On the Allen-Cahn equation in the Grushin plane: a monotone entire solution that is not one-dimensional

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Abstract

We consider solutions of the Allen-Cahn equation in the whole Grushin plane and we show that if they are monotone in the vertical direction, then they are stable and they satisfy a good energy estimate.

However, they are not necessarily one-dimensional, as a counter-example shows.

1 Introduction

We consider here the Grushin plane G (see [**?**]), that is $\mathbb{R}^2$ endowed with the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. We also define $T := [X, Y] = \frac{\partial}{\partial y}$.

The Grushin gradient is then $\nabla_G := (X, Y)$ (with the coordinates taken in the (X, Y)-frame) and the Grushin Laplacian is $\Delta_G := X^2 + Y^2$.

We denote by $< \cdot, \cdot >$ the standard scalar product (when the vectors are taken in the (X, Y)-frame), so that, for a smooth function v we have

$$|\nabla_G v(\zeta)| = \sqrt{< \nabla_G v(\zeta), \nabla_G v(\zeta) >} = \sqrt{(Xv(\zeta))^2 + (Yv(\zeta))^2},$$

for any $\zeta \in \mathbb{R}^2$.

Moreover, the Grushin norm on G is defined as

$$\|(x, y)\| := \sqrt[4]{|x|^4 + 4|y|^2}$$

for any $(x, y) \in \mathbb{R}^2$, and then the Grushin ball of radius $R > 0$ centered at $\zeta \in \mathbb{R}^2$ is

$$B_R(\zeta) := \{\eta \in \mathbb{R}^2 \text{ s.t. } \|\eta - \zeta\| < R\}.$$

The main purpose of this paper is to study solutions of the Allen-Cahn equation in the Grushin plane, that is

$$\Delta_G u(\zeta) + f(u(\zeta)) = 0 \tag{1.1}$$

for any $\zeta \in \mathbb{R}^2$.

We take, for simplicity, $f \in C^1$, though less regularity is also possible to be dealt with. A particular case of interest is when $f = -W'$ and W is a double-well potential.

Namely, through this paper, we denote by W a function with the following properties: $W \in C^2(\mathbb{R})$ is an even function for which $W(\pm 1) = 0 \le W(r)$ for any $r \in \mathbb{R}$, $W''(0) \neq 0$, $W''(\pm 1) \neq 0$, and such that

$$W'(s) = 0 \text{ if and only if } s \in \{-1, 0, +1\}. \tag{1.2}$$

Inspired by a famous conjecture of De Giorgi (see [?]), one may wonder under which conditions the solutions of (??) are one-dimensional, i.e., their level sets are straight lines and so, up to rotation, they depend on only one variable (at least when $f = -W'$).

Natural requirements for such symmetry are monotonicity and stability conditions. Namely, if u is a solution of (??), we say that u is stable if

$$\int_{\mathbb{R}^2} |\nabla_G \phi(x)|^2 - \int_{\mathcal{G}_u} f'(u(x))\big(\phi(x)\big)^2 \, dx \ge 0 \tag{1.3}$$

for any $\phi \in C_0^\infty(\mathbb{R}^2)$.

Stability is a natural condition in the calculus of variation, since it states that the energy functional associated to (??) has non-negative second derivative. The stability condition has thus been widely used in connection with the problems posed by [?] (see, for instance, [?, ?] and references therein).

Also, in the Euclidean setting, the stability condition holds true whenever u is monotone in some direction. The analogy in the Grushin setting is somehow more delicate, since the space is not homogeneous with respect to the choice of a particular direction.

Thus, the monotonicity studied in this paper is the following. We are mostly concerned with solutions that are monotone in the y-direction, that is for which

$$Tu(\zeta) > 0 \text{ for any } \zeta \in \mathbb{R}^2. \tag{1.4}$$

We shall show that (??) implies (??) (see Proposition **??** below).

Symmetry properties for solutions of (??) in the Grushin plane have been recently studied in [?]. For instance, [?] pointed out the following result:

Theorem 1.1. *Let us assume that u is a stable solution of* (??) *in the whole* $\mathbb{R}^2$ *such that*

$$TYuXu(\zeta) - TXuYu(\zeta) \le 0 \qquad \textit{for any } \zeta \in \mathbb{R}^2.$$

Suppose that there exists $C_o \ge 1$ in such a way that

$$\int_{B(0,R)} x^2 |\nabla_G u|^2 \le C_o R^4, \tag{1.5}$$

for any $R \geq C_o$.
Assume also that

$$\nabla_G u(\zeta) \neq 0 \text{ for any } \zeta \in \mathbb{R}^2. \tag{1.6}$$

Then, u depends only on the x-variable.

We observe that (**??**) is also a sort of monotonicity condition, while (**??**) is an energy growth requirement (and energy bounds are often needed in the Euclidean case too, see [**?**]). We shall show that (**??**) implies also (**??**), at least when $f = -W'$ (see Theorem **??** below).

This said, a natural question arises. Namely,

Question 1.2. *Is it true that bounded solutions of* (**??**) *which satisfy* (**??**) *are one-dimensional (at least for $f = -W'$)?*

Note that one may be quite tempted to answer yes to such a question, since (**??**) implies both the stability condition and the good energy growth in (**??**) (again, see for this Proposition **??** and Theorem **??** here below).

The main purpose of this paper is in fact to show that the above question has a *negative answer.* In fact we shall prove more, i.e. that even if u is a global minimizer it does not imply that it is one dimensional.

This will be accomplished in Theorem **??**, by constructing a counter-example which follows the lines of the one in [**?**], i.e. we shall prove the existence of a solution that satisfies (**??**) and tends to 1 and -1 at infinity, and so it is also a global minimizer (see Proposition **??**) but is not one dimensional.

The paper is organized in the following way. After gathering some elementary observations in Section **??**, we point out in Section **??** that the monotonicity condition in (**??**) implies the stability condition in (**??**).

Then, we develop in Section **??** the energy estimates which show that the monotonicity condition in (**??**) also implies the energy growth in (**??**).

Finally, Section **??** contains the construction of the counter-example which shows that Question **??** has a negative answer.

2 Preliminaries

We collect in this section some useful observations.

2.1 General results

We begin with a variation of Green formula, complicated here by the non-homogeneous Grushin scaling.

Lemma 2.1. *Let $u \in \Lambda^2(\mathbb{R}^2)$, $v \in \Lambda^1(\mathbb{R}^2)$. Suppose that $|\nabla_G u| \in L^\infty(\mathbb{R}^2)$ and that $v \geq 0$.*
Then

$$\begin{aligned}\int_{B_R(0)} \Big(< \nabla_G u, \nabla_G v > + \Delta_G u v \Big) \\ \geq -R^2 \, \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} v(RX, R^2 Y) \, d\mathcal{H}^1(X,Y).\end{aligned} \tag{2.1}$$

Proof. We set $U(X,Y) := u(RX, R^2Y)$, $V(X,Y) := v(RX, R^2Y)$. Then, by changing variable, we have

$$\begin{aligned}&\int_{B_R(0)} \Big(< \nabla_G u, \nabla_G v > + \Delta_G u v \Big) \\ = &\int_{B_R(0)} \Big(\partial_x u(x,y)\partial_x v(x,y) + x^2 \partial_y u(x,y)\partial_y v(x,y) \\ &+\partial_{xx} u(x,y) v(x,y) + x^2 \partial_{yy} u(x,y) v(x,y)\Big) \, dx \, dy \\ = & R \int_{B_1(0)} \partial_X U(X,Y)\partial_X V(X,Y) + X^2 \partial_Y U(X,Y)\partial_Y V(X,Y) \\ &+\partial_{XX} U(X,Y) V(X,Y) + X^2 \partial_{YY} U(X,Y) V(X,Y) \, dX \, dY.\end{aligned}$$

Thence, by the standard Euclidean Divergence Theorem, using "$\cdot$" to denote the standard Euclidean scalar product and "ν^E" the standard Euclidean outward normal of $\partial B_1(0)$.

$$\begin{aligned}&\int_{B_R(0)} \Big(< \nabla_G u, \nabla_G v > + \Delta_G u v \Big) \\ = \; & R \int_{B_1(0)} \operatorname{div} \Big[V(X,Y) \Big(\partial_X U(X,Y), X^2 \partial_Y U(X,Y) \Big) \Big] \, dX \, dY \qquad (2.2) \\ = \; & R \int_{\partial B_1(0)} V(X,Y) \Big(\partial_X U(X,Y), X^2 \partial_Y U(X,Y) \Big) \cdot \nu^E(X,Y) \, d\mathcal{H}^1(X,Y) \\ = \; & R^2 \int_{\partial B_1(0)} v(RX, R^2Y) \Big(\partial_x u(RX, R^2Y), RX^2 \partial_y u(RX, R^2Y) \Big) \qquad (2.3) \\ & \cdot \nu^E(X,Y) \, d\mathcal{H}^1(X,Y).\end{aligned}$$

Since the Euclidean norm of ν^E is 1,

$$\begin{aligned}&\int_{B_R(0)} \Big(< \nabla_G u, \nabla_G v > + \Delta_G u v \Big) \\ &\geq -R^2 \int_{\partial B_1(0)} v(RX, R^2Y) \Big| \Big(\partial_x u(RX, R^2Y), RX^2 \partial_y u(RX, R^2Y) \Big) \Big|_E \, d\mathcal{H}^1(X,Y),\end{aligned} \tag{2.4}$$

where $|\cdot|_E$ is the Euclidean norm.
We now observe that, for any $(X,Y)\in\partial B_1(0)$, we have $|X|\le 1$ and

$$\begin{aligned}
& \left|\Big(\partial_x u(RX,R^2Y), RX^2\partial_y u(RX,R^2Y)\Big)\right|_E^2 \\
\le\ & \Big(\partial_x u(RX,R^2Y)\Big)^2 + R^2X^2\Big(\partial_y u(RX,R^2Y)\Big)^2 \\
=\ & |\nabla_G u(RX,R^2Y)| \\
\le\ & \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)}.
\end{aligned}$$

From this and (**??**) we get (**??**). □

We recall the following estimate:

Lemma 2.2. *Let $h\in C^2(\mathbb{R})$. Then,*

$$\|h'\|_{L^\infty(\mathbb{R})}\le 2\Big(\|h\|_{L^\infty(\mathbb{R})}+\|h''\|_{L^\infty(\mathbb{R})}\Big). \tag{2.5}$$

Proof. We may assume that both $\|h\|_{L^\infty(\mathbb{R})}$ and $\|h''\|_{L^\infty(\mathbb{R})}$ are finite, otherwise (**??**) is void. Then, for any $j\in\mathbb{Z}$, there exists $t_j\in[j,j+1]$ in such a way that

$$2\|h\|_{L^\infty(\mathbb{R})}\ge |h(j+1)-h(j)|=|h'(t_j)|.$$

Then, given any $j\in\mathbb{Z}$ and any $t\in[j,j+1]$,

$$\begin{aligned}
|h'(t)| &\le |h'(t_j)| + \left|\int_{t_j}^t h''(s)\,ds\right| \\
&\le 2\|h\|_{L^\infty(\mathbb{R})}+\|h''\|_{L^\infty(\mathbb{R})}|t-t_j| \le 2\|h\|_{L^\infty(\mathbb{R})}+\|h''\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

□

2.2 ODE analysis

The scope of this section is an analysis of the bounded solutions $h\in C^2(\mathbb{R})$ of

$$h''(t)=W'(h(t)) \qquad \text{for any } t\in\mathbb{R}. \tag{2.6}$$

Recall that for any any C^2 solution of (**??**) and any s, $t\in\mathbb{R}$,

$$\frac{|h'(s)|^2}{2}-W(h(s))=\frac{|h'(t)|^2}{2}-W(h(t)). \tag{2.7}$$

Furthermore, from the equation,

$$\|h\|_{C^2(\mathbb{R})}\le C, \tag{2.8}$$

for some $C>0$, possibly depending on W and on $\|h\|_{L^\infty(\mathbb{R})}$.

Lemma 2.3. *For any bounded solution $h \in C^2(\mathbb{R})$ of (??):*

1. *For any $t \in \mathbb{R}$,*

$$-W\Big(\inf_{\mathbb{R}} h\Big) = -W\Big(\sup_{\mathbb{R}} h\Big) = \frac{|h'(t)|^2}{2} - W(h(t)). \tag{2.9}$$

2. *If $h'(t_o) = 0$, then h is symmetric with respect to $t = t_o$, that is $h(t_o - t) = h(t_o + t)$ for any $t \in \mathbb{R}$.*

3. *If h has two or more critical points, then it is periodic.*

4. *If $|h| \leq 1$ and it is not constant, then*

$$\sup_{\mathbb{R}} h = -\inf_{\mathbb{R}} h.$$

5. *If h is not periodic then,*

$$W\Big(\inf_{\mathbb{R}} h\Big) = W\Big(\sup_{\mathbb{R}} h\Big) = 0 \quad \textit{and} \quad \frac{|h'(t)|^2}{2} = W(h(t)). \tag{2.10}$$

6. *Let $a < b$. If h is monotone in (a, b) then*

$$\int_a^b |h'(t)|\, dt \leq 2 \sup_{\mathbb{R}} |h|.$$

7. *Suppose $|h| \leq 1$. Then, if h is not periodic,*

$$\int_{-\infty}^{+\infty} |h'(t)|\, dt \leq 4.$$

8. *Let*

$$\sigma \in \Big\{\inf_{\mathbb{R}} h, \sup_{\mathbb{R}} h\Big\}.$$

   Suppose $|h| \leq 1$. Then,

$$\int_{-\infty}^{+\infty} \frac{|h'(t)|^2}{2} + W(h(t)) - W(\sigma)\, dt \leq C, \tag{2.11}$$

   for a suitable structural constant $C > 0$, unless h is periodic and non-constant.

Proof. 1) Indeed, since $h \in C^2(\mathbb{R})$ and bounded, there exits s_n such that

$$\lim_{n \to +\infty} h(s_n) = \sup_{\mathbb{R}} h, \text{ and } \quad \lim h'(s_n) = 0,$$

hence $\lim_{n\to+\infty} \frac{|h'(s_n)|^2}{2} - W(h(s_n)) = -W(\sup_{\mathbb{R}} h)$. Recalling (**??**), and possibly replacing "sup" with "inf", we get the result.
2) Notice that $h_\pm(t) := h(t_o \pm t)$ satisfy (**??**), with $h_\pm(0) = h(t_o)$ and $h'_\pm(0) = \pm h'(t_o) = 0$. So the claim follows from Cauchy Uniqueness Theorem.
3) Suppose that $h'(a) = h'(b) = 0$ with $b > a$ and let $T := b - a$. Then, using 2.,

$$h(t+T) = h(b+(t-a)) = h(a-(t-b)) = h(a+(t-b)) = h(t-T)$$

for any $t \in \mathbb{R}$, and so h has period $2T$.
4) We argue by contradiction. Namely, we suppose that the set

$$\Big\{ -\inf_{\mathbb{R}} h\,,\ \sup_{\mathbb{R}} h \Big\}$$

consists of two different points, say $s_- < s_+$.
Since h is not constant,

$$\inf_{\mathbb{R}} h \,<\, \sup_{\mathbb{R}} h. \tag{2.12}$$

Also, by (**??**) and recalling that W is even,

$$W(-\inf_{\mathbb{R}} h) = W(\inf_{\mathbb{R}} h) = W(\sup_{\mathbb{R}} h), \tag{2.13}$$

that is,

$$W(s_-) = W(s_+). \tag{2.14}$$

So, by (**??**), (**??**) and Rolle's Theorem, there exists $\eta \in (\inf_{\mathbb{R}} h, \sup_{\mathbb{R}} h) \subset (-1,1)$ such that $W'(\eta) = 0$. From (**??**), we deduce that $\eta \in \{-1, 0, +1\} \cap (-1,1)$, i.e. $\eta = 0$, and so

$$\inf_{\mathbb{R}} h \,<\, 0 \,<\, \sup_{\mathbb{R}} h.$$

As a consequence,

$$\text{both } s_- \text{ and } s_+ \text{ are stricltly positive.} \tag{2.15}$$

On the other hand, by (**??**) and Rolle's Theorem, there exists $\xi \in (s_-, s_+) \subset (-1,1)$ such that $W'(\xi) = 0$. From (**??**), we deduce that $\xi \in \{-1, 0, +1\} \cap (-1,1)$, i.e. $\xi = 0$. This says that $s_- < 0 < s_+$. This is in contadiction with (**??**) and it therefore proves the desired claim.
5) Using 3.,

$$h \text{ has at most one critical point.} \tag{2.16}$$

In particular, h attains either its sup or its inf at either $+\infty$ or $-\infty$. So, let us assume, for definiteness that

$$\sup_{\mathbb{R}} h = \lim_{t\to+\infty} h(t), \tag{2.17}$$

the other cases being analogous.

By (**??**), we obtain that the limit in (**??**) holds in C^1, therefore

$$0 = \lim_{t\to+\infty} \int_{\mathbb{R}} h'(s+t)\phi'(s) + W'(h(s+t))\phi(s)\,ds = \int_{\mathbb{R}} W'\Big(\sup_{\mathbb{R}} h\Big)\phi(s)\,ds,$$

for any $\phi \in C_0^\infty(\mathbb{R})$.
This says that

$$W'\Big(\sup_{\mathbb{R}} h\Big) = 0$$

and so, by (**??**),

$$\sup_{\mathbb{R}} h \in \{-1, 0, +1\}.$$

The case $\sup_{\mathbb{R}} h = 0$ is impossible, because, by 4., we would get $\sup_{\mathbb{R}} h = -\inf_{\mathbb{R}} h = 0$, and so h would be constant and in particular periodic.
If $\sup_{\mathbb{R}} h \in \{-1, +1\}$, then (**??**) holds true, recalling (**??**).
6) We have

$$\int_a^b |h'(t)|\,dt = \left|\int_a^b h'(t)\,dt\right| = |h(b) - h(a)| \le 2\sup_{\mathbb{R}} |h|.$$

7) By 3. either h' never vanish or h' has only one zero.
In any case, there exists $c \in \mathbb{R}$ in such a way that $h'(t) \neq 0$ for any $t \in (-\infty, c) \cup (c, +\infty)$.
Consequently, by 6., for any $a < c < b$,

$$\int_a^b |h'(t)|\,dt = \int_a^c |h'(t)|\,dt + \int_c^b |h'(t)|\,dt \le 2\sup_{\mathbb{R}} |h| + 2\sup_{\mathbb{R}} |h|.$$

Then, the desired result follows by sending $a \to -\infty$ and $b \to +\infty$.
8) Since (**??**) is obvious for h constant, we focus on the case in which h is not periodic. Using 5.,

$$\int_{-\infty}^{+\infty} \frac{|h'(t)|^2}{2} + W(h(t))\,dt = \int_{-\infty}^{+\infty} |h'(t)|^2\,dt \le C \int_{-\infty}^{+\infty} |h'(t)|\,dt.$$

So the desired result follows from 7. □

2.3 A compactness result

We now point out a useful compactness criterion:

Lemma 2.4. *Let u_k be a sequence of solutions of* (**??**) *in the whole $\mathbb{R}^2$. Then, up to subsequence, u_k converges locally uniformly to some u which is also a solution of* (**??**) *in the whole $\mathbb{R}^2$.*

Proof. By Grushin-elliptic regularity (see, e.g., [**?**, **?**, **?**, **?**]), we have that

$$\|u_k\|_{C^\alpha(\mathbb{R}^2)} \leq \bar{C}$$

for some $\bar{C} > 0$, therefore up to subsequence, u_k converges locally uniformly to some u, and so

$$u \text{ is continuous.} \tag{2.18}$$

Moreover, for any $a \in (0,1)$, the Grushin operator Δ_G is uniformly elliptic in $D_a := \{|x| \in (a, 1/a)\}$, therefore standard elliptic estimates give that

$$\|u_k\|_{C^{2,\beta_a}(D_a)} \leq \bar{C}_a$$

and so

$$\Delta_G u(x) = W'(u(x)) \tag{2.19}$$

for any $x \in D_a$.

Since a can be taken arbitrarily small, we have that (**??**) holds for any $x \in \mathbb{R}^2 \setminus \{0\}$. But then, since the map $x \mapsto W'(u(x))$ is continuous, by means of (**??**), it follows that $\Delta_G u$ is continuous too and so (**??**) holds for any $x \in \mathbb{R}^2$. □

2.4 Basic spectral theory

Lemma 2.5. *Fix $a \in (0,1)$, $R \geq 1$. Let $x_o = a + R$ and $\Omega := B_R(x_o, 0)$. Then, there exists $\lambda \in \mathbb{R}$ for which there exists a non-trivial solution ϕ of*

$$\begin{cases} \Delta_G \phi + \lambda \phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Moreover, we can take

$$\lambda \in \left(0, \frac{C}{R^2}\right], \tag{2.21}$$

for a suitable $C > 0$.

Proof. We have

$$\begin{aligned} \int_\Omega |\nabla_G v|^2 &\leq \int_\Omega |\partial_x v|^2 + (x_o + R)^2 |\partial_y v|^2 \\ &\leq (1 + a + 2R)^2 \int_\Omega |\nabla v|^2. \end{aligned} \tag{2.22}$$

Therefore, by standard Poincaré inequality,

$$\int_\Omega |\nabla_G v|^2 \leq C(a,R) \int_\Omega |v|^2, \tag{2.23}$$

for any $v \in C_0^\infty(\Omega)$.

Moreover,

$$\begin{aligned}\int_\Omega |\nabla_G v|^2 &\geq \int_\Omega |\partial_x v|^2 + a^2|\partial_y v|^2 \\ &\geq \min\{1, a^2\} \int_\Omega |\nabla v|^2.\end{aligned} \tag{2.24}$$

From (**??**), we thus follow the standard minimization argument, taking

$$\lambda = \inf_{v\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_G v|^2}{\int_\Omega v^2} \tag{2.25}$$

and we recover compactness from (**??**) and the classical embeddings, proving (**??**).
Now, if λ and ϕ satisfy (**??**), we may suppose that

$$\int_\Omega \phi^2 = 1$$

and so

$$\int_\Omega |\nabla_G \phi|^2 = \lambda$$

which gives that $\lambda > 0$.
Also, from (**??**) and a change of variable,

$$\begin{aligned}\lambda &= \inf_{v\in C_0^\infty(B_R(x_o,0))\setminus\{0\}} \frac{\int_{\{(x-x_o)^4+4y^2\leq R^4\}} |\partial_x v(x,y)|^2 + |x|^2|\partial_y v(x,y)|^2\, d(x,y)}{\int_{\{(x-x_o)^4+4y^2\leq R^4\}} |v(x,y)|^2\, d(x,y)} \\ &= \inf_{\psi\in C_0^\infty(B_1)\setminus\{0\}} \frac{\int_{B_1} R^{-2}(\partial_w\psi(w,z))^2 + R^{-4}|x_o + Rw|^2(\partial_z\psi(w,z))^2\, d(w,z)}{\int_{B_1}\psi^2} \\ &\leq \inf_{\psi\in C_0^\infty(B_1)\setminus\{0\}} \frac{\int_{B_1} R^{-2}(\partial_w\psi(w,z))^2 + R^{-4}(3R)^2(\partial_z\psi(w,z))^2}{\int_{B_1}\psi^2} \\ &\leq 10R^{-2} \inf_{\psi\in C_0^\infty(B_1)\setminus\{0\}} \frac{\int_{B_1}|\nabla\psi|^2}{\int_{B_1}\psi^2}.\end{aligned}$$

This and the classical Poincaré inequality imply (**??**). □

2.5 Extension of bounded harmonic functions

Lemma 2.6. *Let u be Δ_G-harmonic in $B_r\setminus\{0\}$. Suppose that u is bounded in $B_r\setminus\{0\}$. Then, it may be extended to a Δ_G-harmonic function in B_r.*

Proof. The fundamental solution of Δ_G is $\psi(x,y) = (x^4+4y^2)^{\frac{-1}{4}}$ (see Theorem 3.1 of [**?**] for a formula for generalized Grushin operators). Thus, the argument on pages 16–17 of [**?**] may be repeated verbatim. □

3 Monotonicity and stability

We show that (??) is sufficient to prove stability. This is in analogy with the fact that monotonicity in any direction implies stability in the Euclidean setting (see [?]) – but in the Grushin plane the directions do not play the same role, thus (??) somehow selects the good direction for stability.

Proposition 3.1. *Let $u \in C^2(\mathbb{R}^2)$ be a solution of (??) satisfying (??).*
Then, u is stable.

Proof. The argument we present here is a modification of a classical one (see [?] and also Section 7 in [?] for a general result). We recall that we need to prove that for any smooth ϕ, compactly supported

$$0 \leq \int_{\mathbb{R}^n} |\nabla_G \phi|^2 - f'(u)\phi^2 \, dx.$$

For any φ smooth and compactly supported, we have

$$\int_{\mathbb{R}^2} f'(u) T u \varphi = \int_{\mathbb{R}^2} \partial_y \big(f(u)\big) \varphi = - \int_{\mathbb{R}^2} f(u) \partial_y \varphi$$

$$\begin{aligned} \int_{\mathbb{R}^2} \Delta_G u \partial_y \varphi &= - \int_{\mathbb{R}^2} < \nabla_G u, \nabla_G \partial_y \varphi > \\ &= \int_{\mathbb{R}^2} < T \nabla_G u, \nabla_G \varphi > = \int_{\mathbb{R}^2} < \nabla_G (Tu), \nabla_G \varphi > . \end{aligned}$$

Therefore, by taking $\varphi := \phi^2/(Tu)$, and making use of the Cauchy-Schwarz inequality,

$$\begin{aligned} 0 \;=\; & \int_{\mathbb{R}^n} \frac{2\phi < \nabla_G(Tu), \nabla_G \phi >}{Tu} - \frac{\phi^2 |\nabla_G (Tu)|^2}{(Tu)^2} - f'(u)\phi^2 \, dx \\ \leq & \int_{\mathbb{R}^n} |\nabla_G \phi|^2 - f'(u)\phi^2 \, dx. \end{aligned}$$

□

4 Energy estimates

We follow here some ideas of [?] to estimate the energy

$$\mathcal{F}_R(u) := \int_{B_R(0)} \frac{|\nabla_G u(\xi)|^2}{2} + W(u(\xi)) \, d\xi.$$

For this, for any $t \in \mathbb{R}$, we define the translation

$$u^t(x, y) := u(x, y + t)$$

and the translated energy

$$\mathcal{E}_R(t) := \mathcal{F}_R(u^t).$$

Of course, $\mathcal{E}_R(0) = \mathcal{F}_R(u)$.

Lemma 4.1. *Suppose that $u \in C^2(\mathbb{R}^2, [-1,1])$, with $Tu > 0$ and $|\nabla_G u| \in L^\infty(\mathbb{R}^2)$, is a solution of*

$$\Delta_G u(\xi) = W'(u(\xi)) \qquad \text{for any } \xi \in \mathbb{R}^2.$$

Then, there exists a structural constant C in such a way that

$$\mathcal{E}_R(0) \le \mathcal{E}_R(t) + CR^2, \tag{4.1}$$

for any $t \in \mathbb{R}$ and any $R > 0$.

Proof. We prove (**??**) for $t > 0$ (this is enough, since $u(x,-y)$ is also a solution).
We have, recalling Lemma **??**,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_R(t) &= \int_{B_R(0)} < \nabla_G u^t, \nabla_G T u^t > + W'(u^t) T u^t \, d\xi \\
&\ge \int_{B_R(0)} \Big(-\Delta_G (Tu^t) + W'(u^t) \Big) T u^t \, d\xi \\
&\quad - R^2 \|\nabla_G u^t\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} T u^t (RX, R^2 Y) \, d\mathcal{H}^1(X,Y) \\
&= 0 - R^2 \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} \partial_y u(RX, R^2 Y + t) \, d\mathcal{H}^1(X,Y).
\end{aligned}$$

Hence, fixed any $\tau > 0$,

$$\begin{aligned}
\mathcal{E}_R(\tau) - \mathcal{E}_R(0) &= \int_0^\tau \frac{d}{dt}\mathcal{E}_R(t) \, dt \\
&\ge -R^2 \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \int_0^\tau \int_{\partial B_1(0)} \partial_y u(RX, R^2 Y + t) \, d\mathcal{H}^1(X,Y) \, dt \\
&= -R^2 \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} \int_0^\tau \partial_y u(RX, R^2 Y + t) \, dt \, d\mathcal{H}^1(X,Y) \\
&= -R^2 \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} u(RX, R^2 Y + \tau) - u(RX, R^2 Y) \, d\mathcal{H}^1(X,Y) \\
&\ge -2R^2 \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \|u\|_{L^\infty(\mathbb{R}^2)} \, \mathcal{H}^1(\partial B_1(0)),
\end{aligned}$$

which gives (**??**). $\square$

Theorem 4.2. *Suppose that $u \in C^2(\mathbb{R}^2, [-1,1])$, with $|\nabla_G u| \in L^\infty(\mathbb{R}^2)$ is a solution of*

$$\Delta_G u(\xi) = W'(u(\xi)) \qquad \text{for any } \xi \in \mathbb{R}^2.$$

*Assume that (**??**) holds true.*
Then, there exists a structural constant C in such a way that

$$\mathcal{F}_R(u) \le CR^2$$

for any $R > 0$.
*As a consequence, (**??**) holds true.*

Proof. We have that u is bounded and monotone in y, thanks to (**??**). Thus, we may define

$$u^{\pm}(x) := \lim_{y\to\pm\infty} u(x,y). \tag{4.2}$$

Then, from Lemma **??**, we have that

$$\Delta_G u^{\pm}(x) = W'(u^{\pm}(x)) \tag{4.3}$$

In fact, since u does not depend on y, we may write (**??**) as

$$(u^{\pm})''(x) = W'(u^{\pm}(x)) \tag{4.4}$$

and so we may apply to $u^{\pm}$ the ODE analysis developed in Section **??**.
For this, we observe that

$$\text{at least one between } u^+ \text{ and } u^- \text{ is either constant or non-periodic.} \tag{4.5}$$

To prove (**??**), we argue by contradiction, supposing that u^+ and u^- are both periodic and non-constant. In particular, by Cauchy Uniqueness Theorem, $|u^{\pm}| < 1$.
By Lemma **??**(4), we would have that

$$\max_{\mathbb{R}} u^{\pm} = -\min_{\mathbb{R}} u^{\pm}. \tag{4.6}$$

But from (**??**), we know that

$$u^+(x) > u^-(x) \text{ for any } x \in \mathbb{R} \tag{4.7}$$

and so, if we set $x^{\pm}_{\min}$, $x^{\pm}_{\max}$ be such that

$$u^{\pm}(x^{\pm}_{\min}) = \min_{\mathbb{R}} u^{\pm} \quad \text{and} \quad u^{\pm}(x^{\pm}_{\max}) = \max_{\mathbb{R}} u^{\pm},$$

we deduce from (**??**) and (**??**) that

$$\begin{aligned} u^-(x^+_{\min}) \geq u^-(x^-_{\min}) = -u^-(x^-_{\max}) > -u^+(x^-_{\max}) \\ \geq -u^+(x^+_{\max}) = u^+(x^+_{\min}) > u^-(x^+_{\min}). \end{aligned}$$

This contradiction proves (**??**).
We now claim that

$$\text{either } u^+ \text{ or } u^- \text{ is non-periodic or constant but not zero.} \tag{4.8}$$

To prove this, we argue by contradiction. Suppose (**??**) is false. Then, both u^- and u^+ are periodic. Then, at least one, say u^+ is constant, because of (**??**). If u^+ were not equal to zero, then (**??**) would be true, thus we have to say that u^+ is constantly equal to zero and that u^- is periodic. But then u^- cannot be constant, otherwise (**??**), (**??**) and (**??**) would say that u^- is constantly equal to -1 and (**??**) would be true.

Thence, we are forced to the case in which u^+ is identically zero and u^- is periodic and non-constant. Thus, by (**??**),

$$\sup_{\mathbb{R}} u^- \leq 0$$

and so, by Lemma **??**(4),

$$\inf_{\mathbb{R}} u^- = -\sup_{\mathbb{R}} u^- \geq 0 \geq \sup_{\mathbb{R}} u^-.$$

This would say that u^- is constant, while we know it is not the case.

This contradiction proves (**??**).

By means of (**??**), up to a sign change, we may suppose that u^+ is either constant but not zero or it is non-periodic. Consequently, by Lemma **??**(8),

$$\int_{-\infty}^{+\infty} \frac{|(u^+)'(t)|^2}{2} + W(u^+(t)) - W(\sigma^+)\, dt \leq C^+, \tag{4.9}$$

for a suitable $C^+ > 0$, with

$$\sigma^+ \in \Big\{ \inf_{\mathbb{R}} u^+, \sup_{\mathbb{R}} u^+ \Big\}.$$

In fact, (**??**) and (**??**) give that

$$\int_{-\infty}^{+\infty} \frac{|(u^+)'(t)|^2}{2} + W(u^+(t))\, dt \leq C^+. \tag{4.10}$$

Moreover, by (**??**),

$$\begin{aligned}
\mathcal{E}_R(0) - CR^2 &\leq \lim_{t\to+\infty} \mathcal{E}_R(t) \\
&= \lim_{t\to+\infty} \int_{B_R(0)} \frac{|\nabla_G u(x,y+t)|^2}{2} + W(u(x,y+t))\, d(x,y) \\
&= \int_{B_R(0)} \frac{|\nabla_G u^+(x)|^2}{2} + W(u^+(x))\, d(x,y) \\
&\leq \int_{-R^2/2}^{R^2/2} \int_{-\infty}^{+\infty} \frac{|\nabla_G u^+(x)|^2}{2} + W(u^+(x))\, dx\, dy \\
&= R^2 \int_{-\infty}^{+\infty} \frac{|\nabla_G u^+(x)|^2}{2} + W(u^+(x))\, dx.
\end{aligned}$$

Thus, by (**??**),

$$\mathcal{E}_R(0) - CR^2 \leq C^+ R^2.$$

□

We now prove the following minimality result, that completes somehow the stability result of the previous section. Let

$$\mathcal{F}_\Omega(u) := \int_\Omega \frac{|\nabla_G u(\xi)|^2}{2} + W(u(\xi))\, d\xi.$$

Proposition 4.3. *Solutions u of (**??**) that satisfy (**??**) and also*

$$\lim_{y\to-\infty} u(x,y) = -1, \quad \lim_{y\to+\infty} u(x,y) = 1$$

are global minimizers, i.e. for any domain Ω, any $v \in C^2_o(\Omega)$

$$\mathcal{F}_\Omega(u) \le \mathcal{F}_\Omega(u+v).$$

Proof. In order to prove minimality it is enough to prove that given any domain Ω the equation

$$\begin{cases} \Delta_G v(\zeta) + f(v(\zeta)) = 0 & \text{in } \ \Omega \\ v = u & \text{on } \ \partial\Omega \end{cases}$$

has a unique solution. Indeed both the minimal of $\mathcal{F}_\Omega$ and u are solutions of this Dirichlet problem and then they coincide.

Suppose that in fact there are two solutions, i.e. there exists another solution $v \neq u$, hence there exists $x_o \in \Omega$ such that, say, $v(x_o) > u(x_o)$. By maximum principle we get that $|v| < 1$ in Ω.

Let $u_\tau(x,y) = u(x, y+\tau)$; by monotonicity and invariance of the equation u_τ is a solution of (**??**). Also, for any $\tau_1 < \tau_2$, we have $u_{\tau_1} < u_{\tau_2}$, and

$$\lim_{\tau\to\pm\infty} u_\tau(x,y) = \pm 1.$$

Hence for τ sufficiently large

$$u_\tau \ge v \text{ in } \Omega.$$

Let

$$\tau_o = \inf\{\tau > 0, \text{ such that } u_\tau \ge v\}.$$

This implies that $u_{\tau_o} \ge v$ and u_{τ_o} and v touch at an interior point. Observe furthermore that $\tau_o > 0$ otherwise it would contradicts $v(x_o) > u(x_o)$.

By strong maximum principle we would get that that $u_{\tau_o} \equiv v$ but this is false on $\partial\Omega$. We have reached the desired contradiction. □

5 The counter-example

5.1 Monotonicity and Maximum Principle

For any $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$, let

$$T_s\xi := \xi + (0, s).$$

A domain $\Omega \subset \mathbb{R}^2$ is said to be T-convex if for any $\xi_1 \in \Omega$ and any $\alpha > 0$ such that $T_\alpha\xi_1 \in \Omega$ one has that $T_s\xi_1 \in \Omega$ for every $s \in (0, \alpha)$.

That is, Ω is T-convex when vertical segments joining two points of Ω lie in Ω.

Theorem 5.1. *Let Ω be an arbitrary bounded domain of $\mathbb{R}^2$ which is T-convex. Let $u \in \Lambda^2(\Omega) \cap C(\bar{\Omega})$ be a solution of*

$$\left.\begin{array}{rcl} \Delta_G u + f(u) & = & 0 \;\text{ in }\; \Omega \\ u & = & \psi \;\text{ on }\; \partial\Omega \end{array}\right\} \tag{5.1}$$

where f is a Lipschitz continuous function. Assume that for any ξ_1, $\xi_2 \in \partial\Omega$, such that $\xi_2 = T_\alpha \xi_1$ for some $\alpha > 0$, we have, for each $s \in (0, \alpha)$ either

$$\psi(\xi_1) < u(T_s\xi_1) < \psi(\xi_2) \;\text{ if }\; T_s\xi_1 \in \Omega \tag{5.2}$$

or

$$\psi(\xi_1) < \psi(T_s\xi_1) < \psi(\xi_2) \;\text{ if }\; T_s\xi_1 \in \partial\Omega. \tag{5.3}$$

Then u satisfies

$$u(T_{s_1}\xi) < u(T_{s_2}\xi) \tag{5.4}$$

for any $0 < s_1 < s_2 < \alpha$ and for every $\xi \in \Omega$.
*Moreover, u is the unique solution of (**??**) in $\Lambda^2(\Omega) \cap C(\bar{\Omega})$ satisfying (**??**).*

The proof of this result is done through the sliding method introduced in [**?**] for uniformly elliptic equations. This method uses two fundamental ingredients: the Maximum Principle in small domains and the invariance of the operator with respect to "sliding". In [**?**] the equivalent of Theorem **??** was proved for sub-elliptic equations in nilpotent Lie groups. There, the key "new" ingredient being a Hölder estimate for Hörmander type operators proved in [**?**] that allowed to prove the Maximum Principle in small domains.

The operator is invariant by T_s translations and our equation satisfies the hypotheses of [**?**], hence the proof of Theorem **??** proceeds exactly like the one given in [**?**], and we omit it.

5.2 Existence of monotone solutions that are not one-dimensional

The following result shows that Question **??** has a negative answer:

Theorem 5.2. *There exists a solution of*

$$\Delta_G u - W'(u) = 0$$

in $\mathbb{R}^2$ such that

$$Tu = \partial_y u > 0, \quad \textit{and} \quad \lim_{y \to \pm\infty} u(x, y) = \pm 1$$

that is not one-dimensional.[1]

[1] We remark that the u constructed in Theorem **??** is also stable and minimal, see Propositions **??** and **??**.

Proof. We follow the two steps of [**?**].

Step 1. *Construction of a monotone solution in a bounded set.*

Let $M > 0$ be greater then the Lipschitz constant of f, let $g(u) := f(u) + Mu$, $Q_R^+ := (-R, R) \times [0, R^2]$ and $Q_R^- := (-R, R) \times (-R^2, 0)$.

We consider the operator $\mathcal{T}$ on C^α such that $\mathcal{T}v = u$ is the classical solution of

$$\begin{cases} \Delta_G u - Mu = -g(v) & \text{in } Q_R^+ \\ u(x,0) = 0, \ u(x,R^2) = 1, \quad u(-R,y) = \psi(y), \ u(R,y) = \psi(y) \end{cases}$$

where $0 \leq \psi \leq 1$ with $\psi(0) = 0$ and $\psi(R^2) = 1$.

The following properties hold:

(P1) $\mathcal{T}$ is well defined, see [**?**, **?**].

(P2) It is monotone, i.e. $0 \leq v_1 \leq v_2 \leq 1$ implies $\mathcal{T}v_1 \leq \mathcal{T}v_2$. This is just the Maximum Principle, because with our choice of M we get that

$$v_1 \leq v_2 \Longrightarrow g(v_1) \geq g(v_2).$$

(P3) If $0 \leq v \leq 1$ then $0 \leq \mathcal{T}v \leq 1$ (again by Maximum Principle).

(P4) For R sufficiently large there exists $v_o > 0$ in some fixed subset of Q_R^+ such that if $u_k := \mathcal{T}^k(v_o)$ then $u_k \geq v_o$ for any $k \in N$.

Let us prove (P4).

Let

$$l := \lim_{s \to 0} \frac{|W'(s)|}{s}$$

and let R_o be sufficiently large that $Q_{R_o}^+$ contains a ball B such that $B \cap \{x = 0\} = \emptyset$ and λ_o the principal eigenvalue of $-\Delta_G$ in B satisfies

$$\lambda_o \leq \frac{l}{2}.$$

We remark that we can take such a λ_o in the light of Lemma **??**.

Let φ_o be the corresponding eigenfunction normalized by $\sup \varphi_o = 1$. Our choice of λ_o implies that there exists $\varepsilon > 0$ such that

$$\lambda_o \varepsilon \varphi_o \leq |W'(\varepsilon \varphi_o)|. \tag{5.5}$$

Now we define

$$v_o = \begin{cases} \varepsilon \varphi_o & \text{in} \quad B \\ 0 & \text{in} \quad Q_R^+ \setminus B. \end{cases} \tag{5.6}$$

Observe that, in B, the Grushin operator Δ_G is uniformly elliptic, and so by standard estimates we know that $v \in C^\alpha(Q_R^+)$. Using (**??**), (P2) and (P3), we get that $u_o = \mathcal{T}(v_o) \geq v_o$.

So, iteratively, $\mathcal{T}^k(v_o) \geq v_o$ for any $k \in \mathbb{N}$. This proves (P4).

By Lemma **??**, we may and do suppose that $u_k := \mathcal{T}^k(v_o)$ converges to a solution $\tilde{u} = \tilde{u}_R$ of

$$\begin{cases} \Delta_G \tilde{u} - W'(\tilde{u}) = 0 & \text{in } \ Q_R^+ \\ \tilde{u}(x,0) = 0, \ \tilde{u}(x,R^2) = 1 & \tilde{u}(-R,y) = \psi(y), \ \tilde{u}(R,y) = \psi(y) \end{cases}$$

Note that $\tilde{u}$ satisfies $0 \leq \tilde{u} \leq 1$. Hence, using Theorem **??**, we know that

$$\partial_y \tilde{u} > 0. \tag{5.7}$$

Finally, we extend the solution to $Q_R = \overline{Q_R^+} \cup \overline{Q_R^-}$ by taking

$$v_R(x,y) := \begin{cases} \tilde{u}(x,y) & \text{for } (x,y) \in \overline{Q_R^+} \\ -\tilde{u}(x,-y) & \text{for } (x,y) \in \overline{Q_R^-} \end{cases}$$

Clearly, v_R is a solution in $Q_R^+ \cup Q_R^-$. Also the solution u is C^2 up to the boundary for $x \neq 0$. Hence we get that v_R is a solution in $Q_R \setminus \{(0,0)\}$. To check that v_R is a solution in all of Q_R. Observe that the map $\zeta \mapsto W'(v_R(\zeta))$ is in $C^\alpha(Q_R)$, hence there exists $w \in C^{2,\alpha}_{loc}(Q_R) \cap C(\overline{Q_R})$ solution of

$$\Delta_G w - W'(v_R) = 0 \quad \text{in} \quad Q_R.$$

Then $w - v_R$ is Δ_G-harmonic in $Q_R \setminus \{(0,0)\}$ and it is bounded. Thus, by Lemma **??**, it is Δ_G-harmonic in Q_R, and so v_R is a solution in all of Q_R.

Furthermore v_R is monotone in T, in the sense that $Tv_R = \partial_y v_R > 0$, because of (**??**).

Step 2. *Let $R \to \infty$.* Then, by Lemma **??**, v_R locally uniformly converges to some u, which is a solution of

$$\Delta_G u - W'(u) = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Furthermore, in Q_R^+,

$$v_R = \tilde{u} = \lim_{k \to +\infty} u_k \geq v_0,$$

due to (P4) and so, by (**??**), $u \not\equiv 0$ in Q_R^+.

Then, u is monotone, i.e. $\partial_y u > 0$.

We shall now prove that at infinity u tends to ± 1. Observe that since u is monotone we can define

$$v_+(x) = \lim_{y \to +\infty} u(x,y).$$

We want to prove that $v_+(x) \equiv 1$. First observe that, since $u(x,0) = 0$, $v_+(x) > 0$, and it is a solution of (**??**). So we can apply Lemma **??**(4) and get that, if it is not constant,

$$\sup_{\mathbb{R}} v_+ = - \inf_{\mathbb{R}} v_+$$

and this contradicts the fact that it is positive.

Hence v_+ is constant and then, from the equation, it is equal to 1.

The limit at $-\infty$ is treated in the same way.

We shall now prove, u is not one-dimensional; suppose, by contradiction, that there exists a function g such that

$$u(x,y) = g(ax + by),$$

for any $(x,y) \in \mathbb{R}^2$.

Then, the strict monotonicity in T of u implies that $b \neq 0$.

Clearly $g \in W^{2,\infty}(\mathbb{R})$ would be a solution of

$$(a^2 + b^2x^2)g''(ax + by) - W'(g(ax + by)) = 0,$$

for any $(x,y) \in \mathbb{R}^2$.

Since $b \neq 0$, the boundedness of g'' implies that $g'' \equiv 0$. Hence $W'(g(t)) = 0$ for any t and so g would be constant, in contradiction with the fact that $Tu > 0$. $\square$

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