1(\mathbb{R}^3)\big]^2$. Since the operator $-\sigma\cdot A(x)$ is relatively compact with respect to the operator $T_0 := \sigma\cdot D$, and since $\sigma(T_0) = (-\infty, \infty)$, it follows that $\sigma(T_A) = (-\infty, \infty)$. Recalling that

$$H_A = \begin{pmatrix} 0 & T_A \\ T_A & 0 \end{pmatrix} + m\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4.2}$$

we apply Theorem 2.3 to H_A, and get

$$\sigma(H_A) = \sigma_{\mathrm{ess}}(H_A) = (-\infty, -m] \cup [m, +\infty).$$

Hence $\pm m$ are the threshold energies of the operator H_A. Assuming that $\pm m$ are the eigenvalues of H_A, we should emphasize here that the eigenspaces corresponding to the eigenvalues $\pm m$ of H_A are given as the direct sum of $\mathrm{Ker}(T_A)$ and the zero space $\{0\}$ (cf. Corollary 2.1), and that these two eigenspaces themselves as well as their dimensions are independent of m.

Theorem 4.1. *Suppose that Assumption*(SU) *is verified, and that m (resp. $-m$) is an eigenvalue of H_A. Let f be an eigenfunction of H_A corresponding to m (resp. $-m$). Then there exists a zero mode φ^+ (resp. φ^-) of T_A such that for any $\omega \in \mathbb{S}^2$*

$$\lim_{r\to\infty} r^2 f(r\omega) = \begin{pmatrix} u^+(\omega) \\ 0 \end{pmatrix} \quad \Big(\textit{resp.} \begin{pmatrix} 0 \\ u^-(\omega) \end{pmatrix}\Big), \tag{4.3}$$

where

$$u^\pm(\omega) = \frac{i}{4\pi}\int_{\mathbb{R}^3} \big\{\big(\omega\cdot A(y)\big)I_2 + i\sigma\cdot\big(\omega\times A(y)\big)\big\}\varphi^\pm(y)\,dy, \tag{4.4}$$

and the convergence is uniform with respect to ω.

Theorem 4.1 is a direct consequence of Corollary 2.1, together with Saitō and Umeda [22, Theorem 1.2]. Note that under Assumption(SU) every eigenfunction of H_A corresponding to either one of eigenvalues $\pm m$ is a continuous function of x (cf. Saitō and Umeda [23, Theorem 2.1]), therefore the expression $f(r\omega)$ in (4.3) makes sense for each ω.

5. Sparseness of vector potentials yielding threshold eigenfunctions

In this section, we shall discuss the sparseness of the set of vector potentials A which give rise to threshold eigenfunctions of magnetic Dirac operators H_A, in the sprit of Balinsky and Evans [4], [5], where they investigated Pauli operators and Weyl-Dirac operators respectively.

We shall make the following assumption:

Assumption(BE).
$A_j \in L^3(\mathbb{R}^3)$ for $j = 1, 2, 3$.

Under Assumption(BE) Balinsky and Evans [5, Lemma 2] showed that $-\sigma \cdot A$ is infinitesimally small with respect to $T_0 = \sigma \cdot D$ with $\mathrm{Dom}(T_0) = \big[H^1(\mathbb{R}^3)\big]^2$ (see (5.5) below). This fact enables us to define the self-adjoint realization T_A in the Hilbert space $\mathcal{H} = \big[L^2(\mathbb{R}^3)\big]^2$ as the operator sum of T_0 and $-\sigma \cdot A$, thus $\mathrm{Dom}(T_A) = \big[H^1(\mathbb{R}^3)\big]^2$. It turns out that under Assumption(BE) $-\alpha \cdot A$ is infinitesimally small with respect to $H_0 := \alpha \cdot D + m\beta$, and hence we can define the self-adjoint realization H_A in the Hilbert space $\mathcal{K} = \big[L^2(\mathbb{R}^3)\big]^4$ as the operator sum of H_0 and $-\alpha \cdot A$, thus $\mathrm{Dom}(H_A) = \big[H^1(\mathbb{R}^3)\big]^4$. Therefore we can regard H_A as a supersymmetric Dirac operator, and shall apply the results in section 2 to H_A. (Recall (4.2) again.)

Proposition 5.1. *Let Assumption* (BE) *be satisfied. Then* $\sigma(T_A) = (-\infty, \infty)$.

We shall prepare a few lemmas for the proof of Proposition 5.1.

Lemma 5.1. *Let* $z \in \mathbb{C} \setminus (-\infty, \infty)$. *Then* $\langle D\rangle^{1/2}(T_0 - z)^{-1}$ *is a bounded operator in* $\mathcal{H}$. *Moreover we have*

$$\mathrm{Ran}\big(\langle D\rangle^{1/2}(T_0 - z)^{-1}\big) \subset \big[H^{1/2}(\mathbb{R}^3)\big]^2. \tag{5.1}$$

Proof. It is sufficient to show the conclusions of the lemma for $z = -i$. Let $\varphi \in \mathrm{Dom}(T_0)$. Then we have

$$\begin{aligned} \|(T_0 + i)\varphi\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}^3} \big|\big((\sigma \cdot \xi) + iI_2\big)\widehat{\varphi}(\xi)\big|_{\mathbb{C}^2}^2 \, d\xi \\ &= \int_{\mathbb{R}^3} (|\xi|^2 + 1)\big|\widehat{\varphi}(\xi)\big|_{\mathbb{C}^2}^2 \, d\xi \\ &= \|\langle D\rangle\varphi\|_{\mathcal{H}}^2, \end{aligned} \tag{5.2}$$

where we have used the anti-commutation relation $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I_2$ in the second equality. It follows from (5.2) that

$$\|\varphi\|_{\mathcal{H}} = \|\langle D\rangle(T_0 + i)^{-1}\varphi\|_{\mathcal{H}} \tag{5.3}$$

for all $\varphi \in \mathcal{H}$. Furthermore, we see that

$$\begin{aligned} \|\langle D\rangle^{1/2}(T_0+i)^{-1}\varphi\|_{\mathcal{H}} &\le \|\langle D\rangle^{1/2}(T_0+i)^{-1}\varphi\|_{[H^{1/2}(\mathbb{R}^3)]^2} \\ &= \|\langle D\rangle(T_0+i)^{-1}\varphi\|_{\mathcal{H}} \\ &= \|\varphi\|_{\mathcal{H}}. \end{aligned} \tag{5.4}$$

It is evident that (5.4) proves the conclusions of the lemma for $z=-i$. □

Lemma 5.2. *If $\varphi \in \left[H^{1/2}(\mathbb{R}^3)\right]^2$, then $(\sigma\cdot A)\langle D\rangle^{-1/2}\varphi \in \mathcal{H}$.*

Proof. By Balinsky and Evans [5, Lemma 2], we see that for any $\epsilon > 0$, there exists a constant $k_\epsilon > 0$ such that for all $\varphi \in \mathrm{Dom}(T_0)$

$$\|(\sigma\cdot A)\varphi\|_{\mathcal{H}} \le \epsilon\|T_0\varphi\|_{\mathcal{H}} + k_\epsilon\|\varphi\|_{\mathcal{H}}. \tag{5.5}$$

By virtue of the fact that $\langle D\rangle^{-1/2}\varphi \in \mathrm{Dom}(T_0)$ for $\varphi \in \left[H^{1/2}(\mathbb{R}^3)\right]^2$, it follows from (5.5) that

$$\begin{aligned} \|(\sigma\cdot A)\langle D\rangle^{-1/2}\varphi\|_{\mathcal{H}} &\le \epsilon\|(T_0+i)\langle D\rangle^{-1/2}\varphi\|_{\mathcal{H}} + k_\epsilon\|\langle D\rangle^{-1/2}\varphi\|_{\mathcal{H}} \\ &\le \epsilon\|\langle D\rangle^{1/2}\varphi\|_{\mathcal{H}} + k_\epsilon\|\varphi\|_{\mathcal{H}} < +\infty, \end{aligned}$$

where we have used (5.2) and the fact that $\|\langle D\rangle^{1/2}\varphi\|_{\mathcal{H}} = \|\varphi\|_{[H^{1/2}(\mathbb{R}^3)]^2}$. □

Lemma 5.3. *$\langle D\rangle^{-1}(\sigma\cdot A)\langle D\rangle^{-1/2}$ is a compact operator in $\mathcal{H}$.*

Proof. One can make a factorization

$$\begin{aligned} &\langle D\rangle^{-1}(\sigma\cdot A)\langle D\rangle^{-1/2} \\ =&\Big(\frac{|D|^{1/2}}{\langle D\rangle}\Big)\Big(\frac{1}{|D|^{1/2}}(\sigma\cdot A)\frac{1}{|D|^{1/2}}\Big)\Big(\frac{|D|^{1/2}}{\langle D\rangle^{1/2}}\Big). \end{aligned} \tag{5.6}$$

It is obvious that the first term and the last term on the right hand side of (5.6) are bounded operators in $\mathcal{H}$. Then it follows from (5.6) and Balinsky and Evans [5, Lemma 1] that the conclusion of the lemma holds true. □

Lemma 5.4. *Let $z \in \mathbb{C}\setminus(-\infty,\infty)$. Then $(T_A-z)^{-1}\langle D\rangle\Big|_{[H^1(\mathbb{R}^3)]^2}$ can be extended to a bounded operator $\widetilde{R}_A(z)$ in $\mathcal{H}$. Moreover*

$$(T_A-z)^{-1}\varphi = \widetilde{R}_A(z)\langle D\rangle^{-1}\varphi \quad \text{for } \forall\varphi\in\mathcal{H}. \tag{5.7}$$

Proof. We first show that $\langle D\rangle(T-z)^{-1}$ is a closed operator in $\mathcal{H}$. To this end, suppose that $\{\varphi_j\}$ is a sequence in $\mathcal{H}$ such that $\varphi_j \to 0$ in $\mathcal{H}$ and $\langle D\rangle(T-z)^{-1}\varphi_j \to \psi$ in $\mathcal{H}$ as $j\to\infty$. Then $\{(T-z)^{-1}\varphi_j\}$ is a Cauchy sequence in $\left[H^1(\mathbb{R}^3)\right]^2$, hence there exists a $\widetilde{\psi} \in \left[H^1(\mathbb{R}^3)\right]^2$ such that

$$(T-z)^{-1}\varphi_j \to \widetilde{\psi} \ \text{ in } \left[H^1(\mathbb{R}^3)\right]^2 \ \text{ as } j\to\infty. \tag{5.8}$$

Since the topology of $\left[H^1(\mathbb{R}^3)\right]^2$ is stronger than that of $\mathcal{H}$, (5.8) implies that

$$(T-z)^{-1}\varphi_j \to \widetilde{\psi} \ \text{ in } \mathcal{H} \ \text{ as } j\to\infty. \tag{5.9}$$

On the other hand, since $\varphi_j \to 0$ in $\mathcal{H}$, and since $(T-z)^{-1}$ is a bounded operator in $\mathcal{H}$, we see that

$$(T-z)^{-1}\varphi_j \to 0 \text{ in } \mathcal{H} \tag{5.10}$$

as $j \to \infty$. Combining (5.9) and (5.10), we see that $\widetilde{\psi} = 0$. This fact, together with (5.8), $\langle D\rangle(T-z)^{-1}\varphi_j \to 0$ in $\mathcal{H}$ as $j \to \infty$. Hence $\psi = 0$. We have thus shown that $\langle D\rangle(T-z)^{-1}$ is a close operator. Noting that $\mathrm{Dom}(\langle D\rangle(T-z)^{-1}) = \mathcal{H}$, we can conclude from the Banach closed graph theorem that $\langle D\rangle(T-z)^{-1}$ is a bounded operator in $\mathcal{H}$, which will be denoted by $Q_A(z)$.

We now put $\widetilde{R}_A(z) := Q_A(\overline{z})^*$, where $Q_A(\overline{z})^*$ denotes the adjoint operator of $Q_A(\overline{z})$. Then for any $\varphi \in \mathcal{H}$ and any $\psi \in \left[H^1(\mathbb{R}^3)\right]^2$, we have

$$\begin{aligned}(\varphi,\, \widetilde{R}_A(z)\psi)_{\mathcal{H}} &= (Q_A(\overline{z})\varphi,\, \psi)_{\mathcal{H}} \\ &= (\langle D\rangle(T-\overline{z})^{-1}\varphi,\, \psi)_{\mathcal{H}} \\ &= (\varphi,\, (T-z)^{-1}\langle D\rangle\psi)_{\mathcal{H}}.\end{aligned} \tag{5.11}$$

It follows from (5.11) that

$$\widetilde{R}_A(z)\psi = (T-z)^{-1}\langle D\rangle\psi \tag{5.12}$$

for all $\psi \in \left[H^1(\mathbb{R}^3)\right]^2$. Replacing ψ in (5.12) with $\langle D\rangle^{-1}\varphi$, $\varphi \in \mathcal{H}$, we get (5.7). □

Proof of Proposition 5.1 . Since $\sigma(T_0) = \sigma_{\text{ess}}(T_0) = (-\infty,\, \infty)$, it is sufficient to show that

$$\sigma_{\text{ess}}(T_A) = \sigma_{\text{ess}}(T_0). \tag{5.13}$$

To this end, we shall prove that the difference $(T_A + i)^{-1} - (T_0 + i)^{-1}$ is a compact operator in $\mathcal{H}$. Then, this fact implies (5.13); see Reed and Simon [21, p.113, Corollary 1].

We see that

$$\begin{aligned}&(T_A+i)^{-1} - (T_0+i)^{-1} \\ =&(T_A+i)^{-1}(\sigma\cdot A)(T_0+i)^{-1} \\ =&\widetilde{R}_A(-i)\{\langle D\rangle^{-1}(\sigma\cdot A)\langle D\rangle^{-1/2}\}\{\langle D\rangle^{1/2}(T_0+i)^{-1}\},\end{aligned} \tag{5.14}$$

where we have used Lemma 5.4 in (5.14). It follows from Lemmas 5.1–5.4 that (5.14) makes sense as a product of three bounded operators in $\mathcal{H}$ and that the product is a compact operator in $\mathcal{H}$. □

Proposition 5.1, together with Theorem 2.3, gives the following result on the spectrum of the magnetic Dirac operator H_A.

Theorem 5.1. *Let Assumption* (BE) *be satisfied. Then*

$$\sigma(H_A) = \sigma_{\text{ess}}(H_A) = (-\infty,\, -m] \cup [m,\, \infty).$$

We now state the main results in this section, which are concerned with the eigenspaces corresponding to the threshold eigenvalues of magnetic Dirac operators H_A.

Theorem 5.2. *Let Assumption* (BE) *be satisfied. Then*

(i) $\mathrm{Ker}(H_A - m)$ *is non-trivial if and only if* $\mathrm{Ker}(H_A + m)$ *is non-trivial; in other words,*

$$\begin{aligned} &\left\{ A \in \left[L^3(\mathbb{R}^3)\right]^3 \mid \mathrm{Ker}(H_A - m) \neq \{0\} \right\} \\ &\quad = \left\{ A \in \left[L^3(\mathbb{R}^3)\right]^3 \mid \mathrm{Ker}(H_A + m) \neq \{0\} \right\}. \end{aligned}$$

(ii) *There exists a constant c such that*

$$\dim\bigl(\mathrm{Ker}(H_A - m)\bigr) = \dim\bigl(\mathrm{Ker}(H_A + m)\bigr) \leq c \int_{\mathbb{R}^3} |A(x)|^3 \, dx. \tag{5.15}$$

Moreover, the dimension of $\mathrm{Ker}(H_A \mp m)$ *is independent of m.*

(iii) *The set* $\left\{ A \in \left[L^3(\mathbb{R}^3)\right]^3 \mid \mathrm{Ker}(H_A \mp m) = \{0\} \right\}$ *contains an open dense subset of* $\left[L^3(\mathbb{R}^3)\right]^3$.

Proof. By Corollary 2.1, we see that

$$\begin{aligned} \mathrm{Ker}(T_A) \text{ is trivial} &\iff \mathrm{Ker}(H_A - m) \text{ is trivial} \\ &\iff \mathrm{Ker}(H_A + m) \text{ is trivial.} \end{aligned} \tag{5.16}$$

Assertion (i) is equivalent to (5.16). Assertion (ii) follows from Corollary 2.1 and Balinsky and Evans [5, Theorem 3]. Assertion (iii) follows from Corollary 2.1 and Balinsky and Evans [5, Theorem 2]. □

Remark 5.1. Assertions (i) and (ii) of Theorem 5.2 mean the following facts: The threshold energy m is an eigenvalue of H_A if and only if the threshold energy $-m$ is an eigenvalue of H_A. If this is the case, their multiplicity are the same.

Remark 5.2. As for the best constant in (5.15), see Balinsky and Evans [5, Theorem 3].

6. The structure of the set of vector potentials yielding threshold eigenfunctions

In this section, we shall discuss a property of nonlocality of magnetic vector potentials as well as the sparseness of the set of vector potentials A which give rise to threshold eigenfunctions of H_A in the same sprit as in section 5.

We make the following assumption:

Assumption(E).
Each A_j $(j = 1, 2, 3)$ is a real-valued continuous function such that $A_j(x) = o(|x|^{-1})$ as $|x| \to \infty$.

It is straightforward to see that under Assumption(E), $-\sigma\cdot A$ is a bounded self-adjoint operator in the Hilbert space $\mathcal{H} = \left[L^2(\mathbb{R}^3)\right]^2$. Hence we can define the self-adjoint realization T_A with $\mathrm{Dom}(T_A) = \left[H^1(\mathbb{R}^3)\right]^2$ as the operator sum of T_0 and $-\sigma \cdot A$.

Also, it is straightforward to see that $-\alpha \cdot A$ is a bounded self-adjoint operator in the Hilbert space $\mathcal{K} = \left[L^2(\mathbb{R}^3)\right]^4$, hence we can define the self-adjoint realization H_A with $\mathrm{Dom}(H_A) = \left[H^1(\mathbb{R}^3)\right]^4$ in $\mathcal{K}$ as the operator sum of H_0 and $-\alpha\cdot A$. Therefore, in the same way as in section 5, we can regard H_A as a supersymmetric Dirac operator, and apply the results in section 2 to H_A.

We note that under Assumption(E) $(-\sigma \cdot A)(T_0 + i)^{-1}$ is a compact operator in $\mathcal{H}$. Hence in the same way as in the proof of Proposition 5.1 we can show that $\sigma(T_A) = (-\infty, \infty)$. This fact, together with Theorem 2.3, implies the following result.

Theorem 6.1. *Let Assumption* (E) *be satisfied. Then*

$$\sigma(H_A) = \sigma_{\mathrm{ess}}(H_A) = (-\infty, -m] \cup [m, \infty).$$

To state the main results in this section, we need to introduce the following notation:

$$\mathcal{A} := \{\, A \,|\, A \text{ satisfies Assumption(E)} \,\}. \tag{6.1}$$

We regard $\mathcal{A}$ as a Banach space with the norm

$$\|A\|_{\mathcal{A}} = \sup_x \{\langle x\rangle |A(x)|\}$$

Theorem 6.2. *Let Assumption*(E) *be satisfied. Define*

$$\mathcal{Z}_k^{\pm} = \{\, A \in \mathcal{A} \,|\, \dim(\mathrm{Ker}(H \mp m)) = k \,\}$$

for $k = 0, 1, 2, \cdots$. *Then*

(i) $\mathcal{Z}_k^+ = \mathcal{Z}_k^-$ *for all* k.

(ii) $\mathcal{Z}_0^{\pm}$ *is an open and dense subset of* $\mathcal{A}$.

(iii) *For any* k *and any open subset* $\Omega(\neq \emptyset)$ *of* $\mathbb{R}^3$ *there exists an* $A \in \mathcal{Z}_k^{\pm}$ *such that* $A \in \left[C_0^{\infty}(\Omega)\right]^3$.

Proof. Assertion (i) is a direct consequence of Corollary 2.1. Assertions (ii) and (iii) follows from Corollary 2.1 and Elton [10, Theorem 1]. □

It is of some interest to point out a conclusion following from Theorem 4.1 and Assertion (iii) of Theorem 6.2. Namely, there are (at least) countably infinite number of vector potentials A with compact support such that the corresponding Dirac operators H_A have the threshold eigenfunctions f with the property (4.4). The eigenfunction f behave like $|f(x)| \sim |x|^{-2}$ for $|x| \to \infty$, in spite of the fact that the vector potentials and the corresponding magnetic fields vanish outside bounded regions. It is obvious that this phenomenon describes a certain kind of non-locality.

7. A concluding remark

Section 4 is based upon our results on supersymmetric Dirac operators in section 2 of the present paper and those of Saitō and Umeda [23]. Section 5 is based upon our results on supersymmetric Dirac operators in section 2 and those of Balinsky and Evans [5]. Section 6 is based upon our results on supersymmetric Dirac operators and those of Elton [10]. The combinations of our abstract results and those existing works broaden the understandings of the threshold eigenfunctions of the magnetic Dirac operators.

In each section from section 4 to section 6, we have made a different assumption on the vector potential. It is important to compare these assumptions to each other. To this end, mimicking (6.1), we introduce the following notation

$$\begin{aligned} \mathcal{A}_{SU} &:= \{\, A \,|\, A \text{ satisfies Assumption(SU)} \,\}, \\ \mathcal{A}_{BE} &:= \{\, A \,|\, A \text{ satisfies Assumption(BE)} \,\}. \end{aligned}$$

We then have

$$\begin{aligned} &\mathcal{A}_{SU} \subset \mathcal{A}_{BE}, \quad \mathcal{A}_{BE} \setminus \mathcal{A}_{SU} \neq \emptyset, \\ &\mathcal{A} \setminus \mathcal{A}_{SU} \neq \emptyset, \quad \mathcal{A}_{SU} \setminus \mathcal{A} \neq \emptyset, \\ &\mathcal{A} \setminus \mathcal{A}_{BE} \neq \emptyset, \quad \mathcal{A}_{BE} \setminus \mathcal{A} \neq \emptyset. \end{aligned}$$

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