DIRAC–SOBOLEV SPACES AND SOBOLEV SPACES

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Abstract. The aim of this work is to study the first order Dirac-Sobolev spaces in L^p norm on an open subset of $\mathbb{R}^3$ to clarify its relationship with the corresponding Sobolev spaces. It is shown that for $1 < p < \infty$, they coincide, while for $p = 1$, the latter spaces are proper subspaces of the former.

Keywords: Sobolev spaces, Dirac oprerator.

2000 Mathematics Subject Classification: 46E35, 46E40.

1. **Introduction**

In the recent work [2], Balinski-Evans-Saitō introduced an L^p-seminorm $\|(\alpha \cdot p)f\|_{p,\Omega}$ of a $\mathbb{C}^4$-valued function f in an open subset Ω of $\mathbb{R}^3$, relevant to a massless Dirac operator

$$\alpha \cdot p = \sum_{j=1}^{3} \alpha_j(-i\partial_j) \qquad (\partial_j = \partial/\partial x_j), \tag{1.1}$$

where $p = -i\nabla$, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix} \qquad (j = 1, 2, 3) \tag{1.2}$$

with the 2×2 zero matrix 0_2 and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They used this seminorm to give a group of inequalities called *Dirac–Sobolev inequalities* in order to obtain L^p-estimates of the *zero modes*, i.e. eigenfunctions associated with the eigenvalue $\lambda = 0$, of the Dirac operator $(\alpha \cdot p) + Q$, where $Q(x)$ is a 4×4 Hermitian matrix-valued potential decaying at infinity. We believe that our above notation "p" for the differential operator $-i\nabla$ will not be confused with another "p" which appears as the superscript $1 \leq p < \infty$ of the space L^p.

Let Ω be an open subset of $\mathbb{R}^3$ and let the first order Dirac–Sobolev space $\mathbb{H}_0^{1,p}(\Omega)$, $1 \le p < \infty$, be the completion of $[C_0^\infty(\Omega)]^4$ with respect to the norm

(1.3)
$$\|f\|_{D,1,p,\Omega} := \left\{ \int_\Omega (|f(x)|_p^p + |(\alpha\cdot p)f(x)|_p^p)\,dx \right\}^{1/p} = \big(\|f\|_{p,\Omega}^p + \|(\alpha\cdot p)f\|_{p,\Omega}^p\big)^{1/p},$$

where $f(x) = {}^t(f_1(x), f_2(x), f_3(x), f_4(x))$, the norm of a vector $a = {}^t(a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ being denoted by

$$|a|_p = \Big[\sum_{k=1}^4 |a_k|^p\Big]^{1/p}. \tag{1.4}$$

As one of the simplest Dirac–Sobolev inequalities ([2], Corollary 2), they showed: If Ω is a bounded open subset of $\mathbb{R}^3$ and $f \in \mathbb{H}_0^{1,p}(\Omega)$ with $1 \le p < \infty$, then for $1 \le k < p(p+3)/3$ there exists a positive constant C such that

$$\|f\|_{k,\Omega} \le C\|(\alpha\cdot p)f\|_{p,\Omega}\,, \tag{1.5}$$

where $\|g\|_{p,\Omega}$ stands for the norm of $g = {}^t(g_1, g_2, g_3, g_4) \in [L^p(\Omega)]^4$ given by

$$\|g\|_{p,\Omega} = \left\{\int_\Omega |g(x)|_p^p\,dx\right\}^{1/p} = \left\{\int_\Omega \sum_{j=1}^4 |g_j(x)|^p\,dx\right\}^{1/p}. \tag{1.6}$$

Now let β be the fourth Dirac matrix given by

$$\beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}, \tag{1.7}$$

where 1_2 is the 2×2 unit matrix. It has been known that the free massless Dirac operator $\alpha\cdot p$ or the free Dirac operator $\alpha\cdot p + m\beta$ with positive mass m and the relativistic Schrödinger operator $\sqrt{m^2-\Delta}$ may bring similar properties in L^2 sometimes but not necessarily in L^p with $p \ne 2$. On the other hand, the following two norms are equivalent: for $1 < p < \infty$,

$$\begin{cases} (\|\psi\|_p^p + \|\nabla\psi\|_p^p)^{1/p}, \\ \|\sqrt{1-\Delta}\,\psi\|_p. \end{cases}$$

where ψ is a scalar-valued function in $\mathbb{R}^3$ (Stein [6], p.135, Theorem 3 or p.136, Lemma 3). However, for $p = 1$ or $p = \infty$, these two norms are not equivalent, in fact, the one does not dominate the other ([6], p.160, 6.6).

For an open subset Ω of $\mathbb{R}^3$ and $1 \le p < \infty$, let $\mathcal{A}_p(\Omega)$ be all C^∞ functions ψ on Ω such that ψ and $\nabla\psi$ belong to $L^p(\Omega)$ and let $H^{1,p}(\Omega)$ be the completion of $\mathcal{A}_p(\Omega)$

with respect to the norm given by

$$\|\psi\|_{1,p,\Omega} = \left\{ \int_\Omega (|\psi(x)|^p + |\nabla\psi(x)|^p\,)\,dx \right\}^{1/p} = \{\|\psi\|_{p,\Omega}^p + \|\nabla\psi\|_{p,\Omega}^p\}^{1/p}, \tag{1.8}$$

where $|\nabla\psi(x)|^p = \sum_{j=1}^3 |\partial_j\psi(x)|^p$. Let $C_0^\infty(\Omega)$ be the space of all C^∞ functions ϕ on Ω such that support of ϕ is contained in Ω and let $H_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm (1.8). The space $H_0^{1,p}(\Omega)$ is a closed subspace of $H^{1,p}(\Omega)$. The norm $\|\cdot\|_{S,1,p,\Omega}$ of the Banach space $[H^{1,p}(\Omega)]^4$ (and $[H_0^{1,p}(\Omega)]^4$) is given by

$$\|f\|_{S,1,p,\Omega} = \left\{ \int_\Omega \left(|f(x)|_p^p + |\nabla f(x)|_p^p\,\right) dx \right\}^{1/p}, \tag{1.9}$$

where $f = {}^t(f_1, f_2, f_3, f_4) \in [H^{1,p}(\Omega)]^4$ and

$$\begin{cases} |f(x)|_p^p = \displaystyle\sum_{k=1}^4 |f_k(x)|^p, \\ |\nabla f(x)|_p^p = \displaystyle\sum_{j=1}^3 |\partial_j f(x)|_p^p = \sum_{j=1}^3 \sum_{k=1}^4 |\partial_j f_k(x)|^p \end{cases} \tag{1.10}$$

(cf. (1.4)).

Definition 1.1. Let $\mathcal{A}_{p,D}(\Omega)$ be all $[C^\infty(\Omega)]^4$ functions f on Ω such that f and $(\alpha\cdot p)f$ belong to $[L^p(\Omega)]^4$. Then the Dirac-Sobolev spaces $\mathbb{H}^{1,p}(\Omega)$ and $\mathbb{H}_0^{1,p}(\Omega)$ are the completion of $\mathcal{A}_{p,D}(\Omega)$ and $[C_0^\infty(\Omega)]^4$ with respect to the norm

$$\|f\|_{D,1,p,\Omega} = \left\{ \int_\Omega (|f(x)|_p^p + |(\alpha\cdot p)f(x)|_p^p\,)\,dx \right\}^{1/p}, \tag{1.11}$$

respectively, where $f(x) = {}^t(f_1(x), f_2(x), f_3(x), f_4(x))$, and

$$\begin{cases} |f(x)|_p^p = \displaystyle\sum_{k=1}^4 |f_k(x)|^p\,, \\ |(\alpha\cdot p)f(x)|_p^p = |\displaystyle\sum_{j=1}^3 \alpha_j p_j f(x)|_p^p = \sum_{k=1}^4 \Big|\Big(\sum_{j=1}^3 -i\alpha_j\partial_j f\Big)_k(x)\Big|_p^p \end{cases} \tag{1.12}$$

It should be noted that in the paper [2] the space $\mathbb{H}_0^{1,1}(\Omega)$ of our paper was denoted by $\mathbb{H}^{1,1}(\Omega)$ without subscript '0'. We have adopted this notation, following the usual Sobolev space convention.

Remark 1.2. (i) As in the case of Sobolev spaces, we have $\mathbb{H}^{1,p}(\mathbb{R}^3) = \mathbb{H}_0^{1,p}(\mathbb{R}^3)$ since $[C_0^\infty(\mathbb{R}^3)]^4$ is dense in $\mathcal{A}_{p,D}(\mathbb{R}^3)$ with respect to the norm (1.11).

(ii) Let $\mathbb{W}^{1,p}(\Omega)$ be defined by

$$\mathbb{W}^{1,p}(\Omega) = \{f \in [L^p(\Omega)]^4\,;\, (\alpha \cdot p)f \in [L^p(\Omega)]^4\},$$

where $(\alpha \cdot p)f$ is taken in the sense of distributions. As in the case of Sobolev spaces (see, e.g., Adams-Fournier [1], Theorem 3.17), by approximating elements in $\mathbb{W}^{1,p}(\Omega)$ using the molifier, we have $\mathbb{W}^{1,p}(\Omega) = \mathbb{H}^{1,p}(\Omega)$, where $1 \leq p < \infty$ and Ω is an open subset of $\mathbb{R}^3$.

In this work we are going to investigate the relationship of the Dirac–Sobolev spaces $\mathbb{H}^{1,p}(\Omega)$ and the ordinary Sobolev spaces $[H^{1,p}(\Omega)]^4$ as well as the relationship of the Dirac–Sobolev spaces $\mathbb{H}_0^{1,p}(\Omega)$ and the ordinary Sobolev spaces $[H_0^{1,p}(\Omega)]^4$.

To proceed, we note the inclusions $[H^{1,p}(\Omega)]^4 \subset \mathbb{H}^{1,p}(\Omega)$ and $[H_0^{1,p}(\Omega)]^4 \subset \mathbb{H}_0^{1,p}(\Omega)$ to hold, which we shall see precisely later in the next section, Proposition 2.2. So for an open subset Ω of $\mathbb{R}^3$, define the linear map J_Ω and $J_{0,\Omega}$ by

$$\begin{cases} J_\Omega \,:\, [H^{1,p}(\Omega)]^4 \ni f \mapsto J_\Omega f = f \in \mathbb{H}^{1,p}(\Omega), \\ J_{0,\Omega} \,:\, [H_0^{1,p}(\Omega)]^4 \ni f \mapsto J_{0,\Omega} f = f \in \mathbb{H}_0^{1,p}(\Omega). \end{cases} \tag{1.13}$$

Our main result is as follows:

Theorem 1.3. *Let Ω be an open subset of $\mathbb{R}^3$ and let J_Ω and $J_{0,\Omega}$ be as above.*

(i) *Then, for $1 \leq p < \infty$, the map both J_Ω and $J_{0,\Omega}$ are one-to-one and continuous. The Sobolev space $[H_0^{1,p}(\Omega)]^4$ is a dense subspace of the Dirac–Sobolev spaces $\mathbb{H}_0^{1,p}(\Omega)$.*

(ii) *Let $1 < p < \infty$. Then $[H_0^{1,p}(\Omega)]^4 = \mathbb{H}_0^{1,p}(\Omega)$, i.e., the map $J_{0,\Omega}$ is not only one-to-one and continuous, but also they are onto with continuous inverse map $J_{0,\Omega}^{-1}$.*

(iii) *For $p = 1$ the map neither J_Ω nor $J_{0,\Omega}$ are onto, or we have $[H^{1,1}(\Omega)]^4 \subsetneqq \mathbb{H}^{1,1}(\Omega)$ and $[H_0^{1,1}(\Omega)]^4 \subsetneqq \mathbb{H}_0^{1,1}(\Omega)$. For $p = 1$, the norms $\|\cdot\|_{D,1,1,\Omega}$ and $\|\cdot\|_{S,1,1,\Omega}$ of these two spaces are not equivalent; $\|\cdot\|_{D,1,1,\Omega}$ is dominated by $\|\cdot\|_{S,1,1,\Omega}$, but not conversely.*

Remark 1.4. We don't know in (ii) whether or not it holds for a proper open subset Ω that $[H^{1,p}(\Omega)]^4 = \mathbb{H}^{1,p}(\Omega)$, i.e., that the map J_Ω is onto with continuous inverse map J_Ω^{-1} when $\Omega \subsetneqq \mathbb{R}^3$, although it holds by (ii) for $\Omega = \mathbb{R}^3$ that $[H^{1,p}(\mathbb{R}^3)]^4 = \mathbb{H}^{1,p}(\mathbb{R}^3)$, because this space coincides with $[H_0^{1,p}(\mathbb{R}^3)]^4 = \mathbb{H}_0^{1,p}(\mathbb{R}^3)$.

For the proof we shall use a method of classical analysis rather than a subtle pseudo-differentical calculus, in particular, in the case $p = 1$.

In Section 2 we shall prove Theorem 1.3, (i) (Proposition 2.2). In Section 3 we are going to give the proof of Theorem 1.3, (ii) by first dealing with J_Ω and then $J_{0,\Omega}$. Theorem 1.3, (iii), the case that $p = 1$, will be discussed and proved in Section 4.

2. Continuity of the map $\mathbf{J}_\Omega$

In this section we are going to prove Theorem 1.3, (i). Let α_j, $j = 1, 2, 3$, be the Dirac matrices given in (1.2). Then we have

Lemma 2.1. *Let $a = {}^t(a_1, a_2, a_3, a_4) \in \mathbb{C}^4$. Then, for $p \in [1, \infty)$,*

$$|\alpha_j a|_p = |a|_p \qquad (j = 1, 2, 3), \tag{2.1}$$

where the norm $|\cdot|_p$ is given by (1.4).

Proof. By the definition of α_1 we have

$$\alpha_1 a = {}^t(a_4, a_3, a_2, a_1),$$

and hence

$$|\alpha_1 a|_p^p = |a_4|^p + |a_3|^p + |a_2|^p + |a_1|^p = |a|_p^p\,.$$

In quite a similar manner (2.1) can be proved for $j = 2, 3$. □

Now we are in a position to show the continuity of the map J_Ω and $J_{0,\Omega}$ given by (1.13).

Proposition 2.2. *Let Ω be an open subset of $\mathbb{R}^3$ and let $f \in [H^{1,p}(\Omega)]^4$. Then, for $p \in [1, \infty)$, we have $f \in \mathbb{H}^{1,p}(\Omega)$ and there exists a positive constant $C = C_p$, depending only on p, not on Ω, such that*

$$\|f\|_{D,1,p,\Omega} \le C\|f\|_{S,1,p,\Omega}, \tag{2.2}$$

where the norms $\|f\|_{D,1,p,\Omega}$ and $\|f\|_{S,1,p,\Omega}$ are given in (1.11) and (1.9), respectively. Thus the identity maps J_Ω on $[H^{1,p}(\Omega)]^4$ and $J_{0,\Omega}$ on $[H_0^{1,p}(\Omega)]^4$ are continuous, one-to-one maps from $[H^{1,p}(\Omega)]^4$ into $\mathbb{H}^{1,p}(\Omega)$, and $[H_0^{1,p}(\Omega)]^4$ into $\mathbb{H}_0^{1,p}(\Omega)$. Further $[H_0^{1,p}(\Omega)]^4$ is a dense subset of $\mathbb{H}_0^{1,p}(\Omega)$.

Proof. Let $f = {}^t(f_1, f_2, f_3, f_4) \in [H^{1,p}(\Omega)]^4$. By using Lemma 2.1 and Hölder's inequality for $p > 1$ or the triangle inequality for $p = 1$, we have

$$|(\alpha\cdot p)f|_p^p = |(\sum_{l=1}^{3} -\alpha_l i\partial_l)f|_p^p \le \big(\sum_{l=1}^{3} |\alpha_l i\partial_l f|_p\big)^p = \big(\sum_{\ell=1}^{3} |\partial_l f|_p\big)^p$$
$$\le \big[\big(\sum_{l=1}^{3} 1^q\big)^{1/q}\big(\sum_{\ell=1}^{3} |\partial_l f|_p^p\big)^{1/p}\big]^p = 3^{p-1}|\nabla f|_p^p,$$

where $p^{-1} + q^{-1} = 1$ and see (1.10) for the definition of $|\nabla f|_p$. It follows that

$$\|(\alpha\cdot p)f\|_{p,\Omega} = \Big(\int_\Omega |(\alpha\cdot p)f|_p^p\, dx\Big)^{1/p} \le 3^{\frac{p-1}{p}}\Big(\int_\Omega |\nabla f|_p^p\, dx\Big)^{1/p} = 3^{\frac{p-1}{p}}\|\nabla f\|_{p,\Omega}, \tag{2.3}$$

where $\|(\alpha\cdot p)f\|_{p,\Omega}$ is the norm of $(\alpha\cdot p)f \in [L^p(\Omega)]^4$ given by (1.6), and $\|\nabla f\|_{p,\Omega}$ is given by

$$\|\nabla f\|_{p,\Omega} = \Big\{\int_\Omega |\nabla f|_p^p\, dx\Big\}^{1/p}.$$

Then it is easy to see that (2.3) implies (2.2). The map J is one-to-one since, for $f_j \in [H^{1,p}(\Omega)]^4$, $j = 1, 2$, we have

$$J_\Omega f_1 = J_\Omega f_2 \ \text{ in } \mathbb{H}^{1,p}(\Omega) \Longrightarrow f_1 = f_2 \ \text{ in } [L^p(\Omega)]^4 \Longrightarrow f_1 = f_2 \ \text{ in } [H^{1,p}(\Omega)]^4. \tag{2.4}$$

Using (2.2) and proceeding as in (2.4), we see that the identity map $J_{0,\Omega}$ on $[H_0^{1,p}(\Omega)]^4$ is also continuous and one-to-one on $[H_0^{1,p}(\Omega)]^4$. Since $[C_0^\infty(\Omega)]^4$ is dense in both $[H_0^{1,p}(\Omega)]^4$ and $\mathbb{H}_0^{1,p}(\Omega)$, $[H_0^{1,p}(\Omega)]^4$ is a dense subset of $\mathbb{H}_0^{1,p}(\Omega)$. This completes the proof. □

3. Range of the map $\mathbf{J}_{0,\Omega}$

In this section we are going to prove Theorem 1.3, (ii).

Proposition 3.1. *Let $1 < p < \infty$. Then the map $J_{0,\mathbb{R}^3}$ is onto $\mathbb{H}_0^{1,p}(\mathbb{R}^3)$.*

The proof will be given after the following two lemmas.

Lemma 3.2. *Let $1 < q < \infty$. Then $\Delta(C_0^\infty(\mathbb{R}^3))$ is dense in $L^q(\mathbb{R}^3)$.*

Proof of Lemma 3.2. Suppose that $f \in L^r(\mathbb{R}^3)$ with $1/q + 1/r = 1$ satisfies

$$\langle f, \Delta\phi\rangle = \int f(x)\Delta\phi(x)dx = 0, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^3).$$

Then we obtain in the sense of distributions $\Delta f = 0$, namely, Δ annihilates f. By elliptic regularity, we see that f must be C^∞, and hence $f(x)$ is a polynomial of x. Since $f(x)$ should belong to $L^r(\mathbb{R}^3)$, we have $f = 0$. This proves Lemma 3.2. $\square$

Remark 3.3. Lemma 3.2 does not hold for $q = 1$, since, in this case, the Laplacian Δ always annihilates a constant $C \neq 0$ which is a nonzero element of $L^\infty(\mathbb{R}^3) = L^1(\mathbb{R}^3)^*$.

Lemma 3.4. *Let $1 < q < \infty$. Let $\Omega \subset \mathbb{R}^3$ be an open set. Then, for each pair (j, k), $j, k = 1, 2, 3$, there exists a positive constant $C = C_{jk}$ such that*

$$\|\partial_j \partial_k \phi\|_{q,\Omega} \leq C \|\Delta \phi\|_{q,\Omega} \qquad (\phi \in C_0^\infty(\Omega)). \tag{3.1}$$

Proof. By Stein [6], p.59, Proposition 3, there exists a positive constant $C = C_{jk}$ such that

$$\|\partial_j \partial_k \phi\|_q \leq C \|\Delta \phi\|_q, \qquad \phi \in C_0^\infty(\mathbb{R}^3).$$

Of course, this holds for $\phi \in C_0^\infty(\mathbb{R}^3)$. $\square$

Proof of Proposition 3.1. The proof will be divided into four steps.

(I) Let $f = {}^t(f_1, f_2, f_3, f_4) \in \mathbb{H}_0^{1,p}(\mathbb{R}^3)$. Then $f \in [L^p(\mathbb{R}^3)]^4$, and

$$g := (\alpha \cdot p) f = -i[\alpha_1 \partial_1 f + \alpha_2 \partial_2 f + \alpha_3 \partial_3 f]$$

belongs to $[L^p(\mathbb{R}^3)]^4$. Using the definition (1.2) of the Dirac matrices α_j, $j = 1, 2, 3$, we can rewrite this with $g = {}^t(g_1, g_2, g_3 . g_4)$ as

$$\begin{cases} ig_1 = (\partial_1 - i\partial_2) f_4 + \partial_3 f_3, \\ ig_2 = (\partial_1 + i\partial_2) f_3 - \partial_3 f_4, \\ ig_3 = (\partial_1 - i\partial_2) f_2 + \partial_3 f_1, \\ ig_4 = (\partial_1 + i\partial_2) f_1 - \partial_3 f_2. \end{cases} \tag{3.2}$$

Then from the first and second equations of (3.2) we have

$$\begin{aligned} (\partial_1 + i\partial_2) ig_1 &= (\partial_1^2 + \partial_2^2) f_4 + \partial_3 (\partial_1 + i\partial_2) f_3 \\ &= (\partial_1^2 + \partial_2^2) f_4 + \partial_3 (ig_2 + \partial_3 f_4), \end{aligned}$$

so that

$$\Delta f_4 = (\partial_1^2 + \partial_2^2 + \partial_3^3) f_4 = (\partial_1 + i\partial_2)(ig_1) - \partial_3 (ig_2),$$

and hence, by applying ∂_j to both sides of the above equation, we have, for $j = 1, 2, 3$,

$$\Delta \partial_j f_4 = (\partial_j \partial_1 + i\partial_j \partial_2)(ig_1) - \partial_j \partial_3 (ig_2). \tag{3.3}$$

Similarly we have from (3.2)

$$\begin{cases} \Delta\partial_j f_3 = (\partial_j\partial_1 - i\partial_j\partial_2)(ig_2) - \partial_j\partial_3(ig_1), \\ \Delta\partial_j f_2 = (\partial_j\partial_1 + i\partial_j\partial_2)(ig_3) - \partial_j\partial_3(ig_4), \\ \Delta\partial_j f_1 = (\partial_j\partial_1 - i\partial_j\partial_2)(ig_4) - \partial_j\partial_3(ig_3). \end{cases} \tag{3.4}$$

The equalities in (3.3) and (3.4) should be interpreted as equalities in the space $\mathcal{D}'(\mathbb{R}^3)$ of distributions on $\mathbb{R}^3$.

(II) Our first goal is to show that each distribution $\partial_j f_k$ actually belongs to $L^p(\mathbb{R}^3)$, where $j = 1, 2, 3$ and $k = 1, 2, 3, 4$, namely, for each j and k there exists $F_{jk} \in L^p(\mathbb{R}^3)$ such that

$$\langle \partial_j f_k,\ \phi\rangle = \int_{\mathbb{R}^3} F_{jk}(x)\phi(x)\,dx \tag{3.5}$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$, where the left-hand side is a bilinear form on $\mathcal{D}'(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3)$. This will show that f belongs to $[H^{1,p}(\mathbb{R}^3)]^4$. We shall prove (3.5) for $k = 4$ and $j = 1, 2, 3$ since other cases can be proved in a similar manner. After that, finally we show that f belongs to $[H_0^{1,p}(\mathbb{R}^3)]^4$ to complete our proof.

(III) Let q be the conjugate of p or let q satisfy $p^{-1} + q^{-1} = 1$. We see from (3.3) that for $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \langle \partial_j f_4, \Delta\phi\rangle &= \langle \Delta\partial_j f_4, \phi\rangle = \langle (\partial_j\partial_1 + i\partial_j\partial_2)(ig_1) - \partial_j\partial_3(ig_2), \phi\rangle \\ &= \langle ig_1, (\partial_j\partial_1 + i\partial_j\partial_2)\phi\rangle - \langle ig_2, \partial_j\partial_3\phi\rangle. \end{aligned}$$

Hence by Lemma 3.5 we have

$$\begin{aligned} |\langle \partial_j f_4, \Delta\phi\rangle| &\le \|g_1\|_p \|(\partial_j\partial_1 + i\partial_j\partial_2)\phi\|_q + \|g_2\|_p \|\partial_j\partial_3\phi\|_q \\ &\le (C_{j1} + C_{j2})\|g_1\|_p\|\Delta\phi\|_q + C_{j3}\|g_2\|_p\|\Delta\phi\|_q \\ &= [(C_{j1} + C_{j2})\|g_1\|_p + C_{j3}\|g_2\|_p]\|\Delta\phi\|_q. \end{aligned} \tag{3.6}$$

Since $\Delta C_0^\infty(\mathbb{R}^3))$ is dense in $L^q(\mathbb{R}^3)$ as has been shown in Lemma 3.2, the inequality (3.6) is extended uniquely to a continuous linear form on $L^q(\mathbb{R}^3)$. Since $L^p(\mathbb{R}^3)$ is the dual space of $L^q(\mathbb{R}^3)$, there exists a function $F_{j4} \in L^p(\mathbb{R}^3)$ such that $\langle \partial_j f_4,\ \psi\rangle = \int_{\mathbb{R}^3} F_{j4}(x)\psi(x)\,dx$, $\psi \in L^q(\mathbb{R}^3)$, which implies (3.5) with $k = 4$ and $j = 1, 2, 3$. In particular, we have also shown that

$$\|\partial_j f_k\|_p \le C_0\{\Sigma_{k=1}^4 \|g_k\|_p^p\}^{1/p} = C_0\|g\|_p, \tag{3.7}$$

with a positive constant C_0 for all $j = 1, 2, 3$ and $k = 1, 2, 3, 4$.

(IV) Finally, since f is $\mathbb{H}_0^{1,p}(\mathbb{R}^3)$, by definition there exist a sequence $\{f_n\}_{n=1}^{\infty}$ in $C_0^{\infty}(\mathbb{R}^3)$ with $f_n = (f_{n,1}, f_{n,2}, f_{n,3}, f_{n,4})$ such that with $g_n = (g_{n,1}, g_{n,2}, g_{n,3}, g_{n,4}) := (\alpha \cdot p) f_n$,

$$\begin{aligned} \|f_n - f\|_{D,1,p,\mathbb{R}^3}^p &= \|f_n - f\|_p^p + \|(\alpha \cdot p)(f_n - f)\|_p^p \\ &= \|f_n - f\|_p^p + \|g_n - g\|_p^p \to 0, \qquad n \to \infty. \end{aligned}$$

Since by the same argument used to get (3.7) we have $\|\partial_j(f_{n,k} - f_k)\|_p \leq C_0 \|g_n - g\|_p$ for all $j = 1, 2, 3$ and $k = 1, 2, 3, 4$, it follows that $\|f_n - f\|_{S,1,p,\mathbb{R}^3}^p \to 0$ as $n \to \infty$, so that $f \in [H_0^{1,p}(\mathbb{R}^3)]^4$. This completes the proof of Proposition 3.1. □

Proposition 3.5. *Let $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^3$ be an open set and $J_{0,\Omega}$ be given in* (1.13). *Then the map $J_{0,\Omega}$ is onto $\mathbb{H}_0^{1,p}(\Omega)$. Further, the inverse map $J_{0,\Omega}^{-1}$ is well-defined as a bounded linear operator.*

Proof. (I) We have seen in Propositions 3.1 that $[H_0^{1,p}(\mathbb{R}^3)]^4 = \mathbb{H}_0^{1,p}(\mathbb{R}^3)$ as sets and there exist positive constants C_1 and C_2 such that

$$C_1 \|f\|_{D,1,p} \leq \|f\|_{S,1,p} \leq C_2 \|f\|_{D,1,p} \tag{3.8}$$

for $f \in [H_0^{1,p}(\mathbb{R}^3)]^4 = \mathbb{H}_0^{1,p}(\mathbb{R}^3)$.

(II) Let $f \in [H_0^{1,p}(\Omega)]^4$. Then there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset [C_0^{\infty}(\Omega)]^4$ such that

$$\|f - \phi_n\|_{S,1,p,\Omega} \to 0 \qquad (n \to \infty). \tag{3.9}$$

Since each ϕ_n can be naturally extended to be an element of $[C_0^{\infty}(\mathbb{R}^3)]^4$ by setting $\phi_n(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$, we have

$$\|f - \phi_n\|_{S,1,p} \to 0 \qquad (n \to \infty), \tag{3.10}$$

where f is also extended to be a function on $\mathbb{R}^3$ by setting 0 outside Ω, and hence $f \in [H^{1,p}(\mathbb{R}^3)]^4$ with support in the closure of Ω. Therefore $f \in \mathbb{H}^{1,p}(\mathbb{R}^3)$ and f satisfies (3.8). Then, (3.10) is combined with (3.8) to yield

$$\|f - \phi_n\|_{D,1,p} \to 0 \qquad (n \to \infty), \tag{3.11}$$

which implies, together with the fact that and ϕ_n have support in Ω, that

$$\|f - \phi_n\|_{D,1,p,\Omega} \to 0 \qquad (n \to \infty). \tag{3.12}$$

Thus we have $f \in \mathbb{H}_0^{1,p}(\Omega)$, and we obtain from (3.8)

$$C_1 \|f\|_{D,1,p,\Omega} \leq \|f\|_{S,1,p,\Omega} \leq C_2 \|f\|_{D,1,p,\Omega}. \tag{3.13}$$

(III) Let $f \in \mathbb{H}_0^{1,p}(\Omega)$. Then, starting with $f \in \mathbb{H}_0^{1,p}(\Omega)$ proceeding as in (II), we can show that $f \in [H_0^{1,p}(\Omega)]^4$ and the estimates (3.13) are satisfied, which completes the proof. ☐

Proof of Theorem 1.3, (ii). Theorem 1.3, (ii) follows from Propositions 3.1 and 3.5. ☐

4. **The case** $p = 1$

The goal of this section is to prove Theorem 1.3, (iii), that is, to prove $[H^{1,1}(\Omega)]^4$ and $[H_0^{1,1}(\Omega)]^4$ are proper subspaces of $\mathbb{H}^{1,1}(\Omega)$ and $\mathbb{H}_0^{1,1}(\Omega)$, respectively. First, we are going to show, for $\Omega = \mathbb{R}^3$, that $[H_0^{1,1}(\mathbb{R}^3)]^4 = [H^{1,1}(\mathbb{R}^3)]^4$ is a proper subspace of $\mathbb{H}_0^{1,1}(\mathbb{R}^3) = \mathbb{H}^{1,1}(\mathbb{R}^3)$ (Proposition 4.4). Then all other statements in Theorem 1.3, (iii) will follow from Proposition 4.4. In the following, when speaking of $\mathbb{H}_0^{1,1}(\mathbb{R}^3)$ or $\mathbb{H}^{1,1}(\mathbb{R}^3)$, and $[H_0^{1,1}(\mathbb{R}^3)]^4$ or $[H^{1,1}(\mathbb{R}^3)]^4$, we shall use the latter, namely, $\mathbb{H}^{1,1}(\mathbb{R}^3)$ and $[H^{1,1}(\mathbb{R}^3)]^4$. As easily seen, $\mathbb{H}^{1,p}(\mathbb{R}^3)$ is also the subspace of $[L^p(\mathbb{R}^3)]^4$ consisting of all $f \in [L^p(\mathbb{R}^3)]^4$ such that $(\alpha \cdot p + \beta)f \in [L^p(\mathbb{R}^3)]^4$ instead of $(\alpha \cdot p)f \in [L^p(\mathbb{R}^3)]^4$, where β is the fourth Dirac matrix β given by (1.7).

Lemma 4.1. *The map*

$$(\alpha \cdot p) + \beta \,:\, \mathbb{H}^{1,1}(\mathbb{R}^3) \ni f \mapsto (\alpha \cdot p + \beta)f \in [L^1(\mathbb{R}^3)]^4$$

maps $\mathbb{H}^{1,1}(\mathbb{R}^3)$ *one-to-one and onto* $[L^1(\mathbb{R}^3)]^4$.

Proof. (I) We define the Dirac operator $H_0 = (\alpha \cdot p) + \beta$ as a linear operator in $[L^1(\mathbb{R}^3)]^4$ with domain $D(H_0) = \mathbb{H}^{1,1}(\mathbb{R}^3)$. It is easy to see that H_0 is a closed operator in $[L^1(\mathbb{R}^3)]^4$. Let the operator $H = (\alpha \cdot p) + \beta$ be defined as a pseudodifferential operator acting on $[\mathcal{S}'(\mathbb{R}^3)]^4$, the dual space of $[\mathcal{S}(\mathbb{R}^3)]^4$, with 4×4 matrix symbol

$$\sigma_H(\xi) = \alpha \cdot \xi + \beta = \sum_{j=1}^{3} \xi_j \alpha_j + \beta. \tag{4.1}$$

Then the operator H_0 can be viewed as the restriction of the operator H to $\mathbb{H}^{1,1}(\mathbb{R}^3)$. Let B be a pseudodifferential operator acting on $[\mathcal{S}'(\mathbb{R}^3)]^4$ with symbol

$$\sigma_B(\xi) = (1 + |\xi|^2)^{-1}\sigma_H(\xi) = \sigma_H(\xi)[(1 + |\xi|^2)^{-1}I_4].$$

By the anti-commutative relation

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_4 \qquad (j, k = 1, 2, 3, 4, \alpha_4 = \beta),$$

where I_4 is the 4×4 unit matrix, we see that

$$\sigma_H(\xi)\sigma_B(\xi) = \sigma_B(\xi)\sigma_H(\xi) = I_4,$$

which implies that

$$HBf = BHf = f \qquad (f \in [\mathcal{S}'(\mathbb{R}^3)]^4), \tag{4.2}$$

i.e., the operator B is the inverse operator of H on $[\mathcal{S}'(\mathbb{R}^3)]^4$.

(II) Note that

$$B = HB^{(1)}, \tag{4.3}$$

where the symbol $\sigma_{B^{(1)}}(\xi)$ of $B^{(1)}$ is given by $\sigma_B^{(1)}(\xi) = (|\xi|^2+1)^{-1}I_4$. Let $\sigma_0(\xi) = (|\xi|^2+1)^{-1}$ and let $\sigma_0(p)$ be the pseudodifferential operator on $\mathcal{S}'(\mathbb{R}^3)$ with symbol $\sigma_0(\xi)$. The symbol $\sigma_0(\xi)$ is a C^∞ function on $\mathbb{R}^3_\xi$ and bounded together with all their derivatives. Then, by noting that the integrand $(|\xi|^2+1)^{-1}(\mathcal{F}\phi)(\xi)$ is a function in $\mathcal{S}(\mathbb{R}^3_\xi)$ for $\phi \in \mathcal{S}(\mathbb{R}^3)$, we have

$$\begin{aligned}(\sigma_0(p)\phi)(x) &= \lim_{R\to\infty}(2\pi)^{-3/2}\int_{|\xi|<R} e^{ix\cdot\xi}(|\xi|^2+1)^{-1}(\mathcal{F}\phi)(\xi)\,d\xi \\ &= (2\pi)^{-3}\lim_{R\to\infty}\int_{\mathbb{R}^3}\Big\{\int_{|\xi|<R} e^{i(x-y)\cdot\xi}(|\xi|^2+1)^{-1}\,d\xi\Big\}\phi(y)\,dy. \\ &= (2\pi)^{-3}\lim_{R\to\infty}\int_{\mathbb{R}^3}\Big\{\int_{|\xi|<R} e^{iy\cdot\xi}(|\xi|^2+1)^{-1}\,d\xi\Big\}\phi(x-y)\,dy.\end{aligned}$$

for $\phi \in \mathcal{S}(\mathbb{R}^3)$. Since $(|\xi|^2+1)^{-1} \in L^2(\mathbb{R}^3_\xi)$, the integral $\int_{|\xi|<R} e^{iy\cdot\xi}(|\xi|^2+1)^{-1}\,d\xi$ converges in $L^2(\mathbb{R}^3_y)$ as $R \to \infty$. At the same time it is known that the limit

$$\lim_{R\to\infty}(2\pi)^{-3}\int_{|\xi|<R} e^{iy\cdot\xi}(|\xi|^2+1)^{-1}\,d\xi = G(y)$$

exist for $y \neq 0$ with

$$G(y) = \frac{e^{-|y|}}{4\pi|y|},$$

which is the Green function of the operator $1-\Delta$. Thus we have

$$(\sigma_0(p)\phi)(x) = (G*\phi)(x) \qquad (\phi \in \mathcal{S}(\mathbb{R}^3)),$$

where $G*\phi$ denotes the convolution of G and ϕ, which implies that

$$\sigma_B^{(1)}(p)\phi(x) = (G*\phi)(x) = {}^t((G*\phi_1)(x), (G*\phi_2)(x), (G*\phi_3)(x), (G*\phi_4)(x)) \tag{4.4}$$

for $\phi = {}^t(\phi_1, \phi_2, \phi_3, \phi_4) \in [\mathcal{S}(\mathbb{R}^3)]^4$.

(III) Let B be the pseudodifferential operator as in (I). It follows from (4.4) that

$$B\phi(x) = \sigma_H(p)\sigma_B^{(1)}(p)\phi(x) = \Big(\sum_{j=1}^{3} -i\alpha_j\partial_j + \beta\Big)(G * \phi)(x) \tag{4.5}$$

for $\phi(x) \in [\mathcal{S}(\mathbb{R}^3)]^4$, where $\partial_j = \partial/\partial x_j$. Define a 4×4 matrix-valued function $K(x) = (K_{k\ell}(x))_{1\le k,\ell\le 4}$ on $\mathbb{R}^3$ by

$$K(x) = \Big(\sum_{j=1}^{3} -i\alpha_j\partial_j + \beta\Big)G(x) \qquad (G(x) = {}^t(G(x), G(x), G(x), G(x)).$$

Therefore we have

$$K(x) = \frac{1}{4\pi}\Big[\sum_{j=1}^{3} i\alpha_j\Big(\frac{x_j}{|x|^3} + \frac{x_j}{|x|^2}\Big) + \beta\frac{1}{|x|}\Big]e^{-|x|}, \tag{4.6}$$

and each element $K_{k\ell}(x)$ of $K(x)$ belongs to $L^1(\mathbb{R}^3)$. Thus the k-th component $(B\phi)_k$, $k = 1, 2, 3, 4$, of $B\phi$ is expressed as

$$(B\phi)_k(x) = \sum_{\ell=1}^{4}(K_{k\ell} * \phi_\ell)(x),$$

which allows us to apply Young's inequality to see that

$$\|B\phi\|_1 \le C\|\phi\|_1 \qquad (\phi \in [\mathcal{S}(\mathbb{R}^3)]^4)$$

with a positive constant C. Therefore B restricted on $[\mathcal{S}(\mathbb{R}^3)]^4$ is uniquely extended to a bounded linear operator on $[L^1(\mathbb{R}^3)]^4$ which will be denoted by B_0. The operator B_0 is actually the restriction of B to $[L^1(\mathbb{R}^3)]^4$.

(IV) Let $g \in [L^1(\mathbb{R}^3)]^4$. Let $\{g_m\}_{m=1}^\infty \subset [\mathcal{S}(\mathbb{R}^3)]^4$ be a sequence such that $g_m \to g$ in $[L^1(\mathbb{R}^3)]^4$ as $m \to \infty$. It follows from (4.2) that

$$H_0B_0g_m = (\alpha\cdot p + \beta)B_0g_m = g_m \qquad (m = 1, 2, \cdots).$$

Therefore, recalling that B_0 is a bounded operator on $[L^1(\mathbb{R}^3)]^4$, we have

$$\begin{cases} B_0g_m \to B_0g, \\ (\alpha\cdot p + \beta)B_0g_m = g_m \to g \end{cases} \tag{4.7}$$

in $[L^1(\mathbb{R}^3)]^4$ as $m \to \infty$. Since the operator $H_0 = \alpha\cdot p + \beta$ defined on $\mathbb{H}^{1,1}(\mathbb{R}^3)$ is a closed operator, we see from (4.7) that $B_0g \in \mathbb{H}^{1,1}(\mathbb{R}^3)$ and $H_0B_0g = g$, which implies that $H_0 = \alpha\cdot p + \beta$ is onto $[L^1(\mathbb{R}^3)]^4$.

(V) Suppose that $f \in \mathbb{H}^{1,1}(\mathbb{R}^3)$ such that $H_0 f = (\alpha\cdot p+\beta)f = 0$. Let $\{f_m\}_{m=1}^{\infty} \subset [C_0^{\infty}(\mathbb{R}^3)]^4$ such that $f_m \to f$ in $\mathbb{H}^{1,1}(\mathbb{R}^3)$ as $m \to \infty$. Thus we have

$$f_m \to f, \quad (\alpha\cdot p+\beta)f_m \to (\alpha\cdot p+\beta)f \quad \text{in} \quad [L^1(\mathbb{R}^3)]^4. \tag{4.8}$$

On the other hand, we have from (4.2)

$$B_0(\alpha\cdot p+\beta)f_m = (\alpha\cdot p+\beta)B_0 f_m = f_m \qquad (m = 1, 2, \cdots). \tag{4.9}$$

Letting $m \to \infty$ in (4.9), noting that B_0 is a bounded operator and using (4.8), we see that

$$0 = B_0(\alpha\cdot p+\beta)f = \lim_{m\to\infty}(\alpha\cdot p+\beta)B_0 f_m = \lim_{m\to\infty} f_m = f$$

in $[L^1(\mathbb{R}^3)]^4$, which implies that H_0 is one-to-one. This completes the proof of Lemma 4.1. □

Remark 4.2. As has been seen, the key element of the proof of the above Lemma 4.1 is to show that the pseudodifferential operator B given by (4.3) is a bounded linear operator on $[L^1(\mathbb{R}^3)]^4$. Actually it can be shown that B is a bounded linear operator on $[L^p(\mathbb{R}^3)]^4$ for $1 \le p < \infty$. Thus we can prove that Lemma 4.1 holds for any $1 \le p < \infty$. In fact, for $1 < p < \infty$, a theorem in Fefferman [3] (Theorem, a), p.414) can be applied to show that B is a bounded linear operator on $[L^p(\mathbb{R}^n)]^4$ with $n = 3$. Let $\sigma(x, p)$ be a pseudodifferential operator in $\mathbb{R}^n$ whose symbol $\sigma(x, \xi)$ belongs to the Hörmander class $S^{-b}_{1-a,\delta}(\mathbb{R}^n)$ with $0 \le \delta < 1 - a < 1$. Then it follows from the above theorem by Fefferman that $\sigma(x, p)$ is a bounded operator on $L^p(\mathbb{R}^n)$ if $b < na/2$ and

$$\left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{b}{n}\left[\frac{\frac{n}{2}+\lambda}{b+\lambda}\right] \qquad \left(\lambda = \frac{\frac{na}{2} - b}{1-a}\right).$$

By taking $n = 3$, $b = 1$, $\delta = 0$, the above two condition becomes

$$3\left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{1}{a} < \frac{3}{2}. \tag{4.10}$$

For $p > 1$ there exists $a \in (0, 1)$ which satisfies (4.10). For $p = 1$, however, there is no a which satisfies (4.10) since both sides of (4.10) become $3/2$. Indeed, our pseudodifferential operator B has symbol $\sigma_B(\xi)$ belonging to the Hörmander class $S^{-1}_{1-a,0}(\mathbb{R}^3)$. To prove Lemma 4.1, which is the case $p = 1$, we have discussed the integral kernel of the Dirac operator.

To proceed, we need some facts on the local Hardy space $h^1(\mathbb{R}^3)$, which is introduced in Goldberg [5] in connection with the Hardy space $H^1(\mathbb{R}^3)$. The Hardy space is (see e.g. Fefferman-Stein [4]) the proper subspace of $L^1(\mathbb{R}^3)$ consisting of the functions $f \in L^1(\mathbb{R}^3)$ such that $R_j f \in L^1(\mathbb{R}^3)$ for $j = 1, 2, 3$, where $R_j := \partial_j \cdot (-\Delta)^{-1/2}$ are the Riesz transforms, having symbols $i\xi_j/|\xi|$. Let φ be a fixed function in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ such that $\varphi = 1$ in a neighborhood of the origin. By definition a distribution f belongs to $h^1(\mathbb{R}^3)$ if and only if $f \in L^1(\mathbb{R}^3)$ and $r_j f \in L^1(\mathbb{R}^3)$ for $j = 1, 2, 3$, where r_j, $j = 1, 2, 3$, are pseudodifferential operators with symbol $\sigma_{r_j}(\xi) = (1 - \varphi(\xi))(i\xi_j/|\xi|)$ ([5], Theorem 2 (p.33)). The definition is independent of the choice of φ. It is a Banach space with norm $\|f\|_{h^1} = \|f\|_{L^1} + \sum_{j=1}^3 \|r_j f\|_{L^1}$. The space $h^1(\mathbb{R}^3)$ is a proper subspace of $L^1(\mathbb{R}^3)$, which is strictly larger than the Hardy space $H^1(\mathbb{R}^3)$ (see e.g. [5], p.33, just after Theorem 3).

Now, we are introducing the following operator

$$r'_j = \partial_j(1-\Delta)^{-1/2} = \frac{\partial_j}{(-\Delta)^{1/2}} \frac{(-\Delta)^{1/2}}{(1-\Delta)^{1/2}} = R_j \cdot \frac{(-\Delta)^{1/2}}{(1-\Delta)^{1/2}}, \tag{4.11}$$

where we note that the pseudodifferential operator $(-\Delta)^{1/2}/(1-\Delta)^{1/2}$ is a bounded operator on $L^1(\mathbb{R}^3)$ (see Stein [6], p.133, Eq.(31)).

The proof of the lemma below was inspired by the proof of [5], Theorem 2 (p.33).

Lemma 4.3. *A distribution f in $\mathbb{R}^3$ belongs to $h^1(\mathbb{R}^3)$ if and only if $f \in L^1(\mathbb{R}^3)$ and $r'_j f \in L^1(\mathbb{R}^3)$ for $j = 1, 2, 3$.*

Proof. (I) It is sufficient to show that $r_j - r'_j$, $j = 1, 2, 3$, are bounded linear operators on $L^1(\mathbb{R}^3)$ (or, more exactly, the pseudodifferential operator $r_j - r'_j$ defined on $\mathcal{S}(\mathbb{R}^3)$ can be uniquely extended to a bounded linear operator on $L^1(\mathbb{R}^3)$). Note that the operators r_j and r'_j have symbols

$$\sigma_{r_j}(\xi) = \frac{(1-\varphi(\xi))i\xi_j}{|\xi|}, \quad \sigma_{r'_j}(\xi) = \frac{i\xi_j}{(1+|\xi|^2)^{1/2}}, \tag{4.12}$$

and both symbols are C^∞ functions in $\mathbb{R}^3_\xi$ and bounded together with all their derivatives, and we have

$$\begin{aligned}\sigma_{r_j}(\xi) - \sigma_{r'_j}(\xi) = \frac{(1-\varphi(\xi))i\xi_j}{|\xi|}\Big(1 - \frac{|\xi|}{(1+|\xi|^2)^{1/2}}\Big) - \frac{\varphi(\xi)i\xi_j}{(1+|\xi|^2)^{1/2}} \\ =: \sigma_{1j}(\xi) + \sigma_{2j}(\xi).\end{aligned}$$

As in the proof of Lemma 4.1, we are going to show that, for each $j = 1, 2, 3$, the pseudodifferential operator with symbols σ_{1j} and σ_{2j} have integral kernels belonging to $L^1(\mathbb{R}^3)$, in other words, that their inverse Fourier transforms $\overline{\mathcal{F}}\sigma_{1j}(x)$ and $\overline{\mathcal{F}}\sigma_{2j}(x)$

belong to $L^1(\mathbb{R}^3)$, where $\overline{\mathcal{F}}$ is given by

$$\overline{\mathcal{F}}\phi(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\phi(\xi)\, d\xi.$$

It is easy to see that $\overline{\mathcal{F}}\sigma_{2j} \in L^1(\mathbb{R}^3)$, because σ_{2j} belongs to $\mathcal{S}(\mathbb{R}^3)$, so that $\overline{\mathcal{F}}\sigma_{2j}$ belongs to $\mathcal{S}(\mathbb{R}^3)$ and hence it belongs to $L^1(\mathbb{R}^3)$. In the rest of the proof we are going to show that $\overline{\mathcal{F}}\sigma_{1j} \in L^1(\mathbb{R}^3)$.

(II) By definition we have

$$\sigma_{1j}(\xi) = i(1-\varphi(\xi))\frac{\xi_j}{|\xi|(1+|\xi|^2)^{1/2}(|\xi| + (1+|\xi|^2)^{1/2})},$$

and hence, $\sigma_{1j}(\xi) = O(|\xi|^{-2})$ as $|\xi| \to \infty$. Thus, by noting that $1-\varphi(\xi)$ is 0 around the origin $\xi = 0$, we see that $\sigma_{1j} \in L^2(\mathbb{R}^3)$. Therefore $I_j(x) = (\overline{\mathcal{F}}\sigma_{1j})(x)$ exists as a function in $L^2(\mathbb{R}^3_x)$. Let $\rho(t)$ be a real-valued C^∞ function on $[0,\infty)$ such that

$$\rho(t) = 1 \quad (0 \le t \le 1), \quad = 0 \quad (t \ge 2).$$

Then, since $\sigma_{1j}(\xi)\rho(\epsilon|\xi|)$, $\epsilon > 0$, converges to $\sigma_{1j}(\xi)$ in $L^2(\mathbb{R}^3_\xi)$ as $\epsilon \downarrow 0$, we have, by setting

$$I_j(x,\epsilon) := \overline{\mathcal{F}}(\sigma_{1j}(\xi)\rho(\epsilon|\xi|)),$$

$I_j(x,\epsilon)$ converges to $I_j(x)$ in $L^2(\mathbb{R}^3_x)$ as $\epsilon \downarrow 0$, and hence there exists a decreasing sequence

$$1 \ge \epsilon_2 > \epsilon_2 > \cdots > \epsilon_m > \to 0$$

such that $I_j(\epsilon_m, x) \to I_j(x)$ a.e. x as $m \to \infty$. For the sake of the simplicity of notations, we shall use $\epsilon \le 1$ instead of ϵ_m.

(III) Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index. Then we have

$$|\partial_\xi^\alpha \sigma_{1j}(\xi)| \le C_{j,\alpha}(1+|\xi|)^{-2-|\alpha|} \qquad (\xi \in \mathbb{R}^3_\xi) \tag{4.13}$$

with a constant $C_{j,\alpha} > 0$, where we should note that $1-\varphi(\xi)$ is bounded and

$$\partial_\xi^\alpha(1-\varphi(\xi)) = -\partial_\xi^\alpha\varphi(\xi) \in \mathcal{S}(\mathbb{R}^3_\xi)$$

for $\alpha \neq 0$. Let ℓ be a positive integer and $k = 1,2,3$. Then, by integration by parts,

$$\begin{aligned} (2\pi)^{3/2}x_k^\ell I_j(x,\epsilon) &= \int_{\mathbb{R}^3} \big\{(-i\partial_{\xi_k}^\ell)e^{ix\cdot\xi}\big\}\sigma_{1j}(\xi)\rho(\epsilon|\xi|)\, d\xi \\ &= (-i)^\ell(-1)^\ell \int_{\mathbb{R}^3} e^{ix\cdot\xi}\partial_{\xi_k}^\ell\big\{\sigma_{1j}(\xi)\rho(\epsilon|\xi|)\big\}\, d\xi. \end{aligned} \tag{4.14}$$

Here, by the Leibniz formula, we have

$$\partial_{\xi_k}^{\ell}\{\sigma_{1j}(\xi)\rho(\epsilon|\xi|)\} = \big(\partial_{\xi_k}^{\ell}\sigma_{1j}(\xi)\big)\rho(\epsilon|\xi|) + \sum_{m=1}^{\ell} {}_{\ell}C_m\big(\partial_{\xi_k}^{\ell-m}\sigma_{1j}(\xi)\big)\big(\partial_{\xi_k}^{m}\rho(\epsilon|\xi|)\big)$$
$$=: J_0(\xi,\epsilon) + J_1(\xi,\epsilon)$$

with ${}_{\ell}C_m = \ell/(m!(\ell-m)!)$. For $m = 1, 2, \cdots, \ell$, we have

$$\xi \in \ \mathrm{supp}(\partial_{\xi_k}^{m}\rho(\epsilon|\xi|)) \Longrightarrow 1 \le \epsilon|\xi| \le 2 \Longrightarrow \epsilon \le \frac{2}{|\xi|},$$

where $\mathrm{supp}(f)$ denotes the support of f. Thus we can replace ϵ in $\partial_{\xi_k}^{m}\rho(\epsilon|\xi|)$ by $2|\xi|^{-1}$ when we evaluate $|\partial_{\xi_k}^{m}\rho(\epsilon|\xi|)|$. Therefore it follows that

$$|\partial_{\xi_k}^{m}\rho(\epsilon|\xi|)| \le c(1+|\xi|)^{-m}\chi_\epsilon(\xi), \tag{4.15}$$

where $c = c_{j,k,m}$ is a positive constant and $\chi_\epsilon(\xi)$ is the characteristic function of the set $A_\epsilon = \{\xi \,:\, \epsilon^{-1} \le |\xi| \le 2\epsilon^{-1}\}$. Since it is supposed that $\epsilon \le 1$, we have $A_\epsilon \subset \{\xi \,:\, |\xi| \ge 1\}$. The inequalities (4.13) and (4.15) are combined to give

$$|J_1(\xi,\epsilon)| \le C(1+|\xi|)^{-2-\ell}\chi_\epsilon(\xi) \qquad (\xi \in \mathbb{R}^3_\xi)$$

with positive constant $C = C_{j,k,\ell}$. Let $\ell \ge 2$. Then it is seen that $|J_1(\xi,\epsilon)|$ is dominated by $C(1+|\xi|)^{-2-\ell}$, which is in $L^1(\mathbb{R}^3_\xi)$, and $J_1(\xi,\epsilon) \to 0$ for each $\xi \in \mathbb{R}^3_\xi$ as $\epsilon \to 0$, and hence, by the Lebesgue convergence theorem, we have

$$\int_{\mathbb{R}^3} e^{ix\cdot\xi}J_1(\xi,\epsilon)\,d\xi \to 0 \quad (\epsilon \to 0).$$

Similarly, since $|J_0(\xi,\epsilon)|$ is dominated by $|\partial_{\xi_k}^{\ell}\sigma_{1j}(\xi)|$ which is in $L^1(\mathbb{R}^3_\xi)$, and $J_0(\xi,\epsilon)$ converges to $\partial_{\xi_k}^{\ell}\sigma_{1j}(\xi)$ for each $\xi \in \mathbb{R}^3_\xi$ as $\epsilon \to 0$, we have

$$\int_{\mathbb{R}^3} e^{ix\cdot\xi}J_0(\xi,\epsilon)\,d\xi \to \int_{\mathbb{R}^3} e^{ix\cdot\xi}\partial_{\xi_k}^{\ell}\sigma_{1j}(\xi)\,d\xi \quad (\epsilon \to 0).$$

Therefore, by letting $\epsilon \to 0$ in (4.14), we obtain

$$x_k^{\ell}I_j(x) = x_k^{\ell}(\overline{\mathcal{F}}\sigma_{1j})(x) = i^{\ell}(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\partial_{\xi_k}^{\ell}\sigma_{1j}(\xi)\,d\xi \tag{4.16}$$

a.e. $x \in \mathbb{R}^3$ for $\ell \ge 2$ and $j, k = 1, 2, 3$. Here the right-hand side is uniformly bounded for $x \in \mathbb{R}^3$. Thus, by considering the case $\ell = 2$ and $\ell = 4$, it follows that

$$|(\overline{\mathcal{F}}\sigma_{1j})(x)| \le C_j \min(|x|^{-2}, |x|^{-4}) \quad (j = 1, 2, 3)$$

with a positive constant C_j, which implies that $\overline{\mathcal{F}}\sigma_{1j} \in L^1(\mathbb{R}^3)$. This completes the proof of Lemma 4.3. □

Now we are going to prove that $[H^{1,1}(\mathbb{R}^3)]^4$ is a proper subspace of $\mathbb{H}^{1,1}(\mathbb{R}^3)$, which is the most crucial part of Theorem 1.3, (iii) in the following strategy: Let $g \in [L^1(\mathbb{R}^3)]^4 \setminus [h^1(\mathbb{R}^3)]^4$, where $h^1(\mathbb{R}^3)$ is the local Hardy space which can be defined, by Lemma 4.3, as the space of all distributions f such that $f \in L^1(\mathbb{R}^3)$ and $r'_j f \in L^1(\mathbb{R}^3)$ for $j = 1, 2, 3$, where r'_j is given by (4.11). Set $f = (\alpha \cdot p + \beta)^{-1} g$. Then, by using Lemma 4.1, we have $f \in \mathbb{H}^{1,1}(\mathbb{R}^3)$. Then we shall be able to show that $f \notin [H^{1,1}(\mathbb{R}^3)]^4$.

Proposition 4.4. $\mathbb{H}^{1,1}(\mathbb{R}^3)$ *is strictly larger than* $[H^{1,1}(\mathbb{R}^3)]^4$.

Proof. (I) It follows from Lemma 4.1 that, for every $g \in [L^1(\mathbb{R}^3)]^4$ there exists a unique $f \in \mathbb{H}^{1,1}(\mathbb{R}^3)$ such that $g = (\alpha \cdot p + \beta) f$. The equation $g = (\alpha \cdot p + \beta) f$ can be written as

$$\begin{cases} g_1 = -i(\partial_1 - i\partial_2) f_4 - i\partial_3 f_3 + f_1, \\ g_2 = -i(\partial_1 + i\partial_2) f_3 + i\partial_3 f_4 + f_2, \\ g_3 = -i(\partial_1 - i\partial_2) f_2 - i\partial_3 f_1 - f_3, \\ g_4 = -i(\partial_1 + i\partial_2) f_1 + i\partial_3 f_2 - f_4. \end{cases}$$

Solving the above equation for f_1, f_2, f_3 and f_4, we obtain

$$\begin{cases} (1-\Delta) f_1 = -i(\partial_1 - i\partial_2) g_4 - i\partial_3 g_3 + g_1, \\ (1-\Delta) f_2 = -i(\partial_1 + i\partial_2) g_3 - i\partial_3 g_4 + g_2, \\ (1-\Delta) f_3 = -i(\partial_1 - i\partial_2) g_2 - i\partial_3 g_1 - g_3, \\ (1-\Delta) f_4 = -i(\partial_1 + i\partial_2) g_1 - i\partial_3 g_2 - g_4, \end{cases} \tag{4.17}$$

where $\partial_j = \partial/\partial x_j$. Here each equation in (4.17) should be viewed as equations in $\mathcal{S}'(\mathbb{R}^3)$. As has been shown in the proof of Lemma 4.1, the differential operator $1-\Delta$ has the inverse $(1-\Delta)^{-1}$ as a pseudodifferential operator with symbol $(1+|\xi|^2)^{-1}$, and hence, by applying $(1-\Delta)^{-1}$ and ∂_j to each of the equations in (4.17), it follows that

$$\begin{cases} \partial_j f_1 = -i(\partial_j\partial_1 - i\partial_j\partial_2)(1-\Delta)^{-1} g_4 - i\partial_j\partial_3 (1-\Delta)^{-1} g_3 + \partial_j (1-\Delta)^{-1} g_1, \\ \partial_j f_2 = -i(\partial_j\partial_1 + i\partial_j\partial_2)(1-\Delta)^{-1} g_3 + i\partial_j\partial_3 (1-\Delta)^{-1} g_4 + \partial_j (1-\Delta)^{-1} g_2, \\ \partial_j f_3 = -i(\partial_j\partial_1 - i\partial_j\partial_2)(1-\Delta)^{-1} g_2 - i\partial_j\partial_3 (1-\Delta)^{-1} g_1 - \partial_j (1-\Delta)^{-1} g_3, \\ \partial_j f_4 = -i(\partial_j\partial_1 + i\partial_j\partial_2)(1-\Delta)^{-1} g_1 + i\partial_j\partial_3 (1-\Delta)^{-1} g_2 - \partial_j (1-\Delta)^{-1} g_4. \end{cases} \tag{4.18}$$

(II) By Lemma 4.3 we can choose $g_0 \in L^1(\mathbb{R}^3) \setminus h^1(\mathbb{R}^3)$ such that $r'_3 g_0 = \partial_3 (1-\Delta)^{-1/2} g_0 \notin L^1(\mathbb{R}^3)$. Then define $g = {}^t(g_1, g_2, g_3, g_4) \in [L^1(\mathbb{R}^3)]^4$ by

$$g_1(x) = g_3(x) = g_4(x) = 0 \quad \text{and} \quad g_2(x) = g_0(x).$$

Then we have from (4.18)

$$\partial_j f_4 = i\partial_j \partial_3 (1-\Delta)^{-1} g_2 \qquad (j=1,2,3). \tag{4.19}$$

Since $r'_3 g_2 \notin L^1(\mathbb{R}^3)$, we have necessarily $r'_3 g_2 \notin h^1$. Then $\big(r'_1(r'_3 g_2), r'_2(r'_3 g_2), r'_3(r'_3 g_2)\big)$ does not belong to $[L^1(\mathbb{R}^3)]^3$. It follows from (4.11) that the symbol $s_{j3}(\xi)$ of $r'_j r'_3$ is given by

$$s_{j3}(\xi) = \frac{i\xi_j}{(1+|\xi|^2)^{1/2}} \frac{i\xi_3}{(1+|\xi|^2)^{1/2}} = (i\xi_j)(i\xi_3)(1+|\xi|^2)^{-1},$$

and hence by using (4.11) again, we see that

$$r'_j r'_3 g_2 = (\partial_j (1-\Delta)^{-1/2})(\partial_3(1-\Delta)^{-1/2}) g_2 = \partial_j \partial_3 (1-\Delta)^{-1} g_2 \tag{4.20}$$

for $j=1,2,3$. Thus we have from (4.19) and (4.20)

$$(\partial_1 f_4, \partial_2 f_4, \partial_3 f_4) = i(r'_1 r'_3 g_2, r'_2 r'_3 g_2, r'_3 r'_3 g_2) \notin [L^1(\mathbb{R}^3)]^3,$$

which implies that $f \notin [H^{1,1}(\mathbb{R}^3)]^4$. This completes the proof of Proposition 4.4. □

Proposition 4.5. *Let Ω be an open subset of $\mathbb{R}^3$. Then*

(i) $[H_0^{1,1}(\Omega)]^4$ *is a proper subspace of* $\mathbb{H}_0^{1,1}(\Omega)$.

(ii) $[H^{1,1}(\Omega)]^4$ *is a proper subspace of* $\mathbb{H}^{1,1}(\Omega)$.

Proof. (I) We are going to show the norms $\|f\|_{S,1,1,\Omega}$ of $[H_0^{1,1}(\Omega)]^4$ and $\|f\|_{D,1,1,\Omega}$ of $\mathbb{H}_0^{1,1}(\Omega)$ are not equivalent on $[C_0^\infty(\Omega)]^4$ (see (1.9) and (1.11) for the definition of these norms). To this end we use Proposition 4.4. Without loss of generality, we may assume that Ω contains the unit ball $\{x : |x| \le 1\}$ with center at the origin. As in (1.6) (with $p=1$), we denote the norm of $[L^1(\Omega)]^4$ by $\|f\|_{1,\Omega}$, *i.e.*,

$$\|f\|_{1,\Omega} = \int_\Omega \sum_{j=1}^4 |f_j(x)|\, dx \qquad (f(x) = {}^t(f_1(x), f_2(x), f_3(x), f_4(x)).$$

By Proposition 4.4 and the fact that $[C_0^\infty(\mathbb{R}^3)]^4$ is dense in both $[H^{1,1}(\mathbb{R}^3)]^4$ and $\mathbb{H}^{1,1}(\mathbb{R}^3)$ (Theorem 1.3, (i)), the norms $\|f\|_{S,1,1,\mathbb{R}^3}$ and $\|f\|_{D,1,1,\mathbb{R}^3}$ are not equivalent on $[C_0^\infty(\mathbb{R}^3)]^4$. Therefore, by taking note of Proposition 2.2, (2.2) with $\Omega = \mathbb{R}^3$, which says that the norm $\|f\|_{D,1,1,\mathbb{R}^3}$ is dominated by the norm $\|f\|_{S,1,1,\mathbb{R}^3}$, there exists a sequence $\{f_n\}_{n=1}^\infty$ of functions in $[C_0^\infty(\mathbb{R}^3)]^4$ such that $f_n \ne 0$ and $\|f_n\|_{S,1,1,\mathbb{R}^3} \ge (n+1)\|f_n\|_{D,1,1,\mathbb{R}^3}$, or

$$\|f_n\|_{1,\mathbb{R}^3} + \|\nabla f_n\|_{1,\mathbb{R}^3} \ge (n+1)[\|f_n\|_{1,\mathbb{R}^3} + \|(\alpha \cdot p) f_n\|_{1,\mathbb{R}^3}] \tag{4.21}$$

for each $n = 1, 2, \cdots$. Each f_n has support in some ball $\{x : |x| \le R_n\}$ with radius $R_n > 0$ and center at the origin. We may assume with no loss of generality that

$R_n \geq 1$ $(n = 1, 2, \cdots)$. Put $g_n(x) = R_n^3 f_n(R_n x)$. Then g_n has support in the unit ball $\{x : |x| \leq 1\}$, and hence in Ω, so that $\{g_n\} \subset [C_0^\infty(\Omega)]^4$ for each n. We have

$$\|g_n\|_{1,\Omega} = \|f_n\|_{1,\mathbb{R}^3} \quad \text{and} \quad \|\partial_j g_n\|_{1,\Omega} = R_n \|\partial_j f_n\|_{1,\mathbb{R}^3}$$

for $j = 1, 2, 3$. Then by (4.21) we have

$$\|g_n\|_{1,\Omega} + \frac{1}{R_n}\|\nabla g_n\|_{1,\Omega} \geq (n+1)[\|g_n\|_{1,\Omega} + \frac{1}{R_n}\|(\alpha \cdot p)g_n\|_{1,\Omega}],$$

and hence, by noting $R_n \geq 1$

$$\begin{aligned} \frac{1}{R_n}\|\nabla g_n\|_{1,\Omega} &\geq n\|g_n\|_{1,\Omega} + \frac{n+1}{R_n}\|(\alpha \cdot p)g_n\|_{1,\Omega} \\ &\geq \frac{n}{R_n}[\|g_n\|_{1,\Omega} + \|(\alpha \cdot p)g_n\|_{1,\Omega}\|]. \end{aligned}$$

Therefore

$$\|\nabla g_n\|_{1,\Omega} \geq n[\|g_n\|_{1,\Omega} + \|(\alpha \cdot p)g_n\|_{1,\Omega}],$$

which implies that $\|g_n\|_{S,1,1,\Omega} \geq n\|g_n\|_{D,1,1,\Omega}$ for $n = 1, 2, \cdots$. This proves that the norms $\|f\|_{S,1,1,\Omega}$ and $\|f\|_{D,1,1\Omega}$ are not equivalent on $[C_0^\infty(\Omega)]^4$, showing (i).

(II) As we have seen in the proof of (i), the norms of $[H_0^{1,1}(\Omega)]^4$ and $\mathbb{H}_0^{1,1}(\Omega)$ are not equivalent. In fact, there exists a sequence $\{g_n\} \subset [C_0^\infty(\Omega)]^4$ such that

$$\|g_n\|_{S,1,1,\Omega} \geq n\|g_n\|_{D,1,1,\Omega}, \; n = 1, 2, \ldots.$$

Since clearly this sequence is also contained in $[H^{1,1}(\Omega)]^4$, this implies that the norms of $[H^{1,1}(\Omega)]^4$ and $\mathbb{H}^{1,1}(\Omega)$ are not equivalent. This shows (ii), completing the proof of Proposition 4.5. □

Proof of Theorem 1.3, (iii). Theorem 1.3, (iii) follows from Propositions 4.5. □

Acknowledgement. Being deeply grieved that our respected friend, Professor Tetsuro Miyakawa, suddenly passed away in February 2009, we gratefully remember nice useful discussion with him about the Riesz transform at the early stage of the present work. Thanks are also due to Professor Shuichi Sato for valuable discussion about the Hardy space and local Hardy space. The research of T.I. is supported in part by JSPS Grant-in-Aid for Sientific Research No. 17654030. We wish to thank the referee for his kindly pointing out a flaw in an assertion in Theorem 1.3(ii) of our original version of the paper.

References

[1] R. Adams and J. Fournier, *Sobolev Spaces, 2nd edition*, Academic Press, 2003.
[2] A. Balinsky, W. D. Evans and Y. Saitō, *Dirac–Sobolev inequalities and estimates for the zero modes of massless Dirac operators*, J. Math. Physics **49**, (2008), 043524-1 – 043524-10.
[3] C. Fefferman, *L^p bounds for pseudo-differential operators*, Israel J. Math. **1**4 (1973), 413–417.
[4] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193.
[5] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J., **46** (1979), 27–42.
[6] E. M. Stein, *Singular Integrals and Differntiability Propoerties of Functions*, Princeton University Press, 1970.

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