Spatial Structures and Generalized Travelling Waves for an Integro-Differential Equation

N. Apreutesei[1], N. Bessonov[2], V. Volpert[3], V. Vougalter[4]

[1] Department of Mathematics, Technical University of Iasi, Iasi, Romania
[2] Institute of Mechanical Engineering Problems, 199178 Saint Petersburg, Russia
[3] Institute of mathematics, University Lyon 1, UMR 5208 CNRS
69622 Villeurbanne, France
[4]University of Toronto, Department of Mathematics, Toronto, ON, M5S 2E4, Canada

Abstract. Some models in population dynamics with intra-specific competition lead to inegro-differential equations where the integral term corresponds to nonlocal consumption of resources [8], [9]. The principal difference of such equations in comparison with traditional reaction-diffusion equation is that homogeneous in space solutions can lose their stability resulting in emergence of spatial or spatio-temporal structures [4]. We study the existence and global bifurcations of such structures. In the case of unbounded domains, transition between stationary solutions can be observed resulting in propagation of generalized travelling waves (GTW). GTWs are introduced in [18] for reaction-diffusion systems as global in time propagating solutions. In this work their existence and properties are studied for the integro-differential equation. Similar to the reaction-diffusion equation in the monostable case, we prove the existence of generalized travelling waves for all values of the speed greater or equal to the minimal one. We illustrate these results by numerical simulations in one and two space dimensions and observe a variety of structures of GTWs.

AMS Subject Classification: 35K57, 45K05, 92D25

1 Introduction

This work is devoted to the integro-differential equation

$$\frac{\partial u}{\partial t} = \Delta u + ku(1 - au - bJ) \tag{1.1}$$

where k and b are some positive constants, $a \geq 0$,

$$J(x,t) = \int_\Omega \phi(x-y)u(y,t)dy,$$

$\Omega \subset \mathbb{R}^n$ is a bounded or unbounded domain, $\phi(x)$ is a nonnegative function. It is defined in the whole $\mathbb{R}^n$ and has a finite support. We will assume that $\int_{\mathbb{R}^n} \phi(y)dy = 1$.

If $\Omega = \mathbb{R}^n$, then it can be easily verified that equation (1.1) has two homogeneous in space stationary solutions, $u_0 = 0$ and $u_1 = 1/(a+b)$. If we replace $\phi(x)$ by the Dirac delta-function and consider for simplicity the 1D case, then we obtain the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \tag{1.2}$$

where $F(u) = ku(1-pu)$, $p = a+b$. This is the KPP equation [11] describing some problems in population dynamics and other applications [21]. One of the important properties of this equation is the existence of travelling waves, that is of solutions $u(x,t) = w(x-ct)$. Here c is a constant, the wave speed. It is known that the waves exist for all positive c. If $c \geq 2\sqrt{k}$, then they are monotone in space and stable, while for $0 < c < 2\sqrt{k}$ they are nonmonotone and unstable (see, e.g., [11], [21]). These waves have limits at infinity: $w(+\infty) = 0$, $w(-\infty) = 1/p$. The first point is unstable with respect to the equation $du/dt = F(u)$, the second point is stable. As a consequence, this case is called monostable.

The principal difference of equation (1.1) with respect to equation (1.2) is that the homogeneous in space stationary solution u_1 can lose its stability [4], [8]-[10]. If this is the case, then some spatial or temporal-spatial structures can bifurcate from it. Therefore, instead of travelling waves connecting u_0 and u_1 in the case of the reaction-diffusion equation we can expect the existence of some other solutions connecting u_0 at $+\infty$ with some structures at $-\infty$. We call such solutions *generalized travelling waves* (GTW) and study them in this work.

The notion of GTW was first introduced in [18] for reaction-diffusion systems. In order to explain it, let us consider the perturbed reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) + \epsilon g(x,u), \tag{1.3}$$

where ϵ is a small parameter, $g(x,u)$ is some given function and $F(u)$ is a bistable nonlinearity, that is $F(w_\pm) = 0$ for some w_+ and w_-, and $F'(w_\pm) < 0$. If $\epsilon = 0$, then under certain conditions on the function F there exists a unique up to translation in space travelling wave solution $u(x,t) = w(x-ct)$ of this equation. If $\epsilon \neq 0$ and $g(x,u)$ depends on x, then travelling wave solution of this equation does not exist. However, we can expect that similar in some sense solutions may exist at least for small values of ϵ. Such solutions can be characterized by two main properties:

(1) They exist for all t from $-\infty$ to $+\infty$. The solution of the Cauchy problem for equation (1.3) exists for positive time and, generally speaking, cannot be extended for all negative t. It appears that the existence of such global solutions, which exist for all $t \in \mathbb{R}$, can be proved. Moreover, under certain conditions such solution can be unique and stable.

(2) These are propagating solutions. The property of propagation can be explained as follows. Let q be a constant, $w_+ < q < w_-$. For each t fixed consider the equation $u(x,t) = q$

with respect to x. Denote by $m_q^+(t)$ its maximal solution (if it exists) and by $m_q^-(t)$ its minimal solution. If $m_q^\pm(t)/t \to c$ as $t \to \infty$, then we say that this solution propagates with the speed c.

Thus, GTW are global propagating solutions. Their existence and structure are studied in [18]-[20] for reaction-diffusion systems of equations. We note that this definition applies also for autonomous equations. In particular, various oscillating solutions, which appear when a usual travelling wave loses its stability, are particular examples of GTW. Periodic travelling waves in a nonhomogeneous medium provide another example [3], [15], [23].

In this work, we study GTW for the integro-differential equation (1.1). Equations of this type arise in population dynamics with nonlocal consumption of resources (intra-specific competition) [8], [9] and in other biological applications [6], [12], [17]. Models in population dynamics with nonlocal boundary conditions are discussed in [5]. If the support of the function ϕ under the integral is small, the integro-differential equation is in some sense close to the reaction-diffusion equation. The wave existence in this case is proved for the monostable case [7] and for the bistable case [1], [2]. An approach based on the Hamilton-Jacobi equation is developed in [16].

If the support of ϕ is not small, then numerical simulations show the existence of periodic travelling waves and of propagating solutions with a more complex structure. The main goal of this work is to prove the existence of such generalized travelling waves. We begin with the analysis of the existence of nonhomogeneous in space stationary solutions in bounded or unbounded domains (Section 2). We use here the methods based on the topological degree theory and linear stability analysis.

We then study the existence of solutions $u(x,t)$ of the GTW type such that $u(x,t) \to 0$ as $x \to \infty$ (Section 3). The main result of this work asserts the existence of positive GTW for all values c of the velocity grater or equal to the minimal velocity c_0. This result generalizes the existence of travelling waves for the KPP equation. To the best of our knowledge, this is the first result on the existence of GTW with multiple speeds. The method of proof is based on some a priori estimates of solutions. We cannot specify the behavior of GTW as $x \to -\infty$. We can expect that they converge to the nonhomogeneous solutions the existence of which is discussed above.

The last section of this work is devoted to numerical simulations. We observe GTW with various structures and with various speeds of propagation. In the 2D case they depend on the form of the support of the function ϕ.

2 Existence of stationary solutions

We begin the study of the integro-differential equation (1.1) with the existence of its stationary solutions. The method of proof is based on the analysis of continuous branches of solutions which start either with the trivial solution $u = 0$ or with the constant solution $u = 1/(a+b)$. We investigate the first case in more detail. In the second case we restrict ourselves to the linear stability analysis of the homogeneous in space solution when the domain is the whole $\mathbb{R}^n$. This simple analysis shows an important property of this equation: it can lose its stability resulting in the appearance of spatial structures. This stability analysis and the subsequent investigation of the branches of solutions can also be carried out for bounded or unbounded domains. The existence of spatial structures makes it possible a transition between them, that is the existence of generalized travelling waves studied in Sections 3 and 4.

2.1 Formulation and a priori estimates

Consider the integro-differential equation

$$\Delta u + ku(1 - au - bJ(x)) = 0 \tag{2.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary condition:

$$u|_{\partial\Omega} = 0. \tag{2.2}$$

Here k and b are some positive constants, $a \geq 0$,

$$J(x) = \int_\Omega \phi(x-y)u(y)dy, \tag{2.3}$$

where $\phi(x)$ is a nonnegative function. By $(x-y)$ we understand $(x_1 - y_1, ..., x_n - y_n)$. We introduce the operator A corresponding to the left-hand side of (2.1) and suppose that it acts from the space

$$E_0 = \{u \in C^{2+\alpha}(\bar{\Omega}), \ \ u|_{\partial\Omega} = 0\}$$

into $E = C^\alpha(\bar{\Omega})$. The boundary of the domain belongs to $C^{2+\alpha}$.

Lemma 2.1. *The operator $A(u) : E_0 \to E$ is bounded and continuous. Its Fréchet derivative*

$$A'(u_0)v = \Delta v + p_1(x)v - p_2(x)\int_\Omega \phi(x-y)v(y)dy,$$

where

$$p_1(x) = k(1 - 2au_0 - bJ_0(x)), \quad p_2(x) = kbu_0(x), \quad J_0(x) = \int_{-\infty}^{\infty} \phi(x-y)u_0(y)dy$$

is a bounded operator from E_0 into E and continuous with respect to u_0 in the operator norm.

The proof of this lemma is standard and it is omitted. We note that $J_0(x)$ in the formulation of the lemma is obtained from $J(x)$ in (2.3) if we replace u by u_0.

Suppose that problem (2.1), (2.2) has a positive in Ω solution $u_0(x)$. Then

$$0 \leq u_0(x) \leq \frac{1}{a}, \quad x \in \bar{\Omega}. \tag{2.4}$$

Indeed, if this is not the case, then $u_0(x)$ attains its maximum at some point $x_0 \in \Omega$ and $u_0(x_0) > 1/a$. Since $J(x_0) \geq 0$, we obtain a contradiction in signs in equation (2.1) at the point $x = x_0$. From estimate (2.4) it follows that

$$0 \leq J(x) \leq \frac{1}{a} \int_{\Omega} \phi(x-y)dy \leq M, \quad x \in \bar{\Omega} \tag{2.5}$$

for some positive constant M. Hence

$$-bM\frac{k}{a} \leq \ ku_0(x)(1 - au_0(x) - bJ(x)) \ \leq \frac{k}{a}, \quad x \in \bar{\Omega}. \tag{2.6}$$

Denote

$$f(x) = -ku_0(x)(1 - au_0(x) - bJ(x))$$

and consider the problem

$$\Delta u = f, \quad u|_{\Omega} = 0. \tag{2.7}$$

By virtue of (2.6), $f \in L^p(\Omega)$ for any p. Hence, we obtain for the solution u_0 of this problem the estimate

$$\|u_0\|_{W^{2,p}(\Omega)} \leq K\|f\|_{L^p(\Omega)},$$

where K is the norm of the inverse to the Laplace operator. For $p > n$ we obtain from the embedding theorem an estimate of the norm $\|u_0\|_{C^{1+\alpha}(\bar{\Omega})}$. This allows us to estimate the norm $\|f\|_{C^{1+\alpha}(\bar{\Omega})}$ and, from problem (2.7), the norm $\|u_0\|_{C^{2+\alpha}(\bar{\Omega})}$. In fact, we obtain the estimate of the stronger norm $\|u_0\|_{C^{3+\alpha}(\bar{\Omega})}$ but we will not use it. Thus, we have proved the following lemma.

Lemma 2.2. *Any positive solution u of problem (2.1), (2.2) admits the estimate*

$$\|u\|_{C^{2+\alpha}(\bar{\Omega})} \leq K,$$

where K depends on a, b, k, M and on the domain Ω and does not depend on the solution.

2.2 Existence of solutions in bounded domains

In this section we will use the topological degree theory in order to prove existence of solutions of problem (2.1), (2.2). We begin with the case $b = 0$. Problem (2.1), (2.2) becomes a semilinear elliptic problem:

$$\Delta u + ku(1 - au) = 0 \tag{2.8}$$

$$u|_{\partial\Omega} = 0. \tag{2.9}$$

Consider the eigenvalue problem

$$\Delta u = \lambda u, \quad u|_{\partial\Omega} = 0. \tag{2.10}$$

We denote by λ_0 its principal eigenvalue and by λ_1 the next eigenvalue. Then $\lambda_1 < \lambda_0 < 0$. The principal eigenvalue is simple.

We now increase k from $k = 0$. For $k = |\lambda_0|$, there is a bifurcation of two new solutions of problem (2.8), (2.9) from the trivial solution $u = 0$. One solution, $w_+(x)$ is positive in Ω, another one, $w_-(x)$ is negative.

We fix the value of k, $|\lambda_0| < k < |\lambda_1|$ and consider positive values of b. Let now b be sufficiently small. Lemma 2.1 allows us to apply the implicit function theorem. We obtain the following result.

Lemma 2.3. *Suppose that the operator linearized about $w_+(w_-)$ does not have zero eigenvalue. Then for all $b > 0$ sufficiently small, there exists a positive (negative) in Ω solution $u \in C^{2+\alpha}(\bar{\Omega})$ of problem (2.1), (2.2), which converges to w_+ (w_-) as $b \to 0$.*

The positiveness (negativeness) of the solution follows from the fact that it is close to $w_+(x)$ $(w_-(x))$ in the $C^{2+\alpha}(\bar{\Omega})$ norm and the normal derivatives of $w_\pm(x)$ at the boundary are different from zero.

Lemma 2.4. *Suppose that w_b is a solution of problem (2.1), (2.2) continuous with respect to b in the $C^{2+\alpha}(\bar{\Omega})$ norm. If for some $b = b_0$, $w_b(x)$ is positive, negative or have a variable sign, then it preserves this property for all positive b.*

Proof. Suppose that $w_{b_0}(x) > 0$ for $x \in \Omega$ and $w_b(x)$ has some negative values for some x and $b > b_0$. Then there exists $b_1 > b_0$ such that the positiveness of the solution is preserved for $b_0 < b < b_1$ and it is not preserved for $b > b_1$. Then $w_{b_1}(x) \geq 0$ for $x \in \bar{\Omega}$ and there are three possible cases:

1. $w_b(x) \not\equiv 0$ and $w_{b_1}(x_0) = 0$ for some $x_0 \in \Omega$,
2. $w_b(x) \not\equiv 0$ and $\partial w_{b_1}(x_0)/\partial n = 0$ for some $x_0 \in \partial\Omega$.
3. $w_b(x) \equiv 0$.

In the first two cases, this contradicts the maximum principle if we consider w_b as a solution of the problem

$$\Delta u + s(x)u = 0, \quad u|_{\partial\Omega} = 0,$$

where

$$s(x) = k\left(1 - aw_b(x) - bq(x)\int_\Omega \phi(x-y)w_b(y)dy.\right)$$

This coefficient is not necessarily positive.

The third case is not possible since the problem linearized about $u = 0$ does not have zero eigenvalue and, as a consequence, the trivial solution is isolated.

The prove remains the same in the case of negative solutions. If a solution of a variable sign become positive (negative), then we can vary b in the opposite direction. In this case, a positive (negative) solution becomes with variable sign and we can repeat the arguments above. The lemma is proved.

Consider the following set:

$$D = \{u \in E_0, \ \ \|u\|_{C^{2+\alpha}(\bar{\Omega})} < K_1, \ \ u(x) > \epsilon(x), \ \ x \in \Omega, \ \frac{\partial u}{\partial n} < 0, \ \ x \in \partial\Omega\},$$

where $K_1 > K$ (Lemma 2.2), $\epsilon(x)$ is a non-negative C^∞ function with the support strictly inside Ω. Obviously, it is an open bounded set in E_0.

Lemma 2.5. *Let $b_0 > 0$. The function $\epsilon(x)$ can be chosen in such a way that there are no solutions of problem (2.1), (2.2) at ∂D for any $b \in [0, b_0]$.*

Proof. Consider the sequence of functions $\epsilon_n(x)$ such that $\Omega_n = supp \ \epsilon_n \subset \Omega$, $\epsilon_n(x) > 0$ for $x \in \Omega_n$. Let the distance from the boundaries $\partial\Omega_n$ to the boundary $\partial\Omega$ tend to zero as $n \to \infty$. Moreover, we suppose that $\epsilon_n \to 0$ in $C^1(\bar{\Omega})$.

Denote

$$D_n = \{u \in E_0, \ \ \|u\|_{C^{2+\alpha}(\bar{\Omega})} < K_1, \ \ u(x) > \epsilon_n(x), \ \ x \in \Omega, \ \frac{\partial u}{\partial n} < 0, \ \ x \in \partial\Omega\}.$$

We should prove that there exists such n that there are no solutions of problem (2.1), (2.2) at ∂D_n. Suppose that this is not the case. Then there is a sequence $u_n(x)$ of solutions of this problem such that for each of them

$$\epsilon_n(x) \le u_n(x) \le K_1, \ \ x \in \Omega, \quad \frac{\partial u_n}{\partial n} \le 0, \ \ x \in \partial\Omega$$

and one of the following conditions is satisfied: a). $u_n(x_n) = K_1$ for some $x_n \in \Omega$, b). $u_n(x_n) = \epsilon_n(x_n)$ for some $x_n \in \Omega$, c). $\partial u_n(x_n)/\partial n = 0$ for some $x_n \in \partial\Omega$.

We show that this assumption leads to a contradiction. Indeed, the case a) is not possible due to Lemma 2.2, while the case c) contradicts the Hopf lemma which asserts that the normal derivative at the boundary is negative. It remains to consider the case b). We can choose a convergent subsequence from the sequence x_n. Moreover, it follows from Lemma 2.2 that we can choose a convergent subsequence from the sequence $u_n(x)$. The limiting function $u_0(x)$ is nonnegative and satisfies problem (2.1), (2.2).

If the limiting point x_0 of x_n is inside Ω, then $u_0(x_0) = 0$ and we obtain a contradiction with the maximum principle. Suppose now that $x_0 \in \partial\Omega$. Then

$$\frac{\partial u_0}{\partial n} \mid_{x_0} = 0. \tag{2.11}$$

Indeed, $u_n(x_n) = \epsilon_n(x_n)$ and $u_n(x) \ge \epsilon_n(x)$ for $x \in \Omega$. Hence $\nabla u_n(x_n) = \nabla\epsilon_n(x_n)$. By virtue of the convergence $\epsilon_n \to 0$ in $C^1(\bar{\Omega})$, $|\nabla u_n(x_n)| \to 0$. This convergence proves (2.11). However, this equality contradicts the Hopf lemma. The lemma is proved.

Consider the operator $A : E_0 \to E$ introduced in the previous section. The topological degree can be defined for it similarly to elliptic operators without the integral terms [22]. We note that this degree is defined for Fredholm and proper operators with the zero index. Its construction is different in comparison with the Leray-Schauder degree defined for compact perturbations of the identity operator. We can now apply the Leray-Schauder method to prove the existence of solutions. We will use the notation A_b to show the dependence on the parameter b. It follows from the previous lemma that the degree $\gamma(A_b, D)$ is independent of b. It remains to verify that it is different from zero for $b = 0$.

Lemma 2.6. $\gamma(A_0, D) = 1$.

Proof. For $b = 0$ and k crossing the bifurcation point $|\lambda_0|$, there is a single positive solution w_+ of problem (2.8), (2.9). Therefore, $\gamma(A_0, D) = \gamma(A_0, U(w_+))$, where $U(w_+)$ is a small neighborhood in E_0 of w_+. On the other hand, $\gamma(A_0, U(w_+)) = (-1)^\nu$, where ν is the number of positive eigenvalues of the operator linearized about w_+. It remains to note that $\nu = 0$. The lemma is proved.

We have proved the following theorem.

Theorem 2.7. *For any k, $|\lambda_0| < k < |\lambda_1|$ and $b \geq 0$, there is a positive solution $u(x)$ of problem (2.1), (2.2). It satisfies the estimate $u(x) \leq 1/a$ in $\bar{\Omega}$.*

Remark 2.8. Each eigenvalue λ_i of the Laplace operator is a bifurcation point of problem (2.8), (2.9). This means that when the value of k passes for example $|\lambda_1|$, there are two other solutions of this problem that bifurcate from the trivial solution. These solutions have a variable sign for k close to the critical value since only the principal eigenfunction is positive. Due to Lemma 2.4, these solutions have variable sign under further increase of k and for positive b. Therefore, there are no other solutions which can enter the domain D. For the same reason, positive solutions from D cannot merge with the trivial solution. Hence the assertion of the theorem remains true for all positive k.

2.3 Problem in $\mathbb{R}^n$

Existence of solutions in unbounded domains can be proved approximating them by a sequence of bounded domains and passing to the limit. In this section we consider a problem in the whole space and present a linear stability analysis which shows that the homogeneous in space solution can lose its stability. In this case some periodic in space solutions appear. They determine the behavior of generalized travelling waves studied below.

Consider equation (2.1) in $\mathbb{R}^n$:

$$\Delta u + ku \left(1 - au - b \int_{\mathbb{R}^n} \phi(x-y)u(y)dy \right) = 0. \tag{2.12}$$

It has a constant solution $u_0(x) \equiv 1/(a+b)$. The eigenvalue problem for the operator linearized about $u_0(x)$ has the form:

$$\Delta v - k_1 v - k_2 \int_{\mathbb{R}^n} \phi(x-y)v(y)dy = \lambda v, \tag{2.13}$$

where

$$k_1 = \frac{ka}{a+b}, \quad k_2 = \frac{kb}{a+b}.$$

We consider the linearized operator $\mathcal{L}$ on $L^2(\mathbb{R}^n)$, such that

$$\mathcal{L}u := -\Delta u + k_1 u + k_2 \int_{\mathbb{R}^n} \phi(x-y)u(y)dy, \quad k_1 \geq 0, \quad k_2 > 0$$

with $\phi(x) \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. The spectrum of this operator is denoted $\sigma(\mathcal{L})$. The problem

$$\mathcal{L}u = \lambda u$$

is considered to be *spectrally stable* if $\sigma(\mathcal{L}) \subset [0, +\infty)$ and *spectrally unstable* if there exists $\lambda < 0,\ \lambda \in \sigma(\mathcal{L})$. For the quadratic form of this operator we easily obtain

$$(\mathcal{L}u, u) = \int_{\mathbb{R}^n} [p^2 + k_1 + k_2(2\pi)^{\frac{n}{2}}\widehat{\phi}(p)]|\widehat{u}(p)|^2 dp. \tag{2.14}$$

Here and further down $p^2 = p_1^2 + ... + p_n^2$ and the hat symbol stands for the Fourier transform, such that

$$\widehat{u}(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x)e^{-ipx}dx,$$

where $px = p_1x_1 + ... + p_nx_n$. Let us introduce

$$\Phi(p) := p^2 + k_1 + k_2(2\pi)^{\frac{n}{2}}\widehat{\phi}(p).$$

We give here some simple qualitative properties of the spectrum. Its more detailed analysis in the particular case $k_1 = 0$ is presented in the appendix.

1. First let us consider the situation in one dimension, such that

$$\phi(x) = \frac{1}{2N}\chi_{[-N,N]}(x),\ x \in \mathbb{R},\ N > 0$$

and χ stands for the characteristic function of a set. A trivial computation yields

$$\widehat{\phi}(p) = \frac{1}{\sqrt{2\pi}}\frac{sinpN}{pN},\ p \in \mathbb{R}$$

and therefore, $\Phi(p) = p^2 + k_1 + k_2\frac{sinpN}{pN}$. Qualitatively speaking, when $\frac{5\pi}{4N} < p < \frac{7\pi}{4N}$, k_2 is large enough and k_1 is small, we have $\Phi(p) < 0$. Thus by choosing the trial function $\widehat{u}(p)$ with support on the interval $[\frac{5\pi}{4N}, \frac{7\pi}{4N}]$ by means of (2.14) we obtain $(\mathcal{L}u, u) < 0$ which by means of the min-max principle (see [14]) implies the existence of the negative spectrum for the operator $\mathcal{L}$ and, therefore, instability. On the other hand, $\Phi(p)$ is a continuous function with limit at zero equal to $k_1 + k_2 > 0$, the term $\frac{sinpN}{pN}$ is bounded. Thus when k_1 is large enough and k_2 is small we have the positivity of $\Phi(p)$ on the whole real line and therefore, stability.

2. In two dimensions we consider

$$\phi(x, y) = \frac{1}{4N^2}\chi_{\{(x,y)\in\mathbb{R}^2 | -N\leq x\leq N,\ -N\leq y\leq N\}},\ N > 0.$$

By applying the Fourier transform we easily obtain

$$\widehat{\phi}(p) = \frac{1}{2\pi N^2}\frac{sin(p_1N)}{p_1}\frac{sin(p_2N)}{p_2},\ p = (p_1, p_2) \in \mathbb{R}^2.$$

Thus $\Phi(p) = p^2 + k_1 + \frac{k_2}{N^2}\frac{sin(p_1N)}{p_1}\frac{sin(p_2N)}{p_2}$. Let us choose the trial function $\widehat{u}(p)$ with support in the domain $[\frac{5\pi}{4N}, \frac{7\pi}{4N}] \times [\frac{\pi}{4N}, \frac{3\pi}{4N}] \in \mathbb{R}^2$. When k_2 is large enough and k_1 is small, we have $(\mathcal{L}u, u) < 0$ and therefore, spectral instability. The function $\Phi(p)$ has a limit at the origin equal to $k_1 + k_2 > 0$, the term $\dfrac{1}{N^2}\dfrac{sin(p_1N)}{p_1}\dfrac{sin(p_2N)}{p_2}$ is bounded and continuous and therefore, when k_1 is large enough and k_2 is small, $\Phi(p)$ is positive in $\mathbb{R}^2$ and we have spectral stability.

We conclude this section with a more detailed analysis of the spectrum of the linearized operator in a particular case. We consider the linearized operator $\mathcal{L}$ on $L^2(\mathbb{R}^n),\ n \in \mathbb{N}$, with $k_1 = 0$, such that

$$\mathcal{L}u := -\Delta u + k_2 \int_{\mathbb{R}^n} \phi(x - y)u(y)dy, \quad k_2 > 0.$$

Clearly its quadratic form is given by $(\mathcal{L}u, u) = \int_{\mathbb{R}^n} \Phi(p)|\widehat{u}(p)|^2dp$, where $\Phi(p) = p^2 + (2\pi)^{\frac{n}{2}}k_2\widehat{\phi}(p)$ and $p \in \mathbb{R}^n$ and the spectral problem for it on $L^2(\mathbb{R}^n)$ is

$$\mathcal{L}u = \lambda u. \tag{2.15}$$

The first two examples of the kernel function are in two dimensions. We choose

$$\phi(x, y) = \frac{1}{2N}\chi_{[-N,N]}(x)\sqrt{\frac{\alpha}{\pi}}e^{-\alpha y^2}, \quad x, y \in \mathbb{R} \tag{2.16}$$

and

$$\phi(x, y) = \frac{1}{2N}\chi_{[-N,N]}(x)\frac{\alpha}{2}e^{-\alpha|y|}, \quad x, y \in \mathbb{R}, \tag{2.17}$$

where α and N are positive parameters. Clearly $\phi(x, y) \ge 0,\ x, y \in \mathbb{R}$ and an easy computation yields $\int_{\mathbb{R}^2} \phi(x, y)dxdy = 1$ in both cases. The third example of the nonnegative kernel used in the nonlocal term of the operator $\mathcal{L}$ is the four dimensional generalization of the second one, namely

$$\phi(x, y) = \frac{1}{2N}\chi_{[-N,N]}(x)\frac{\alpha^3}{8\pi}e^{-\alpha|y|}, \quad x \in \mathbb{R}, y \in \mathbb{R}^3, \quad \alpha, N > 0. \tag{2.18}$$

A straightforward computation gives $\int_{\mathbb{R}^4} \phi(x, y)dxdy = 1$. We have the following statement on the regions of stability and instability for the spectral problem (2.15) in all of these examples.

Theorem 2.9 *Let the kernel ϕ be given either by (2.16), (2.17) or (2.18) . Then the problem (2.15) is spectrally stable if $\dfrac{1}{k_2} \ge -N^2\dfrac{sinz_1}{z_1^3}$ and spectrally unstable if $\dfrac{1}{k_2} < -N^2\dfrac{sinz_1}{z_1^3}$.*

The proof of the theorem is given in the appendix.

3 Generalized travelling waves

Equation (2.1) can have a nontrivial solution which bifurcates either from the trivial solution $u = 0$ (Section 2.2) or from the constant solution $u = 1/(a + b)$ (Section 2.3). Therefore, we can expect the existence of propagating solutions providing the transition between $u = 0$ and the nontrivial solution. Such propagating solutions are called generalized travelling waves. We prove here their existence in the 1D case. Similar results can be obtained for multi-dimensional equations in the whole space or in cylindrical domains.

3.1 Some properties of the parabolic problem

Consider the problem

$$\frac{\partial u}{\partial t} = \Delta u + ku \left(1 - au - b \int_{\Omega} \phi(x-y) u(y,t) dy \right) \tag{3.1}$$

with the boundary condition

$$u|_{\partial\Omega} = 0 \tag{3.2}$$

and initial conditions

$$u(x, 0) = u^0(x), \tag{3.3}$$

where the function $u^0(x)$ satisfies (3.2). The domain Ω can be bounded or unbounded. Its boundary is $C^{2+\alpha}$. By standard arguments we can easily prove the following results.

Lemma 3.1. *Let $u^0(x) \in C^0(\bar{\Omega})$ and $0 \le u^0(x) \le 1/a$ for all $x \in \Omega$. If the solution $u(x,t)$ of problem (3.1)-(3.3) exists, then it satisfies the estimate*

$$0 \le u(x,t) \le \frac{1}{a}\,, \quad x \in \bar{\Omega}, \ \ t \ge 0.$$

For an arbitrary $T > 0$, let $C^{1+\alpha/2, 2+\alpha}(Q_T)$ be the space of all functions of class $C^{1+\alpha/2}$ with respect to t and of class $C^{2+\alpha}$ with respect to x, defined on Q_T.

Theorem 3.2. *If $0 \le u^0(x) \le 1/a$, $(\forall)\ x \in \Omega$, then there exists a global solution of problem (3.1)-(3.3), $u \in C^{1+\alpha/2,2+\alpha}(Q_T)$, with $Q_T = \bar{\Omega} \times [0, T]$, for any arbitrary $T > 0$.*

3.2 Generalized travelling waves in the 1D case

Consider the Cauchy problem for the $1D$ equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + ku \left(1 - au - b \int_{-\infty}^{\infty} \phi(x-y) u(y,t) dy \right), \tag{3.4}$$

where $x \in \mathbb{R}$. Here $c \geq 2\sqrt{k}$ is a given constant. Assume that the initial condition $u(x,0) = u^0(x)$ is nonnegative. Then the solution $u(x,t)$ exists and is also nonnegative for all $t \geq 0$. (This can be proved like in Theorem 3.2.) Therefore

$$I(x,t) \equiv \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy \geq 0.$$

Hence

$$d(x,t) \equiv 1 - au - bI(x,t) \leq 1.$$

We can write equation (3.4) in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + kd(x,t)u. \tag{3.5}$$

Consider also the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + c\frac{\partial v}{\partial x} + kv \tag{3.6}$$

and denote $z = v - u$. Taking the difference of equations (3.6) and (3.5), we obtain

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + c\frac{\partial z}{\partial x} + kz + k(1 - d\,(x,t))u. \tag{3.7}$$

Since the last term in the right-hand side of this equation is nonnegative, then from the inequality $z(x,0) \geq 0$ for all $x \in \mathbb{R}$ it follows that $z(x,t) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. Hence,

$$u(x,0) \leq v(x,0),\ x \in \mathbb{R} \ \Rightarrow\ u(x,t) \leq v(x,t),\ x \in \mathbb{R},\ t \geq 0. \tag{3.8}$$

The functions

$$v_c^1(x) = k_1 e^{-\sigma_1 x},\ v_c^2(x) = k_1 e^{-\sigma_2 x},\ \sigma_1 = \frac{c}{2} - \sqrt{\frac{c^2}{4} - k},\ \sigma_2 = \frac{c}{2} + \sqrt{\frac{c^2}{4} - k}$$

are stationary solutions of equation (3.6) if $c > 2\sqrt{k}$. In the special case when $c = 2\sqrt{k}$, we have $\sigma_1 = \sigma_2 = \sqrt{k}$, so

$$v_c^1(x) = k_1 e^{-\sqrt{k}x},\ v_c^2(x) = k_1 x e^{-\sqrt{k}x}.$$

Denote by $v_c\,(x)$ any of them. For $x \in \mathbb{R}$, from (3.8) it follows

$$u(x,0) \leq v_c(x),\ x \in \mathbb{R} \ \Rightarrow\ u(x,t) \leq v_c(x),\ x \in \mathbb{R},\ t \geq 0. \tag{3.9}$$

Thus, we obtain an estimate from above of the solution of the Cauchy problem associated to (3.5). We now estimate it from below. From the last inequality,

$$I(x,t) \leq \int_{-\infty}^{\infty} \phi(x-y)v_c(y)dy \equiv J(x).$$

Consider the equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial x} + kw(1 - av_c - bJ(x)) \tag{3.10}$$

and denote $s = u - w$. Then

$$\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} + c\frac{\partial s}{\partial x} + ks(1 - av_c - bJ(x)) + kbu(J(x) - I(x,t)) + kau(v_c - u). \tag{3.11}$$

Since the last two terms in the right-hand side of this equation are nonnegative, then

$$w(x,0) \le u(x,0),\ x \in \mathbb{R} \ \Rightarrow\ w(x,t) \le u(x,t),\ x \in \mathbb{R},\ t \ge 0. \tag{3.12}$$

We will take as w a stationary solution $w_c(x)$ of equation (3.10). It verifies the equation

$$w'' + cw' + kw(1 - av_c - bJ(x)) = 0. \tag{3.13}$$

Lemma 3.3. *If $c \ge 2\sqrt{k}$, there exists a solution $w_c(x)$ of* (3.13) *such that $w_c(x^0) = 0$ for some x^0, $w_c(x) > 0$ for $x > x^0$, $w_c(x) < 0$ for $x < x^0$ and*

$$w_c(x) \sim k_2 e^{-\sigma_1 x}\ (c > 2\sqrt{k}) \text{ or } w_c(x) \sim k_2 x e^{-\sqrt{k}x}\ (c = 2\sqrt{k}) \text{ as } x \to +\infty. \tag{3.14}$$

Proof. Since

$$1 - av_c - bJ(x) \to 0 \text{ as } x \to +\infty,$$

then the solutions of (3.13) are exponentially decaying at $+\infty$. More exactly,

$$w_c(x) \sim k_2 e^{-\sigma_1 x} \text{ or } w_c(x) \sim k_2 e^{-\sigma_2 x},\ x \to +\infty,\ \text{for } c > 2\sqrt{k}, \tag{3.15}$$

and

$$w_c(x) \sim k_2 e^{-\sqrt{k}x} \text{ or } w_c(x) \sim k_2 x e^{-\sqrt{k}x},\ x \to +\infty,\ \text{for } c = 2\sqrt{k}, \tag{3.16}$$

respectively, with some real k_2. So we can take them positive.

Then we prove the existence of a solution which has a zero. Denote $g(x) = 1 - av_c - bJ(x)$ and let $I = (-\infty, \omega)$ be the interval where $g(x) \le 0$. Then any nonzero solution of (3.13) has at most one zero in I. Indeed, let $w(x)$ be solution of (3.13) and x^0 one of its zeros. Put

$$W(x) = e^{c\left(x - x^0\right)} w(x)\, w'(x),\ x \in I.$$

Taking into account the equation (3.13), we have

$$W'(x) = e^{c\left(x - x^0\right)} \left[(w')^2 - g(x)\, w^2\right],\ x \in I,$$

so W is nondecreasing on I. If W has another zero $x^1 \in I$, then $W(x) = 0$ on $[x^0, x^1]$. Thus w is constant on $[x^0, x^1]$, namely $w = 0$ (from (3.13)) on $[x^0, x^1]$ and consequently on I. The contradiction shows that w has at most one zero in I.

Let w_c be a solution of (3.13), satisfying (3.15) (for $c > 2\sqrt{k}$) or (3.16) (for $c = 2\sqrt{k}$).

Case 1. If w_c has a unique zero in I, say x^0, we may suppose that $w_c(x) < 0$, for $x < x^0$. If it is not the case, we change the sign of the solution and thus the same inequality is verified.

Case 2. If w_c preserves a constant sign on I, say $w_c(x) > 0$, then the general solution of (3.13) is given by

$$w(x) = w_c(x)\left[c_1 + c_2 \int_{a_0}^{x} \frac{e^{-c(t-a_0)}}{w_c^2(t)} dt\right], \ x \in \mathbb{R},$$

where a_0 is an arbitrary fixed number and c_1, c_2 are real constants. Denote by $h(x)$ the square bracket above. If $c_2 > 0$, then h is strictly increasing on $\mathbb{R}$. We can chose $c_1 < 0$ and $c_2 > 0$ such that h has an only zero x^0 and $h(x) < 0$ for $x < x^0$, $x \in I$. Then, w has the same property, i. e. $w(x) < 0$ for $x < x^0$. In addition, $w(x)$ behaves as $k_2 e^{-\sigma_1 x}$, if $c > 2\sqrt{k}$ and like $k_2 x e^{-x}$, if $c = 2\sqrt{k}$. Indeed, we have the following situations.

1. If $c > 2\sqrt{k}$ and $w_c(x) \sim e^{-\sigma_2 x}$ as $x \to +\infty$, then the integral grows and $w(x) \sim e^{-\sigma_1 x}$, that is w decays slower than the corresponding solution $v_c(x) = k_1^{-\sigma_2 x}$ of (3.6). Hence we cannot have the estimate $w(x) \leq u(x, 0) \leq v_c(x)$ for the initial condition.

2. If $c > 2\sqrt{k}$ and $w_c(x) \sim e^{-\sigma_1 x}$ as $x \to +\infty$, then the integral is bounded and $w(x) \sim e^{-\sigma_1 x}$. We can have the estimate $w(x) \leq u(x, 0) \leq v_c(x) = k_1 e^{-\sigma_1 x}$ for the initial condition.

3. If $c = 2\sqrt{k}$ and $w_c(x) \sim e^{-\sigma_1 x} = e^{-\sqrt{k}x}$, the integral grows like x, $w(x) \sim x e^{-\sqrt{k}x}$; there is no appropriate estimate for the initial condition.

4. If $c = 2\sqrt{k}$ and $w_c(x) \sim x e^{-\sigma_1 x} = x e^{-\sqrt{k}x}$, the integral is bounded and $w(x) \sim x e^{-\sqrt{k}x}$. In this case we can have the estimate for the initial condition.

Thus we have shown that in all cases there exists a solution of (3.13). Denote it again by $w_c(x)$, such that $w_c(x) < 0$ for $x < x^0$, $w_c(x) > 0$ for $x > x^0$ and (3.14) holds. The claim is proved.

Lemma 3.4. *Let* $z_1(x) = \max(0, w_c(x))$ *and* $z_2(x_1) = \min(1/a, v_c(x))$. *If*

$$z_1(x) \leq u^0(x) \leq z_2(x), \ \ x \in \mathbb{R},$$

then the solution of the Cauchy problem for equation (3.4) with the initial condition $u^0(x)$ *satisfies the estimate*

$$z_1(x) \leq u(x,t) \leq z_2(x), \ \ x \in \mathbb{R},$$

for all $t \geq 0$.

The proof of this lemma is based on the maximum principle. It is standard and we omit it.

Definition 3.5. Generalized travelling wave (GTW) of equation (3.1) is a nontrivial solution $u(x,t)$ of this equation defined for all $t \in \mathbb{R}$. If for some $a > 0$, the maximal solution $x = m_a(t)$ of the equation $u(x,t) = a$ is defined, $m_a(t)/t \to c$ as $t \to \infty$ and for any $b \neq a$, $\overline{\lim}_{t\to\infty} m_b(t)/t \leq c$, then the generalized travelling wave has the speed c.

Theorem 3.6. *There exist positive GTW solutions of equation (3.1) for all $c \geq 2\sqrt{k}$. Positive GTW converging to zero as $x \to \infty$ do not exist for $c < 2\sqrt{k}$.*
Proof. The existence of GTWs for all $c \geq 2\sqrt{k}$ follows from the previous lemma. Indeed, consider solution of equation (3.1) in the form $u(x,t) = w(x - ct, t)$. Then

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial x} + kw \left(1 - aw - b\int_{-\infty}^{\infty} \phi(x-y)w(y,t)dy\right). \qquad (3.17)$$

It follows from Lemma 3.4 that there exists an ω-limit solution $w_c(x,t)$ of equation (3.17) such that

$$z_1(x) \leq w_c(x,t) \leq z_2(x), \quad x \in \mathbb{R}, \qquad (3.18)$$

for all $t \in \mathbb{R}$. In order to construct this solution, consider the solution $w(x,t)$ of equation (3.17) with an initial condition $w_0(x)$ which satisfies the inequality $z_1(x) \leq w_0(x) \leq z_2(x)$ for all x. Let $t_n \to \infty$ as $n \to \infty$. Consider next solutions $w_n(x,t)$ with the initial conditions $w_0^n = w(x,t_n)$. Obviously, each of them is defined for $t \geq -t_n$. A locally convergent subsequence of the sequence of functions $w_n(x,t)$ is a solution of equation (3.17) defined for all $t \in \mathbb{R}$. It satisfies inequality (3.18). It can be easily verified that it is a GTW with the speed c.

Suppose now that there exists a positive GTW $w_c(x,t)$, converging to 0 as $x \to \infty$, with a speed $c < 2\sqrt{k}$. Then $w_c(x-ct,t)$ satisfies equation (3.17). Let us take $c < c_0 < 2\sqrt{k}$ and consider the equation

$$w'' + c_0 w' + kw = 0.$$

It has a solution $w_0(x) = \exp(-c_0 x/2)\sin(ax)$, where $a = \sqrt{|c_0^2/4 - k|}$. Therefore, equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial x} + kw \qquad (3.19)$$

has a solution $w_*(x,t) = \epsilon w_0(x - (c_0 - c)t)$, where ϵ is a positive constant. Let $x = N_1$ and $x = N_2$ be two consecutive zeros of the function $w_0(x)$ such that w_0 is positive between them. Then $w_*(x,t)$ is a solution of the initial boundary value problem for equation (3.19) in the domain

$$N_1 + (c_0 - c)t \leq x \leq N_2 + (c_0 - c)t$$

with the zero boundary conditions. For ϵ small enough, similarly to (3.12) we can obtain the inequality

$$w_*(x,t) < w_c(x - ct, t), \quad N_1 + (c_0 - c)t \leq x \leq N_2 + (c_0 - c)t.$$

Since $c_0 > c$ and $w_c(x,t)$ converges to zero as $x \to \infty$, then the last inequality contradicts the assumption that $w_c(x,t)$ is a GTW with the speed c. Indeed, if $m_a(t)$ is the maximal solution of the equation

$$w_c(x,t) = a, \quad 0 < a < \max_{N_1+(c_0-c)t \leq x_1 \leq N_2+(c_0-c)t} w_*(x,t),$$

then

$$\overline{\lim}_{t\to\infty} m_a(t)/t \geq c_0 > c.$$

This contradiction proves the theorem.

4 Numerical simulations

In this section we present the results of numerical simulations of equation (1.1) with $a = 0$ in one and two space dimensions. We begin with the 1D case. The function $\phi(x)$ is piece-wise constant with the support $I = [\xi_1, \xi_2]$, that is $\phi(x) = 1/(\xi_2 - \xi_1)$ inside this interval and zero outside.

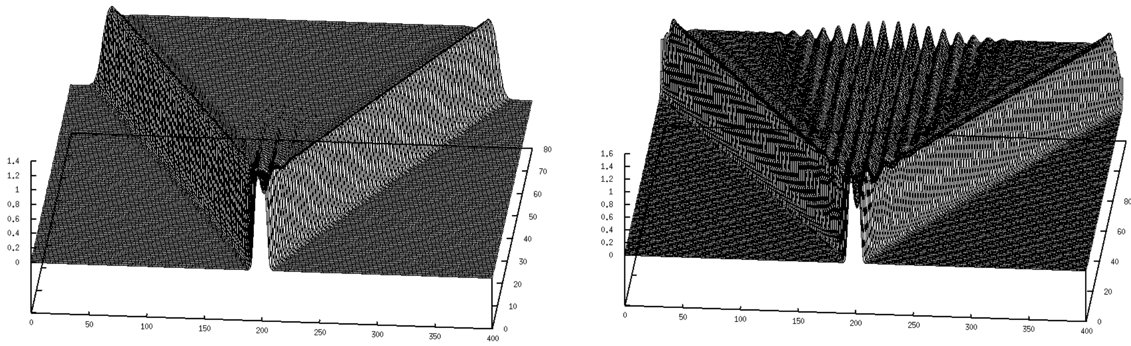

Figure 1: Travelling wave (left) and periodic travelling wave (right) in 1D.

If the support of ϕ is symmetric and sufficiently small, then there is a usual travelling wave propagating with a constant speed. Figure 1 (left) shows the solution $u(x,t)$ of equation (1.1) with the initial condition which has a bounded support. The solution represents two waves propagating in the opposite directions. It is interesting to note that the wave is not monotone with respect to x. Such waves can exist for the scalar reaction-diffusion equations but they are unstable. If we increase the support of the function ϕ, then the homogeneous in space stationary solution $u_1 = 1/(a+b)$ loses its stability and a periodic in space structure appears. In this case we observe propagation of a periodic wave (Figure 1, right).

The structure of solution is different if the function ϕ is not symmetric. Figure 2 shows the solution $u(x,t)$ (left) and its level lines (right). We observe spatio-temporal oscillations behind the wave front. This behavior is related to the fact that loss of stability of the homogeneous in space solution u_1 results in this case in the emergence of oscillating in time spatially distributed solutions [4]. Qualitatively, they can behave as $\sin(sx - \omega t)$. Such sinusoidal waves interact with the wave front possibly creating more complex structures.

The initial condition $u(x,0)$ in the simulations described above is a function with a finite support. In the case of the scalar reaction-diffusion equation, solutions of the Cauchy problem converge in this case to the waves with a minimal speed. In order to obtain the convergence to other waves, the initial condition should decay exponentially at infinity with

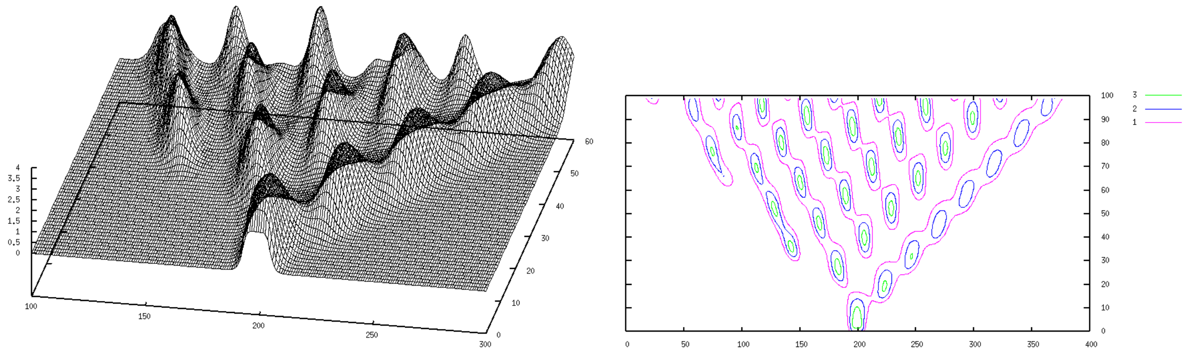

Figure 2: Generalized travelling wave in the case of asymmetric function ϕ. Function $u(x,t)$ (left) and its level lines (right).

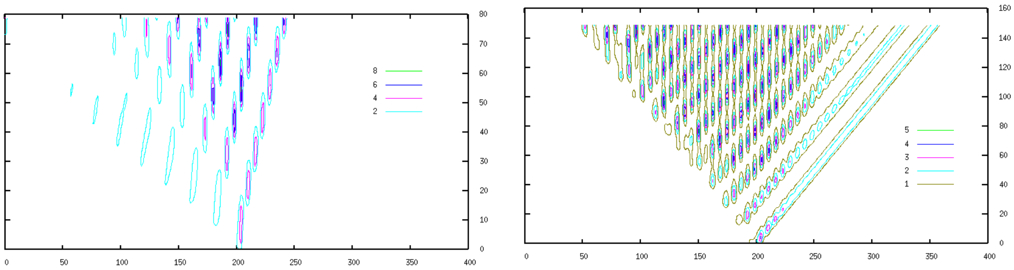

Figure 3: Generalized travelling waves with exponential initial conditions. Level lines of the function $u(x,t)$ with different initial conditions.

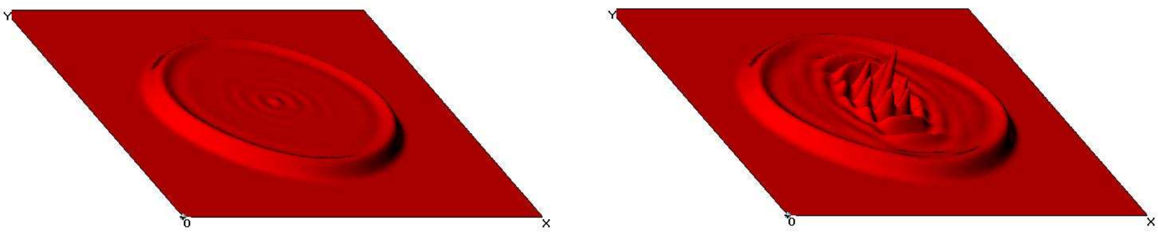

Figure 4: Snapshot of the solution in the case of circular (left) and elliptic (right) support of the function ϕ.

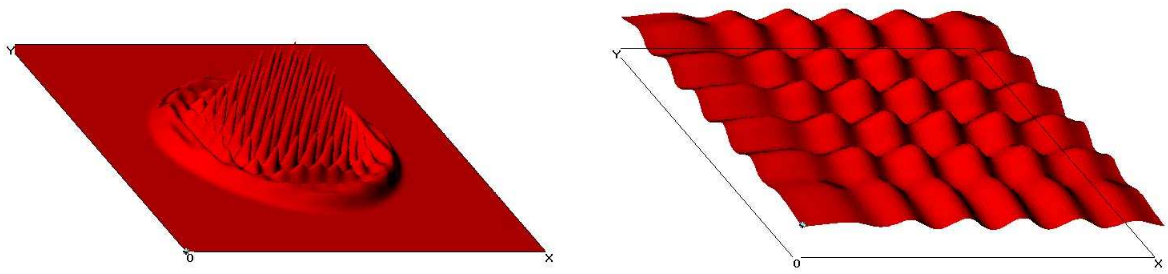

Figure 5: Snapshot of the solution in the case of square support of the function ϕ (left); asymmetric support of ϕ (right).

the same exponent as the wave. Similar behavior is observed for the integro-differential equation. We have proved in the previous section that positive GTW exist for all values of the speed greater or equal to some minimal speed. Figure 3 shows the level lines of the solution $u(x,t)$ with the same values of parameters as in Figure 2 but with the exponentially decaying initial conditions. Though we consider in numerical simulations a finite interval, if it is sufficiently large, then the solution can approach the corresponding GTW. We recall that the function ϕ is not even. Therefore, even if the initial condition is even, the solution is not (Figure 3, right). The speeds of the left and of the right waves are the same. They are different if the decay rates of the initial condition at the left and at the right differ from each other (Figure 3, left).

Let us now discuss the results of two-dimensional simulations. As in the 1D case, the function ϕ is piece-wise constant. However, we need now to specify the form of its support. It appears that it influences the properties of the GTW. Figure 4 shows the solution $u(x,t)$ at some fixed t. The initial condition has the support in the center of the computation domain. In the case where the support of the function ϕ s circular (Figure 4, left), the wave front is also circular and there is a weak circular structure behind the front. If the support is elliptic (Figure 4, right), then the wave front remains circular. However, there are some elliptic structures behind the front followed by the region with strongly pronounced picks. In the case of the square support and with the same values of parameters (Figure 5, left), the wave front remains circular with square structures and even more pronounced picks behind.

If the support of the function ϕ is not symmetric, then the structure emerging after the wave propagation is not stationary. In the case of the square support, it is shown in Figure 5 (right) at some fixed moment of time. Observing its evolution in time we can notice that it moves along the diagonal of the computational domain. The direction of its motion is determined by the support of ϕ, which is shifted along the diagonal from its symmetric configuration.

To conclude the description of the numerical simulations, we note that they confirm the theoretical results presented in the previous sections. The nonuniqueness of GTW and the variety of their structures, revealed in this work, require further investigations.

Acknowledgement

The first author's work was supported by the grant ID 342/2009, CNCSIS, Romania.

References

[1] N. Apreutesei, A. Ducrot, V. Volpert, Travelling waves for integro-differential equations in population dynamics, Discrete Cont. Dynam. Systems Ser. B, 11 (2009), no. 3, 541-561.

[2] N. Apreutesei, A. Ducrot, V. Volpert, Competition of species with intra-specific competition, Math. Model. Nat. Phenom., 3 (2008), no. 4, 1-27.

[3] H. Berestycki, F. Hamel. Front propagation in periodic excitable media. Comm. Pure Appl. Math. 55 (2002), No. 8, 949–1032.

[4] N.F. Britton. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. SIAM J. Appl. Math., 50 (1990), No. 6, 1663-1688.

[5] V. Capasso. Mathematical structure of epidemic systems. Lecture Notes in Biomathematics, **97**, Springer-Verlag, Heidelberg, 1993.

[6] L. Desvillettes, C. Prevost, R. Ferrieres. *Infinite Dimensional Reaction-Diffusion for Population Dynamics.* Preprint No. 2003-04 du CMLA, ENS Cachan.

[7] A. Ducrot. *Travelling wave solutions for a scalar age-structured equation*, Discrete Contin. Dyn. Syst. Ser. B , **7** (2007), 251–273.

[8] S. Genieys, V. Volpert, P. Auger. Pattern and waves for a model in population dynamics with nonlocal consumption of resources. Mathem. Modelling of Natural Phenomena, (2006), **1**, No. 1.

[9] S. Genieys, V. Volpert, P. Auger. Adaptive dynamics: modelling Darwin's divergence principle. Comptes Rendus Biologies, 329 (11), 876-879 (2006).

[10] S. A. Gourley, *Travelling front solutions of a nonlocal Fisher equation*, J. Math. Biol., **41** (2000), 272–284.

[11] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov. *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique,* B. Univ. d'Etat à Moscou, Sér. Intern. A 1 (1937), pp.1-26.

[12] R. Lefever and O. Lejeune, *On the origin of tiger bush*, Bull. Math. Biology, **59** (1997), 163–294.

[13] E. Lieb, M. Loss: "Analysis", Graduate studies in Mathematics, 14, American Mathematical Society, Providence, RI, (1997)

[14] E. Lieb, B. Simon, A. Wightman. Studies in Mathematical Physics: Essays in Honor of Valentine Bargmann, Princeton University Press, 1976.

[15] H. Matano, K.I. Nakamura, B. Lou. Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit. NHN, 4 (2006), 537-568.

[16] B. Perthame and S. Genieys, *Concentration in the nonlocal Fisher equation: The Hamilton-Jacobi limit*, Math. Model. Nat. Phenom., **4** (2007), 135–151.

[17] S. Ruan. Spatial-temporal dynamics in nonlocal epidemiological models, in "Mathematics for Life Science and Medicine", Y. Iwasa, K. Sato and Y. Takeuchi, eds., Springer-Verlag, New York, 2007, pp. 99-122.

[18] S. Vakulenko, V. Volpert. *Generalized travelling waves for perturbed monotone reaction-diffusion systems.* Nonlinear Analysis. TMA 2001 (46) 757-776.

[19] S. Vakulenko, V. Volpert. *New effects in propagation of waves for reaction-diffusion systems.* Asymptotic Analysis, 38(1) (2004),11-33.

[20] S. Vakulenko, V. Volpert. Complex wave dynamics for reaction-diffusion systems. "Patterns and Waves", A. Abramian, S. Vakulenko, V. Volpert, Eds. St. Petersburg, 2003, pp. 221-239.

[21] A. Volpert, Vit. Volpert, Vl. Volpert. *Traveling wave solutions of parabolic systems.* Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence, 1994.

[22] V. Volpert, A. Volpert. Properness and topological degree for general elliptic operators. Abstract and Applied Analysis, 2003, No. 3, pp. 129-182.

[23] J. Xin, Front propagation in heterogeneous media. Siam Rev 42 (2000), 161-230.

5 Appendix. Proof of Theorem 2.9

We precede the proof of the theorem by an auxiliary result. Consider the following equation

$$tanz = \frac{1}{3}z. \tag{5.1}$$

We have the following technical statement concerning its solutions.

Lemma A1 *On each interval* $(\pi j - \frac{\pi}{2}, \pi j + \frac{\pi}{2}),\ j \in \mathbb{N}$ *the equation (5.1) has a unique solution* $z_j \in (\pi j, \pi j + \frac{\pi}{2})$. *Moreover, for odd values of* j *the following inequality holds*

$$-\frac{sin(z_{2n+1})}{z_{2n+1}^3} > -\frac{sin(z_{2n+3})}{z_{2n+3}^3}, \quad n \in \mathbb{N} \cup \{0\}. \tag{5.2}$$

Proof. First of all, the equation (5.1) does not have any solutions on the subintervals $(\pi j - \dfrac{\pi}{2}, \pi j],\ j \in N$ since the left and the right sides of the equation (5.1) have the opposite signs.

On a subinterval $(\pi j, \pi j + \dfrac{\pi}{2}),\ j \in \mathbb{N}$ consider the function $f(z) := tanz - \frac{1}{3}z$, which is continuous, monotonically increasing, negative near the left corner and positive near the right corner. Therefore, $f(z)$ has a unique zero $z_j \in (\pi j, \pi j + \dfrac{\pi}{2}),\ j \in \mathbb{N}$.

Let us turn our attention to the odd values of $j = 2n+1,\ n \in \mathbb{N} \cup \{0\}$. We write $z_{2n+3} = z_{2n+1} + 2\pi + \Delta z_{2n+1}$ with the term $\Delta z_{2n+1} > 0$. Indeed, $f(z_{2n+1} + 2\pi) = -\dfrac{2\pi}{3} < 0$, the point $z_{2n+3} \in (3\pi + 2\pi n, 3\frac{1}{2}\pi + 2\pi n)$, the monotonically increasing and continuous function $f(z)$ on this subinterval is positive near $7\pi/2 + 2\pi n$. Therefore, $z_{2n+3} > z_{2n+1} + 2\pi$.

An easy computation shows that proving the inequality (5.2) is equivalent to proving the positivity of the fraction

$$\frac{sin(z_{2n+1} + \Delta z_{2n+1}) - sin(z_{2n+1})\Big(1 + \frac{2\pi + \Delta z_{2n+1}}{z_{2n+1}}\Big)^3}{(z_{2n+1} + 2\pi + \Delta z_{2n+1})^3}.$$

We write its numerator as

$$-cos(z_{2n+1})\Big[-\frac{1}{3}z_{2n+1}cos(\Delta z_{2n+1}) - sin(\Delta z_{2n+1}) + \frac{1}{3}z_{2n+1}\Big(1 + \frac{2\pi + \Delta z_{2n+1}}{z_{2n+1}}\Big)^3\Big]$$

such that $cos(z_{2n+1}) < 0$ and the expression in square brackets can be estimated below as

$$\frac{1}{3}z_{2n+1}(1 - cos(\Delta z_{2n+1})) + \frac{2\pi}{3} + \frac{\Delta z_{2n+1}}{3} - sin(\Delta z_{2n+1}) > 0.$$

$\square$

Armed with this technical statement we prove the propositions about the regions of stability and instability mentioned above.

Proof of Theorem 2.9. *Step I.* A straightforward computation yields that the Fourier transform of the kernel given by (2.16) equals to $\widehat{\phi}(p) = \dfrac{1}{2\pi}\dfrac{sin(p_1 N)}{p_1 N}e^{-\frac{p_2^2}{4\alpha}},\ p = (p_1, p_2) \in \mathbb{R}^2$ and therefore, $\Phi(p) = p^2 + k_2\dfrac{sin(p_1 N)}{p_1 N}e^{-\frac{p_2^2}{4\alpha}}$. This function is smooth, bounded below, growing to infinity at infinity, attains the value of $k_2 > 0$ at the origin, positive if the coefficient k_2 is small enough and sign indefinite for values of k_2 sufficiently large. In the critical case it has the minimal value of zero, such that the following system of three equations is satisfied at its critical point

$$\begin{cases} \Phi(p) = 0, \\ \frac{\partial \Phi}{\partial p_1}(p) = 0 \\ \frac{\partial \Phi}{\partial p_2}(p) = 0\ . \end{cases} \tag{5.3}$$

Thus we have

$$\begin{cases} p^2 + k_2 \frac{sin(p_1 N)}{p_1 N} e^{-\frac{p_2^2}{4\alpha}} = 0 \\ 2p_1 + k_2 \left[\frac{cos(p_1 N)}{p_1} - \frac{sin(p_1 N)}{p_1^2 N} \right] e^{-\frac{p_2^2}{4\alpha}} = 0 \\ 2p_2 - k_2 \frac{sin(p_1 N)}{p_1 N} e^{-\frac{p_2^2}{4\alpha}} \frac{p_2}{2\alpha} = 0 \ . \end{cases}$$

The necessary condition for the compatibility of the first and the third equations in the system above is $p_2 = 0$, which reduces the system to

$$\begin{cases} p_1^2 + k_2 \frac{sin(p_1 N)}{p_1 N} = 0 \\ 2p_1 + k_2 \left[\frac{cos(p_1 N)}{p_1} - \frac{sin(p_1 N)}{p_1^2 N} \right] = 0 \ . \end{cases} \tag{5.4}$$

Since the function $\Phi(p)$ is even in the first variable, we consider only $p_1 > 0$. Let us introduce the new variable $z := p_1 N > 0$. The system of the two equations above easily implies, that it must satisfy the equation (5.1) with $sin z < 0$. To satisfy this condition we need to consider only the solutions z_j with odd values of $j = 2n+1, \ n \in \mathbb{N} \cup \{0\}$, such that $z_{2n+1} \in (\pi + 2\pi n, \dfrac{3\pi}{2} + 2\pi n)$. Therefore, the set of critical points of the function $\Phi(p), \ p \in \mathbb{R}^2, \ p_1 > 0$ satisfying the system (5.3) is given by

$$\left(\frac{z_{2n+1}}{N}, 0 \right), \ n \in \mathbb{N} \cup \{0\} \ . \tag{5.5}$$

Let us compute the second derivatives of $\Phi(p)$ at these points. Using the first equation in (5.4) along with (5.1) we obtain the Hessian matrix at (5.5):

$$\begin{pmatrix} 6 + z_{2n+1}^2 & 0 \\ 0 & 2 + \frac{z_{2n+1}^2}{2\alpha N^2} \end{pmatrix} ,$$

which is positive definite. This confirms that (5.5) are the minimal points of $\Phi(p)$. From the first equation of the system (5.4) we easily obtain the relation

$$\frac{1}{k_2} = -N^2 \frac{sin(z_{2n+1})}{z_{2n+1}^3}, \ n \in \mathbb{N} \cup \{0\} \ ,$$

which produces the family of parabolas on the $(N, \dfrac{1}{k_2})$ plane. For a fixed value of N the function $\Phi(p)$ is positive in a neighborhood of the point $\left(\dfrac{z_{2n+1}}{N}, 0 \right)$ if $\dfrac{1}{k_2} > -N^2 \dfrac{sin(z_{2n+1})}{z_{2n+1}^3}$ and negative near this point if $\dfrac{1}{k_2} < -N^2 \dfrac{sin(z_{2n+1})}{z_{2n+1}^3}$. By choosing the trial function $\widehat{u}(p)$ compactly supported in the region where $\Phi(p) < 0$ we show that the quadratic form of the operator $\mathcal{L}$ can attain the negative values which by means of the min-max principle yields the existence of the negative spectrum for the problem (2.15) and therefore, spectral instability.

Due to the statement of the Lemma A1 when increasing k_2 for a fixed N the transition to instability occurs at the boundary curve

$$\frac{1}{k_2} = -N^2 \frac{sin(z_1)}{z_1^3} \; . \tag{5.6}$$

Step II. When the kernel function is given by (2.17), its Fourier transform is equal to $\widehat{\phi}(p) = \frac{1}{2\pi} \frac{sin(p_1 N)}{p_1 N} \frac{\alpha^2}{p_2^2 + \alpha^2}$, $p = (p_1, p_2) \in \mathbb{R}^2$. Thus $\Phi(p) = p^2 + k_2 \frac{sin(p_1 N)}{p_1 N} \frac{\alpha^2}{p_2^2 + \alpha^2}$. Clearly, it has the qualitative properties analogous to those in the Step I. In the critical case by means of (5.3) we obtain the system

$$\begin{cases} p^2 + k_2 \frac{sin(p_1 N)}{p_1 N} \frac{\alpha^2}{p_2^2+\alpha^2} = 0 \\ 2p_1 + k_2 \left[\frac{cos(p_1 N)}{p_1} - \frac{sin(p_1 N)}{p_1^2 N} \right] \frac{\alpha^2}{p_2^2+\alpha^2} = 0 \\ 2p_2 - k_2 \frac{sin(p_1 N)}{p_1 N} \frac{2\alpha^2 p_2}{(p_2^2+\alpha^2)^2} = 0 \; . \end{cases}$$

The first and the third equations in it are compatible only if $p_2 = 0$, which implies the system (5.4). By the same argument as in the Step I, the sequence of critical points of $\Phi(p)$, $p \in \mathbb{R}^2$, $p_1 > 0$ satisfying the system (5.3) is given by (5.5). A straightforward computation yields the Hessian matrix for $\Phi(p)$ at these points

$$\begin{pmatrix} 6 + z_{2n+1}^2 & 0 \\ 0 & 2 + \frac{2z_{2n+1}^2}{\alpha^2 N^2} \end{pmatrix}, \; n \in \mathbb{N} \cup \{0\} \; .$$

Due to its positive definiteness the critical points (5.5) of the function $\Phi(p)$ are the points of local minima. By the argument analogous to the one in the Step I, the region of the spectral instability for the problem is located below the parabola (5.6) on the $\left(N, \frac{1}{k_2}\right)$ plane and the complement of this set is the stability region.

Step III. The proof for the last example about the regions of stability and instability has a lot in common with the previous two, with the principal exception that the problem now is four dimensional.

Having the formula for the heat kernel of the root of the Laplacian handy (see e.g. p.169 of [13]), we easily obtain

$$\widehat{e^{-\alpha|x|}}(p) = \frac{(2\pi)^{\frac{3}{2}} \alpha}{\pi^2 [\alpha^2 + p^2]^2}, \; p \in \mathbb{R}^3 \; .$$

Therefore, the Fourier transform of the function (2.18) equals to

$$\widehat{\phi}(p) = \frac{1}{(2\pi)^2} \frac{sin(p_1 N)}{p_1 N} \frac{\alpha^4}{[\alpha^2 + p_2^2]^2}, \; p = (p_1, p_{2,1}, p_{2,2}, p_{2,3}) \in \mathbb{R}^4$$

such that

$$\Phi(p) = p^2 + k_2 \frac{sin(p_1 N)}{p_1 N} \frac{\alpha^4}{[\alpha^2 + p_2^2]^2} \ .$$

It possesses the properties similar to the ones discussed in the previous two steps such that in the critical case we have the system

$$\begin{cases} \Phi(p) = 0 \\ \frac{\partial \Phi}{\partial p_1}(p) = 0 \\ \frac{\partial \Phi}{\partial p_{2,i}}(p) = 0, \ i = 1, 2, 3 \ . \end{cases} \tag{5.7}$$

Hence we arrive at

$$\begin{cases} p^2 + k_2 \frac{sin(p_1 N)}{p_1 N} \frac{\alpha^4}{[\alpha^2+p_2^2]^2} = 0 \\ 2p_1 + k_2 \left[\frac{cos(p_1 N)}{p_1} - \frac{sin(p_1 N)}{p_1^2 N} \right] \frac{\alpha^4}{[\alpha^2+p_2^2]^2} = 0 \\ 2p_{2,i} - k_2 \frac{sin(p_1 N)}{p_1 N} \frac{4\alpha^4 p_{2,i}}{[\alpha^2+p_2^2]^3} = 0, \ i = 1, 2, 3 \ . \end{cases}$$

The first and the third equations in this system are incompatible unless $p_2 = 0$, which reduces the system to (5.4). By the same reasoning as in the previous two steps the set of critical points of the function $\Phi(p), \ p \in \mathbb{R}^4, \ p_1 > 0$ satisfying the system (5.7) is given by

$$\left(\frac{z_{2n+1}}{N}, 0, 0, 0 \right), \ n \in \mathbb{N} \cup \{0\} \ . \tag{5.8}$$

Evaluating the Hessian matrix for $\Phi(p)$ at these points we arrive at

$$\begin{pmatrix} 6 + z_{2n+1}^2 & 0 & 0 & 0 \\ 0 & 2 + \frac{4z_{2n+1}^2}{\alpha^2 N^2} & 0 & 0 \\ 0 & 0 & 2 + \frac{4z_{2n+1}^2}{\alpha^2 N^2} & 0 \\ 0 & 0 & 0 & 2 + \frac{4z_{2n+1}^2}{\alpha^2 N^2} \end{pmatrix}, \ n \in \mathbb{N} \cup \{0\} \ .$$

The positive definiteness of the Hessian implies that (5.8) are the points of local minima of the function $\Phi(p)$. By the same reasoning as in the two previous steps the points on the $\left(N, \frac{1}{k_2} \right)$ plane located on the parabola (5.6) and above it correspond to spectral stability and below it to instability.

□