

# Non-cooperative Equilibria of Fermi Systems With Long Range Interactions

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## Abstract

We define a Banach space  $\mathcal{M}_1$  of models for fermions or quantum spins in the lattice with long range interactions and explicit the structure of (generalized) equilibrium states for any  $\mathfrak{m} \in \mathcal{M}_1$ . In particular, we give a first answer to an old open problem in mathematical physics – first addressed by Ginibre in 1968 within a different context – about the validity of the so-called Bogoliubov approximation on the level of states. Depending on the model  $\mathfrak{m} \in \mathcal{M}_1$ , our method provides a systematic way to study all its correlation functions and can thus be used to analyze the physics of long range interactions. Furthermore, we show that the thermodynamics of long range models  $\mathfrak{m} \in \mathcal{M}_1$  is governed by the non-cooperative equilibria of a zero-sum game, called here thermodynamic game.

**Keywords:** Long range interaction, Choquet theorem, equilibrium state, non-cooperative equilibrium, two-person zero-sum game, Bogoliubov approximation, approximating Hamiltonian method, fermion system, quantum spin system.

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## 1. Introduction

States are the positive and normalized linear functionals on a  $*$ –algebra  $\mathcal{U}$  and forms a convex set  $E$ . This set is weak $^*$ –compact in the case where  $\mathcal{U}$  is a  $C^*$ –algebra and it is even metrizable if  $\mathcal{U}$  is separable, cf. [1, Theorem 3.16]. The structure of the set  $E$  of states is then satisfactorily described by Choquet theorem [2, 3]: Any state has a unique decomposition as an integral on extreme states of  $E$ .

Special subsets  $\Omega \subset E$  of states on  $\mathcal{U}$  are of particular importance in statistical physics, for instance, if  $\mathcal{U}$  is the observable ( $C^*$ -) algebra of Fermi or quantum spin systems on a lattice  $\mathbb{Z}^d$  ( $d \geq 1$ ). In this case, one of the main issues is to understand the limit  $l \rightarrow \infty$  of sequences of (local) Gibbs equilibrium states

$$\rho_l(\cdot) = \frac{\text{Trace}(\cdot e^{-\beta U_l})}{\text{Trace}(e^{-\beta U_l})}$$

defined, for all  $\beta > 0$  and  $l \in \mathbb{N}$ , from self-adjoint operators  $U_l \in \mathcal{U}$ . In quantum statistical mechanics,  $\beta > 0$  is the inverse temperature,  $U_l$  represents the energy observable of particles enclosed in a finite box  $\Lambda_l \subset \mathbb{Z}^d$ , and the limit  $l \rightarrow \infty$  is such that  $\Lambda_l \nearrow \mathbb{Z}^d$  (thermodynamic limit). As  $E$  is weak\*-compact, any sequence of states  $\rho_l \in E$  converges – along a subsequence – towards a so-called *equilibrium state*  $\omega \in E$  as  $l \rightarrow \infty$ . An explicit characterization of the limit state  $\omega \in E$  is a rather difficult issue in most interesting cases.

Taking  $U_l$  from *local* (i.e., short range) translation invariant interactions  $\Phi$  and by conveniently choosing boundary conditions, the limit state  $\omega \in E$  is found to be a solution of a variational problem on the (convex and weak\*-compact) set  $E_1 \subset E$  of translation invariant states, i.e., it minimizes a weak\*-lower semi-continuous functional  $f_\Phi$  on  $E_1$ . This result is standard for quantum spin systems, see e.g. [4, Chapter II] or [5, Section 6.2].  $f_\Phi$  and its minimizers are called, respectively, the free-energy density functional and equilibrium states of the system under consideration.

Fermion systems on a lattice correspond to choose the  $C^*$ -algebra  $\mathcal{U}$  as the inductive limit of the net of complex Clifford algebras  $\mathcal{U}_\Lambda$ ,  $\Lambda \subset \mathbb{Z}^d$ ,  $|\Lambda| < \infty$ , generated by the elements<sup>1</sup>  $a_{x,s}$  and  $a_{x,s}^+$  satisfying the so-called canonical anti-commutation relations (CAR) for  $x \in \Lambda$  and  $s \in S$ . Here, the finite set  $S$  corresponds to the internal degrees of freedom (spin) of particles. Quantum spin systems on a lattice are described by infinite tensor products of finite dimensional  $C^*$ -algebras attached to each site  $x \in \mathbb{Z}^d$ . As a consequence, in contrast to lattice quantum spins, elements  $A \in \mathcal{U}_\Lambda$  and  $B \in \mathcal{U}_{\Lambda'}$  in disjoint regions of the lattice ( $\Lambda \cap \Lambda' = \emptyset$ ) do *not* generally commute with each other. A study of equilibrium states of lattice fermions similar to the one for lattice quantum spins is hence more involved and was only<sup>2</sup> performed in 2004 by Araki and Moriya [8]. In particular, the limit state  $\omega \in E$  is again a minimizer of a weak\*-lower semi-continuous functional  $f_\Phi$  on  $E_1$ .

All these results [4, 5, 8] use Banach spaces of *local* interactions. Unfortunately, these Banach spaces are too small to include all physically interesting

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<sup>1</sup> $a_{x,s}$  and  $a_{x,s}^+$  are the annihilation and creation operators of a particle at lattice position  $x$ .

<sup>2</sup>There are some results on the level of the pressure [6, 7] by using the quantum spin representation of fermion systems for a specific class of models

systems. Indeed, physically speaking, local interaction means that the interaction between particles is *short* range, i.e., it has to decrease sufficiently fast as the inter-particle distance increases. Nevertheless, *long* range interactions are also fundamental as they explain important physical phenomena like conventional superconductivity.

In this paper, we construct a Banach space  $\mathcal{M}_1$  of translation invariant Fermi models including a class of long range interactions on the lattice. We restrict our analysis on translation invariant Fermi systems, but we emphasize that all our studies can also be performed for quantum spins<sup>3</sup> as well as for (not necessarily translation invariant, but only) periodically invariant systems. Then we generalize<sup>4</sup> some previous results of [4, 5, 8] to the larger space  $\mathcal{M}_1$ . By conveniently choosing boundary conditions, we show, in particular, that the sequence of Gibbs states  $\rho_l \in E$  defined from any  $\mathbf{m} \in \mathcal{M}_1$  converges along a subsequence to a minimizer of the  $\Gamma$ -regularization  $\Gamma(f_{\mathbf{m}}^{\sharp})$  of the so-called free energy density functional  $f_{\mathbf{m}}^{\sharp}$ . Note that  $f_{\mathbf{m}}^{\sharp}$  is affine, but *possibly not* weak\*-lower semi-continuous. Nevertheless, we prove that all weak\*-limit points of any sequence  $\{\rho_i\}_{i=1}^{\infty}$  of its approximating minimizers<sup>5</sup> belong to the closed, convex, and weak\*-compact set  $\Omega_{\mathbf{m}}^{\sharp}$  of minimizers of  $\Gamma(f_{\mathbf{m}}^{\sharp})$ . Observe that  $\Gamma(f_{\mathbf{m}}^{\sharp})$  is the largest convex and weak\*-lower semi-continuous minorant of  $f_{\mathbf{m}}^{\sharp}$  and its minimizers are called *generalized equilibrium states*. Minimizers of  $f_{\mathbf{m}}^{\sharp}$  are (usual) equilibrium states and form a subset of  $\Omega_{\mathbf{m}}^{\sharp}$ .

If the long range component of the interaction is purely attractive then  $\Omega_{\mathbf{m}}^{\sharp}$  is always a face of  $E_1$ . However, in the general case,  $\Omega_{\mathbf{m}}^{\sharp}$  is only a *subset* of a non-trivial face of  $E_1$ . From Choquet theorem (see, e.g., [3, p. 14]), any generalized equilibrium state  $\omega \in \Omega_{\mathbf{m}}^{\sharp}$  of an arbitrary long range model  $\mathbf{m} \in \mathcal{M}_1$  has a decomposition in terms of extreme states of  $\Omega_{\mathbf{m}}^{\sharp} \subset E_1$ . As  $E_1$  is known to be a Choquet simplex, this decomposition is unique whenever  $\Omega_{\mathbf{m}}^{\sharp}$  is a face. Additionally, extreme states are shown to be minimizers of an explicitly given weak\*-lower semi-continuous (possibly neither convex nor concave) functional  $g_{\mathbf{m}}$ . We also show that – exactly as in the case of local interactions – the set  $\Omega_{\mathbf{m}}^{\sharp}$  of generalized equilibrium states can be identified with the set of all continuous tangent functionals at the point  $\mathbf{m} \in \mathcal{M}_1$  of a convex and continuous functional  $P_{\mathbf{m}}^{\sharp}$ , the so-called pressure, on the Banach space  $\mathcal{M}_1$ .

Note that non-uniqueness of generalized equilibrium states corresponds to the existence of phase transitions for the considered model. Physically important phase transitions are those for which the minimizers of  $\Gamma(f_{\mathbf{m}}^{\sharp})$  break initial symmetries of the system. This case is called *spontaneous symmetry*

<sup>3</sup>In fact, quantum spin systems are easier to analyze than fermion systems.

<sup>4</sup>In [8] the authors use a slightly different Banach space of local interactions, see Remark 2.26.

<sup>5</sup>It means that  $\lim_{i \rightarrow \infty} f_{\mathbf{m}}(\rho_i) = \inf f_{\mathbf{m}}(E_1)$

*breaking*. For concrete local interactions, such a phenomenon is usually difficult to prove in the quantum case, whereas there are many explicit models  $\mathfrak{m} \in \mathcal{M}_1$  where it can easily be seen, see, e.g., [9, 10].

Observe that, in general, the solutions of the variational problems given in [4, 5, 8] for local interactions cannot be computed explicitly. The variational problem

$$(1.1) \quad P_{\mathfrak{m}}^{\sharp} = -\inf f_{\mathfrak{m}}^{\sharp}(E_1) = -\inf \Gamma(f_{\mathfrak{m}}^{\sharp})(E_1)$$

generalizing previous results on local interactions to any  $\mathfrak{m} \in \mathcal{M}_1$  is, a priori, even more difficult. We prove, however, that (1.1) can be explicitly analyzed from variational problems with local interactions. This strong simplification is related to an old open problem in mathematical physics – first addressed by Ginibre [11, p. 28] in 1968 within a different context – about the validity of the so-called Bogoliubov approximation on the level of states. Indeed, we give a first answer to this problem in the special class  $\mathcal{M}_1$  of models  $\mathfrak{m}$  by showing that any extreme generalized equilibrium state  $\omega \in \Omega_{\mathfrak{m}}^{\sharp}$  is an equilibrium state of an effective local interaction  $\Phi_{\omega}$ . Such extreme generalized equilibrium states satisfy Euler–Lagrange equations called *gap equations* in the Physics literature. In fact, when the correlation functions of the effective local interaction  $\Phi_{\omega}$  turn out to be accessible, our method provides a systematic way to analyze, at once, *all* correlation functions of the given long range model  $\mathfrak{m} \in \mathcal{M}_1$ . Applications of our method include: A full analysis (postponed to separated papers) of equilibrium states of BCS-type models, the explicit description of models showing qualitatively the same density dependency of the critical temperature observed in high- $T_c$  superconductors [9, 10], etc.

One important consequence of the detailed analysis of the set  $\Omega_{\mathfrak{m}}^{\sharp}$  of generalized equilibrium states is the fact that the thermodynamics of models  $\mathfrak{m} \in \mathcal{M}_1$  is governed by the following *two-person zero-sum game*: For any model  $\mathfrak{m}$ , we define a functional

$$f_{\mathfrak{m}}^{\text{ext}} : L_{-}^2 \times C(L_{-}^2, L_{+}^2) \rightarrow \mathbb{R}.$$

Here,  $L_{\pm}^2$  are two orthogonal sub-spaces of a Hilbert space  $L^2(\mathcal{A}, \mathbb{C})$  and  $C(L_{-}^2, L_{+}^2)$  is the set of continuous maps from  $L_{-}^2$  to  $L_{+}^2$ . The set  $L_{-}^2$  is seen as the set of strategies of the “attractive” player with loss function  $f_{\mathfrak{m}}^{\text{ext}}$  and  $C(L_{-}^2, L_{+}^2)$  is the set of strategies of the “repulsive” player with loss function  $-f_{\mathfrak{m}}^{\text{ext}}$ . This game has a non-cooperative equilibrium and the value of the game is precisely  $-P_{\mathfrak{m}}^{\sharp}$ . Moreover, for any  $\mathfrak{m} \in \mathcal{M}_1$ , equilibria of this game classify extreme generalized equilibrium states in  $\Omega_{\mathfrak{m}}^{\sharp}$  in the following sense: There is a set  $\{\mathfrak{e}_a\}_{a \in \mathcal{A}}$  of observables such that, for any extreme state  $\hat{\omega} \in \Omega_{\mathfrak{m}}^{\sharp}$ , there is a non-cooperative equilibrium

$$(d_{a,-}, r_{+}) \in L_{-}^2 \times C(L_{-}^2, L_{+}^2)$$

with

$$\hat{\omega}(\epsilon_a) = d_{a,-} + r_+(d_{a,-}) \in L^2(\mathcal{A}, \mathbb{C}).$$

Conversely, for each non-cooperative equilibrium

$$(d_{a,-}, r_+) \in L_-^2 \times C(L_-^2, L_+^2),$$

there is a – not necessarily extreme –  $\omega \in \Omega_{\mathfrak{m}}^{\sharp}$  satisfying the above equation.

To conclude, this paper is organized as follows. In Section 2, we briefly explain the mathematical framework of Fermi systems on a lattice. Then the main results concerning the thermodynamic study of any  $\mathfrak{m} \in \mathcal{M}_1$  are formulated in Section 3. A discussion on previous results related to the ones presented here is given in Section 4. In order to keep the main issues as transparent as possible, we reduce the technical aspects to a minimum in Sections 2–4. Section 5 is an account on periodic boundary conditions which ensure the weak\*-convergence as  $l \rightarrow \infty$  of the Gibbs equilibrium states  $\rho_l$  to a generalized equilibrium state  $\omega \in \Omega_{\mathfrak{m}}^{\sharp}$ . In Section 6 we describe in details the set of periodic states. Up to Sections 6.3 and 6.6, this analysis is only an adaptation of known results for quantum spin systems. Section 7 explains permutation invariant models in relation with Størmer theorem [9, 12] for permutation invariant states on the CAR algebra because they are technically important for the derivation of the variational problem (1.1) for the pressure. Sections 8–11 give the detailed proofs of the main theorems about the game theoretical issues and generalized equilibrium states of long range models. In particular, we analyze in details in Sections 10–11 the relation between the thermodynamics of general long range models  $\mathfrak{m} \in \mathcal{M}_1$  and effective local interactions  $\Phi_{\omega}$ . This is related to the so-called approximating Hamiltonian method used on the level of the pressure in [13, 14, 15, 16]. We give in Section 12 a short review on this subject as well as on Gibbs equilibrium states, compact convex sets, Choquet simplices, tangent functionals, the  $\Gamma$ -regularization, the Legendre–Fenchel transform, and on two-person zero-sum games. All the material in Section 12, up to Lemma 12.32 and Theorems 12.37–12.38, can be found in standard textbooks. These topics are concisely discussed here to make our results accessible to a wide audience, since various fields of theoretical physics and mathematics are concerned (non-linear analysis, game theory, convex analysis, and statistical mechanics).

## 2. Fermi Systems on Lattices

### 2.1. Local fermion algebras

Let  $\mathcal{L} := \mathbb{Z}^d$  be the  $d$ -dimensional cubic lattice and  $\mathcal{H}$  be a finite dimensional Hilbert space with orthonormal basis  $\{e_s\}_{s \in \mathcal{S}}$ .

NOTATION 2.1. Here, we use the convention that  $\mathfrak{L}$  stands for  $\mathbb{Z}^d$  as seen as a set (lattice), whereas with  $\mathbb{Z}^d$  the abelian group  $(\mathbb{Z}^d, +)$  is meant.

REMARK 2.2 (Lattices  $\mathfrak{L}$ ).

The lattice  $\mathfrak{L}$  is taken to be a cubic one because it is technically easier, but this choice is not necessary for our proofs.

For any set  $M$ , we define  $\mathcal{P}_f(M)$  to be the set of all finite subsets of  $M$ . In the special case where  $M = \mathfrak{L}$  we use below the sequence of cubic boxes

$$(2.2) \quad \Lambda_l := \{x \in \mathfrak{L} : |x_i| \leq l, i = 1, \dots, d\} \in \mathcal{P}_f(\mathfrak{L})$$

of the lattice  $\mathfrak{L}$  with volume  $|\Lambda_l| = (2l + 1)^d$  for  $l \in \mathbb{N}$ .

REMARK 2.3 (Van Hove nets).

The sequence  $\{\Lambda_l\}_{l \in \mathbb{N}}$  is used to define the thermodynamic limit. It is a technically convenient choice, but it is not necessary in our proofs. The minimal requirement on any net  $\{\Lambda_i\}_{i \in I}$  of finite boxes is that the volume  $|\partial\Lambda_i|$  of the boundaries<sup>6</sup>  $\partial\Lambda_i \subset \Lambda_i \in \mathcal{P}_f(\mathfrak{L})$  must be negligible with respect to (w.r.t.) the volume  $|\Lambda_i|$  of  $\Lambda_i$  at “large”  $i \in I$ , i.e.,  $\lim_I \{|\partial\Lambda_i|/|\Lambda_i|\} = 0$ . Such families  $\{\Lambda_i\}_{i \in I}$  of subsets are known as Van Hove nets, see, e.g., [8].

For any  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , let  $\mathcal{U}_\Lambda$  be the complex Clifford algebra with identity  $\mathbf{1}$  and generators  $\{a_{x,s}, a_{x,s}^+\}_{x \in \Lambda, s \in S}$  satisfying the so-called canonical anti-commutation relations (CAR):

$$(2.3) \quad \begin{cases} a_{x,s} a_{x',s'} + a_{x',s'} a_{x,s} = 0, \\ a_{x,s}^+ a_{x',s'}^+ + a_{x',s'}^+ a_{x,s}^+ = 0, \\ a_{x,s} a_{x',s'}^+ + a_{x',s'}^+ a_{x,s} = \delta_{x,x'} \delta_{s,s'} \mathbf{1}. \end{cases}$$

The set  $\mathcal{U}_\Lambda$  is a  $C^*$ -algebra because it is isomorphic to the algebra  $B(\wedge \mathcal{H}_\Lambda)$  of all bounded linear operators on the fermion Fock space  $\wedge \mathcal{H}_\Lambda$ , where

$$\mathcal{H}_\Lambda := \bigoplus_{x \in \Lambda} \mathcal{H}_x.$$

For any  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ ,  $\mathcal{U}_\Lambda$  is called the *local fermion (field) algebras* of the lattice  $\mathfrak{L}$ . Indeed, in quantum statistical mechanics  $a_{x,s}^+ = (a_{x,s})^*$  and  $a_{x,s}$  are interpreted, respectively, as the creation and annihilation of a fermion with spin  $s \in S$  at the position  $x \in \mathfrak{L}$  of the lattice, and the CAR (2.3) implement the Pauli principle.

For any  $\Lambda \subset \Lambda' \subset \Lambda'' \in \mathcal{P}_f(\mathfrak{L})$ , there are canonical inclusions  $j_{\Lambda, \Lambda'} : \mathcal{U}_\Lambda \rightarrow \mathcal{U}_{\Lambda'}$  satisfying  $j_{\Lambda', \Lambda''} \circ j_{\Lambda, \Lambda'} = j_{\Lambda, \Lambda''}$  and  $j_{\Lambda, \Lambda'}(a_{x,s}) = a_{x,s}$  for any  $x \in \Lambda$  and

<sup>6</sup>By fixing  $m \geq 1$ , the boundary  $\partial\Lambda$  of any  $\Lambda \subset \Gamma$  is defined by  $\partial\Lambda := \{x \in \Lambda : \exists y \in \Gamma \setminus \Lambda \text{ with } d(x, y) \leq m\}$ , see (2.14) below for the definition of the metric  $d(x, y)$ .



$s \in S$ . The inductive limit of local algebras  $\{\mathcal{U}_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathcal{L})}$  is the  $C^*$ -algebra  $\mathcal{U}$ , called the *fermion (field) algebra* (also known as the *CAR algebra*). A dense subset of  $\mathcal{U}$  is given by the  $*$ -algebra

$$\mathcal{U}_0 := \bigcup_{\Lambda \in \mathcal{P}_f(\mathcal{L})} \mathcal{U}_\Lambda$$

of local elements, which implies the separability of  $\mathcal{U}$  as  $\mathcal{U}_\Lambda$  is a finite dimensional space for any  $\Lambda \in \mathcal{P}_f(\mathcal{L})$ .

REMARK 2.4 (Quantum spin systems).

*For quantum spin systems,  $\mathcal{U}$  would be the infinite tensor product of finite dimensional  $C^*$ -algebras attached to each site  $x \in \mathbb{Z}^d$ . All results of this papers hold in this case, but we concentrate our attention on fermion algebras as they are more difficult to handle because of the non-commutativity of its elements on different lattice sites.*

For any fixed  $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ , the condition

$$(2.4) \quad \sigma_\theta(a_{x,s}) = e^{-i\theta} a_{x,s}$$

defines a unique automorphism  $\sigma_\theta$  of the algebra  $\mathcal{U}$ . A special role is played by  $\sigma_\pi$ . Elements  $A, B \in \mathcal{U}$  satisfying  $\sigma_\pi(A) = A$  and  $\sigma_\pi(B) = -B$  are respectively called *even* and *odd*, whereas elements  $A \in \mathcal{U}$  satisfying  $\sigma_\theta(A) = A$  for any  $\theta \in [0, 2\pi)$  are called *gauge invariant*. The set

$$(2.5) \quad \mathcal{U}^+ := \{A \in \mathcal{U} : A - \sigma_\pi(A) = 0\} \subset \mathcal{U}$$

of all even elements and the set

$$(2.6) \quad \mathcal{U}^\circ := \bigcap_{\theta \in \mathbb{R}/(2\pi\mathbb{Z})} \{A \in \mathcal{U} : A = \sigma_\theta(A)\} \subset \mathcal{U}^+$$

of all gauge invariant elements are  $*$ -algebras. By continuity of  $\sigma_\theta$ , it follows that  $\mathcal{U}^+$  and  $\mathcal{U}^\circ$  are closed and hence  $C^*$ -algebras, respectively called *sub-algebra of even elements* and *fermion observable algebra*.

REMARK 2.5 (Gauge invariant projection).

*By density of the  $*$ -algebra  $\mathcal{U}_0$  of local elements, for any  $A \in \mathcal{U}$ , the map  $\theta \mapsto \sigma_\theta(A)$  is continuous. Thus, for any  $A \in \mathcal{U}$ , the Riemann integral*

$$\sigma^\circ(A) := \frac{1}{2\pi} \int_0^{2\pi} \sigma_\theta(A) \, d\theta$$

*defines a linear map  $\sigma^\circ : \mathcal{U} \rightarrow \mathcal{U}^\circ$ , which is a projection on the fermion observable algebra  $\mathcal{U}^\circ$ , i.e.,  $\sigma^\circ \circ \sigma^\circ = \sigma^\circ$ .*

NOTATION 2.6 (Gauge invariant objects).

*Any symbol with a circle  $\circ$  as an exponent (as for instance  $\sigma^\circ$ ) is, by definition, an object related to gauge invariance.*

## 2.2. States of Fermi systems on lattices

As  $\mathcal{U}$  is a Banach space, by Corollary 12.9, its dual  $\mathcal{U}^*$  is a locally convex real space<sup>7</sup> with respect to (w.r.t.) the weak\*-topology, which is Hausdorff. Moreover, as  $\mathcal{U}$  is separable, by Theorem 12.10, the weak\*-topology is metrizable on any weak\*-compact subset of  $\mathcal{U}^*$  as, for instance, on the weak\*-compact convex set  $E \subset \mathcal{U}^*$  of all *states* on  $\mathcal{U}$ .

States are linear functionals  $\rho \in \mathcal{U}^*$  which are positive, i.e., for all  $A \in \mathcal{U}$ ,  $\rho(A^*A) \geq 0$ , and normalized, i.e.,  $\rho(\mathbf{1}) = 1$ . Equivalently,  $\rho \in \mathcal{U}^*$  is a state iff  $\rho(\mathbf{1}) = 1$  and  $\|\rho\| = 1$  which clearly means that  $E$  is a subset of the unit ball of  $\mathcal{U}^*$ . Note that any  $\rho \in E$  is continuous and Hermitian, i.e., for all  $A \in \mathcal{U}$ ,  $\rho(A^*) = \overline{\rho(A)}$ , and defines by restriction a state on the sub-algebras  $\mathcal{U}^+$ ,  $\mathcal{U}^\circ$ , and  $\mathcal{U}_\Lambda$ . For any  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , we use  $\rho_\Lambda$  and  $E_\Lambda$  to denote, respectively, the restriction of any  $\rho \in E$  on the local sub-algebra  $\mathcal{U}_\Lambda$  and the set of all states  $\rho_\Lambda$  on  $\mathcal{U}_\Lambda$ .

NOTATION 2.7 (States).

The letters  $\rho$ ,  $\varrho$ , and  $\omega$  are exclusively reserved to denote states.

Invariant states under the action of groups  $G$  play a crucial role in the sequel. In the special case where  $G = (\mathbb{Z}^d, +)$ , the condition

$$(2.7) \quad \alpha_x(a_{y,s}) = a_{y+x,s}, \quad \forall y \in \mathbb{Z}^d, \forall s \in S,$$

defines a homomorphism  $x \mapsto \alpha_x$  from  $\mathbb{Z}^d$  to the group of \*-automorphisms of  $\mathcal{U}$ . In other words, the family of \*-automorphisms  $\{\alpha_x\}_{x \in \mathfrak{L}}$  represents here the action of the group of lattice translations on  $\mathcal{U}$ . Consider now the sub-groups  $G = (\vec{\ell}\mathbb{Z}^d, +) \subset (\mathbb{Z}^d, +)$  with

$$\vec{\ell}\mathbb{Z}^d := \ell_1\mathbb{Z} \times \dots \times \ell_d\mathbb{Z}$$

for  $\vec{\ell} \in \mathbb{N}^d$ . Any state  $\rho \in E$  satisfying  $\rho \circ \alpha_x = \rho$  for all  $x \in \vec{\ell}\mathbb{Z}^d$  is called  $\vec{\ell}\mathbb{Z}^d$ -invariant on  $\mathcal{U}$  or  $\vec{\ell}$ -periodic. The set of all  $\vec{\ell}\mathbb{Z}^d$ -invariant states is denoted by

$$(2.8) \quad E_{\vec{\ell}} := \bigcap_{x \in \vec{\ell}\mathbb{Z}^d, A \in \mathcal{U}} \{\rho \in \mathcal{U}^* : \rho(\mathbf{1}) = 1, \rho(A^*A) \geq 0 \text{ with } \rho = \rho \circ \alpha_x\}.$$

Note that  $E_{\vec{1}} := E_{(1, \dots, 1)}$  corresponds to the set of all *translation invariant* (t.i.) states. The  $\vec{\ell}$ -periodicity of states yields a crucial property (see Corollary 6.3):

LEMMA 2.8 ( $\vec{\ell}$ -periodic states are even).

Any  $\vec{\ell}\mathbb{Z}^d$ -invariant state  $\rho$  is even, i.e.,  $\rho = \rho \circ \sigma_\pi$  with the automorphism  $\sigma_\pi$  defined by (2.4) for  $\theta = \pi$ .

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<sup>7</sup>We use here Rudin's definition, see Definition 12.7.

In other words, all  $\vec{\ell}.\mathbb{Z}^d$ -invariant states  $\rho \in E_{\vec{\ell}}$  must be the zero functional on the sub-space of odd elements of  $\mathcal{U}$ . This symmetry property is a necessary ingredient to study thermodynamics of Fermi systems.

The set  $E_{\vec{\ell}}$  is clearly convex and weak\*-compact. So, Krein–Milman theorem (Theorem 12.11) tells us that it is the weak\*-closure of the convex hull of the (non-empty) set  $\mathcal{E}_{\vec{\ell}}$  of its extreme points. (Here,  $\mathcal{E}_1 := \mathcal{E}_{(1, \dots, 1)}$  is the set of extreme points of the set  $E_1$  of t.i. states.) Since  $E_{\vec{\ell}}$  is also metrizable (Theorem 12.10), from Choquet theorem (Theorem 12.18), each state  $\rho \in E_{\vec{\ell}}$  has a decomposition in terms of extreme states  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  of  $E_{\vec{\ell}}$ . This decomposition is unique and norm preserving by Lemma 6.4.

**THEOREM 2.9** (Ergodic decomposition of states in  $E_{\vec{\ell}}$ ).

*For any  $\rho \in E_{\vec{\ell}}$ , there is a unique probability measure  $\mu_\rho$  on  $E_{\vec{\ell}}$  such that*

$$\mu_\rho(\mathcal{E}_{\vec{\ell}}) = 1 \quad \text{and} \quad \rho = \int_{E_{\vec{\ell}}} d\mu_\rho(\hat{\rho}) \hat{\rho}.$$

*Furthermore, the map  $\rho \mapsto \mu_\rho$  is an isometry in the norm of linear functionals, i.e.,  $\|\rho - \rho'\| = \|\mu_\rho - \mu_{\rho'}\|$  for any  $\rho, \rho' \in E_1$ .*

**REMARK 2.10** (Barycenters).

*The integral written in Theorem 2.9 only means here that  $\rho \in E_{\vec{\ell}}$  is the (unique) barycenter of the probability measure, i.e., the normalized positive Borel regular measure,  $\mu_\rho \in M_1^+(E_{\vec{\ell}})$  on  $E_{\vec{\ell}}$ , see Definition 12.15 and Theorem 12.16.*

**NOTATION 2.11** (Extreme states).

*Extreme points of  $E_{\vec{\ell}}$  are written as  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  or sometime  $\hat{\omega} \in \mathcal{E}_{\vec{\ell}}$ .*

The uniqueness of the probability measure  $\mu_\rho$  given in Theorem 2.9 implies, by Theorem 12.22, that  $E_{\vec{\ell}}$  is a (Choquet) *simplex* (see Definition 12.21), which is in fact a consequence of Lemma 2.8 together with the asymptotic abelianess (6.12) of the even sub-algebra  $\mathcal{U}^+$  (2.5), see [17, Corollary 4.3.11.]. Observe also that the simplex  $E_{\vec{\ell}}$  has a fairly complicated geometrical structure: For any  $\vec{\ell} \in \mathbb{N}^d$ ,  $\mathcal{E}_{\vec{\ell}}$  is a weak\*-dense  $G_\delta$  subset in  $E_{\vec{\ell}}$ , see Corollary 6.6. In fact, up to an affine homeomorphism the set  $E_{\vec{\ell}}$  is the *Poulsen simplex*, see Theorem 12.26:

**THEOREM 2.12** ( $E_{\vec{\ell}}$  and the Poulsen simplex).

*The Choquet simplices  $\{E_{\vec{\ell}}\}_{\vec{\ell} \in \mathbb{N}^d}$  are all affinely homeomorphic to the Poulsen simplex, i.e.,  $E_{\vec{\ell}}$  is unique up to an affine homeomorphism.*

Note that the simplex  $E_{\vec{\ell}}$  can also be seen as a simplexoid, i.e., a compact convex set in which all closed proper faces are simplices. An example of a closed face of  $E_{\vec{\ell}}$ , for any  $\vec{\ell} \in \mathbb{N}^d$ , is given by the Bauer simplex  $E_\Pi \subset E_{\vec{\ell}}$  of permutation invariant states described in Section 7.1.

REMARK 2.13 (Gauge invariant t.i. states).

An important subset of  $E_1$  is the convex and weak\*-compact set

$$E_1^\circ := \{\rho \in E_1 : \rho = \rho \circ \sigma^\circ\} \subset E_1$$

of translation and gauge invariant states, cf. Remark 2.5. States describing physical systems generally belong to  $E_1^\circ$  which is again the Poulsen simplex (up to an affine homeomorphism). This can be proven by identifying  $E_1^\circ$  with the set of all t.i. states on  $\mathcal{U}^\circ$  (2.6) which is an asymptotically abelian  $C^*$ -algebra.

The result of Theorem 2.12 is standard in statistical mechanics, in particular for lattice quantum spin systems [17, p. 405–406, 464]. It means that the complicated geometrical structure of the simplices  $E_{\vec{\ell}}$  is, in a sense, universal and in fact, physically natural. Indeed, the set  $\mathcal{E}_{\vec{\ell}}$  of extreme points of  $E_{\vec{\ell}}$  can be characterized through a (physically natural) condition related to space-averaging as follows.

For any  $A \in \mathcal{U}$ ,  $L \in \mathbb{N}$  and  $\vec{\ell} \in \mathbb{N}^d$ , let  $A_{L,\vec{\ell}} \in \mathcal{U}$  be defined by the space-average

$$(2.9) \quad A_{L,\vec{\ell}} := \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{x \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} \alpha_x(A).$$

By definition,  $A_L := A_{L,\vec{\ell}}$  for  $\vec{\ell} = (1, \dots, 1)$ . This sequence  $\{A_{L,\vec{\ell}}\}_{L \in \mathbb{N}}$  of operators in  $\mathcal{U}$  defines space-averaging functionals:

DEFINITION 2.14 (Space-averaging functionals).

For any  $A \in \mathcal{U}$  and  $\vec{\ell} \in \mathbb{N}^d$ , the ( $\vec{\ell}$ -) space-averaging functional is the map

$$\rho \mapsto \Delta_{A,\vec{\ell}}(\rho) := \lim_{L \rightarrow \infty} \rho(A_{L,\vec{\ell}}^* A_{L,\vec{\ell}})$$

from  $E_{\vec{\ell}}$  to  $\mathbb{R}$ . Here,  $\Delta_A := \Delta_{A,(1,\dots,1)}$ .

The functional  $\Delta_{A,\vec{\ell}}$  is well-defined, for all  $A \in \mathcal{U}$  and  $\vec{\ell} \in \mathbb{N}^d$ , and we give in Section 2.3 a complete description of  $\Delta_A$ . This map is pivotal as it is used to define *ergodic* states in the following way:

DEFINITION 2.15 (Ergodic states).

A  $\vec{\ell}$ -periodic state  $\hat{\rho} \in E_{\vec{\ell}}$  is ( $\vec{\ell}$ -) ergodic iff, for all  $A \in \mathcal{U}$ ,  $\Delta_{A,\vec{\ell}}(\hat{\rho}) = |\hat{\rho}(A)|^2$ .

The equality in this definition says that space fluctuations of measures on a system described by a  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant state  $\hat{\rho}$  are small when it is ergodic: For any observable  $A$ , we are able to determine  $\hat{\rho}(A)$  through space-averaging over the sub-lattice  $\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d$  at large  $L$ . We can view this result as a non-commutative version of law of large numbers.

The unique decomposition expressed in Theorem 2.9 of any  $\rho \in E_{\vec{\ell}}$  in terms of extreme states  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  of  $E_{\vec{\ell}}$  is also called the *ergodic decomposition*.

Indeed, we prove in Section 6.2 that any ergodic state is an extreme state in  $E_{\vec{\ell}}$  and conversely, see Lemmata 6.5 and 6.8 together with Corollary 6.9.

**THEOREM 2.16** (Extremality = Ergodicity).

Any extreme state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  of  $E_{\vec{\ell}}$  is ergodic and conversely. Additionally, any extreme state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  is strongly clustering, i.e., for all  $A, B \in \mathcal{U}$ ,

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{y \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} \hat{\rho}(\alpha_x(A) \alpha_y(B)) = \hat{\rho}(A) \hat{\rho}(B)$$

uniformly in  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ .

Observe that a strongly clustering state  $\rho \in E_{\vec{\ell}}$  is not necessarily *strongly mixing* which means that

$$(2.10) \quad \lim_{|x| \rightarrow \infty} \rho(A \alpha_x(B)) = \rho(A) \rho(B)$$

for all  $A, B \in \mathcal{U}$ . The converse is trivial: Any strongly mixing state satisfies the ergodicity property.

**REMARK 2.17** (Gauge invariant states and ergodicity).

From Remark 2.13, a state  $\hat{\rho} \in E_1^\circ$  is extreme in  $E_1^\circ$  iff  $\hat{\rho} \in E_1^\circ$  is ergodic w.r.t. the sub-algebra  $\mathcal{U}^\circ \subset \mathcal{U}$ , i.e., for all  $A \in \mathcal{U}^\circ$ ,

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|^2} \sum_{x, y \in \Lambda_L} \hat{\rho}(\alpha_x(A^*) \alpha_y(A)) = |\hat{\rho}(A)|^2.$$

### 2.3. The space-averaging functional $\Delta_A$

The set of translation invariant (t.i.) states  $E_1 := E_{(1, \dots, 1)}$  and the space-averaging functional  $\Delta_A := \Delta_{A, (1, \dots, 1)}$  play a central role below as we concentrate our attention on the thermodynamics of translation invariant (t.i.) Fermi systems. However, our analysis can easily be generalized to the  $(\vec{\ell})$  space-averaging functional  $\Delta_{A, \vec{\ell}}$  for any  $\vec{\ell} \in \mathbb{N}^d$ , see Definition 2.14.

First, by Lemma 6.10, the space-averaging functional  $\Delta_A$  is well-defined for all  $\vec{\ell}$ -periodic states  $\rho \in E_{\vec{\ell}}$  at any  $\vec{\ell} \in \mathbb{N}^d$ . In this case,

$$(2.11) \quad \rho \mapsto \Delta_A(\rho) := \lim_{L \rightarrow \infty} \rho(A_L^* A_L) \in [|\rho(A_{\vec{\ell}})|^2, \|A\|^2],$$

with

$$(2.12) \quad A_{\vec{\ell}} := \frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i-1\}} \alpha_x(A)$$

for any  $\vec{\ell} \in \mathbb{N}^d$ .

As explained in the previous section, extremality of t.i. states can be characterized by means of the space-averaging functional  $\Delta_A$ . Indeed, the set

of t.i. states  $\rho \in E_1$  which fulfil  $\Delta_A(\rho) = |\rho(A)|^2$  for any  $A \in \mathcal{U}$ , i.e., the set of ergodic states (Definition 2.15), is the set  $\mathcal{E}_1$  of extreme states of  $E_1$ , see Theorem 2.16. Nevertheless, this functional has never gained much attention before beyond the fact that it can be used to characterize extremality of states. It turns out that other properties of the space-averaging functional are also crucial in the analysis of thermodynamic effects of long range interactions. Its basic properties – proven in Lemmata 6.11 and 6.12 – are listed in the following theorem:

**THEOREM 2.18** (Properties of the functional  $\Delta_A$  on  $E_{\vec{\ell}}$ ).

- (i) *At fixed  $A \in \mathcal{U}$ , the map  $\rho \mapsto \Delta_A(\rho)$  from  $E_{\vec{\ell}}$  to  $\mathbb{R}_0^+$  is a weak\*-upper semi-continuous affine functional. It is also t.i., i.e., for all  $x \in \mathbb{Z}^d$  and  $\rho \in E_{\vec{\ell}}$ ,  $\Delta_A(\rho \circ \alpha_x) = \Delta_A(\rho)$ .*
- (ii) *At fixed  $\rho \in E_{\vec{\ell}}$  and for all  $A, B \in \mathcal{U}$ ,*

$$|\Delta_A(\rho) - \Delta_B(\rho)| \leq (\|A\| + \|B\|)\|A - B\|,$$

*i.e., the map  $A \mapsto \Delta_A(\rho)$  from  $\mathcal{U}$  to  $\mathbb{R}_0^+$  is locally Lipschitz continuous.*

The affinity and the translation invariance of  $\Delta_A$ , as well as (ii), are immediate consequences of its definition (see Lemmata 6.11 and 6.12). Its weak\*-upper semi-continuity follows from the fact that  $\Delta_A$  is the infimum of a family of weak\*-continuous functionals (see Lemmata 6.10 and 6.11).

Note that  $\Delta_A$  is not weak\*-continuous for all  $A \in \mathcal{U}$ , even on the set  $E_1$ . Indeed, if  $\Delta_A$  is weak\*-continuous on  $E_1$  then  $\Delta_A(\rho) = |\rho(A)|^2$  for all  $\rho \in E_1$  because of Theorem 2.16 and the weak\*-density of the set  $\mathcal{E}_1$  in  $E_1$  (Corollary 6.6). Therefore, there is  $A \in \mathcal{U}$  such that  $\Delta_A$  is not weak\*-upper semi-continuous. Otherwise, any state  $\rho \in E_1$  would be ergodic and hence, an extreme point of  $E_1$  by Theorem 2.16. A more detailed study on the weak\*-continuity of the space-averaging functional  $\Delta_A$  on the set  $E_1$  of t.i. states is given by the following theorem:

**THEOREM 2.19** (Properties of the map  $\rho \mapsto \Delta_A(\rho)$  on  $E_1$  at fixed  $A \in \mathcal{U}$ ).

- (i)  *$\Delta_A$  is weak\*-continuous on  $E_1$  iff the affine map  $\rho \mapsto |\rho(A)|$  from  $E_1$  to  $\mathbb{C}$  is a constant map.*
- (ii)  *$\Delta_A$  is weak\*-discontinuous on a weak\*-dense subset of  $E_1$  unless  $\rho \mapsto |\rho(A)|$  is a constant map from  $E_1$  to  $\mathbb{C}$ .*
- (iii)  *$\Delta_A$  is continuous on the  $G_\delta$  weak\*-dense subset  $\mathcal{E}_1$  of extreme states of  $E_1$ . In particular, the set of all states of  $E_1$  where  $\Delta_A$  is weak\*-discontinuous is meager.*
- (iv)  *$\Delta_A$  can be decomposed in terms of an integral on the set  $\mathcal{E}_1$ , i.e., for all  $\rho \in E_1$ ,*

$$\Delta_A(\rho) = \int_{\mathcal{E}_1} d\mu_\rho(\hat{\rho}) |\hat{\rho}(A)|^2$$

with the probability measure  $\mu_\rho$  defined by Theorem 2.9.

(v) Its  $\Gamma$ -regularization  $\Gamma_{E_1}(\Delta_A)$  on  $E_1$  is the weak\*-continuous convex map  $\rho \mapsto |\rho(A)|^2$ .

Recall that the  $\Gamma$ -regularization of functionals is defined in Definition 12.27. For more details, we recommend Section 12.5 as well as Corollary 12.30 in Section 12.6.

The continuity properties (i)–(iii) result partially from Theorems 2.9 and 2.16, for more details see Proposition 6.13. The assertion (iv) is a direct consequence of Theorem 2.18 (i) and Lemma 12.17 combined with Theorems 2.9 and 2.16. The last statement (v) is deduced from the density of the set  $\mathcal{E}_1$  in  $E_1$  (Corollary 6.6) together with Theorems 2.16 and standard arguments from convex analysis, see Lemma 6.14.

REMARK 2.20 ( $\Delta_A$  and Jensen's inequality).

The inequality  $\Delta_A(\rho) \geq |\rho(A)|^2$  can directly be deduced from Theorem 2.19 (iv) and Jensen's inequality (Lemma 12.33) as  $\mu_\rho$  is a probability measure.

REMARK 2.21 (Trivial space-averaging functional  $\Delta_A$  on  $E_1$ ).

If the affine map  $\rho \mapsto |\rho(A)|$  from  $E_1$  to  $\mathbb{C}$  is a constant map then from Theorem 2.19 (iv),  $\Delta_A(\rho) = |\rho(A)|^2$  for any  $\rho \in E_1$ . An example of such trivial behavior is given by choosing  $A = \lambda 1 + B - \alpha_x(B)$  for any  $\lambda \in \mathbb{C}$ ,  $B \in \mathcal{U}$ , and  $x \in \mathbb{Z}^d$ . Recall that the translation  $\alpha_x$  is the \*-automorphism defined by (2.7).

## 2.4. Local interactions and internal energies

An *interaction* is defined via a family of even and self-adjoint local elements  $\Phi_\Lambda$  and it is associated with *internal energies* as follows:

DEFINITION 2.22 (Interactions and internal energies).

- (i) An *interaction* is a family  $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathcal{L})}$  of even and self-adjoint local elements  $\Phi_\Lambda = \Phi_\Lambda^* \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$  with  $\Phi_\emptyset = 0$ .
- (ii) For any  $\Lambda \in \mathcal{P}_f(\mathcal{L})$ , its *internal energy* is the local Hamiltonian

$$U_\Lambda^\Phi := \sum_{\Lambda' \in \mathcal{P}_f(\mathcal{L}), \Lambda' \subset \Lambda} \Phi_{\Lambda'} \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda.$$

NOTATION 2.23 (Interactions).

The letters  $\Phi$  and  $\Psi$  are exclusively reserved to denote interactions.

An interaction  $\Phi$  is per definition *translation invariant* (t.i.) iff, for all  $x \in \mathbb{Z}^d$  and  $\Lambda \in \mathcal{P}_f(\mathcal{L})$ ,  $\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda)$  with

$$(2.13) \quad \Lambda + x := \{x' + x \in \mathcal{L} : x' \in \Lambda\}.$$

Another important symmetry of Fermi models, which appears together with the translation invariance in most physically relevant situations, is the gauge symmetry (or the particle number conservation). An interaction  $\Phi$  is said to be *gauge invariant* (i.e.,  $\Phi$  conserves the particle number) iff  $\Phi_\Lambda \in \mathcal{U}^\circ$  for all  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , see (2.6).

Observe now that an interaction  $\Phi$  may have finite range. This property is defined via the usual metric  $d : \mathfrak{L} \times \mathfrak{L} \rightarrow [0, \infty)$  defined by

$$(2.14) \quad d(x, x') := \sqrt{|x_1 - x'_1|^2 + \cdots + |x_d - x'_d|^2}$$

on the lattice  $\mathfrak{L} := \mathbb{Z}^d$  together with the function

$$(2.15) \quad \phi(\Lambda) := \max_{x, x' \in \Lambda} \{d(x, x')\} \quad \text{for any } \Lambda \in \mathcal{P}_f(\mathfrak{L}).$$

Indeed, we say that the interaction  $\Phi$  has *finite range* iff there is some  $R < \infty$  such that  $\phi(\Lambda) > R$  implies  $\Phi_\Lambda = 0$ .

The set of all interactions can be endowed with a real vector space structure:

$$(\lambda_1 \Phi + \lambda_2 \Psi)_\Lambda := \lambda_1 \Phi_\Lambda + \lambda_2 \Psi_\Lambda$$

for any interactions  $\Phi, \Psi$ , and any real numbers  $\lambda_1, \lambda_2$ . So, we can define a Banach space  $\mathcal{W}_1$  of t.i. interactions by using a specific norm:

DEFINITION 2.24 (Banach space  $\mathcal{W}_1$  of t.i. interactions).

*The real Banach space  $\mathcal{W}_1$  is the set of all t.i. interactions  $\Phi$  with finite norm*

$$\|\Phi\|_{\mathcal{W}_1} := \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}), 0 \in \Lambda} |\Lambda|^{-1} \|\Phi_\Lambda\| < \infty.$$

The norm  $\|\cdot\|_{\mathcal{W}_1}$  plays here an important role because its finiteness implies, among other things, the existence of the pressure in the thermodynamic limit. The set  $\mathcal{W}_1^f$  of all finite range t.i. interactions is dense in  $\mathcal{W}_1$ . In particular, the set  $\mathcal{W}_1$  is a separable Banach space because, for all  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , local algebras  $\mathcal{U}_\Lambda$  are finite dimensional.

By Corollary 12.9, its dual  $\mathcal{W}_1^*$  is a locally convex real space<sup>8</sup> w.r.t. the weak\*-topology. The weak\*-topology is Hausdorff and, by Theorem 12.10, it is metrizable on any weak\*-compact subset of  $\mathcal{W}_1^*$  as, for instance, on the weak\*-compact convex set  $E_1$  seen as a subset of  $\mathcal{W}_1^*$ , see Section 6.5 for more details.

REMARK 2.25 (Generalizations of the norm  $\|\cdot\|_{\mathcal{W}_1}$ ).

*The norm in Definition 2.24 is only a specific example of the general class of*

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<sup>8</sup>We use here Rudin's definition, see Definition 12.7.



norms for t.i. interactions:

$$\|\Phi\|_{\varkappa} := \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}), 0 \in \Lambda} \varkappa(|\Lambda|, \phi(\Lambda)) \|\Phi_{\Lambda}\| \quad \text{with } \varkappa(x, y) > 0.$$

REMARK 2.26 (Banach space of standard potentials).

In [8, Definition 5.10] the authors use another kind of norm for t.i. interactions. Their norm is not equivalent to  $\|\cdot\|_{\mathcal{W}_1}$  and also defines a Banach space of the so-called translation covariant potentials, see [8, Proposition 8.8.]. In fact, in contrast to the potentials of [8, Section 5.5] we cannot associate a symmetric derivation<sup>9</sup>, as it is done in [8, Theorem 5.7], to all t.i. interactions of  $\mathcal{W}_1$ . But no dynamical questions – as, for instance, the existence and characterization of KMS-states done in [8] – are addressed in the present paper. That is why we can use here weaker norms leading to more general classes of t.i. local interactions than in [8].

## 2.5. Energy and entropy densities

As far as the thermodynamics of Fermi systems is concerned, there are two other important functionals associated with any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  on  $\mathcal{U}$ : The *entropy density* functional  $\rho \mapsto s(\rho)$  and the *energy density* functional  $\rho \mapsto e_{\Phi}(\rho)$  w.r.t. to a local t.i. interaction  $\Phi \in \mathcal{W}_1$ . We start with the entropy density functional which is defined as follows:

DEFINITION 2.27 (Entropy density functional  $s$ ).

The entropy density functional  $s : E_{\vec{\ell}} \rightarrow \mathbb{R}_0^+$  is defined by

$$s(\rho) := - \lim_{l \rightarrow \infty} \left\{ \frac{1}{|\Lambda_L|} \text{Trace} \left( d_{\rho_{\Lambda_L}} \ln d_{\rho_{\Lambda_L}} \right) \right\},$$

where  $\rho_{\Lambda_L}$  is the restriction of any  $\rho \in E_{\vec{\ell}}$  on the sub-algebra  $\mathcal{U}_{\Lambda_L}$  and  $d_{\rho_{\Lambda_L}} \in \mathcal{U}_{\Lambda_L}$  is the (uniquely defined) density matrix representing the state  $\rho_{\Lambda_L}$  as a trace:

$$\rho_{\Lambda_L}(\cdot) = \text{Trace} \left( \cdot d_{\rho_{\Lambda_L}} \right).$$

The entropy density is therefore given as the so-called von Neumann entropy per unit volume in the thermodynamic limit, cf. Section 6.4. The functional  $s$  is well-defined on the set  $E_{\vec{\ell}}$  of  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant states and it has the following properties:

LEMMA 2.28 (Properties of the entropy density functional  $s$ ).

(i) The map  $\rho \mapsto s(\rho)$  from  $E_{\vec{\ell}}$  to  $\mathbb{R}_0^+$  is a weak\*-upper semi-continuous affine

<sup>9</sup>A symmetric derivation  $A \mapsto \delta(A)$  is a linear map satisfying  $\delta(AB) = \delta(A)B + A\delta(B)$  and  $\delta(A^*) = \delta(A)^*$  for any  $A, B \in \mathcal{D}_{\delta}$  with its domain  $\mathcal{D}_{\delta}$  being a dense \*-sub-algebra of  $\mathcal{U}$ .

functional. It is also t.i., i.e., for all  $x \in \mathbb{Z}^d$  and  $\rho \in E_{\vec{\ell}}$ ,  $s(\rho \circ \alpha_x) = s(\rho)$ .

(ii) For any t.i. state  $\rho \in E_1$ , there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty \subset \mathcal{E}_1$  of ergodic states converging in the weak\*-topology to  $\rho$  and such that

$$s(\rho) = \lim_{n \rightarrow \infty} s(\hat{\rho}_n).$$

(iii) The map  $\rho \mapsto s(\rho)$  from  $E_{\vec{\ell}}$  to  $\mathbb{R}_0^+$  is Lipschitz continuous in the norm topology of states: For any  $\rho, \varrho \in E_{\vec{\ell}}$ ,

$$|s(\rho) - s(\varrho)| \leq C_{|S|} \|\rho - \varrho\| \quad \text{with} \quad \|\rho\| := \sup_{A \in \mathcal{U}, A=A^*, \|A\|=1} |\rho(A)|.$$

Here,  $C_{|S|}$  is a finite constant depending on the size  $|S|$  of the spin set  $S$ .

The assertions (i) and (iii) are two standard results, see, e.g., [8, Theorem 10.3. and Corollary 10.5.]. The proof of (i) is shortly checked in Lemma 6.15 but we omit the proof of (iii) which is only used in Remark 2.29. However, the second one (ii) does not seem to have been observed before although it is not difficult to prove, see Lemma 6.16. This property turns out to be *crucial* because it allows us to go around the lack of weak\*-continuity of the entropy density functional  $s$ . The map  $\rho \mapsto s(\rho)$  is, indeed, not weak\*-continuous but only norm continuous as expressed by (iii), see, e.g., [18, 19]. Note that (ii) uses the fact that the set  $\mathcal{E}_1$  of extreme states is a dense subset of  $E_1$  as explained after Notation 2.11, see also Corollary 6.6.

REMARK 2.29 (Boundedness of the entropy density functional  $s$ ).

The third assertion (iii) of Lemma 2.28 is given for information as it is only used in the paper to see that  $s(\rho) \in [0, 2C_{|S|}]$  for all  $\rho \in E_1$  because there is  $\varrho \in E_1$  such that  $s(\varrho) = 0$  and  $\|\rho - \varrho\| \leq \|\rho\| + \|\varrho\| = 2$ . Similarly, for quantum spin systems (cf. Remark 2.4) the entropy density functional belongs to  $[0, D_{|S|}]$  with  $D_{|S|} < \infty$ . In particular, it is still bounded from below.

The energy density is the thermodynamic limit of the internal energy  $U_\Lambda^\Phi$  (Definition 2.22) per unit volume associated with any fixed local interaction  $\Phi \in \mathcal{W}_1$ :

DEFINITION 2.30 (Energy density functional  $e_\Phi$ ).

The energy density of any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  w.r.t. a t.i. local interaction  $\Phi \in \mathcal{W}_1$  is defined by

$$e_\Phi(\rho) := \lim_{L \rightarrow \infty} \frac{\rho(U_{\Lambda_L}^\Phi)}{|\Lambda_L|} < \infty.$$

The existence of the energy density  $e_\Phi(\rho)$  can easily be checked for all  $\Phi \in \mathcal{W}_1$ , see Lemma 6.17. Actually,  $e_\Phi(\rho) = \rho(\epsilon_{\Phi, \vec{\ell}})$  with

$$(2.16) \quad \epsilon_{\Phi, \vec{\ell}} := \frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i-1\}} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}), x \in \Lambda} \frac{\Phi_\Lambda}{|\Lambda|}$$

for any  $\Phi \in \mathcal{W}_1$ . Per definition,  $\mathbf{e}_\Phi := \mathbf{e}_{\Phi, (1,1,\dots,1)}$ . The operator  $\mathbf{e}_{\Phi, \vec{\ell}} \in \mathcal{U}^+$  is called the *energy observable* associated with the t.i. local interaction  $\Phi \in \mathcal{W}_1$  for the set  $E_{\vec{\ell}}$  of  $\vec{\ell}$ -periodic states. Remark that  $\mathbf{e}_{\Phi, \vec{\ell}} \in \mathcal{U}^+$  results from the fact that, for all  $\Lambda \in \mathcal{P}_f(\mathcal{L})$ ,  $\Phi_\Lambda \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$  and

$$(2.17) \quad \|\mathbf{e}_{\Phi, \vec{\ell}}\| \leq \|\Phi\|_{\mathcal{W}_1} < \infty.$$

Observe additionally that

$$\mathbf{e}_{\Phi, \vec{\ell}} = \frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i-1\}} \alpha_x(\mathbf{e}_\Phi).$$

It is then straightforward to prove the following properties of the energy density functional  $e_\Phi$  (see also [8, Theorem 9.5]):

LEMMA 2.31 (Properties of the energy density functional  $e_\Phi$ ).

- (i) For any  $\Phi \in \mathcal{W}_1$ , the map  $\rho \mapsto e_\Phi(\rho)$  from  $E_{\vec{\ell}}$  to  $\mathbb{R}$  is a weak\*-continuous affine functional. It is also t.i., i.e., for all  $x \in \mathbb{Z}^d$ ,  $e_\Phi(\rho \circ \alpha_x) = e_\Phi(\rho)$ .
- (ii) At fixed  $\rho \in E_{\vec{\ell}}$  and for all  $\Phi, \Psi \in \mathcal{W}_1$ ,

$$|e_\Phi(\rho) - e_\Psi(\rho)| \leq \|\Phi - \Psi\|_{\mathcal{W}_1},$$

i.e., the linear map  $\Phi \mapsto e_\Phi(\rho)$  from  $\mathcal{W}_1$  to  $\mathbb{R}$  is Lipschitz continuous.

Note that the entropy density functional  $s$  and the energy density functional  $e_\Phi$  define the so-called *free-energy density functional*  $f_\Phi$ :

DEFINITION 2.32 (Free-energy density functional  $f_\Phi$ ).

For  $\beta \in (0, \infty]$ , the free-energy density functional  $f_\Phi$  w.r.t. the t.i. interaction  $\Phi \in \mathcal{W}_1$  is the map

$$\rho \mapsto f_\Phi(\rho) := e_\Phi(\rho) - \beta^{-1}s(\rho)$$

from  $E_{\vec{\ell}}$  to  $\mathbb{R}$ .

From Lemmata 2.28 (i) and 2.31 (i), the functional  $f_\Phi$  is weak\*-lower semi-continuous, t.i., and affine. Moreover, by Lemma 2.28 (ii), for any  $\rho \in E_1$ , there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty \subset \mathcal{E}_1$  of ergodic states converging in the weak\*-topology to  $\rho$  and such that

$$(2.18) \quad f_\Phi(\rho) = \lim_{n \rightarrow \infty} f_\Phi(\hat{\rho}_n).$$

REMARK 2.33 (Temperature of Fermi systems).

All assertions in the sequel depend on the fixed positive parameter  $\beta > 0$  which is often omitted to simplify the notation. However, we keep it in all definitions because  $\beta \in (0, \infty]$  is interpreted in Physics as the inverse temperature of the system.  $\beta = \infty$  corresponds to the zero-temperature for which the (thermal) entropy density disappears. In fact, the free-energy density corresponds to the maximum energy which can be extracted from a thermodynamical system at fixed temperature  $\beta^{-1}$ .

### 3. Fermi Systems with Long Range Interactions

#### 3.1. Fermi systems with long range interactions

Let  $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$  be a *separable* measure space with  $\mathfrak{A}$  and  $\mathfrak{a} : \mathfrak{A} \rightarrow \mathbb{R}_0^+$  being respectively some  $\sigma$ -algebra on  $\mathcal{A}$  and some measure on  $\mathfrak{A}$ . The separability of  $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$  means, per definition, that the space  $L^2(\mathcal{A}, \mathbb{C}) := L^2(\mathcal{A}, \mathfrak{a}, \mathbb{C})$  of square integrable complex valued functions on  $\mathcal{A}$  is a separable Hilbert space. This property is assumed here because, by Theorem 12.10 together with Banach–Alaoglu theorem, it yields the metrizable of the weak topology on any norm bounded subset  $B \subset L^2(\mathcal{A}, \mathbb{C})$ , which is a useful property in the sequel.

Then, as  $\mathcal{W}_1$  is a Banach space (Definition 2.24), we can follow the construction done in Section 12.3 for  $\mathcal{X} = \mathcal{W}_1$  to define the space  $\mathcal{L}^2(\mathcal{A}, \mathcal{W}_1)$  of  $\mathcal{L}^2$ -interactions which in turn is used to define models with long interactions as follows:

DEFINITION 3.1 (Banach space  $\mathcal{M}_1$  of long range models).

The set of long range models is given by

$$\mathcal{M}_1 := \mathcal{W}_1 \times \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1) \times \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1)$$

and is equipped with the semi-norm

$$\|\mathfrak{m}\|_{\mathcal{M}_1} = \|\Phi\|_{\mathcal{W}_1} + \|\Phi_a\|_2 + \|\Phi'_a\|_2$$

for any  $\mathfrak{m} := (\Phi, \{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}) \in \mathcal{M}_1$ . We identify in  $\mathcal{M}_1$  models  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  whenever  $\|\mathfrak{m}_1 - \mathfrak{m}_2\|_{\mathcal{M}_1} = 0$ , i.e., whenever  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  belong to the same equivalence class of models. For convenience, we ignore the distinction between models and their equivalence classes and see  $\mathcal{M}_1$  as a Banach space of long range models with norm  $\|\cdot\|_{\mathcal{M}_1}$ .

NOTATION 3.2 (Models).

The symbol  $\mathfrak{m}$  is exclusively reserved to denote elements of  $\mathcal{M}_1$ .

An important sub-space of  $\mathcal{M}_1$  is the set  $\mathcal{M}_1^f$  of finite range models defined as follows:

$$\mathfrak{m} := (\Phi, \{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}) \in \mathcal{M}_1$$

has *finite range* iff  $\Phi$  is finite range and  $\{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}$  are finite range almost everywhere (a.e.). The sub-space  $\mathcal{M}_1^f$  of all finite range models is dense in  $\mathcal{M}_1$  because of Lebesgue's dominated convergence theorem and the density of set  $\mathcal{W}_1^f$  of all finite range t.i. interactions in  $\mathcal{W}_1$ . Another dense<sup>10</sup>

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<sup>10</sup>This follows from the density of step functions in  $\mathcal{L}^2(\mathcal{A}, \mathcal{W}_1)$ .

sub-space of  $\mathcal{M}_1$  is given by the set  $\mathcal{M}_1^{\text{d}}$  of *discrete* elements  $\mathfrak{m}$ , i.e., elements for which the set

$$\{\Phi_a : a \in \mathcal{A}\} \cup \{\Phi'_a : a \in \mathcal{A}\}$$

has a finite number of interactions. Therefore, the sub-space  $\mathcal{M}_1^{\text{df}} := \mathcal{M}_1^{\text{d}} \cap \mathcal{M}_1^{\text{f}}$  is also clearly dense in  $\mathcal{M}_1$ . It is an important dense sub-space used to prove Theorem 3.12 in Section 8.

Like t.i. local interactions  $\Phi \in \mathcal{W}_1$  (cf. Definition 2.22), any long range model  $\mathfrak{m} \in \mathcal{M}_1$  is associated with a family of internal energies as follows:

**DEFINITION 3.3** (Internal energy with long range interactions).

For any  $\mathfrak{m} \in \mathcal{M}_1$  and  $l \in \mathbb{N}$ , its internal energy in the box  $\Lambda_l$  is defined by

$$U_l := U_{\Lambda_l}^{\Phi} + \frac{1}{|\Lambda_l|} \int_{\mathcal{A}} \gamma_a (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})^* (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}) \, \mathfrak{d}\mathfrak{a}(a),$$

with  $\gamma_a \in \{-1, 1\}$  being a fixed measurable function.

The internal energy  $U_l$  is well-defined. Indeed, by continuity of the linear map  $\Phi \mapsto U_{\Lambda_l}^{\Phi}$ , for any  $\mathfrak{m} \in \mathcal{M}_1$ , the map  $a \mapsto \gamma_a U_{\Lambda_l}^{\Phi_a}$  from  $\mathcal{A}$  to  $\mathcal{U}_{\Lambda_l}$  belongs to  $\mathcal{L}^2(\mathcal{A}, \mathcal{U}_{\Lambda_l})$  (see Section 12.3 for the definition of the space  $\mathcal{L}^p(\mathcal{A}, \mathcal{U}_{\Lambda_l})$ ). Then, as  $\mathcal{U}$  is a  $C^*$ -algebra, the map

$$a \mapsto \gamma_a (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})^* (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})$$

belongs to the space  $\mathcal{L}^1(\mathcal{A}, \mathcal{U}_{\Lambda_l})$  and  $\mathfrak{m} \mapsto U_l$  is a well-defined functional from the Banach space  $\mathcal{M}_1$  to the  $C^*$ -algebra  $\mathcal{U}$ . By (8.1), this map is even continuous w.r.t. to the norms of  $\mathcal{M}_1$  and  $\mathcal{U}$ .

The long range character of Fermi models  $\mathfrak{m} \in \mathcal{M}_1$  with local internal energy  $U_l$  – as compared to the usual models defined from local interactions  $\Phi \in \mathcal{W}_1$  only – can be seen as follows. For each fixed  $\epsilon \in (0, 1)$ , we define the long range truncation of the internal energy  $U_{\Lambda_l}^{\Phi}$  (Definition 2.22) associated with the local part  $\Phi$  of  $\mathfrak{m}$  by

$$U_{l,\epsilon}^{\Phi} := \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}), \Lambda \subset \Lambda_l, \varphi(\Lambda) > \epsilon l} \Phi_{\Lambda},$$

where the function  $\varphi(\Lambda)$  is the diameter of  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , see (2.15). Analogously, the long range truncation of the internal energy  $(U_l - U_{\Lambda_l}^{\Phi})$  associated with the long rang part of  $\mathfrak{m}$  is per definition equal to

$$U_{l,\epsilon} := \frac{1}{|\Lambda_l|} \sum_{\Lambda, \Lambda' \in \mathcal{P}_f(\mathfrak{L}), \Lambda \subset \Lambda_l, \Lambda' \subset \Lambda_l, \varphi(\Lambda \cup \Lambda') > \epsilon l} \int_{\mathcal{A}} \gamma_a (\Phi_{a,\Lambda} + i\Phi'_{a,\Lambda})^* (\Phi_{a,\Lambda'} + i\Phi'_{a,\Lambda'}) \, \mathfrak{d}\mathfrak{a}(a).$$

Then, because  $\Phi \in \mathcal{W}_1$ , one can generally check for any  $\epsilon \in (0, 1)$  that

$$\lim_{l \rightarrow \infty} \frac{\|U_{l,\epsilon}^{\Phi}\|}{\|U_{l,\epsilon}\|} = 0$$

provided that  $\mathfrak{m} \neq (\Phi, 0, 0)$ . In other words, the long range part  $(U_l - U_{\Lambda_l}^\Phi)$  of the internal energy  $U_l$  generally dominates the interaction at long distances for large  $l \in \mathbb{N}$ .

The aim of the paper is the study of the thermodynamic behavior of any models  $\mathfrak{m} \in \mathcal{M}_1$  with long range interactions. In the thermodynamic limit, long range interactions act completely differently depending whether they are positive long range interactions, i.e., *long range repulsions*, or negative long range interactions, i.e., *long range attractions*. These two types of long range interactions are defined via the negative and positive parts

$$(3.1) \quad \gamma_{a,\pm} := 1/2(|\gamma_a| \pm \gamma_a) \in \{0, 1\}$$

of the fixed measurable function

$$\gamma_a = \gamma_{a,+} - \gamma_{a,-} \in \{-1, 1\}$$

as follows:

DEFINITION 3.4 (Long range attractions and repulsions).

(−) *The long range attractions of any  $\mathfrak{m} \in \mathcal{M}_1$  are the  $\mathcal{L}^2$ -interactions*

$$\{\Phi_{a,-} := \gamma_{a,-}\Phi_a\}_{a \in \mathcal{A}} \in \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1) \quad \text{and} \quad \{\Phi'_{a,-} := \gamma_{a,-}\Phi'_a\}_{a \in \mathcal{A}} \in \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1).$$

(+) *The long range repulsions of any  $\mathfrak{m} \in \mathcal{M}_1$  are the  $\mathcal{L}^2$ -interactions*

$$\{\Phi_{a,+} := \gamma_{a,+}\Phi_a\}_{a \in \mathcal{A}} \in \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1) \quad \text{and} \quad \{\Phi'_{a,+} := \gamma_{a,+}\Phi'_a\}_{a \in \mathcal{A}} \in \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1).$$

It is important to observe that our class of models  $\mathfrak{m} \in \mathcal{M}_1$  includes Fermi systems

$$(\Phi, \{\Phi_a^1\}_{a \in \mathcal{A}}, \{\Phi_a^2\}_{a \in \mathcal{A}}, \{\Phi_a^3\}_{a \in \mathcal{A}}, \{\Phi_a^4\}_{a \in \mathcal{A}}) \in \mathcal{M}_1 \times \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1) \times \mathcal{L}^2(\mathcal{A}, \mathcal{W}_1)$$

with internal energies of the type

$$V_l := U_{\Lambda_l}^\Phi + \frac{1}{|\Lambda_l|} \int_{\mathcal{A}} (U_{\Lambda_l}^{\Phi_a^1} + iU_{\Lambda_l}^{\Phi_a^2})^* (U_{\Lambda_l}^{\Phi_a^3} + iU_{\Lambda_l}^{\Phi_a^4}) d\mathfrak{a}(a) + \text{h.c.}$$

because

$$2(A^*B + B^*A) = (A + B)^*(A + B) - (A - B)^*(A - B).$$

In other words, such Fermi systems correspond to models  $\mathfrak{m} \in \mathcal{M}_1$  with long range attractions and repulsions together.

### 3.2. Free-energy densities and existence of thermodynamics

As for local interactions (see, e.g., [8, Theorem 11.4.]), the analysis of the thermodynamics of long range Fermi systems in the grand-canonical ensemble is related to an important functional associated with any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  on  $\mathcal{U}$ : The *free-energy density functional*  $f_{\mathfrak{m}}^\#$  of the long range model

$$\mathfrak{m} := (\Phi, \{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}) \in \mathcal{M}_1.$$

This functional is the sum of the local free–energy density functional  $f_\Phi$  (Definition 2.32) and the long range energy densities defined from the space–averaging functional  $\Delta_A$  (Definition 2.14) for  $A = \mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}$ , see (2.16). Using that

$$(3.2) \quad \Delta_{a,\pm}(\rho) := \gamma_{a,\pm} \Delta_{\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}}(\rho) \in [0, \|\Phi_a\|_{\mathcal{W}_1}^2 + \|\Phi'_a\|_{\mathcal{W}_1}^2]$$

(cf. (2.11) and (2.17)) with  $\gamma_{a,\pm} \in \{0, 1\}$  being the negative and positive parts (3.1) of the fixed measurable function  $\gamma_a \in \{-1, 1\}$ , we define the free–energy density functional  $f_{\mathbf{m}}^\sharp$  as follows:

DEFINITION 3.5 (Free–energy density functional  $f_{\mathbf{m}}^\sharp$ ).

For  $\beta \in (0, \infty]$ , the free–energy density functional  $f_{\mathbf{m}}^\sharp$  w.r.t. any  $\mathbf{m} \in \mathcal{M}_1$  is the map from  $E_{\bar{\ell}}$  to  $\mathbb{R}$  defined by

$$\rho \mapsto f_{\mathbf{m}}^\sharp(\rho) := \|\Delta_{a,+}(\rho)\|_1 - \|\Delta_{a,-}(\rho)\|_1 + e_\Phi(\rho) - \beta^{-1}s(\rho).$$

By Corollary 6.20 (i), this functional is well–defined on  $E_{\bar{\ell}}$ . It is also t.i. and affine. Moreover, on the dense set  $\mathcal{E}_1$  of extreme states of  $E_1$ , i.e., on the dense set of ergodic states (see Definition 2.15, Theorem 2.16 and Corollary 6.6),  $f_{\mathbf{m}}^\sharp$  equals the reduced free–energy density functional  $g_{\mathbf{m}}$  defined on  $E_{\bar{\ell}}$  as follows:

DEFINITION 3.6 (Reduced free–energy density functional  $g_{\mathbf{m}}$ ).

For  $\beta \in (0, \infty]$ , the reduced free–energy density functional  $g_{\mathbf{m}}$  w.r.t. any  $\mathbf{m} \in \mathcal{M}_1$  is the map from  $E_{\bar{\ell}}$  to  $\mathbb{R}$  defined by

$$\rho \mapsto g_{\mathbf{m}}(\rho) := \|\gamma_{a,+}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 - \|\gamma_{a,-}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 + e_\Phi(\rho) - \beta^{-1}s(\rho).$$

This functional is an essential ingredient of the paper. By Corollary 6.20 (ii), it is well–defined and by using Lemmata 2.28 and 2.31 (i) as well as the weak\*–continuity of the maps

$$(3.3) \quad \rho \mapsto \|\gamma_{a,\pm}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 \in [0, \|\Phi_a\|_2^2 + \|\Phi'_a\|_2^2]$$

defined for all  $\rho \in E_{\bar{\ell}}$  (cf. (2.17)), it has the following properties<sup>11</sup>:

LEMMA 3.7 (Properties of the reduced free–energy density functional  $g_{\mathbf{m}}$ ).

(i) The map  $\rho \mapsto g_{\mathbf{m}}(\rho)$  from  $E_{\bar{\ell}}$  to  $\mathbb{R}$  is a weak\*–lower semi–continuous functional.

(ii) For any t.i. state  $\rho \in E_1$ , there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty \subset \mathcal{E}_1$  of ergodic states converging in the weak\*–topology to  $\rho$  and such that

$$g_{\mathbf{m}}(\rho) = \lim_{n \rightarrow \infty} g_{\mathbf{m}}(\hat{\rho}_n).$$

<sup>11</sup>The proof of (i) uses the weak\*–continuity of  $\rho \mapsto |\gamma_{a,\pm}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})|^2$ , the inequality  $|\gamma_{a,\pm}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})| \leq \|\Phi_a\|_{\mathcal{W}_1} + \|\Phi'_a\|_{\mathcal{W}_1}$ , and Lebesgue’s dominated convergence theorem as  $\mathbf{m} \in \mathcal{M}_1$ .

However, since the maps (3.3) are generally not affine, the reduced free-energy density functional  $g_m$  has, in general, a *geometrical* drawback:

- (−)  $g_m$  is *generally not* convex provided that  $\Phi_{a,-} \neq 0$  (a.e.) or  $\Phi'_{a,-} \neq 0$  (a.e.), see Definition 3.4.

This does not occur (w.r.t. the set  $E_1$  of t.i. states) if the long range attractions  $\Phi_{a,-}$  and  $\Phi'_{a,-}$  are *trivial* on  $E_1$ , i.e., if

$$\rho \mapsto |\gamma_{a,-}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})|$$

is (a.e.) a constant map on  $E_1$ , see Remark 2.21. The property (−) represents a problem for our study because we are interested in the set of t.i. minimizers of  $g_m$ , see Theorem 3.12 (i) and Section 3.3.

By contrast, since by Lemmata 2.28 (i), 2.31 (i) and 6.19, the functionals  $s$ ,  $e_\Phi$ , and the maps

$$(3.4) \quad \rho \mapsto \|\Delta_{a,\pm}(\rho)\|_1$$

are all affine, the free-energy density functional  $f_m^\sharp$  is affine. In fact, by using Theorem 2.9 and Lemma 12.17 on each functional  $s$ ,  $e_\Phi$ , and (3.4), we can decompose, for any t.i. state  $\rho \in E_1$ , the free-energy density functional  $f_m^\sharp$  in terms of an integral on the set  $\mathcal{E}_1$ :

LEMMA 3.8 (Properties of the free-energy density functional  $f_m^\sharp$ ).

- (i) *The map  $\rho \mapsto f_m^\sharp(\rho)$  from  $E_{\bar{\ell}}$  to  $\mathbb{R}$  is an affine functional. It is also t.i., i.e., for all  $x \in \mathbb{Z}^d$  and  $\rho \in E_{\bar{\ell}}$ ,  $f_m^\sharp(\rho \circ \alpha_x) = f_m^\sharp(\rho)$ .*  
(ii) *The map  $\rho \mapsto f_m^\sharp(\rho)$  from  $E_1$  to  $\mathbb{R}$  can be decomposed in terms of an integral on the set  $\mathcal{E}_1$  of extreme states of  $E_1$ , i.e., for all  $\rho \in E_1$ ,*

$$f_m^\sharp(\rho) = \int_{\mathcal{E}_1} d\mu_\rho(\hat{\rho}) g_m(\hat{\rho}),$$

with the probability measure  $\mu_\rho$  defined by Theorem 2.9.

However, since the maps (3.4) are generally not weak\*-continuous (see, e.g., Theorem 2.19), the free-energy density functional  $f_m^\sharp$  has, in general, a *topological* drawback:

- (+)  $f_m^\sharp$  is *generally not* weak\*-lower semi-continuous on  $E_1$  provided that  $\Phi_{a,+} \neq 0$  (a.e.) or  $\Phi'_{a,+} \neq 0$  (a.e.), see Definition 3.4.

This does not appear (w.r.t. the set  $E_1$  of t.i. states) if the long range repulsions  $\Phi_{a,+}$  and  $\Phi'_{a,+}$  are *trivial* on  $E_1$ , i.e., if

$$\rho \mapsto |\gamma_{a,+}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})|$$



is (a.e.) a constant map on  $E_1$ , see Theorem 2.19 (i) and Remark 2.21. The problem (+) is serious for our study because we are interested in t.i. minimizers of  $f_m^\sharp$ , see Theorem 3.12 (i) and Section 3.3.

Neither the free-energy density functional  $f_m^\sharp$  nor the reduced free-energy density functional  $g_m$  has the usual good properties to analyze their infimum and minimizers over t.i. states. However, the corresponding variational problems coincide:

LEMMA 3.9 (Minimum of the free-energy densities).

For any  $m \in \mathcal{M}_1$ ,

$$\inf_{\rho \in E_1} f_m^\sharp(\rho) = \inf_{\hat{\rho} \in \mathcal{E}_1} f_m^\sharp(\hat{\rho}) = \inf_{\hat{\rho} \in \mathcal{E}_1} g_m(\hat{\rho}) = \inf_{\rho \in E_1} g_m(\rho) > -\infty$$

with  $\mathcal{E}_1$  being the dense set of extreme states of  $E_1$ .

*Proof.* First, as  $m \in \mathcal{M}_1$ , note that all infima in this lemma are finite because of Remark 2.29, (2.17), Lemma 2.31 (ii), (3.2), and (3.3).

Now, the maps (3.4) are both weak\*-upper semi-continuous affine functionals (Lemma 6.19) and the map

$$(3.5) \quad \rho \mapsto -\|\Delta_{a,-}(\rho)\|_1 + e_\Phi(\rho) - \beta^{-1}s(\rho)$$

from  $E_1$  to  $\mathbb{R}$  is affine and weak\*-lower semi-continuous (cf. Lemmata 2.28 (i) and 2.31 (i)). Therefore,  $f_m^\sharp$  is the sum of a concave weak\*-lower semi-continuous functional and a concave weak\*-upper semi-continuous functional, whereas  $E_1$  is weak\*-compact and convex. Applying Lemma 12.32, we obtain that

$$\inf_{\rho \in E_1} f_m^\sharp(\rho) = \inf_{\hat{\rho} \in \mathcal{E}_1} f_m^\sharp(\hat{\rho}).$$

Since  $f_m^\sharp = g_m$  on  $\mathcal{E}_1$ , it remains to prove the equality

$$(3.6) \quad \inf_{\rho \in E_1} g_m(\rho) = \inf_{\hat{\rho} \in \mathcal{E}_1} g_m(\hat{\rho}).$$

In fact, using the weak\*-lower semi-continuity of  $g_m$  (Lemma 3.7 (i)), the functional  $g_m$  has, at least, one minimizer  $\omega$  over  $E_1$  and by Lemma 3.7 (ii) there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty \subset \mathcal{E}_1$  of ergodic states converging in the weak\*-topology to  $\omega$  with the property that  $g_m(\hat{\rho}_n)$  converges to  $g_m(\omega)$  as  $n \rightarrow \infty$ . The latter yields Equality (3.6). ■

REMARK 3.10 (Extension of the Bauer maximum principle).

*Lemma 12.32 is an extension of the Bauer maximum principle (Lemma 12.31) which does not seem to have been observed before. This lemma can be useful to do similar studies for more general long range interactions as it is defined in [20, 21] for quantum spin systems (see Remark 2.4).*

Lemma 3.9 might be surprising as *no inequality* between  $g_{\mathbf{m}}(\rho)$  and  $f_{\mathbf{m}}^{\sharp}(\rho)$  is generally valid for all t.i. states  $\rho \in E_1$ . In fact, it is a pivotal result because the variational problems of Lemma 3.9 are found in the analysis of the thermodynamics of all models  $\mathbf{m} \in \mathcal{M}_1$  at fixed inverse temperature  $\beta \in (0, \infty)$  in the grand-canonical ensemble.

Indeed, the first task on the thermodynamics of long range models is the analysis of the thermodynamic limit  $l \rightarrow \infty$  of the finite volume pressure

$$(3.7) \quad p_l = p_{l,\mathbf{m}} := \frac{1}{\beta|\Lambda_l|} \ln \text{Trace}_{\mathcal{H}_{\Lambda}}(e^{-\beta U_l})$$

associated with the internal energy  $U_l$  for  $\beta \in (0, \infty)$  and any  $\mathbf{m} \in \mathcal{M}_1$ , see Definition 3.3. This limit defines a map  $\mathbf{m} \mapsto P_{\mathbf{m}}^{\sharp}$  from  $\mathcal{M}_1$  to  $\mathbb{R}$ :

DEFINITION 3.11 (Pressure  $P_{\mathbf{m}}^{\sharp}$ ).

For  $\beta \in (0, \infty)$ , the (infinite volume) pressure is the map from  $\mathcal{M}_1$  to  $\mathbb{R}$  defined by

$$\mathbf{m} \mapsto P_{\mathbf{m}}^{\sharp} := \lim_{l \rightarrow \infty} \{p_{l,\mathbf{m}}\}.$$

The pressure  $P_{\mathbf{m}}^{\sharp}$  is well-defined for any  $\mathbf{m} \in \mathcal{M}_1$  and can be written as an infimum of either the free-energy density functional  $f_{\mathbf{m}}^{\sharp}$  or the reduced free-energy density functional  $g_{\mathbf{m}}$  over states (see Lemma 3.9):

THEOREM 3.12 (Pressure  $P_{\mathbf{m}}^{\sharp}$  as a variational problem on states).

(i) For any  $\mathbf{m} \in \mathcal{M}_1$ ,

$$P_{\mathbf{m}}^{\sharp} = - \inf_{\rho \in E_1} f_{\mathbf{m}}^{\sharp}(\rho) = - \inf_{\rho \in E_1} g_{\mathbf{m}}(\rho) < \infty.$$

(ii) The map  $\mathbf{m} \mapsto P_{\mathbf{m}}^{\sharp}$  from  $\mathcal{M}_1$  to  $\mathbb{R}$  is locally Lipschitz continuous.

The proof of this theorem uses many arguments and to simplify its understanding we break it in several Lemmata and postpone it until Section 8, see Theorem 8.8. In fact, some arguments generalize those of [8, Theorem 11.4] to non-standard potentials  $\Phi \in \mathcal{W}_1$  but others are new, in particular, the ones related to the long range interaction

$$\frac{1}{|\Lambda_l|} \int_{\mathcal{A}} \gamma_a(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})^*(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}) da(a).$$

Note that one argument concerning the long range interaction uses permutation invariant states described in Section 7. This method turns out to be similar to the one used in [20, Theorem 3.4] and [21, Lemma 6.1] for quantum spin systems (see Remark 2.4).

Indeed, for t.i. quantum spin systems with long range components, the (infinite volume) pressure was recently proven to be given by a variational problem over states in [20, 21]. In [20] the long range part of one-dimensional

models has the form  $|\Lambda_L|g(A_L)$  with  $g$  being any real continuous function (and with a stronger norm than  $\|\cdot\|_{\mathcal{W}_1}$ ), whereas in [21] there is no restriction on the dimension and the long range part is  $|\Lambda_L|g(A_L, B_L)$  for some “non-commutative polynomial”  $g$ . Here,  $A_L$  and  $B_L$  are space-averages (defined similarly as in (2.9)) for (not necessarily commuting) self-adjoint operators  $A$  and  $B$  of the quantum spin algebra described in Remark 2.4.

However, Theorem 3.12 for t.i. Fermi models with long range interactions has not been obtained before. Note that a certain type of t.i. Fermi models with long range components (e.g., reduced BCS models) has been analyzed in [7] via the quantum spin representation of fermions, *which we never use here* as it generally breaks the translation invariance of interactions of  $\mathcal{W}_1$ . Nevertheless, because of the technical approach used in [7], the (infinite volume) pressure is given in [7, II.2 Theorem] through two variational problems (\*) and (\*\*) over states on a much larger algebra than the original observable algebra of the model. By [7, II.2 Theorem and II.3 Proposition (1)], both variational problems (\*) and (\*\*) have non-empty compact sets – respectively  $M_*$  and  $M_{**}$  – of minimizers, but the link between them and Gibbs equilibrium states is unclear. Moreover, by [7, II.3 Proposition (1)], extreme states of the convex and compact set  $M_*$  are constructed from minimizers of the second variational problem (\*\*) which, as the authors wrote in [7, p. 642], “can pose a formidable task”.

In fact, Theorem 3.12 (i) also gives the pressure as two variational problems. We prove in Theorem 3.21 (ii) that extreme states of the convex and weak\*-compact set  $\Omega_m^\sharp$  of all weak\*-limit points of approximating minimizers of  $f_m^\sharp$  over  $E_1$  (cf. Definition 3.15 and Lemma 3.16) are likewise minimizers of the second variational problem, i.e., elements of the weak\*-compact set  $\hat{M}_m$  defined below by (3.10), see also Lemma 3.19 (i). Meanwhile, the second variational problem can be analyzed and interpreted as a two-person zero-sum game, see Section 3.6. In particular, in contrast to [7] and the sets  $M_*$  and  $M_{**}$ ,  $\hat{M}_m$  can be explicitly characterized for all  $m \in \mathcal{M}_1$ , see Theorem 3.39, whereas the set  $\Omega_m^\sharp$  is related to Gibbs equilibrium states in the sense of Theorem 3.29, see also Theorem 5.13. Before going into such results, we need first to discuss the definitions and properties of the sets  $\Omega_m^\sharp$  and  $\hat{M}_m$  in the next subsection.

### 3.3. Generalized t.i. equilibrium states

We discuss now a special class of states: The (possibly generalized) equilibrium states which are supposed to describe physical systems at thermodynamic equilibrium. These states are always defined in relation to a given interaction which describes the energy density for a given state as well as the microscopical dynamics. We define here (possibly generalized) equilibrium states via a

variational principle (Definitions 3.13 and 3.15). However, this is not the only reasonable way of defining equilibrium states. At fixed interaction they can also be defined as tangent functionals to the corresponding pressure (Definition 3.27) or other conditions like: The local stability condition, the Gibbs condition, or the Kubo–Martin–Schwinger (KMS) condition. These definitions are generally not equivalent to each other. For more details, see [8].

From Theorem 3.12 (i), the pressure  $P_m^\sharp$  is given by the infimum of the free–energy density functional  $f_m^\sharp$  over t.i. states  $\rho \in E_1$ . When  $\mathbf{m} = (\Phi, 0, 0) \in \mathcal{M}_1$  the map

$$(3.8) \quad \rho \mapsto f_m^\sharp(\rho) = f_\Phi(\rho) := e_\Phi(\rho) - \beta^{-1}s(\rho)$$

is weak\*–lower semi–continuous and affine, see Lemmata 2.28 (i), 2.31 (i) and Definition 2.32. In particular, it has minimizers in the set  $E_1$  of t.i. states. The corresponding set  $M_\Phi$  of all t.i. minimizers is a (non–empty) closed face of the Poulsen simplex  $E_1$ . Then, similarly to what is done for translation invariant quantum spin systems (see, e.g., [5, 22]), *t.i. equilibrium states* are defined as follows:

DEFINITION 3.13 (Set of t.i. equilibrium states).

For  $\beta \in (0, \infty)$  and any  $\mathbf{m} \in \mathcal{M}_1$ , the set  $M_m^\sharp$  of t.i. equilibrium states is the set

$$M_m^\sharp := \left\{ \omega \in E_1 : f_m^\sharp(\omega) = \inf_{\rho \in E_1} f_m^\sharp(\rho) \right\}$$

of all minimizers of the free–energy density functional  $f_m^\sharp$  over the set  $E_1$ .

The set  $M_m^\sharp$  is convex and in fact, a face by affinity of the free–energy density functional  $f_m^\sharp$  (Lemma 3.8 (i)):

LEMMA 3.14 (Properties of non–empty sets  $M_m^\sharp$ ).

If  $\mathbf{m} \in \mathcal{M}_1$  is such that  $M_m^\sharp$  is non–empty then  $M_m^\sharp$  is a (possibly not closed) face of  $E_1$ .

Nevertheless,  $M_m^\sharp$  is not necessarily weak\*–compact and depending on the model  $\mathbf{m} \in \mathcal{M}_1$ , it *could even be empty*. Indeed, the situation is more involved in the case of long range models of  $\mathcal{M}_1$  than for t.i. interactions of  $\mathcal{W}_1$  as  $f_m^\sharp$  is generally not weak\*–lower semi–continuous: As explained above, if the long range repulsions  $\Phi_{a,+} \neq 0$  or  $\Phi'_{a,+} \neq 0$  then the functional  $f_m^\sharp$  is a sum of the maps (3.4) (with +) and (3.5) which are, respectively, weak\*–upper and weak\*–lower semi–continuous functionals, see Lemmata 2.28 (i), 2.31 (i) and 6.19. In particular, the existence of minimizers of  $f_m^\sharp$  over  $E_1$  is unclear unless  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.).

Therefore, we shall consider any sequence  $\{\rho_n\}_{n=1}^\infty$  of *approximating t.i. minimizers*, that is, any sequence  $\{\rho_n\}_{n=1}^\infty$  in  $E_1$  such that

$$(3.9) \quad \lim_{n \rightarrow \infty} f_{\mathbf{m}}^\sharp(\rho_n) = \inf_{\rho \in E_1} f_{\mathbf{m}}^\sharp(\rho).$$

Such sequences clearly exist and since  $E_1$  is sequentially weak\*-compact<sup>12</sup>, they converge in the weak\*-topology – along subsequences – towards t.i. states  $\omega \in E_1$ . Thus, *generalized t.i. equilibrium states* are naturally defined as follows:

DEFINITION 3.15 (Set of generalized t.i. equilibrium states).

For  $\beta \in (0, \infty]$  and any  $\mathbf{m} \in \mathcal{M}_1$ , the set  $\Omega_{\mathbf{m}}^\sharp$  of generalized t.i. equilibrium states is the (non-empty) set

$$\Omega_{\mathbf{m}}^\sharp := \left\{ \omega \in E_1 : \exists \{\rho_n\}_{n=1}^\infty \subset E_1 \text{ with weak}^* \text{-limit point } \omega \right. \\ \left. \text{such that } \lim_{n \rightarrow \infty} f_{\mathbf{m}}^\sharp(\rho_n) = \inf_{\rho \in E_1} f_{\mathbf{m}}^\sharp(\rho) \right\}$$

of all weak\*-limit points of approximating minimizers of the free-energy density functional  $f_{\mathbf{m}}^\sharp$  over the set  $E_1$ .

In contrast to the convex set  $M_{\mathbf{m}}^\sharp$  which may be either empty or not weak\*-compact, the set  $\Omega_{\mathbf{m}}^\sharp \subset E_1$  is always a (non-empty) weak\*-compact convex set:

LEMMA 3.16 (Properties of the set  $\Omega_{\mathbf{m}}^\sharp$  for  $\mathbf{m} \in \mathcal{M}_1$ ).

The set  $\Omega_{\mathbf{m}}^\sharp$  is a (non-empty) convex and weak\*-compact subset of  $E_1$ .

*Proof.* The convexity of the set  $\Omega_{\mathbf{m}}^\sharp$  results from the affinity of  $f_{\mathbf{m}}^\sharp$ . Since  $E_1$  is a weak\*-compact subset of  $\mathcal{U}^*$ , the weak\*-topology is metrizable on  $E_1$  (Theorem 12.10) and  $\Omega_{\mathbf{m}}^\sharp \subset E_1$  is weak\*-compact by Lemma 12.36. ■

NOTATION 3.17 (Generalized t.i. equilibrium states).

The letter  $\omega$  is exclusively reserved to denote generalized t.i. equilibrium states. Extreme points of  $\Omega_{\mathbf{m}}^\sharp$  are usually written as  $\hat{\omega} \in \mathcal{E}(\Omega_{\mathbf{m}}^\sharp)$  (cf. Theorem 12.11).

Obviously,  $M_{\mathbf{m}}^\sharp \subset \Omega_{\mathbf{m}}^\sharp$  and for any  $\Phi \in \mathcal{W}_1$ , i.e.,  $\mathbf{m} = (\Phi, 0, 0) \in \mathcal{M}_1$ ,  $M_\Phi = \Omega_\Phi$ . Conversely, ergodic generalized t.i. equilibrium states  $\hat{\omega} \in \Omega_{\mathbf{m}}^\sharp \cap \mathcal{E}_1$  are always contained in  $M_{\mathbf{m}}^\sharp$ :

LEMMA 3.18 (Ergodic generalized t.i. equilibrium states are minimizers).

For any  $\mathbf{m} \in \mathcal{M}_1$ ,  $\Omega_{\mathbf{m}}^\sharp \cap \mathcal{E}_1 = M_{\mathbf{m}}^\sharp \cap \mathcal{E}_1$ . In particular, if  $\Omega_{\mathbf{m}}^\sharp$  is a face then it is the weak\*-closure of the non-empty set  $M_{\mathbf{m}}^\sharp$  of minimizers of  $f_{\mathbf{m}}^\sharp$  over  $E_1$ .

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<sup>12</sup> $E_1$  is sequentially weak\*-compact because it is weak\*-compact and metrizable in the weak\*-topology (Theorem 12.10).

*Proof.* Because of Definition 3.15, the proof is a direct consequence of the continuity of the space-averaging functional  $\Delta_A$  at any ergodic state  $\rho \in \mathcal{E}_1$  together with Lebesgue's dominated convergence theorem and the weak\*-lower semi-continuity of the (local) free-energy density functional  $f_\Phi$  (3.8), see Theorem 2.19 (iii), Lemmata 2.28 (i) and 2.31 (i).

Additionally, if  $\Omega_m^\sharp$  is a face then by affinity of  $f_m^\sharp$ , any state of the convex hull of  $\Omega_m^\sharp \cap \mathcal{E}_1$  is a minimizer of  $f_m^\sharp$ . By Krein-Milman theorem (Theorem 12.11),  $\Omega_m^\sharp$  is contained in the weak\*-closure of the set of all minimizers of  $f_m^\sharp$ . On the other hand, as any minimizer of  $f_m^\sharp$  is contained in the closed set  $\Omega_m^\sharp$ , the weak\*-closure of the set of all minimizers is obviously included in  $\Omega_m^\sharp$ . ■

We observe now that Definition 3.15 is not the only natural way of defining generalized t.i. equilibrium states. Indeed, Theorem 3.12 (i) says that the pressure  $P_m^\sharp$  is also given (up to a minus sign) by the infimum of the reduced free-energy density functional  $g_m$  over  $E_1$ . The functional  $g_m$  from Definition 3.6 is a weak\*-lower semi-continuous map (Lemma 3.7 (i)) and has only (usual) minimizers in the set  $E_1$  as any sequence  $\{\rho_n\}_{n=1}^\infty \subset E_1$  of approximating t.i. minimizers of  $g_m$  converges to a minimizer of  $g_m$  over  $E_1$ . Minimizers of  $g_m$  over  $E_1$  form a non-empty set denoted by

$$(3.10) \quad \hat{M}_m := \left\{ \omega \in E_1 : g_m(\omega) = \inf_{\rho \in E_1} g_m(\rho) \right\}.$$

This set is weak\*-compact and included in the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states:

LEMMA 3.19 (Properties of the set  $\hat{M}_m$  for  $m \in \mathcal{M}_1$ ).

- (i) *The set  $\hat{M}_m$  is a (non-empty) weak\*-compact subset of  $E_1$ .*
- (ii) *The weak\*-closed convex hull of  $\hat{M}_m$  is included in  $\Omega_m^\sharp$ , i.e.,*

$$\overline{\text{co}(\hat{M}_m)} \subset \Omega_m^\sharp.$$

*Proof.* The assertion (i) is a direct consequence of the weak\*-lower semi-continuity of the functional  $g_m$  (Lemma 3.7 (i)) together with the weak\*-compactity of  $E_1$ . The second one results from Lemmata 3.7 (ii), 3.9 and 3.16. Indeed, by Lemma 3.7 (ii), for any  $\omega \in \hat{M}_m$ , there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty \subset \mathcal{E}_1$  of ergodic states converging in the weak\*-topology to  $\omega$  with the property that  $g_m(\hat{\rho}_n) = f_m^\sharp(\hat{\rho}_n)$  converges to  $g_m(\omega)$  as  $n \rightarrow \infty$ . Since by Lemma 3.9,  $g_m(\omega)$  is also the infimum of the functional  $f_m^\sharp$  over  $E_1$ , we obtain that  $\omega \in \Omega_m^\sharp$ , see Definition 3.15. As a consequence, the second assertion (ii) holds because  $\Omega_m^\sharp$  is convex and weak\*-compact by Lemma 3.16. ■

Definition 3.15 seems to be a more reasonable way of defining generalized t.i. equilibrium states. Indeed,  $\hat{M}_m$  is generally not convex because the functional  $g_m$  is generally not convex provided that  $\Phi_{a,-} \neq 0$  (a.e.) or  $\Phi'_{a,-} \neq 0$

(a.e.). Hence, we have, in general, only one inclusion:  $\hat{M}_{\mathbf{m}} \subset \Omega_{\mathbf{m}}^{\sharp}$ . In fact, we show in Theorem 3.21 (i) that the weak\*-closed convex hull of  $\hat{M}_{\mathbf{m}}$  equals  $\Omega_{\mathbf{m}}^{\sharp}$ . The equality  $\hat{M}_{\mathbf{m}} = \Omega_{\mathbf{m}}^{\sharp}$  holds for purely repulsive long range models for which  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.), see Theorem 3.25 (+).

REMARK 3.20 (Generalized t.i. ground states).

*All results concerning generalized t.i. equilibrium states are performed at finite temperature, i.e., at fixed  $\beta \in (0, \infty)$ . However, each weak\*-limit point  $\omega$  of the sequence of states  $\omega^{(n)} \in \Omega_{\mathbf{m}_n}^{\sharp}$  of models  $\{\mathbf{m}_n\}_{n \in \mathbb{N}}$  in  $\mathcal{M}_1$  such that  $\beta_n \rightarrow \infty$  and  $\mathbf{m}_n \rightarrow \mathbf{m} \in \mathcal{M}_1$  can be seen as a generalized t.i. ground state of  $\mathbf{m}$ . An analysis of generalized t.i. ground states is not performed here, but it essentially uses the same kind of arguments as for  $\Omega_{\mathbf{m}}^{\sharp}$ , see, e.g., [9, Section 6.2].*

### 3.4. Structure of the set $\Omega_{\mathbf{m}}^{\sharp}$ of generalized t.i. equilibrium states

By Lemma 3.8 (i) recall that the free-energy density functional  $f_{\mathbf{m}}^{\sharp}$  is affine but generally not weak\*-lower semi-continuous, even on the set  $E_1$  of t.i. states as explained in Sections 3.2 and 3.3. The variational problem

$$P_{\mathbf{m}}^{\sharp} = - \inf_{\rho \in E_1} f_{\mathbf{m}}^{\sharp}(\rho)$$

given in Theorem 3.12 (i) is, however, not as difficult as it may look like provided it is attacked in the right way.

Indeed, since we are interested in *global* (possibly approximating) t.i. minimizers of  $f_{\mathbf{m}}^{\sharp}$  (cf. Definition 3.15), it is natural to introduce its  $\Gamma$ -regularization  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  on  $E_1$ , see Definition 12.27 in Section 12.5. Indeed, for all  $\mathbf{m} \in \mathcal{M}_1$ ,

$$\inf_{\rho \in E_1} f_{\mathbf{m}}^{\sharp}(\rho) = \inf_{\rho \in E_1} \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})(\rho),$$

see Theorem 12.37 (i). The functional  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  has the advantage to be a weak\*-lower semi-continuous convex functional, see Section 12.5. As a consequence,  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  has minimizer and only (usual) minimizers over the set  $E_1$  as any sequence of approximating t.i. minimizers of  $f_{\mathbf{m}}^{\sharp}$  automatically converges to a minimizer of this functional over  $E_1$ . In fact, the set of minimizers of  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  coincides with the set  $\Omega_{\mathbf{m}}^{\sharp}$  of generalized minimizers of  $f_{\mathbf{m}}^{\sharp}$ , see Lemma 3.16 and Theorem 12.37 (ii). Hence, we shall describe it in more details.

The free-energy density functional  $f_{\mathbf{m}}^{\sharp}$  is the sum of maps (3.4) (with +) and (3.5). From Theorem 2.19 (v), the  $\Gamma$ -regularization of  $\Delta_{a,+}$  on  $E_1$  is the weak\*-lower semi-continuous convex map

$$(3.11) \quad \rho \mapsto |\gamma_{a,+} \rho (\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi_a'})|_2^2,$$

(cf. (2.16)), whereas the  $\Gamma$ -regularization of the map (3.5) on  $E_1$  is itself because (3.5) is a weak\*-lower semi-continuous convex functional (cf. Corollary

12.30). Therefore, we could try to replace the functional  $\Delta_{a,+}$  in  $f_m^\sharp$  by its  $\Gamma$ -regularization (3.11). Doing this we denote by  $f_m^b$  the real functional defined by

$$(3.12) \quad f_m^b(\rho) := \|\gamma_{a,+}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 - \|\Delta_{a,-}(\rho)\|_1 + e_\Phi(\rho) - \beta^{-1}s(\rho)$$

for all  $\rho \in E_1$ . However, we can *not* expect that the functional  $f_m^b$  is, in all cases<sup>13</sup>, equal to the  $\Gamma$ -regularization  $\Gamma_{E_1}(f_m^\sharp)$  of  $f_m^\sharp$  because the  $\Gamma$ -regularization  $\Gamma(h_1 + h_2)$  of the sum of two functionals  $h_1$  and  $h_2$  is generally not equal to the sum  $\Gamma(h_1) + \Gamma(h_2)$ .

In fact, the  $\Gamma$ -regularization  $\Gamma_K(h)$  of any functional  $h$  is its largest lower semi-continuous and convex minorant on  $K$  (Corollary 12.30) and as  $f_m^b$  is a convex weak\*-lower semi-continuous functional (cf. Lemmata 2.28 (i), 2.31 (i) and 6.19), we have the inequalities

$$(3.13) \quad f_m^b(\rho) \leq \Gamma_{E_1}(f_m^\sharp)(\rho) \leq f_m^\sharp(\rho)$$

for all  $\rho \in E_1$ . The first inequality is *generally strict*. This can easily be seen by using, for instance, any model  $\mathbf{m} \in \mathcal{M}_1$  such that

$$\|\Delta_{a,-}(\rho)\|_1 = \|\Delta_{a,+}(\rho)\|_1$$

for all  $\rho \in E_1$ . As a consequence, the variational problem

$$(3.14) \quad P_m^b := - \inf_{\rho \in E_1} f_m^b(\rho)$$

is only a upper bound of the pressure  $P_m^\sharp$ , i.e.,  $P_m^b \geq P_m^\sharp$ .

Nevertheless,  $P_m^b$  is still an interesting variational problem because it has a direct interpretation in terms of the max-min variational problem  $F_m^b$  of the thermodynamic game defined in Definition 3.35, see Theorem 3.36 (b). Moreover, as  $\Delta_A(\hat{\rho}) = |\hat{\rho}(A)|^2$  for any ergodic state  $\hat{\rho} \in \mathcal{E}_1$  and  $A \in \mathcal{U}$ , we have that

$$f_m^b(\hat{\rho}) = g_m(\hat{\rho}) = f_m^\sharp(\hat{\rho})$$

for all extreme states  $\hat{\rho} \in \mathcal{E}_1$ . By (3.13), it follows that  $\Gamma_{E_1}(f_m^\sharp)$  coincides on  $\mathcal{E}_1$  with the explicit weak\*-lower semi-continuous functional  $g_m$  defined in Definition 3.6:

$$(3.15) \quad \Gamma_{E_1}(f_m^\sharp)(\hat{\rho}) = g_m(\hat{\rho}) = f_m^b(\hat{\rho}) = f_m^\sharp(\hat{\rho})$$

for any  $\mathbf{m} \in \mathcal{M}_1$  and all  $\hat{\rho} \in \mathcal{E}_1$ .

Since the set  $\mathcal{E}_1$  of extreme points of  $E_1$  is dense (cf. Corollary 6.6), Equality (3.15) is a strong property on the functional  $\Gamma_{E_1}(f_m^\sharp)$ . Indeed, by combining (3.15) with Lemma 2.28, Lemma 3.8, Corollary 12.30, Lemma 12.33, and Theorem 12.37, we arrive at a fundamental characterization of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states:

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<sup>13</sup>In fact,  $f_m^b = g_m = \Gamma_{E_1}(f_m^\sharp)$  when  $\Phi_{a,-} = 0$  (a.e.), see proof of Theorem 3.21.



THEOREM 3.21 (Structure of the set  $\Omega_{\mathbf{m}}^{\sharp}$  for any  $\mathbf{m} \in \mathcal{M}_1$ ).

(i) *The weak\*-compact and convex set  $\Omega_{\mathbf{m}}^{\sharp}$  is the weak\*-closed convex hull of the weak\*-compact set  $\hat{M}_{\mathbf{m}}$  (3.10), i.e.,*

$$\Omega_{\mathbf{m}}^{\sharp} = \overline{\text{co}(\hat{M}_{\mathbf{m}})}.$$

(ii) *The set  $\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})$  of extreme states of  $\Omega_{\mathbf{m}}^{\sharp}$  is included in  $\hat{M}_{\mathbf{m}}$ , i.e.,*

$$\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}) \subseteq \hat{M}_{\mathbf{m}}.$$

(iii) *For any  $\omega \in \Omega_{\mathbf{m}}^{\sharp}$ , there is a probability measure  $v_{\omega}$  on  $\Omega_{\mathbf{m}}^{\sharp}$  such that*

$$v_{\omega}(\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})) = 1 \quad \text{and} \quad \omega = \int_{\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})} dv_{\omega}(\hat{\omega}) \hat{\omega}.$$

*Proof.* We first prove that  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp}) = \Gamma_{E_1}(g_{\mathbf{m}})$  on  $E_1$ . We start by showing that  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  is a lower bound for  $\Gamma_{E_1}(g_{\mathbf{m}})$ .

For any  $\rho \in E_1$ , there is, by Lemma 3.7 (ii), a sequence  $\{\hat{\rho}_n\}_{n=1}^{\infty} \subset \mathcal{E}_1$  of ergodic states converging in the weak\*-topology to  $\rho$  and such that  $g_{\mathbf{m}}(\hat{\rho}_n)$  converges to  $g_{\mathbf{m}}(\rho)$ . By (3.15), it follows that  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})(\hat{\rho}_n)$  also converges to  $g_{\mathbf{m}}(\rho)$ . Moreover, as  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  is weak\*-lower semi-continuous on  $E_1$ ,

$$(3.16) \quad \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})(\rho) \leq \lim_{n \rightarrow \infty} \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})(\hat{\rho}_n) = g_{\mathbf{m}}(\rho)$$

for any  $\rho \in E_1$ . Applying Corollary 12.30 we deduce from (3.16) that

$$(3.17) \quad \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})(\rho) \leq \Gamma_{E_1}(g_{\mathbf{m}})(\rho)$$

for all  $\rho \in E_1$ . We show next the converse inequality.

Since the functional  $\Gamma_{E_1}(g_{\mathbf{m}})$  is convex, by using Theorem 2.9 together with Jensen's inequality (Lemma 12.33), Lemma 3.8 (ii), and Equality (3.15), we obtain that

$$\Gamma_{E_1}(g_{\mathbf{m}})(\rho) \leq \int_{\mathcal{E}_1} d\mu_{\rho}(\hat{\rho}) \Gamma_{E_1}(g_{\mathbf{m}})(\hat{\rho}) = f_{\mathbf{m}}^{\sharp}(\rho)$$

for all  $\rho \in E_1$ , which, by Corollary 12.30, implies the inequality

$$(3.18) \quad \Gamma_{E_1}(g_{\mathbf{m}})(\rho) \leq \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})(\rho)$$

for all  $\rho \in E_1$ . Therefore, Inequalities (3.17) and (3.18) yield  $\Gamma_{E_1}(g_{\mathbf{m}}) = \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  on  $E_1$ .

We apply now Theorem 12.37 to  $K = E_1$  and  $h = f_{\mathbf{m}}^{\sharp}$  to show that the set of minimizers of  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  over  $E_1$  is the weak\*-closed convex hull of  $\Omega_{\mathbf{m}}^{\sharp}$ . By Lemma 3.16,  $\Omega_{\mathbf{m}}^{\sharp}$  is a convex and weak\*-compact set. Hence, the set of minimizers of  $\Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  over  $E_1$  equals  $\Omega_{\mathbf{m}}^{\sharp}$ . Then, as  $\Gamma_{E_1}(g_{\mathbf{m}}) = \Gamma_{E_1}(f_{\mathbf{m}}^{\sharp})$  on  $E_1$ ,  $\Omega_{\mathbf{m}}^{\sharp}$  is also the set of minimizers of  $\Gamma_{E_1}(g_{\mathbf{m}})$  over  $E_1$  and by applying again Theorem 12.37 (i)–(ii) and also Theorem 12.38 (i) to  $K = E_1$  and  $h = g_{\mathbf{m}}^{\sharp}$  we get the assertions (i)–(ii).

The third statement (iii) is a consequence of Choquet theorem (see Theorem 12.18) because the set  $\Omega_{\mathbf{m}}^{\sharp}$  is convex, weak\*-compact (Lemma 3.16), and metrizable by Theorem 12.10. In particular, the equality

$$\omega = \int_{\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})} dv_{\omega}(\hat{\omega}) \hat{\omega}$$

means, per definition, that  $\omega \in \Omega_{\mathbf{m}}^{\sharp}$  is the barycenter of the probability measure, i.e., the normalized positive Borel regular measure,  $v_{\omega}$  on  $\Omega_{\mathbf{m}}^{\sharp}$ , see Definition 12.15 and Theorem 12.16. ■

REMARK 3.22 (Minimization of real functionals).

*Theorem 3.21 (i)–(ii) can be proven without Theorems 12.37–12.38 by using Lemma 3.19 combined with Lanford III – Robinson theorem [23, Theorem 1] (Theorem 12.46) and Lemma 3.9. However, Theorems 12.37–12.38 – which do not seem to have been proven before – are very useful results to analyze variational problems with non-convex functionals on a compact convex set  $K$ . Indeed, the minimization of any real functional  $h$  over  $K$  can be done in this case by analyzing a variational problem related to a convex lower semi-continuous functional  $\Gamma_K(h)$  for which various methods are available.*

Note that the integral representation (iii) in Theorem 3.21 may not be unique, i.e.,  $\Omega_{\mathbf{m}}^{\sharp}$  may not be a Choquet simplex (Definition 12.23) in contrast to all sets  $E_{\vec{\ell}}$  for all  $\vec{\ell} \in \mathbb{N}^d$ , see Theorems 2.9 and 2.12. In Theorem 3.46 we give some special (but yet physically relevant) cases for which the sets  $\Omega_{\mathbf{m}}^{\sharp}$  are simplices.

REMARK 3.23 (Pure thermodynamic phases).

*From Theorem 3.21, we have in  $\Omega_{\mathbf{m}}^{\sharp}$  a notion of pure and mixed thermodynamic phases (equilibrium states) by identifying purity with extremality. If  $\Omega_{\mathbf{m}}^{\sharp}$  turns out to be a face in  $E_1$  (see, e.g., Theorem 3.25 (–)) then purity corresponds to ergodicity as  $\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}) = \Omega_{\mathbf{m}}^{\sharp} \cap \mathcal{E}_1$  in this special case.*

REMARK 3.24 (Gauge invariant t.i. equilibrium states).

*If the model  $\mathbf{m} \in \mathcal{M}_1$  is gauge invariant, which means that  $U_l \in \mathcal{U}^{\circ}$  (cf. (2.6)), then the set  $\Omega_{\beta}^{\sharp, \circ} := \Omega_{\mathbf{m}}^{\sharp} \cap E_1^{\circ}$  of gauge invariant t.i. equilibrium states of  $\mathbf{m}$  is the weak\*-closed convex hull of the (non-empty) set  $\hat{M}_{\mathbf{m}} \cap E_1^{\circ}$  and its set of extreme points equals*

$$\mathcal{E}(\Omega_{\beta}^{\sharp, \circ}) = \mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}) \cap E_1^{\circ} \subset \hat{M}_{\mathbf{m}} \cap E_1^{\circ},$$

*cf. Remark 2.13. This follows by using Theorem 3.21 together with elementary arguments. We omit the details.*

We conclude now this subsection by analyzing some effects of negative and repulsive long range interactions on the thermodynamics of models  $\mathbf{m} \in \mathcal{M}_1$ ,

see Definition 3.4. In particular, we observe that long range attractions  $\Phi_{a,-}$  and  $\Phi'_{a,-}$  have no important effect on the structure of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states which is, for all purely local models  $(\Phi, 0, 0) \in \mathcal{M}_1$ , a (non-empty) closed face of  $E_1$ . By contrast, long range repulsions  $\Phi_{a,+}$  and  $\Phi'_{a,+}$  have generally a *geometrical* effect by possibly breaking the face structure of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states. Indeed, we have the following statements:

**THEOREM 3.25** ( $\Omega_m^\sharp$  when  $\Phi_{a,+} = \Phi'_{a,+} = 0$  or  $\Phi_{a,-} = \Phi'_{a,-} = 0$ ).

(-) If  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.) then  $P_m := P_m^\sharp = P_m^b$  and  $\Omega_m^\sharp = M_m^\sharp$  is a closed face of the Poulsen simplex  $E_1$ .

(+) If  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) then  $P_m := P_m^\sharp = P_m^b$  and  $\Omega_m^\sharp = \hat{M}_m$  is the set of minimizers of the convex functional  $g_m$  over  $E_1$ , cf. (3.10).

*Proof.* In any case,  $f_m^b$  is weak\*-lower semi-continuous, see (3.12). If  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.) then  $f_m^b = f_m^\sharp = \Gamma_{E_1}(f_m^\sharp)$  is also affine, see Definition 3.5 and (3.13). Then the first assertion (-) is obvious.

If  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) then  $f_m^b = g_m$  and, by Theorem 3.12 (i),  $P_m := P_m^\sharp = P_m^b$ . Moreover, the weak\*-lower semi-continuous functional  $g_m$  becomes convex when  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.), see Definition 3.6 and (3.12). As a consequence, the set  $\hat{M}_m$  of minimizers of  $f_m^b = g_m$  over  $E_1$  is convex and also weak\*-compact because of Lemma 3.19 (i). Then applying Theorem 3.21 (i) we arrive at the second assertion (+). ■

If  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) then  $g_m = f_m^b$  can be strictly convex. As a consequence, its set  $\hat{M}_m$  of minimizers over  $E_1$  is, in general, not a face, see Lemma 11.8 in Section 11.2. This geometrical effect can lead to a *long range order* (LRO) implied by long range repulsions, see Section 3.8.

### 3.5. Gibbs states versus generalized equilibrium states

The Gibbs equilibrium state is defined in Definition 12.1 and equals the explicitly given state  $\rho_l := \rho_{\Lambda_l, U_l}$  (12.2) because of Theorem 12.2, see Section 12.1. The physical relevance of such a finite volume equilibrium state is based – among other things – on the minimum free energy principle and the second law of thermodynamics as explained in Section 12.1:  $\rho_l$  is a finite volume thermal state at equilibrium. In the same way, a generalized t.i. equilibrium state  $\omega \in \Omega_m^\sharp$  represents an infinite volume thermal state at equilibrium. There are, however, important differences between the finite volume system and its thermodynamic limit:

- *Non-uniqueness of generalized t.i. equilibrium states.* The Gibbs equilibrium state is the unique minimizer in  $E_\Lambda$  of the finite volume free-energy

density (Theorem 12.2) but at infinite volume,  $\omega \in \Omega_{\mathfrak{m}}^{\sharp}$  may not be unique, see, e.g., [9, Section 6.2]. Such a phenomenon is found in symmetry broken quantum phases like the superconducting phase. Mathematically, it is related to the fact that we leave the Fock space representation of models to go to a representation-free formulation of thermodynamic phases. Doing so we take advantage of the non-uniqueness of the representation of the  $C^*$ -algebra  $\mathcal{U}$ , as stressed for instance in [24, 25, 26] for the BCS model in infinite volume. This property is, indeed, necessary to get non-unique generalized equilibrium states which imply phase transitions.

- *Space symmetry of generalized equilibrium states.* The Gibbs equilibrium state minimizes the finite volume free-energy density functional over the set  $E$  of all states (Theorem 12.2). Observe that the Gibbs equilibrium state may eventually not converge to a t.i. state in the thermodynamic limit. By contrast, generalized t.i. equilibrium states  $\omega \in \Omega_{\mathfrak{m}}^{\sharp}$  are weak\*-limit points of approximating minimizers of the free-energy density functional  $f_{\mathfrak{m}}^{\sharp}$  over the subset  $E_1 \subset E$  of t.i. states (Theorem 3.12 (i)). Indeed, the functional  $f_{\mathfrak{m}}^{\sharp}$  is, a priori, only well-defined on the set  $E_{\vec{\ell}}$  (cf. Definition 3.5). Therefore, it only makes sense to speak about generalized  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant equilibrium states. The translation invariance property of interactions in every model  $\mathfrak{m} \in \mathcal{M}_1$  ensures the existence of generalized t.i. equilibrium states ( $\Omega_{\mathfrak{m}}^{\sharp} \neq \emptyset$ ), but it does not exclude the existence of generalized  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant equilibrium states for  $\vec{\ell} \neq (1, \dots, 1)$ . In other words, a t.i. (physical) system can lead to periodic (non-translation invariant) structures. This phenomenon can be an explanation of the appearance of periodic superconducting phases as observed recently, see, e.g., [27, 28]. No comprehensive theory is available to explain such a phenomenon and we will investigate this question in another paper by using the present formalism, in particular the decomposition of generalized t.i. equilibrium states w.r.t. generalized  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant equilibrium states. Observe further that, by Theorem 8.8, there is a natural extension  $\mathfrak{F}_{\mathfrak{m}}^{\sharp}$  (8.7) of  $f_{\mathfrak{m}}^{\sharp}$  on  $E$  such that

$$P_{\mathfrak{m}}^{\sharp} = - \inf_{\rho \in E} \mathfrak{F}_{\mathfrak{m}}^{\sharp}(\rho) = - \inf_{\rho \in E_{\vec{\ell}}} f_{\mathfrak{m}}^{\sharp}(\rho) = - \inf_{\rho \in E_1} f_{\mathfrak{m}}^{\sharp}(\rho).$$

So, the first equality could be used to define non-periodic generalized equilibrium states for long range systems.

REMARK 3.26 (Generalized  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant equilibrium states).

*Using periodically invariant interactions the set of generalized  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant equilibrium states can be analyzed in the same way we study  $\Omega_{\mathfrak{m}}^{\sharp}$ . In fact, we restrict our analysis on t.i. Fermi systems, but all our studies can also be done for models constructed from periodically invariant interactions.*

The Gibbs equilibrium state  $\rho_l$ , seen as defined either on the local algebra  $\mathcal{U}_{\Lambda_l}$  or on the whole algebra  $\mathcal{U}$  by periodically extending<sup>14</sup> it (with period  $(2l + 1)$  in each direction of the lattice  $\mathfrak{L}$ ), should converge to a minimum of the functional  $\mathfrak{F}_m^\sharp$  (8.7) over  $E$  but it may not converge to a generalized t.i. equilibrium state  $\omega \in \Omega_m^\sharp$ . By contrast, the *space-averaged* t.i. Gibbs state

$$(3.19) \quad \hat{\rho}_l := \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \rho_l \circ \alpha_x \in E_1$$

constructed from  $\rho_l := \rho_{\Lambda_l, U_l}$  (12.2) and the  $*$ -automorphisms  $\{\alpha_x\}_{x \in \mathbb{Z}^d}$  defined on  $\mathcal{U}$  by (2.7) always converges in the weak\*-topology to a generalized t.i. equilibrium state, see Theorem 3.29.

This can be seen by using a characterization of generalized t.i. equilibrium states as *tangent functionals* to the pressure  $P_m^\sharp$ . Indeed, by Definition 3.11, the pressure  $P_m^\sharp$  is a map from  $\mathcal{M}_1$  to  $\mathbb{R}$  and, as a consequence, it defines by restriction a map

$$(3.20) \quad \Phi \mapsto P_m^\sharp(\Phi) := P_{m+(\Phi, 0, 0)}^\sharp$$

from the real Banach space  $\mathcal{W}_1$  of t.i. interactions to  $\mathbb{R}$  at any fixed  $m \in \mathcal{M}_1$ . By Theorem 3.12 (ii), the map  $\Phi \mapsto P_m^\sharp(\Phi)$  is (norm) continuous and also convex because it is the supremum over the family  $\{\mathbb{A}(\rho)\}_{\rho \in E_1}$  of affine maps

$$\Phi \mapsto \mathbb{A}(\rho)(\Phi) := -\|\Delta_{a,+}(\rho)\|_1 + \|\Delta_{a,-}(\rho)\|_1 - e_\Phi(\rho) + \beta^{-1}s(\rho)$$

from  $\mathcal{W}_1$  to  $\mathbb{R}$ . Therefore, by applying Theorem 12.47 we observe that the pressure  $P_m^\sharp$  has on each point  $\Phi \in \mathcal{W}_1$ , at least, one continuous *tangent linear functional* in  $\mathcal{W}_1^*$ , see Definition 12.43 in Section 12.6.

By a slight abuse of notation, note that the set  $E_1 \subset \mathcal{U}^*$  of t.i. states can be seen as included in  $\mathcal{W}_1^*$ . Indeed, the energy density functional  $e_\Phi$  defines an affine weak\*-homeomorphism  $\rho \mapsto \mathbb{T}(\rho)$  from  $E_1$  to  $\mathcal{W}_1^*$  which is a norm-isometry defined for any  $\rho \in E_1$  by the linear continuous map

$$\Phi \mapsto \mathbb{T}(\rho)(\Phi) := -e_\Phi(\rho)$$

from  $\mathcal{W}_1$  to  $\mathbb{R}$ . For more details, we recommend Section 6.5, in particular Lemma 6.18. For convenience, we ignore the distinction between  $E_1 \subset \mathcal{U}^*$  and  $\mathbb{T}(E_1) \subset \mathcal{W}_1^*$ .

Using this view point, Theorem 3.12 (i) says that the map  $\Phi \mapsto P_m^\sharp(\Phi)$  is the *Legendre-Fenchel transform* of the free-energy density functional  $f_m^\sharp$  extended over the whole space  $\mathcal{W}_1^*$ , i.e.,

$$(3.21) \quad P_m^\sharp(\Phi) := P_{m+(\Phi, 0, 0)}^\sharp = (f_m^\sharp)^*(\Phi),$$

<sup>14</sup>By the definition of interactions,  $\rho_l$  is an even state and hence, products of translates of  $\rho_l$  are well-defined, see [8, Theorem 11.2].

see Definitions 12.28 and 12.40. Of course, the free–energy density functional  $f_{\mathbf{m}}^{\sharp}$  is seen here as a map from  $E_1 \subset \mathcal{W}_1^*$  to  $\mathbb{R}$ . As a consequence, the pressure  $P_{\mathbf{m}}^{\sharp}$  is the Legendre–Fenchel transform  $(f_{\mathbf{m}}^{\sharp})^*(0)$  of  $f_{\mathbf{m}}^{\sharp}$  at  $\Phi = 0$  and it is thus natural to identify the set of all continuous tangent functionals to  $\Phi \mapsto P_{\mathbf{m}}^{\sharp}(\Phi)$  at 0 with a set of t.i. states:

**DEFINITION 3.27** (Set of tangent states to the pressure).

*For  $\beta \in (0, \infty)$  and any  $\mathbf{m} \in \mathcal{M}_1$ , we define  $T_{\mathbf{m}}^{\sharp} \subset E_1$  to be the set of t.i. states which are continuous tangent functionals<sup>15</sup> to the map  $\Phi \mapsto P_{\mathbf{m}}^{\sharp}(\Phi)$  at the point  $0 \in \mathcal{W}_1$ .*

Definitions 3.15 and 3.27 are, a priori, not equivalent to each other. In the special case of purely local interactions  $\Phi$ , i.e., when  $\mathbf{m} = (\Phi, 0, 0)$ , it is already known that

$$(3.22) \quad M_{\Phi} := M_{\mathbf{m}}^{\sharp} = \Omega_{\mathbf{m}}^{\sharp} = T_{\mathbf{m}}^{\sharp} =: T_{\Phi}$$

for translation covariant potentials  $\Phi$ , see Remark 2.26 and [8, Theorem 12.10.].

In fact, upon choosing  $h = f_{\mathbf{m}}^{\sharp}$  and  $K = E_1$  for which  $\Omega(f_{\mathbf{m}}^{\sharp}, E_1) = \Omega_{\mathbf{m}}^{\sharp}$  is convex and weak\*–compact (Lemma 3.16), Corollary 12.48 says that the set  $T_{\mathbf{m}}^{\sharp}$  of all continuous tangent functionals equals the set  $\Omega_{\mathbf{m}}^{\sharp}$  of generalized t.i. equilibrium states. In other words, Definitions 3.15 and 3.27 turn out to be equivalent:

**THEOREM 3.28** (Generalized t.i. equilibrium states as tangent states).

*For all  $\mathbf{m} \in \mathcal{M}_1$ ,  $T_{\mathbf{m}}^{\sharp} = \Omega_{\mathbf{m}}^{\sharp}$ .*

The equivalence of Definitions 3.15 and 3.27 – in the special case of local models  $\mathbf{m} = (\Phi, 0, 0)$  – has been proven, for instance, in [8, Theorem 12.10.] or in [5, Proof of Theorem 6.2.42.] for quantum spin systems by using two results of convex analysis: Mazur theorem [29] and Lanford III – Robinson theorem [23, Theorem 1], see Theorems 12.44 and 12.46. This method is standard, but highly non trivial. In fact, as observed in [30, Theorem I.6.6], the approach of Theorem 12.47, which uses the Legendre–Fenchel transform, is much easier.

Mazur theorem [29] (Theorem 12.44) has an interesting consequence on the instability of coexisting thermodynamic phases. Indeed, thermodynamic phases are identified here with generalized t.i. equilibrium states. From Theorem 12.44 and Remark 12.45 combined with Theorem 3.28, the set of t.i. interactions in  $\mathcal{W}_1$  having exactly one generalized t.i. equilibrium state is dense. Hence, coexistence of thermodynamic phases is unstable in the sense that they can be destroyed by arbitrarily small (w.r.t. to the norm  $\|\cdot\|_{\mathcal{W}_1}$ ) perturbations of the local interaction  $\Phi$  of  $\mathbf{m} \in \mathcal{M}_1$ . This phenomenon is well–

<sup>15</sup>Recall that we identify  $\rho \in E_1$  with  $\mathbb{T}(\rho) \in \mathcal{W}_1^*$ , cf. Lemma 6.18.

known within the case of purely local models, see, e.g., [5, Observation 2, p. 303] for the case of quantum spin systems.

We are now in position to prove that the space-averaged t.i. Gibbs state  $\hat{\rho}_l$  defined by (3.19) always converges in the weak\*-topology to a generalized t.i. equilibrium state:

**THEOREM 3.29** (Weak\*-limit of space-averaged t.i. Gibbs states).

*For any  $\mathfrak{m} \in \mathcal{M}_1$ , the weak\*-accumulation points of the sequence  $\{\hat{\rho}_l\}_{l \in \mathbb{N}}$  of ergodic states  $\hat{\rho}_l \in \mathcal{E}_1$  belong to the set  $\Omega_{\mathfrak{m}}^{\sharp}$  of generalized t.i. equilibrium states.*

*Proof.* Note that  $\hat{\rho}_l \in \mathcal{E}_1$  is an ergodic state, see the proof of Corollary 6.6. Because  $E_1$  is weak\*-compact and metrizable, the t.i. state  $\hat{\rho}_l$  converges in the weak\*-topology – along a subsequence – towards  $\omega \in E_1$ . Therefore, since by Theorem 3.28  $T_{\mathfrak{m}}^{\sharp} = \Omega_{\mathfrak{m}}^{\sharp}$ , we need to prove that  $\omega \in T_{\mathfrak{m}}^{\sharp}$  is a continuous tangent functional to the map  $\Phi \mapsto P_{\mathfrak{m}}^{\sharp}(\Phi)$  (3.20) at the point  $0 \in \mathcal{W}_1$ .

For any t.i. interaction  $\Phi \in \mathcal{W}_1$ , we use Theorem 12.2 (passivity of Gibbs states) to obtain the inequality

$$(3.23) \quad p_{l, \mathfrak{m}+(\Phi, 0, 0)} - p_{l, \mathfrak{m}} \geq -\frac{1}{|\Lambda_l|} \rho_l(U_{\Lambda_l}^{\Phi}).$$

If  $\Phi \in \mathcal{W}_1^f$  is a finite range interaction then Lemma 8.6 tells us that the mean internal energy per volume  $\rho_l(U_{\Lambda_l}^{\Phi})/|\Lambda_l|$  and the energy density  $e_{\Phi}(\hat{\rho}_l)$  converge as  $l \rightarrow \infty$  to the same limit which is  $e_{\Phi}(\omega)$  because of the weak\*-continuity of  $e_{\Phi}$  (Lemma 2.31 (i)). Therefore, by combining (3.23) with Definition 3.11 and Lemma 8.6 one gets that for all  $\Phi \in \mathcal{W}_1^f$  and  $\mathfrak{m} \in \mathcal{M}_1$ ,

$$(3.24) \quad P_{\mathfrak{m}+(\Phi, 0, 0)}^{\sharp} - P_{\mathfrak{m}}^{\sharp} \geq -e_{\Phi}(\omega).$$

By density of the space  $\mathcal{W}_1^f$  in  $\mathcal{W}_1$  together with the continuity of the maps  $\Phi \mapsto e_{\Phi}(\rho)$  (Lemma 2.31 (ii)) and  $\Phi \mapsto P_{\mathfrak{m}}^{\sharp}(\Phi)$  (cf. (3.20) and Theorem 3.12 (ii)), we extend the inequality (3.24) to all t.i. interactions  $\Phi \in \mathcal{W}_1$ , which means that  $\omega \in T_{\mathfrak{m}}^{\sharp} = \Omega_{\mathfrak{m}}^{\sharp}$  (Theorem 3.28). ■

A sufficient condition to obtain the weak\*-convergence of the Gibbs equilibrium state  $\rho_l$  is to have a permutation invariant model, see Section 7, in particular Definition 7.6 and Corollary 7.9. In fact, the convergence or non-convergence of the Gibbs equilibrium state  $\rho_l$  drastically depends on the boundary conditions on the box  $\Lambda_l$  which can break the translation invariance of the infinite volume system. If periodic boundary conditions (see Section 5) are imposed, i.e., the internal energy  $\tilde{U}_l$  (Definition 5.7) is defined to be translation invariant on the torus  $\Lambda_l$ , then the Gibbs equilibrium state  $\tilde{\rho}_l := \rho_{\Lambda_l, \tilde{U}_l}$  (12.2) with periodic boundary conditions and its space-average  $\hat{\rho}_l$  have the same weak\*-limit point and  $\tilde{\rho}_l$  converges in the weak\*-topology to a generalized t.i. equilibrium state  $\omega \in \Omega_{\mathfrak{m}}^{\sharp}$ , see Theorem 5.13.

We conclude now by another interesting consequence – already observed by Israel [4, Theorem V.2.2.] for quantum spin systems with purely local interactions – of Theorem 3.28. Indeed, we deduce from Theorem 3.28 that any finite set of extreme t.i. states can be seen as a subset of  $\Omega_{\mathfrak{m}}^{\sharp}$  for some model  $\mathfrak{m}$ :

**COROLLARY 3.30** (Generalized t.i. equilibrium ergodic states).

*Let  $\mathfrak{m} \in \mathcal{M}_1$  such that  $\Omega_{\mathfrak{m}+(\Phi,0,0)}^{\sharp}$  is a face for all  $\Phi \in \mathcal{W}_1$ . Then, for any subset  $\{\hat{\omega}_1, \dots, \hat{\omega}_n\}$  of  $\mathcal{E}_1$ , there is  $\Phi \in \mathcal{W}_1$  such that  $\{\hat{\omega}_1, \dots, \hat{\omega}_n\} \subset \Omega_{\mathfrak{m}+(\Phi,0,0)}^{\sharp}$ .*

*Proof.* The corollary follows from Bishop–Phelps’ theorem together with Theorem 2.9. The arguments are exactly those of Israel. Therefore, for more details, we recommend [4, Theorem V.2.2.]. ■

Note that the assumption of Corollary 3.30 is satisfied, for instance, if the long range part of the model  $\mathfrak{m} \in \mathcal{M}_1$  is purely attractive, i.e.,  $\Phi_{a,+} = 0$  (a.e.), see Theorem 3.25 (–).

### 3.6. Thermodynamics and game theory

Effects of the long range attractions  $\Phi_{a,-}, \Phi'_{a,-}$  and repulsions  $\Phi_{a,+}, \Phi'_{a,+}$  defined in Definition 3.4 are *not symmetric* w.r.t. thermodynamics as everything depends on variational problems given by infima, see Theorem 3.12 (i). For instance, the long range attractions  $\Phi_{a,-}$  and  $\Phi'_{a,-}$  only reinforce the weak\*–lower semi–continuity of the free–energy density functional  $f_{\mathfrak{m}}^{\sharp}$ . In particular, if  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.) then  $\Omega_{\mathfrak{m}}^{\sharp}$  is, as for models  $(\Phi, 0, 0) \in \mathcal{M}_1$ , a (non–empty) closed face of  $E_1$ , see Theorem 3.25 (–). By contrast, the long range repulsions  $\Phi_{a,+}$  and  $\Phi'_{a,+}$  have a stronger effect. Indeed,  $\Phi_{a,+}$  and  $\Phi'_{a,+}$  generally break the weak\*–lower semi–continuity of the functional  $f_{\mathfrak{m}}^{\sharp}$  on  $E_1$  which, by elementary arguments, yields, in general, to a non–affine functional  $\Gamma_{E_1}(f_{\mathfrak{m}}^{\sharp})$ . As a consequence,  $\Omega_{\mathfrak{m}}^{\sharp}$  is generally not anymore a closed face of  $E_1$ , see Theorem 3.25 (+) and Lemma 11.8 in Section 11.2.

To understand this in more details, we use the view point of game theory and interpret in Definition 3.35 the long range attractions  $\Phi_{a,-}, \Phi'_{a,-}$  and repulsions  $\Phi_{a,+}, \Phi'_{a,+}$  of any model  $\mathfrak{m} \in \mathcal{M}_1$  as attractive and repulsive players, respectively. This approach is strongly related with the validity of the so–called Bogoliubov approximation. In the context of the analysis of the thermodynamic pressure of models  $\mathfrak{m} \in \mathcal{M}_1^{\text{df}} \subset \mathcal{M}_1$  (cf. Section 3.1) with discrete long range part, it is known as the *approximating Hamiltonian method* [13, 14, 15, 16], see Sections 4.2 and 12.2. Beside our interpretation of thermodynamics in terms of game theory, this method gives a natural way to compute, from local interactions, the variational problems given in Theorem 3.12 (i) for the pressure  $P_{\mathfrak{m}}^{\sharp}$ .



We show below that the pressure  $P_m^\sharp$  can be studied for any models

$$\mathbf{m} := (\Phi, \{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}) \in \mathcal{M}_1$$

via a (Bogoliubov) min–max variational problem on the Hilbert space  $L^2(\mathcal{A}, \mathbb{C})$  of square integrable functions, which is interpreted as the result of a two–person zero–sum game. Our proof establishes, moreover, a clear link between the Bogoliubov min–max principle for the pressure of long range models and von Neumann min–max theorem. Functions  $c_a \in L^2(\mathcal{A}, \mathbb{C})$  are related to *approximating interactions* defined as follows:

DEFINITION 3.31 (Approximating interactions).

*Approximating interactions of any model  $\mathbf{m} \in \mathcal{M}_1$  are t.i. interactions defined, for each  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , by*

$$\Phi(c_a) = \Phi_{\mathbf{m}}(c_a) := \Phi + 2 \operatorname{Re} \left\{ \langle \Phi_a + i\Phi'_a, \gamma_a c_a \rangle \right\} \in \mathcal{W}_1$$

with  $\langle \cdot, \cdot \rangle$  being the scalar product constructed in Section 12.3 for  $\mathcal{X} = \mathcal{W}_1$  and  $\gamma_a \in \{-1, 1\}$  a fixed measurable function.

Then, by Definition 2.22, the internal energy  $U_{\Lambda_l}^{\Phi(c_a)}$  associated with the t.i. interaction  $\Phi(c_a)$  equals

$$(3.25) \quad U_l(c_a) := U_{\Lambda_l}^{\Phi} + \int_{\mathcal{A}} \gamma_a \left\{ c_a (U_{\Lambda}^{\Phi_a} + iU_{\Lambda}^{\Phi'_a})^* + \bar{c}_a (U_{\Lambda}^{\Phi_a} + iU_{\Lambda}^{\Phi'_a}) \right\} d\mathbf{a}(a).$$

In particular, for any generalized t.i. equilibrium state  $\omega \in \Omega_{\mathbf{m}}^\sharp$  and any  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ ,

$$(3.26) \quad \begin{aligned} & |\Lambda_l|^{-1} \omega(U_l - U_l(c_a)) + \|c_{a,+}\|_2^2 - \|c_{a,-}\|_2^2 \\ &= \int_{\mathcal{A}} \gamma_a \left| \left( |\Lambda_l|^{-1} \omega(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}) - c_a \right) \right|^2 d\mathbf{a}(a) \end{aligned}$$

with  $c_{a,\pm} := \gamma_{a,\pm} c_a$ , where  $\gamma_{a,\pm} \in \{0, 1\}$  are the negative and positive parts (3.1) of the fixed measurable function  $\gamma_a$ . Upon choosing

$$(3.27) \quad c_a = d_a := |\Lambda_l|^{-1} \omega(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}) + o(1) \quad (\text{a.e.})$$

we observe that the energy densities

$$|\Lambda_l|^{-1} \omega(U_l) \quad \text{and} \quad |\Lambda_l|^{-1} \omega(U_l(d_a))$$

only differ in the thermodynamic limit  $l \rightarrow \infty$  by the explicit constant

$$(\|d_{a,-}\|_2^2 - \|d_{a,+}\|_2^2).$$

In particular, by using the Bogoliubov (convexity) inequality [31, Corollary D.4], we can expect that the approximating interaction  $\Phi(d_a) \in \mathcal{W}_1$  highlights the thermodynamic properties of models  $\mathbf{m} \in \mathcal{M}_1$ .

REMARK 3.32. *Even if the order parameter  $d_a \in L^2(\mathcal{A}, \mathbb{C})$  is shown to be generally not unique, these heuristic arguments are confirmed by Theorem 3.36 on the level of pressure, and by Theorems 3.39 on the level of states.*

Therefore, in order to understand the variational problems on the set  $E_1$  given by Theorem 3.12 (i) and more particularly the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states (Definition 3.15), we introduce the concept of *approximating free-energy density functionals* whose definition needs some preliminaries.

First, for any  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , the finite volume pressure

$$(3.28) \quad p_l(c_a) := \frac{1}{\beta|\Lambda_l|} \ln \text{Trace}_{\wedge \mathcal{H}_{\Lambda}}(e^{-\beta U_l(c_a)})$$

associated with the internal energy  $U_l(c_a)$  (3.25) converges as  $l \rightarrow \infty$  to a well-defined (infinite volume) pressure

$$(3.29) \quad P_m(c_a) = - \inf_{\rho \in E_1} f_m(\rho, c_a)$$

given by a variational problem over t.i. states, see Theorem 3.12 (i) or Proposition 9.1 in Section 9.1. In comparison with the pressure  $P_m^\sharp$  for all  $m \in \mathcal{M}_1$ ,  $P_m(c_a)$  is, in practice, easier to compute because it is associated with the (purely local) approximating interaction  $\Phi(c_a)$  (Definition 3.31). Indeed,  $P_m(c_a)$  is the pressure  $P_{(\Phi(c_a), 0, 0)}$  and the free-energy density functional  $f_{\Phi(c_a)}$  (see Definition 2.32) is equal in this case to

$$(3.30) \quad f_m(\rho, c_a) := 2 \text{Re} \left\{ \langle e_{\Phi_a}(\rho) + i e_{\Phi'_a}(\rho), \gamma_a c_a \rangle \right\} + e_{\Phi}(\rho) - \beta^{-1} s(\rho)$$

for all  $c_a \in L^2(\mathcal{A}, \mathbb{C})$  and  $\rho \in E_1$ .

From Lemmata 2.28 (i) and 2.31 (i), the map  $\rho \mapsto f_m(\rho, c_a)$  from  $E_1$  to  $\mathbb{R}$  is weak\*-lower semi-continuous and affine. This implies that the variational problem (3.29) leading to the pressure  $P_m(c_a)$  has a closed face of minimizers (cf. Definition 3.13):

LEMMA 3.33 (Equilibrium states of approximating interactions).

*For any  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , the set  $M_{\Phi(c_a)} = \Omega_{\Phi(c_a)}$  of t.i. equilibrium states of the approximating interaction  $\Phi(c_a)$  is a (non-empty) closed face of the Poulsen simplex  $E_1$ .*

For more details concerning the map  $(\rho, c_a) \mapsto f_m(\rho, c_a)$ , see Proposition 9.1 in Section 9.1.

Second, we recall again that the thermodynamics of any model  $m \in \mathcal{M}_1$  drastically depends on the sign of the coupling constant

$$\gamma_a = \gamma_{a,+} - \gamma_{a,-} \in \{-1, 1\}, \quad \text{where } \gamma_{a,\pm} := 1/2(|\gamma_a| \pm \gamma_a),$$

see also (3.1). Thus, we define two Hilbert spaces corresponding respectively to the long range repulsions  $\Phi_{a,+}$ ,  $\Phi'_{a,+}$  and attractions  $\Phi_{a,-}$ ,  $\Phi'_{a,-}$  of any model

$\mathfrak{m} \in \mathcal{M}_1$ :

$$(3.31) \quad L_{\pm}^2(\mathcal{A}, \mathbb{C}) := \{c_{a,\pm} \in L^2(\mathcal{A}, \mathbb{C}) : c_{a,\pm} = \gamma_{a,\pm} c_{a,\pm}\}.$$

Note that we obviously have the equality

$$L^2(\mathcal{A}, \mathbb{C}) = L_+^2(\mathcal{A}, \mathbb{C}) \oplus L_-^2(\mathcal{A}, \mathbb{C}).$$

Then we define the approximating free–energy density functional  $\mathfrak{f}_{\mathfrak{m}}$  as follows:

DEFINITION 3.34 (Approximating free–energy density functional).

The approximating free–energy density functional is the map

$$\mathfrak{f}_{\mathfrak{m}} : L_-^2(\mathcal{A}, \mathbb{C}) \times L_+^2(\mathcal{A}, \mathbb{C}) \rightarrow \mathbb{R}$$

defined for any  $c_{a,\pm} \in L_{\pm}^2(\mathcal{A}, \mathbb{C})$  by

$$\mathfrak{f}_{\mathfrak{m}}(c_{a,-}, c_{a,+}) := -\|c_{a,+}\|_2^2 + \|c_{a,-}\|_2^2 - P_{\mathfrak{m}}(c_{a,-} + c_{a,+}).$$

This functional is analyzed in Lemma 10.1 and is used to define the (two–person zero–sum) *thermodynamic game* with the so–called *conservative values*  $F_{\mathfrak{m}}^{\flat}$  and  $F_{\mathfrak{m}}^{\sharp}$ :

DEFINITION 3.35 (Thermodynamic game).

The thermodynamic game is the two–person zero–sum game defined from the functional  $\mathfrak{f}_{\mathfrak{m}}$  with conservative values

$$F_{\mathfrak{m}}^{\flat} := \sup_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})} \mathfrak{f}_{\mathfrak{m}}^{\flat}(c_{a,+}) \quad \text{and} \quad F_{\mathfrak{m}}^{\sharp} := \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \mathfrak{f}_{\mathfrak{m}}^{\sharp}(c_{a,-}),$$

where

$$\mathfrak{f}_{\mathfrak{m}}^{\flat}(c_{a,+}) := \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \mathfrak{f}_{\mathfrak{m}}(c_{a,-}, c_{a,+}), \quad \mathfrak{f}_{\mathfrak{m}}^{\sharp}(c_{a,-}) := \sup_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})} \mathfrak{f}_{\mathfrak{m}}(c_{a,-}, c_{a,+}).$$

Any function  $c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})$  (resp.  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ ) is interpreted as a *strategy* of the *repulsive* (resp. *attractive*) player.  $\mathfrak{f}_{\mathfrak{m}}^{\flat}$  is the *least gain* functional of the attractive player, whereas  $\mathfrak{f}_{\mathfrak{m}}^{\sharp}$  is called the *worst loss* functional of the repulsive player. Minimizers (resp. maximizers), if there are any, of  $\mathfrak{f}_{\mathfrak{m}}^{\sharp}$  (resp.  $\mathfrak{f}_{\mathfrak{m}}^{\flat}$ ) are the *conservative strategies* of the attractive (resp. repulsive) player. For more details concerning two–person zero–sum games, see Section 12.7.

In Section 10.1, we prove that both optimization problems  $F_{\mathfrak{m}}^{\flat}$  and  $F_{\mathfrak{m}}^{\sharp}$  are finite and the two optimizations of  $\mathfrak{f}_{\mathfrak{m}}(c_{a,-}, c_{a,+})$  can be restricted to balls in  $L_{\pm}^2(\mathcal{A}, \mathbb{C})$  of radius  $R < \infty$ , see Lemma 10.4. Moreover, the sup and inf, both in  $F_{\mathfrak{m}}^{\flat}$  and  $F_{\mathfrak{m}}^{\sharp}$ , are attained, i.e., they are respectively a max and a min and the sets

$$(3.32) \quad \begin{aligned} \mathcal{C}_{\mathfrak{m}}^{\flat} &:= \{d_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C}) : F_{\mathfrak{m}}^{\flat} = \mathfrak{f}_{\mathfrak{m}}^{\flat}(d_{a,+})\}, \\ \mathcal{C}_{\mathfrak{m}}^{\sharp} &:= \{d_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C}) : F_{\mathfrak{m}}^{\sharp} = \mathfrak{f}_{\mathfrak{m}}^{\sharp}(d_{a,-})\} \end{aligned}$$

of conservative strategies of the repulsive and attractive players, respectively, are non-empty. In fact, by Lemma 10.4, the set  $\mathcal{C}_m^b$  has exactly one element  $d_{a,+}$  if  $\gamma_{a,+} \neq 0$  (a.e.), whereas  $\mathcal{C}_m^\sharp$  is non-empty, norm bounded, and weakly compact.

The conservative values  $F_m^b$  and  $F_m^\sharp$  of the thermodynamic game turn out to be extremely useful to understand the thermodynamics of models  $m \in \mathcal{M}_1$  as they have a direct interpretation in terms of variational problems over the set  $E_1$ . Indeed, we prove in Section 10.2 (cf. Lemmata 10.5 (i) and 10.7) the following theorem:

**THEOREM 3.36** (Thermodynamics as a two-person zero-sum game).

(b)  $P_m^b = -F_m^b$  with the pressure  $P_m^b$  defined, for  $m \in \mathcal{M}_1$ , by the minimization of the functional  $f_m^b$  over  $E_1$ , see (3.14).

( $\sharp$ )  $P_m^\sharp = -F_m^\sharp$  with the pressure  $P_m^\sharp$  given, for  $m \in \mathcal{M}_1$ , by the minimization of the functional  $f_m^\sharp$  over  $E_1$ , see Definition 3.11 and Theorem 3.12 (i).

The proof of this theorem uses neither Ginibre inequalities [11, Eq. (2.10)] nor the Bogoliubov (convexity) inequality [31, Corollary D.4] w.r.t.  $U_l$  and  $U_l(c_a)$  (3.25). In particular, we *never use* Equality (3.26). Consequently, the proof given in this paper is essentially different from those of [13, 14, 15, 16]. Additionally, the equality  $P_m^b = -F_m^b$  is a new result and we do not need additional assumptions as in [13, 14, 15, 16] when  $\gamma_{a,+} \neq 0$  (a.e.), see Condition (A4) and Theorem 12.3 in Section 12.2. Our proof uses, instead, Theorem 3.12 (i) together with a fine analysis of the corresponding variational problems over the set  $E_1$ .

It follows from Theorem 3.36 that  $P_m := P_m^\sharp = P_m^b$  whenever either  $\Phi_{a,-} = 0$  (a.e.) or  $\Phi_{a,+} = 0$  (a.e.), as explained in Theorem 3.25. However, in the general case, one only has  $F_m^b \leq F_m^\sharp$ , i.e.,  $P_m^b \geq P_m^\sharp$ , see, e.g., (3.13). In fact, generally,  $P_m^b > P_m^\sharp$ , i.e.,  $F_m^b < F_m^\sharp$ . This fact is, indeed, not surprising as a sup and a inf do not generally commute.

As an example, take  $A = A^* \in \mathcal{U}_0$  and two ergodic states  $\omega_1, \omega_2 \in \mathcal{E}_1$  such that  $\omega_1(A) \neq \omega_2(A)$ . From Corollary 3.30, there is  $\Phi \in \mathcal{W}_1$  such that the t.i. states  $\omega_1$  and  $\omega_2$  belong to the closed face  $\Omega_\Phi = M_\Phi$  of t.i. equilibrium states of the (local) model  $(\Phi, 0, 0) \in \mathcal{M}_1$ . In other words, for any  $\lambda \in [0, 1]$ , the convex sum

$$\lambda\omega_1(A) + (1 - \lambda)\omega_2(A)$$

is a minimizer of the free-energy density functional  $f_\Phi$  defined in Definition 2.32. Consequently, by using (6.17) (see Section 6.3) we obtain that

$$\begin{aligned} \inf_{\rho \in E_1} f_\Phi(\rho) &= \inf_{\rho \in E_1} \{ \Delta_A(\rho) - \Delta_A(\rho) + f_\Phi(\rho) \} \\ &> \inf_{\rho \in E_1} \left\{ |\rho(A)|_2^2 - \Delta_A(\rho) + f_\Phi(\rho) \right\}. \end{aligned}$$

Combined with Theorem 3.36 this strict inequality gives a trivial example where  $P_m^b > P_m^\sharp$ , i.e.,  $F_m^b < F_m^\sharp$ , because, for any  $A = A^* \in \mathcal{U}_0$ , there exists a finite range interaction  $\Phi^A \in \mathcal{W}_1$  satisfying  $\|\Phi^A\|_{\mathcal{W}_1} = \|A\|$  and  $e_{\Phi^A}(\rho) = \rho(\mathbf{e}_{\Phi^A}) = \rho(A)$ . Other less trivial examples can also be found by directly showing that  $F_m^b < F_m^\sharp$ . Use, for instance, the strong coupling BCS–Hubbard Hamiltonian described in [9]. Therefore, in general, there is no *saddle points* (Definition 12.49) in the thermodynamic game defined in Definition 3.35.

The non–existence of saddle points in the thermodynamic game is an *important* observation. It reflects the fact that repulsive and attractive long range forces  $\Phi_{a,\pm}, \Phi'_{a,\pm}$  (Definition 3.4) are not in “duality” in which concerns thermodynamics properties of a given long range model  $\mathbf{m} \in \mathcal{W}_1$ . Indeed, the long range attractions  $\Phi_{a,-}, \Phi'_{a,-}$  and repulsions  $\Phi_{a,+}, \Phi'_{a,+}$  act on the thermodynamics of  $\mathbf{m} \in \mathcal{M}_1$  as the attractive and repulsive players, respectively. Since the result of the thermodynamic game is the conservative value  $F_m^\sharp = -P_m^\sharp$ , the attractive player minimizes the functional  $f_m^\sharp(c_{a,-})$ , i.e., he optimizes his worse loss  $f_m(c_{a,-}, c_{a,+})$  without knowing the choice  $d_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})$  of the repulsive player. By contrast, the repulsive player determines his strategy after having full information on the choice of the attractive player. In other words, as in general  $F_m^b < F_m^\sharp$ , there is a strong asymmetry between both players, i.e., between the role of the two kinds of long range interactions  $\Phi_{a,-}, \Phi'_{a,-}$  and  $\Phi_{a,+}, \Phi'_{a,+}$ .

The thermodynamic game of any given long range model  $\mathbf{m}$  can be extended [33, Ch. 7, Section 7.2] to another two–person zero–sum game with *exchange of information* which has the advantage to have, at least, one *non–cooperative equilibrium*, also called saddle point in this context. This can be seen as follows.

First, it is instructive to analyze the variational problems respectively given by  $f_m^b(c_{a,+})$  and  $f_m^\sharp(c_{a,-})$  at fixed  $c_{a,\pm} \in L_\pm^2(\mathcal{A}, \mathbb{C})$ . So, we introduced their sets

$$(3.33) \quad \begin{aligned} \mathcal{C}_m^b(c_{a,+}) &:= \left\{ d_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C}) : f_m^b(c_{a,+}) = f_m(d_{a,-}, c_{a,+}) \right\}, \\ \mathcal{C}_m^\sharp(c_{a,-}) &:= \left\{ d_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C}) : f_m^\sharp(c_{a,-}) = f_m(c_{a,-}, d_{a,+}) \right\} \end{aligned}$$

of, respectively, minimizers and maximizers for any  $c_{a,\pm} \in L_\pm^2(\mathcal{A}, \mathbb{C})$ . We prove in Lemma 10.3 that, for all  $c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})$ , the set  $\mathcal{C}_m^b(c_{a,+})$  is non–empty, norm bounded, and weakly compact, whereas, for all  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ , the set  $\mathcal{C}_m^\sharp(c_{a,-})$  has exactly one element  $r_+(c_{a,-})$  provided that  $\gamma_{a,\pm} \neq 0$  (a.e.). Therefore, we would like to use Theorem 12.51 to extend the strategy set  $L_+^2(\mathcal{A}, \mathbb{C})$  of the thermodynamic game to the set  $C(L_-^2, L_+^2)$  of continuous mappings from  $L_-^2(\mathcal{A}, \mathbb{C})$  to  $L_+^2(\mathcal{A}, \mathbb{C})$  with  $L_-^2(\mathcal{A}, \mathbb{C})$  and  $L_+^2(\mathcal{A}, \mathbb{C})$  equipped with the weak and norm topologies, respectively.

In this context  $C(L_-^2, L_+^2)$  is called the set of *continuous decision rules* of the repulsive player. If  $\gamma_{a,\pm} \neq 0$  (a.e.) then an important continuous decision rule is given by the unique solution  $r_+(c_{a,-})$  of the variational problem  $f_m^\sharp(c_{a,-})$ , see Lemma 10.3 ( $\sharp$ ). Indeed, the map  $r_+$  from  $L_-^2(\mathcal{A}, \mathbb{C})$  to  $L_+^2(\mathcal{A}, \mathbb{C})$  defined by

$$(3.34) \quad r_+ : c_{a,-} \mapsto r_+(c_{a,-}) \in \mathcal{C}_m^\sharp(c_{a,-})$$

belongs to  $C(L_-^2, L_+^2)$  because of Lemma 10.8. The functional  $r_+$  is called the *thermodynamic decision rule* of the model  $\mathbf{m} \in \mathcal{M}_1$ .

We define now, for any long range model  $\mathbf{m} \in \mathcal{M}_1$ , a map  $f_m^{\text{ext}}$  from  $L_-^2(\mathcal{A}, \mathbb{C})$  to  $C(L_-^2, L_+^2)$  by

$$f_m^{\text{ext}}(c_{a,-}, \tilde{r}_+) := f_m(c_{a,-}, \tilde{r}_+(c_{a,-}))$$

for all  $\tilde{r}_+ \in C(L_-^2, L_+^2)$ . This functional is called the *loss-gain* function of the *extended thermodynamic game* of the model  $\mathbf{m}$ . In contrast to the thermodynamic game defined in Definition 3.35, this extended game has the main advantage to have, at least, one *non-cooperative equilibrium*:

**THEOREM 3.37** (Non-cooperative equilibrium of the extended game).

*Let  $\gamma_{a,+} \neq 0$  (a.e.). Then any  $d_{a,-} \in \mathcal{C}_m^\sharp$  and the map  $r_+ \in C(L_-^2, L_+^2)$  defined by (3.34) form a saddle point of the extended thermodynamic game defined by*

$$\begin{aligned} F_m^\sharp &= \sup_{\tilde{r}_+ \in C(L_-^2, L_+^2)} \left\{ \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} f_m^{\text{ext}}(c_{a,-}, \tilde{r}_+) \right\} \\ &= \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \left\{ \sup_{\tilde{r}_+ \in C(L_-^2, L_+^2)} f_m^{\text{ext}}(c_{a,-}, \tilde{r}_+) \right\}. \end{aligned}$$

*Proof.* The map  $r_+$  is well-defined because of Lemma 10.3 ( $\sharp$ ) and, by Lemma 10.8, it is continuous w.r.t. the weak topology in  $L_-^2(\mathcal{A}, \mathbb{C})$  and the norm topology in  $L_+^2(\mathcal{A}, \mathbb{C})$ , i.e.,  $r_+ \in C(L_-^2, L_+^2)$ .

By Lemma 10.4 ( $\sharp$ ), the non-empty set  $\mathcal{C}_m^\sharp \subset L_-^2(\mathcal{A}, \mathbb{C})$  of conservative strategies of the attractive player (cf. (3.32)) is norm bounded and weakly compact, whereas, by Lemma 10.3 ( $\sharp$ ), the set  $\mathcal{C}_m^\sharp(c_{a,-})$  (cf. (3.33)) has exactly one element  $r_+(c_{a,-})$  at any fixed  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ . As a consequence, the infimum and supremum of  $F_m^\sharp < \infty$  can be restricted to balls  $\mathcal{B}_R(0)$  in  $L_\pm^2(\mathcal{A}, \mathbb{C})$  of radius  $R < \infty$ . Therefore, by using Lemma 10.1, we can apply Theorem 12.51 to get

$$(3.35) \quad F_m^\sharp = \sup_{\tilde{r}_+ \in C(L_-^2, L_+^2)} \left\{ \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} f_m^{\text{ext}}(c_{a,-}, \tilde{r}_+) \right\}.$$

The inf and sup in the r.h.s. of the last equality trivially commute, i.e.,

$$(3.36) \quad F_m^\sharp = \inf_{c_{a,-} \in L^2(\mathcal{A}, \mathbb{C})} \left\{ \sup_{\tilde{r}_+ \in C(L_-^2, L_+^2)} f_m^{\text{ext}}(c_{a,-}, \tilde{r}_+) \right\},$$

because

$$\sup_{\tilde{r}_+ \in C(L_-^2, L_+^2)} f_m^{\text{ext}}(c_{a,-}, \tilde{r}_+) = f_m^{\text{ext}}(c_{a,-}, r_+) = \sup_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})} f_m(c_{a,-}, c_{a,+}).$$

In particular, for any  $d_{a,-} \in \mathcal{C}_m^\sharp$ ,

$$F_m^\sharp = f_m^{\text{ext}}(d_{a,-}, r_+) = \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} f_m^{\text{ext}}(c_{a,-}, r_+) = \sup_{\tilde{r}_+ \in C(L_-^2, L_+^2)} f_m^{\text{ext}}(d_{a,-}, \tilde{r}_+)$$

which combined with (3.35)–(3.36) implies that  $(d_{a,-}, r_+)$  is a saddle point of  $f_m^{\text{ext}}$ . ■

**REMARK 3.38** (Thermodynamics as a three–person zero–sum game).

Since the pressure  $P_m(c_a)$  in Definition 3.34 of the approximating free–energy density  $f_m$  equals the variational problem (3.29) over t.i. states, we could also see the equality  $F_m^\sharp = -P_m^\sharp$  of Theorem 3.36 as the result of a three–person zero–sum game. By (10.7) and (10.8), note that the infimum over t.i. states and the supremum over  $L_+^2(\mathcal{A}, \mathbb{C})$  commute with each other, see the proof of Lemma 10.5 for more details.

### 3.7. Gap equations and effective theories

The structure of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states (Definition 3.15) w.r.t. to the thermodynamic game can be now discussed in details. It is based on Section 11.1 which gives a rigorous justification, on the level of generalized t.i. equilibrium states, of the heuristics discussed in the beginning of Section 3.6. In particular, we prove that Equality (3.27) must be satisfied in the thermodynamic limit for any extreme point of  $\Omega_m^\sharp$ .

More precisely, for all functions  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , we define the (possibly empty) set

$$(3.37) \quad \Omega_m^\sharp(c_a) := \{\omega \in M_{\Phi(c_a)} : \gamma_a(e_{\Phi_a}(\omega) + ie_{\Phi_a'}(\omega)) = c_a \text{ (a.e.)}\}$$

with  $M_{\Phi(c_a)}$  being the closed face described in Lemma 3.33. Then we obtain *Euler–Lagrange equations* for the approximating interactions (cf. Remark 3.42) – also called *gap equations* in the Physics literature (cf. Remark 3.43) – which say that any extreme point of  $\Omega_m^\sharp$  must belong to a set

$$(3.38) \quad \Omega_m^\sharp(d_{a,-} + r_+(d_{a,-}))$$

with  $d_{a,-} \in \mathcal{C}_m^\sharp$ ,  $r_+ \in C(L_-^2, L_+^2)$  defined by (3.34), and where  $\mathcal{C}_m^\sharp$  is the non–empty, norm bounded, and weakly compact set defined by (3.32), see Lemma 10.4 (♯). Indeed, we obtain the following statements:

THEOREM 3.39 (Gap equations for  $\mathbf{m} \in \mathcal{M}_1 - \mathbf{I}$ ).

(i) The set  $\hat{M}_{\mathbf{m}}$  (3.10) of minimizers of the functional  $g_{\mathbf{m}}$  over  $E_1$  equals

$$\hat{M}_{\mathbf{m}} = \bigcup_{d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}} \Omega_{\mathbf{m}}^{\sharp}(d_{a,-} + r_+(d_{a,-})).$$

(ii) The set  $\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})$  of extreme points of  $\Omega_{\mathbf{m}}^{\sharp}$  is included in the union for all  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  of the sets of all extreme points of the non-empty, disjoint, convex and weak\*-compact sets (3.38), i.e.,

$$\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}) \subseteq \bigcup_{d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}} \mathcal{E}\left(\Omega_{\mathbf{m}}^{\sharp}(d_{a,-} + r_+(d_{a,-}))\right)$$

*Proof.* The first assertion (i) corresponds to Theorem 11.4, see Section 11.1. By Corollary 11.3, we also observe that

$$\left\{ \Omega_{\mathbf{m}}^{\sharp}(d_{a,-} + r_+(d_{a,-})) \right\}_{d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}}$$

is a family of disjoint subsets of  $E_1$  which are all non-empty, convex, and weak\*-compact. Using (i) and Theorem 3.21 (ii) we arrive at the second assertion (ii) with the set

$$\mathcal{E}\left(\Omega_{\mathbf{m}}^{\sharp}(d_{a,-} + r_+(d_{a,-}))\right) \neq \emptyset$$

of all extreme points of (3.38) being non-empty for any  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  because of Theorem 12.11 (i). ■

REMARK 3.40 (The set  $\hat{M}_{\mathbf{m}}$  for purely repulsive/attractive models).

If  $\Phi_{a,-} = 0$  (a.e.) and  $\Phi_{a,+} \neq 0$  (a.e.) then Theorem 3.39 reads as follows:  $\hat{M}_{\mathbf{m}} = \Omega_{\mathbf{m}}^{\sharp}(d_{a,+})$  with  $d_{a,+} \in \mathcal{C}_{\mathbf{m}}^{\flat}$  defined by (3.32), see Lemma 11.2. In particular,  $\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}) = \mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}(d_{a,+}))$ . If  $\Phi_{a,+} = 0$  (a.e.) then  $\Omega_{\mathbf{m}}^{\sharp} = M_{\mathbf{m}} = \overline{\text{co}(\hat{M}_{\mathbf{m}})}$  is a closed face. In particular,

$$\mathcal{E}(\Omega_{\mathbf{m}}^{\sharp}) = \bigcup_{d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}} \mathcal{E}\left(\Omega_{\mathbf{m}}^{\sharp}(d_{a,-} + r_+(d_{a,-}))\right).$$

Theorem 3.39 is less useful in this last situation.

Theorem 3.39 (ii) implies that, for any  $\hat{\omega} \in \mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})$ , there is  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  satisfying the Euler–Lagrange equations (cf. Remark 3.42) – or gap equations in Physics (cf. Remark 3.43) –

$$(3.39) \quad d_a := d_{a,-} + r_+(d_{a,-}) = \gamma_a(e_{\Phi_a}(\omega) + ie_{\Phi'_a}(\omega)) \text{ (a.e.)}.$$

Conversely, for any  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$ , there is some  $\omega \in \hat{M}_{\mathbf{m}}$  satisfying the Euler–Lagrange equations but  $\omega$  is *not necessarily* an extreme point of  $\Omega_{\mathbf{m}}^{\sharp}$ . Observe, however, that if  $\omega \notin \mathcal{E}(\Omega_{\mathbf{m}}^{\sharp})$  then we have a strong constraint on the set  $\mathcal{C}_{\mathbf{m}}^{\sharp}$ :



**THEOREM 3.41** (Gap equations for  $\mathfrak{m} \in \mathcal{M}_1 - \text{II}$ ).

For any  $d_{a,-} \in \mathcal{C}_m^\sharp$  such that there exists  $\omega \in \mathcal{E}(\Omega_m^\sharp(d_{a,-} + r_+(d_{a,-})))$  satisfying  $\omega \notin \mathcal{E}(\Omega_m^\sharp)$ , there is a probability measure  $\nu_{d_{a,-}}$  on  $\mathcal{C}_m^\sharp$  not concentrated on  $d_{a,-}$  such that (a.e.)

$$d_{a,-} = \int_{\mathcal{C}_m^\sharp} \hat{d}_{a,-} \, d\nu_{d_{a,-}}(\hat{d}_{a,-}) \quad \text{and} \quad r_+(d_{a,-}) = \int_{\mathcal{C}_m^\sharp} r_+(\hat{d}_{a,-}) \, d\nu_{d_{a,-}}(\hat{d}_{a,-}).$$

*Proof.* If

$$\omega \in \mathcal{E}(\Omega_m^\sharp(d_{a,-} + r_+(d_{a,-}))) \subset \hat{M}_m \subset \Omega_m^\sharp$$

and  $\omega \notin \mathcal{E}(\Omega_m^\sharp)$  then, by Theorem 3.21 (iii), there is a probability measure  $v_\omega$  on  $\Omega_m^\sharp$  not concentrated on the convex weak\*-compact set  $\Omega_m^\sharp(d_{a,-} + r_+(d_{a,-}))$  such that

$$(3.40) \quad v_\omega(\mathcal{E}(\Omega_m^\sharp)) = 1 \quad \text{and} \quad \omega = \int_{\mathcal{E}(\Omega_m^\sharp)} dv_\omega(\hat{\omega}) \hat{\omega}.$$

Recall that  $e_\Phi$  is affine and weak\*-continuous (Lemma 2.31 (i)) and applying (3.40) on the energy observable  $\epsilon_{\Phi_a} + i\epsilon_{\Phi'_a}$  (cf. (2.16)) we obtain that

$$(3.41) \quad d_{a,-} + r_+(d_{a,-}) = \int_{\mathcal{E}(\Omega_m^\sharp)} dv_\omega(\hat{\omega}) \gamma_a(e_{\Phi_a}(\hat{\omega}) + ie_{\Phi'_a}(\hat{\omega})) \quad (\text{a.e.})$$

because of Lemma 12.17. Hence, the theorem results from (3.39) and (3.41).

■

Because of this last theorem we expect the equality

$$(3.42) \quad \mathcal{E}(\Omega_m^\sharp) = \bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} \mathcal{E}(\Omega_m^\sharp(d_{a,-} + r_+(d_{a,-})))$$

to hold not only for purely repulsive or purely attractive models (see Remark 3.40), but in a much larger class of long rang models. In fact, for most relevant models coming from Physics, like BCS-type models, Equality (3.42) clearly holds.

**REMARK 3.42** (Euler–Lagrange equations).

Equations (3.39) are Euler–Lagrange equations of the min–max variational problem  $F_m^\sharp$  defined in Definition 3.35. We observe, however, that the pressure  $P_m(c_{a,-} + c_{a,+})$  in Definition 3.34 is generally not Fréchet differentiable w.r.t. to either  $c_{a,-}$  or  $c_{a,+}$  as the variational problem (3.29) can have several t.i. equilibrium states (cf. Lemma 3.30). In fact, Theorem 12.44 and Remark 12.45 only ensure the Fréchet differentiability of the convex and continuous map  $c_a \mapsto P_m(c_a)$  from  $L^2(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  on a dense subset.

**REMARK 3.43** (Gap equations in Physics).

Equations (3.39) are also called gap equations by analogy with the Bardeen–Cooper–Schrieffer (BCS) theory for conventional superconductors [34, 35, 36].

Indeed, within this theory, the existence of a non-zero solution  $d_{a,-} \in \mathcal{C}_m^\sharp$  implies a superconducting state as well as a gap in the spectrum of the effective (approximating) BCS Hamiltonian. The equations satisfied by  $d_{a,-}$  are called gap equations in the Physics literature because of this property.

Recall now that the integral representation (iii) in Theorem 3.21 may not be unique, i.e.,  $\Omega_m^\sharp$  may not be a Choquet simplex (Definition 12.23) as one may conjecture from Theorem 3.39 (ii). For models with purely attractive long range interactions for which  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.), observe that  $\Omega_m^\sharp$  cannot generally be homeomorphic to the Poulsen simplex in contrast to all sets  $\{E_{\vec{\ell}}\}_{\vec{\ell} \in \mathbb{N}^d}$ , see Theorem 2.12. Indeed, the Poulsen simplex has a dense set of extreme points whereas we have the following assertion (cf. Theorems 3.21 (i), 12.37 (ii) and 12.38 (ii)):

**THEOREM 3.44** (Density of  $\mathcal{E}(\Omega_m^\sharp)$  yields convexity of  $\hat{M}_m$ ).  
*If the compact set  $\hat{M}_m$  is not convex then  $\mathcal{E}(\Omega_m^\sharp)$  is not dense in  $\Omega_m^\sharp$ .*

Note that the convexity of  $\hat{M}_m$  is *only* a necessary condition to obtain a dense set  $\mathcal{E}(\Omega_m^\sharp)$  of extreme points of  $\Omega_m^\sharp$  in  $\Omega_m^\sharp$ .

The convexity of the set  $\hat{M}_m$  can only be broken by the long range attractions  $\Phi_{a,-}$  and  $\Phi'_{a,-}$ , see discussions following Lemma 3.7. Note further that sets of generalized t.i. equilibrium states are simplices for purely attractive long range models ( $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.)) as  $\Omega_m^\sharp$  is a closed face of  $E_1$  in this case, see Theorem 3.25 (-). Additionally, by using Theorem 3.39  $\Omega_m^\sharp$  is even a Bauer simplex (Definition 12.24) if the following assumption holds:

**HYPOTHESIS 3.45.**

*For any  $d_{a,-} \in \mathcal{C}_m^\sharp$ , the set  $M_{\Phi(d_{a,-} + r_+(d_{a,-}))}$  of t.i. equilibrium states of the approximating interaction  $\Phi(d_{a,-} + r_+(d_{a,-}))$  contains exactly one state.*

**THEOREM 3.46** (The set  $\Omega_m^\sharp$  for  $m \in \mathcal{M}_1$  as a simplex).

(-) *If  $\Phi_{a,+} = 0$  (a.e.) then the face  $\Omega_m^\sharp$  is a Choquet simplex.*

( $\exists!$ ) *Under Hypothesis 3.45  $\Omega_m^\sharp$  is a face and a Bauer simplex.*

*Proof.* The first assertion is trivial. Indeed, by Theorem 2.9, the set  $E_1$  is a Choquet simplex and, by Theorem 12.22, its closed faces are Choquet simplices. Then the assertion (-) results from Theorem 3.25 (-).

Assume now that Hypothesis 3.45 holds. Then, as

$$\Omega_{\Phi(d_{a,-} + r_+(d_{a,-}))}^\sharp = M_{\Phi(d_{a,-} + r_+(d_{a,-}))}$$

is a face of  $E_1$  (Lemma 3.33), its unique element has to be ergodic and thus extreme in  $\Omega_m^\sharp$ . Hence, using Theorem 3.39,  $\hat{M}_m \subset \mathcal{E}(\Omega_m^\sharp)$ . By Theorem 3.21 (ii),  $\mathcal{E}(\Omega_m^\sharp) \subset \hat{M}_m$  and hence,  $\mathcal{E}(\Omega_m^\sharp) = \hat{M}_m$  is a closed set as  $\hat{M}_m$  is weak\*-

compact (cf. Lemma 3.19 (i)). In particular, because  $\hat{M}_{\mathfrak{m}} \subset \mathcal{E}_1$ ,  $\Omega_{\mathfrak{m}}^{\sharp}$  is a closed face of  $E_1$  and it is thus a Bauer simplex. ■

If  $\Omega_{\mathfrak{m}}^{\sharp}$  is a Bauer simplex (for instance if Hypothesis 3.45 holds) then, by Theorem 12.25, the generalized t.i. equilibrium states of  $\mathfrak{m}$  can be – affinely and homeomorphically – identified with states on the commutative  $C^*$ -algebra  $C(\mathcal{E}(\Omega_{\mathfrak{m}}^{\sharp}))$ . For instance, Hypothesis 3.45 is satisfied if, for any  $d_{a,-} \in \mathcal{C}_{\mathfrak{m}}^{\sharp}$ , the approximating interaction

$$\Phi(d_{a,-} + r_+(d_{a,-})) \in \mathcal{W}_1$$

is either quadratic in the annihilation and creation operators  $a_{x,s}, a_{x',s'}^+$  in any dimension ( $d \geq 1$ ) or corresponds to a finite range one-dimensional ( $d = 1$ ) Fermi system. These conditions hold for many relevant models coming from Physics, like BCS-type models.

This case has also a specific interpretation in terms of game theory as  $\mathcal{C}_{\mathfrak{m}}^{\sharp}$  (3.32) is the set of conservative strategies of the attractive player of the corresponding thermodynamic game defined by Definition 3.35:

**THEOREM 3.47** (Mixed conservative strategies of the attractive player).

*For any  $\mathfrak{m} \in \mathcal{M}_1$  satisfying Hypothesis 3.45, there is an affine homeomorphism between  $\Omega_{\mathfrak{m}}^{\sharp}$  and the set of states of the commutative  $C^*$ -algebra  $C(\mathcal{C}_{\mathfrak{m}}^{\sharp})$  of continuous functions on the (weakly compact) set  $\mathcal{C}_{\mathfrak{m}}^{\sharp}$ . Here, the homeomorphism concerns the weak\*-topologies in the sets  $\Omega_{\mathfrak{m}}^{\sharp}$  and  $C(\mathcal{C}_{\mathfrak{m}}^{\sharp})$ .*

*Proof.* This result is a direct consequence of Theorems 3.46 and 12.25 combined with Corollary 11.6. ■

This last result can be interpreted from the point of view of game theory as follows. By Riesz–Markov theorem, the set of states on  $C(\mathcal{C}_{\mathfrak{m}}^{\sharp})$  is the same as the set of probability measures on the set  $\mathcal{C}_{\mathfrak{m}}^{\sharp}$  of conservative strategies of the attractive player. As discussed above, the best the attractive player can do – as she/he has no access to the choice of strategy of the repulsive one – is to choose some conservative strategy in order to minimize her/his loss in the game. She/he could also do this in a non-deterministic way. I.e., she/he determines with which probability distribution the different conservative strategies have to be chosen. This kind of procedure is called *mixed strategy* in game theory. Hence, the set of all generalized t.i. equilibrium states is – in the situation of Theorem 3.47 above – (even affinely) the same as the set of all mixed conservative strategies of the attractive player of the thermodynamic game.

Now, we observe that Theorem 3.36 (‡) tell us that the conservative value  $F_{\mathfrak{m}}^{\sharp}$  for the thermodynamic game defined in Definition 3.35 leads to the pressure  $P_{\mathfrak{m}}^{\sharp}$  (up to a minus sign) for any model  $\mathfrak{m} \in \mathcal{M}_1$ . In other words, the approximating Hamiltonian method [13, 14, 15, 16] (see Section 12.2) extended to all  $\mathfrak{m} \in \mathcal{M}_1$  is still an efficient technique to obtain the pressure. On the other

hand, the min–max variational problem  $F_m^\sharp$  is related via (3.28) and (3.29) to the family

$$\{\Phi(d_{a,-} + r_+(d_{a,-}))\}_{d_{a,-} \in \mathcal{C}_m^\sharp}$$

of approximating interactions (Definition 3.31) with  $r_+ \in C(L_-^2, L_+^2)$  defined by (3.34). Therefore, for any model  $\mathbf{m} \in \mathcal{M}_1$ , one could, a priori, think that the weak\*–closed convex hull of the union of the family

$$\{M_{\Phi(d_{a,-} + r_+(d_{a,-}))}\}_{d_{a,-} \in \mathcal{C}_m^\sharp}$$

of sets of t.i. equilibrium states (cf. Lemma 3.33) equals the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states. This fact is generally wrong, i.e., the approximating Hamiltonian method does not generally lead to an *effective local theory*.

To explain this, we define more precisely the notion of *theories* as follows:

DEFINITION 3.48 (Theory for  $\mathbf{m} \in \mathcal{M}_1$ ).

A theory for  $\mathbf{m} \in \mathcal{M}_1$  is any subset  $\mathfrak{T}_m \subset \mathcal{M}_1$ .

Of course, a good theory  $\mathfrak{T}_m$  for  $\mathbf{m} \in \mathcal{M}_1$  means that elements of  $\mathfrak{T}_m$  are simplified models in comparison with  $\mathbf{m} \in \mathcal{M}_1$  and that it allows the complete description of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states. This last property corresponds to have an *effective theory* in the following sense:

DEFINITION 3.49 (Effective theory).

A theory  $\mathfrak{T}_m$  for  $\mathbf{m} \in \mathcal{M}_1$  is said to be effective at  $\beta \in (0, \infty)$  iff

$$\overline{\text{co}\left(\bigcup_{\hat{\mathbf{m}} \in \mathfrak{T}_m} \Omega_{\hat{\mathbf{m}}}^\sharp\right)} = \Omega_m^\sharp \quad \text{and} \quad \mathcal{E}(\Omega_m^\sharp) \subset \bigcup_{\hat{\mathbf{m}} \in \mathfrak{T}_m} \Omega_{\hat{\mathbf{m}}}^\sharp.$$

The closure is taken in the weak\*–topology and  $\text{co}(M)$  denotes as usual the convex hull of a set  $M \subset \mathcal{U}^*$ .

The second condition in the above definition means that any pure generalized equilibrium state of  $\mathbf{m}$  should be a generalized t.i. equilibrium state of  $\hat{\mathbf{m}}$  for some  $\hat{\mathbf{m}} \in \mathfrak{T}_m$  in the theory  $\mathfrak{T}_m$ . By Theorem 12.13 (ii) (Milman theorem), this holds if the union  $\bigcup_{\hat{\mathbf{m}} \in \mathfrak{T}_m} \Omega_{\hat{\mathbf{m}}}^\sharp$  is closed w.r.t. the weak\*–topology. This is the case in the examples of effective theories discussed here. Two general classes of theories are of particular importance w.r.t. models  $\mathbf{m} \in \mathcal{M}_1$ : The *repulsive* and *local* theories defined below.

DEFINITION 3.50 (Repulsive theory).

For  $\mathbf{m} \in \mathcal{M}_1$ , a theory  $\mathfrak{T}_m$  is said to be repulsive iff the subset  $\mathfrak{T}_m \subset \mathcal{M}_1$  has only models with purely repulsive long range interactions, i.e., models for which  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.), see Definition 3.4.

An example of repulsive theory is given by using partially the approximating Hamiltonian method: For any model  $\mathbf{m} \in \mathcal{M}_1$  and all  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ , we

define the approximating repulsive model

$$(3.43) \quad \mathfrak{m}(c_{a,-}) := (\Phi(c_{a,-}), \{\Phi_{a,+}\}_{a \in \mathcal{A}}, \{\Phi'_{a,+}\}_{a \in \mathcal{A}}) \in \mathcal{M}_1.$$

Here,  $\Phi_{a,+} := \gamma_{a,+} \Phi_a$  and  $\Phi'_{a,+} := \gamma_{a,+} \Phi'_a$  (cf. Definition 3.4), whereas  $\Phi(c_{a,-})$  is defined in Definition 3.31. Since  $\mathfrak{m}(c_{a,-})$  is a model with purely repulsive long range interactions for all  $c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})$ , it can be used to define a repulsive theory as follows:

DEFINITION 3.51 (The min repulsive theory).

At  $\beta \in (0, \infty)$ , the min repulsive theory for  $\mathfrak{m} \in \mathcal{M}_1$  is the subset

$$\mathfrak{T}_m^+ := \bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} \mathfrak{m}(d_{a,-}) \subset \mathcal{M}_1$$

with the set  $\mathcal{C}_m^\sharp$  of conservative strategies of the attractive player defined by (3.32).

Observe that  $\mathfrak{m}(d_{a,-})$  has a local (effective) interaction  $\Phi(d_{a,-})$  non-trivially depending on the inverse temperature  $\beta > 0$  of the system (cf. Remark 2.33). In other words, the min repulsive theory  $\mathfrak{T}_m^+$  is *temperature-dependent*.

*Local* theories are made of subsets of the real Banach space  $\mathcal{W}_1$  of t.i. interactions  $\Phi$ , see Definition 2.24.

DEFINITION 3.52 (Local theories).

A theory  $\mathfrak{T}_m$  for  $\mathfrak{m} \in \mathcal{M}_1$  is said to be local iff  $\mathfrak{T}_m \subset \mathcal{W}_1$ , where  $\mathcal{W}_1$  is seen as a sub-space of  $\mathcal{M}_1$ .

The min-max variational problem  $F_m^\sharp$  of the thermodynamic game defined by Definition 3.35 leads to an important example of local theories: The *min-max* local theory, which is also a *temperature-dependent* theory.

DEFINITION 3.53 (The min-max local theory).

At  $\beta \in (0, \infty)$ , the min-max local theory for  $\mathfrak{m} \in \mathcal{M}_1$  is the subset

$$\mathfrak{T}_m^\sharp := \bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} \Phi(d_{a,-} + r_+(d_{a,-})) \subset \mathcal{W}_1,$$

where the set  $\mathcal{C}_m^\sharp$  is defined by (3.32) and the map  $r_+$  by (3.34).

To get an effective local theory  $\mathfrak{T}_m$  for a model  $\mathfrak{m} \in \mathcal{M}_1$ , the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states must be a face. It is a necessary condition as the weak\*-closed convex hull of the union

$$\bigcup_{\Phi \in \mathfrak{T}_m} \Omega_\Phi = \bigcup_{\Phi \in \mathfrak{T}_m} M_\Phi$$

of faces in  $E_1$  is again a face in  $E_1$  if

$$\overline{\text{co}\left(\bigcup_{\Phi \in \mathfrak{T}_m} \Omega_\Phi^\sharp\right)} = \Omega_m^\sharp \quad \text{and} \quad \mathcal{E}(\Omega_m^\sharp) \subset \bigcup_{\Phi \in \mathfrak{T}_m} \Omega_\Phi^\sharp.$$

Indeed, for all  $\Phi \in \mathcal{W}_1$ , the set  $M_\Phi = \Omega_\Phi$  is a face by weak\*-lower semi-continuity and affinity of the functional  $f_\Phi$ , see Lemmata 2.28 (i), 2.31 (i) and Definition 2.32. Lemma 11.8 says that  $\Omega_m^\sharp$  is generally not a face in  $E_1$ . As a consequence, we obtain the following result:

**THEOREM 3.54** (Breakdown of effective local theories).

*At fixed  $\beta \in (0, \infty)$ , there are uncountably many  $\mathbf{m} \in \mathcal{M}_1$  with no effective local theory.*

In particular, the equality  $P_m^\sharp = -F_m^\sharp$  of Theorem 3.36 ( $\sharp$ ) does not necessarily imply that the min-max local theory  $\mathfrak{T}_m^\sharp$  (Definition 3.53) is an effective theory, see Definition 3.49. By contrast, the min repulsive theory (Definition 3.51) is always an effective theory:

**THEOREM 3.55** (Effectiveness of the min repulsive theory  $\mathfrak{T}_m^+$ ).

*$\mathfrak{T}_m^+$  is an effective repulsive theory for any  $\mathbf{m} \in \mathcal{M}_1$ , i.e.,*

$$\overline{\text{co}\left(\bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} \Omega_{\mathbf{m}(d_{a,-})}^\sharp\right)} = \Omega_m^\sharp \quad \text{and} \quad \mathcal{E}(\Omega_m^\sharp) \subset \bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} \Omega_{\mathbf{m}(d_{a,-})}^\sharp.$$

*Proof.* This follows from Lemmata 11.1 and 11.2 which yield in particular the equality

$$\Omega_{\mathbf{m}(d_{a,-})}^\sharp = \Omega_m^\sharp(d_{a,-} + r_+(d_{a,-}))$$

for all  $d_{a,-} \in \mathcal{C}_m^\sharp$ . See also Theorems 3.21 (i) and 3.39 (i). ■

Therefore, the breakdown of effective local theories results from long range repulsions  $\Phi_{a,+}$ ,  $\Phi'_{a,+}$  and not from long range attractions  $\Phi_{a,-}$ ,  $\Phi'_{a,-}$ , see Definition 3.4. This is another strong asymmetry between both long range interactions. To illustrate this, observe that for models  $\mathbf{m}$  with  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.), the min repulsive and the min-max local theories are the same, i.e.,  $\mathfrak{T}_m^+ = \mathfrak{T}_m^\sharp$ , see Definitions 3.51 and 3.53 together with Definition 3.31 and (3.43). In this purely attractive case, for all  $d_{a,-} \in \mathcal{C}_m^\sharp$ ,  $\Omega_{\mathbf{m}(d_{a,-})}^\sharp = M_{\Phi(d_{a,-})}$  is always a face in  $E_1$  and so is the set  $\Omega_m^\sharp$  by Theorem 3.55. In other words, if the long range repulsions  $\Phi_{a,+}$  and  $\Phi'_{a,+}$  are switched off, there is always an effective local theory.

In the general case, the min-max local theory  $\mathfrak{T}_m^\sharp$  (Definition 3.53) is not accurate enough. It means that the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states is only included in (but generally not equal to) the weak\*-closed convex hull of the set

$$(3.44) \quad M(\mathfrak{T}_m^\sharp) := \bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} M_{\Phi(d_{a,-} + r_+(d_{a,-}))}^\sharp.$$

This result is a simple corollary of Theorems 3.21 (i) and 3.39 (i):

COROLLARY 3.56 (Accuracy of the min–max local theory  $\mathfrak{T}_m^\sharp$ ).

For any  $m \in \mathcal{M}_1$ ,

$$\Omega_m^\sharp \subseteq \overline{\text{co}(M(\mathfrak{T}_m^\sharp))}.$$

Indeed, by Theorems 3.21 (i) and 3.39 (i),  $\Omega_m^\sharp$  is the weak\*–closed convex hull of the set of states in  $M(\mathfrak{T}_m^\sharp)$  satisfying the Euler–Lagrange equations (3.39).

REMARK 3.57. If  $\Omega_m^\sharp$  is not a face then there is, at least, one ergodic state<sup>16</sup>  $\hat{\omega} \in M(\mathfrak{T}_m^\sharp) \cap \mathcal{E}_1$  which does not satisfy the Euler–Lagrange equations (3.39).

REMARK 3.58 (Max attractive theory  $\mathfrak{T}_m^-$  and max–min local theory  $\mathfrak{T}_m^b$ ). In the same way we define the min repulsive theory  $\mathfrak{T}_m^+$  (Definition 3.51) and the min–max local theory  $\mathfrak{T}_m^\sharp$  (Definition 3.53), one could define the max attractive theory  $\mathfrak{T}_m^-$  and the max–min local theory  $\mathfrak{T}_m^b$  for any  $m \in \mathcal{M}_1$ . In the same way we have Theorems 3.21, 3.39 and 3.55, such theories  $\mathfrak{T}_m^-$  and  $\mathfrak{T}_m^b$  shall be related to the (non–empty) set  $M_m^b$  of minimizers of the functional  $f_m^b$  over  $E_1$ , see (3.12), (3.14) and Theorem 3.36 (b).

### 3.8. Long range interactions and long range order (LRO)

The solution  $d_a \in L^2(\mathcal{A}, \mathbb{C})$  defined by (3.39) has a direct interpretation as the mean energy density of long range interactions  $\Phi_a$  and  $\Phi'_a$ . Moreover, it is related to the so–called *long range order* (LRO) property. In particular, models with non–zero  $d_{a,-} \in \mathcal{C}_m^\sharp$  show an *off diagonal long range order* (ODLRO), a property proposed by Yang [37] to define super–conducting phases. The latter can be seen as a consequence of the following theorem:

THEOREM 3.59 (Off diagonal long range order).

For any  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , let  $A_{c_a} := \langle \mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}, \gamma_a c_a \rangle$ . Then, for any  $\omega \in \Omega_m^\sharp$ ,

$$\Delta_{A_{c_a}}(\omega) := \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|^2} \sum_{x, y \in \Lambda_L} \omega(\alpha_x(A_{c_a}^*) \alpha_y(A_{c_a}))$$

satisfies the inequality

$$\Delta_{A_{c_a}}(\omega) \geq \min_{d_{a,-} \in \mathcal{C}_m^\sharp} \{ |\langle d_{a,-} + r_+(d_{a,-}), \gamma_a c_a \rangle|^2 \}.$$

*Proof.* By Definition 2.14, Remark 2.20, and Theorem 3.39 (ii), for any extreme state  $\hat{\omega} \in \mathcal{E}(\Omega_m^\sharp)$ , there is  $d_{a,-} \in \mathcal{C}_m^\sharp$  such that

$$\Delta_{A_{c_a}}(\hat{\omega}) \geq |\hat{\omega}(A_{c_a})|^2 = |\langle d_a, \gamma_a c_a \rangle|^2$$

<sup>16</sup>Note that  $M(\mathfrak{T}_m^\sharp) \cap \mathcal{E}_1 \neq \emptyset$  because  $M(\mathfrak{T}_m^\sharp)$  is a union of non–empty closed faces by (3.44), see also Lemma 3.33.

with  $d_a := d_{a,-} + r_+(d_{a,-})$ . Then via Theorem 3.21 (iii) combined with Lemma 12.17 one gets the assertion. ■

REMARK 3.60. *By using similar arguments as above, if all extreme generalized t.i. equilibrium states  $\hat{\omega} \in \mathcal{E}(\Omega_m^\sharp)$  are strongly mixing (see (2.10)) then*

$$\Delta_{A_{c_a}}(\omega) = \lim_{|y-x| \rightarrow \infty} \omega(\alpha_x(A_{c_a}^*)\alpha_y(A_{c_a})) \geq \min_{d_{a,-} \in \mathcal{C}_m^\sharp} \{|\langle d_{a,-} + r_+(d_{a,-}), \gamma_a c_a \rangle|^2\}$$

for all  $\omega \in \Omega_m^\sharp$ .

Theorem 3.59 implies ODLRO in the following sense. Take any gauge invariant model  $\mathfrak{m} \in \mathcal{M}_1$  – which means that  $U_l \in \mathcal{U}^\circ$  (cf. (2.6)) – such that its set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states contains at least one state from  $\mathcal{E}(E_1^\circ)$ , i.e.,  $\Omega_m^\sharp \cap \mathcal{E}(E_1^\circ) \neq \emptyset$ . This is the case, for instance, if the long range interactions of  $\mathfrak{m}$  are purely attractive (i.e.,  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.)) as, in this situation,  $\Omega_m^{\sharp,\circ} := \Omega_m^\sharp \cap E_1^\circ$  is a face in  $E_1^\circ$ , see also Remark 3.24. Suppose that  $c_a$  is chosen such that<sup>17</sup>

$$\sigma^\circ(A_{c_a}) = 0 \quad \text{and} \quad \min_{d_{a,-} \in \mathcal{C}_m^\sharp} \{|\langle d_{a,-} + r_+(d_{a,-}), \gamma_a c_a \rangle|^2\} > 0,$$

see Remark 2.5. Choose now any gauge invariant t.i. equilibrium state  $\hat{\omega} \in \Omega_m^{\sharp,\circ}$ , which is extreme in the set  $E_1^\circ$  of t.i. and gauge invariant states (cf. Remarks 2.13, 2.17, and 3.24). Then, by the assumptions, for all  $A^\circ \in \mathcal{U}^\circ$  such that  $\hat{\omega}(A^\circ) = 0$ ,

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|^2} \sum_{x,y \in \Lambda_L} \hat{\omega}(\alpha_x(A^\circ)^* \alpha_y(A^\circ)) = |\hat{\omega}(A^\circ)|^2 = 0.$$

However,

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|^2} \sum_{x,y \in \Lambda_L} \hat{\omega}(\alpha_x(A_{c_a})^* \alpha_y(A_{c_a})) > 0$$

in spite of the fact that  $\hat{\omega}(A_{c_a}) = \hat{\omega} \circ \sigma^\circ(A_{c_a}) = 0$ . Indeed, any quadratic element  $A^\circ = A_1 A_2 \in \mathcal{U}^\circ$  with  $A_1, A_2 \in \mathcal{U}^\circ$  is called “diagonal”, whereas elements of the form  $B^\circ = B_1 B_2 \in \mathcal{U}^\circ$  with  $B_1, B_2 \in \mathcal{U} \setminus \mathcal{U}^\circ$  – as, for instance, the elements  $\alpha_x(A_{c_a})^* \alpha_y(A_{c_a})$  considered above – are called “off-diagonal” w.r.t. to the algebra  $\mathcal{U}^\circ$ , see, e.g., [22, Section 5.2].

In the general case, the order parameter  $d_{a,-}$  is, a priori, not unique since the non-empty set  $\mathcal{C}_m^\sharp$  (3.32) of conservative strategies of the attractive player is only weakly compact, see Lemma 10.4 (#). Non-uniqueness of solutions of

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<sup>17</sup>Both assumption can easily be verified in various long range gauge invariant models, see, e.g., [9].



the min–max variational problem  $F_m^\sharp$  of the thermodynamic game defined in Definition 3.35 ensures the existence of a non–zero  $d_{a,-} \in \mathcal{C}_m^\sharp$  which should be related to ODLRO as explained above. In particular, ODLRO w.r.t. elements of the form  $A_{c_a}$  as defined above is usually related to long range attractions  $\Phi_{a,-}, \Phi'_{a,-}$  (Definition 3.4). As an example, we recommend to have a look on the strong coupling BCS–Hubbard model analyzed in [9].

By contrast, the solution  $r_+(d_{a,-}) \in \mathcal{C}_m^\sharp(d_{a,-})$  of the variational problem  $f_m^\sharp(d_{a,-})$  defined in Definition 3.35 is always unique, see Lemma 10.3 ( $\sharp$ ). In particular, if the model  $\mathfrak{m} \in \mathcal{M}_1$  has purely repulsive long range interactions, i.e.,  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.), then no first order phase transition (related to observables of the form  $A_{c_a}$ ) can appear. If, additionally,  $\mathfrak{m}$  is also gauge invariant – which means that  $U_l \in \mathcal{U}^\circ$  – then

$$d_{a,+} = \omega(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}) = \omega(\sigma^\circ(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}))$$

for all  $\omega \in \Omega_m^\sharp$ , see again (2.6) and Remark 2.5 for definitions of the set  $\mathcal{U}^\circ$  of all gauge invariant elements and the gauge invariant projection  $\sigma^\circ$  respectively. In particular, for all  $a \in \mathcal{A}$  such that  $\sigma^\circ(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}) = 0$ , the unique  $d_{a,+}$  must be zero.

However, the existence of a non–zero order parameter  $d_{a,-}$  is, a priori, *not necessary* to get LRO. Take again some gauge invariant model  $\mathfrak{m} \in \mathcal{M}_1$  with purely repulsive long range interactions ( $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.)). In this case, two different conclusions w.r.t. the LRO can be drawn depending on the structure of its set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states (cf. Remark 3.40):

- $\Omega_m^\sharp$  is a face in  $E_1$ , i.e.,  $\mathcal{E}(\Omega_m^\sharp) = \Omega_m^\sharp \cap \mathcal{E}_1$ . Then, by the uniqueness of solution  $d_{a,+}$  of the variational problem  $f_m^\sharp(c_{a,-})$  (Definition 3.35) combined with Theorem 2.19 (iv), and Theorem 3.39 (ii), we obtain that

$$(3.45) \quad \Delta_{A_{c_a}}(\omega) := \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|^2} \sum_{x,y \in \Lambda_L} \omega(\alpha_x(A_{c_a})^* \alpha_y(A_{c_a})) = |\omega(A_{c_a})|^2$$

for all  $\omega \in \Omega_m^\sharp$  and with  $A_{c_a} \in \mathcal{U}$  defined as above. In particular, the condition  $\sigma^\circ(A_{c_a}) = 0$  implies for  $\omega \in \Omega_m^{\sharp,\circ}$  (cf. Remark 3.24) that  $\omega(A_{c_a}) = \omega \circ \sigma^\circ(A_{c_a}) = 0$  and thus,  $\Delta_{A_{c_a}}(\omega) = 0$ . In other words, no ODLRO w.r.t. elements of the form  $A_{c_a}$  can be observed.

- $\Omega_m^\sharp$  is not a face in  $E_1$ , see Lemma 11.8 in Section 11.2. Then there is an extreme generalized t.i. equilibrium states  $\hat{\omega}_0 \in \mathcal{E}(\Omega_m^\sharp)$  which is not ergodic. Since  $M(\mathfrak{T}_m^\sharp)$  is a face of  $E_1$  (cf. Lemma 3.33 and (3.44)), by Theorem 2.19 (iv) and Corollary 3.56,  $\hat{\omega}_0 \in M(\mathfrak{T}_m^\sharp)$  and

$$(3.46) \quad \Delta_{A_{c_a}}(\hat{\omega}_0) = \int_{M(\mathfrak{T}_m^\sharp) \cap \mathcal{E}_1} d\mu_{\hat{\omega}_0}(\hat{\rho}) |\langle \gamma_a c_a, e_{\Phi_a}(\hat{\rho}) + i e_{\Phi'_a}(\hat{\rho}) \rangle|^2.$$

In particular, there is  $c_a \in L^2(\mathcal{A}, \mathbb{C})$  and a non-ergodic state  $\hat{\omega}_0 \in \mathcal{E}(\Omega_{\mathfrak{m}}^{\sharp})$  such that

$$\Delta_{A_{c_a}}(\hat{\omega}_0) > 0.$$

The latter holds even if  $d_{a,+} = 0$ , which implies that

$$|\hat{\omega}(A_{c_a})|^2 = |\langle d_{a,+}, \gamma_a c_a \rangle|^2 = 0$$

for all  $\hat{\omega} \in \mathcal{E}(\Omega_{\mathfrak{m}}^{\sharp})$  and  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , see (3.39) and Lemma 10.3 (#). Indeed, assume that  $d_{a,+} = 0$  and  $\Delta_{A_{c_a}}(\hat{\omega}_0) = 0$  for all  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ . By (3.46), this would imply for  $\hat{\rho} \mu_{\hat{\omega}_0}$ -a.e. that

$$e_{\Phi_a}(\hat{\rho}) + ie_{\Phi'_a}(\hat{\rho}) = 0 \text{ (a.e.)},$$

i.e.,  $\hat{\rho} \in M(\mathfrak{T}_{\mathfrak{m}}^{\sharp}) \cap \mathcal{E}_1$  solves the Euler-Lagrange equations (3.39) and thus  $\hat{\rho} \in \Omega_{\mathfrak{m}}^{\sharp}$ . Since the measure  $\mu_{\hat{\omega}_0}$  is not concentrated on  $\hat{\omega}_0 \notin \mathcal{E}_1$ , this would imply that  $\hat{\omega}_0$  is decomposable within  $\Omega_{\mathfrak{m}}^{\sharp}$  contradicting the fact that  $\hat{\omega}_0 \in \mathcal{E}(\Omega_{\mathfrak{m}}^{\sharp})$ .

In other words, LRO w.r.t. elements of the form  $A_{c_a}$  may still be observed. The existence of such a model (i.e., a model  $\mathfrak{m}$  such that  $d_{a,+} = 0$  and  $\Omega_{\mathfrak{m}}^{\sharp}$  is not a face in  $E_1$ ) follows easily from the construction done in Lemmata 11.7 and 11.8.

In conclusion, both long range interactions  $\Phi_{a,-}, \Phi'_{a,-}$  and  $\Phi_{a,+}, \Phi'_{a,+}$  (Definition 3.4) can produce a LRO, usually at high enough inverse temperatures  $\beta > 0$ . Nevertheless, long range attractions  $\Phi_{a,-}, \Phi'_{a,-}$  and repulsions  $\Phi_{a,+}, \Phi'_{a,+}$  act in a completely different way. Long range attractions  $\Phi_{a,-}, \Phi'_{a,-}$  imply ODLRO by producing non-uniqueness of conservative strategies of the attractive player (i.e.  $|\mathcal{C}_{\mathfrak{m}}^{\sharp}| > 1$ ), whereas long range repulsions  $\Phi_{a,+}, \Phi'_{a,+}$  produce LRO by breaking the face structure of the set  $\Omega_{\mathfrak{m}}^{\sharp}$ .

## 4. Concluding Remarks

In this section, we explain our achievements in the light of previous results. We review – on a formal level – in Section 4.1 the original idea of the Bogoliubov approximation, which was so successfully used in theoretical physics. Section 4.2 compares our results with the approximating Hamiltonian method defined by Bogoliubov Jr., Brankov, Kurbatov, Tonchev, and Zagrebnov. In order to be as short as possible we reduce the technical aspects to an absolute minimum in all this section, hoping that it is still understandable.

### 4.1. The Bogoliubov approximation

Roughly speaking, the Bogoliubov approximation consists in replacing specific operators appearing in the Hamiltonian of a given physical system by constants

which are determined as solutions of some self-consistency equation or some associated variational problem. One important issue is the way such substitutions should be performed. To be successful, it depends much on the system under consideration. In order to highlight this aspect, we discuss below three different situations where Bogoliubov's method is usually applied.

Within his celebrated microscopic theory of superfluidity [38] of Helium 4, Bogoliubov proposed in 1947 his famous "trick", the so-called Bogoliubov approximation, by observing the following:

- (i) For the considered Hamiltonian modelling a Bose gas in weak interaction inside a finite box  $\Lambda$ , the annihilation and creation operators<sup>18</sup>  $b_0$  and  $b_0^*$  of bosons only appear in the form  $b_0|\Lambda|^{-1/2}$  and  $b_0^*|\Lambda|^{-1/2}$ .
- (ii) Because of the Canonical Commutation Relations (CCR),  $b_0|\Lambda|^{-1/2}$  and  $b_0^*|\Lambda|^{-1/2}$  almost commute at large volume  $|\Lambda|$ .
- (iii) The operators  $b_0$  and  $b_0^*$  are *unbounded*.

Based on (i)–(iii) Bogoliubov suggested that  $b_0$  (resp.  $b_0^*$ ) can be replaced by a complex number  $c_\Lambda = \mathcal{O}(|\Lambda|^{1/2})$  (resp.  $\bar{c}_\Lambda$ ) to be determined *self-consistently*. For a detailed description of the Bogoliubov theory of superfluidity, we recommend the review [31].

The Bogoliubov approximation in this precise situation was rigorously justified in 1968 by Ginibre [11] on the level of the grand-canonical pressure in the thermodynamic limit. See also [39, 40]. Actually, the (infinite volume) pressure is given through a supremum over complex numbers and the constant  $c := c_\Lambda|\Lambda|^{-1/2}$  in the substitution must be a solution of this variational problem. Up to additional technical arguments this proof [11, 39] is based on Laplace's method together with the completeness of the family of coherent vectors  $\{|c\rangle\}_{c \in \mathbb{C}}$  whose elements satisfy  $b_0|c\rangle = c|c\rangle$ . In fact, in which concerns the (infinite volume) pressure, the Bogoliubov approximation is exact for the (stable) Bose gas even if the number  $n_\Lambda$  of boson operators  $\{b_j\}_{j=1}^{n_\Lambda}$  replaced by a constant is large, provided that  $n_\Lambda = o(|\Lambda|)$ , see [39]. Observe that the validity of the Bogoliubov approximation on the level of the pressure has nothing to do with the existence, or not, of a Bose condensation. However, this approximation becomes useful when the expectation value of either  $b_0$  (resp.  $b_0^*$ ) or  $b_0^*b_0$  becomes macroscopic, i.e., in the case of a Bose condensation.

REMARK 4.1. *In the case considered above, the validity of the replacement of operators by (possibly non zero) complex numbers depends on the unboundedness of boson operators, whose corresponding expectation value can possibly*

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<sup>18</sup>In Bogoliubov's theory,  $b_0$  and  $b_0^*$  are the annihilation/creation operators w.r.t. the constant function  $|\Lambda|^{-1/2}$  acting on the boson Fock space.

become macroscopic (which means  $c \neq 0$ ). Observe that the same kind of argument cannot work for Fermi systems since the corresponding annihilation and creation operators  $a_j$  and  $a_j^+$  are bounded in norm.

Another kind of Bogoliubov approximation can be applied on a large class of (superstable) Bose gases having the long range interaction  $\lambda N_\Lambda^2/|\Lambda|$  with  $\lambda > 0$ , see [41, 42]. Here,  $N_\Lambda$  is the particle number operator inside a finite box  $\Lambda$  acting on the boson Fock space. Its expectation value per unit volume is always a finite number, i.e., the particle density, since it is a *space-average*. This observation is not depending on the fact that  $N_\Lambda$  is unbounded. It is therefore natural to replace, in the long range interaction  $\lambda N_\Lambda^2/|\Lambda|$ , the term  $N_\Lambda/|\Lambda|$  by a positive real number  $\rho > 0$  in order to get an effective approximating model in the thermodynamic limit. This approximation is proven in [41] to be exact on the level of the pressure provided that it is done in an appropriated manner. Indeed, the (infinite volume) pressure, in this case, is the infimum over strictly positive real parameters  $\rho$  of pressures of approximating models, use  $\rho = (\mu - \alpha)/2\lambda$  in [41, Eq. (3.4)]. Observe that the constants replacing operators in the corresponding Bogoliubov approximations must be a solution of that variational problem, see [41, Theorem 4.1]. However, the approximating model leading to this variational problem is derived by replacing  $\lambda N_\Lambda^2/|\Lambda|$  with  $\lambda(2\rho N_\Lambda - |\Lambda|\rho^2)$ , i.e., one term  $\lambda\rho N_\Lambda$  for each choice of  $N_\Lambda$  in  $\lambda N_\Lambda^2/|\Lambda|$ . See again [41, Eq. (3.4)] with the choice  $\rho = (\mu - \alpha)/2\lambda > 0$  because of [41, Theorem 4.1]. This kind of Bogoliubov approximation could also be called a *Bogoliubov linearization*.

A similar observation holds of course for our class of Fermi models, see Definition 3.31 and Theorem 3.36 (#). Indeed, our long range interaction (Definition 3.3) is a sum of products

$$(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})^*(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}),$$

where the expectation value of  $(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})$  per unit volume is always a finite number (a mean energy density) as it is also a *space-average*. Similar to [41, 42] for the real case, from our results the following replacement has to be done:

$$\begin{aligned} & \frac{1}{|\Lambda_l|} (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})^*(U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}) \\ \longrightarrow & \bar{c}_a (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a}) + (U_{\Lambda_l}^{\Phi_a} + iU_{\Lambda_l}^{\Phi'_a})^* c_a - |\Lambda_l| |c_a|^2, \quad c_a \in \mathbb{C}. \end{aligned}$$

The relative universality of this phenomenon comes – in the case of models considered here – from the law of large numbers, whose representative in our setting is von Neumann ergodic theorem (Theorem 6.2). It leads again to an approximating model by appropriately replacing an operator by a complex number.

All mathematical results on Bogoliubov approximations are only performed on the level of the pressure and eventually quasi-means provided the

pressure is known to be differentiable w.r.t. to suitable parameters. Some conjectures have been done on the level of states (see, e.g., [43, Definition 3.2]). As far as we know, this paper is a first result describing the validity of the Bogoliubov approximation on the level of (generalized) equilibrium states. See, e.g., Theorems 3.21 and 3.39.

Indeed, Ginibre [11, p. 28] addressed as an important open problem the question of the validity of the Bogoliubov approximation (or Bogoliubov linearization) in the thermodynamic limit on the level of (generalized) equilibrium states. Theorems 3.21 and 3.39 give a first answer to this question, at least for the class of models treated here. We prove that the Bogoliubov approximation is in general not exact on the level of equilibrium states in the presence of *non-trivial* long range repulsions  $\Phi_{a,+}, \Phi'_{a,+} \neq 0$  (a.e.), see Definition 3.4, Theorem 3.54 and Corollary 3.56. This is so in spite of the fact that the Bogoliubov approximation is exact for *any* long range model on the level of the pressure. In the situation where the long range component of the interaction is *purely attractive*, i.e., when  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.), the Bogoliubov approximation turns out to be always exact also on the level of generalized t.i. equilibrium states as the min repulsive and the min–max local theories are the same, i.e.,  $\mathfrak{T}_m^+ = \mathfrak{T}_m^\sharp$ , see Definitions 3.51 and 3.53 together with Theorem 3.55.

## 4.2. Comparison with the approximating Hamiltonian method

The Bogoliubov approximation was already used for Fermi systems on lattices in 1957 to derive the celebrated Bardeen–Cooper–Schrieffer (BCS) theory for conventional type I superconductors [34, 35, 36]. The authors were of course inspired by Bogoliubov and his revolutionary paper [38]. A rigorous justification of this theory was given on the level of ground states by Bogoliubov in 1960 [44]. Then a method for analyzing the Bogoliubov approximation in a systematic way – on the level of the pressure – was introduced by Bogoliubov Jr. in 1966 [13, 32] and by Brankov, Kurbatov, Tonchev, Zagrebnov during the seventies and eighties [14, 15, 16]. This method is known in the literature as the *approximating Hamiltonian method* and leads – on the class of Hamiltonians it applies – to a rigorous proof of the exactness of the Bogoliubov approximation on the level of the pressure, provided it is done in an appropriated manner, see discussions in Section 4.1 about Bogoliubov linearization. For more details, we recommend [15] as well as Section 12.2.

The class of lattice models on which the approximating Hamiltonian method is applied belongs to the sub–space  $\mathcal{M}_1^d \subset \mathcal{M}_1$  of Fermi (or quantum spin) systems with discrete long range part, see Section 12.2. Within our framework, it means that there is a finite family of interactions  $\{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N$  defining  $\mathfrak{m}$  (cf. Section 3.1). Observe that in [15] the Hamiltonian  $H_\Lambda$  (see (12.3)) can describe particles on lattices or on  $\mathbb{R}^d$  as its local part  $T_\Lambda$  could be unbounded.

However, restricted to models of  $\mathcal{M}_1$ , our result is more general – even on the level of the pressure – in many aspects: We prove that the ergodicity condition (A4) formulated in Section 12.2 and needed in [15] is, by far, unnecessary (cf. Remark 12.5). Moreover, by inspection of explicit examples and using the triangle inequality of the operator norm, the commutator inequalities (A3) are very unlikely to hold – in general – for all models of  $\mathcal{M}_1$  (cf. Remark 12.6). Technically and conceptually speaking, our study is performed in a different framework not included in [15] and allows any Fermi systems  $\mathfrak{m} \in \mathcal{M}_1$ .

Additionally, the method discussed here gives new and *deeper* results on the level of states. In contrast to the approximating Hamiltonian method, which can be applied for the pressure and eventually quasi-means only, our method leads to a natural notion of (generalized) equilibrium and ground states and, depending on the model  $\mathfrak{m} \in \mathcal{M}_1$ , it allows the direct analysis of *all correlation functions*. This is the main and crucial difference between the approximating Hamiltonian method and our approach using the structure of sets of states.

## 5. Periodic Boundary Conditions and Gibbs Equilibrium States

We have shown in Theorem 3.12 (i) that the pressure of Fermi systems with long range interactions is given in the thermodynamic limit by two different variational problems on the set  $E_1$  of t.i. states. We also present in Sections 3.4 and 3.7 a detailed study of generalized t.i. equilibrium states. The weak\*-convergence of Gibbs equilibrium states (cf. Section 12.1) to generalized t.i. equilibrium states is, a priori, not clear. In fact, Gibbs equilibrium states do not generally converge to a generalized t.i. equilibrium state, see Section 3.5. This depends on boundary conditions.

We introduce periodic boundary conditions and show in this particular case that the Gibbs equilibrium state does converge in the weak\*-topology towards a generalized t.i. equilibrium state, see Section 5.4 (Theorem 5.13). On the level of the pressure, periodic boundary conditions are “universal” in the sense that, for any  $\mathfrak{m} \in \mathcal{M}_1$ , the thermodynamic limit of the pressure (3.7) can be studied via models with periodic boundary conditions, see Section 5.3 (Theorem 5.11). Note that it is convenient to use *interaction kernels* to use internal energies with periodic boundary conditions as defined in Section 5.1. Fermi systems with periodic boundary conditions are then defined in Section 5.2 by means of such interaction kernels.

NOTATION 5.1 (Periodic boundary conditions).

*Any symbol with a tilde on the top (as for instance  $\tilde{p}$ ) is, by definition, an object related to periodic boundary conditions.*

### 5.1. Interaction kernels

It is useful to describe interactions in terms of interaction kernels. This requires some preliminary definitions.

Let  $\mathbb{X}_{\mathcal{L}} = \{+, -\} \times S \times \mathcal{L}$ , where we recall that  $\mathcal{L} := \mathbb{Z}^d$  and  $S$  is a finite set defining a finite dimensional Hilbert space  $\mathcal{H}$  of spins with orthonormal basis  $\{e_s\}_{s \in S}$ . Elements of  $\mathbb{X}_{\mathcal{L}}$  are written as  $X = (\nu, s, x)$  and we define  $\bar{X} := (\bar{\nu}, s, x)$  with the convention  $\bar{+} := -$  and  $\bar{-} := +$ . Then *interaction kernels* are defined as follows:

DEFINITION 5.2 (Interaction kernels).

An interaction kernel is a family  $\varphi = \{\varphi_n\}_{n \in \mathbb{N}_0}$  of anti-symmetric functions  $\varphi_n : \mathbb{X}_{\mathcal{L}}^n \rightarrow \mathbb{C}$  satisfying  $\varphi_n = 0$  for  $n \notin 2\mathbb{N}_0$  as well as the self-adjointness property: For any  $X_1, \dots, X_n \in \mathbb{X}_{\mathcal{L}}$ ,

$$\varphi_n(X_1, \dots, X_n) = \overline{\varphi_n(\bar{X}_n, \dots, \bar{X}_1)}.$$

The set of all interaction kernels is denoted by  $\mathcal{K}$ .

NOTATION 5.3 (Interaction kernel).

The letter  $\varphi$  is exclusively reserved to denote interaction kernels.

Note that any  $\varphi \in \mathcal{K}$  can be associated with an interaction  $\Phi(\varphi)$  (Definition 2.22) with

$$\begin{aligned} \Phi_{\Lambda}(\varphi) &= \sum_{\{X_i=(\nu_i, s_i, x_i) \in \mathbb{X}_{\mathcal{L}}\}_{i=1}^n, \{x_1, \dots, x_n\}=\Lambda} \varphi_n(X_1, \dots, X_n) : a_{x_1, s_1}^{\nu_1} \dots a_{x_n, s_n}^{\nu_n} : \\ (5.1) \quad &= \sum_{\{X_i=(\nu_i, s_i, x_i) \in \mathbb{X}_{\mathcal{L}}\}_{i=1}^n, \{x_1, \dots, x_n\}=\Lambda} \varphi_n(X_1, \dots, X_n) : a(X_1) \dots a(X_n) : \end{aligned}$$

Here,

$$a_{x, s}^- := a_{x, s} \quad \text{and} \quad a(X) := a_{x, s}^{\nu}$$

for  $X = (\nu, s, x)$ . The notation

$$(5.2) \quad : a_{x_1, s_1}^{\nu_1} \dots a_{x_n, s_n}^{\nu_n} : := (-1)^{\zeta} a_{x_{\zeta(1)}, s_{\zeta(1)}}^{\nu_{\zeta(1)}} \dots a_{x_{\zeta(n)}, s_{\zeta(n)}}^{\nu_{\zeta(n)}}$$

stands for the normal ordered product defined via any permutation  $\zeta$  of the set  $\{1, \dots, n\}$  moving all creation operators in the product  $a_{x_{\zeta(1)}, s_{\zeta(1)}}^{\nu_{\zeta(1)}} \dots a_{x_{\zeta(n)}, s_{\zeta(n)}}^{\nu_{\zeta(n)}}$  to the left of all annihilation operators. This permutation is of course not unique. The operator defined by the normal ordering is nevertheless uniquely defined because of the factor  $(-1)^{\zeta}$  in (5.2) and because of the CAR (2.3).

We use below the following convention: For any interaction kernel  $\varphi$ ,  $\Phi = \Phi(\varphi)$  is always an interaction as an operator valued map on  $\mathcal{P}_f(\mathcal{L})$  which is formally written as

$$(5.3) \quad \Phi(\varphi) =: \sum_{X_1, \dots, X_n \in \mathbb{X}_{\mathcal{L}}} \varphi_n(X_1, \dots, X_n) : a(X_1) \dots a(X_n) : .$$

The map  $\varphi \mapsto \Phi(\varphi)$  is not injective and hence, the choice of kernels  $\{\varphi_n\}$  for a given interaction  $\Phi$  is not unique. Note that (5.3) is only a formal notation since infinite sums over all  $\mathfrak{L}$  do not appear in the definition of interactions, see (5.1). We can now transpose all properties of interactions  $\Phi$  in terms of interaction kernels  $\varphi \in \mathcal{K}$ .

First, we say that the interaction kernel  $\varphi$  has *finite range* iff there is a positive real number  $d_{\max}$  such that  $d(x, x') > d_{\max}$  (cf. (2.14)) implies

$$\varphi_n((\nu_1, s_1, x), (\nu_2, s_2, x'), X_3, \dots, X_n) = 0$$

for any integer  $n \geq 2$ , any  $(\nu_1, s_1), (\nu_2, s_2) \in \{+, -\} \times S$ , and all  $X_3, \dots, X_n \in \mathbb{X}_{\mathfrak{L}}$ . Because of the CAR (2.3) we can assume without loss of generality that, for any finite range interaction  $\varphi$ , there is  $N \in \mathbb{N}$  such that  $\varphi_n = 0$  for all  $n \geq N$ . Clearly, if the interaction kernel  $\varphi$  is finite range then the corresponding interaction  $\Phi = \Phi(\varphi)$  is also finite range.

An interaction kernel  $\varphi \in \mathcal{K}$  is *translation invariant* (t.i.) iff  $\alpha_x(\varphi) = \varphi$  for any  $x \in \mathbb{Z}^d$ . Here,  $\alpha_x$  is action of the group of lattice translations on the set  $\mathcal{K}$  defined by

$$\alpha_x(\varphi)_n((\nu_1, s_1, x_1), \dots, (\nu_n, s_n, x_n)) := \varphi_n((\nu_1, s_1, x_1 - x), \dots, (\nu_n, s_n, x_n - x)).$$

Note that the notation  $\alpha_x$  is also used to define via (2.7) the action of the group of lattice translations on  $\mathcal{U}$ . If the interaction kernel  $\varphi$  is t.i. then the interaction  $\Phi = \Phi(\varphi)$  is obviously translation invariant.

Additionally, the *gauge invariance* of interactions  $\Phi$  is translated in terms of interaction kernels  $\varphi \in \mathcal{K}$  via the following property: For any  $n \in 2\mathbb{N}$  and  $X_1 = (x_1, s_1, \nu_1), \dots, X_n = (x_n, s_n, \nu_n) \in \mathbb{X}_{\mathfrak{L}}$ ,

$$|\{k : \nu_k = +\}| \neq |\{k : \nu_k = -\}| \quad \text{implies} \quad \varphi_n(X_1, \dots, X_n) = 0.$$

Here,  $|\mathcal{X}|$  denotes the size (or cardinality) of a finite set  $\mathcal{X}$ .

To conclude, we introduce  $\ell_1$ -type norms in the case of t.i. interaction kernels. Observe that usual  $\ell_1$ -norms would have no meaning for t.i. functions as it would be either infinite or zero. Indeed, we define the norm  $\|\cdot\|_{1,\infty}$  on the space of t.i. anti-symmetric functions  $f_n$  on  $\mathbb{X}_{\mathfrak{L}}^n$  to be

$$\|f_n\|_{1,\infty} := \max_{X_1 \in \mathbb{X}_{\mathfrak{L}}} \sum_{X_2, \dots, X_n \in \mathbb{X}_{\mathfrak{L}}} |f_n(X_1, \dots, X_n)|.$$

Then via this norm we can mimic on interaction kernels  $\varphi$  norms of the form  $\|\cdot\|_{\mathcal{K}}$  introduced for t.i. interactions in Remark 2.25.

**DEFINITION 5.4** (The Banach space  $\mathcal{K}_1$  of t.i. interaction kernels).

*The real Banach space  $\mathcal{K}_1$  is the set of all t.i. interaction kernels  $\varphi$  with finite norm*

$$\|\varphi\|_{\mathcal{K}_1} := |\varphi_0| + \sum_{n=1}^{\infty} n \|\varphi_n\|_{1,\infty} < \infty.$$



Note that the set of finite range interaction kernels is dense in  $\mathcal{K}_1$ . In particular,  $\mathcal{K}_1$  is separable. One can also verify the following relations between the norms  $\|\cdot\|_{\mathcal{W}_1}$  and  $\|\cdot\|_{\mathcal{K}_1}$ :

LEMMA 5.5 (Relationship between  $\mathcal{K}_1$  and  $\mathcal{W}_1$  ).

- (i) For all  $\varphi \in \mathcal{K}_1$ ,  $\|\Phi(\varphi)\|_{\mathcal{W}_1} \leq 2|S| \|\varphi\|_{\mathcal{K}_1}$  with the size  $|S| \in \mathbb{N}$  of the finite set  $S$  being the dimension of the Hilbert space  $\mathcal{H}$  of spins.
- (ii) The set  $\{\Phi(\varphi) : \varphi \in \mathcal{K}_1\}$  of t.i. interactions formally defined by (5.3) is dense in  $\mathcal{W}_1$ .

A typical example of an interaction  $\Phi(\varphi) \in \mathcal{W}_1$  defined via an interaction kernel  $\varphi \in \mathcal{K}_1$  which is gauge invariant is the Hubbard model  $\Phi_{\text{Hubb}}$  defined as follows:  $S = \{\uparrow, \downarrow\}$  (because electrons have spin 1/2) and

$$\begin{aligned} \Phi_{\text{Hubb}} := & t \sum_{x,y \in \mathcal{L}, d(x,y)=1, s \in S} a_{x,s}^+ a_{y,s} + t' \sum_{x,y \in \mathcal{L}, d(x,y)=\sqrt{2}, s \in S} a_{x,s}^+ a_{y,s} \\ & - \mu \sum_{(x,s) \in \mathcal{L} \times S} a_{x,s}^+ a_{x,s} + \lambda \sum_{x \in \mathcal{L}} a_{x,\uparrow}^+ a_{x,\downarrow}^+ a_{x,\downarrow} a_{x,\uparrow}. \end{aligned}$$

Here,  $d(x, y)$  is the metric defined by (2.14) and so, the real parameters  $t, t', \mu$  and  $\lambda$  are respectively the nearest neighbor hopping amplitude, the next-to-nearest neighbor hopping amplitude, the chemical potential and the interaction between pairs of particles of different spins at the same site.

## 5.2. Periodic boundary conditions

We are now in position to introduce for any t.i. interaction kernel  $\varphi \in \mathcal{K}_1$  an interaction  $\tilde{\Phi}_l = \tilde{\Phi}_l(\varphi)$  with periodic boundary conditions:

DEFINITION 5.6 (Periodic interactions).

For any t.i. interaction kernel  $\varphi \in \mathcal{K}_1$  and each  $l \in \mathbb{N}$ , we define the interaction  $\tilde{\Phi}_l = \tilde{\Phi}_l(\varphi)$  with periodic boundary conditions as follows:

$$\begin{aligned} \tilde{\Phi}_{l,\Lambda} := & \mathbf{1}_{\{\Lambda \subset \Lambda_l\}} \sum_{\{X_i = (\nu_i, s_i, x_i) \in \mathbb{X}_{\mathcal{L}}\}_{i=1}^n, \{x_1, \dots, x_n\} = \Lambda} \\ & \left( \sum_{x'_2, \dots, x'_n \in \mathcal{L}, \xi_l(x'_i) = x_i} \varphi_n(X_1, X'_2, \dots, X'_n) : a(X_1) a(X_2) \cdots a(X_n) : \right) \end{aligned}$$

with  $X'_i := (\nu_i, s_i, x'_i)$ , the normal ordered product  $: a(X_1) \cdots a(X_n) :$  defined by (5.2), and  $\mathbb{X}_{\mathcal{L}} := \{+, -\} \times S \times \mathcal{L}$ . Here, the map  $\xi_l : \mathcal{L} \rightarrow \Lambda_l$  (cf. (2.2)) is defined, for the  $j^{\text{th}}$  coordinate, by  $\xi_l(x)_j = x_j \bmod 2l + 1$  with  $j = 1, \dots, d$ .

Since  $\varphi \in \mathcal{K}_1$ , observe that the operator  $\tilde{\Phi}_{l,\Lambda}$  is clearly bounded, i.e.,  $\|\tilde{\Phi}_{l,\Lambda}\| < \infty$  for all  $l \in \mathbb{N}$  and all  $\Lambda \in \mathcal{P}_f(\mathcal{L})$ . The subset  $\Lambda_l \subset \mathcal{L}$  can be seen within this

context as the torus  $\mathbb{Z}^d/((2l+1)\mathbb{Z})^d$ . Therefore, we say that the interaction  $\tilde{\Phi}_{l,\Lambda}$  fulfills *periodic boundary conditions* because it is invariant w.r.t. translations in its corresponding torus: For all  $x \in \mathbb{Z}^d$  and all  $\Lambda \subset \Lambda_l$ ,

$$\tilde{\Phi}_{l,\xi_l(\Lambda+x)} = \tilde{\alpha}_{l,x}(\tilde{\Phi}_{l,\Lambda}).$$

Here, the torus translation automorphisms  $\tilde{\alpha}_{l,x} : \mathcal{U}_{\Lambda_l} \rightarrow \mathcal{U}_{\Lambda_l}$ ,  $l \in \mathbb{N}$ ,  $x \in \mathbb{Z}^d$  are defined – uniquely – by the condition

$$\tilde{\alpha}_{l,x}(a_y) = \alpha_{\xi_l(x+y)}$$

for all  $y \in \Lambda_l$ .

Then we construct from the Banach space  $\mathcal{K}_1$  of interaction kernels the space

$$(5.4) \quad \mathcal{N}_1 := \mathcal{K}_1 \times \mathcal{L}^2(\mathcal{A}, \mathcal{K}_1) \times \mathcal{L}^2(\mathcal{A}, \mathcal{K}_1)$$

of (kernel) models as explained in Section 12.3 and define internal energies with periodic boundary conditions as follows:

**DEFINITION 5.7** (Internal energy with periodic boundary conditions).

For any  $\mathbf{n} := (\varphi, \{\varphi_a\}_{a \in \mathcal{A}}, \{\varphi'_a\}_{a \in \mathcal{A}}) \in \mathcal{N}_1$  and any  $l \in \mathbb{N}$ , the internal energy  $\tilde{U}_l$  in the box  $\Lambda_l$  with periodic boundary conditions is defined to be

$$\tilde{U}_l := U_{\Lambda_l}^{\tilde{\Phi}_l} + \frac{1}{|\Lambda_l|} \int_{\mathcal{A}} \gamma_a (U_{\Lambda_l}^{\tilde{\Phi}_{l,a}} + iU_{\Lambda_l}^{\tilde{\Phi}'_{l,a}})^* (U_{\Lambda_l}^{\tilde{\Phi}_{l,a}} + iU_{\Lambda_l}^{\tilde{\Phi}'_{l,a}}) d\mathbf{a}(a),$$

where  $\gamma_a \in \{-1, 1\}$  is a measurable function and with  $\tilde{\Phi}_l = \tilde{\Phi}_l(\varphi)$ ,  $\tilde{\Phi}_{l,a} = \tilde{\Phi}_l(\varphi_a)$ , and  $\tilde{\Phi}'_{l,a} = \tilde{\Phi}_l(\varphi'_a)$  for any  $a \in \mathcal{A}$ .

**NOTATION 5.8** (Model kernels).

The symbol  $\mathbf{n}$  is exclusively reserved to denote elements of  $\mathcal{N}_1$ .

Re-expressing objects in terms of interactions with periodic boundary conditions has the advantage that the notion of translation invariance is locally preserved. This implies, among other things, the translation invariance of the thermodynamic limit of Gibbs equilibrium states (Definition 12.1). It is an essential property to obtain a generalized t.i. equilibrium state in the thermodynamic limit.

**REMARK 5.9.** Any  $\mathbf{n} = (\varphi, \{\varphi_a\}_{a \in \mathcal{A}}, \{\varphi'_a\}_{a \in \mathcal{A}}) \in \mathcal{N}_1$  is identified with the long range model  $(\Phi(\varphi), \{\Phi(\varphi_a)\}_{a \in \mathcal{A}}, \{\Phi(\varphi'_a)\}_{a \in \mathcal{A}}) \in \mathcal{M}_1$  for a given  $\gamma_a$ .

### 5.3. Pressure and periodic boundary conditions

Periodic boundary conditions are very particular and idealized in which concerns the represented physical situations. Dirichlet-like or von Neumann-like boundary conditions are – physically speaking – more natural. In spite of that,

they are extensively used in theoretical or mathematical physics because they allow for the use of Fourier analysis, making computations much easier. In fact, we show the “universality” of periodic boundary conditions on the level of the pressure. This means that, for any  $\mathfrak{m} \in \mathcal{M}_1$ , the thermodynamic limit of the pressure (3.7) can be studied via models with periodic boundary conditions, see Definition 5.7.

Indeed, observe first that periodic boundary conditions do not change the internal energy per volume associated with any t.i. interaction kernel  $\varphi \in \mathcal{K}_1$ :

LEMMA 5.10 (Internal energy and periodic boundary conditions).

For any  $\varphi \in \mathcal{K}_1$ ,

$$\lim_{l \rightarrow \infty} \frac{1}{|\Lambda_l|} \|U_{\Lambda_l}^{\Phi(\varphi)} - U_{\Lambda_l}^{\tilde{\Phi}_l}\| = 0$$

with  $U_{\Lambda}^{\Phi}$ ,  $\Phi(\varphi)$ , and  $\tilde{\Phi}_l$  respectively defined by Definition 2.22, (5.1) (see also (5.3)) and Definition 5.6.

*Proof.* For any  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , let  $\Lambda^c := \mathfrak{L} \setminus \Lambda$  be its complement and we denote by

$$\hat{d}(x, \Lambda) := \min_{x' \in \Lambda} \{d(x, x')\}$$

the distance between any point  $x \in \mathbb{Z}^d$  and the set  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ . The latter is constructed via the metric  $d(x, x')$  defined by (2.14). It follows from Definitions 2.22 and 5.6 together with Equality (5.1) that

$$\begin{aligned} \|U_{\Lambda_l}^{\Phi(\varphi)} - U_{\Lambda_l}^{\tilde{\Phi}_l}\| &\leq \sum_{\{X_i \in \mathbb{X}_{\mathfrak{L}}\}_{i=1}^n, x_1 \in \Lambda_l, \{x_2, \dots, x_n\} \cap \Lambda_l^c \neq \emptyset} n |\varphi_n(X_1, \dots, X_n)| \\ &\leq \sum_{\{X_i \in \mathbb{X}_{\mathfrak{L}}\}_{i=1}^n, x_1 \in \Lambda_l, \{x_2, \dots, x_n\} \cap \Lambda_l^c \neq \emptyset} \left( \mathbf{1}_{\{\hat{d}(x_1, \Lambda_l^c) \leq \sqrt{l}\}} n |\varphi_n(X_1, \dots, X_n)| \right. \\ (5.5) \quad &\left. + \mathbf{1}_{\{\hat{d}(x_1, \Lambda_l^c) > \sqrt{l}\}} n |\varphi_n(X_1, \dots, X_n)| \right). \end{aligned}$$

We observe that

$$(5.6) \quad \lim_{l \rightarrow \infty} \frac{1}{|\Lambda_l|} \sum_{\{X_i \in \mathbb{X}_{\mathfrak{L}}\}_{i=1}^n, x_1 \in \Lambda_l, \{x_2, \dots, x_n\} \cap \Lambda_l^c \neq \emptyset} \mathbf{1}_{\{\hat{d}(x_1, \Lambda_l^c) \leq \sqrt{l}\}} n |\varphi_n(X_1, \dots, X_n)| = 0$$

as  $\|\varphi\|_{\mathcal{K}_1} < \infty$ . Moreover, since, by translation invariance of the interaction kernel  $\varphi$ ,

$$\begin{aligned} &\frac{1}{|\Lambda_l|} \sum_{\{X_i \in \mathbb{X}_{\mathfrak{L}}\}_{i=1}^n, x_1 \in \Lambda_l, \{x_2, \dots, x_n\} \cap \Lambda_l^c \neq \emptyset} \mathbf{1}_{\{\hat{d}(x_1, \Lambda_l^c) > \sqrt{l}\}} n |\varphi_n(X_1, \dots, X_n)| \\ &\leq \sum_{\{X_i \in \mathbb{X}_{\mathfrak{L}}\}_{i=2}^n} \sum_{(\nu, s) \in \{+, -\} \times S} \mathbf{1}_{\{\min\{|x_2|, \dots, |x_n|\} > \sqrt{l}\}} n |\varphi_n(\tilde{X}, X_2, \dots, X_n)| \end{aligned}$$

with  $\tilde{X} := (\nu, s, 0)$ , we use again  $\|\varphi\|_{\mathcal{K}_1} < \infty$  and Lebesgue's dominated convergence theorem to obtain that

$$(5.7) \quad \lim_{l \rightarrow \infty} \frac{1}{|\Lambda_l|} \sum_{\{X_i \in \mathbb{X}_{\mathcal{E}}\}_{i=1}^n, x_1 \in \Lambda_l, \{x_2, \dots, x_n\} \cap \Lambda_l^c \neq \emptyset} \mathbf{1}_{\{\hat{d}(x_1, \Lambda_l^c) > \sqrt{l}\}} n |\varphi_n(X_1, \dots, X_n)| = 0.$$

Therefore, the lemma follows from Inequality (5.5) together with the limits (5.6) and (5.7). ■

To show now that the pressure (3.7) can be studied via models with periodic boundary conditions, we need some preliminary definitions. First, for any  $\mathbf{n} \in \mathcal{N}_1$  and  $l \in \mathbb{N}$ , let

$$(5.8) \quad \tilde{p}_l = \tilde{p}_{l, \mathbf{n}} := \frac{1}{\beta |\Lambda_l|} \ln \text{Trace}_{\wedge \mathcal{H}_{\Lambda}} (e^{-\beta \tilde{U}_l})$$

be the pressure associated with the internal energy  $\tilde{U}_l$  (Definition 5.7). Then we extend the map  $\varphi \mapsto \Phi(\varphi)$  (cf. (5.3)) to a map  $\mathbf{n} \mapsto \mathbf{m}(\mathbf{n})$  from  $\mathcal{N}_1$  to  $\mathcal{M}_1$ . To simplify the notation let

$$(5.9) \quad \mathbf{n} \mapsto \mathbf{m}(\mathbf{n}) \mapsto f_{\mathbf{m}(\mathbf{n})}^{\#} =: f_{\mathbf{n}}^{\#}.$$

$f_{\mathbf{m}}^{\#}$  is seen below as map from  $\mathcal{M}_1$  to the set  $\mathcal{F}_{E_1}$  of affine functionals on  $E_1$  (see Definition 3.5 and Lemma 3.8 (i)), whereas  $f_{\mathbf{n}}^{\#}$  is seen as a map from  $\mathcal{N}_1$  to  $\mathcal{F}_{E_1}$  via (5.9). In the same way we have introduced the dense sub-spaces  $\mathcal{M}_1^f$ ,  $\mathcal{M}_1^d$ , and  $\mathcal{M}_1^{\text{df}}$  in Section 3.1, we finally define the dense sub-spaces  $\mathcal{N}_1^f$  and  $\mathcal{N}_1^d$  of  $\mathcal{N}_1$  to be, respectively, the sets of finite range  $\mathbf{n}$  and discrete elements  $\mathbf{n}$ , see Section 5.1 and (5.4). So,  $\mathcal{N}_1^{\text{df}} := \mathcal{N}_1^d \cap \mathcal{N}_1^f$  is the (dense) sub-space of finite range discrete elements  $\mathbf{n}$ .

We are now in position to give the main theorem of this subsection about the “universality” of periodic boundary conditions w.r.t. the pressure of long range Fermi systems.

**THEOREM 5.11** (Reduction to periodic boundary conditions).

For any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ , there exists  $\mathbf{n} \in \mathcal{N}_1^{\text{df}}$  such that:

$$(i) \lim_{l \rightarrow \infty} \{\tilde{p}_{l, \mathbf{n}} - p_{l, \mathbf{m}}\} = 0; \quad (ii) f_{\mathbf{m}}^{\#} = f_{\mathbf{n}}^{\#}.$$

*Proof.* For any finite range interaction  $\Phi \in \mathcal{W}_1$ , the energy observable  $\mathbf{e}_{\Phi} \in \mathcal{U}^+$  defined by (2.16) belongs to the set  $\mathcal{U}_0$  of local elements and thus, there is a finite range interaction kernel  $\varphi(\Phi)$  such that

$$(5.10) \quad \mathbf{e}_{\Phi} = \mathbf{e}_{\Phi(\varphi(\Phi))} \quad \text{and} \quad \|U_{\Lambda_l}^{\Phi} - U_{\Lambda_l}^{\Phi(\varphi(\Phi))}\| \leq \mathcal{O}(|\partial \Lambda_l|) = \mathcal{O}(l^{d-1})$$

with  $\partial\Lambda_l$  being the boundary<sup>19</sup> of the cubic box  $\Lambda_l$ . Therefore, for any finite range discrete model

$$\mathfrak{m} := \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\text{df}},$$

there exists

$$\mathfrak{n} := \{\varphi(\Phi)\} \cup \{\varphi(\Phi_k), \varphi(\Phi'_k)\}_{k=1}^N \in \mathcal{N}_1^{\text{df}}$$

satisfying (5.10) for each interaction  $\Phi$ ,  $\Phi_k$ , and  $\Phi'_k$ . Any  $\mathfrak{n} \in \mathcal{N}_1^{\text{df}}$  defines an internal energy  $\tilde{U}_l$  with periodic boundary conditions. So, the first statement (i) of the lemma is a consequence of the bound

$$(5.11) \quad |\ln(\text{Trace}_{\wedge\mathcal{H}_\Lambda}(e^A)) - \ln(\text{Trace}_{\wedge\mathcal{H}_\Lambda}(e^B))| \leq \|A - B\|$$

combined with Lemma 5.10 for any t.i. interaction kernel  $\varphi \in \mathcal{K}_1$ . The second statement (ii) is a direct consequence of (5.10). ■

REMARK 5.12. *Note that the restriction  $\mathfrak{m} \in \mathcal{M}_1^{\text{df}}$  in this last theorem is unimportant, see Corollary 8.3.*

#### 5.4. Gibbs and generalized t.i. equilibrium states

Periodic boundary conditions are, on the level of the pressure, universal in the sense described by Theorem 5.11. However, it is important to note that periodic boundary conditions do *not* yield a complete thermodynamic description of long range Fermi systems on the level of equilibrium states. As shown below (Theorem 5.13), any weak\*-convergent sequence of Gibbs equilibrium states (Definition 12.1) of long range Fermi systems with periodic boundary conditions converges to a generalized t.i. equilibrium state. The convergence of arbitrary convergent sequences  $\rho_l$  of (local) Gibbs equilibrium states of t.i. long range models  $\mathfrak{m} \in \mathcal{M}_1$  (defined by  $\rho_l := \rho_{\Lambda_l, U_l}$  (12.2)) towards a (infinite volume) generalized t.i. equilibrium state is, a priori, not clear and could in fact be even wrong in some cases (depending on boundary conditions). Together with Theorem 5.11, this means that the infimum over the set  $E$  of all states given in Theorem 8.8 (i) could also be attained by a sequence of approximating minimizers (cf. (3.9)) with weak\*-limit points not in  $E_1$  as explained in Section 3.5.

Therefore, we study now the convergence of (local) Gibbs equilibrium states only for the particular case of periodic boundary conditions, i.e., the convergence of the states  $\tilde{\rho}_l := \rho_{\Lambda_l, \tilde{U}_l}$  (12.2). Note that this state  $\tilde{\rho}_l$  is as usual seen as defined either on the local algebra  $\mathcal{U}_{\Lambda_l}$  or on the whole algebra  $\mathcal{U}$  by periodically extending it (with period  $(2l + 1)$  in each direction of the lattice

<sup>19</sup>By fixing  $m \geq 1$  the boundary  $\partial\Lambda$  of any  $\Lambda \subset \Gamma$  is defined by  $\partial\Lambda := \{x \in \Lambda : \exists y \in \Gamma \setminus \Lambda \text{ with } d(x, y) \leq m\}$ , see (2.14) for the definition of the metric  $d(x, y)$ .

$\mathfrak{L}$ ). Observe here that, by the definition of interaction kernels,  $\tilde{\rho}_l$  is an even state and hence products of translates of  $\tilde{\rho}_l$  are well-defined (cf. [8, Theorem 11.2.]). The Gibbs equilibrium state  $\tilde{\rho}_l$  is *generally not* translation invariant. We construct the space-averaged t.i. Gibbs state  $\hat{\rho}_l \in E_1$  from  $\tilde{\rho}_l$  as it is done in (3.19), that is,

$$\hat{\rho}_l := \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \tilde{\rho}_l \circ \alpha_x,$$

where we recall that the  $*$ -automorphisms  $\{\alpha_x\}_{x \in \mathbb{Z}^d}$  defined by (2.7) are the action of the group of lattice translations on  $\mathcal{U}$ . Then, from Theorems 3.12 (i) and 3.28, we prove the convergence of local states  $\tilde{\rho}_l$  and  $\hat{\rho}_l$  towards the same generalized t.i. equilibrium state:

**THEOREM 5.13** (Weak\*-limit of Gibbs equilibrium states).

*For any  $\mathfrak{n} \in \mathcal{N}_1$ , the states  $\tilde{\rho}_l$  and  $\hat{\rho}_l$  converge in the weak\*-topology along any convergent subsequence towards the same generalized t.i. equilibrium state  $\omega \in \Omega_{\mathfrak{n}}^{\sharp}$ .*

*Proof.* By weak\*-compactness of  $E_1$ , the space-averaged t.i. Gibbs state  $\hat{\rho}_l$  converges in the weak\*-topology along a subsequence towards  $\omega \in E_1$ . By translation invariance of  $\tilde{\rho}_l$  in the torus  $\Lambda_l$ , it is also easy to see that the sequences of states  $\tilde{\rho}_l$  and  $\hat{\rho}_l$  have the same weak\*-limit points. Then, since Theorem 3.28 says that  $T_{\mathfrak{m}}^{\sharp} = \Omega_{\mathfrak{m}}^{\sharp}$  for all  $\mathfrak{m} \in \mathcal{M}_1$ , we show that  $\omega \in T_{\mathfrak{n}}^{\sharp}$  in the same way we prove Theorem 3.29 because of Lemma 5.10, Theorem 5.11, and the density of the sets  $\mathcal{N}_1^{\text{df}}$  and  $\{\Phi(\varphi)\}_{\varphi \in \mathcal{K}_1}$  respectively in  $\mathcal{N}_1$  and  $\mathcal{W}_1$ . We omit the details. ■

## 6. The Set $E_{\vec{\ell}}$ of $\vec{\ell} \cdot \mathbb{Z}^d$ -Invariant States

In this section, we study in details the structure of the convex and weak\*-compact sets  $E_{\vec{\ell}}$  of  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant states defined by (2.8) for any  $\vec{\ell} \in \mathbb{N}^d$ . The set  $\mathcal{E}_{\vec{\ell}}$  of extreme points of  $E_{\vec{\ell}}$  is intimately related with a property of ergodicity (Definition 2.15). For  $\vec{\ell} = (1, \dots, 1)$ , the ergodicity of states is characterized via the space-averaging functional  $\Delta_A$  defined for any  $A \in \mathcal{U}$  in Definition 2.14.

We discuss in Section 6.2 the main structural properties of the set  $\mathcal{E}_{\vec{\ell}}$  and analyze the map  $\Delta_A$  in Section 6.3. The properties of the entropy density functional  $s$  defined in Definition 2.27 are discussed in Section 6.4. In Section 6.5 we analyze the energy density functional  $e_{\Phi}$  defined, for any t.i. interaction  $\Phi \in \mathcal{W}_1$ , in Definition 2.30. By means of the energy density  $e_{\Phi}$ , each  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  defines a continuous linear functional  $\mathbb{T}(\rho) \in \mathcal{W}_1^*$  on the Banach space  $\mathcal{W}_1$  (Definition 2.24). The map  $\rho \mapsto \mathbb{T}(\rho)$  restricted to the set  $E_1$  of t.i.

states is injective. This allows the identification of states of  $E_1$  with functionals of  $\mathcal{W}_1^*$ .

Note that some important statements presented here are standard (see, e.g., Theorems 6.1 and 6.2). They are given in Section 6.1 for completeness. We start with a preliminary discussion about the Gelfand–Naimark–Segal (GNS) representation of  $G$ -invariant states [17, Corollary 2.3.17] and then about von Neumann ergodic theorem [17, Proposition 4.3.4].

### 6.1. GNS representation and von Neumann ergodic theorem

Any state  $\rho \in E$  has a GNS representation [17, Theorem 2.3.16]: For any  $\rho \in E$ , there exist a Hilbert space  $\mathcal{H}_\rho$ , a representation  $\pi_\rho : \mathcal{U} \rightarrow \mathcal{B}(\mathcal{H}_\rho)$  from  $\mathcal{U}$  to the set  $\mathcal{B}(\mathcal{H}_\rho)$  of bounded operators on  $\mathcal{H}_\rho$ , and a cyclic vector  $\Omega_\rho \in \mathcal{H}_\rho$  w.r.t.  $\pi_\rho(\mathcal{U})$  such that, for all  $A \in \mathcal{U}$ ,

$$\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle.$$

The representation  $\pi_\rho$  is faithful if  $\rho$  is faithful, that is, if  $\rho(A^*A) = 0$  implies  $A = 0$ . The triple  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$  is unique up to unitary equivalence.

Assume now the existence of a group homomorphism  $g \mapsto \alpha_g$  from  $G$  to the group of  $*$ -automorphisms of  $\mathcal{U}$ . The state  $\rho$  is  $G$ -invariant iff  $\rho \circ \alpha_g = \rho$  for any  $g \in G$ . The GNS representation of such a  $G$ -invariant state  $\rho$  carries this symmetry through a uniquely defined family of unitary operators, see [17, Corollary 2.3.17]:

**THEOREM 6.1** (GNS representation of  $G$ -invariant states).

*Let  $\rho$  be a  $G$ -invariant state with GNS representation  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ . Then there is a uniquely defined family of unitary operators  $\{U_g\}_{g \in G}$  in  $\mathcal{B}(\mathcal{H}_\rho)$  with invariant vector  $\Omega_\rho$ , i.e.,  $\Omega_\rho = U_g\Omega_\rho$  for any  $g \in G$ , and such that  $\pi_\rho(\alpha_g(A)) = U_g\pi_\rho(A)U_g^*$  for any  $g \in G$  and  $A \in \mathcal{U}$ .*

Since we study the set  $E_{\vec{\ell}}$  (2.8) of  $\vec{\ell}$ -periodic states, the special cases we are interested in are  $G = (\vec{\ell} \cdot \mathbb{Z}^d, +)$  for all  $\vec{\ell} \in \mathbb{N}^d$ . The group homomorphism  $g \mapsto \alpha_g$  from  $G$  to the group of  $*$ -automorphisms of  $\mathcal{U}$  corresponds, in this case, to the group  $\{\alpha_x\}_{x \in \mathbb{Z}^d}$  (2.7) of lattice translations on  $\mathcal{U}$ . Within this framework, an essential ingredient of our analysis is von Neumann ergodic theorem [17, Proposition 4.3.4] which is a representative of the law of large numbers:

**THEOREM 6.2** (von Neumann ergodic theorem).

*Let  $x \mapsto U_x$  be a representation of the abelian group  $(\vec{\ell} \cdot \mathbb{Z}^d, +)$  by unitary operators on a Hilbert space  $\mathcal{H}$  and the set*

$$I := \bigcap_{x \in \vec{\ell} \cdot \mathbb{Z}^d} \{\psi \in \mathcal{H} : \psi = U_x(\psi)\}$$

be the closed sub-space of all invariant vectors. For any  $L \in \mathbb{N}$ , define the contraction

$$P^{(L)} := \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{x \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} U_x \in \mathcal{B}(\mathcal{H})$$

and denote the orthogonal projection on  $I$  by  $P$ . Then, for all  $L \in \mathbb{N}$ ,  $PP^{(L)} = P^{(L)}P = P$  and the operator  $P^{(L)}$  converges strongly to  $P$  as  $L \rightarrow \infty$ .

*Proof.* The proof of this statement is standard, see, e.g., [4, Theorem IV.2.2]. It is given here for completeness. Note that the property  $PP^{(L)} = P^{(L)}P = P$  is, in general, not explicitly given in the versions of von Neumann ergodic theorem found in textbooks.

Without loss of generality, assume that  $\vec{\ell} = (1, \dots, 1)$ . For any  $i \in \{1, \dots, d\}$ , let us consider the unitary operators  $U_i := U_{(\delta_{i,1}, \dots, \delta_{i,d})}$  with  $\delta_{i,j} = 0$  for any  $i \neq j$  and  $\delta_{i,i} = 1$ . Since  $\mathbb{Z}^d$  is abelian, the normal operators  $U_i$  for  $i \in \{1, \dots, d\}$  commute with each other. Their joint spectrum is contained in the  $d$ -dimensional torus

$$\mathbb{T}_d := \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_i| = 1, i = 1, \dots, d\}$$

and the spectral theorem ensures the existence of a projection-valued measure  $dP$  on the torus  $\mathbb{T}_d$  such that

$$(6.1) \quad P^{(L)} = \int_{\mathbb{T}_d} f_L(z_1, \dots, z_d) dP(z_1, \dots, z_d)$$

for any  $L \in \mathbb{N}$ , where

$$f_L(z_1, \dots, z_d) := \frac{1}{|\Lambda_L|} \sum_{(x_1, \dots, x_d) \in \Lambda_L} z_1^{x_1} \cdots z_d^{x_d}.$$

Observe that  $f_L$  converges point-wise as  $L \rightarrow \infty$  to the characteristic function of the set  $\{(1, \dots, 1)\} \subset \mathbb{T}_d$ , i.e.,

$$(6.2) \quad f_\infty(z_1, \dots, z_d) := \lim_{L \rightarrow \infty} f_L(z_1, \dots, z_d) = \begin{cases} 1 & \text{if } (z_1, \dots, z_d) = (1, \dots, 1). \\ 0 & \text{else.} \end{cases}$$

Hence, from (6.1), the operator  $P^{(L)}$  converges strongly to

$$(6.3) \quad P^{(\infty)} := \int_{\mathbb{T}_d} f_\infty(z_1, \dots, z_d) dP(z_1, \dots, z_d).$$

Note that the operator  $P^{(\infty)}$  is an orthogonal projection because of (6.2)–(6.3). Additionally,  $I \subset P^{(\infty)}(\mathcal{H})$  per definition and  $P^{(\infty)}(\mathcal{H}) \subset I$  by using (6.1)–(6.3) combined with  $U_{x+y} = U_y U_x$  for any  $x, y \in \mathbb{Z}^d$ . Therefore,  $P^{(\infty)} = P$  and from (6.3) together with  $f_L f_\infty = f_\infty$  we deduce that  $PP^{(L)} = P^{(L)}P = P$  for any  $L \in \mathbb{N}$ . ■



For any  $\rho \in E_{\vec{\ell}}$  with GNS representation  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ , we define  $P_\rho$  to be the strong limit of contractions  $P^{(L)}$  defined in Theorem 6.2 w.r.t. the unitary operators  $\{U_x\}_{x \in \vec{\ell} \cdot \mathbb{Z}^d}$  on the Hilbert space  $\mathcal{H}_\rho$  of Theorem 6.1 for  $G = (\vec{\ell} \cdot \mathbb{Z}^d, +)$ . By using the projection  $P_\rho$  and the equality  $\Omega_\rho = P_\rho \Omega_\rho$ , it is then easy to check that all  $\vec{\ell}$ -periodic states  $\rho \in E_{\vec{\ell}}$  are even (see, e.g., [5, Example 5.2.21]):

**COROLLARY 6.3** ( $\vec{\ell}$ -periodic states are even).

Let  $\rho \in E_{\vec{\ell}}$  with GNS representation  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ . Then, for all odd elements  $A \in \mathcal{U}$ ,  $P_\rho \pi_\rho(A) P_\rho = 0$ .

*Proof.* Since  $\rho$  is  $\vec{\ell}$ -periodic, by Theorem 6.1, there are unitary operators  $\{U_x\}_{x \in \vec{\ell} \cdot \mathbb{Z}^d}$  acting on  $\mathcal{H}_\rho$  and defining a representation of  $(\vec{\ell} \cdot \mathbb{Z}^d, +)$  such that  $U_x \Omega_\rho = \Omega_\rho$  and  $\pi_\rho(\alpha_x(A)) = U_x \pi_\rho(A) U_x^*$  for all  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ . The  $*$ -automorphism  $\alpha_x$  is defined by (2.7). If  $A \in \mathcal{U}$  is odd, i.e.,  $\sigma_\pi(A) = -A$  (cf. (2.4)), then

$$\lim_{|x| \rightarrow \infty} (A^* \alpha_x(A) + \alpha_x(A) A^*) = 0.$$

Consequently, by using Theorem 6.2 and observing that  $U_x P_\rho = P_\rho U_x = P_\rho$ , for any  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ ,

$$(P_\rho \pi_\rho(A)^* P_\rho)(P_\rho \pi_\rho(A) P_\rho) + (P_\rho \pi_\rho(A) P_\rho)(P_\rho \pi_\rho(A)^* P_\rho) = 0.$$

Both terms on the l.h.s. of the last equality are positive. Therefore, if  $A \in \mathcal{U}$  is odd then  $P_\rho \pi_\rho(A) P_\rho = 0$ . ■

The set  $E_{\vec{\ell}}$  is clearly convex, weak\*-compact, and also metrizable, by Theorem 12.10. By using Choquet theorem (Theorem 12.18), each state  $\rho \in E_{\vec{\ell}}$  has a decomposition in terms of states in the (non-empty) set  $\mathcal{E}_{\vec{\ell}}$  of extreme points of  $E_{\vec{\ell}}$ . The Choquet decomposition is, generally, not unique. However, in the particular case of the convex set  $E_{\vec{\ell}}$  the uniqueness of this decomposition follows from von Neumann ergodic theorem (Theorem 6.2):

**LEMMA 6.4** (Uniqueness of the Choquet decomposition in  $E_{\vec{\ell}}$ ).

For any  $\rho \in E_{\vec{\ell}}$ , the probability measure  $\mu_\rho$  given by Theorem 12.18 is unique and norm preserving in the sense that  $\|\rho - \rho'\| = \|\mu_\rho - \mu_{\rho'}\|$  for any  $\rho, \rho' \in E_{\vec{\ell}}$ . Here,  $\|\rho - \rho'\|$  and  $\|\mu_\rho - \mu_{\rho'}\|$  stand for the norms of  $(\rho - \rho')$  and  $(\mu_\rho - \mu_{\rho'})$  seen as linear functionals.

*Proof.* Observe that the map  $\rho \mapsto \mu_\rho$  is norm preserving, by [4, Theorem IV.4.1]. See also [4, Corollary IV.4.2] for the special case of spin systems. To prove the uniqueness of  $\mu_\rho$ , we adapt here the proof given in [4, Theorem IV.3.3] for quantum spin systems to our case of Fermi systems. For all  $A \in \mathcal{U}$ , let the (affine) weak\*-continuous map

$$\rho \mapsto \hat{A}(\rho) := \rho(A)$$

from the set  $E_{\vec{\ell}}$  to  $\mathbb{C}$ . The family  $\{\hat{A}\}_{A \in \mathcal{U}}$  of continuous functionals separates states, i.e., for all  $\rho, \rho' \in E_{\vec{\ell}}$  with  $\rho \neq \rho'$ , there is  $A \in \mathcal{U}$  such that  $\hat{A}(\rho) \neq \hat{A}(\rho')$ . Thus, by the Stone–Weierstrass theorem, the uniqueness of the probability measure  $\mu_\rho$  of Theorem 12.18 is equivalent to the uniqueness of the complex numbers

$$(6.4) \quad \mu_\rho(\hat{A}_1 \cdots \hat{A}_n) = \int_{E_{\vec{\ell}}} d\mu_\rho(\hat{\rho}) \hat{\rho}(A_1) \cdots \hat{\rho}(A_n), \quad A_1, \dots, A_n \in \mathcal{U}, \quad n \in \mathbb{N}.$$

By von Neumann ergodic theorem (Theorem 6.2), for any  $\rho \in E_{\vec{\ell}}$ ,  $A_1, \dots, A_n \in \mathcal{U}$  and  $n \in \mathbb{N}$ ,

$$\lim_{L \rightarrow \infty} \rho \left( (A_1)_{L, \vec{\ell}} \cdots (A_n)_{L, \vec{\ell}} \right) = \langle \Omega_\rho, \pi_\rho(A_1) P_\rho \pi_\rho(A_1) P_\rho \cdots P_\rho \pi_\rho(A_1) \Omega_\rho \rangle.$$

Recall that  $A_{L, \vec{\ell}}$  is defined by (2.9) for any  $A \in \mathcal{U}$ ,  $L \in \mathbb{N}$ , and any  $\vec{\ell} \in \mathbb{N}^d$ . By Lemma 6.8 below, the projection  $P_\rho$  is one-dimensional with  $\text{ran } P_\rho = \mathbb{C} \Omega_\rho$  whenever  $\rho \in \mathcal{E}_{\vec{\ell}}$  is extreme in  $E_{\vec{\ell}}$ . In particular, for all extreme states  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  and all  $A_1, \dots, A_n \in \mathcal{U}$ ,  $n \in \mathbb{N}$ ,

$$\hat{\rho}(A_1) \cdots \hat{\rho}(A_n) = \lim_{L \rightarrow \infty} \hat{\rho} \left( (A_1)_{L, \vec{\ell}} \cdots (A_n)_{L, \vec{\ell}} \right).$$

Hence, as  $\mu_\rho(E_{\vec{\ell}} \setminus \mathcal{E}_{\vec{\ell}}) = 0$  (Theorem 12.18), by using (6.4) together with Lebesgue's dominated convergence, it follows that, for any  $A_1, \dots, A_n \in \mathcal{U}$  with  $n \in \mathbb{N}$ , the complex number

$$\begin{aligned} \mu_\rho(\hat{A}_1 \cdots \hat{A}_n) &= \lim_{L \rightarrow \infty} \int_{E_{\vec{\ell}}} d\mu_\rho(\hat{\rho}) \hat{\rho} \left( (A_1)_{L, \vec{\ell}} \cdots (A_n)_{L, \vec{\ell}} \right) \\ &= \lim_{L \rightarrow \infty} \rho \left( (A_1)_{L, \vec{\ell}} \cdots (A_n)_{L, \vec{\ell}} \right) \end{aligned}$$

is uniquely determined. ■

As a consequence, the set  $E_{\vec{\ell}}$  is a (Choquet) simplex, see Definition 12.21 and Theorem 12.22.

## 6.2. The set $\mathcal{E}_{\vec{\ell}}$ of extreme states of $E_{\vec{\ell}}$

We want to prove next that all extreme states are ergodic w.r.t. the space-average (2.9) (Definition 2.15) and conversely. The fact that all ergodic states are extreme is not difficult to verify:

LEMMA 6.5 (Ergodicity implies extremality).

Any ergodic state  $\rho \in E_{\vec{\ell}}$  is extreme in  $E_{\vec{\ell}}$ , i.e.,  $\rho \in \mathcal{E}_{\vec{\ell}}$ .

*Proof.* If  $\rho \notin \mathcal{E}_{\vec{\ell}}$  is not extreme, there are two states  $\rho_1, \rho_2 \in E_{\vec{\ell}}$  with  $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$  and  $\rho_1(A) \neq \rho_2(A)$  for some  $A = A^* \in \mathcal{U}$ . Then

$$(6.5) \quad |\rho(A)|^2 < \frac{1}{2}|\rho_1(A)|^2 + \frac{1}{2}|\rho_2(A)|^2.$$

For all  $\vec{\ell} \in \mathbb{N}^d$  and any state  $\rho \in E_{\vec{\ell}}$  with GNS representation  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ , by Theorem 6.1 for  $G = (\vec{\ell} \cdot \mathbb{Z}^d, +)$  and Theorem 6.2, we get

$$(6.6) \quad \Delta_{A, \vec{\ell}}(\rho) := \lim_{L \rightarrow \infty} \rho(A_{L, \vec{\ell}}^* A_{L, \vec{\ell}}) = \lim_{L \rightarrow \infty} \|P_\rho^{(L)} \pi_\rho(A) \Omega_\rho\|^2 = \|P_\rho \pi_\rho(A) \Omega_\rho\|^2.$$

Using Cauchy–Schwarz inequality together with  $P_\rho \Omega_\rho = \Omega_\rho$  (Theorem 6.2),

$$|\rho(A)|^2 = |\langle \Omega_\rho, P_\rho \pi_\rho(A) \Omega_\rho \rangle|^2 \leq \|P_\rho \pi_\rho(A) \Omega_\rho\|^2 = \Delta_{A, \vec{\ell}}(\rho)$$

for any state  $\rho \in E_{\vec{\ell}}$ . Applying the last inequality to states  $\rho_1$  and  $\rho_2$  we conclude from (6.5) that

$$|\rho(A)|^2 < \frac{1}{2} \Delta_{A, \vec{\ell}}(\rho_1) + \frac{1}{2} \Delta_{A, \vec{\ell}}(\rho_2) = \Delta_{A, \vec{\ell}}(\rho).$$

It follows that  $\rho \notin \mathcal{E}_{\vec{\ell}}$  is not ergodic. ■

The last lemma is elementary, but it implies an essential topological property of the set  $\mathcal{E}_{\vec{\ell}}$  of extreme points of the convex and weak\*-compact set  $E_{\vec{\ell}}$ :

**COROLLARY 6.6** (Density of the set  $\mathcal{E}_{\vec{\ell}}$  of extreme points of  $E_{\vec{\ell}}$ ).

For any  $\vec{\ell} \in \mathbb{N}^d$ , the set  $\mathcal{E}_{\vec{\ell}}$  is a  $G_\delta$  weak\*-dense subset of  $E_{\vec{\ell}}$ .

*Proof.* The proof of this lemma is a slight adaptation of the proof of [4, Lemma IV.3.2.] for quantum spin systems to the case of even states over the fermion algebra  $\mathcal{U}$ . It is a *pivotal* proof in the sequel.

The set  $\mathcal{E}_{\vec{\ell}}$  of extreme points of  $E_{\vec{\ell}}$  is a  $G_\delta$  set, by Theorem 12.13 (i), as  $E_{\vec{\ell}}$  is metrizable. Thus, it suffices to prove that  $\mathcal{E}_{\vec{\ell}}$  is dense in  $E_{\vec{\ell}}$ . For any  $\rho \in E_{\vec{\ell}}$ , we define the state  $\tilde{\rho}_n$  to be the restriction  $\rho_{\Lambda_n} \in E_{\Lambda_n}$  on the box

$$(6.7) \quad \Lambda_{n, \vec{\ell}} := \left\{ x = (x_1, \dots, x_d) \in \mathbb{Z}^d : |x_i| \leq n\ell_i \right\}$$

seen as a  $(2n+1)\vec{\ell}$ -periodic state. This is possible, by [8, Theorem 11.2.], because any  $\vec{\ell}$ -periodic state is even, by Corollary 6.3. From the state  $\tilde{\rho}_n \in E_{(2n+1)\vec{\ell}}$  we define next the  $\vec{\ell}$ -periodic state

$$(6.8) \quad \hat{\rho}_n := \frac{1}{|\Lambda_{n, \vec{\ell}} \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{x \in \Lambda_{n, \vec{\ell}} \cap \vec{\ell} \cdot \mathbb{Z}^d} \tilde{\rho}_n \circ \alpha_x \in E_{\vec{\ell}}.$$

Clearly, the space-averaged state  $\hat{\rho}_n$  converges towards  $\rho \in E_{\vec{\ell}}$  w.r.t. the weak\*-topology and we prove below that  $\hat{\rho}_n \in \mathcal{E}_{\vec{\ell}}$  by using Lemma 6.5.

Indeed, for any  $A \in \mathcal{U}_0$ , there is a positive constant  $C > 0$  such that

$$\tilde{\rho}_n(\alpha_x(A^*) \alpha_y(A)) = \tilde{\rho}_n(\alpha_x(A^*)) \tilde{\rho}_n(\alpha_y(A))$$

whenever  $d(x, y) \geq C$ . Here,  $d : \mathfrak{L} \times \mathfrak{L} \rightarrow [0, \infty)$  is the usual metric defined on the lattice  $\mathfrak{L} := \mathbb{Z}^d$  by (2.14). Using the space-average  $A_{L, \vec{\ell}}$  defined by (2.9)

we then deduce that

$$(6.9) \quad \tilde{\rho}_n(A_{L,\vec{\ell}}^* A_{L,\vec{\ell}}) = \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|^2} \sum_{x,y \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} \tilde{\rho}_n(\alpha_x(A^*)) \tilde{\rho}_n(\alpha_y(A)) + \mathcal{O}(L^{-d}).$$

Since  $\hat{\rho}_n \in E_{\vec{\ell}}$  is a  $\vec{\ell}$ -periodic state, for any  $A \in \mathcal{U}_0$ , one has that

$$\frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{x \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} \tilde{\rho}_n(\alpha_x(A)) = \hat{\rho}_n(A) + \mathcal{O}(L^{-1})$$

which combined with the asymptotics (6.9) implies that

$$\lim_{L \rightarrow \infty} \tilde{\rho}_n(A_{L,\vec{\ell}}^* A_{L,\vec{\ell}}) = |\hat{\rho}_n(A)|^2.$$

Using this last equality we then obtain from (6.8) that, for any  $A \in \mathcal{U}_0$ ,

$$(6.10) \quad \lim_{L \rightarrow \infty} \hat{\rho}_n(A_{L,\vec{\ell}}^* A_{L,\vec{\ell}}) = |\hat{\rho}_n(A)|^2$$

because  $\hat{\rho}_n \in E_{\vec{\ell}}$  and

$$(6.11) \quad \alpha_x(A_{L,\vec{\ell}}^* A_{L,\vec{\ell}}) = (\alpha_x(A))_{L,\vec{\ell}}^* (\alpha_x(A))_{L,\vec{\ell}}$$

for all  $x \in \mathbb{Z}^d$ . Since the set  $\mathcal{U}_0$  is dense in the fermion algebra  $\mathcal{U}$ , we can extend (6.10) to any  $A \in \mathcal{U}$  which shows that the state  $\hat{\rho}_n \in E_{\vec{\ell}}$  is ergodic and thus extreme in  $E_{\vec{\ell}}$ , by Lemma 6.5. ■

We show now the converse of Lemma 6.5 which is not as obvious as the proof of Lemma 6.5. Take, for instance, the trivial action of the group  $(\vec{\ell} \cdot \mathbb{Z}^d, +)$  on the  $C^*$ -algebra  $\mathcal{U}$  given by  $\tilde{\alpha}_x : A \mapsto A$  for all  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ . Observe that w.r.t. this choice, the set of invariant states is simply the set  $E$  of all states. Then, by the proof of Lemma 6.5, any ergodic state w.r.t. to this action is again an extreme point of the set of all states. But, generally, extreme states are not ergodic w.r.t. to the trivial action of  $\vec{\ell} \cdot \mathbb{Z}^d$ : Consider for simplicity the case of quantum spin systems (cf. Remark 2.4). For a given element  $A \in \mathcal{U}$  such that  $A^*A \neq A$ , we can always find a state  $\rho$  satisfying  $\rho(A^*A) \neq |\rho(A)|^2$  and thus, because the set  $\mathcal{E}$  of extreme states of  $E$  is weak\*-dense in  $E$  (see [17, Example 4.1.31.]), there is an extreme state with this property.

In order to get the equivalence between ergodicity and extremality of states, the *asymptotic abelianess* of the even sub-algebra  $\mathcal{U}^+$  (2.5), i.e., the fact that

$$(6.12) \quad \lim_{|x| \rightarrow \infty} [A, \alpha_x(B)] = 0 \quad \text{for any } A, B \in \mathcal{U}^+,$$

is crucial.

Indeed, for any state  $\rho \in E_{\vec{\ell}}$  with GNS representation  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ , let us first consider the von Neumann algebra

$$\mathfrak{A}_\rho := [\pi_\rho(\mathcal{U}) \cup \{U_x\}_{x \in \vec{\ell} \cdot \mathbb{Z}^d}]'' \subset B(\mathcal{H}_\rho).$$

Here,  $\{U_x\}_{x \in \vec{\ell}\mathbb{Z}^d}$  are the unitary operators of Theorem 6.1 with  $G = (\vec{\ell}\mathbb{Z}^d, +)$ . This von Neumann algebra is related to the projection  $P_\rho := P$  defined via Theorem 6.2 for  $\mathcal{H} = \mathcal{H}_\rho$ :

LEMMA 6.7 (Properties of the von Neumann algebra  $\mathfrak{R}_\rho$ ).

For any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$ ,  $P_\rho \in \mathfrak{R}_\rho$  and  $P_\rho \mathfrak{R}_\rho P_\rho$  is an abelian von Neumann algebra on  $P_\rho \mathcal{H}_\rho$ .

*Proof.* On the one hand, by Theorem 6.2, the projection  $P_\rho$  is the strong limit of linear combinations of unitary operators  $U_x$  for  $x \in \vec{\ell}\mathbb{Z}^d$  and so,  $P_\rho \in \mathfrak{R}_\rho$ . On the other hand, if  $\mathfrak{M}$  is a von Neumann algebra on a Hilbert space  $\mathcal{H}$  and  $P$  is any projection from  $\mathfrak{M}$ , the set  $P\mathfrak{M}P$  is a von Neumann algebra on  $P\mathcal{H}$ . See, e.g., [4, Lemma IV.2.5]. Therefore, it remains to show that  $P_\rho \mathfrak{R}_\rho P_\rho$  is abelian. To prove it we adapt now the proof of [4, Lemma IV.2.6] – performed for quantum spin systems – to the case where  $\mathcal{U}$  is a fermion algebra. In particular, we show first that  $P_\rho \mathfrak{R}_\rho P_\rho = [P_\rho \pi_\rho(\mathcal{U}) P_\rho]''$  and then the abelianness of  $[P_\rho \pi_\rho(\mathcal{U}) P_\rho]''$ .

Since  $U_x \pi_\rho(A) = \pi_\rho(\alpha_x(A)) U_x$ , it follows that each element  $B \in \mathfrak{R}_\rho$  is the strong limit as  $n \rightarrow \infty$  of a sequence of elements of the form

$$B_n := \sum_j U_{x_j} \pi_\rho(A_j)$$

with  $x_j \in \vec{\ell}\mathbb{Z}^d$  and  $A_j \in \mathcal{U}$ . In particular, by using Theorem 6.2, each element of  $P_\rho \mathfrak{R}_\rho P_\rho$  is the strong limit as  $n \rightarrow \infty$  of elements of the form

$$P_\rho B_n P_\rho = P_\rho \pi_\rho\left(\sum_j A_j\right) P_\rho.$$

In other words, since  $P_\rho \pi_\rho(\mathcal{U}) P_\rho \subset P_\rho \mathfrak{R}_\rho P_\rho$  is clear, we deduce from the last equality that  $P_\rho \mathfrak{R}_\rho P_\rho = [P_\rho \pi_\rho(\mathcal{U}) P_\rho]''$  as  $[P_\rho \pi_\rho(\mathcal{U}) P_\rho]''$  is the strong closure of  $P_\rho \pi_\rho(\mathcal{U}) P_\rho$ .

Take now two local even elements  $A, B \in \mathcal{U}_\Lambda \cap \mathcal{U}^+$  with  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ . Then via Theorem 6.2, for all  $\xi \in \mathcal{H}_\rho$ ,

$$\begin{aligned} (6.13) \quad & [(P_\rho \pi_\rho(A) P_\rho) (P_\rho \pi_\rho(B) P_\rho) - (P_\rho \pi_\rho(B) P_\rho) (P_\rho \pi_\rho(A) P_\rho)] \xi \\ &= \lim_{L \rightarrow \infty} \left[ \left( P_\rho \pi_\rho(A) P_\rho^{(L)} \pi_\rho(B) P_\rho \right) - \left( P_\rho \pi_\rho(B) P_\rho^{(L)} \pi_\rho(A) P_\rho \right) \right] \xi \\ &= \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L \cap \vec{\ell}\mathbb{Z}^d|} \sum_{x \in \Lambda_L \cap \vec{\ell}\mathbb{Z}^d} P_\rho [A, \alpha_x(B)] P_\rho = 0 \end{aligned}$$

because  $[A, \alpha_x(B)] = 0$  for any  $x \in \mathbb{Z}^d$  such that  $d(x, 0) \geq 2|\Lambda|$ , see (2.14) for the definition of the metric  $d$ . From Corollary 6.3, recall that  $P_\rho \pi_\rho(A) P_\rho = 0$  for any odd element  $A \in \mathcal{U}$ . Therefore, by combining this with the density of the  $*$ -algebra  $\mathcal{U}_0 \subset \mathcal{U}$  of local elements, we can extend the equality (6.13) to

any  $A, B \in \mathcal{U}$ , i.e., for all  $A, B \in \mathcal{U}$ ,

$$[P_\rho \pi_\rho(A) P_\rho, P_\rho \pi_\rho(B) P_\rho] = 0.$$

In other words,  $P_\rho \pi_\rho(\mathcal{U}) P_\rho$  is abelian. Since  $P_\rho \pi_\rho(\mathcal{U}) P_\rho$  is strongly dense in  $[P_\rho \pi_\rho(\mathcal{U}) P_\rho]'' = P_\rho \mathfrak{A}_\rho P_\rho$ , the von Neumann algebra  $P_\rho \mathfrak{A}_\rho P_\rho$  is itself abelian.

■

We are now in position to show that all extreme points  $\rho \in \mathcal{E}_{\vec{\ell}}$  of  $E_{\vec{\ell}}$  are ergodic.

LEMMA 6.8 (Extremality implies ergodicity).

For any extreme state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  of  $E_{\vec{\ell}}$ ,  $P_{\hat{\rho}}$  is the orthogonal projection on the one-dimensional sub-space generated by  $\Omega_{\hat{\rho}}$ . In particular, any state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  is ergodic.

*Proof.* For any  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$ , observe that the von Neumann algebra  $\mathfrak{A}_{\hat{\rho}}$  is irreducible, i.e.,  $\mathfrak{A}'_{\hat{\rho}} = \mathbb{C} \mathbf{1}$ . Indeed, by contradiction, assume that  $\mathfrak{A}'_{\hat{\rho}}$  is strictly larger than its sub-algebra  $\mathbb{C} \mathbf{1}$ . Then there is at least one non-trivial (orthogonal) projection  $P \in \mathfrak{A}'_{\hat{\rho}}$ . By cyclicity of  $\Omega_{\hat{\rho}}$  w.r.t.  $\mathfrak{A}'_{\hat{\rho}}$ ,  $P \Omega_{\hat{\rho}} \neq 0$  and thus  $\langle \Omega_{\hat{\rho}}, P \Omega_{\hat{\rho}} \rangle = \|P \Omega_{\hat{\rho}}\|_2^2 > 0$ . Similarly,  $\langle \Omega_{\hat{\rho}}, (\mathbf{1} - P) \Omega_{\hat{\rho}} \rangle > 0$ . Define the following continuous linear functionals on  $\mathcal{U}$ :

$$\begin{aligned} \rho_1(A) &:= \langle \Omega_{\hat{\rho}}, P \Omega_{\hat{\rho}} \rangle^{-1} \langle \Omega_{\hat{\rho}}, P \pi_{\hat{\rho}}(A) \Omega_{\hat{\rho}} \rangle, \\ \rho_2(A) &:= \langle \Omega_{\hat{\rho}}, (\mathbf{1} - P) \Omega_{\hat{\rho}} \rangle^{-1} \langle \Omega_{\hat{\rho}}, (\mathbf{1} - P) \pi_{\hat{\rho}}(A) \Omega_{\hat{\rho}} \rangle. \end{aligned}$$

Observe that, by cyclicity of  $\Omega_{\hat{\rho}}$  w.r.t.  $\pi_{\hat{\rho}}(\mathcal{U})$ ,  $\rho_1 \neq \rho_2$ . Since  $U_x \Omega_{\hat{\rho}} = \Omega_{\hat{\rho}}$  and  $P$  commutes per definition with  $\pi_{\hat{\rho}}(A)$  and  $U_x$  for all  $A \in \mathcal{U}$  and  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ , the functionals  $\rho_1$  and  $\rho_2$  belong to  $E_{\vec{\ell}}$ , whereas

$$\hat{\rho} = \langle \Omega_{\hat{\rho}}, P \Omega_{\hat{\rho}} \rangle \rho_1 + \langle \Omega_{\hat{\rho}}, (\mathbf{1} - P) \Omega_{\hat{\rho}} \rangle \rho_2.$$

Since  $\langle \Omega_{\hat{\rho}}, (\mathbf{1} - P) \Omega_{\hat{\rho}} \rangle > 0$  and  $\langle \Omega_{\hat{\rho}}, P \Omega_{\hat{\rho}} \rangle > 0$ , this last equality contradicts the fact that  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$ . Therefore,  $\mathfrak{A}'_{\hat{\rho}} = \mathbb{C} \mathbf{1}$  whenever  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$ .

Observe now that

$$(6.14) \quad [P_{\hat{\rho}} \mathfrak{A}_{\hat{\rho}} P_{\hat{\rho}}]' = P_{\hat{\rho}} \mathfrak{A}'_{\hat{\rho}} P_{\hat{\rho}} = \mathbb{C} P_{\hat{\rho}}.$$

Here we use that, for any von Neumann algebra  $\mathfrak{M}$  and any orthogonal projection  $P \in \mathfrak{M}$ ,  $[P \mathfrak{M} P]' = P \mathfrak{M}' P$ , see, e.g., [4, Lemma IV.2.5]. By Lemma 6.7, the von Neumann algebra  $P_{\hat{\rho}} \mathfrak{A}_{\hat{\rho}} P_{\hat{\rho}}$  is abelian. In particular, from (6.14),

$$P_{\hat{\rho}} \mathfrak{A}_{\hat{\rho}} P_{\hat{\rho}} \subset P_{\hat{\rho}} \mathfrak{A}'_{\hat{\rho}} P_{\hat{\rho}} = \mathbb{C} P_{\hat{\rho}}$$

which implies that  $P_{\hat{\rho}} \mathfrak{A}_{\hat{\rho}} P_{\hat{\rho}} = \mathbb{C} P_{\hat{\rho}}$ . This yields

$$P_{\hat{\rho}} \pi_{\hat{\rho}}(A) \Omega_{\hat{\rho}} = P_{\hat{\rho}} \pi_{\hat{\rho}}(A) P_{\hat{\rho}} \Omega_{\hat{\rho}} \in \mathbb{C} P_{\hat{\rho}} \Omega_{\hat{\rho}} = \mathbb{C} \Omega_{\hat{\rho}}$$

for any  $A \in \mathcal{U}$ . In other words, by cyclicity of  $\Omega_{\hat{\rho}}$ ,  $P_{\hat{\rho}}\mathcal{H}_{\hat{\rho}} = \mathbb{C}\Omega_{\hat{\rho}}$  and thus

$$\begin{aligned} \|P_{\hat{\rho}}\pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}}\|^2 &= \langle P_{\hat{\rho}}\pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}}, P_{\hat{\rho}}\pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}} \rangle \\ &= \langle P_{\hat{\rho}}\pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}}, \Omega_{\hat{\rho}} \rangle \langle \Omega_{\hat{\rho}}, P_{\hat{\rho}}\pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}} \rangle \\ &= \langle \pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}}, \Omega_{\hat{\rho}} \rangle \langle \Omega_{\hat{\rho}}, \pi_{\hat{\rho}}(A)\Omega_{\hat{\rho}} \rangle \end{aligned}$$

implying, by (6.6), that any state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  is ergodic.  $\blacksquare$

As we can relate the ergodicity with the so-called *strongly clustering* property [17, Section 4.3.2], we deduce from Lemma 6.8 that any extreme state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  is strongly clustering:

**COROLLARY 6.9** (Extreme states are strongly clustering).

*Any extreme state  $\hat{\rho} \in \mathcal{E}_{\vec{\ell}}$  is strongly clustering, i.e., for all  $A, B \in \mathcal{U}$ ,*

$$(6.15) \quad \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{y \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} \hat{\rho}(\alpha_x(A)\alpha_y(B)) = \hat{\rho}(A)\hat{\rho}(B)$$

*uniformly in  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ .*

*Proof.* This corollary can directly be seen from Lemma 6.8 combined with Theorem 6.2 because

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \sum_{y \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d} \hat{\rho}(\alpha_x(A)\alpha_y(B)) &= \lim_{L \rightarrow \infty} \langle U_x \pi_{\hat{\rho}}(A^*) \Omega_{\hat{\rho}}, P_{\hat{\rho}}^{(L)} \pi_{\hat{\rho}}(B) \Omega_{\hat{\rho}} \rangle \\ &= \langle U_x \pi_{\hat{\rho}}(A^*) \Omega_{\hat{\rho}}, P_{\hat{\rho}} \pi_{\hat{\rho}}(B) \Omega_{\hat{\rho}} \rangle \\ &= \langle \Omega_{\hat{\rho}}, \pi_{\hat{\rho}}(A) \Omega_{\hat{\rho}} \rangle \langle \Omega_{\hat{\rho}}, \pi_{\hat{\rho}}(B) \Omega_{\hat{\rho}} \rangle \end{aligned}$$

for any  $A, B \in \mathcal{U}$  and  $x \in \vec{\ell} \cdot \mathbb{Z}^d$ . By using Cauchy–Schwarz inequality, note that the limit  $L \rightarrow \infty$  is uniform in  $x \in \vec{\ell} \cdot \mathbb{Z}^d$  because  $P_{\hat{\rho}}^{(L)}$  converges strongly to the projection  $P_{\hat{\rho}}$ . See Theorem 6.2.  $\blacksquare$

Therefore, Theorem 2.16 is a consequence of Lemmata 6.5 and 6.8 together with Corollary 6.9.

### 6.3. Properties of the space-averaging functional $\Delta_A$

We characterize now the properties of the space-averaging functional  $\Delta_A$  defined in Definition 2.14 for any  $A \in \mathcal{U}$  because it is intimately related with the structure of the set  $E_1$  of t.i. states. We start by proving that this functional is well-defined, even for  $\vec{\ell}$ -periodic states  $\rho \in E_{\vec{\ell}}$ :

**LEMMA 6.10** (Well-definiteness of the map  $\rho \mapsto \Delta_A(\rho)$ ).

*For any  $A \in \mathcal{U}$ , the space-averaging functional  $\Delta_A$  is well-defined on the set  $E_{\vec{\ell}}$  of  $\vec{\ell}$ -periodic states for any  $\vec{\ell} \in \mathbb{N}^d$  and it satisfies*

$$\Delta_A(\rho) = \inf_{(L, \dots, L) \in \vec{\ell} \cdot \mathbb{N}^d} \{ \rho(A_L^* A_L) \} \in [|\rho(A_{\vec{\ell}})|^2, \|A\|^2].$$

*Proof.* Assume that  $(L, \dots, L) \in \vec{\ell}\mathbb{N}^d$ . In the same way we prove (6.6), for any state  $\rho \in E_{\vec{\ell}}$  with GNS representation  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ , we obtain, by using Theorem 6.1 for  $G = (\vec{\ell}\mathbb{Z}^d, +)$  and Theorem 6.2 for  $\mathcal{H} = \mathcal{H}_\rho$ , that

$$(6.16) \quad \lim_{L \rightarrow \infty} \rho(A_L^* A_L) = \lim_{L \rightarrow \infty} \|P_\rho^{(L)} \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2 = \|P_\rho \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2 \leq \|A\|^2.$$

The inequality  $\Delta_A(\rho) \geq |\rho(A_{\vec{\ell}})|^2$  then follows by using the Cauchy–Schwarz inequality and  $P_\rho \Omega_\rho = \Omega_\rho$ . Additionally, by using again Theorem 6.2 we see that, for all  $(L, \dots, L) \in \vec{\ell}\mathbb{N}^d$ ,

$$\|P_\rho^{(L)} \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2 \geq \|P_\rho P_\rho^{(L)} \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2 = \|P_\rho \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2.$$

Therefore, the functional  $\Delta_A$  is an infimum over  $(L, \dots, L) \in \vec{\ell}\mathbb{N}^d$  as claimed in the lemma.

Now, there is a constant  $C < \infty$  such that, for all  $L' \in \mathbb{N}$ , there is  $L \in \mathbb{N}$  such that  $|L - L'| \leq C$  and  $(L, \dots, L) \in \vec{\ell}\mathbb{N}^d$ . It follows that

$$\rho(A_{L'}^* A_{L'}) = \rho(A_L^* A_L) + \mathcal{O}(L^{-1}),$$

which implies, for any diverging sequence  $\{L_n\}_{n=1}^\infty$  of natural numbers, that

$$\lim_{n \rightarrow \infty} \rho(A_{L_n}^* A_{L_n}) = \|P_\rho \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2 \in [|\rho(A_{\vec{\ell}})|^2, \|A\|^2]$$

because of (6.16). ■

From Lemma 6.10 we deduce now the main properties of the functional  $\Delta_A$ :

LEMMA 6.11 (Weak\*–upper semi–continuity, t.i., and affinity of  $\Delta_A$ ).

*For any  $A \in \mathcal{U}$ , the space–averaging functional  $\Delta_A$  on the set  $E_{\vec{\ell}}$  of  $\vec{\ell}$ –periodic states is affine, t.i., and weak\*–upper semi–continuous.*

*Proof.* Because the map  $\rho \mapsto \rho(A)$  is affine,  $\Delta_A$  is also affine. Moreover, by using (2.12), (6.11), and (6.16) we obtain, for all  $x \in \mathbb{Z}^d$ , that

$$\Delta_A(\rho \circ \alpha_x) = \Delta_{\alpha_x(A)}(\rho) = \|P_\rho \pi_\rho(\alpha_x(A_{\vec{\ell}})) \Omega_\rho\|^2 = \|P_\rho \pi_\rho(A_{\vec{\ell}}) \Omega_\rho\|^2 = \Delta_A(\rho)$$

because  $\rho \in E_{\vec{\ell}}$ . In other words, the map  $\rho \mapsto \Delta_A(\rho)$  is t.i. on  $E_{\vec{\ell}}$ . Finally, by Lemma 6.10,  $\Delta_A$  is an infimum over weak\*–continuous functionals and is therefore weak\*–upper semi–continuous. The latter is completely standard to verify. Indeed, by Lemma 6.10,

$$M_r := \{\rho \in E_{\vec{\ell}} : \Delta_A(\rho) < r\} = \bigcup_{(L, \dots, L) \in \vec{\ell}\mathbb{N}^d} \{\rho \in E_{\vec{\ell}} : \rho(A_L^* A_L) < r\}$$

for any constant  $r \in \mathbb{R}_0^+$ . Since, for any  $A \in \mathcal{U}$ , the map  $\rho \mapsto \rho(A)$  is weak\*–continuous,  $M_r$  is the union of open sets which implies the weak\*–upper semi–continuity of  $\Delta_A$ . ■



LEMMA 6.12 (Locally Lipschitz continuity of the map  $A \mapsto \Delta_A(\rho)$ ).

For all  $\rho \in E_{\bar{c}}$  and all  $A, B \in \mathcal{U}$ ,

$$|\Delta_A(\rho) - \Delta_B(\rho)| \leq (\|A\| + \|B\|)\|A - B\|.$$

*Proof.* This proof is straightforward. Indeed, observe that

$$\|\rho(A_L^* A_L) - \rho(B_L^* B_L)\| \leq \|A^* - B^*\| \|A\| + \|B^*\| \|A - B\|$$

from which we deduce the lemma. ■

We analyze now the space-averaging functional  $\rho \mapsto \Delta_A(\rho)$  seen as a map from the set  $E_1$  of t.i. states to  $\mathbb{R}$ .

PROPOSITION 6.13 (Continuity/Discontinuity of  $\Delta_A$  on  $E_1$ ).

- (i)  $\Delta_A$  is continuous on  $E_1$  iff the affine map  $\rho \mapsto |\rho(A)|$  from  $E_1$  to  $\mathbb{C}$  is a constant map.
- (ii) For all  $A \in \mathcal{U}$  such that  $\rho \mapsto |\rho(A)|$  is not constant,  $\Delta_A$  is discontinuous on a weak\*-dense subset of  $E_1$ .
- (iii)  $\Delta_A$  is weak\*-continuous on the  $G_\delta$  weak\*-dense subset  $\mathcal{E}_1$  of ergodic states in  $E_1$ . In particular, the set of all points in  $E_1$  where this functional is discontinuous is meager.

*Proof.* We start by proving the statements (i)–(ii). From Lemmata 6.8, 6.11 and 12.17 combined with Theorem 2.9,  $\Delta_A$  can be decomposed in terms of an integral on the set  $\mathcal{E}_1$ , see Theorem 2.19 (iv). As a consequence, if  $\rho \mapsto |\rho(A)|$  is a constant map on  $E_1$  then the functional  $\Delta_A$  is clearly constant on  $E_1$  and hence continuous. Take now  $A \in \mathcal{U}$  such that the map  $\rho \mapsto |\rho(A)|$  is not constant. Then, for any  $\rho \in E_1$ , there is at least one state  $\varrho_\rho \in E_1$  such that  $|\rho(A)| \neq |\varrho_\rho(A)|$ . For all  $\rho \in E_1$ , we define the subset  $I(\rho) \subset E_1$  by

$$I(\rho) := \{\lambda\rho + (1 - \lambda)\varrho_\rho \text{ for any } \lambda \in (0, 1)\}.$$

Finally, let us consider the subset

$$D := \bigcup_{\rho \in E_1} I(\rho) \subset E_1 \setminus \mathcal{E}_1.$$

By continuity of the map  $\lambda \mapsto \lambda\rho$  for  $\lambda \in \mathbb{C}$  and  $\rho \in E_1$ , the set  $D$  is dense in  $E_1$  w.r.t. the weak\*-topology. Moreover, the map  $\rho \mapsto \Delta_A(\rho)$  is discontinuous at any  $\rho \in D$ . This can be seen as follows.

Recall that any  $\rho \in D$  is of the form

$$\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$$

for some  $\lambda \in (0, 1)$  and states  $\rho_1, \rho_2 \in E_1$  with  $|\rho_1(A)| \neq |\rho_2(A)|$ . From Corollary 6.6, the set  $\mathcal{E}_1$  of extreme states is weak\*-dense in  $E_1$ . So, for any

$\rho \in \mathcal{D}$ , there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty \subset \mathcal{E}_1$  of extreme states converging w.r.t. the weak\*-topology to  $\rho$ . Then, by Lemma 6.8, it follows that

$$(6.17) \quad \lim_{n \rightarrow \infty} \Delta_A(\hat{\rho}_n) = |\hat{\rho}_n(A)|^2 = |\lambda\rho_1(A) + (1-\lambda)\rho_2(A)|^2 < \lambda|\rho_1(A)|^2 + (1-\lambda)|\rho_2(A)|^2 \leq \Delta_A(\rho)$$

because  $\rho \mapsto |\rho(A)|^2$  is weak\*-continuous,  $\lambda \in (0, 1)$ ,  $\Delta_A(\rho)$  is affine, and  $\Delta_A(\rho) \geq |\rho(A)|^2$  for any  $\rho \in E_1$ .

We conclude this proof by showing that  $\Delta_A$  is weak\*-continuous for any  $\hat{\rho} \in \mathcal{E}_1$  which yields (iii), by Corollary 6.6. Take  $\hat{\rho} \in \mathcal{E}_1$  and consider any sequence  $\{\rho_n\}_{n=1}^\infty$  of states of  $E_1$  converging w.r.t. the weak\*-topology to  $\hat{\rho}$ . The functional  $\Delta_A$  is weak\*-upper semi-continuous, whereas, for all  $\rho \in E_1$ ,  $\Delta_A(\rho) \geq |\rho(A)|^2$  with equality whenever  $\rho \in \mathcal{E}_1$  (see Lemma 6.8). Therefore,

$$|\hat{\rho}(A)|^2 = \Delta_A(\hat{\rho}) \geq \lim_{n \rightarrow \infty} \Delta_A(\rho_n) \geq \lim_{n \rightarrow \infty} |\rho_n(A)|^2 = |\hat{\rho}(A)|^2.$$

In other words, the functional  $\Delta_A$  is weak\*-continuous on  $\mathcal{E}_1$ . ■

Note that the map  $\rho \mapsto |\rho(A)|^2$  is a weak\*-continuous convex minorant of the space-averaging functional  $\Delta_A$ , see Lemma 6.10 for  $\vec{\ell} = (1, \dots, 1)$ . From Definitions 2.14, 2.15, and Theorem 2.16 (or Lemma 6.8),  $\Delta_A(\hat{\rho}) = |\hat{\rho}(A)|^2$  for any extreme state  $\hat{\rho} \in \mathcal{E}_1$ . Since, by Corollary 6.6, the set  $\mathcal{E}_1$  of extreme states is weak\*-dense in  $E_1$ , these last properties suggest that the map  $\rho \mapsto |\rho(A)|^2$  is the largest weak\*-lower semi-continuous convex minorant of  $\Delta_A$ . This is proven in our last lemma on the functional  $\Delta_A$ .

LEMMA 6.14 ( $\Gamma$ -regularization of  $\Delta_A$ ).

The  $\Gamma$ -regularization on  $E_1$  of the functional  $\Delta_A$  is the weak\*-continuous convex functional  $\rho \mapsto |\rho(A)|^2$ . In particular,  $\rho \mapsto |\rho(A)|^2$  is the largest weak\*-lower semi-continuous convex minorant of  $\Delta_A$  on  $E_1$ .

*Proof.* Recall that the  $\Gamma$ -regularization of functionals are defined by Definition 12.27. By Lemmata 6.8 and 6.10 for  $\vec{\ell} = (1, \dots, 1)$ ,  $\Delta_A(\hat{\rho}) = |\hat{\rho}(A)|^2$  for any  $\hat{\rho} \in \mathcal{E}_1$ , whereas, for all  $\rho \in E_1$ ,  $\Delta_A(\rho) \geq |\rho(A)|^2$ . Since the map  $\rho \mapsto |\rho(A)|^2$  from  $E_1$  to  $\mathbb{R}$  is a weak\*-continuous convex functional, by Corollary 12.30, the  $\Gamma$ -regularization  $\Gamma_{E_1}(\Delta_A)$  of  $\Delta_A$  is bounded from below on  $E_1$  by the map  $\rho \mapsto |\rho(A)|^2$ , whereas, for any extreme state  $\hat{\rho} \in \mathcal{E}_1$ ,  $\Gamma_{E_1}(\Delta_A)(\hat{\rho}) = |\hat{\rho}(A)|^2$ . Because of the weak\*-density of  $\mathcal{E}_1$  in  $E_1$  (Corollary 6.6), we deduce by using the weak\*-lower semi-continuity of the functional  $\Gamma_{E_1}(\Delta_A)$  that  $\Gamma_{E_1}(\Delta_A)(\rho) = |\rho(A)|^2$  for all  $\rho \in E_1$ . ■

#### 6.4. Von Neumann entropy and entropy density of $\vec{\ell}$ -periodic states

For any local state  $\rho_\Lambda \in E_\Lambda$ , there exists a unique density matrix  $d_{\rho_\Lambda} \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$  satisfying  $\rho_\Lambda(A) = \text{Trace}(d_{\rho_\Lambda} A)$  for all  $A \in \mathcal{U}_\Lambda$ . The von Neumann entropy is

then defined, for any local state  $\rho_\Lambda$  with density matrix  $d_{\rho_\Lambda}$ , by

$$(6.18) \quad S(\rho_\Lambda) := \text{Trace} (\eta(d_{\rho_\Lambda})) \geq 0.$$

Here,  $\eta(x) := -x \log(x)$ . Observe that  $\mathcal{U}_\Lambda$  is isomorphic to some (finite dimensional) matrix algebra  $B(\mathbb{C}^{N_\Lambda})$ . The linear functional  $\text{Trace} : \mathcal{U}_\Lambda \rightarrow \mathbb{C}$  is defined by  $\text{Trace} := \text{Tr} \circ \varphi$  with  $\varphi$  being an arbitrary  $*$ -isomorphism  $\mathcal{U}_\Lambda \rightarrow B(\mathbb{C}^{N_\Lambda})$  and  $\text{Tr}$  being the usual trace for linear operators on  $\mathbb{C}^{N_\Lambda}$ . Note further that  $\text{Trace}$  does not depend on the choice of the isomorphism  $\varphi$ . The von Neumann entropy has the following well-known properties:

**S1** It is  $\vec{\ell}$ -periodic in the sense that, for any  $\rho \in E_{\vec{\ell}}$ ,  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , and  $x \in \vec{\ell}\mathbb{Z}^d$ ,

$$S(\rho_\Lambda) = S(\rho_{\Lambda+x})$$

with the local state  $\rho_\Lambda$  being the restriction of the  $\vec{\ell}$ -periodic state  $\rho$  on the sub-algebra  $\mathcal{U}_\Lambda \subset \mathcal{U}$  and with  $\Lambda + x$  defined by (2.13).

**S2** It is strongly sub-additive, i.e., for any  $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\mathfrak{L})$  and any local state  $\rho_{\Lambda_1 \cup \Lambda_2}$  on  $\mathcal{U}_{\Lambda_1 \cup \Lambda_2}$ ,

$$S(\rho_{\Lambda_1 \cup \Lambda_2}) - S(\rho_{\Lambda_1}) - S(\rho_{\Lambda_2}) + S(\rho_{\Lambda_1 \cap \Lambda_2}) \leq 0,$$

see [8, Theorems 3.7 and 10.1].

**S3** It is concave, i.e., for any  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , any states  $\rho_{\Lambda,1}, \rho_{\Lambda,2}$  on  $\mathcal{U}_\Lambda$ , and  $\lambda \in [0, 1]$ ,

$$S(\lambda \rho_{\Lambda,1} + (1 - \lambda) \rho_{\Lambda,2}) \geq \lambda S(\rho_{\Lambda,1}) + (1 - \lambda) S(\rho_{\Lambda,2}),$$

see [5, Proposition 6.2.28].

**S4** It is approximately convex, i.e., for any  $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ , any states  $\rho_{\Lambda,1}, \rho_{\Lambda,2}$  on  $\mathcal{U}_\Lambda$ , and  $\lambda \in [0, 1]$ ,

$$S(\lambda \rho_{\Lambda,1} + (1 - \lambda) \rho_{\Lambda,2}) \leq \lambda S(\rho_{\Lambda,1}) + (1 - \lambda) S(\rho_{\Lambda,2}) + \eta(\lambda) + \eta(1 - \lambda),$$

see [5, Proposition 6.2.28].

**S1–S4** ensure the existence as well as some basic properties of the entropy density  $s : E_{\vec{\ell}} \rightarrow \mathbb{R}_0^+$  defined in Definition 2.27:

LEMMA 6.15 (Existence and properties of the entropy density).

The map  $\rho \mapsto s(\rho)$  from  $E_{\vec{\ell}}$  to  $\mathbb{R}$  equals

$$s(\rho) := \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} S(\rho_{\Lambda_L}) = \inf_{(L, \dots, L) \in \vec{\ell}\mathbb{N}^d} \frac{1}{|\Lambda_L|} S(\rho_{\Lambda_L}).$$

It is an affine, t.i., and weak\*-upper semi-continuous functional.

*Proof.* This lemma is standard, see, e.g., [8, Section 3]. Indeed, the existence of the entropy density is a direct consequence of properties S1–S2 because one deduces from these properties that

$$s(\rho) = \inf_{(L, \dots, L) \in \vec{\ell} \cdot \mathbb{N}^d} \frac{1}{|\Lambda_L|} S(\rho_{\Lambda_L}).$$

This equation implies the weak\*-upper semi-continuity of the entropy density functional  $s$  as the map  $\rho \mapsto S(\rho_{\Lambda_L})$  is weak\*-continuous for any  $L \in \mathbb{N}$ , see similar arguments performed in the proof of Lemma 6.11. By using the property S3, the functional  $s$  is convex, whereas from S4 one deduces that it is also concave. Therefore,  $\rho \mapsto s(\rho)$  defines a weak\*-upper semi-continuous affine functional on  $E_{\vec{\ell}}$ . The translation invariance of  $s$  follows from the strong sub-additivity S2 together with standard estimates. ■

Observe that the entropy density functional  $s$  is not weak\*-continuous but only norm continuous. These properties are well known, see, e.g., [18, 19]. Nevertheless, the entropy density functional  $s$  has still an interesting weak\*-“pseudo-continuity” property w.r.t. to specific sequences of ergodic states. This property is important in the following and reads as follows:

LEMMA 6.16 (Weak\*-pseudo-continuity of the entropy density).

For any t.i. state  $\rho \in E_1$ , there is a sequence  $\{\hat{\rho}_n\}_{n=1}^\infty$  of ergodic states converging in the weak\*-topology to  $\rho$  and such that

$$s(\rho) = \lim_{n \rightarrow \infty} s(\hat{\rho}_n).$$

*Proof.* The states  $\hat{\rho}_n$  defined by (6.8) with  $\vec{\ell} = (1, \dots, 1)$  for any  $\rho \in E_1$  and all  $n \in \mathbb{N}$  are ergodic, i.e.,  $\hat{\rho}_n \in \mathcal{E}_1$ , see (6.10) (extended by density of  $\mathcal{U}_0$  to all  $A \in \mathcal{U}$ ). Moreover, the sequence  $\{\hat{\rho}_n\}_{n=1}^\infty$  converges in the weak\*-topology towards  $\rho$ . On the other hand, by translation invariance and affinity of the entropy (Lemma 6.15),

$$s(\hat{\rho}_n) = s(\tilde{\rho}_n) = \frac{1}{|\Lambda_n|} S(\rho_{\Lambda_n})$$

with  $S$  being the von Neumann entropy (6.18) and  $\tilde{\rho}_n$  the  $(2n+1)(1, \dots, 1)$ -periodic continuation of the restriction  $\rho_{\Lambda_n} \in E_{\Lambda_n}$  of the state  $\rho \in E_1$  on the box  $\Lambda_n = \Lambda_{n, (1, \dots, 1)}$  (defined by (6.7)). Therefore the entropy density  $s(\hat{\rho}_n)$  converges to  $s(\rho)$  as  $n \rightarrow \infty$ , see Definition 2.27. ■

## 6.5. The set $E_1$ as a subset of the dual space $\mathcal{W}_1^*$

Another important thermodynamic quantity associated with any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  on  $\mathcal{U}$  is the energy density  $\rho \mapsto e_\Phi(\rho)$  defined for any t.i. interaction  $\Phi \in \mathcal{W}_1$ . It is the thermodynamic limit of the internal energy  $\rho(U_\Lambda^\Phi)$

(Definition 2.22 (ii)) per unit volume associated with any fixed local interaction  $\Phi$ , see Definition 2.30. This last definition makes sense as soon as  $\Phi \in \mathcal{W}_1$ . Indeed, this basically follows from Lebesgue's dominated convergence theorem:

LEMMA 6.17 (Well-definiteness of the energy density).

The energy density  $e_\Phi(\rho)$  of any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  w.r.t.  $\Phi \in \mathcal{W}_1$  equals  $e_\Phi(\rho) = \rho(\mathbf{e}_{\Phi, \vec{\ell}})$  with  $\mathbf{e}_{\Phi, \vec{\ell}}$  being defined by (2.16) for any  $\vec{\ell} \in \mathbb{N}^d$ .

*Proof.* For any t.i. interaction  $\Phi \in \mathcal{W}_1$ , its internal energy equals

$$(6.19) \quad U_\Lambda^\Phi := \sum_{\Lambda' \in \mathcal{P}_f(\mathfrak{L})} \mathbf{1}_{\{\Lambda' \subset \Lambda\}} \Phi_{\Lambda'} = \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i-1\}} \sum_{y \in \Lambda \cap \vec{\ell} \cdot \mathbb{Z}^d, x+y \in \Lambda} \sum_{\Lambda' \in \mathcal{P}_f(\mathfrak{L}), 0 \in \Lambda'} \mathbf{1}_{\{\Lambda' \subset (\Lambda - x - y)\}} \frac{\Phi_{(x+y)+\Lambda'}}{|\Lambda'|}.$$

Then, for any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  and any  $L \in \mathbb{R}$ ,

$$\begin{aligned} \frac{\rho(U_{\Lambda_L}^\Phi)}{|\Lambda_L|} &= \frac{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|}{|\Lambda_L|} \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i-1\}} \sum_{\Lambda' \in \mathcal{P}_f(\mathfrak{L}), 0 \in \Lambda'} \rho \left( \frac{\Phi_{x+\Lambda'}}{|\Lambda'|} \right) \\ &\quad \times \frac{1}{|\Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d|} \left( \sum_{y \in \Lambda_L \cap \vec{\ell} \cdot \mathbb{Z}^d, x+y \in \Lambda_L} \mathbf{1}_{\{\Lambda' \subset (\Lambda_L - x - y)\}} \right). \end{aligned}$$

As  $\|\Phi\|_{\mathcal{W}_1} < \infty$ , we can perform the limit  $L \rightarrow \infty$  in this last equality by using Lebesgue's dominated convergence theorem in order to show that

$$e_\Phi(\rho) := \lim_{L \rightarrow \infty} \frac{\rho(U_{\Lambda_L}^\Phi)}{|\Lambda_L|} = \rho(\mathbf{e}_{\Phi, \vec{\ell}}).$$

■

The functional  $e_\Phi$  can be seen either as the affine map  $\rho \mapsto e_\Phi(\rho)$  at fixed  $\Phi \in \mathcal{W}_1$  or as the linear functional  $\Phi \mapsto e_\Phi(\rho)$  at fixed  $\rho \in E_{\vec{\ell}}$ . In this section we use the second point of view to identify the set  $E_1$  of all t.i. states on  $\mathcal{U}$  with a weak\*-compact set of norm one functionals on the Banach space  $\mathcal{W}_1$  (Definition 2.24). Indeed, we define the map  $\rho \mapsto \mathbb{T}(\rho)$  from  $E_{\vec{\ell}}$  to the dual space  $\mathcal{W}_1^*$  which associates to any  $\vec{\ell}$ -periodic state  $\rho \in E_{\vec{\ell}}$  on  $\mathcal{U}$  the affine continuous functional  $\mathbb{T}(\rho) \in \mathcal{W}_1^*$  defined on the Banach space  $\mathcal{W}_1$  by

$$(6.20) \quad \Phi \mapsto \mathbb{T}(\rho)(\Phi) := -e_\Phi(\rho).$$

The functional  $\mathbb{T}(\rho)$  is clearly continuous and linear for any  $\rho \in E_{\vec{\ell}}$  since

$$|e_\Phi(\rho)| \leq \|\Phi\|_{\mathcal{W}_1} \quad \text{and} \quad e_{(\lambda_1 \Phi + \lambda_2 \Psi)}(\rho) = \lambda_1 e_\Phi(\rho) + \lambda_2 e_\Psi(\rho)$$

for any  $\lambda_1, \lambda_2 \in \mathbb{R}$  and any  $\Phi, \Psi \in \mathcal{W}_1$ , see Lemma 6.17. Observe that the minus sign in the definition (6.20) is arbitrary. It is used only for convenience

when we have to deal with tangent functionals (Definition 12.43) of the pressure (3.20), see Section 3.5. The map  $\mathbb{T}$  restricted on the set  $E_1$  has some interesting topological properties:

LEMMA 6.18 (Properties of  $\mathbb{T}$  on  $E_1$ ).

*The affine map  $\mathbb{T} : E_1 \rightarrow \mathbb{T}(E_1) \subset \mathcal{W}_1^*$  is a homeomorphism in the weak\*-topology and an isometry in the norm topology, i.e.,  $\|\mathbb{T}(\rho) - \mathbb{T}(\rho')\| = \|\rho - \rho'\|$  for all  $\rho, \rho' \in E_1$ .*

*Proof.* The functional  $\mathbb{T}$  is weak\*-continuous because the map  $\rho \rightarrow e_\Phi(\rho)$  is weak\*-continuous, by Lemma 2.31 (i). As  $E_1$  is compact w.r.t. the weak\*-topology and the dual space  $\mathcal{W}_1^*$  is Hausdorff w.r.t. the weak\*-topology (cf. Corollary 12.9), it is a homeomorphism from  $E_1$  to  $\mathbb{T}(E_1)$  if it is an injection from  $E_1$  to  $\mathcal{W}_1^*$ .

In fact, for any  $\rho, \rho' \in E_{\tilde{c}}$ , observe that

$$(6.21) \quad \|\mathbb{T}(\rho) - \mathbb{T}(\rho')\| := \sup_{\Phi \in \mathcal{W}_1, \|\Phi\|_{\mathcal{W}_1} = 1} |\rho(\epsilon_\Phi) - \rho'(\epsilon_\Phi)| \leq \|\rho - \rho'\|.$$

Therefore, in order to show that the functional  $\mathbb{T}$  on  $E_1$  is an isometry, which yields its injectivity, it suffices to prove the opposite inequality.

For any  $A = A^* \in \mathcal{U}_0$ , there exists a finite range interaction  $\Phi^A \in \mathcal{W}_1$  with  $\|\Phi^A\|_{\mathcal{W}_1} = \|A\|$  such that, for any  $\rho \in E_1$ ,

$$e_{\Phi^A}(\rho) = \rho(A).$$

For  $A = A^* \in \mathcal{U}_\Lambda$ , choose, for instance,  $\Phi^A(\Lambda') = \alpha_x(A)$  if  $\Lambda' = \Lambda + x$  and  $\Phi^A(\Lambda') = 0$  else. It follows that, for any  $A = A^* \in \mathcal{U}_0$ ,

$$(6.22) \quad |\rho(A) - \rho'(A)| \leq \|\mathbb{T}(\rho) - \mathbb{T}(\rho')\| \|A\|.$$

The difference  $(\rho - \rho')$  of states  $\rho, \rho' \in E_1$  is a Hermitian functional on a  $C^*$ -algebra which implies that

$$\|\rho - \rho'\| = \sup_{A \in \mathcal{U}, A = A^*, \|A\| = 1} |\rho(A) - \rho'(A)|.$$

Since the algebra  $\mathcal{U}_0$  of local elements is dense in  $\mathcal{U}$ , this last equality together with (6.21) and (6.22) implies that, for all  $\rho, \rho' \in E_1$ ,

$$\|\mathbb{T}(\rho) - \mathbb{T}(\rho')\| = \|\rho - \rho'\|.$$

■

As a consequence, we can identify any t.i. state  $\rho \in E_1$  with the continuous linear functional  $\mathbb{T}(\rho) \in \mathcal{W}_1^*$ .

### 6.6. Well-definiteness of the free-energy densities on $E_{\vec{\ell}}$

Two crucial functionals related to the thermodynamics of long range models

$$\mathbf{m} := (\Phi, \{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}) \in \mathcal{M}_1$$

are the free-energy density functional  $f_{\mathbf{m}}^{\sharp}$  defined on the set  $E_{\vec{\ell}}$  of  $\vec{\ell}$ -periodic states by

$$f_{\mathbf{m}}^{\sharp}(\rho) := \|\Delta_{a,+}(\rho)\|_1 - \|\Delta_{a,-}(\rho)\|_1 + e_{\Phi}(\rho) - \beta^{-1}s(\rho)$$

and the reduced free-energy density functional  $g_{\mathbf{m}}$  defined on  $E_{\vec{\ell}}$  by

$$g_{\mathbf{m}}(\rho) := \|\gamma_{a,+}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 - \|\gamma_{a,-}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 + e_{\Phi}(\rho) - \beta^{-1}s(\rho),$$

see Definitions 3.5 and 3.6. Here,  $\Delta_{a,\pm}(\rho)$  is defined by (3.2), that is,

$$\Delta_{a,\pm}(\rho) := \gamma_{a,\pm}\Delta_{\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}}(\rho) \in [0, \|\Phi_a\|_{\mathcal{W}_1}^2 + \|\Phi'_a\|_{\mathcal{W}_1}^2]$$

(cf. (2.17) and Lemma 6.10) with

$$\gamma_{a,\pm} := 1/2(|\gamma_a| \pm \gamma_a) \in \{0, 1\}$$

being the negative and positive parts (3.1) of the fixed measurable function  $\gamma_a \in \{-1, 1\}$ .

Both functionals  $f_{\mathbf{m}}^{\sharp}$  and  $g_{\mathbf{m}}$  are well-defined. Indeed, the entropy density functional  $s$  as well as the energy density functional  $e_{\Phi}$  are both well-defined, see Lemmata 6.15 and 6.17. Moreover, for any  $\rho \in E_{\vec{\ell}}$  and any  $\mathbf{m} \in \mathcal{M}_1$ , the maps  $a \mapsto \Delta_{a,\pm}(\rho)$  are measurable and  $\|\Delta_{a,\pm}(\rho)\|_1 < \infty$ :

LEMMA 6.19 (Long range energy densities for  $\mathbf{m} \in \mathcal{M}_1$ ).

The maps  $\rho \mapsto \|\Delta_{a,\pm}(\rho)\|_1$  from  $E_{\vec{\ell}}$  to  $\mathbb{R}_0^+$  are well-defined affine, t.i., and weak\*-upper semi-continuous functionals which equal

(6.23)

$$\|\Delta_{a,\pm}(\rho)\|_1 = \inf_{(L,\dots,L) \in \vec{\ell}\mathbb{N}^d} \left\{ \int_{\mathcal{A}} \gamma_{a,\pm}\rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) \mathbf{d}\mathbf{a}(a) \right\} \leq \|\Phi_a\|_2^2 + \|\Phi'_a\|_2^2$$

for any  $\rho \in E_{\vec{\ell}}$ , where

$$\mathbf{u}_{L,a} := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \alpha_x(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}) \in \mathcal{U}.$$

*Proof.* The maps  $a \mapsto \Delta_{a,\pm}(\rho)$  are measurable and

$$\|\Delta_{a,\pm}(\rho)\|_1 \leq \|\Phi_a\|_2^2 + \|\Phi'_a\|_2^2 < \infty$$

for any  $\mathbf{m} \in \mathcal{M}_1$  and  $\rho \in E_{\vec{\ell}}$ . It is a consequence of (2.17) and Lemma 6.10 which also implies that

$$\|\Delta_{a,\pm}(\rho)\|_1 = \int_{\mathcal{A}} \gamma_{a,\pm}\Delta_{a,\pm}(\rho) \mathbf{d}\mathbf{a}(a) = \int \gamma_{a,\pm} \left\{ \inf_{(L,\dots,L) \in \vec{\ell}\mathbb{N}^d} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) \right\} \mathbf{d}\mathbf{a}(a)$$

for any  $\rho \in E_{\vec{\ell}}$ . Thus, by the monotonicity of integrals,

$$(6.24) \quad \|\Delta_{a,\pm}(\rho)\|_1 \leq \inf_{(L,\dots,L) \in \vec{\ell} \cdot \mathbb{N}^d} \left\{ \int_{\mathcal{A}} \gamma_{a,\pm} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) d\mathbf{a}(a) \right\}.$$

By (2.17), note that, for all  $L \in \mathbb{N}$ ,

$$\rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) \leq 2\|\Phi_a\|_{\mathcal{W}_1}^2 + 2\|\Phi'_a\|_{\mathcal{W}_1}^2.$$

Therefore, using that

$$\Delta_{a,\pm}(\rho) = \inf_{(L,\dots,L) \in \vec{\ell} \cdot \mathbb{N}^d} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) = \lim_{L \rightarrow \infty} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a})$$

and Lebesgue's dominated convergence we obtain that

$$\|\Delta_{a,\pm}(\rho)\|_1 = \lim_{L \rightarrow \infty} \int_{\mathcal{A}} \gamma_{a,\pm} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) d\mathbf{a}(a) = \liminf_{L \rightarrow \infty} \int_{\mathcal{A}} \gamma_{a,\pm} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) d\mathbf{a}(a).$$

In particular, we have that

$$\|\Delta_{a,\pm}(\rho)\|_1 \geq \inf_{(L,\dots,L) \in \vec{\ell} \cdot \mathbb{N}^d} \left\{ \int_{\mathcal{A}} \gamma_{a,\pm} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) d\mathbf{a}(a) \right\}$$

which combined with (6.24) implies Equality (6.23).

By Lebesgue's dominated convergence theorem, the map

$$\rho \mapsto \int_{\mathcal{A}} \gamma_{a,\pm} \rho(\mathbf{u}_{L,a}^* \mathbf{u}_{L,a}) d\mathbf{a}(a)$$

is weak\*-continuous for any  $L \in \mathbb{N}$ . So, the weak\*-upper semi-continuity of the maps  $\rho \mapsto \|\Delta_{a,\pm}(\rho)\|_1$  results from (6.23), see similar arguments in the proof of Lemma 6.11. Additionally, the maps  $\rho \mapsto \|\Delta_{a,\pm}(\rho)\|_1$  inherit the t.i. and affinity of the space-averaging functionals  $\Delta_{a,\pm}$ , see again Lemma 6.11.

■

Therefore, combining Lemmata 6.15 and 6.17 with Lemma 6.19, we obtain the well-definiteness of the functionals  $f_{\mathfrak{m}}^{\sharp}$  and  $g_{\mathfrak{m}}$ :

COROLLARY 6.20 (Well-definiteness of the functionals  $f_{\mathfrak{m}}^{\sharp}$  and  $g_{\mathfrak{m}}$ ).

- (i)  $\rho \mapsto f_{\mathfrak{m}}^{\sharp}(\rho)$  is a well-defined map from  $E_{\vec{\ell}}$  to  $\mathbb{R}$ .
- (ii)  $\rho \mapsto g_{\mathfrak{m}}(\rho)$  is a well-defined map from  $E_{\vec{\ell}}$  to  $\mathbb{R}$ .

## 7. Permutation Invariant Fermi Systems

By using the so-called passivity of Gibbs states (Theorem 12.2) the pressure  $p_l = p_{l,\mathfrak{m}}$  defined by (3.7) for  $l \in \mathbb{N}$  and any *discrete* model

$$\mathfrak{m} = \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^d \subset \mathcal{M}_1$$



(see Section 3.1) can easily be bounded from below, for all states  $\rho \in E$ , by

$$(7.1) \quad p_l \geq - \sum_{k=1}^N \frac{\gamma_k}{|\Lambda_l|^2} \rho \left( (U_{\Lambda_l}^{\Phi_k} + iU_{\Lambda_l}^{\Phi'_k})^* (U_{\Lambda_l}^{\Phi_k} + iU_{\Lambda_l}^{\Phi'_k}) \right) - \frac{1}{|\Lambda_l|} \rho(U_{\Lambda_l}^{\Phi}) + \frac{1}{\beta|\Lambda_l|} S(\rho_{\Lambda_l})$$

with  $S$  being the von Neumann entropy defined by (6.18). Furthermore, Theorem 12.2 tells us that the equality in (7.1) is only satisfied for the Gibbs equilibrium state  $\rho_l = \rho_{\Lambda_l, U_l}$  (12.2), i.e.,

$$(7.2) \quad p_l = - \sum_{k=1}^N \frac{\gamma_k}{|\Lambda_l|^2} \rho_l \left( (U_{\Lambda_l}^{\Phi_k} + iU_{\Lambda_l}^{\Phi'_k})^* (U_{\Lambda_l}^{\Phi_k} + iU_{\Lambda_l}^{\Phi'_k}) \right) - \frac{1}{|\Lambda_l|} \rho_l(U_{\Lambda_l}^{\Phi}) + \frac{1}{\beta|\Lambda_l|} S(\rho_l).$$

Therefore, in order to prove Theorem 3.12 for any discrete models, one has to control each term in (7.1) and (7.2) as  $l \rightarrow \infty$ . Unfortunately, it is not clear how to perform this program directly, even if we concentrate on discrete long range models. In fact, as it is originally done in [20] and subsequently in [21] for quantum spin systems (Remark 2.4), we first need to understand *permutation invariant* models  $\mathfrak{m} \in \mathcal{M}_1$  to be able to prove Theorem 3.12.

This specific class of models is defined and analyzed in Section 7.2. Indeed, such a study requires a preliminary analysis, done in Section 7.1, of the set  $E_{\Pi} \subset E_1$  of permutation invariant states. This corresponds to a direct extension of our results [9] on the strong coupling BCS–Hubbard model to general permutation invariant systems and is given for completeness as well as a kind of “warm up” for the non–expert reader. Among other things, we shortly establish Størmer theorem, a non–commutative version of the celebrated de Finetti theorem for permutation invariant states on the fermion algebra  $\mathcal{U}$  as it is proven in [9].

### 7.1. The set $E_{\Pi}$ of permutation invariant states

Let  $\Pi$  be the set of all bijective maps from  $\mathfrak{L}$  to  $\mathfrak{L}$  which leaves all but finitely many elements invariant. It is a group w.r.t. the composition of maps. The condition

$$(7.3) \quad \alpha_{\pi} : a_{x,s} \mapsto a_{\pi(x),s}, \quad s \in S, x \in \mathfrak{L},$$

defines a group homomorphism  $\pi \mapsto \alpha_{\pi}$  from  $\Pi$  to the group of  $*$ –automorphisms of  $\mathcal{U}$ . The set of all permutation invariant states is then defined by

$$E_{\Pi} := \bigcap_{\pi \in \Pi, A \in \mathcal{U}} \{ \rho \in \mathcal{U}^* : \rho(\mathbf{1}) = 1, \rho(A^*A) \geq 0 \text{ with } \rho = \rho \circ \alpha_{\pi} \}.$$

Since obviously

$$E_{\Pi} \subset E_1 \subset \bigcap_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}},$$

every permutation invariant state  $\rho \in E_{\Pi}$  is even, by Lemma 2.8. Furthermore,  $E_{\Pi}$  is clearly convex and weak\*-compact and, by Krein–Milman theorem (Theorem 12.11), it is the weak\*-closure of the convex hull of the (non-empty) set  $\mathcal{E}_{\Pi}$  of its extreme points.

The set  $\mathcal{E}_{\vec{\ell}}$  of extreme states of  $E_{\vec{\ell}}$  is characterized by Theorem 2.16 and  $\mathcal{E}_{\Pi}$  can likewise be precisely characterized by Størmer theorem for permutation invariant states on the fermion algebra  $\mathcal{U}$ . This theorem is a non-commutative version of the celebrated de Finetti theorem from (classical) probability theory and it is proven in the case of even states on the fermion algebra  $\mathcal{U}$  in [9]. Indeed, extreme permutation invariant states  $\rho \in \mathcal{E}_{\Pi}$  are product states defined as follows.

Let  $\rho_{\{0\}} \in E_{\mathcal{U}_{\{0\}}}$  be any *even* state on the one-site  $C^*$ -algebra  $\mathcal{U}_{\{0\}}$ , i.e.,  $\rho_{\{0\}} = \rho_{\{0\}} \circ \sigma_{\pi}$  with  $\sigma_{\pi}$  defined by (2.4) for  $\theta = \pi$ . Then, from [8, Theorem 11.2.], there is a unique even state  $\hat{\rho} \in E_{\Pi}$  satisfying

$$\hat{\rho}(\alpha_{x_1}(A_1) \cdots \alpha_{x_n}(A_n)) = \rho_{\{0\}}(A_1) \cdots \rho_{\{0\}}(A_n)$$

for all  $A_1 \dots A_n \in \mathcal{U}_{\{0\}}$  and all  $x_1, \dots, x_n \in \mathbb{Z}^d$  such that  $x_i \neq x_j$  for  $i \neq j$ . The set of all states  $\hat{\rho}$  of this form, called product states, is denoted by  $E_{\otimes}$  which is nothing else but the set  $\mathcal{E}_{\Pi}$  of extreme points of  $E_{\Pi}$ :

**THEOREM 7.1** (Størmer theorem, lattice CAR-algebra version).

*Extreme permutation invariant states  $\hat{\rho} \in \mathcal{E}_{\Pi}$  are product states and conversely, i.e.,  $\mathcal{E}_{\Pi} = E_{\otimes}$ .*

This theorem was proven by Størmer [12] for the case of lattice quantum spin systems (cf. Remark 2.4). Its corresponding version for permutation invariant states on the fermion algebra  $\mathcal{U}$  follows from [9, Lemmata 6.6–6.8]. Observe that the proof of Theorem 7.1 is performed in [9] for a spin set  $S = \{\uparrow, \downarrow\}$ . It can easily be extended to the general case of Theorem 7.1.

It follows from Theorem 7.1 that all permutation invariant states  $\hat{\rho} \in \mathcal{E}_{\Pi}$  are strongly mixing which means (2.10). They are, in particular, strongly clustering and thus ergodic w.r.t. to any sub-group  $\vec{\ell} \cdot \mathbb{Z}^d$  of  $\mathbb{Z}^d$ , where  $\vec{\ell} \in \mathbb{N}^d$ . In other words, for all  $\vec{\ell} \in \mathbb{N}^d$ ,  $\mathcal{E}_{\Pi} = E_{\otimes} \subset \mathcal{E}_{\vec{\ell}}$  and the set  $E_{\Pi} \subset E_{\vec{\ell}}$  is hence a closed metrizable face of  $E_{\vec{\ell}}$ . Therefore, by using Theorem 2.9 and Theorem 7.1, we obtain the existence of a unique decomposition of states  $\rho \in E_{\Pi}$  in terms of product states:

**THEOREM 7.2** (Unique decomposition of permutation invariant states).  
 For any  $\rho \in E_{\Pi}$ , there is a unique probability measure  $\mu_{\rho}$  on  $E_{\Pi}$  such that

$$\mu_{\rho}(E_{\otimes}) = 1 \quad \text{and} \quad \rho = \int_{E_{\Pi}} d\mu_{\rho}(\hat{\rho}) \hat{\rho}.$$

Furthermore, the map  $\rho \mapsto \mu_{\rho}$  is an isometry in the norm of linear functionals, i.e.,  $\|\rho - \rho'\| = \|\mu_{\rho} - \mu_{\rho'}\|$  for any  $\rho, \rho' \in E_{\Pi}$ .

From Theorem 2.12, for all  $\vec{\ell} \in \mathbb{N}^d$ , the sets  $E_{\vec{\ell}}$  are affinely homeomorphic to the Poulsen simplex, but the set  $E_{\Pi}$  of all permutation invariant states do not share this property. Indeed,  $E_{\Pi}$  is a Bauer simplex (Definition 12.24), i.e., a (Choquet) simplex whose set of extreme points is closed:

**THEOREM 7.3** ( $E_{\Pi}$  is a Bauer simplex).

The set  $E_{\Pi}$  is a Bauer simplex. In particular, the map  $\rho \mapsto \mu_{\rho}$  of Theorem 7.2 from  $E_{\Pi}$  to the set  $M_1^+(\mathcal{E}_{\Pi}) = M_1^+(E_{\otimes})$  of probability measures on  $\mathcal{E}_{\Pi} = E_{\otimes}$  is an affine homeomorphism w.r.t. to the weak\*-topologies on  $E_{\Pi}$  and  $M_1^+(\mathcal{E}_{\Pi})$ .

*Proof.* As explained above, for all  $\vec{\ell} \in \mathbb{N}^d$ ,  $E_{\Pi}$  is a closed face of  $E_{\vec{\ell}}$  (and thus a closed simplex) with set  $\mathcal{E}_{\Pi}$  of extreme points being the set  $E_{\otimes}$  of product states, i.e.,  $\mathcal{E}_{\Pi} = E_{\otimes} \subset \mathcal{E}_{\vec{\ell}}$ , see Theorem 7.1. Since the set  $E_{\otimes}$  is obviously closed in the weak\*-topology, it is a Bauer simplex which, combined with Theorem 12.25, implies the statement. ■

Therefore, the simplex  $E_{\Pi}$  has a much simpler geometrical structure than all simplices  $\{E_{\vec{\ell}}\}_{\vec{\ell} \in \mathbb{N}^d}$  and it is easier to use in practice, see, e.g., [9]. For instance, for any fixed element  $A$  of the one-site  $C^*$ -algebra  $\mathcal{U}_{\{0\}}$ , the space-averaging functional  $\Delta_A$  described in Sections 2.3 and 6.3 has a very explicit representation on the Bauer simplex  $E_{\Pi}$ :

**LEMMA 7.4** (The space-averaging functional  $\Delta_A$  on  $E_{\Pi}$ ).

At fixed  $A \in \mathcal{U}_{\{0\}}$ , the restriction on  $E_{\Pi}$  of the functional  $\Delta_A$  equals, for any  $x \in \mathbb{Z}^d \setminus \{0\}$ , the weak\*-continuous affine map  $\rho \mapsto \rho(A^* \alpha_x(A))$  from  $E_{\Pi}$  to  $\mathbb{R}_0^+$ .

*Proof.* This lemma follows from elementary combinatorics, see, e.g., [9, Lemma 6.2]. ■

Permutation invariance is, however, a too restrictive condition in general. Indeed, most of models coming from Physics are only translation invariant. In particular, the general set of states to be considered in these cases is the Poulsen simplex (up to an affine homeomorphism), which is in a sense *complementary* to the Bauer simplices, see [2, p. 164] or [45, Section 5].

## 7.2. Thermodynamics of permutation invariant Fermi systems

*Permutation invariant* interactions form a subset of the real Banach space  $\mathcal{W}_1$  of all t.i. interactions  $\Phi$ , see Definition 2.24. They are naturally defined as follows:

DEFINITION 7.5 (Permutation invariant interactions).

*A t.i. interaction  $\Phi \in \mathcal{W}_1$  is permutation invariant if  $\Phi_\Lambda = 0$  whenever  $|\Lambda| \neq 1$ .*

*Permutation invariant* Fermi systems  $\mathfrak{m} \in \mathcal{M}_1$  with long range interactions (see Definition 3.1) are then defined from permutation invariant interactions as follows:

DEFINITION 7.6 (Permutation invariant models).

*A long range model  $\mathfrak{m} := (\Phi, \{\Phi_a\}_{a \in \mathcal{A}}, \{\Phi'_a\}_{a \in \mathcal{A}}) \in \mathcal{M}_1$  is permutation invariant whenever the interactions  $\Phi$ ,  $\Phi_a$  and  $\Phi'_a$  are permutation invariant for all (a.e.)  $a \in \mathcal{A}$ .*

If the model  $\mathfrak{m} \in \mathcal{M}_1$  is permutation invariant then the corresponding internal energies  $U_l$  defined for  $l \in \mathbb{N}$  in Definition 3.3 are invariant w.r.t. to permutations of lattice sites inside the boxes  $\Lambda_l$ . More precisely: For all  $l \in \mathbb{N}$  and all  $\pi \in \Pi$  such that  $\pi|_{\mathfrak{L} \setminus \Lambda_l} = \text{id}|_{\mathfrak{L} \setminus \Lambda_l}$ ,  $\alpha_\pi(U_l) = U_l$ . Here,  $\text{id} \in \Pi$  is the neutral element of the group  $\Pi$ , i.e., the identity map  $\mathfrak{L} \rightarrow \mathfrak{L}$ . As a consequence, for any permutation invariant  $\mathfrak{m} \in \mathcal{M}_1$ , the thermodynamic limit

$$P_{\mathfrak{m}}^\sharp := \lim_{l \rightarrow \infty} \{p_l\}$$

of the pressure  $p_l = p_{l, \mathfrak{m}}$  (3.7) associated with the internal energy  $U_l$  can be computed via the minimization of the affine free-energy functional  $f_{\mathfrak{m}}^\sharp$  on the subset  $E_\Pi \subset E_1$  of permutation invariant states, see Definitions 3.5, 3.11 and Lemma 3.8 (i).

THEOREM 7.7 (Thermodynamics as a variational problem on  $E_\Pi$ ).

*For any permutation invariant  $\mathfrak{m} \in \mathcal{M}_1$ ,*

$$P_{\mathfrak{m}}^\sharp = - \inf_{\rho \in E_\Pi} f_{\mathfrak{m}}^\sharp(\rho) = - \inf_{\rho \in E_\otimes} f_{\mathfrak{m}}^\sharp(\rho).$$

*Here, the restriction of  $f_{\mathfrak{m}}^\sharp$  on the weak\*-compact convex set  $E_\Pi$  equals, for any  $x \in \mathbb{Z}^d \setminus \{0\}$ , the weak\*-lower semi-continuous affine map*

$$(7.4) \quad \rho \mapsto \int_{\mathcal{A}} \gamma_a \rho((\mathbf{e}_{\Phi_a} - i\mathbf{e}_{\Phi'_a})\alpha_x(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})) \, d\mathbf{a}(a) + e_\Phi(\rho) - \beta^{-1}s(\rho)$$

*from  $E_\Pi$  to  $\mathbb{R}$ , see (2.16) for the definition of  $\mathbf{e}_\Phi$ .*

*Proof.* Observe first that the equality between  $f_{\mathfrak{m}}^\sharp$  and the weak\*-lower semi-continuous affine map (7.4) (cf. Lemmata 2.28 (i) and 2.31 (i)) is a direct

consequence of Lemma 7.4 because  $\mathfrak{m}$  is permutation invariant. By the Bauer maximum principle (Lemma 12.31), it follows that the minimization of  $f_{\mathfrak{m}}^{\sharp}$  on the weak\*-compact convex set  $E_{\Pi}$  can be restricted to the subset  $\mathcal{E}_{\Pi}$  of extreme points which by Theorem 7.1 equals the set  $E_{\otimes}$  of product states.

We analyze now the thermodynamic limit  $l \rightarrow \infty$  of the pressure  $p_l = p_{l,\mathfrak{m}}$ . We concentrate our study on *discrete* permutation invariant models

$$\mathfrak{m} = \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\text{df}} \subset \mathcal{M}_1^{\text{d}} \subset \mathcal{M}_1$$

only. The extension of this proof to any permutation invariant models  $\mathfrak{m} \in \mathcal{M}_1$  is performed by using the density of the set of discrete permutation invariant models in the set of permutation invariant models, see similar arguments performed in Section 3.1 as well as in Section 8.1.

The lower bound on the pressure  $p_l = p_{l,\mathfrak{m}}$  for discrete models  $\mathfrak{m} \in \mathcal{M}_1^{\text{df}}$  follows from the passivity of Gibbs states (Theorem 12.2). Indeed, note that  $\mathfrak{e}_{\Phi} \in \mathcal{U}_{\{0\}}$  for any permutation invariant interaction  $\Phi \in \mathcal{W}_1$ . Therefore, as  $\mathfrak{m}$  is permutation invariant, straightforward estimates show, for all  $\rho \in E_{\Pi}$  and any  $x \in \mathbb{Z}^d \setminus \{0\}$ , that

$$(7.5) \quad \lim_{l \rightarrow \infty} \left\{ \frac{1}{|\Lambda_l|^2} \rho \left( (U_{\Lambda_l}^{\Phi_k} + iU_{\Lambda_l}^{\Phi'_k})^* (U_{\Lambda_l}^{\Phi_k} + iU_{\Lambda_l}^{\Phi'_k}) \right) \right\} = \rho(\mathfrak{e}_{\Phi_k}^* \alpha_x(\mathfrak{e}_{\Phi_k})).$$

Therefore, from (7.1) and (7.5) combined with Definitions 2.27 and 2.30, we deduce that

$$(7.6) \quad \liminf_{l \rightarrow \infty} p_l \geq - \inf_{\rho \in E_{\Pi}} f_{\mathfrak{m}}^{\sharp}(\rho).$$

So, we concentrate now our analysis on the upper bound.

Let  $\rho_l \in E_{\Lambda_l}$  be the Gibbs equilibrium state (12.2) w.r.t. to the internal energy  $U_{\Lambda_l}$ . We define as usual a space-averaged t.i. Gibbs state  $\hat{\rho}_l \in E_1$  by using (3.19) with the even state  $\rho_l$  seen as a periodic state on the whole  $C^*$ -algebra  $\mathcal{U}$ . Observe that the sequences  $\{\rho_l\}_{l \in \mathbb{N}}$  and  $\{\hat{\rho}_l\}_{l \in \mathbb{N}}$  have the same weak\*-accumulation points. Since  $\mathfrak{m}$  is permutation invariant, the internal energy  $U_l$  is invariant w.r.t. permutations of lattice sites inside the boxes  $\Lambda_l$  which in turn implies the invariance of the state  $\rho_l \in E$  under permutations  $\pi \in \Pi$  such that  $\pi|_{\mathcal{E} \setminus \Lambda_l} = \text{id}|_{\mathcal{E} \setminus \Lambda_l}$ . This invariance property of  $\rho_l$  yields that the weak\*-accumulation points of sequences  $\{\rho_l\}_{l \in \mathbb{N}}$  and  $\{\hat{\rho}_l\}_{l \in \mathbb{N}}$  belong to  $E_{\Pi}$ . As a consequence, there is  $\rho_{\infty} \in E_{\Pi}$  and a diverging subsequence  $\{l_n\}_{n \in \mathbb{N}}$  such that both  $\rho_{l_n}$  and  $\hat{\rho}_{l_n}$  converge in the weak\*-topology to the permutation invariant state  $\rho_{\infty}$ .

As  $\mathfrak{e}_{\Phi} \in \mathcal{U}_{\{0\}}$  for any permutation invariant model  $\mathfrak{m} \in \mathcal{M}_1$ , observe, by Lemma 2.31 (i), that

$$(7.7) \quad \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_{l_n}|} \rho_{l_n}(U_{l_n}^{\Phi}) = \lim_{n \rightarrow \infty} \rho_{l_n}(\hat{\mathfrak{e}}_{\Phi, l_n}) = \lim_{n \rightarrow \infty} e_{\Phi}(\rho_{l_n}) = e_{\Phi}(\rho_{\infty}),$$

where

$$(7.8) \quad \widehat{\mathbf{e}}_{\Phi,L} := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \alpha_x(\mathbf{e}_\Phi) = \widehat{\mathbf{e}}_{\Phi,L}^*.$$

By combining the symmetry of the state  $\rho_l \in E$  under permutations of lattice sites inside the boxes  $\Lambda_l$  with elementary combinatorics,

$$(7.9) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \left\{ \frac{1}{|\Lambda_{l_n}|^2} \rho_{l_n} \left( (U_{\Lambda_{l_n}}^{\Phi_k} + iU_{\Lambda_{l_n}}^{\Phi'_k})^* (U_{\Lambda_{l_n}}^{\Phi_k} + iU_{\Lambda_{l_n}}^{\Phi'_k}) \right) \right\} \\ &= \lim_{n \rightarrow \infty} \rho_{l_n}((\mathbf{e}_{\Phi_k} - i\mathbf{e}_{\Phi'_k})\alpha_x(\mathbf{e}_{\Phi_k} + i\mathbf{e}_{\Phi'_k})) = \rho_\infty((\mathbf{e}_{\Phi_k} - i\mathbf{e}_{\Phi'_k})\alpha_x(\mathbf{e}_{\Phi_k} + i\mathbf{e}_{\Phi'_k})) \end{aligned}$$

for any  $x \in \mathbb{Z}^d \setminus \{0\}$ . Furthermore, by using Lemma 2.28 (i), the periodicity of  $\rho_l$  and the additivity of the von Neumann entropy for product states,

$$(7.10) \quad s(\widehat{\rho}_l) = \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} s(\rho_l \circ \alpha_x) = s(\rho_l) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_l^{(n)}|} S(\rho_l|_{\mathcal{U}_{\Lambda_l^{(n)}}}) = \frac{1}{|\Lambda_l|} S(\rho_l)$$

with the definition

$$(7.11) \quad \Lambda_l^{(n)} := \bigcup_{x \in \Lambda_n} \{\Lambda_l + (2l+1)x\}.$$

Therefore, by using (7.2) combined with (7.7), (7.9), (7.10), and Lemma 7.4,

$$(7.12) \quad \limsup_{l \rightarrow \infty} p_l \leq - \lim_{n \rightarrow \infty} f_m^\sharp(\rho_{l_n}) \leq -f_m^\sharp(\rho_\infty)$$

because the entropy density functional  $s$  is a weak\*-upper semi-continuous functional on  $E_1$  (Lemma 2.28 (i)).

Since  $\rho_\infty \in E_{\text{II}}$ , the theorem follows from (7.6) and (7.12) combined with the density of the set of discrete permutation invariant models in the set of permutation invariant models, see, e.g., Corollary 8.3. ■

As a consequence, the thermodynamics of any permutation invariant model  $\mathbf{m} \in \mathcal{M}_1$  can be related to a weak\*-continuous free-energy density functional over one-site states:

COROLLARY 7.8 (Variational problem on one-site states).

For any permutation invariant  $\mathbf{m} \in \mathcal{M}_1$ , the (infinite volume) pressure equals

$$P_m^\sharp = - \inf_{\rho_{\{0\}} \in E_{\{0\}}} \left\{ \int_{\mathcal{A}} \gamma_a |\rho_{\{0\}}(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})|^2 d\mathbf{a}(a) + \rho_{\{0\}}(\mathbf{e}_\Phi) - \beta^{-1} S(\rho_{\{0\}}) \right\}$$

with the weak\*-continuous functional  $S$  being the von Neumann entropy defined by (6.18).

*Proof.* By Lemma 7.4, for any permutation invariant model  $\mathbf{m} \in \mathcal{M}_1$ ,  $x \in \mathbb{Z}^d \setminus \{0\}$  and all product states  $\rho \in E_\otimes$ ,

$$\Delta_{\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}}(\rho) = \rho((\mathbf{e}_{\Phi_a} - i\mathbf{e}_{\Phi'_a})\alpha_x(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})) = |\rho_{\{0\}}(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})|^2$$

with the state  $\rho_{\{0\}} \in E_{\{0\}}$  being the restriction of  $\rho \in E_{\otimes}$  on the local sub-algebra  $\mathcal{U}_{\{0\}}$ . Furthermore, observe that, for any product state  $\rho \in E_{\otimes}$ ,  $s(\rho) = S(\rho_{\{0\}})$ . Therefore, Corollary 7.8 is a direct consequence of Theorem 7.7. ■

The map (7.4) is a weak\*-lower semi-continuous affine map from  $E_{\Pi}$  to  $\mathbb{R}$ . So, from Theorem 7.7, all generalized permutation invariant equilibrium states are (usual) equilibrium states as

$$\Omega_{\mathfrak{m}}^{\sharp} \cap E_{\Pi} = M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi} \neq \emptyset.$$

Moreover,  $\Omega_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}$  is a *face* of  $E_{\Pi}$  (cf. Definitions 3.13 and 3.15). Since  $E_{\Pi}$  is a Bauer simplex (Theorem 7.3) with its set  $\mathcal{E}_{\Pi}$  of extreme points being the set  $E_{\otimes}$  of product states (Theorem 7.1),  $M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}$  is also a simplex and, by using Choquet theorem (cf. Theorems 12.18 and 12.22), each permutation invariant equilibrium state  $\omega \in M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}$  has a *unique* decomposition in terms of states of the set

$$\mathcal{E}(M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}) = \mathcal{E}(M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}) \cap E_{\otimes}$$

of extreme states of  $M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}$ . In fact, Theorem 7.7 and Corollary 7.8 make a detailed analysis of the set  $M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}$  of permutation invariant equilibrium states possible. As an example we recommend [9], where a complete description of permutation invariant equilibrium states for a class of physically relevant models is performed.

Note that  $\Omega_{\mathfrak{m}}^{\sharp} \setminus E_{\Pi}$  may not be empty, i.e., the existence of a generalized t.i. equilibrium state which is not permutation invariant, is, a priori, not excluded. However, for permutation invariant models  $\mathfrak{m}$ , this set  $\Omega_{\mathfrak{m}}^{\sharp} \setminus E_{\Pi}$  is not relevant as soon as the weak\*-limit of Gibbs states is concerned:

**COROLLARY 7.9** (Weak\*-limit of Gibbs equilibrium states).

*For any permutation invariant  $\mathfrak{m} \in \mathcal{M}_1$ , the weak\*-accumulation points of Gibbs equilibrium states  $\{\rho_l\}_{l \in \mathbb{N}}$  belong to the set  $M_{\mathfrak{m}}^{\sharp} \cap E_{\Pi}$  of permutation invariant equilibrium states.*

*Proof.* As explained in the proof of Theorem 7.7, the state  $\rho_l \in E_{\Lambda_l}$  (12.2) associated with  $U_{\Lambda_l}$  allows us to define a space-averaged t.i. Gibbs state  $\hat{\rho}_l \in E_1$ . The sequences  $\{\rho_l\}_{l \in \mathbb{N}}$  and  $\{\hat{\rho}_l\}_{l \in \mathbb{N}}$  have the same weak\*-accumulation points which all belong to  $E_{\Pi}$  because  $\rho_l$  is invariant under permutations  $\pi \in \Pi$  such that  $\pi|_{\mathcal{E} \setminus \Lambda_l} = \text{id}|_{\mathcal{E} \setminus \Lambda_l}$ . Therefore, the corollary is a direct consequence of Theorem 7.7 combined with Equation (7.12) extended to any permutation invariant model  $\mathfrak{m} \in \mathcal{M}_1$  (instead of discrete models only). ■

## 8. Analysis of the Pressure via t.i. States

The aim of this section is to prove Theorem 3.12. This proof is broken in several lemmata. We first show in Section 8.1 that one can reduce the computation

of the thermodynamic limit of (3.7), for any  $\mathbf{m} \in \mathcal{M}_1$ , to discrete finite range models

$$\{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\text{df}} := \mathcal{M}_1^{\text{d}} \cap \mathcal{M}_1^{\text{f}} \subset \mathcal{M}_1,$$

see Corollary 8.3. Then in Section 8.2 we use the so-called passivity of Gibbs states (Theorem 12.2) to find the thermodynamic limit of (3.7), for any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ , from which we deduce Theorem 3.12, see Theorem 8.8.

### 8.1. Reduction to discrete finite range models

From the density of the set of finite range interactions in  $\mathcal{W}_1$ , recall that the sub-space  $\mathcal{M}_1^{\text{df}} := \mathcal{M}_1^{\text{d}} \cap \mathcal{M}_1^{\text{f}}$  of discrete finite range models is dense in  $\mathcal{M}_1$ . As a consequence, the thermodynamic limit of (3.7), for any  $\mathbf{m} \in \mathcal{M}_1$ , can be found by using a sequence  $\{\mathbf{m}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_1^{\text{df}}$  of discrete finite range models converging to  $\mathbf{m}$ . This result follows from the next two lemmata:

LEMMA 8.1 (Equicontinuity of the map  $\mathbf{m} \mapsto p_{\ell, \mathbf{m}}$ ).

*The family of maps  $\mathbf{m} \mapsto p_{l, \mathbf{m}}$  is equicontinuous<sup>20</sup> for  $l \in \mathbb{N}$ . In particular,  $\mathbf{m} \mapsto P_{\mathbf{m}}^{\#}$  (Definition 3.11) is a locally Lipschitz continuous map from  $\mathcal{M}_1$  to  $\mathbb{R}$ .*

*Proof.* For any  $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}_1$  observe that the corresponding internal energies  $U_{l,1}$  and  $U_{l,2}$  (Definition 3.3) satisfy the bound

$$(8.1) \quad \|U_{l,1} - U_{l,2}\| \leq |\Lambda_l| \|\mathbf{m}_1 - \mathbf{m}_2\|_{\mathcal{M}_1} (1 + \|\mathbf{m}_1\|_{\mathcal{M}_1} + \|\mathbf{m}_2\|_{\mathcal{M}_1}).$$

In particular, the map  $\mathbf{m} \mapsto U_l$  is continuous. For each sequence  $\{\mathbf{m}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_1$  converging to  $\mathbf{m}$ , we obtain from the bound (5.11) combined with (8.1) that

$$|p_{l, \mathbf{m}_n} - p_{l, \mathbf{m}}| \leq \|\mathbf{m}_n - \mathbf{m}\|_{\mathcal{M}_1} (1 + \|\mathbf{m}_n\|_{\mathcal{M}_1} + \|\mathbf{m}\|_{\mathcal{M}_1}).$$

This bound leads to the equicontinuity of the family of maps  $\mathbf{m} \mapsto p_l$  for  $l \in \mathbb{N}$  and the locally Lipschitz continuity of the map  $\mathbf{m} \mapsto P_{\mathbf{m}}^{\#}$ . ■

LEMMA 8.2 (Equicontinuity of the map  $\mathbf{m} \mapsto f_{\mathbf{m}}^{\#}(\rho)$ ).

*The family of maps  $\mathbf{m} \mapsto f_{\mathbf{m}}^{\#}(\rho)$  is equicontinuous for  $\rho \in E_1$ . In particular, for any sequence  $\{\mathbf{m}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_1$  converging to  $\mathbf{m} \in \mathcal{M}_1$ ,*

$$\inf_{\rho \in E_1} f_{\mathbf{m}}^{\#}(\rho) = \lim_{n \rightarrow \infty} \inf_{\rho \in E_1} f_{\mathbf{m}_n}^{\#}(\rho).$$

*Proof.* This lemma is a consequence of the norm equicontinuity of the family of maps  $\Phi \mapsto e_{\Phi}(\rho)$  and

$$\mathbf{m} \mapsto \|\Delta_{a,+}(\rho)\|_1 - \|\Delta_{a,-}(\rho)\|_1$$

<sup>20</sup>For each sequence  $\{\mathbf{m}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_1$  converging to  $\mathbf{m}$ ,  $p_{\ell, \mathbf{m}_n}$  converges uniformly in  $\ell \in \mathbb{N}$  to  $p_{\ell, \mathbf{m}}$ .



for  $\rho \in E_1$ . Indeed, for all  $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}_1$  and  $\rho \in E_1$ , the corresponding functionals  $\Delta_{a,\pm}^{(1)}$  and  $\Delta_{a,\pm}^{(2)}$  satisfy the inequality

$$\begin{aligned} & \left| \int_{\mathcal{A}} \Delta_{a,\pm}^{(1)}(\rho) \, d\mathbf{a}(a) - \int_{\mathcal{A}} \Delta_{a,\pm}^{(2)}(\rho) \, d\mathbf{a}(a) \right| \\ & \leq \|\mathbf{m}_1 - \mathbf{m}_2\|_{\mathcal{M}_1} (1 + \|\mathbf{m}_1\|_{\mathcal{M}_1} + \|\mathbf{m}_2\|_{\mathcal{M}_1}) \end{aligned}$$

for all  $\rho \in E_1$ . ■

Therefore, by using Lemmata 8.1–8.2, we can assume, without loss of generality, that

$$\mathbf{m} := \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\text{df}} := \mathcal{M}_1^{\text{d}} \cap \mathcal{M}_1^{\text{f}} \subset \mathcal{M}_1$$

in order to prove Theorem 3.12. Indeed, using the density of the set  $\mathcal{M}_1^{\text{df}}$  in  $\mathcal{M}_1$ , we deduce from Lemmata 8.1 and 8.2 the following corollary:

**COROLLARY 8.3** (Reduction to discrete finite range models).

For any  $\mathbf{m} \in \mathcal{M}_1$ , there exists a sequence  $\{\mathbf{m}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_1^{\text{df}}$  converging to  $\mathbf{m} \in \mathcal{M}_1$  such that

$$P_{\mathbf{m}}^{\sharp} = \lim_{n \rightarrow \infty} P_{\mathbf{m}_n}^{\sharp} \quad \text{and} \quad \inf_{\rho \in E_1} f_{\mathbf{m}}^{\sharp}(\rho) = \lim_{n \rightarrow \infty} \inf_{\rho \in E_1} f_{\mathbf{m}_n}^{\sharp}(\rho).$$

## 8.2. Passivity of Gibbs states and thermodynamics

From Theorem 12.2, the pressure  $p_l$  (3.7) of any finite range discrete model  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$  is bounded from below, for all states  $\rho \in E$ , by (7.1) with Equality (7.2) for  $\rho = \rho_l$ . Recall that  $\rho_l := \rho_{\Lambda_l, U_l}$  is the Gibbs equilibrium state (12.2) with internal energy  $U_l$  defined in Definition 3.3 for any  $\mathbf{m} \in \mathcal{M}_1$  and  $l \in \mathbb{N}$ . This even state  $\rho_l$  is seen as defined either on the local algebra  $\mathcal{U}_{\Lambda_l}$  or on the whole algebra  $\mathcal{U}$  by periodically extending it (with period  $(2l+1)$  in each direction of the lattice  $\mathfrak{L}$ ).

Thus, for any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ , the lower bound on the pressure  $p_l$  in the thermodynamic limit is found by studying the r.h.s. of (7.1) as  $l \rightarrow \infty$ :

**LEMMA 8.4** (Thermodynamic limit of the pressure  $p_l$  – lower bound).

For any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ ,

$$\liminf_{l \rightarrow \infty} p_l \geq - \inf_{\rho \in E_1} f_{\mathbf{m}}^{\sharp}(\rho),$$

with the free-energy density functional  $f_{\mathbf{m}}^{\sharp}$  defined in Definition 3.5.

*Proof.* The first term in the r.h.s. of (7.1) is the only one we really need to control. To this purpose, observe that, for any  $\Phi \in \mathcal{W}_1$  and  $l \in \mathbb{N}$ , the space-average  $\widehat{\mathbf{e}}_{\Phi, l}$  (7.8) of the energy observable  $\mathbf{e}_{\Phi}$  (2.16) is obviously a bounded operator. Hence, by using

$$\widehat{\mathbf{e}}_{\Phi, l} - |\Lambda_l|^{-1} U_{\Lambda_l}^{\Phi} = \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}), 0 \in \Lambda} \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \mathbf{1}_{\{\Lambda \not\subseteq (\Lambda_l - x)\}} \frac{\Phi_{\Lambda+x}}{|\Lambda|},$$

$\|\Phi_{\Lambda+x}\| = \|\Phi_\Lambda\|$ ,  $\|\Phi\|_{\mathcal{W}_1} < \infty$ , and Lebesgue's dominated convergence theorem, we have that

$$(8.2) \quad \lim_{l \rightarrow \infty} \|\widehat{\mathfrak{e}}_{\Phi,l} - |\Lambda_l|^{-1} U_{\Lambda_l}^\Phi\| = 0.$$

Therefore, by using the definition  $\mathfrak{C}_l := U_{\Lambda_l}^\Phi + iU_{\Lambda_l}^{\Phi'}$  for any  $l \in \mathbb{N}$  and any finite range interaction  $\Phi \in \mathcal{W}_1^f$ , we obtain

$$(8.3) \quad \lim_{l \rightarrow \infty} \left\{ \frac{1}{|\Lambda_l|^2} \rho(\mathfrak{C}_l^* \mathfrak{C}_l) - \rho((\widehat{\mathfrak{e}}_{\Phi,l} + i\widehat{\mathfrak{e}}_{\Phi',l})^* (\widehat{\mathfrak{e}}_{\Phi,l} + i\widehat{\mathfrak{e}}_{\Phi',l})) \right\} = 0$$

uniformly in  $\rho \in E$ . Consequently, the lower bound on the pressure  $p_l$  as  $l \rightarrow \infty$  follows from (7.1) combined with Definitions 2.14, 2.27, 2.30, and (8.3). ■

To obtain the upper bound on the pressure  $p_l$ , as in the proof of Theorem 7.7, one needs to control each term in (7.2) when  $l \rightarrow \infty$ . Observe that  $\rho_l$  is *generally not t.i.* even if  $\mathfrak{m} \in \mathcal{M}_1^{\text{df}}$  does per definition. But, we can canonically construct a space-averaged t.i. Gibbs state  $\hat{\rho}_l$  from  $\rho_l$ , see (3.19). If we restrict ourselves to the case of models with purely repulsive long range interactions (i.e.  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.)), we can analyze each term in (7.2) as a function of  $\hat{\rho}_l \in E_1$  in the limit  $l \rightarrow \infty$ . The mean entropy per volume as a function of the t.i. state  $\hat{\rho}_l$  (3.19) is already given in the proof of Theorem 7.7 by Equality (7.10). The analysis of the other terms is, however, more involved than for permutation invariant models (Definition 7.6). The first term of the r.h.s. of (7.2) being the most problematic one if we tries to use the space-averaged t.i. Gibbs state  $\hat{\rho}_l$  as test states.

In the following we treat first – as a kind of “warming up” – the special case of purely repulsive models by using the sequence  $\hat{\rho}_l \in E_1$  of test states. The general case – for which the test states have to be much more carefully chosen – is considered afterwards.

We now prove that, at large  $l$ , the internal energy computed from a large box  $\Lambda_l^{(n)}$  (7.11) is the same as the one for  $|\Lambda_n|$  copies of boxes of volume  $|\Lambda_l|$ . This is a standard method often used in statistical mechanics to prove the existence of the thermodynamic limit.

LEMMA 8.5 (Internal energy).

For any finite range t.i. interaction  $\Phi \in \mathcal{W}_1^f$ ,

$$\sup_{n \in \mathbb{N}} \left\{ \frac{1}{|\Lambda_l^{(n)}|} \|U_{\Lambda_l^{(n)}}^\Phi - \sum_{x \in \Lambda_n} U_{\Lambda_l + (2l+1)x}^\Phi\| \right\} = \mathcal{O}(l^{-1}).$$

*Proof.* From Definition 2.22, it is straightforward to check, for any t.i. finite range interaction  $\Phi$ , that

$$\begin{aligned} \frac{1}{|\Lambda_l^{(n)}|} \|U_{\Lambda_l^{(n)}}^\Phi - \sum_{x \in \Lambda_n} U_{\Lambda_l + (2l+1)x}^\Phi\| &\leq \frac{|\Lambda_n|}{|\Lambda_l^{(n)}|} \sum_{\Lambda \subset \partial \Lambda_l} \|\Phi_\Lambda\| \\ &\leq \frac{|\partial \Lambda_l|}{|\Lambda_l|} \|\Phi\|_{\mathcal{W}_1} = \mathcal{O}(l^{-1}) \end{aligned}$$

with  $\partial \Lambda_l$  being the boundary<sup>21</sup> of the cubic box  $\Lambda_l$  defined for large enough  $m \geq 1$ . ■

As a consequence, as far as the limit  $l \rightarrow \infty$  is concerned one can use, for all  $\Phi \in \mathcal{W}_1$ , the energy density  $e_\Phi(\hat{\rho}_l)$  instead of the mean internal energy per volume  $\rho_l(U_{\Lambda_l}^\Phi)/|\Lambda_l|$ . (Recall that  $\hat{\rho}_l \in E_1$  is the t.i. state (3.19).) Indeed, one deduces from Lemma 8.5 the following result:

LEMMA 8.6 (Mean internal energy per volume as  $l \rightarrow \infty$ ).

For any  $\mathfrak{m} \in \mathcal{M}_1$  and all finite range interactions  $\Phi \in \mathcal{W}_1^f$ ,

$$\left| e_\Phi(\hat{\rho}_l) - \frac{\rho_l(U_{\Lambda_l}^\Phi)}{|\Lambda_l|} \right| = \mathcal{O}(l^{-1})$$

with the energy density  $e_\Phi(\rho)$  defined by Definition 2.30.

*Proof.* By  $(2l+1)\mathbb{Z}^d$ -invariance of Gibbs equilibrium states  $\rho_l$ , it follows that

$$\sum_{x \in \Lambda_n} \rho_l(U_{\Lambda_l + (2l+1)x}^\Phi) = |\Lambda_n| \rho_l(U_{\Lambda_l}^\Phi).$$

Consequently, by using Lemma 8.5 and the limit  $n \rightarrow \infty$ , one obtains that

$$(8.4) \quad \left| e_\Phi(\rho_l) - \frac{\rho_l(U_{\Lambda_l}^\Phi)}{|\Lambda_l|} \right| = \mathcal{O}(l^{-1}).$$

The functional  $\rho \mapsto e_\Phi(\rho)$  is affine and t.i., see Lemma 2.31 (i). Therefore  $e_\Phi(\hat{\rho}_l) = e_\Phi(\rho_l)$  which combined with (8.4) implies the lemma. ■

The next step to find the upper bound is now to study the first term in the r.h.s of (7.2) because the others terms can be controlled by using (7.10) and Lemma 8.6. The relationship of this term with  $\Delta_{\mathfrak{e}_\Phi + i\mathfrak{e}_{\Phi'}}(\hat{\rho}_l)$  at large  $l$  is problematic (recall that  $\mathfrak{e}_\Phi := \mathfrak{e}_{\Phi, (1, \dots, 1)}$  and  $\Delta_A$  are respectively defined by (2.16) and Definition 2.14): On the one hand, we cannot expect the limit

$$\lim_{l \rightarrow \infty} \left( \frac{1}{|\Lambda_l|^2} \rho_l((U_{\Lambda_l}^\Phi + iU_{\Lambda_l}^{\Phi'})^* (U_{\Lambda_l}^\Phi + iU_{\Lambda_l}^{\Phi'})) - |\rho_l(\mathfrak{e}_\Phi + i\mathfrak{e}_{\Phi'})|^2 \right) = 0$$

<sup>21</sup>By fixing  $m \geq 1$  the boundary  $\partial \Lambda$  of any  $\Lambda \subset \Gamma$  is defined by  $\partial \Lambda := \{x \in \Lambda : \exists y \in \Gamma \setminus \Lambda \text{ with } d(x, y) \leq m\}$ , see (2.14) for the definition of the metric  $d(x, y)$ .

to hold in general. Otherwise it would follow – at least w.r.t. the observables  $\mathbf{e}_\Phi$  and  $\mathbf{e}_{\Phi'}$  – the absence of long range order (LRO). On the other hand, we know – as  $\hat{\rho}_l$  are ergodic states – that:

$$\Delta_{\mathbf{e}_\Phi + i\mathbf{e}_{\Phi'}}(\hat{\rho}_l) = |\hat{\rho}_l(\mathbf{e}_\Phi + i\mathbf{e}_{\Phi'})|^2.$$

In the case of purely repulsive long range coupling constants where  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) (cf. Definition 3.4), the arguments become easier because from the GNS representation of  $\rho_l$  combined with (8.3) for  $\rho = \rho_l$  we obtain that, for any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ ,

$$\liminf_{l \rightarrow \infty} \left\{ \frac{1}{|\Lambda_l|^2} \rho_l \left( (U_{\Lambda_l}^\Phi + iU_{\Lambda_l}^{\Phi'})^* (U_{\Lambda_l}^\Phi + iU_{\Lambda_l}^{\Phi'}) \right) - \Delta_{\mathbf{e}_\Phi + i\mathbf{e}_{\Phi'}}(\hat{\rho}_l) \right\} \geq 0.$$

This last limit combined with (7.2), (7.10), and Lemma 8.6, yields the desired upper bound when  $\Phi_{a,-} = 0$  (a.e.), i.e., for purely repulsive long range models.

However, as soon as we have long range attractions  $\Phi_{a,-}, \Phi'_{a,-} \neq 0$  (a.e.), the proof of the upper bound on the pressure requires Corollary 7.8 as a key ingredient to obtain a more convenient sequence of test states  $\hat{\rho}_l \in \mathcal{E}_1$ . ( $\rho_l, \hat{\rho}_l$ , and  $\hat{\rho}_l$  have not necessarily the same weak\*-accumulation points.) In fact, similar arguments was first used in [20] and subsequently in [21] for translation invariant quantum spin systems (Remark 2.4). Following their strategy [20, 21] combined with Corollary 7.8, we obtain the desired upper bound for any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ :

LEMMA 8.7 (Thermodynamic limit of the pressure  $p_l$  – upper bound).

For any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ , there is a sequence  $\{\hat{\rho}_l\}_{l \in \mathbb{N}} \subset \mathcal{E}_1$  of ergodic states such that

$$\limsup_{l \rightarrow \infty} p_{l,\mathbf{m}} = - \lim_{l \rightarrow \infty} g_{\mathbf{m}}(\hat{\rho}_l) = - \lim_{l \rightarrow \infty} f_{\mathbf{m}}^\sharp(\hat{\rho}_l) \leq - \inf_{\rho \in E_1} f_{\mathbf{m}}^\sharp(\rho)$$

with the functional  $g_{\mathbf{m}}$  defined by Definition 3.6.

*Proof.* We use here, at any fixed  $l \in \mathbb{N}$  and any interaction  $\Phi \in \mathcal{W}_1$ , the self-adjoint elements

$$U_{l,n}^\Phi := \sum_{x \in \Lambda_n} \alpha_{(2l+1)x}(U_{\Lambda_l}^\Phi)$$

for  $n \in \mathbb{N}$ , whereas  $U_{l,0}^\Phi := U_{\Lambda_l}^\Phi$ . Then, for any  $l, n \in \mathbb{N}$  and any discrete finite range model

$$\mathbf{m} := \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\text{df}},$$

we define the internal energy  $U_{l,n}$  by

$$U_{l,n} := U_{l,n}^\Phi + \sum_{k=1}^N \frac{\gamma_k}{|\Lambda_l^{(n)}|} (U_{l,n}^{\Phi_k} + iU_{l,n}^{\Phi'_k})^* (U_{l,n}^{\Phi_k} + iU_{l,n}^{\Phi'_k})$$

with  $\Lambda_l^{(n)}$  defined by (7.11). The pressure associated with  $U_{l,n}$  is as usual defined, for  $\beta \in (0, \infty)$ , by

$$p_{l,m}(n, \beta) := \frac{1}{\beta |\Lambda_l^{(n)}|} \ln \text{Trace}_{\mathcal{H}_{\Lambda_l}}(e^{-\beta U_{l,n}}).$$

Now, by using Lemma 8.5 together with (5.11) observe that

$$(8.5) \quad \lim_{l \rightarrow \infty} \left\{ \limsup_{n \rightarrow \infty} |p_{l,m}(n, \beta) - p_{2l n + n + l, m}| \right\} = 0$$

for any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ . The pressure  $p_{l,m}(n, \beta)$  can be seen as a pressure of a *permutation invariant* model  $\mathbf{m}_l$  defined as follows. Recall that the  $C^*$ -algebra  $\mathcal{U}$  is the fermion algebra defined in Section 2.1 with a spin set  $S$ . Then the space  $\mathcal{M}_1 = \mathcal{M}_1(\mathcal{U})$  defined by Definition 3.1 is the Banach space of long range models constructed from  $\mathcal{U}$ . Now, for each  $l \in \mathbb{N}$ , we define the  $C^*$ -algebra  $\mathcal{U}_l$  to be the fermion algebra with spin set  $S \times \Lambda_l$ , and in the same way  $\mathcal{M}_1$  is defined, we construct from  $\mathcal{U}_l$  the Banach space  $\mathcal{M}_1(\mathcal{U}_l)$  of long range models. For any  $x \in \Lambda_n$ , note that the sub-algebra  $(\mathcal{U}_l)_{\{x\}}$  of  $\mathcal{U}_l$  can be canonically identified with the sub-algebra  $\mathcal{U}_{\Lambda_l + (2l+1)x}$  of  $\mathcal{U}$ . At  $l \in \mathbb{N}$  and for any  $\mathbf{m} \in \mathcal{M}_1^{\text{df}}$ , the permutation invariant discrete long range model  $\mathbf{m}_l$  is the element

$$\mathbf{m}_l := \{\Phi^{(l)}\} \cup \{\Phi_k^{(l)}, (\Phi^{(l)})'_k\}_{k=1}^N \in \mathcal{M}_1(\mathcal{U}_l)$$

uniquely defined by the conditions

$$\Phi_{\{0\}}^{(l)} := |\Lambda_l|^{-1} U_{\Lambda_l}^{\Phi}, \quad (\Phi_k^{(l)})_{\{0\}} := |\Lambda_l|^{-1} U_{\Lambda_l}^{\Phi_k}, \quad ((\Phi^{(l)})'_k)_{\{0\}} := |\Lambda_l|^{-1} U_{\Lambda_l}^{\Phi'_k}$$

with  $\Phi_{\Lambda}^{(l)} = (\Phi_k^{(l)})_{\Lambda} = ((\Phi^{(l)})'_k)_{\Lambda} = 0$  whenever  $|\Lambda| \neq 1$ .

By using these definitions, we have

$$p_{l,m}(n, \beta) = p_{n, \mathbf{m}_l}(0, \beta_l)$$

with  $\beta_l := |\Lambda_l| \beta$ . Therefore, we are in position to use Corollary 7.8 in order to compute the thermodynamic limit  $n \rightarrow \infty$  of the permutation invariant discrete model  $\mathbf{m}_l \in \mathcal{M}_1(\mathcal{U}_l)$ :

$$\begin{aligned} \lim_{n \rightarrow \infty} p_{l,m}(n, \beta) &= \lim_{n \rightarrow \infty} p_{n, \mathbf{m}_l}(0, \beta_l) = - \inf_{\rho_{\Lambda_l} \in E_{\Lambda_l}} \left\{ \sum_{k=1}^N \gamma_k |\Lambda_l|^{-2} |\rho_{\Lambda_l}(U_{\Lambda_l}^{\Phi_k} + i U_{\Lambda_l}^{\Phi'_k})|^2 \right. \\ &\quad \left. + |\Lambda_l|^{-1} \rho_{\Lambda_l}(U_{\Lambda_l}^{\Phi}) - (\beta |\Lambda_l|)^{-1} S(\rho_{\Lambda_l}) \right\} \end{aligned}$$

with the weak\*-continuous functional  $S$  being the von Neumann entropy defined by (6.18). This variational problem is a minimization of a weak\*-continuous functional over the set  $E_{\Lambda_l}$  of all (local) states on the finite dimensional algebra  $\mathcal{U}_{\Lambda_l}$ . Therefore, for each  $l \in \mathbb{N}$ , it has a minimizer  $\varrho_l \in E_{\Lambda_l}$  which can also be seen as a state on the whole algebra  $\mathcal{U}$  by periodically extending it (with

period  $(2l + 1)$  in each direction of the lattice  $\mathfrak{L}$ ). We define from  $\varrho_l \in E$  the t.i. space-averaged (ergodic) state

$$\hat{\varrho}_l := \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \varrho_l \circ \alpha_x \in \mathcal{E}_1$$

(compare this definition with (6.8) for  $\vec{\ell} = (1, \dots, 1)$ ). Then, by using  $\Delta_A(\hat{\varrho}_l) = |\hat{\varrho}_l(A)|^2$  (see Theorem 2.19 (iv)), Equality (7.10) and Lemma 8.6 applied to states  $\varrho_l \in E$  and  $\hat{\varrho}_l \in \mathcal{E}_1$ , we obtain that

$$(8.6) \quad \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} p_{l,m}(n, \beta) = - \lim_{l \rightarrow \infty} g_m(\hat{\varrho}_l) = - \lim_{l \rightarrow \infty} f_m^\sharp(\hat{\varrho}_l),$$

see also Lemma 3.8 (ii). Therefore, the limits (8.5) and (8.6) yield the lemma.

■

Consequently, Theorem 3.12 is a direct consequence of Lemmata 3.9, 8.1, 8.4, and 8.7 together with Corollary 8.3. In fact, we obtain a bit more than Theorem 3.12. Indeed, by combining Theorem 3.12 with Theorem 12.2, (8.3) and the fact that the space-average  $\widehat{\mathbf{e}}_{\Phi,l}$  (7.8) is uniformly bounded by  $\|\Phi\|_{\mathcal{W}_1}$  for  $l \in \mathbb{N}$ , we show that the map

$$(8.7) \quad \rho \mapsto \mathfrak{F}_m^\sharp(\rho) := \limsup_{l \rightarrow \infty} \left\{ \int_{\mathcal{A}} \gamma_a \rho((\widehat{\mathbf{e}}_{\Phi_a,l} + i\widehat{\mathbf{e}}_{\Phi'_a,l})^*(\widehat{\mathbf{e}}_{\Phi_a,l} + i\widehat{\mathbf{e}}_{\Phi'_a,l})) da(a) + \frac{1}{|\Lambda_l|} \rho(U_{\Lambda_l}^\Phi) - \frac{1}{\beta|\Lambda_l|} S(\rho_{\Lambda_l}) \right\}$$

from  $E$  to  $\mathbb{R}$  makes sense, as the quantity in the lim sup above is uniformly bounded in  $l \in \mathbb{N}$ . Furthermore, for any  $\rho \in E_{\vec{\ell}}$ ,  $\mathfrak{F}_m^\sharp(\rho) = f_m^\sharp(\rho)$  because the lim sup in the definition of  $\mathfrak{F}_m^\sharp$  above can be changed into a lim on the set  $E_{\vec{\ell}}$  of  $\vec{\ell}\mathbb{Z}^d$ -invariant states for any  $\vec{\ell} \in \mathbb{N}^d$ . See also Corollary 6.20 (i). By deriving upper and lower bounds for the pressure w.r.t.  $\mathfrak{F}_m^\sharp(\rho)$ , exactly in the same way we did for  $f_m^\sharp(\rho)$ , we get the following theorem:

**THEOREM 8.8** (Pressure  $P_m^\sharp$  as variational problems on states).

(i) For  $\vec{\ell} \in \mathbb{N}^d$  and any  $\mathbf{m} \in \mathcal{M}_1$ ,

$$P_m^\sharp := \lim_{l \rightarrow \infty} \{p_l\} = - \inf_{\rho \in E} \mathfrak{F}_m^\sharp(\rho) = - \inf_{\rho \in E_{\vec{\ell}}} f_m^\sharp(\rho) = - \inf_{\rho \in E_1} f_m^\sharp(\rho) < \infty.$$

(ii) The map  $\mathbf{m} \mapsto P_m^\sharp$  from  $\mathcal{M}_1$  to  $\mathbb{R}$  is locally Lipschitz continuous.

The two infima, respectively over the set  $E$  and  $E_{\vec{\ell}}$  of Theorem 8.8 (i), are not really used in the sequel as we concentrate our attention on t.i. states. These results are only discussed in Section 3.5.

**REMARK 8.9** (Convexity of the functional  $\mathfrak{F}_m^\sharp$ ).

As  $\mathfrak{F}_m^\sharp(\rho)$  is defined by a lim sup, by using the property **S4** of the von Neumann entropy, it is easy to check that the map  $\rho \mapsto \mathfrak{F}_m^\sharp(\rho)$  from  $E$  to  $\mathbb{R}$  is a convex functional.

## 9. Purely Attractive Long Range Fermi Systems

Recall that generalized t.i. equilibrium states are defined to be weak\*-limit points of approximating minimizers of the free-energy density functional  $f_m^\sharp$ , see Definition 3.15. It is, a priori, not clear that the first variational problem

$$P_m^\sharp = - \inf_{\rho \in E_1} f_m^\sharp(\rho)$$

given in Theorem 3.12 (i) has any minimizer. The problem comes from the fact that  $f_m^\sharp$  is generally not weak\*-lower semi-continuous because of the long range repulsions, see discussions after Lemma 3.8. As a consequence, models without long range repulsions (Definition 3.4 (+)), i.e., with  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.), are the easiest case to handle. This specific case is analyzed in this section also because it is necessary to understand the variational problem  $F_m^b$  of the thermodynamic game defined in Definition 3.35 and studied in Section 10.1.

Thermodynamics of models without long range repulsions is then discussed in Section 9.2. We start, indeed, in Section 9.1 with some preliminary results about the thermodynamics of approximating interactions of long range models, see Definition 3.31.

### 9.1. Thermodynamics of approximating interactions

As a preliminary step, we describe the thermodynamic limit

$$P_m(c_a) := \lim_{l \rightarrow \infty} \{p_l(c_a)\}$$

of the pressure  $p_l(c_a)$  (3.28) associated with the internal energy  $U_l(c_a) := U_{\Lambda_l}^{\Phi_{\Lambda}(c_a)}$  (3.25) for any  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ . This question is already solved by Theorem 3.12 for all  $m \in \mathcal{M}_1$ , and so, in particular for  $(\Phi(c_a), 0, 0) \in \mathcal{M}_1$ , see Definition 3.31. We give this result together with additional properties as a proposition:

**PROPOSITION 9.1** (Pressure of approximating interactions of  $m \in \mathcal{M}_1$ ).

(i) For any  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ ,

$$P_m(c_a) = - \inf_{\rho \in E_1} f_m(\rho, c_a) = - \inf_{\hat{\rho} \in \mathcal{E}_1} f_m(\hat{\rho}, c_a)$$

with the map  $(\rho, c_a) \mapsto f_m(\rho, c_a)$  defined by (3.30), see also (9.1) just below.

(ii) The map  $c_a \mapsto P_m(c_a)$  from  $L^2(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  is convex and Lipschitz norm continuous as, for all  $c_a, c'_a \in L^2(\mathcal{A}, \mathbb{C})$ ,

$$|P_m(c_a) - P_m(c'_a)| \leq 2(\|\Phi_a\|_2 + \|\Phi'_a\|_2)\|c_a - c'_a\|_2.$$

It is also continuous w.r.t. to the weak topology on any ball  $\mathcal{B}_R(0) \subset L^2(\mathcal{A}, \mathbb{C})$  of arbitrary radius  $R > 0$  centered at 0.

*Proof.* The first assertion (i) is just Lemma 3.9 and Theorem 3.12 (i) applied to the (local) model  $(\Phi(c_a), 0, 0) \in \mathcal{M}_1$  because, for all  $c_a \in L^2(\mathcal{A}, \mathbb{C})$  and  $\rho \in E_1$ ,

$$(9.1) \quad f_{\Phi(c_a)} = f_{\mathfrak{m}}(\rho, c_a) := 2 \operatorname{Re} \left\{ \langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), \gamma_a c_a \rangle \right\} + e_{\Phi}(\rho) - \beta^{-1} s(\rho),$$

see Definition 2.32. The definition of  $\langle \cdot, \cdot \rangle$  is given in Section 12.3. Thus, the Lipschitz norm continuity of the map  $c_a \mapsto P_{\mathfrak{m}}(c_a)$  is a direct consequence of (i) together with the Cauchy–Schwarz inequality and the uniform upper bound of Lemma 2.31 (ii). Knowing (i), the convexity of  $c_a \mapsto P_{\mathfrak{m}}(c_a)$  is also easy to deduce because the map  $c_a \mapsto f_{\mathfrak{m}}(\rho, c_a)$  is obviously real linear for any  $\rho \in E_1$ . The proof of the continuity of  $c_a \mapsto P_{\mathfrak{m}}(c_a)$  w.r.t. to the weak topology on any ball  $\mathcal{B}_R(0)$  results from the weak equicontinuity of the family

$$(9.2) \quad \{c_a \mapsto f_{\mathfrak{m}}(\rho, c_a)\}_{\rho \in E_1}$$

of real linear functionals on  $\mathcal{B}_R(0)$ . The latter is proven as follows.

If  $\mathfrak{m} = \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\mathfrak{d}}$  is a discrete model then the family (9.2) of maps is weakly equicontinuous on  $L^2(\mathcal{A}, \mathbb{C})$ . This follows from the (uniform) upper bound

$$\left| \langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), \gamma_a c'_a \rangle \right| \leq \sum_{k=1}^N (\|\Phi_k\|_{\mathcal{W}_1} + \|\Phi'_k\|_{\mathcal{W}_1}) |\langle c'_a, \mathbf{1}_{I_k} \rangle|,$$

satisfied for all  $\rho \in E_1$ , where  $I_k \in \mathfrak{A}$  are conveniently chosen subsets of  $\mathcal{A}$  such that  $\mathfrak{a}(I_k) < \infty$  for  $k \in \{1, \dots, N\}$ . Let  $\varepsilon, R > 0$  and  $\mathfrak{m} \in \mathcal{M}_1$ . From the density of  $\mathcal{M}_1^{\mathfrak{d}}$  in  $\mathcal{M}_1$  and the uniform upper bound of Lemma 2.31 (ii) combined with the Cauchy–Schwarz inequality, there is  $\mathfrak{m}' \in \mathcal{M}_1^{\mathfrak{d}}$  such that, for all  $c_a \in \mathcal{B}_R(0)$  and  $\rho \in E_1$ ,

$$|f_{\mathfrak{m}}(\rho, c_a) - f_{\mathfrak{m}'}(\rho, c_a)| \leq \frac{\varepsilon}{3}.$$

By the equicontinuity on  $L^2(\mathcal{A}, \mathbb{C})$  of the family (9.2) of maps for any discrete models, for all  $c_a \in \mathcal{B}_R(0)$  there is a weak neighborhood  $\mathcal{V}_\varepsilon$  of  $c_a$  such that, for all  $c'_a \in \mathcal{V}_\varepsilon$  and all  $\rho \in E_1$ ,

$$|f_{\mathfrak{m}'}(\rho, c_a) - f_{\mathfrak{m}'}(\rho, c'_a)| \leq \frac{\varepsilon}{3}.$$

Therefore, for all  $c_a \in \mathcal{B}_R(0)$ , there is a weak neighborhood  $\mathcal{V}_\varepsilon$  of  $c_a$  such that, for all  $c'_a \in \mathcal{V}_\varepsilon$  and all  $\rho \in E_1$ ,

$$|f_{\mathfrak{m}}(\rho, c'_a) - f_{\mathfrak{m}}(\rho, c_a)| \leq \varepsilon.$$

In other words, for any  $\mathfrak{m} \in \mathcal{M}_1$ , the family (9.2) of maps is weakly equicontinuous on  $\mathcal{B}_R(0)$  which yields the continuity of the map  $c_a \mapsto P_{\mathfrak{m}}(c_a)$  in the weak topology on  $\mathcal{B}_R(0)$ . ■



## 9.2. Structure of the set $M_m^\sharp = \Omega_m^\sharp$ of t.i. equilibrium states

We analyze models without long range repulsions (Definition 3.4 (+)), i.e.,  $m \in \mathcal{M}_1$  satisfying  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.). Their (infinite volume) pressure

$$P_m := P_m^\sharp = P_m^b$$

defined in Definition 3.11 (see also (3.14)) is already given by Theorem 3.12 and we first prove Theorem 3.36. In fact, by using the simple inequality

$$(9.3) \quad |\rho(A - c)|^2 = |\rho(A)|^2 - 2 \operatorname{Re} \{ \rho(A) \bar{c} \} + |c|^2 \geq 0$$

for any  $c \in \mathbb{C}$  and  $A \in \mathcal{U}$ , Theorem 3.36 for models without long range repulsions is not difficult to show. Indeed, (9.3) yields the following lemma:

LEMMA 9.2 ( $c_{a,\pm}$ -approximation of  $\|\gamma_{a,\pm} \rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2$ ).

For any  $m \in \mathcal{M}_1$  and all  $\rho \in E_1$ ,

$$\sup_{c_{a,\pm} \in L_{\pm}^2(\mathcal{A}, \mathbb{C})} \left\{ -\|c_{a,\pm}\|_2^2 + 2 \operatorname{Re} \{ \langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), c_{a,\pm} \rangle \} \right\} = \|\gamma_{a,\pm} \rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2$$

with unique maximizer  $d_{a,\pm}(\rho) = \gamma_{a,\pm}(e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho))$  (a.e.).

*Proof.* This lemma is a direct consequence of (9.3). In particular, the solution  $d_{a,\pm}(\rho) \in L_{\pm}^2(\mathcal{A}, \mathbb{C})$  of the variational problem satisfies, for all  $c_{a,-} \in L_{-}^2(\mathcal{A}, \mathbb{C})$ , the Euler–Lagrange equations

$$\operatorname{Re} \{ \langle d_{a,\pm}(\rho), c_{a,\pm} \rangle \} = \operatorname{Re} \{ \langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), c_{a,\pm} \rangle \}.$$

■

Then Theorem 3.36 for models without long range repulsions is a direct consequence of Theorem 3.12 (i) together with Proposition 9.1 and Lemma 9.2.

PROPOSITION 9.3 (Pressure of models without long range repulsions).

For any  $m \in \mathcal{M}_1$  satisfying  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.),

$$P_m = -F_m^\sharp = -F_m^b = - \inf_{c_{a,-} \in \mathcal{B}_{R,-}} f_m(c_{a,-}, 0) =: -F_m$$

with  $f_m(c_{a,-}, 0)$  defined by Definition 3.34 and  $\mathcal{B}_{R,-} \subset L_{-}^2(\mathcal{A}, \mathbb{C})$  (3.31) being a closed ball of sufficiently large radius  $R > 0$  centered at 0.

*Proof.* If  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.) then, for all extreme states  $\hat{\rho} \in \mathcal{E}_1$ ,

$$f_m^\sharp(\hat{\rho}) = g_m(\hat{\rho}) = - \|\gamma_{a,-} \hat{\rho}(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 + e_{\Phi}(\hat{\rho}) - \beta^{-1} s(\hat{\rho}),$$

see Lemma 3.8 (ii). From Lemma 9.2 it follows that

$$(9.4) \quad \inf_{\hat{\rho} \in \mathcal{E}_1} f_m^\sharp(\hat{\rho}) = \inf_{\hat{\rho} \in \mathcal{E}_1} \left\{ \inf_{c_{a,-} \in L_{-}^2(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,-}\|_2^2 + f_m(\hat{\rho}, c_{a,-}) \right\} \right\}$$

with  $f_m(\rho, c_{a,-})$  defined by (9.1) for  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.). The infima in Equality (9.4) obviously commute with each other and, by doing this, we get via Theorem 3.12 (i) and Proposition 9.1 (i) that

$$P_m = \sup_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} \left\{ -\|c_{a,-}\|_2^2 + P_m(c_{a,-}) \right\} = - \inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} f_m(c_{a,-}, 0) < \infty.$$

Finally, the existence of a radius  $R > 0$  such that

$$\inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} f_m(c_{a,-}, 0) = \inf_{c_{a,-} \in \mathcal{B}_{R,-}} f_m(c_{a,-}, 0)$$

directly follows from the upper bound of Proposition 9.1 (ii). ■

The description of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states (Definition 3.15) is also easy to perform when there is no long range repulsions. Indeed, the free-energy density functional  $f_m^\sharp$  becomes weak\*-lower semi-continuous when  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.), see discussions after Lemma 3.8. In particular, the variational problem

$$P_m = - \inf_{\rho \in E_1} f_m^\sharp(\rho)$$

has t.i. minimizers, i.e.,  $\Omega_m^\sharp = M_m^\sharp$  (Definition 3.13). Recall that  $\Omega_m^\sharp$  is convex and weak\*-compact, by Lemma 3.16, and since  $M_m^\sharp = \Omega_m^\sharp$  in this case, the non-empty set  $\Omega_m^\sharp$  is a closed face of  $E_1$  by Lemma 3.14. Therefore, to extract the structure of the set  $\Omega_m^\sharp = M_m^\sharp$ , it suffices to describe extreme states  $\hat{\omega} \in \Omega_m^\sharp \cap \mathcal{E}_1$  which are directly related with the solutions  $d_{a,-} \in \mathcal{C}_m^\sharp \subset L^2_-(\mathcal{A}, \mathbb{C})$  of the variational problem given in Proposition 9.3:

PROPOSITION 9.4 (Gap equations).

Let  $m \in \mathcal{M}_1$  be a model without long range repulsions:  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.).

(i) For all ergodic states  $\hat{\omega} \in \Omega_m^\sharp \cap \mathcal{E}_1$ ,

$$(9.5) \quad d_{a,-} = \gamma_{a,-}(e_{\Phi_a}(\hat{\omega}) + ie_{\Phi'_a}(\hat{\omega})) \in \mathcal{C}_m^\sharp$$

and  $\hat{\omega} \in M_{\Phi(d_{a,-})}$  with  $M_{\Phi(d_{a,-})}$  being described in Lemma 3.33.

(ii) Conversely, for any fixed  $d_{a,-} \in \mathcal{C}_m^\sharp$ ,  $M_{\Phi(d_{a,-})} \cap \mathcal{E}_1 \subset \Omega_m^\sharp \cap \mathcal{E}_1$  and all states  $\omega \in M_{\Phi(d_{a,-})}$  satisfy (9.5).

*Proof.* (i) Recall that  $\Omega_m^\sharp = M_m^\sharp$ . Any  $\hat{\omega} \in \Omega_m^\sharp \cap \mathcal{E}_1$  is a solution of the l.h.s. of (9.4) and the solution  $d_{a,-} = d_{a,-}(\hat{\omega})$  of the variational problem

$$\inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,-}\|_2^2 + f_m(\hat{\omega}, c_{a,-}) \right\}$$

satisfies the Euler-Lagrange equations (9.5), by Lemma 9.2. The two infima in (9.4) commute with each other. It is what it is done above to prove Proposition 9.3. Therefore,  $d_{a,-}(\hat{\omega}) \in \mathcal{C}_m^\sharp$  and, by (9.1),  $\hat{\omega}$  belongs to the set  $M_{\Phi(d_{a,-})} = \Omega_{\Phi(d_{a,-})}$  of t.i. equilibrium states of the approximating interaction  $\Phi(d_{a,-})$ .

(ii) Any  $d_{a,-} \in \mathcal{C}_m^\sharp$  is solution of the variational problem given in Proposition 9.3, that is,

$$(9.6) \quad \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,-}\|_2^2 + \inf_{\rho \in E_1} f_m(\rho, c_{a,-}) \right\},$$

see Proposition 9.1 (i). Since the two infima in (9.6) commute with each other as before, any t.i. equilibrium state  $\omega \in M_{\Phi(d_{a,-})}$  satisfies (9.5) because of Lemma 9.2, and  $M_{\Phi(d_{a,-})} \cap \mathcal{E}_1 \subset \Omega_m^\sharp \cap \mathcal{E}_1$  because of (9.4). ■

Therefore, since the convex and weak\*-compact set  $\Omega_m^\sharp = M_m^\sharp$  is a closed face of  $E_1$  in this case, Proposition 9.4 leads to an exact characterization of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states via the closed faces  $M_{\Phi(d_{a,-})}$  for  $d_{a,-} \in \mathcal{C}_m^\sharp$ :

**COROLLARY 9.5** (Structure of  $\Omega_m^\sharp$  through approximating interactions). *For any model  $m \in \mathcal{M}_1$  such that  $\Phi_{a,+} = \Phi'_{a,+} = 0$  (a.e.), the closed face  $\Omega_m^\sharp$  is the weak\*-closed convex hull of*

$$\bigcup_{d_{a,-} \in \mathcal{C}_m^\sharp} M_{\Phi(d_{a,-})}.$$

## 10. The max–min and min–max Variational Problems

The thermodynamics of any model  $m \in \mathcal{M}_1$  is given on the level of the pressure by Theorem 3.12. This result is not satisfactory enough because we also would like to have access to generalized t.i. equilibrium states from local theories (cf. Definition 3.52). The additional information we need for this purpose is Theorem 3.36. In particular, it is necessary to relate the thermodynamics of models  $m \in \mathcal{M}_1$  with their approximating interactions through the thermodynamic games defined in Definition 3.35.

As a preliminary step of the proof of Theorem 3.36, we need to analyze more precisely the max–min and min–max variational problems  $F_m^b$  and  $F_m^\sharp$ . This is performed in Section 10.1 and the proof of Theorem 3.36 is postponed until Section 10.2, see Lemmata 10.5 and 10.7.

### 10.1. Analysis of the conservative values $F_m^b$ and $F_m^\sharp$

We start by giving important properties of the map

$$(c_{a,-}, c_{a,+}) \mapsto \mathfrak{f}_m(c_{a,-}, c_{a,+}) := -\|c_{a,+}\|_2^2 + \|c_{a,-}\|_2^2 - P_m(c_{a,-} + c_{a,+})$$

from  $L_-^2(\mathcal{A}, \mathbb{C}) \times L_+^2(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$ , see Definition 3.34.

LEMMA 10.1 (Approximating free–energy density  $f_m$  for  $m \in \mathcal{M}_1$ ).

(+) At any fixed  $c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})$ , the map  $c_{a,+} \mapsto f_m(c_{a,-}, c_{a,+})$  from  $L^2_+(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  is upper semi–continuous in the weak topology and strictly concave ( $\gamma_{a,+} \neq 0$  (a.e.)).

(–) At any fixed  $c_{a,+} \in L^2_+(\mathcal{A}, \mathbb{C})$ , the map  $c_{a,-} \mapsto f_m(c_{a,-}, c_{a,+})$  from  $L^2_-(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  is lower semi–continuous in the weak topology.

*Proof.* The maps  $c_{a,\pm} \mapsto \|c_{a,\pm}\|_2^2$  from  $L^2_\pm(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  are lower semi–continuous in the weak topology and, as soon as  $\gamma_{a,\pm} \neq 0$  (a.e.), strictly convex. By Proposition 9.1 (ii), the map  $c_a \mapsto P_m(c_a)$  is weakly continuous on any ball  $\mathcal{B}_R(0) \subset L^2(\mathcal{A}, \mathbb{C})$  of radius  $R < \infty$  and convex. Therefore, the map  $c_{a,+} \mapsto f_m(c_{a,-}, c_{a,+})$  is upper semi–continuous and strictly concave if  $\gamma_{a,\pm} \neq 0$  (a.e.), whereas  $c_{a,-} \mapsto f_m(c_{a,-}, c_{a,+})$  is lower semi–continuous. ■

We continue our analysis of the conservative values  $F_m^\flat$  and  $F_m^\sharp$  by studying the functionals  $f_m^\flat$  and  $f_m^\sharp$  of the thermodynamic game defined in Definition 3.35.

LEMMA 10.2 (Properties of functionals  $f_m^\flat$  and  $f_m^\sharp$  for  $m \in \mathcal{M}_1$ ).

(b) The map  $c_{a,+} \mapsto f_m^\flat(c_{a,+})$  from  $L^2_+(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  is upper semi–continuous in the weak topology and strictly concave ( $\gamma_{a,+} \neq 0$  (a.e.)).

(♯) The map  $c_{a,-} \mapsto f_m^\sharp(c_{a,-})$  from  $L^2_-(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  is lower semi–continuous in the weak topology.

*Proof.* By Proposition 9.1 (ii), we first observe that there is  $R > 0$  such that

$$f_m^\flat(c_{a,+}) = \inf_{c_{a,-} \in \mathcal{B}_{R,-}} f_m(c_{a,-}, c_{a,+}) \quad \text{and} \quad f_m^\sharp(c_{a,-}) = \sup_{c_{a,+} \in \mathcal{B}_{R,+}} f_m(c_{a,-}, c_{a,+}),$$

where  $\mathcal{B}_{R,\pm} \subset L^2_\pm(\mathcal{A}, \mathbb{C})$  are the closed balls of radius  $R$  centered at 0. In other words,  $f_m^\flat(c_{a,+})$  and  $f_m^\sharp(c_{a,-})$  are well–defined for any  $c_{a,\pm} \in L^2_\pm(\mathcal{A}, \mathbb{C})$ .

(b) From Proposition 9.3, there exists also  $R < \infty$  such that

$$(10.1) \quad P_{m(c_{a,+})} = \sup_{c_{a,-} \in \mathcal{B}_{R,-}} \left\{ -\|c_{a,-}\|_2^2 + P_m(c_{a,-} + c_{a,+}) \right\}$$

is the pressure of the Fermi system

$$(10.2) \quad m(c_{a,+}) := (\Phi(c_{a,+}), \{\Phi_{a,-}\}_{a \in \mathcal{A}}, \{\Phi'_{a,-}\}_{a \in \mathcal{A}}) \in \mathcal{M}_1.$$

Here,  $\Phi_{a,-} := \gamma_{a,-} \Phi_a$  and  $\Phi'_{a,-} := \gamma_{a,-} \Phi'_a$ , whereas  $\Phi(c_{a,+}) = \Phi_m(c_{a,+})$  is defined in Definition 3.31.

By using similar arguments as in the proof of Proposition 9.1 (ii), one obtains that the family

$$(10.3) \quad \{c_{a,+} \mapsto f_m(\rho, c_{a,+} + c_{a,-})\}_{\rho \in E_1, c_{a,-} \in \mathcal{B}_{R,-}}$$

of real linear functionals is weakly equicontinuous on the ball  $\mathcal{B}_{R,+}$ . It follows from Proposition 9.1 (i) and (10.1) that the map  $c_{a,+} \mapsto \mathbb{P}_{\mathfrak{m}(c_{a,+})}$  is weakly continuous on the ball  $\mathcal{B}_{R,+}$ . Additionally,  $c_{a,+} \mapsto \|c_{a,+}\|_2^2$  is lower semi-continuous in the weak topology. Therefore, the map

$$(10.4) \quad c_{a,+} \mapsto \mathfrak{f}_{\mathfrak{m}}^{\flat}(c_{a,+}) = -\|c_{a,+}\|_2^2 - \mathbb{P}_{\mathfrak{m}(c_{a,+})}$$

is upper semi-continuous in the weak topology. As soon as  $\gamma_{a,+} \neq 0$  (a.e.), the functional  $\mathfrak{f}_{\mathfrak{m}}^{\flat}$  is also strictly concave: For all  $\lambda \in (0, 1)$  and  $c_{a,+}^{(1)}, c_{a,+}^{(2)} \in L_+^2(\mathcal{A}, \mathbb{C})$  such that  $c_{a,+}^{(1)} \neq c_{a,+}^{(2)}$  (a.e.),

$$\lambda \mathfrak{f}_{\mathfrak{m}}^{\flat}(c_{a,+}^{(1)}) + (1 - \lambda) \mathfrak{f}_{\mathfrak{m}}^{\flat}(c_{a,+}^{(2)}) < \mathfrak{f}_{\mathfrak{m}}^{\flat}(\lambda c_{a,+}^{(1)} + (1 - \lambda) c_{a,+}^{(2)}).$$

( $\sharp$ ) The functional  $\mathfrak{f}_{\mathfrak{m}}^{\sharp}$  is lower semi-continuous w.r.t. the weak topology because it is the supremum of a family

$$\{c_{a,-} \mapsto \mathfrak{f}_{\mathfrak{m}}(c_{a,-}, c_{a,+})\}_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})}$$

of lower semi-continuous functionals, see Lemma 10.1 (–). ■

For all  $c_{a,\pm} \in L_{\pm}^2(\mathcal{A}, \mathbb{C})$ , we study now the sets  $\mathcal{C}_{\mathfrak{m}}^{\flat}(c_{a,+})$  and  $\mathcal{C}_{\mathfrak{m}}^{\sharp}(c_{a,-})$  related to the solutions of the variational problems  $\mathfrak{f}_{\mathfrak{m}}^{\flat}$  and  $\mathfrak{f}_{\mathfrak{m}}^{\sharp}$  and defined by (3.33).

LEMMA 10.3 (Solutions of variational problems  $\mathfrak{f}_{\mathfrak{m}}^{\flat}$  and  $\mathfrak{f}_{\mathfrak{m}}^{\sharp}$ ).

(b) For all  $c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})$ , the set  $\mathcal{C}_{\mathfrak{m}}^{\flat}(c_{a,+})$  is non-empty, norm bounded and weakly compact.

( $\sharp$ ) If  $\gamma_{a,+} \neq 0$  (a.e.) then, for all  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ , the set  $\mathcal{C}_{\mathfrak{m}}^{\sharp}(c_{a,-})$  has exactly one element  $r_+(c_{a,-})$ .

*Proof.* Fix  $c_{a,\pm} \in L_{\pm}^2(\mathcal{A}, \mathbb{C})$ . From Proposition 9.1 (ii), there is  $R < \infty$  such that  $\mathcal{C}_{\mathfrak{m}}^{\flat}(c_{a,+}) \subset \mathcal{B}_{R,-}$  and  $\mathcal{C}_{\mathfrak{m}}^{\sharp}(c_{a,-}) \subset \mathcal{B}_{R,+}$  with  $\mathcal{B}_{R,\pm} \subset L_{\pm}^2(\mathcal{A}, \mathbb{C})$  being the closed balls of radius  $R$  centered at 0.

(b) We first observe that, by the separability assumption on the measure space  $(\mathcal{A}, \mathfrak{a})$ , the weak topology of any weakly compact set is metrizable, by Theorem 12.10. Therefore, since, by Banach–Alaoglu theorem, balls  $\mathcal{B}_{R,-}$  are weakly compact, they are metrizable and we can restrict ourself on sequences instead of more general nets. Take now any sequence  $\{c_{a,-}^{(n)}\}_{n=1}^{\infty}$  of approximating minimizers in  $\mathcal{B}_{R,-}$  such that

$$\mathfrak{f}_{\mathfrak{m}}^{\flat}(c_{a,+}) = \lim_{n \rightarrow \infty} \mathfrak{f}_{\mathfrak{m}}(c_{a,-}^{(n)}, c_{a,+}).$$

By compactness and metrizability of balls  $\mathcal{B}_{R,-}$  in the weak topology, we can assume without loss of generality that  $\{c_{a,-}^{(n)}\}_{n=1}^{\infty}$  converges weakly towards  $d_{a,-} \in \mathcal{B}_{R,-}$ .

The map  $c_{a,-} \mapsto \mathfrak{f}_m(c_{a,-}, c_{a,+})$  is lower semi-continuous in the weak topology, see Lemma 10.1 (-). It follows that

$$\mathfrak{f}_m^b(c_{a,+}) = \mathfrak{f}_m(d_{a,-}, c_{a,+}).$$

In other words, for all  $c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})$ , the set  $\mathcal{C}_m^b(c_{a,+}) \subset \mathcal{B}_{R,-}$  is non-empty and norm bounded. Again by weakly lower semi-continuity of the map  $c_{a,-} \mapsto \mathfrak{f}_m(c_{a,-}, c_{a,+})$ , for any sequence  $\{d_{a,-}^{(n)}\}_{n=1}^\infty$  in  $\mathcal{C}_m^b(c_{a,+})$  converging weakly towards  $d_a^{(\infty)} \in L_-^2(\mathcal{A}, \mathbb{C})$  as  $n \rightarrow \infty$ , it is also clear that  $d_a^{(\infty)} \in \mathcal{C}_m^b(c_{a,+})$  is weakly compact. Thus  $\mathcal{C}_m^b(c_{a,+})$  is weakly compact because it is a weakly closed subset of a weakly compact set.

(#) Similarly as in (b), the set  $\mathcal{C}_m^\sharp(c_{a,-}) \subset \mathcal{B}_{R,+}$  is non-empty because the map  $c_{a,+} \mapsto \mathfrak{f}_m(c_{a,-}, c_{a,+})$  is upper semi-continuous in the weak topology, by Lemma 10.1 (+). The uniqueness of  $r_+(c_{a,-})$  in the  $L_+^2(\mathcal{A}, \mathbb{C})$ -sense for any fixed  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$  follows from the strict concavity of the functional  $c_{a,+} \mapsto \mathfrak{f}_m(c_{a,-}, c_{a,+})$ , see again Lemma 10.1 (+). ■

Then we conclude the analysis of the two optimization problems  $F_m^b$  and  $F_m^\sharp$  of the thermodynamic game defined in Definition 3.35 with a study of their sets  $\mathcal{C}_m^b$  and  $\mathcal{C}_m^\sharp$  of conservative strategies, see (3.32).

LEMMA 10.4 (The set of optimizers for  $m \in \mathcal{M}_1$ ).

- (b) If  $\gamma_{a,+} \neq 0$  (a.e.), the set  $\mathcal{C}_m^b \subset L_+^2(\mathcal{A}, \mathbb{C})$  has exactly one element  $d_{a,+}$ .  
 (#) The set  $\mathcal{C}_m^\sharp \subset L_-^2(\mathcal{A}, \mathbb{C})$  is non-empty, norm bounded, and weakly compact.

*Proof.* From Proposition 9.1 (ii), there is  $R < \infty$  such that  $\mathcal{C}_m^b \subset \mathcal{B}_{R,+}$  and  $\mathcal{C}_m^\sharp \subset \mathcal{B}_{R,-}$  with  $\mathcal{B}_{R,\pm} \subset L_\pm^2(\mathcal{A}, \mathbb{C})$  being the closed balls of radius  $R$  centered at 0. In particular,  $-\infty < F_m^b \leq F_m^\sharp < \infty$ .

(b) From Lemma 10.2 (b),  $F_m^b$  is a supremum of a weakly upper semi-continuous functional  $\mathfrak{f}_m^b$  and  $\mathcal{C}_m^b$  is the set of its maximizers. Therefore, in the same way we prove (b) in Lemma 10.3,  $\mathcal{C}_m^b \subset \mathcal{B}_{R,+}$  is non-empty and weakly compact. Moreover, Lemma 10.2 (b) also tells us that  $\mathfrak{f}_m^b$  is strictly concave as soon as  $\gamma_{a,+} \neq 0$  (a.e.). Therefore, there is actually a unique solution  $d_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})$  of the variational problem

$$F_m^b := \sup_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})} \mathfrak{f}_m^b(c_{a,+}).$$

(#) To prove the second statement, we use similar arguments as in (b). Indeed, one uses Lemma 10.2 (#). Observe, however, that  $\mathfrak{f}_m^\sharp$  is not strictly convex and so, the solution of the variational problem

$$F_m^\sharp := \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \mathfrak{f}_m^\sharp(c_{a,-})$$

may not be unique. ■

## 10.2. $F_m^b$ and $F_m^\sharp$ as variational problems over states: Proof of Theorem 3.36

Theorem 3.36 (b), i.e.,

$$P_m^b := - \inf_{\rho \in E_1} f_m^b(\rho) = -F_m^b,$$

follows from Lemma 9.2 together with von Neumann min–max theorem (Theorem 12.50) which also give us additional information about the non–empty set

$$(10.5) \quad M_m^b := \left\{ \varrho \in E_1 : f_m^b(\varrho) = \inf_{\rho \in E_1} f_m^b(\rho) \right\}$$

of t.i. minimizers of the weak\*–lower semi–continuous convex functional  $f_m^b$  (3.12). This is proven in the next lemma.

LEMMA 10.5 ( $F_m^b$  and gap equations).

For any  $\mathbf{m} \in \mathcal{M}_1$ ,  $P_m^b = -F_m^b$  and there is  $\omega \in \Omega_{\mathbf{m}(d_{a,+})}^\sharp \cap M_m^b$  satisfying

$$(10.6) \quad d_{a,+} = \gamma_{a,+}(e_{\Phi_a}(\omega) + ie_{\Phi'_a}(\omega)) \text{ (a.e.)}$$

with  $d_{a,+} \in \mathcal{C}_m^b$  and  $\Omega_{\mathbf{m}(d_{a,+})}^\sharp = M_{\mathbf{m}(d_{a,+})}^\sharp$  being the set of generalized t.i. equilibrium states of the model  $\mathbf{m}(d_{a,+}) \in \mathcal{M}_1$  with purely attractive long range interactions defined by (10.2). Compare with Proposition 9.4 and Corollary 9.5.

*Proof.* On the one hand, by using Lemma 9.2, observe that

$$(10.7) \quad \inf_{\rho \in E_1} f_m^b(\rho) = \inf_{\rho \in E_1} \left\{ \sup_{c_{a,+} \in \mathcal{B}_{R,+}} \left\{ -\|c_{a,+}\|_2^2 + 2 \operatorname{Re} \left\{ \langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), c_{a,+} \rangle \right\} \right. \right. \\ \left. \left. - \|\Delta_{a,-}(\rho)\|_1 + e_\Phi(\rho) - \beta^{-1}s(\rho) \right\} \right\}$$

with  $\mathcal{B}_{R,+} \subset L_+^2(\mathcal{A}, \mathbb{C})$  being a closed ball of sufficiently large radius  $R > 0$  centered at 0.

On the other hand, the set  $\mathcal{C}_m^b \subset \mathcal{B}_{R,+}$  of conservative strategies of  $F_m^b$  defined by (3.32) has a unique element (Lemma 10.4 (b)) and, by using Proposition 9.3 and (10.4),

$$(10.8) \quad F_m^b = \sup_{c_{a,+} \in \mathcal{B}_{R,+}} \left\{ \inf_{\rho \in E_1} \left\{ -\|c_{a,+}\|_2^2 + 2 \operatorname{Re} \left\{ \langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), c_{a,+} \rangle \right\} \right. \right. \\ \left. \left. - \|\Delta_{a,-}(\rho)\|_1 + e_\Phi(\rho) - \beta^{-1}s(\rho) \right\} \right\}$$

provided the radius  $R > 0$  is taken sufficiently large.

Now, the real functional

$$(\rho, c_{a,+}) \mapsto -\|c_{a,+}\|_2^2 + 2 \operatorname{Re}\{e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), c_{a,+}\} \\ - \|\Delta_{a,-}(\rho)\|_1 + e_{\Phi}(\rho) - \beta^{-1}s(\rho)$$

is convex and weak\*-lower semi-continuous w.r.t.  $\rho \in E_1$ , but concave and weakly upper semi-continuous w.r.t.  $c_{a,+} \in L^2_+(\mathcal{A}, \mathbb{C})$ . Additionally, the sets  $E_1$  and  $\mathcal{B}_{R,+}$  are clearly convex and compact, in the weak\* and weak topologies respectively. Therefore, from von Neumann min-max theorem (Theorem 12.50), there is a saddle point  $(\omega, d_{a,+}) \in E_1 \times L^2_+(\mathcal{A}, \mathbb{C})$  which yields  $P_m^b = -F_m^b$ , see Definition 12.49. In particular, by Lemma 9.2, there are  $\omega \in \Omega_m^\sharp(d_{a,+}) \cap M_m^b$  and  $d_{a,+} \in C_m^b$  satisfying the Euler-Lagrange equations (10.6), which are also called gap equations in Physics (Remark 3.43). ■

Note that (10.8) can be interpreted as a two-person zero-sum game with a non-cooperative equilibrium defined by the saddle point  $(\omega, d_{a,+})$ . Observe also that Lemma 10.5 combined with Theorem 3.25 (+) directly yields Theorem 3.36 (‡) for purely repulsive long range interactions:

COROLLARY 10.6 (Thermodynamics game and pressure – I).

For any  $m \in \mathcal{M}_1$  and under the condition that  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.),

$$P_m := P_m^\sharp = P_m^b = -F_m$$

with  $F_m := F_m^\sharp = F_m^b$ , see Definition 3.35.

We are now in position to prove Theorem 3.36 (‡) in the general case.

LEMMA 10.7 (Thermodynamics game and pressure – II).

For any  $m \in \mathcal{M}_1$ ,  $P_m^\sharp = -F_m^\sharp$  with the pressure  $P_m^\sharp$  given for  $m \in \mathcal{M}_1$  by the minimization of the free-energy density functional  $f_m^\sharp$  over  $E_1$ , see Definition 3.11 and Theorem 3.12 (i).

*Proof.* From Theorem 3.12 (i) combined with Lemmata 3.9 and 9.2,

$$(10.9) \quad -P_m^\sharp = \inf_{\hat{\rho} \in \mathcal{E}_1} \left\{ \inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,-}\|_2^2 + f_{m(c_{a,-})}^\sharp(\hat{\rho}) \right\} \right\} \\ = \inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} \left\{ \inf_{\hat{\rho} \in \mathcal{E}_1} \left\{ \|c_{a,-}\|_2^2 + f_{m(c_{a,-})}^\sharp(\hat{\rho}) \right\} \right\}$$

with

$$f_{m(c_{a,-})}^\sharp(\rho) := -2 \operatorname{Re}\{e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), c_{a,-}\} + \|\Delta_{a,+}(\rho)\|_1 + e_{\Phi}(\rho) - \beta^{-1}s(\rho)$$

for all  $\rho \in E_1$ . By using again Lemma 3.9 and Theorem 3.12 (i), for all  $c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})$ ,

$$P_{m(c_{a,-})}^\sharp = - \inf_{\hat{\rho} \in \mathcal{E}_1} f_{m(c_{a,-})}^\sharp(\hat{\rho}) = - \inf_{\rho \in E_1} f_{m(c_{a,-})}^\sharp(\rho)$$



is the pressure associated with the purely repulsive long range model

$$(10.10) \quad \mathbf{m}(c_{a,-}) := (\Phi(c_{a,-}), \{\Phi_{a,+}\}_{a \in \mathcal{A}}, \{\Phi'_{a,+}\}_{a \in \mathcal{A}}) \in \mathcal{M}_1,$$

where  $\Phi_{a,+} := \gamma_{a,+} \Phi_a$  and  $\Phi'_{a,+} := \gamma_{a,+} \Phi'_a$ , see (3.43). In particular,

$$(10.11) \quad -P_{\mathbf{m}}^{\sharp} = \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,-}\|_2^2 - P_{\mathbf{m}(c_{a,-})}^{\sharp} \right\}.$$

Therefore, applying Corollary 10.6 on the model  $\mathbf{m}(c_{a,-})$  with purely repulsive long range interactions, one gets from (10.11) that

$$-P_{\mathbf{m}}^{\sharp} = \inf_{c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})} \left\{ \sup_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})} f_{\mathbf{m}}(c_{a,-}, c_{a,+}) \right\} = F_{\mathbf{m}}^{\sharp}$$

for any  $\mathbf{m} \in \mathcal{M}_1$ . ■

Observe that treating first the positive part of the model  $\mathbf{m} \in \mathcal{M}_1$  in  $P_{\mathbf{m}}^{\sharp}$  by using Lemma 9.2 is not necessarily useful in the general case unless  $F_{\mathbf{m}}^{\sharp} = F_{\mathbf{m}}^{\flat}$ . Indeed, we approximate first the long range attractions  $\Phi_{a,-}$  and  $\Phi'_{a,-}$  because we can then commute in (10.9) two infima. If we would have first approximated the long range repulsions  $\Phi_{a,+}$  and  $\Phi'_{a,+}$ , by using Lemma 9.2, we would have to commute a sup and a inf, which is generally not possible because we would have obtained  $P_{\mathbf{m}}^{\flat}$  and not  $P_{\mathbf{m}}^{\sharp} \geq P_{\mathbf{m}}^{\flat}$ , see Lemma 10.5.

Finally, we conclude by giving an interesting lemma about the continuity of the thermodynamic decision rule

$$r_+ : c_{a,-} \mapsto r_+(c_{a,-}) \in \mathcal{C}_{\mathbf{m}}^{\sharp}(c_{a,-})$$

(cf. (3.34)) with  $r_+(c_{a,-})$  being the unique element of the set  $\mathcal{C}_{\mathbf{m}}^{\sharp}(c_{a,-})$  defined by (3.33) for all  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ , cf. Lemma 10.3 (#). This lemma follows from Lemma 10.7.

LEMMA 10.8 (Weak–norm continuity of the map  $r_+$ ).

If  $\gamma_{a,+} \neq 0$  (a.e.) then the map

$$r_+ : c_{a,-} \mapsto r_+(c_{a,-}) \in \mathcal{C}_{\mathbf{m}}^{\sharp}(c_{a,-})$$

from  $L_-^2(\mathcal{A}, \mathbb{C})$  to  $L_+^2(\mathcal{A}, \mathbb{C})$  is continuous w.r.t. the weak topology in  $L_-^2(\mathcal{A}, \mathbb{C})$  and the norm topology in  $L_+^2(\mathcal{A}, \mathbb{C})$ .

*Proof.* First, recall that  $\mathbf{m}(c_{a,-}) \in \mathcal{M}_1$  is the model with purely repulsive long range interactions defined by (10.10) for any  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$ . From Lemma 10.7, its pressure equals

$$(10.12) \quad P_{\mathbf{m}(c_{a,-})} = \inf_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,+}\|_2^2 + P_{\mathbf{m}}(c_{a,-} + c_{a,+}) \right\} = \|c_{a,-}\|_2^2 - f_{\mathbf{m}}^{\sharp}(c_{a,-}).$$

Take any sequence  $\{c_{a,-}^{(n)}\}_{n=1}^{\infty}$  converging to  $c_{a,-} \in L_-^2(\mathcal{A}, \mathbb{C})$  in the weak topology. From the uniform boundedness principle (Banach–Steinhaus theorem), it follows that any weakly convergent sequence in  $L_-^2(\mathcal{A}, \mathbb{C})$  is norm bounded. In particular, the sequence  $\{c_{a,-}^{(n)}\}_{n=1}^{\infty}$  belongs to a ball  $\mathcal{B}_{R,-} \subset L_-^2(\mathcal{A}, \mathbb{C})$  of sufficiently large radius  $R$  centered at 0. By Proposition 9.1 (ii), the family

$$\{c_{a,-} \mapsto P_{\mathfrak{m}}(c_{a,-} + c_{a,+})\}_{c_{a,+} \in L_+^2(\mathcal{A}, \mathbb{C})}$$

of functionals is weakly equicontinuous on the ball  $\mathcal{B}_{R,-} \subset L_-^2(\mathcal{A}, \mathbb{C})$ . It follows that

$$(10.13) \quad \lim_{n \rightarrow \infty} P_{\mathfrak{m}(c_{a,-}^{(n)})} = P_{\mathfrak{m}(c_{a,-})}.$$

For all  $n \in \mathbb{N}$ , the unique  $r_+(c_{a,-}^{(n)}) \in \mathcal{C}_{\mathfrak{m}}^{\sharp}(c_{a,-}^{(n)})$  satisfies

$$(10.14) \quad P_{\mathfrak{m}(c_{a,-}^{(n)})} = \|r_+(c_{a,-}^{(n)})\|_2^2 + P_{\mathfrak{m}}(c_{a,-}^{(n)} + r_+(c_{a,-}^{(n)})).$$

By (10.12), we obtain that, for all  $n \in \mathbb{N}$ ,

$$(10.15) \quad \|r_+(c_{a,-}^{(n)})\|_2^2 \leq P_{\mathfrak{m}}(c_{a,-}^{(n)}) - P_{\mathfrak{m}}(c_{a,-}^{(n)} + r_+(c_{a,-}^{(n)})).$$

Using Proposition 9.1 (ii), one also gets that, for all  $n \in \mathbb{N}$ ,

$$P_{\mathfrak{m}}(c_{a,-}^{(n)}) - P_{\mathfrak{m}}(c_{a,-}^{(n)} + r_+(c_{a,-}^{(n)})) \leq 2(\|\Phi_a\|_2 + \|\Phi'_a\|_2) \|r_+(c_{a,-}^{(n)})\|_2.$$

Combined with (10.15), the previous inequality yields the existence of a closed ball  $\mathcal{B}_{R,+} \subset L_+^2(\mathcal{A}, \mathbb{C})$  of radius  $R$  centered at 0 such that

$$\{r_+(c_{a,-}^{(n)})\}_{n=1}^{\infty} \in \mathcal{B}_{R,+}.$$

By compactness and metrizable of  $\mathcal{B}_{R,+}$  in the weak topology (cf. Banach–Alaoglu theorem and Theorem 12.10), we can then assume that  $r_+(c_{a,-}^{(n)})$  weakly converges to  $d_{a,+}^{\infty} \in L_+^2(\mathcal{A}, \mathbb{C})$  as  $n \rightarrow \infty$ .

The map  $c_{a,+} \mapsto \|c_{a,+}\|_2^2$  from  $L_+^2(\mathcal{A}, \mathbb{C})$  to  $\mathbb{R}$  is weakly lower semi-continuity and, by Proposition 9.1 (ii),

$$c_{a,+} \mapsto P_{\mathfrak{m}}(c_{a,-} + c_{a,+})$$

is weakly continuous on  $\mathcal{B}_{R,+}$ . It follows that

$$\lim_{n \rightarrow \infty} \left\{ \|r_+(c_{a,-}^{(n)})\|_2^2 + P_{\mathfrak{m}}(c_{a,-}^{(n)} + r_+(c_{a,-}^{(n)})) \right\} \geq \|d_{a,+}^{\infty}\|_2^2 + P_{\mathfrak{m}}(c_{a,-} + d_{a,+}^{\infty}).$$

Combined with (10.12), (10.13), and (10.14), the previous inequality implies that  $d_{a,+}^{\infty} \in \mathcal{C}_{\mathfrak{m}}^{\sharp}(c_{a,-})$  and

$$(10.16) \quad \lim_{n \rightarrow \infty} \|r_+(c_{a,-}^{(n)})\|_2^2 = \|d_{a,+}^{\infty}\|_2^2$$

because of Proposition 9.1 (ii). As a consequence,

$$d_{a,+}^{\infty} = r_+(c_{a,-}) \in \mathcal{C}_{\mathfrak{m}}^{\sharp}(c_{a,-}),$$

cf. Lemma 10.3 (#). Moreover, since

$$\|r_+(c_{a,-}^{(n)}) - d_{a,+}^\infty\|_2^2 = \|r_+(c_{a,-}^{(n)})\|_2^2 + \|d_{a,+}^\infty\|_2^2 - 2\operatorname{Re}\{\langle r_+(c_{a,-}^{(n)}), d_{a,+}^\infty \rangle\},$$

the limit (10.16) and the weak convergence of the sequence  $\{r_+(c_{a,-}^{(n)})\}_{n=1}^\infty$  to  $d_{a,+}^\infty$  imply that  $r_+(c_{a,-}^{(n)})$  converges in the norm topology to  $d_{a,+}^\infty \in L_+^2(\mathcal{A}, \mathbb{C})$  as  $n \rightarrow \infty$ . ■

## 11. Bogoliubov Approximation and Effective Theories

The precise characterization of the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states defined in Definition 3.15 is performed in Theorem 3.21. It is the weak\*-closed convex hull of the set

$$\hat{M}_m := \left\{ \omega \in E_1 : g_m(\omega) = \inf_{\rho \in E_1} g_m(\rho) \right\}$$

of t.i. minimizers of the reduced free-energy density functional defined by (11.1)

$$g_m(\rho) := \|\gamma_{a,+,\rho}(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 - \|\gamma_{a,-,\rho}(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 + e_\Phi(\rho) - \beta^{-1}s(\rho)$$

for all  $\rho \in E_1$ , see Definition 3.6 and (3.10). Thus the first aim of the present section is to characterize the weak\*-compact set  $\hat{M}_m$  (see Lemma 3.19 (i)).

A key information to analyze the set  $\hat{M}_m$  is given by Theorem 3.36. It establishes a relation between the thermodynamics of models  $m \in \mathcal{M}_1$  and the thermodynamics of their approximating interactions through thermodynamic games. Combining this with some additional arguments we prove that  $\hat{M}_m$  is a subset of the set  $\overline{\operatorname{co}(M(\mathfrak{T}_m^\sharp))}$  (3.44) of convex combinations of t.i. equilibrium states coming from the min-max local theory  $\mathfrak{T}_m^\sharp$  (Definition 3.53). This last result is proven in Section 11.1 and gives a first answer to an old open problem in mathematical physics – first addressed by Ginibre [11, p. 28] in 1968 within a different context – about the validity of the so-called Bogoliubov approximation (see Section 4.1) on the level of states. Then in Section 11.2 we show that the set  $\Omega_m^\sharp$  of generalized t.i. equilibrium states is not a face for an uncountable set of models of  $\mathcal{M}_1$ . This last fact implies that  $\Omega_m^\sharp$  is strictly smaller than  $\overline{\operatorname{co}(M(\mathfrak{T}_m^\sharp))}$ , i.e.,  $\Omega_m^\sharp \subsetneq \overline{\operatorname{co}(M(\mathfrak{T}_m^\sharp))}$ , preventing such models to have effective local theories, see Definitions 3.49 and 3.52.

### 11.1. Gap equations

From Lemma 9.2, we have that

$$(11.2) \quad \inf_{\rho \in E_1} g_{\mathbf{m}}(\rho) = \inf_{\rho \in E_1} \left\{ \inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} \left\{ \|c_{a,-}\|_2^2 + f_{\mathbf{m}(c_{a,-})}^b(\rho) \right\} \right\}$$

$$(11.3) \quad = \inf_{c_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})} \left\{ \inf_{\rho \in E_1} \left\{ \|c_{a,-}\|_2^2 + f_{\mathbf{m}(c_{a,-})}^b(\rho) \right\} \right\}$$

for any  $\mathbf{m} \in \mathcal{M}_1$ , where the model  $\mathbf{m}(c_{a,-})$  with *purely repulsive* long range interactions  $\Phi_{a,+} := \gamma_{a,+} \Phi_a$  and  $\Phi'_{a,+} := \gamma_{a,+} \Phi'_a$  is defined by (10.10) in Section 10.2 or by (3.43) in Section 3.7.

It is thus natural to relate the set  $\hat{M}_{\mathbf{m}}$  of t.i. minimizers of the functional  $g_{\mathbf{m}}$  with the sets  $\Omega_{\mathbf{m}(d_{a,-})}^{\sharp}$  of generalized t.i. equilibrium states of models  $\mathbf{m}(d_{a,-})$  for all  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  (3.32). In fact, we verify below that the set  $\hat{M}_{\mathbf{m}}$  is the union of the sets  $\Omega_{\mathbf{m}(d_{a,-})}^{\sharp}$  for all  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$ :

LEMMA 11.1 ( $\hat{M}_{\mathbf{m}}$  and generalized t.i. equilibrium states of  $\mathbf{m}(d_{a,-})$ ).

(i) For any  $\mathbf{m} \in \mathcal{M}_1$ ,

$$\hat{M}_{\mathbf{m}} = \bigcup_{d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}} \Omega_{\mathbf{m}(d_{a,-})}^{\sharp}.$$

(ii) For any state  $\omega \in \hat{M}_{\mathbf{m}}$ , there is  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  such that  $\omega \in \Omega_{\mathbf{m}(d_{a,-})}^{\sharp}$  and

$$(11.4) \quad d_{a,-} = \gamma_{a,-}(e_{\Phi_a}(\omega) + ie_{\Phi'_a}(\omega)) \quad (\text{a.e.}).$$

(iii) Conversely, for any  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$ , all states  $\omega \in \Omega_{\mathbf{m}(d_{a,-})}^{\sharp} \subset \hat{M}_{\mathbf{m}}$  satisfy (11.4).

*Proof.* By using Lemma 9.2, any minimizer  $\omega \in \hat{M}_{\mathbf{m}}$  is solution of the variational problem (11.2) with  $d_{a,-} \in L^2_-(\mathcal{A}, \mathbb{C})$  satisfying the Euler–Lagrange equations (11.4). Since the two infima commute in (11.2),  $(\omega, d_{a,-})$  is also solution of the variational problem (11.3), i.e.,  $\omega \in \Omega_{\mathbf{m}(d_{a,-})}^{\sharp}$  and  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$ .

Conversely, for any  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  and all  $\omega \in \Omega_{\mathbf{m}(d_{a,-})}^{\sharp}$ ,  $(\omega, d_{a,-})$  is solution of the variational problem (11.3). The latter implies that  $(\omega, d_{a,-})$  is a minimum of (11.2), i.e., by Lemma 9.2,  $\omega \in \hat{M}_{\mathbf{m}}$  and  $d_{a,-} \in \mathcal{C}_{\mathbf{m}}^{\sharp}$  satisfies the Euler–Lagrange equations (11.4). ■

It now remains to characterize the set  $\Omega_{\mathbf{m}(d_{a,-})}^{\sharp}$  of generalized t.i. equilibrium states for the model  $\mathbf{m}(d_{a,-})$  (10.10) with purely repulsive long range interactions. So, the next step is to analyze the set  $\Omega_{\mathbf{m}}^{\sharp}$  for any arbitrary model without long range attractions, that is,  $\mathbf{m} \in \mathcal{M}_1$  such that  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.), see Definition 3.4. In this case we can relate  $\Omega_{\mathbf{m}}^{\sharp}$  to the set

$$\Omega_{\mathbf{m}}^{\sharp}(d_{a,+}) := \left\{ \omega \in M_{\Phi(d_{a,+})} : \gamma_{a,+}(e_{\Phi_a}(\omega) + ie_{\Phi'_a}(\omega)) = d_{a,+} \quad (\text{a.e.}) \right\}$$

defined by (3.37) for the unique element  $c_a = d_{a,+} \in \mathcal{C}_m^b$  (see (3.32) and Lemma 10.4 (b)), where  $M_{\Phi(d_{a,+})}$  is the closed face described in Lemma 3.33. In fact we show below that the sets  $\Omega_m^\sharp$  and  $\Omega_m^\sharp(d_{a,+})$  coincide for any model with purely repulsive long range interactions:

LEMMA 11.2 ( $\Omega_m^\sharp$  for models without long range attractions).

For any  $\mathbf{m} \in \mathcal{M}_1$  such that  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) and  $\gamma_{a,+} \neq 0$ ,

$$\Omega_m^\sharp = \hat{M}_m = \Omega_m^\sharp(d_{a,+})$$

with  $d_{a,+} \in \mathcal{C}_m^b$  being unique.

*Proof.* If  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) then, by (3.12) and (11.1),  $f_m^b = g_m$  on  $E_1$  and, by Theorem 3.25 (+),

$$\Omega_m^\sharp = \hat{M}_m = M_m^b,$$

where  $M_m^b$  is the non-empty set of t.i. minimizers of  $f_m^b$ , see (10.5). Therefore, since  $\mathbf{m}(d_{a,+}) = \Phi(d_{a,+})$  when  $\Phi_{a,-} = \Phi'_{a,-} = 0$  (a.e.) (cf. (10.2)), applying Lemma 10.5 we have a t.i. equilibrium state  $\omega \in M_{\Phi(d_{a,+})} \cap \Omega_m^\sharp$  satisfying the Euler–Lagrange equations

$$(11.5) \quad \gamma_{a,+}(e_{\Phi_a}(\omega) + ie_{\Phi'_a}(\omega)) = d_{a,+} \text{ (a.e.)},$$

where  $d_{a,+} \in \mathcal{C}_m^b$  is the unique element of the set  $\mathcal{C}_m^b$ , see Lemma 10.4 (b).

We now observe that

$$(11.6) \quad \begin{aligned} & 2 \operatorname{Re}\{\langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), d_{a,+} \rangle\} \\ &= \|\gamma_{a,+}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 + \|d_{a,+}\|_2^2 - \|\gamma_{a,+}\rho(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}) - d_{a,+}\|_2^2 \end{aligned}$$

and since  $\omega \in M_{\Phi(d_{a,+})} \cap \Omega_m^\sharp$  satisfies (11.5), we obtain that

$$(11.7) \quad \begin{aligned} & \inf_{\rho \in E_1} \{2 \operatorname{Re}\{\langle e_{\Phi_a}(\rho) + ie_{\Phi'_a}(\rho), d_{a,+} \rangle\} + e_\Phi(\rho) - \beta^{-1}s(\rho)\} \\ &= \|\gamma_{a,+}\omega(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a})\|_2^2 + e_\Phi(\omega) - \beta^{-1}s(\omega) + \|d_{a,+}\|_2^2 \\ &= g_m(\omega) + \|d_{a,+}\|_2^2 \end{aligned}$$

$$(11.8) \quad = \inf_{\rho \in E_1} g_m(\rho) + \|d_{a,+}\|_2^2.$$

Going backwards from (11.8) to (11.7) and using then (11.6), we obtain, for any generalized t.i. equilibrium state  $\omega \in \Omega_m^\sharp = \hat{M}_m$ , the inequality

$$g_m(\omega) + \|d_{a,+}\|_2^2 \leq g_m(\omega) - \|\gamma_{a,+}\omega(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}) - d_{a,+}\|_2^2 + \|d_{a,+}\|_2^2,$$

i.e.,

$$\|\gamma_{a,+}\omega(\mathbf{e}_{\Phi_a} + i\mathbf{e}_{\Phi'_a}) - d_{a,+}\|_2^2 \leq 0.$$

As a consequence, any generalized t.i. equilibrium state  $\omega \in \Omega_m^\sharp = \hat{M}_m$  satisfies the Euler–Lagrange equations (11.5) with  $d_{a,+} \in \mathcal{C}_m^b$ . Combining this with (11.6) it follows that  $\Omega_m^\sharp \subset \Omega_m^\sharp(d_{a,+})$ .

Conversely, take any  $\omega \in M_{\Phi(d_{a,+})}$  satisfying the Euler–Lagrange equations (11.5) with  $d_{a,+} \in \mathcal{C}_m^b$ . Such a state  $\omega \in M_{\Phi(d_{a,+})}$  is a solution of the variational problem (11.7) and we easily deduce that  $\omega \in \Omega_m^\# = \hat{M}_m$ . ■

Applying Lemma 11.2 to the model  $\mathbf{m}(d_{a,-})$  (10.10) with purely repulsive long range interactions, we obtain the following corollary:

**COROLLARY 11.3** (Generalized t.i. equilibrium states of  $\mathbf{m}(d_{a,-})$ ).

For any  $\mathbf{m} \in \mathcal{M}_1$  and all  $d_{a,-} \in \mathcal{C}_m^\#$ ,

$$\Omega_{\mathbf{m}(d_{a,-})}^\# = \Omega_{\mathbf{m}(d_{a,-})}^\#(\mathbf{r}_+(d_{a,-})) = \Omega_{\mathbf{m}}^\#(d_{a,-} + \mathbf{r}_+(d_{a,-}))$$

are (non-empty) convex and weak\*-compact subsets of  $E_1$  satisfying

$$\Omega_{\mathbf{m}(d_{a,-})}^\# \cap \Omega_{\mathbf{m}(d'_{a,-})}^\# = \emptyset$$

whenever  $d_{a,-} \neq d'_{a,-}$  with  $d_{a,-}, d'_{a,-} \in \mathcal{C}_m^\#$ . Here,  $\mathbf{r}_+$  is the thermodynamic decision rule defined by (3.34) and  $\Omega_{\mathbf{m}}^\#(d_{a,-} + \mathbf{r}_+(d_{a,-}))$  is defined by (3.37).

*Proof.* First,  $\Omega_{\mathbf{m}(d_{a,-})}^\#$  is a (non-empty) convex and weak\*-compact subset of  $E_1$  for any  $d_{a,-} \in \mathcal{C}_m^\#$ , by Lemma 3.16. By Lemma 11.1 (iii), all states  $\omega \in \Omega_{\mathbf{m}(d_{a,-})}^\#$  must satisfy (11.4). On the other hand, by Corollary 11.2 applied to the model  $\mathbf{m}(d_{a,-})$  (10.10) without long range attractions, we have

$$\Omega_{\mathbf{m}(d_{a,-})}^\# = \Omega_{\mathbf{m}(d_{a,-})}^\#(\mathbf{r}_+(d_{a,-})),$$

see (3.34). Therefore, by combining (11.4) with the last equality, we deduce that

$$\Omega_{\mathbf{m}(d_{a,-})}^\# = \Omega_{\mathbf{m}}^\#(d_{a,-} + \mathbf{r}_+(d_{a,-})),$$

which in turn implies that

$$\Omega_{\mathbf{m}(d_{a,-})}^\# \cap \Omega_{\mathbf{m}(d'_{a,-})}^\# = \emptyset$$

when  $d_{a,-} \neq d'_{a,-}$  with  $d_{a,-}, d'_{a,-} \in \mathcal{C}_m^\#$ . ■

As a consequence, by combining Lemma 11.1 (i) with Corollary 11.3, we finally obtain the following theorem:

**THEOREM 11.4** (Characterization of the set  $\hat{M}_m$ ).

For any  $\mathbf{m} \in \mathcal{M}_1$ ,

$$\hat{M}_m = \bigcup_{d_{a,-} \in \mathcal{C}_m^\#} \Omega_{\mathbf{m}}^\#(d_{a,-} + \mathbf{r}_+(d_{a,-})),$$

where  $\mathbf{r}_+$  is the thermodynamic decision rule defined by (3.34).

For many relevant models coming from Physics, like, for instance, BCS type models, the set  $M_{\Phi(c_a)}$  contains exactly one state. (Actually it is enough to have  $|M_{\Phi(c_a)}| = 1$  for  $c_a = d_{a,-} + r_+(d_{a,-})$  with  $d_{a,-} \in \mathcal{C}_m^\sharp$ .) This special case has an interesting interpretation in terms of game theory as explained in Section 3.7 after Theorem 3.47. We conclude this section by proving Theorem 3.47.

First, observe that, in this case, there is an injective and continuous map  $d_{a,-} \mapsto \omega_{d_{a,-}}$  from  $\mathcal{C}_m^\sharp$  to  $\mathcal{E}_1$ :

LEMMA 11.5 (Properties of the map  $d_{a,-} \mapsto \hat{\omega}_{d_{a,-}}$  from  $\mathcal{C}_m^\sharp$  to  $\mathcal{E}_1$ ).

For any  $m \in \mathcal{M}_1$  and all  $d_{a,-} \in \mathcal{C}_m^\sharp$ , assume that  $M_{\Phi(d_{a,-} + r_+(d_{a,-}))}$  contains exactly one state denoted by  $\hat{\omega}_{d_{a,-}}$ . Then the map  $d_{a,-} \mapsto \hat{\omega}_{d_{a,-}}$  from  $\mathcal{C}_m^\sharp$  to  $\mathcal{E}_1$  is injective and continuous w.r.t. to the weak topology on  $\mathcal{C}_m^\sharp$  and the weak\*-topology on the set  $\mathcal{E}_1$  of ergodic states.

*Proof.* By the assumptions,  $\hat{\omega}_{d_{a,-}}$  is ergodic as  $M_{\Phi(d_{a,-} + r_+(d_{a,-}))}$  is always a face of  $E_1$ , see Lemma 3.33. If  $d_{a,-} \neq d'_{a,-}$  then  $\hat{\omega}_{d'_{a,-}} \neq \hat{\omega}_{d_{a,-}}$  because of Corollary 11.3. Thus the map  $d_{a,-} \mapsto \hat{\omega}_{d_{a,-}}$  is injective. The Hilbert space  $L^2(\mathcal{A}, \mathbb{C})$  is separable and  $\mathcal{C}_m^\sharp$  is weakly compact and, therefore, closed in the weak topology. By Theorem 12.10, the weak topology in  $\mathcal{C}_m^\sharp$  is metrizable and we can restrict ourself to sequences instead of more general nets.

Take any sequence  $\{d_{a,-}^{(n)}\}_{n=0}^\infty \subset \mathcal{C}_m^\sharp$  converging in the weak topology to  $d_{a,-} \in \mathcal{C}_m^\sharp$  as  $n \rightarrow \infty$ . The thermodynamic decision rule  $r_+$  is weak-norm continuous, by Lemma 10.8, and, from the definition of  $\Phi(c_a)$ , the map  $c_a \mapsto \Phi(c_a)$  from  $L^2(\mathcal{A}, \mathbb{C})$  to  $\mathcal{W}_1$  is continuous w.r.t. to the weak topology of  $L^2(\mathcal{A}, \mathbb{C})$  and the norm topology of  $\mathcal{W}_1$ . It follows that the sequence

$$\left\{ \Phi(d_{a,-}^{(n)} + r_+(d_{a,-}^{(n)})) \right\}_{n=0}^\infty \subset \mathcal{W}_1$$

converges in norm to  $\Phi(d_{a,-} + r_+(d_{a,-})) \in \mathcal{W}_1$ . The map  $\Phi \mapsto P_{(\Phi, 0, 0)}^\sharp$  from  $\mathcal{W}_1$  to  $\mathbb{R}$  is (norm) continuous, by Theorem 3.12 (ii). Therefore,

$$(11.9) \quad P_{(\Phi(d_{a,-} + r_+(d_{a,-})), 0, 0)}^\sharp = \lim_{n \rightarrow \infty} P_{(\Phi(d_{a,-}^{(n)} + r_+(d_{a,-}^{(n)})), 0, 0)}^\sharp$$

By Theorem 3.12 (i) and Lemma 3.33,

$$P_{(\Phi(d_{a,-}^{(n)} + r_+(d_{a,-}^{(n)})), 0, 0)}^\sharp = f_m^\sharp(\hat{\omega}_{d_{a,-}^{(n)}})$$

with  $\hat{\omega}_{d_{a,-}^{(n)}} \in M_{\Phi(d_{a,-}^{(n)} + r_+(d_{a,-}^{(n)}))}$ . Combined with (11.9) and Lemma 3.33 for the t.i. interaction  $\Phi(d_{a,-} + r_+(d_{a,-}))$ , the last equality implies that any accumulation point of the sequence  $\{\hat{\omega}_{d_{a,-}^{(n)}}\}_{n=0}^\infty$  converges in the weak\*-topology to a t.i. equilibrium state  $\hat{\omega}_{d_{a,-}} \in M_{\Phi(d_{a,-} + r_+(d_{a,-}))}$  which is assumed to be unique and is thus ergodic. ■

Notice that, by Lemma 11.1 (i) and Corollary 11.3, for all  $d_{a,-} \in \mathcal{C}_m^\sharp$ , the sets  $\Omega_m^\sharp(d_{a,-} + r_+(d_{a,-}))$  are never empty. As a consequence, by Theorem 3.21 (ii), the map  $d_{a,-} \mapsto \omega_{d_{a,-}}$  of Lemma 11.5 is bijective from  $\mathcal{C}_m^\sharp$  to the set  $\mathcal{E}(\Omega_m^\sharp)$  of extreme generalized t.i. equilibrium states. Since  $\mathcal{C}_m^\sharp$  is weakly compact, it is a homeomorphism:

**COROLLARY 11.6** (The map  $d_{a,-} \mapsto \omega_{d_{a,-}}$  from  $\mathcal{C}_m^\sharp$  to  $\mathcal{E}(\Omega_m^\sharp)$ ).

*For any  $m \in \mathcal{M}_1$  and all  $d_{a,-} \in \mathcal{C}_m^\sharp$ , assume that  $M_{\Phi(d_{a,-} + r(d_{a,-}))}$  contains exactly one state denoted by  $\omega_{d_{a,-}}$ . Then the map  $d_{a,-} \mapsto \omega_{d_{a,-}}$  from  $\mathcal{C}_m^\sharp$  to  $\mathcal{E}_1$  defines a homeomorphism between  $\mathcal{C}_m^\sharp$  and  $\mathcal{E}(\Omega_m^\sharp)$  w.r.t. to the weak topology in  $\mathcal{C}_m^\sharp$  and the weak\*-topology in the set  $\mathcal{E}(\Omega_m^\sharp)$ . In particular,  $\mathcal{E}(\Omega_m^\sharp)$  is weak\*-compact.*

Consequently, any continuous function  $f \in C(\mathcal{E}(\Omega_m^\sharp))$  can be identified with a continuous function  $g \in C(\mathcal{C}_m^\sharp)$  through the prescription  $g(d_{a,-}) := f(\omega_{d_{a,-}})$ . This map  $C(\mathcal{E}(\Omega_m^\sharp)) \rightarrow C(\mathcal{C}_m^\sharp)$  clearly defines an isomorphism of  $C^*$ -algebras. Therefore, by combining this with Theorems 12.25 and 3.46, we obtain Theorem 3.47.

## 11.2. Breakdown of effective local theories: Proof of Theorem 3.54

The fact that the approximating Hamiltonian method (Section 12.2) leads to the correct pressure (cf. Theorem 3.36 (#)) does not mean that the min-max local theory  $\mathfrak{T}_m^\sharp$  (Definition 3.53) is an effective theory for  $m \in \mathcal{M}_1$ . In fact, we prove the existence of uncountably many models  $m \in \mathcal{M}_1$  having no effective local theory.

The construction of such models uses the fact, first observed by Israel [4, Theorem V.2.2.] for lattice spin systems with purely local interactions, that any finite set of extreme t.i. states can be seen as t.i. equilibrium states of some t.i. interaction  $\Phi \in \mathcal{W}_1$ :

**LEMMA 11.7** (Ergodic states as t.i. equilibrium states).

*For any finite subset  $\{\hat{\rho}_1, \dots, \hat{\rho}_n\} \subset \mathcal{E}_1$  of ergodic states, there is  $\Phi \in \mathcal{W}_1$  such that  $\{\hat{\rho}_1, \dots, \hat{\rho}_n\} \subset M_\Phi$ .*

*Proof.* For any  $\Phi \in \mathcal{W}_1$ , recall that the map

$$\rho \mapsto f_\Phi(\rho) := e_\Phi(\rho) - \beta^{-1}s(\rho)$$

is weak\*-lower semi-continuous and affine, see Lemmata 2.28 (i), 2.31 (i) and Definition 2.32. In particular,  $\Omega_\Phi = M_\Phi$  is the (non-empty) set of all t.i. minimizers which is a closed face of  $E_1$ . Therefore, the lemma follows from Bishop-Phelps' theorem [4, Theorem V.1.1.] together with Choquet theorem



(Theorem 2.9) and Theorem 3.28 for  $\mathfrak{m} = (\Phi, 0, 0)$ . The arguments are *exactly* those of Israel and we recommend [4, Theorem V.2.2. (a)] for more details. ■

Using this last lemma, we can then construct uncountably many models  $\mathfrak{m} \in \mathcal{M}_1$  such that its set  $\Omega_{\mathfrak{m}}^{\sharp}$  of generalized t.i. equilibrium states is not a face of  $E_1$ .

LEMMA 11.8 (The set  $\Omega_{\mathfrak{m}}^{\sharp}$  is generally not a face).

*There are uncountably many  $\mathfrak{m} \in \mathcal{M}_1$  for which  $\Omega_{\mathfrak{m}}^{\sharp}$  is not a face of  $E_1$ .*

*Proof.* Let

$$\mathcal{U}^- \subset \mathcal{U}_0 \setminus \{A \in \mathcal{U}_0 : A = A^*\}$$

be the (non-empty) set of non self-adjoint local elements of the  $*$ -algebra  $\mathcal{U}_0$  defined by

$$\mathcal{U}^- := \bigcup_{\theta \in \mathbb{R}/(2\pi\mathbb{Z})} \{A \in \mathcal{U}_0 : A = -\sigma_{\theta}(A), \rho(A) \neq 0 \text{ for some } \rho \in E_1\}$$

with  $\sigma_{\theta}$  being the automorphism of the algebra  $\mathcal{U}$  defined by (2.4). Since, for any  $x, y \in \mathfrak{L}$  with  $x \neq y$ , any  $s \in S$ , and any  $\lambda \in \mathbb{R} \setminus \{0\}$ , we have  $\lambda a_{x,s} a_{y,s} \in \mathcal{U}^-$ , the set  $\mathcal{U}^-$  contains uncountably many elements.

By assumption, for any  $A \in \mathcal{U}^-$ , there is  $\hat{\rho}_1 \in E_1$  such that  $\hat{\rho}_1(A) \neq 0$ . By density of the set  $\mathcal{E}_1$  of extreme points of  $E_1$  (Corollary 6.6), we can assume without loss of generality that  $\hat{\rho}_1 \in \mathcal{E}_1$ . As  $A \in \mathcal{U}^-$ , there is  $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$  such that

$$(11.10) \quad \hat{\rho}_1(A) = -\hat{\rho}_2(A) \neq 0$$

with  $\hat{\rho}_2 := \hat{\rho}_1 \circ \sigma_{\theta}$ . Since  $\sigma_{\theta}$  is an automorphism of  $\mathcal{U}$ ,  $\hat{\rho}_2 \neq \hat{\rho}_1$  is clearly a state. As  $\hat{\rho}_1 \in \mathcal{E}_1$ , by using Theorem 2.16 and  $\alpha_x \circ \sigma_{\theta} = \sigma_{\theta} \circ \alpha_x$ , we have that  $\hat{\rho}_2 \in \mathcal{E}_1$ . Now, by Lemma 11.7, there is  $\Phi \in \mathcal{W}_1$  such that  $\{\hat{\rho}_1, \hat{\rho}_2\} \subset M_{\Phi}$ .

Any non self-adjoint local element  $A \in \mathcal{U}^- \subset \mathcal{U}_0$  can be decomposed as  $A = A_R + iA_I$  with  $A_R = A_R^* \in \mathcal{U}_0$  and  $A_I = A_I^* \in \mathcal{U}_0$ . Thus, as explained in the proof of Lemma 6.18, there exists two finite range t.i. interactions  $\Phi^{A_R}, \Phi^{A_I} \in \mathcal{W}_1$  with  $\|\Phi^{A_R}\|_{\mathcal{W}_1} = \|A_R\|$  and  $\|\Phi^{A_I}\|_{\mathcal{W}_1} = \|A_I\|$  such that

$$(11.11) \quad \rho(A) = e_{\Phi^{A_R}}(\rho) + ie_{\Phi^{A_I}}(\rho)$$

for any  $\rho \in E_1$ . For any  $A \in \mathcal{U}^-$ , we define the discrete model

$$\mathfrak{m}_A := (\Phi, \Phi^{A_R}, \Phi^{A_I}) \in \mathcal{M}_1$$

without long range attractions, i.e.,  $\Phi_{a,-} = \Phi'_{a,-} = 0$ ,  $\Phi_{a,+} := \Phi^{A_R}$ , and  $\Phi'_{a,+} := \Phi^{A_I}$ , see Definition 3.4.

As  $\{\hat{\rho}_1, \hat{\rho}_2\} \subset M_{\Phi}$  and by convexity of the set  $M_{\Phi}$ ,

$$(11.12) \quad \omega := \frac{1}{2}\hat{\rho}_1 + \frac{1}{2}\hat{\rho}_2 \in M_{\Phi}.$$

It follows from Definition 3.6 that

$$(11.13) \quad g_{\mathbf{m}_A}(\omega) = f_{\Phi}(\omega) < g_{\mathbf{m}_A}(\hat{\rho}_1) = g_{\mathbf{m}_A}(\hat{\rho}_2)$$

because of (11.10) and (11.11). Therefore,  $\hat{\rho}_1, \hat{\rho}_2 \notin \hat{M}_{\mathbf{m}_A}$  do not belong to the set  $\hat{M}_{\mathbf{m}_A}$  (3.10) of minimizers of  $g_{\mathbf{m}_A}$  over  $E_1$ . However, since  $\mathbf{m}_A$  is a model with purely repulsive long range interactions,  $f_{\Phi} \leq g_{\mathbf{m}_A}$  on  $E_1$  and, by (11.12) and (11.13), we obtain that  $\omega \in \hat{M}_{\mathbf{m}_A}$ . Since  $\Omega_{\mathbf{m}_A}^{\#} = \hat{M}_{\mathbf{m}_A}$ , by Theorem 3.25 (+), we finally get that  $\omega \in \Omega_{\mathbf{m}_A}^{\#}$ , whereas  $\hat{\rho}_1, \hat{\rho}_2 \notin \Omega_{\mathbf{m}_A}^{\#}$  in spite of the decomposition (11.12). In other words, for any  $A \in \mathcal{U}^-$ ,  $\Omega_{\mathbf{m}_A}^{\#}$  is not a face of  $E_1$ . ■

As a consequence, the equality  $P_{\mathbf{m}}^{\#} = -F_{\mathbf{m}}^{\#}$  of Theorem 3.36 (#) does not necessarily imply that the min–max local theory  $\mathfrak{T}_{\mathbf{m}}^{\#}$  (Definition 3.53) is an effective theory, see Definition 3.49. In fact, if  $\Omega_{\mathbf{m}}^{\#}$  is not a face then there is no effective local theory and Lemma 11.8 implies Theorem 3.54.

## 12. Appendix

For the reader's convenience we give here a short review on the following subjects:

- Gibbs equilibrium states (Section 12.1), see, e.g., [5];
- The approximating Hamiltonian method (Section 12.2), see, e.g., [13, 14, 15, 16];
- $\mathcal{L}^p$ -spaces of maps with values in a Banach space (Section 12.3);
- Compact convex sets and Choquet simplices (Section 12.4), see, e.g., [2, 3];
- $\Gamma$ -regularization of real functionals (Section 12.5), see, e.g., [46];
- Legendre–Fenchel transform and tangent functionals (Section 12.6), see, e.g., [30, 47];
- Two–person zero–sum games (Section 12.7), see, e.g., [33, 48].

These subjects are rather standard and can be found in many textbooks. Therefore, we keep the exposition here as short as possible and only concentrate on results used in this paper. It is important to note, however, that we also give two new and useful theorems – Theorems 12.37 and 12.38 – which are general results related to the study of variational problems with non–convex functionals on compact convex sets. Observe further that Lemma 12.32 in Section 12.5 does not seem to have been observed before. In fact, Lemma 12.32 and Theorems 12.37–12.38 are given in this appendix – and not in the main part of the text – as they will be subject of a separate paper.

### 12.1. Gibbs equilibrium states

In quantum statistical mechanics a physical system of fermions on a lattice is first characterized by its energy observables  $U_\Lambda$  for particles enclosed in finite boxes  $\Lambda \in \mathfrak{L}$ . Mathematically speaking,  $U_\Lambda$  are self-adjoint elements of the local algebras  $\mathcal{U}_\Lambda$ . Given any local state  $\rho_\Lambda \in E_\Lambda$  on  $\mathcal{U}_\Lambda$ , the energy observable  $U_\Lambda$  fixes the so-called *finite volume* free-energy density (in the box  $\Lambda \in \mathfrak{L}$ )

$$f_{\Lambda, U_\Lambda}(\rho_\Lambda) := |\Lambda|^{-1} \rho_\Lambda(U_\Lambda) - (\beta|\Lambda|)^{-1} S(\rho_\Lambda),$$

of the physical system at inverse temperature  $\beta > 0$ . The functional  $f_{\Lambda, U_\Lambda}$  can be seen either as a map from  $E_\Lambda$  to  $\mathbb{R}$  or from  $E$  to  $\mathbb{R}$  by taking, for all  $\rho \in E$ , the restriction  $\rho_\Lambda \in E_\Lambda$  on  $\mathcal{U}_\Lambda$ . The first term in  $f_{\Lambda, U_\Lambda}$  is obviously the mean energy per volume of the physical system found in the state  $\rho_\Lambda$ , whereas  $S$  is the von Neumann entropy defined by (6.18) which measures, in a sense, the amount of randomness carried by the state. See Section 6.4 for more details.

The state of a system in thermal equilibrium and at fixed mean energy per volume maximizes the entropy, by the second law of thermodynamics. Therefore, it minimizes the free-energy density functional  $f_{\Lambda, U_\Lambda}$ . Such well-known arguments lead to the study of the variational problem

$$(12.1) \quad \inf_{\rho \in E} f_{\Lambda, U_\Lambda}(\rho) = \inf_{\rho_\Lambda \in E_\Lambda} f_{\Lambda, U_\Lambda}(\rho_\Lambda).$$

As the von Neumann entropy  $S$  is weak\*-continuous, the functional  $f_{\Lambda, U_\Lambda}$  has at least one minimizer on  $E_\Lambda$  which is the local equilibrium state of the physical system, also called *Gibbs equilibrium state*:

DEFINITION 12.1 (Gibbs equilibrium state).

A *Gibbs equilibrium state* is a solution of the variational problem (12.1), i.e., a minimizer of the finite volume free-energy density functional  $f_{\Lambda, U_\Lambda}$ .

The set of solutions of the variational problem (12.1) is, a priori, not unique. But, for  $\beta \in (0, \infty)$ , it is well-known that the maximum of  $-f_{\Lambda, U_\Lambda}$  over  $E$  equals the finite volume pressure

$$p_{\Lambda, U_\Lambda} := \frac{1}{\beta|\Lambda|} \ln \text{Trace}_{\wedge \mathcal{H}_\Lambda} \left( e^{-\beta U_\Lambda} \right)$$

(compare with (3.7)) and is attained for the unique minimizer  $\rho_{\Lambda, U_\Lambda} \in E_\Lambda$  of  $f_{\Lambda, U_\Lambda}$  defined by

$$(12.2) \quad \rho_{\Lambda, U_\Lambda}(A) := \frac{\text{Trace}_{\wedge \mathcal{H}_\Lambda} (A e^{-\beta U_\Lambda})}{\text{Trace}_{\wedge \mathcal{H}_\Lambda} (e^{-\beta U_\Lambda})}, \quad A \in \mathcal{U}_\Lambda.$$

This result is a key ingredient in the proof of Theorem 3.12 (see Sections 7 and 8) and is also known in the literature as the *passivity of Gibbs states*:

**THEOREM 12.2** (Passivity of Gibbs states).

For  $\beta \in (0, \infty)$  and any self-adjoint  $U_\Lambda \in \mathcal{U}_\Lambda$ ,

$$p_{\Lambda, U_\Lambda} = -\inf_{\rho \in E} f_{\Lambda, U_\Lambda}(\rho) = -\inf_{\rho_\Lambda \in E_\Lambda} f_{\Lambda, U_\Lambda}(\rho_\Lambda) = -f_{\Lambda, U_\Lambda}(\rho_{\Lambda, U_\Lambda})$$

with the Gibbs equilibrium state  $\rho_{\Lambda, U_\Lambda} \in E_\Lambda$  being the unique minimizer on  $E_\Lambda$  of the finite volume free-energy density functional  $f_{\Lambda, U_\Lambda}$ .

The proof of this standard theorem is a (non-trivial) consequence of Jensen's inequality, see, e.g., [9, Lemma 6.3] or [5, Proposition 6.2.22] (for quantum spin systems).

## 12.2. The approximating Hamiltonian method

The approximating Hamiltonian method is presented in [13, 14, 15, 16]. This rigorous technique for computing the thermodynamic pressure does not seem to be well-known in the mathematical physics community, unfortunately. Therefore, we give below a brief account on the approximating Hamiltonian method and we compare it to our results.

Let

$$(12.3) \quad H_\Lambda := T_\Lambda + \frac{1}{|\Lambda|} \sum_{k=1}^N \gamma_k (U_{k, \Lambda} + iU'_{k, \Lambda})^* (U_{k, \Lambda} + iU'_{k, \Lambda})$$

be any self-adjoint operator acting on a Hilbert space  $\mathfrak{H}_\Lambda$  of a box  $\Lambda$  with  $\gamma_k = -1$  for any  $k \in \{1, \dots, n\}$  and  $\gamma_k = 1$  for  $k \in \{n+1, \dots, N\}$  ( $n < N$  being fixed). Here,  $T_\Lambda = T_\Lambda^*$  and  $\{U_{k, \Lambda}, U'_{k, \Lambda}\}_{k=1}^N$  are operators acting on  $\mathfrak{H}_\Lambda$ . Then the approximating Hamiltonian method corresponds to use so-called approximating Hamiltonians to compute the finite volume pressure

$$p[H_\Lambda] := \frac{1}{\beta|\Lambda|} \ln \text{Trace}_{\mathfrak{H}_\Lambda}(e^{-\beta H_\Lambda})$$

associated with  $H_\Lambda$ , for any  $\beta \in (0, \infty)$ , in the thermodynamic limit. A minimal requirement on  $H_\Lambda$  to have a thermodynamic behavior is of course to ensure the finiteness of  $p[H_\Lambda]$ . The latter is, in fact, fulfilled because this method is based on operators  $T_\Lambda = T_\Lambda^*$  and  $\{U_{k, \Lambda}, U'_{k, \Lambda}\}_{k=1}^N$  satisfying the following conditions:

(A1) The finite volume pressure of  $T_\Lambda$  exists, i.e.,

$$\left| \ln \text{Trace}_{\mathfrak{H}_\Lambda}(e^{-\beta T_\Lambda}) \right| \leq \beta|\Lambda| C_0.$$

(A2) The operators

$$(U_{k, \Lambda} + iU'_{k, \Lambda})^\# \in \{U_{k, \Lambda} + iU'_{k, \Lambda}, (U_{k, \Lambda} + iU'_{k, \Lambda})^*\}$$

are bounded in operator norm, for any  $k \in \{1, \dots, N\}$ , by  $C_1|\Lambda|$ .

(A3) The following commutators are also bounded for any  $k, q, p \in \{1, \dots, N\}$ :

$$\begin{aligned} \|[U_{k,\Lambda} + iU'_{k,\Lambda}, (U_{q,\Lambda} + iU'_{q,\Lambda})^\#]\| &\leq |\Lambda|C_2. \\ \|[U_{k,\Lambda} + iU'_{k,\Lambda})^\#, [(U_{q,\Lambda} + iU'_{q,\Lambda})^\#, U_{p,\Lambda} + iU'_{p,\Lambda}]]\| &\leq |\Lambda|C_3. \\ \|[U_{k,\Lambda} + iU'_{k,\Lambda})^\#, [U_{q,\Lambda} + iU'_{q,\Lambda}, T_\Lambda]]\| &\leq |\Lambda|C_4. \end{aligned}$$

For all  $k \in \{1, 2, 3, 4\}$ , note that the constants  $C_k$  are finite and do not depend on the box  $\Lambda$ .

Approximating Hamiltonians are then defined from  $H_\Lambda$  by

$$\begin{aligned} H_\Lambda(\vec{c}_-, \vec{c}_+) &:= T_\Lambda - \sum_{k=1}^n (\bar{c}_{k,-} (U_{k,\Lambda} + iU'_{k,\Lambda}) + c_{k,-} (U_{k,\Lambda} + iU'_{k,\Lambda})^*) \\ &\quad + \sum_{k=n+1}^N (\bar{c}_{k,+} (U_{k,\Lambda} + iU'_{k,\Lambda}) + c_{k,+} (U_{k,\Lambda} + iU'_{k,\Lambda})^*) \end{aligned}$$

with  $\vec{c}_- := (c_{1,-}, \dots, c_{n,-}) \in \mathbb{C}^n$ ,  $\vec{c}_+ := (c_{n+1,+}, \dots, c_{N,+}) \in \mathbb{C}^{N-n}$ . Let

$$f_{H,\Lambda}(\vec{c}_-, \vec{c}_+) := -|\vec{c}_+|^2 + |\vec{c}_-|^2 - \frac{1}{\beta|\Lambda|} \ln \text{Trace}_{\mathfrak{H}_\Lambda}(e^{-\beta H_\Lambda(\vec{c}_-, \vec{c}_+)})$$

be the approximating free-energy density and

$$\langle - \rangle_{\vec{c}_-, \vec{c}_+} := \frac{\text{Trace}_{\mathfrak{H}_\Lambda}(-e^{-\beta H_\Lambda(\vec{c}_-, \vec{c}_+)})}{\text{Trace}_{\mathfrak{H}_\Lambda}(e^{-\beta H_\Lambda(\vec{c}_-, \vec{c}_+)})}$$

be the (local) Gibbs equilibrium state associated with  $H_\Lambda(\vec{c}_-, \vec{c}_+)$ , see Section 12.1. From (A1)–(A2) it can be proven that, for any  $\vec{c}_- \in \mathbb{C}^n$ , there is a unique solution  $r_+(\vec{c}_-) := (d_{1,+}, \dots, d_{N,+}) \in \mathbb{C}^{N-n}$  of the (finite volume) gap equations

$$(12.4) \quad |\Lambda|^{-1} \langle U_{k,\Lambda} + iU'_{k,\Lambda} \rangle_{\vec{c}_-, r_+(\vec{c}_-)} = d_{k,+}$$

for all  $k \in \{n+1, \dots, N\}$ . Then let us consider two additional conditions:

(A4) For any  $k \in \{n, \dots, N\}$  with fixed  $n < N$ , the operators  $U_{k,\Lambda}^\#$  satisfy the ergodicity condition

$$\begin{aligned} \lim_{|\Lambda| \rightarrow \infty} \left\{ |\Lambda|^{-2} \left( \langle (U_{k,\Lambda} + iU'_{k,\Lambda})^* (U_{k,\Lambda} + iU'_{k,\Lambda}) \rangle_{\vec{c}_-, r_+(\vec{c}_-)} \right. \right. \\ \left. \left. - |\langle U_{k,\Lambda} + iU'_{k,\Lambda} \rangle_{\vec{c}_-, r_+(\vec{c}_-)}|^2 \right) \right\} = 0 \end{aligned}$$

for all  $\vec{c}_- \in \mathbb{C}^n$ .

(A5) The free-energy density  $f_{H,\Lambda}(\vec{c}_-, \vec{c}_+)$  converges in the thermodynamic limit  $|\Lambda| \rightarrow \infty$  towards

$$\lim_{|\Lambda| \rightarrow \infty} f_{H,\Lambda}(\vec{c}_-, \vec{c}_+) =: f_H(\vec{c}_-, \vec{c}_+)$$

for any  $\vec{c}_- \in \mathbb{C}^n$  and  $\vec{c}_+ \in \mathbb{C}^{N-n}$ .

Bogoliubov Jr. *et al.* have shown [15] the following:

**THEOREM 12.3** (Bogoliubov Jr., Brankov, Zagrebnov, Kurbatov, and Tonchev).  
Under assumptions (A1)–(A4) we obtain:

(i) For any box  $\Lambda$  and at fixed  $\vec{c}_- \in \mathbb{C}^n$ , the solution of the variational problem

$$\sup_{\vec{c}_+ \in \mathbb{C}^{N-n}} \mathfrak{f}_{\mathbb{H}, \Lambda}(\vec{c}_-, \vec{c}_+) = \mathfrak{f}_{\mathbb{H}, \Lambda}(\vec{c}_-, \mathfrak{r}_+(\vec{c}_-))$$

is unique and solution of (12.4), whereas there is  $\vec{d}_- \in \mathbb{C}^n$  such that

$$\inf_{\vec{c}_- \in \mathbb{C}^n} \left\{ \sup_{\vec{c}_+ \in \mathbb{C}^{N-n}} \mathfrak{f}_{\mathbb{H}, \Lambda}(\vec{c}_-, \vec{c}_+) \right\} = \mathfrak{f}_{\mathbb{H}, \Lambda}(\vec{d}_-, \mathfrak{r}_+(\vec{d}_-)).$$

(ii) In the thermodynamic limit

$$\lim_{|\Lambda| \rightarrow \infty} \left\{ p[\mathbb{H}_\Lambda] + \mathfrak{f}_{\mathbb{H}, \Lambda}(\vec{d}_-, \mathfrak{r}_+(\vec{d}_-)) \right\} = 0$$

and if (A5) also holds then

$$\lim_{|\Lambda| \rightarrow \infty} p[\mathbb{H}_\Lambda] = - \inf_{\vec{c}_- \in \mathbb{C}^n} \left\{ \sup_{\vec{c}_+ \in \mathbb{C}^{N-n}} \mathfrak{f}_{\mathbb{H}}(\vec{c}_-, \vec{c}_+) \right\}.$$

The proof of this theorem uses as a key ingredient the Bogoliubov (convexity) inequality [31, Corollary D.4] which can be deduced from Theorem 12.2. Another important technique used by the authors [14, 15, 16] are the Ginibre inequalities [11, Eq. (2.10)]. Their proofs are thus essentially different from ours.

To conclude we analyze Conditions (A1)–(A3) and (A5) for discrete Fermi systems  $\mathfrak{m} \in \mathcal{M}_1^{\text{d}}$  (see Section 3.1).

**LEMMA 12.4** (Conditions (A1)–(A3) and (A5) for  $\mathfrak{m} \in \mathcal{M}_1^{\text{d}}$ ).  
For any discrete Fermi system  $\mathfrak{m} := \{\Phi\} \cup \{\Phi_k, \Phi'_k\}_{k=1}^N \in \mathcal{M}_1^{\text{d}}$ , the self-adjoint operators  $\mathbb{T}_{\Lambda_l} := U_{\Lambda_l}^\Phi$ ,  $U_{k, \Lambda_l} := U_{\Lambda_l}^{\Phi_k}$ , and  $U_{k, \Lambda_l} := U_{\Lambda_l}^{\Phi'_k}$  satisfy Conditions (A1)–(A2) and (A5). (A3) holds whenever  $\mathfrak{m} \in \mathcal{M}_1^{\text{df}}$  is also finite range.

*Proof.* Condition (A1)–(A2) and (A5) are clearly satisfied. Condition (A3) requires direct computations. We omit the details. ■

**REMARK 12.5** (Condition (A4) as a non-necessary assumption).  
Condition (A4) is used in Theorem 12.3 to handle the positive part of long range interactions. It is generally not satisfied for discrete Fermi systems  $\mathfrak{m} \in \mathcal{M}_1^{\text{d}}$ . This condition is shown here to be absolutely not necessary to handle the thermodynamic limit of the pressure of Fermi systems  $\mathfrak{m} \in \mathcal{M}_1$  (see Theorem 3.36).

REMARK 12.6 (Condition (A3) as a non-necessary assumption).

Let  $\Phi, \Phi' \in \mathcal{W}_1$  such that  $\|\Phi_{\Lambda_l}\|, \|\Phi'_{\Lambda_l}\| = \mathcal{O}(l^{-(d+\epsilon)})$  for some small  $\epsilon > 0$ . Such interactions clearly exist. If this is the only information we have about the interactions then the only bound we can give for the commutators  $[U_\Lambda^\Phi, U_\Lambda^{\Phi'}]$  is

$$\|[U_\Lambda^\Phi, U_\Lambda^{\Phi'}]\| \leq \sum_{\Lambda_1, \Lambda'_1 \subset \Lambda, \Lambda_1 \cap \Lambda'_1 \neq \emptyset} 2\|\Phi_{\Lambda_1}\| \|\Phi'_{\Lambda'_1}\|.$$

Depending on  $\epsilon > 0$ , the r.h.s. of the last inequality grows at large  $|\Lambda|$  much faster than the volume  $|\Lambda|$ . Hence, the condition (A3) is very unlikely to hold for all  $\Phi, \Phi' \in \mathcal{W}_1$ .

### 12.3. $\mathcal{L}^p$ -spaces of maps with values in a Banach space

Let  $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$  be a separable measure space with  $\mathfrak{A}$  and  $\mathfrak{a} : \mathfrak{A} \rightarrow \mathbb{R}_0^+$  being respectively some  $\sigma$ -algebra on  $\mathcal{A}$  and some measure on  $\mathfrak{A}$ . Recall that  $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$  being separable means that the space  $L^2(\mathcal{A}, \mathbb{C}) := L^2(\mathcal{A}, \mathfrak{a}, \mathbb{C})$  of square integrable complex valued functions on  $\mathcal{A}$  is a separable Hilbert space. This property implies, in particular, that  $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$  is a  $\sigma$ -finite measure space, see [49, p. 54].

Let  $\mathcal{X}$  be any Banach space with norm  $\|\cdot\|_{\mathcal{X}}$ . We denote by  $\mathcal{S}(\mathcal{A}, \mathcal{X})$  the set of measurable step functions with support of finite measure. For any measurable map  $s_a : \mathcal{A} \rightarrow \mathcal{X}$  and any  $p \geq 1$ , we define the semi-norm

$$\|s\|_p := \int_{\mathcal{A}} \|s_a\|_{\mathcal{X}}^p d\mathfrak{a}(a) \in [0, \infty].$$

Let  $s_a^{(n)}$  be any  $\mathcal{L}^p$ -Cauchy sequence of measurable maps, i.e.,  $\|s_a^{(n)}\|_p < \infty$  and

$$\lim_{N \rightarrow \infty} \sup_{n, m > N} \|s_a^{(n)} - s_a^{(m)}\|_p = 0.$$

Then there is a measurable function  $s_\infty$  from  $\mathcal{A}$  to  $\mathcal{X}$  with  $\|s_a^{(\infty)}\|_p < \infty$  such that

$$\lim_{n \rightarrow \infty} \|s_a^{(n)} - s_a^{(\infty)}\|_p = 0$$

(Completeness of Banach-valued  $\mathcal{L}^p$ -spaces). Now, define the sub-space

$$\mathcal{L}^p(\mathcal{A}, \mathcal{X}) := \left\{ s_a^{(\infty)} : \text{there is } \{s_a^{(n)}\}_{n=1}^\infty \text{ in } \mathcal{S}(\mathcal{A}, \mathcal{X}) \text{ with } \lim_{n \rightarrow \infty} \|s_a^{(n)} - s_a^{(\infty)}\|_p = 0 \right\}$$

of the space of measurable functions  $\mathcal{A} \rightarrow \mathcal{X}$ . Observe that the semi-norm  $\|\cdot\|_p$  is finite on  $\mathcal{L}^p(\mathcal{A}, \mathcal{X})$ . In other words,  $\mathcal{L}^p(\mathcal{A}, \mathcal{X})$  is the closure of  $\mathcal{S}(\mathcal{A}, \mathcal{X})$  w.r.t. the semi-norm  $\|\cdot\|_p$ .

Define the linear map from  $\mathcal{S}(\mathcal{A}, \mathcal{X})$  to  $\mathcal{X}$  by

$$(12.5) \quad s_a \mapsto \int_{\mathcal{A}} s_a d\mathfrak{a}(a) := \sum_{x \in s_a(\mathcal{A})} x \mathfrak{a}(s_a^{-1}(x)).$$

Obviously, for all  $s_a \in \mathcal{S}(\mathcal{A}, \mathcal{X})$ ,

$$(12.6) \quad \left\| \int_{\mathcal{A}} s_a d\mathbf{a}(a) \right\|_{\mathcal{X}} \leq \|s_a\|_1.$$

Now, for each function  $c_a \in L^2(\mathcal{A}, \mathbb{C})$ , let us consider the linear map  $s_a \mapsto \langle s_a, c_a \rangle$  from  $\mathcal{S}(\mathcal{A}, \mathcal{X})$  to  $\mathcal{X}$  defined by

$$(12.7) \quad \langle s_a, c_a \rangle := \sum_{x \in s_a(\mathcal{A})} x \int_{s_a^{-1}(x)} \bar{c}_a d\mathbf{a}(a).$$

From the (finite dimensional) Cauchy–Schwarz inequality note that, for all  $s_a \in \mathcal{S}(\mathcal{A}, \mathcal{X})$ ,

$$(12.8) \quad \|\langle s_a, c_a \rangle\|_{\mathcal{X}} \leq \|s_a\|_2 \|c_a\|_2.$$

By using Hahn–Banach theorem and the density of  $\mathcal{S}(\mathcal{A}, \mathcal{X})$  in  $\mathcal{L}^p(\mathcal{A}, \mathcal{X})$ , we obtain the existence and uniqueness of linear extensions of the maps (12.5) and (12.7) respectively to the spaces  $\mathcal{L}^1(\mathcal{A}, \mathcal{X})$  and  $\mathcal{L}^2(\mathcal{A}, \mathcal{X})$ . In particular, the linear extensions of (12.5) and (12.7) satisfy (12.6) and (12.8), respectively.

#### 12.4. Compact convex sets and Choquet simplices

The theory of compact convex subsets of a locally convex (topological vector) space  $\mathcal{X}$  is standard. For more details, see, e.g., [2, 3]. Note, however, that the definitions of topological vector spaces found in the literature differ slightly from each other. Those differences mostly concern the *Hausdorff* property. Here, we use Rudin’s definition [1, Section 1.6]:

DEFINITION 12.7 (Topological vector spaces).

*A topological vector space  $\mathcal{X}$  is a vector space equipped with a topology  $\tau$  for which the vector space operations of  $\mathcal{X}$  are continuous and every point of  $\mathcal{X}$  defines a closed set.*

The fact that every point of  $\mathcal{X}$  is a closed set is usually not part of the definition of a topological vector space in many textbooks. It is used here because it is satisfied in most applications – including those of this paper – and, in this case, the space  $\mathcal{X}$  is automatically Hausdorff by [1, Theorem 1.12]. Examples of topological vector spaces used in this paper are the dual spaces (cf. [1, Theorem 3.10]):

THEOREM 12.8 (Dual space of a topological vector space).

*The dual space  $\mathcal{X}^*$  of a (topological vector) space  $\mathcal{X}$  is a locally convex space in the  $\sigma(\mathcal{X}^*, \mathcal{X})$ -topology – known as the *weak\**-topology – and its dual is  $\mathcal{X}$ .*

Since any Banach space is a topological vector space in the sense of Definition 12.7, the dual space of a Banach space is a locally convex space:



COROLLARY 12.9 (Dual space of a Banach space).

*The dual space  $\mathcal{X}^*$  of a Banach space  $\mathcal{X}$  is a locally convex space in the  $\sigma(\mathcal{X}^*, \mathcal{X})$ -topology – known as the weak\*-topology – and its dual is  $\mathcal{X}$ .*

It follows that the dual spaces  $\mathcal{U}^*$  and  $\mathcal{W}_1^*$  respectively of the Banach spaces  $\mathcal{U}$  and  $\mathcal{W}_1$  (cf. Section 2.1 and Definition 2.24) are both locally convex real spaces w.r.t. the weak\*-topology. Note that  $\mathcal{U}$  and  $\mathcal{W}_1$  are separable. This property yields the metrizability of any weak\*-compact subset  $K$  of their dual spaces (cf. [1, Theorem 3.16]):

THEOREM 12.10 (Metrizability of weak\*-compact sets).

*Let  $K \subset \mathcal{X}^*$  be any weak\*-compact subset of the dual  $\mathcal{X}^*$  of a separable topological vector space  $\mathcal{X}$ . Then  $K$  is metrizable in the weak\*-topology.*

One important observation concerning locally convex spaces  $\mathcal{X}$  is that any compact convex subset  $K \subset \mathcal{X}$  is the closure of the convex hull of the (non-empty) set  $\mathcal{E}(K)$  of its extreme points, i.e., of the points which cannot be written as – non-trivial – convex combinations of other elements in  $K$ . This is Krein–Milman theorem (see, e.g., [1, Theorems 3.4 (b) and 3.23]):

THEOREM 12.11 (Krein–Milman).

*Let  $K \subset \mathcal{X}$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$ . Then we have that:*

- (i) *The set  $\mathcal{E}(K)$  of its extreme points is non-empty.*
- (ii) *The set  $K$  is the closed convex hull of  $\mathcal{E}(K)$ .*

REMARK 12.12.  *$\mathcal{X}$  being a topological vector space on which its dual space  $\mathcal{X}^*$  separates points is the only condition necessary on  $\mathcal{X}$  in Krein–Milman theorem. For more details, see, e.g., [1, Theorem 3.23].*

In fact, the set  $\mathcal{E}(K)$  of extreme points is even a  $G_\delta$  set if the compact convex set  $K \subset \mathcal{X}$  is metrizable. Moreover, among all subsets  $Z \subset K$  generating  $K$ ,  $\mathcal{E}(K)$  is – in a sense – the smallest one (see, e.g., [3, Proposition 1.5]):

THEOREM 12.13 (Properties of the set  $\mathcal{E}(K)$ ).

*Let  $K \subset \mathcal{X}$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$ . Then we have that:*

- (i) *If  $K$  is metrizable then the set  $\mathcal{E}(K)$  of extreme points of  $K$  forms a  $G_\delta$  set.*
- (ii) *If  $K$  is the closed convex hull of  $Z \subset K$  then  $\mathcal{E}(K)$  is included in the closure of  $Z$ .*

Property (i) can be found in [3, Proposition 1.3] and only needs that  $\mathcal{X}$  is a topological vector space, whereas the second statement (ii) is a classical result obtained by Milman, see [3, Proposition 1.5].

Theorem 12.11 restricted to finite dimensions is a classical result of Minkowski which, for any  $x \in K$  in (non-empty) compact convex subset  $K \subset \mathcal{X}$ , states the existence of a finite number of extreme points  $\hat{x}_1, \dots, \hat{x}_k \in \mathcal{E}(K)$  and positive numbers  $\mu_1, \dots, \mu_k \geq 0$  with  $\sum_{j=1}^k \mu_j = 1$  such that

$$(12.9) \quad x = \sum_{j=1}^k \mu_j \hat{x}_j.$$

To this simple decomposition we can associate a probability measure, i.e., a *normalized positive Borel regular measure*,  $\mu$  on  $K$ .

Indeed, the Borel sets of any set  $K$  are elements of the  $\sigma$ -algebra  $\mathfrak{B}$  generated by closed – or open – subsets of  $K$ . Positive Borel regular measures are the positive countably additive set functions  $\mu$  over  $\mathfrak{B}$  satisfying

$$\mu(B) = \sup \{ \mu(C) : C \subset B, C \text{ closed} \} = \inf \{ \mu(O) : B \subset O, O \text{ open} \}$$

for any Borel subset  $B \in \mathfrak{B}$  of  $K$ . If  $K$  is compact then any positive Borel regular measure  $\mu$  corresponds (one-to-one) to an element of the set  $M^+(K)$  of Radon measures with  $\mu(K) = \|\mu\|$  and we write

$$(12.10) \quad \mu(h) = \int_K d\mu(\hat{x}) h(\hat{x})$$

for any continuous function  $h$  on  $K$ . A probability measure  $\mu \in M_1^+(K)$  is per definition a positive Borel regular measure  $\mu \in M^+(K)$  which is *normalized*:  $\|\mu\| = 1$ .

REMARK 12.14. *The set  $M_1^+(K)$  of probability measures on  $K$  can also be seen as the set of states on the commutative  $C^*$ -algebra  $C(K)$  of continuous functionals on the compact set  $K$ , by the Riesz–Markov theorem.*

Therefore, using the probability measure  $\mu_x \in M_1^+(K)$  on  $K$  defined by

$$\mu_x = \sum_{j=1}^k \mu_j \delta_{\hat{x}_j}$$

with  $\delta_y$  being the Dirac – or point – mass<sup>22</sup> at  $y$ , Equation (12.9) can be seen as an integral defined by (12.10) for the probability measure  $\mu_x \in M_1^+(K)$ :

$$(12.11) \quad x = \int_K d\mu_x(\hat{x}) \hat{x}.$$

The point  $x$  is in fact the *barycenter* of the probability measure  $\mu_x$ . This notion is defined in the general case as follows (cf. [3, p. 1]):

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<sup>22</sup> $\delta_y$  is the Borel measure such that for any Borel subset  $B \in \mathfrak{B}$  of  $K$ ,  $\delta_y(B) = 1$  if  $y \in B$  and  $\delta_y(B) = 0$  if  $y \notin B$ .

DEFINITION 12.15 (Barycenters of probability measures in convex sets).  
 Let  $K \subset \mathcal{X}$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$  and let  $\mu \in M_1^+(K)$  be a probability measure on  $K$ . We say that  $x \in K$  is the barycenter<sup>23</sup> of  $\mu$  if, for all continuous linear<sup>24</sup> functionals  $h$  on  $\mathcal{X}$ ,

$$h(x) = \int_K d\mu(\hat{x}) h(\hat{x}).$$

Barycenters are well-defined for all probability measures in convex compact subsets of locally convex spaces (cf. [3, Propositions 1.1 and 1.2]):

THEOREM 12.16 (Well-definiteness and uniqueness of barycenters).

Let  $K \subset \mathcal{X}$  be any (non-empty) compact subset of a locally convex space  $\mathcal{X}$  such that  $\overline{\text{co}(K)}$  is also compact. Then we have that:

(i) For any probability measure  $\mu \in M_1^+(K)$  on  $K$ , there is a unique barycenter  $x_\mu \in \overline{\text{co}(K)}$ . In particular, if  $K$  is convex then, for any  $\mu \in M_1^+(K)$ , there is a unique barycenter  $x_\mu \in K$ . Moreover, the map  $\mu \mapsto x_\mu$  from  $M_1^+(K)$  to  $\overline{\text{co}(K)}$  is affine and weak\*-continuous.

(ii) Conversely, for any  $x \in \overline{\text{co}(K)}$ , there is a probability measure  $\mu_x \in M_1^+(K)$  on  $K$  with barycenter  $x$ .

Therefore, we write the barycenter  $x_\mu$  of any probability measure  $\mu$  in  $K$  as

$$x_\mu = \int_K d\mu(\hat{x}) \hat{x},$$

where the integral has to be understood in the weak sense. By Definition 12.15, it means that  $h(x_\mu)$  can be decomposed by the probability measure  $\mu \in M_1^+(K)$  provided  $h$  is a continuous linear functional. In fact, this last property can also be extended to all affine upper semi-continuous functionals on  $K$ , see, e.g., [17, Corollary 4.1.18.] together with [1, Theorem 1.12]:

LEMMA 12.17 (Barycenters and affine maps).

Let  $K \subset \mathcal{X}$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$ . Then, for any probability measure  $\mu \in M_1^+(K)$  on  $K$  with barycenter  $x_\mu \in K$  and for any affine upper semi-continuous functional  $h$  on  $K$ ,

$$h(x) = \int_K d\mu_x(\hat{x}) h(\hat{x}).$$

It is natural to ask whether, for any  $x \in K$  in a convex set  $K$ , there is a (possibly not unique) probability measure  $\mu_x$  on  $K$  supported on  $\mathcal{E}(K)$  with

<sup>23</sup>Other terminology existing in the literature: “ $x$  is represented by  $\mu$ ”, “ $x$  is the resultant of  $\mu$ ”.

<sup>24</sup>Barycenters can also be defined in the same way via affine functionals instead of linear functionals, see [17, Proposition 4.1.1.].

barycenter  $x$ . Equation (12.11) already gives a first positive answer to that problem in the finite dimensional case. The general case has been proven by Choquet, whose theorem is a remarkable refinement of Krein–Milman theorem (see, e.g., [3, p. 14]):

**THEOREM 12.18** (Choquet).

*Let  $K \subset \mathcal{X}$  be any (non-empty) metrizable compact convex subset of a locally convex space  $\mathcal{X}$ . Then, for any  $x \in K$ , there is a probability measure  $\mu_x \in M_1^+(K)$  on  $K$  such that*

$$\mu_x(\mathcal{E}(K)) = 1 \quad \text{and} \quad x = \int_K d\mu_x(\hat{x}) \hat{x}.$$

*Recall that the integral above means that  $x \in K$  is the barycenter of  $\mu_x$ .*

**REMARK 12.19** (Choquet theorem and affine maps).

*By Lemma 12.17, Choquet theorem can be used to decompose any affine upper semi-continuous functional defined on the metrizable compact convex subset  $K \subset \mathcal{X}$  w.r.t. extreme points of  $K$ .*

**REMARK 12.20** (Choquet theorem for non metrizable  $K$ ).

*If the (non-empty) compact convex subset  $K \subset \mathcal{X}$  is not metrizable then  $\mathcal{E}(K)$  may not form a Borel set. Choquet theorem (Theorem 12.18) stays, however, valid under the modification that  $\mu_x$  is pseudo-supported by  $\mathcal{E}(K)$  which means that  $\mu_x(B) = 1$  for all Baire sets  $B \supseteq \mathcal{E}(K)$ . This result is known as the Choquet–Bishop–de Leeuw theorem, see [3, p. 17].*

Note that the probability measure  $\mu_x$  of Theorem 12.18 is a priori not unique. For instance, in the 2-dimensional plane, simplices (points, segments, and triangles) are uniquely decomposed in terms of their extreme points, i.e., they are uniquely represented by a convex combination of extreme points. But this decomposition is not anymore unique for a square. In fact, uniqueness of the decomposition given in Theorem 12.18 is related to the theory of simplices.

To define them in the general case, let  $S$  be a compact convex set of a locally convex real space  $\mathcal{X}$ . Without loss of generality assume that the compact convex set  $S$  is included in a closed hyper-plane which does not contain the origin<sup>25</sup>. Let

$$\mathfrak{K} := \{\alpha x : \alpha \geq 0, x \in S\}$$

be the cone with base  $S$ . Recall that the cone  $\mathfrak{K}$  induces a partial ordering on  $\mathcal{X}$  by using the definition  $x \geq y$  iff  $x - y \in \mathfrak{K}$ . A *least upper bound* for  $x$  and  $y$  is an element  $x \vee y \geq x, y$  satisfying  $w \geq x \vee y$  for all  $w$  with  $w \geq x, y$ . Then a simplex is defined as follows:

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<sup>25</sup>Otherwise, we embed  $\mathcal{X}$  as  $\mathcal{X} \times \{1\}$  in  $\mathcal{X} \times \mathbb{R}$ .

DEFINITION 12.21 (Simplices).

The (non-empty) compact convex set  $S$  is a simplex whenever  $\mathfrak{K}$  is a lattice with respect to the partial ordering  $\geq$ . This means that each pair  $x, y \in \mathfrak{K}$  has a least upper bound  $x \vee y \in \mathfrak{K}$ .

Observe that a simplex can also be defined for non-compact convex sets but we are only interested here in compact simplices. Such simplices are particular examples of *simplexoids*, i.e., compact convex sets whose closed proper faces are simplices.

The definition of simplices above agrees with the usual definition in finite dimensions as the  $n$ -dimensional simplex  $\{(\lambda_1, \lambda_2, \dots, \lambda_{n+1}), \sum_j \lambda_j = 1\}$  is the base of the  $(n+1)$ -dimensional cone  $\{(\lambda_1, \lambda_2, \dots, \lambda_{n+1}), \lambda_j \geq 0\}$ . In fact, for all metrizable simplices, the probability measure  $\mu_x$  of Theorem 12.18 is unique and conversely, if  $\mu_x$  is always uniquely defined then the corresponding metrizable compact convex set is a simplex (see, e.g., [3, p. 60]):

THEOREM 12.22 (Choquet).

Let  $S \subset \mathcal{X}$  be any (non-empty) closed convex metrizable subset of a locally convex space  $\mathcal{X}$ . Then  $S$  is a simplex iff, for any  $x \in S$ , there is a unique probability measure  $\mu_x \in M_1^+(S)$  on  $S$  such that

$$\mu_x(\mathcal{E}(S)) = 1 \quad \text{and} \quad x = \int_S d\mu_x(\hat{x}) \hat{x}.$$

Compact and metrizable convex sets for which the integral representation in Theorem 12.22 is unique are also called *Choquet simplices*:

DEFINITION 12.23 (Choquet simplex).

The simplex  $S$  is a Choquet simplex whenever the decomposition of  $S$  on  $\mathcal{E}(S)$  given by Theorem 12.18 (see also Remark 12.20) is unique.

In this paper we are only interested in metrizable compact convex set on which Theorem 12.22 is applied. Therefore, all our examples of simplices are in fact Choquet simplices.

Two further special types of simplices are of particular importance: The *Bauer* and the *Poulsen* simplices. The first one is defined as follows:

DEFINITION 12.24 (Bauer simplex).

The simplex  $S$  is a Bauer simplex whenever its set  $\mathcal{E}(S)$  of extreme points is closed.

A compact Bauer simplex  $S$  has the interesting property that it is affinely homeomorphic to the set of states on the commutative  $C^*$ -algebra  $C(\mathcal{E}(S))$  (see, e.g., [2, Corollary II.4.2]):

THEOREM 12.25 (Bauer).

Let  $S \subset \mathcal{X}$  be any compact Bauer Simplex of a locally convex space  $\mathcal{X}$ . Then the map  $x \mapsto \mu_x$  defined by Theorem 12.22 from  $S$  to the set  $M_1^+(\mathcal{E}(S))$  of probability measures<sup>26</sup> on  $\mathcal{E}(S)$  is an affine homeomorphism.

Bauer simplices are special simplices as the set of  $\mathcal{E}(S)$  of extreme points of a simplex  $S$  may not be closed. In fact, E. T. Poulsen [50] constructed in 1961 an example of a metrizable simplex  $S$  with  $\mathcal{E}(S)$  being dense in  $S$ . This simplex is now well-known as the Poulsen simplex because it is unique [45, Theorem 2.3.] up to an affine homeomorphism:

THEOREM 12.26 (Lindenstrauss–Olsen–Sternfeld).

Every (non-empty) compact metrizable simplex  $S$  with  $\mathcal{E}(S)$  being dense in  $S$  is affinely homeomorphic to the Poulsen simplex.

The original example given by Poulsen [50] is not explained here as we give in Section 2.2 a prototype of the Poulsen simplex: The set  $E_{\vec{\ell}} \subset \mathcal{U}^*$  of all  $\vec{\ell} \cdot \mathbb{Z}^d$ -invariant states defined by (2.8) for any  $\vec{\ell} \in \mathbb{N}^d$ , see Theorem 2.12.

For more details on the Poulsen simplex we recommend [45] where its specific properties are described. They also show that the Poulsen simplex is, in a sense, complementary to the Bauer simplices, see [2, p. 164] or [45, Section 5].

## 12.5. $\Gamma$ -regularization of real functionals

The  $\Gamma$ -regularization of real functionals on a subset  $K \subset \mathcal{X}$  is defined from the space  $A(\mathcal{X})$  of all affine continuous real valued functionals on  $\mathcal{X}$  as follows (cf. [46, Definition 2.1.1]):

DEFINITION 12.27 ( $\Gamma$ -regularization of real functionals).

For any real functional  $h$  defined from a locally convex space  $\mathcal{X}$  to  $(-\infty, \infty]$ , its  $\Gamma$ -regularization  $\Gamma_K(h)$  on a subset  $K \subset \mathcal{X}$  is the functional defined as the supremum over all affine and continuous minorants from  $\mathcal{X}$  to  $\mathbb{R}$  of  $h|_K$ , i.e., for all  $x \in \mathcal{X}$ ,

$$\Gamma_K(h)(x) := \sup \{m(x) : m \in A(\mathcal{X}) \text{ and } m|_K \leq h|_K\}.$$

If a functional  $h$  is only defined on a subset  $K \subset \mathcal{X}$  of a locally convex space  $\mathcal{X}$  then we compute  $\Gamma_{\mathcal{X}}(h)$  by extending  $h$  to the locally convex space  $\mathcal{X}$  as follows:

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<sup>26</sup>I.e. the set of states on the commutative  $C^*$ -algebra  $C(\mathcal{E}(S))$  of continuous functionals on the compact set  $\mathcal{E}(S)$ .

DEFINITION 12.28 (Extension of functionals on a locally convex space  $\mathcal{X}$ ). Any functional  $h : K \subset \mathcal{X} \rightarrow (-\infty, \infty]$  is seen as a map from  $\mathcal{X}$  to  $(-\infty, \infty]$  by the definition

$$h(x) := \begin{cases} h(x) & , \text{ for } x \in K. \\ +\infty & , \text{ for } x \in \mathcal{X} \setminus K. \end{cases}$$

If  $h$  is convex and weak\*-lower semi-continuous on the closed and convex subset  $K \subset \mathcal{X}$  then its extension on  $\mathcal{X}$  is also convex and weak\*-lower semi-continuous. Moreover, in this case,  $\Gamma_{\mathcal{X}}(h) = \Gamma_K(h)$  on  $\mathcal{X}$ .

Since the  $\Gamma$ -regularization  $\Gamma_K(h)$  of a real functional  $h$  is a supremum of continuous functionals,  $\Gamma_K(h)$  is a convex and lower semi-continuous functional on  $\mathcal{X}$ . In fact, every convex and lower semi-continuous functional on  $K$  equals its  $\Gamma$ -regularization on  $K$  (cf. [46, Proposition 2.1.2]):

PROPOSITION 12.29 ( $\Gamma$ -regularization of lower semi-cont. convex maps). Let  $h$  be any functional from a (non-empty) closed convex subset  $K \subset \mathcal{X}$  of a locally convex space  $\mathcal{X}$  to  $(-\infty, \infty]$ . Then the following statements are equivalent:

- (i)  $\Gamma_K(h) = h$  on  $K$ .
- (ii)  $h$  is a lower semi-continuous convex functional on  $K$ .

This proposition is a standard result which can directly be proven without using the fact that the  $\Gamma$ -regularization  $\Gamma_K(h)$  of a functional  $h$  on  $K$  equals its twofold Legendre-Fenchel transform – also called the biconjugate (functional) of  $h$ . Indeed,  $\Gamma_K(h)$  is the largest lower semi-continuous and convex minorant of  $h$ :

COROLLARY 12.30 (Largest lower semi-continuous convex minorant of  $h$ ). Let  $h$  be any functional from a (non-empty) closed convex subset  $K \subset \mathcal{X}$  of a locally convex space  $\mathcal{X}$  to  $(-\infty, \infty]$ . Then its  $\Gamma$ -regularization  $\Gamma_K(h)$  is its largest lower semi-continuous and convex minorant on  $K$ .

*Proof.* For any lower semi-continuous convex functional  $f$  satisfying  $f \leq h$  on  $K$ , we have, by Proposition 12.29, that

$$f(x) = \sup \{m(x) : m \in \mathbf{A}(\mathcal{X}) \text{ and } m|_K \leq f|_K \leq h|_K\} \leq \Gamma_K(h)(x)$$

for any  $x \in K$ . ■

In particular, if  $(\mathcal{X}, \mathcal{X}^*)$  is a dual pair and  $h$  is any functional from  $\mathcal{X}$  to  $(-\infty, \infty]$  then  $\Gamma_{\mathcal{X}}(h) = h^{**}$ , by using Theorem 12.41 together with Corollary 12.30. See Corollary 12.42.

Proposition 12.29 has further interesting consequences. The first one we would like to mention is an extension of the Bauer maximum principle [17, Lemma 4.1.12], that is:

LEMMA 12.31 (Bauer maximum principle).

Let  $\mathcal{X}$  be a topological vector space. An upper semi-continuous convex real functional  $h$  over a compact convex subset  $K \subset \mathcal{X}$  attains its maximum at an extreme point of  $K$ , i.e.,

$$\sup_{x \in K} h(x) = \max_{\hat{x} \in \mathcal{E}(K)} h(\hat{x}).$$

Here,  $\mathcal{E}(K)$  is the (non-empty) set of extreme points of  $K$ , cf. Theorem 12.11.

Indeed, by combining Proposition 12.29 with Lemma 12.31 it is straightforward to check the following statement which does not seem to have been observed before:

LEMMA 12.32 (Extension of the Bauer maximum principle).

Let  $h_{\pm}$  be two convex real functionals from a locally convex space  $\mathcal{X}$  to  $(-\infty, \infty]$  such that  $h_{-}$  and  $h_{+}$  are, respectively, lower and upper semi-continuous. Then the supremum of the sum  $h := h_{-} + h_{+}$  over a compact convex subset  $K \subset \mathcal{X}$  can be reduced to the (non-empty) set  $\mathcal{E}(K)$  of extreme points of  $K$ , i.e.,

$$\sup_{x \in K} h(x) = \sup_{\hat{x} \in \mathcal{E}(K)} h(\hat{x}).$$

*Proof.* We first use Proposition 12.29 in order to write  $h_{-} = \Gamma_K(h_{-})$  as a supremum over affine and continuous functionals. Then we commute this supremum with the one over  $K$  and apply the Bauer maximum principle to obtain that

$$\sup_{x \in K} h(x) = \sup \left\{ \sup_{\hat{x} \in \mathcal{E}(K)} \{m(\hat{x}) + h_{+}(\hat{x})\} : m \in \mathbf{A}(\mathcal{X}) \text{ and } m|_K \leq h_{-}|_K \right\}.$$

The lemma follows by commuting once again both suprema and by using  $h_{-} = \Gamma_K(h_{-})$ . ■

Observe, however, that, under the conditions of the lemma above the supremum of  $h = h_{-} + h_{+}$  is, in general, not attained on  $\mathcal{E}(K)$ .

Another consequence of Proposition 12.29 is Jensen's inequality for convex lower semi-continuous real functionals on a compact convex sets  $K$ .

LEMMA 12.33 (Jensen's inequality on compact convex sets).

Let  $\mathcal{X}$  be a locally convex space,  $h$  be any lower semi-continuous convex real functional over a compact convex subset  $K \subset \mathcal{X}$  and  $\mu_x \in M_1^+(K)$  be any probability measure with barycenter  $x \in K$  (Definition 12.15). Assume the existence of some positive and  $\mu_x$ -integrable upper bound  $\mathfrak{h}$  for  $h$ , i.e., some measurable functional  $\mathfrak{h}$  from  $K$  to  $\mathbb{R}_0^+$  satisfying

$$\int_K d\mu_x(\hat{x}) \mathfrak{h}(\hat{x}) < \infty \quad \text{and} \quad h \leq \mathfrak{h} \quad \mu_x\text{-a.e. on } K.$$



Then

$$h(x) \leq \int_K d\mu_x(\hat{x}) h(\hat{x}).$$

Jensen’s inequality is of course a well-known result stated in various situations including functionals taking value in a topological vector space. We give here a rather simple proof, in our case, by using Proposition 12.29.

*Proof.* As  $h$  is convex and lower semi-continuous, by Proposition 12.29,

$$h(x) = \sup \{m(x) : m \in A(\mathcal{X}) \text{ and } m|_K \leq h|_K\}$$

for any  $x \in K$ . We further observe that, for any affine continuous real functional  $m$  and any probability measure  $\mu_x$  with barycenter  $x \in K$ ,

$$m(x) = \int_K d\mu_x(\hat{x}) m(\hat{x}),$$

see Lemma 12.17. Thus

$$(12.12) \quad h(x) = \sup \left\{ \int_K d\mu_x(\hat{x}) m(\hat{x}) : m \in A(\mathcal{X}) \text{ and } m|_K \leq h|_K \right\}.$$

Since there is a positive and  $\mu_x$ -integrable upper bound  $\mathfrak{h}$  for  $h$ , we have that

$$\int_K d\mu_x(\hat{x}) \max \{h(\hat{x}), 0\} < \infty.$$

Hence, by (12.12) together with the monotonicity of integrals,

$$h(x) \leq \int_K d\mu_x(\hat{x}) h(\hat{x}) < \infty.$$

■

We give now an interesting property concerning the  $\Gamma$ -regularization of real functionals in relation with compact convex sets (cf. [2, Corollary I.3.6.]):

**THEOREM 12.34** ( $\Gamma$ -regularization of continuous maps).

Let  $K \subset \mathcal{X}$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$  and  $h : K \rightarrow (-\infty, \infty]$  be a continuous real functional. Then, for any  $x \in K$ , there is a probability measure  $\mu_x \in M_1^+(K)$  on  $K$  with barycenter  $x$  such that

$$\Gamma_K(h)(x) = \int_K d\mu_x(\hat{x}) h(\hat{x}).$$

This theorem is a useful result to study variational problems – at least the ones appearing in this paper. Indeed, if  $h$  is a continuous functional from a compact convex set  $K$  to  $[k, \infty]$  with  $k \in \mathbb{R}$  then extreme points of the compact set of minimizers of  $\Gamma_K(h)$  on  $K$  are minimizers of  $h$ . This can be seen – in a more general setting – as follows.

Let  $K$  be a compact convex subset of a locally convex space  $\mathcal{X}$  and  $h : K \rightarrow (-\infty, \infty]$  be any real functional. Then  $\{x_i\}_{i \in I} \subset K$  is – per definition – a net of *approximating minimizers* when

$$\lim_I h(x_i) = \inf_{x \in K} h(x).$$

Note that nets  $\{x_i\}_{i \in I} \subset K$  converges along a subnet as  $K$  is compact. Then we define the set of generalized minimizers of  $h$  as follows:

DEFINITION 12.35 (Set of generalized minimizers).

Let  $K$  be a (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$  and  $h : K \rightarrow (-\infty, \infty]$  be any real functional. Then the set  $\Omega(h, K) \subset K$  of generalized minimizers of  $h$  is the (non-empty) set

$$\Omega(h, K) := \left\{ y \in K : \exists \{x_i\}_{i \in I} \subset K \text{ converging to } y \text{ with } \lim_I h(x_i) = \inf_K h \right\}$$

of all limit points of approximating minimizers of  $h$ .

Note that the non-empty set  $\Omega(h, K)$  is compact when  $K$  is metrizable:

LEMMA 12.36 (Properties of the set  $\Omega(h, K)$ ).

Let  $K$  be a compact, convex, and metrizable subset of a locally convex space  $\mathcal{X}$  and  $h : K \rightarrow (-\infty, \infty]$  be any real functional. Then the set  $\Omega(h, K)$  of generalized minimizers of  $h$  over  $K$  is compact.

*Proof.* Since  $K$  is compact,  $\Omega(h, K) \subset K$  is compact if it is a closed set. Because it is metrizable,  $K$  is sequentially compact and we can restrict ourself on sequences instead of more general nets. Then the lemma can easily be proven by using any metric  $d_K(x, y)$  on  $K$  generating the topology. Indeed, for any sequence  $\{y_n\}_{n=1}^\infty \subset \Omega(h, K)$  of generalized minimizers converging to  $y$ , there is, by Definition 12.35, a sequence  $\{x_{n,m}\}_{n,m=1}^\infty \subset K$  of approximating minimizers converging, for any  $n \in \mathbb{N}$ , to  $y_n \in \Omega(h, K)$  as  $m \rightarrow \infty$ . In particular, for all  $n \in \mathbb{N}$ , there exists  $N_n > 0$  such that, for all  $m > N_n$ ,

$$d_K(x_{n,m}, y) \leq 2^{-n} + d_K(y_n, y) \quad \text{and} \quad |h(x_{n,m}) - \inf_K h| \leq 2^{-n}.$$

By taking any function  $p(n) \in \mathbb{N}$  satisfying  $p(n) > N_n$  and converging to  $\infty$  as  $n \rightarrow \infty$  we obtain that  $\{x_{n,p(n)}\}_{n=1}^\infty$  is a sequence of approximating minimizers converging to  $y$  as  $n \rightarrow \infty$ . In other words,  $y \in \Omega(h, K)$ . ■

Now, we are in position to give a useful theorem on the minimization of real functionals:

THEOREM 12.37 (Minimization of real functionals – I).

Let  $K$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$

and  $h : K \rightarrow [k, \infty]$  be any real functional with  $k \in \mathbb{R}$ . Then we have that:

(i)

$$\inf h(K) = \inf \Gamma_K(h)(K).$$

(ii) The set  $M$  of minimizers of  $\Gamma_K(h)$  over  $K$  equals the closed convex hull of the set  $\Omega(h, K)$  of generalized minimizers of  $h$  over  $K$ , i.e.,

$$M = \overline{\text{co}(\Omega(h, K))}.$$

*Proof.* The assertion (i) is a standard result. Indeed, by Definition 12.27,  $\Gamma_K(h) \leq h$  on  $K$  and thus

$$\inf \Gamma_K(h)(K) \leq \inf h(K).$$

The converse inequality is derived by restricting the supremum in Definition 12.27 to constant maps  $m$  from  $K$  to  $\mathbb{R}$  with  $k \leq m \leq h$ .

By Definition 12.27, we also observe that  $\Gamma_K(h)$  is a lower semi-continuous functional. This implies that the variational problem  $\inf \Gamma_K(h)(K)$  has minimizers and the set  $M = \Omega(\Gamma_K(h), K)$  of all minimizers of  $\Gamma_K(h)$  is compact. Moreover, again by Definition 12.27, the functional  $\Gamma_K(h)$  is convex which obviously yields the convexity of the set  $M$ .

For any  $y \in \Omega(h, K)$ , there is a net  $\{x_i\}_{i \in I} \subset K$  of approximating minimizers of  $h$  on  $K$  converging to  $y$ . In particular, since the functional  $\Gamma_K(h)$  is lower semi-continuous and  $\Gamma_K(h) \leq h$  on  $K$ , we have that

$$\Gamma_K(h)(y) \leq \liminf_I \Gamma_K(h)(x_i) \leq \lim_I h(x_i) = \inf h(K) = \inf \Gamma_K(h)(K),$$

i.e.,  $y \in M$ . As  $M$  is convex and compact we obtain that

$$(12.13) \quad M \supset \overline{\text{co}(\Omega(h, K))}.$$

So, we prove now the converse inclusion. We can assume without loss of generality that  $\overline{\text{co}(\Omega(h, K))} \neq K$  since there is otherwise nothing to prove. We show next that, for any  $x \in K \setminus \overline{\text{co}(\Omega(h, K))}$ , we have  $x \notin M$ .

As  $\overline{\text{co}(\Omega(h, K))}$  is a closed set of a locally convex space  $\mathcal{X}$ , for any  $x \in K \setminus \overline{\text{co}(\Omega(h, K))}$ , there is an open and convex neighborhood  $\mathcal{V}_x \subset \mathcal{X}$  of  $\{0\} \subset \mathcal{X}$  which is symmetric, i.e.,  $\mathcal{V}_x = -\mathcal{V}_x$ , and which satisfies

$$\mathcal{G}_x \cap [\{x\} + \mathcal{V}_x] = \emptyset$$

with

$$\mathcal{G}_x := K \cap \left[ \overline{\text{co}(\Omega(h, K))} + \mathcal{V}_x \right].$$

This follows from [1, Theorem 1.10] together with the fact that each neighborhood of  $\{0\} \subset \mathcal{X}$  contains some open and convex neighborhood of  $\{0\} \subset \mathcal{X}$  because  $\mathcal{X}$  is locally convex. Observe also that any one-point set  $\{x\} \subset \mathcal{X}$  is compact.

For any neighborhood  $\mathcal{V}_x$  of  $\{0\} \subset \mathcal{X}$  in a locally convex space, there is another convex, symmetric, and open neighborhood  $\mathcal{V}'_x$  of  $\{0\} \subset \mathcal{X}$  such that  $[\mathcal{V}'_x + \mathcal{V}'_x] \subset \mathcal{V}_x$ , see proof of [1, Theorem 1.10]. Let

$$\mathcal{G}'_x := K \cap \left[ \overline{\text{co}(\Omega(h, K))} + \mathcal{V}'_x \right].$$

Then the following inclusions hold:

$$(12.14) \quad \overline{\text{co}(\Omega(h, K))} \subset \mathcal{G}'_x \subset \overline{\mathcal{G}'_x} \subset \mathcal{G}_x \subset \overline{\mathcal{G}_x} \subset K \setminus \{x\}.$$

Since  $K$ ,  $\mathcal{V}_x$ ,  $\mathcal{V}'_x$ , and  $\overline{\text{co}(\Omega(h, K))}$  are all convex sets,  $\mathcal{G}_x$  and  $\overline{\mathcal{G}'_x}$  are also convex. Seen as subsets of  $K$  they are open neighborhoods of  $\overline{\text{co}(\Omega(h, K))}$ .

By Definition 12.7, the set  $\mathcal{X}$  is a Hausdorff space and thus any compact subset  $K$  of  $\mathcal{X}$  is a normal space. By Urysohn lemma, there is a continuous function

$$f_x : K \rightarrow [\inf h(K), \inf h(K \setminus \mathcal{G}'_x)]$$

satisfying  $f_x \leq h$  and

$$f_x(y) = \begin{cases} \inf h(K) & \text{for } y \in \overline{\mathcal{G}'_x}. \\ \inf h(K \setminus \mathcal{G}'_x) & \text{for } y \in K \setminus \mathcal{G}_x. \end{cases}$$

By compacticity of  $K \setminus \mathcal{G}'_x$  and the inclusion  $\Omega(h, K) \subset \mathcal{G}'_x$ , observe that

$$\inf h(K \setminus \mathcal{G}'_x) > \inf h(K).$$

Then we have per construction that

$$(12.15) \quad f_x(\overline{\text{co}(\Omega(h, K))}) = \{\inf h(K)\}$$

and

$$(12.16) \quad f_x^{-1}(\inf h(K)) = \Omega(f_x, K) \subset \mathcal{G}_x$$

for any  $x \in K \setminus \overline{\text{co}(\Omega(h, K))}$ .

We use now the  $\Gamma$ -regularization  $\Gamma_K(f_x)$  of  $f_x$  on the set  $K$  and denote by  $M_x = \Omega(\Gamma_K(f_x), K)$  its non-empty set of minimizers over  $K$ . Applying Theorem 12.34 for any  $y \in M_x$  we have a probability measure  $\mu_y \in M_1^+(K)$  on  $K$  with barycenter  $y$  such that

$$(12.17) \quad \Gamma_K(f_x)(y) = \int_K d\mu_y(z) f_x(z).$$

As  $y \in M_x$ , i.e.,

$$(12.18) \quad \Gamma_K(f_x)(y) = \inf \Gamma_K(f_x)(K) = \inf f_x(K),$$

we deduce from (12.17) that

$$\mu_y(\Omega(f_x, K)) = 1$$

and it follows that  $y \in \overline{\text{co}(\Omega(f_x, K))}$ , by Theorem 12.16. By (12.16) together with the convexity of the open neighborhood  $\mathcal{G}_x$  of  $\text{co}(\Omega(h, K))$ , we thus obtain

$$(12.19) \quad M_x \subset \overline{\text{co}(\Omega(f_x, K))} \subset \overline{\mathcal{G}_x}$$

for any  $x \in K \setminus \overline{\text{co}(\Omega(h, K))}$ .

We remark now that the inequality  $f_x \leq h$  on  $K$  yields  $\Gamma_K(f_x) \leq \Gamma_K(h)$  on  $K$  because of Corollary 12.30. As a consequence, it results from (i) and (12.15) that the set  $M$  of minimizers of  $\Gamma_K(h)$  over  $K$  is included in  $M_x$ , i.e.,  $M \subset M_x$ . Hence, by (12.14) and (12.19), we have the inclusions

$$(12.20) \quad M \subset \overline{\mathcal{G}_x} \subset K \setminus \{x\}.$$

Therefore, we combine (12.13) with (12.20) for all  $x \in K \setminus \overline{\text{co}(\Omega(h, K))}$  to obtain the desired equality in the assertion (ii). ■

This last theorem can be useful to analyze variational problems with non-convex functionals on compact convex sets  $K$ . Indeed, the minimization of a real functional  $h$  over  $K$  can be done in this case by analyzing a variational problem related to a lower semi-continuous convex functional  $\Gamma_K(h)$  for which many different methods of analysis are available.

To conclude, note that extreme points of the compact convex set  $M$  belongs to the set  $\Omega(h, K)$  and the non-convexity of  $\Omega(h, K)$  prevents the set  $M$  from being homeomorphic to the Poulsen simplex:

**THEOREM 12.38** (Minimization of real functionals – II).

*Let  $K$  be any (non-empty) compact convex subset of a locally convex space  $\mathcal{X}$  and  $h : K \rightarrow [k, \infty]$  be any real functional with  $k \in \mathbb{R}$ . Then we have that:*

(i) *Extreme points of the compact convex set  $M$  of minimizers of  $\Gamma_K(h)$  over  $K$  belong to the closure of the set of generalized minimizers of  $h$ , i.e.,  $\mathcal{E}(M) \subset \overline{\Omega(h, K)}$ .*

(ii) *If  $\mathcal{E}(M)$  is dense in  $M$  then  $\overline{\Omega(h, K)} = M$  is a compact and convex set.*

*Proof.* The first statement (i) results from Theorem 12.37 (ii) together with Theorem 12.13 (ii). The second assertion (ii) is also straightforward. Indeed, if  $\mathcal{E}(M)$  is dense in  $M$  then  $\Omega(h, K)$  is also dense in  $M$  as  $\mathcal{E}(M) \subset \Omega(h, K)$ , by (i). As a consequence,  $M = \overline{\Omega(h, K)}$ . ■

Therefore, if  $K$  is metrizable and  $\mathcal{E}(M)$  is dense in  $M$  then, by Lemma 12.36 together with Theorem 12.38 (ii),  $\Omega(h, K) = M$  is a compact and convex set.

## 12.6. The Legendre–Fenchel transform and tangent functionals

In contrast to the  $\Gamma$ -regularization defined in Section 12.5 the notion of Legendre–Fenchel transform requires the use of dual pairs defined as follow:

DEFINITION 12.39 (Dual pairs).

For any locally convex space  $(\mathcal{X}, \tau)$ , let  $\mathcal{X}^*$  be its dual space, i.e., the set of all continuous linear functionals on  $\mathcal{X}$ . Let  $\tau^*$  be any locally convex topology on  $\mathcal{X}^*$ .  $(\mathcal{X}, \mathcal{X}^*)$  is called a dual pair iff, for all  $x \in \mathcal{X}$ , the functional  $x^* \mapsto x^*(x)$  on  $\mathcal{X}^*$  is continuous w.r.t.  $\tau^*$ , and all linear functionals which are continuous w.r.t.  $\tau^*$  have this form.

By Theorem 12.8, a typical example of a dual pair  $(\mathcal{X}, \mathcal{X}^*)$  is given by any locally convex space  $(\mathcal{X}, \tau)$  and  $\mathcal{X}^*$  equipped with the  $\sigma(\mathcal{X}^*, \mathcal{X})$ -topology  $\tau^*$ , i.e., the weak\*-topology. In particular, as  $\mathcal{W}_1$  is a Banach space, by Corollary 12.9,  $(\mathcal{W}_1, \mathcal{W}_1^*)$  is a dual pair w.r.t. the norm and weak\*-topologies. We also observe that if  $(\mathcal{X}, \mathcal{X}^*)$  is a dual pair w.r.t.  $\tau$  and  $\tau^*$  then  $(\mathcal{X}^*, \mathcal{X})$  is a dual pair w.r.t.  $\tau^*$  and  $\tau$ .

The Legendre–Fenchel transform of a functional  $h$  on  $\mathcal{X}$  – also called the conjugate (functional) of  $h$  – is defined as follows:

DEFINITION 12.40 (The Legendre–Fenchel transform).

Let  $(\mathcal{X}, \mathcal{X}^*)$  be a dual pair. For any functional  $h : \mathcal{X} \rightarrow (-\infty, \infty]$ , its Legendre–Fenchel transform  $h^*$  is the convex lower semi-continuous functional from  $\mathcal{X}^*$  to  $(-\infty, \infty]$  defined, for any  $x^* \in \mathcal{X}^*$ , by

$$h^*(x^*) := \sup_{x \in \mathcal{X}} \{x^*(x) - h(x)\}.$$

If a functional  $h$  is only defined on a subset  $K \subset \mathcal{X}$  of a locally convex space  $\mathcal{X}$  then one uses Definition 12.28 to compute its Legendre–Fenchel transform  $h^*$ .

The Legendre–Fenchel transform and the  $\Gamma$ -regularization  $\Gamma_{\mathcal{X}}(h)$  of  $h$  are strongly related to one another. This can be seen in the next theorem which gives an important property – proven, for instance, in [47, Proposition 51.6] – of the double Legendre–Fenchel transform  $h^{**}$ , also called the biconjugate (functional) of  $h$ :

THEOREM 12.41 (Property of the biconjugate).

Let  $(\mathcal{X}, \mathcal{X}^*)$  be a dual pair and  $h : \mathcal{X} \rightarrow (-\infty, \infty]$  be any real functional. Then  $h^{**} \leq h$  and  $h^{**} \leq \hat{h} \leq h$  implies  $h^{**} = \hat{h}$  whenever  $\hat{h}$  is convex and lower semi-continuous.

By using Theorem 12.41 together with Proposition 12.29, we observe that  $h^{**}$  is thus equal to the  $\Gamma$ -regularization  $\Gamma_{\mathcal{X}}(h)$  of  $h$ :

COROLLARY 12.42 (Biconjugate and  $\Gamma$ -regularization of  $h$ ).

Let a dual pair  $(\mathcal{X}, \mathcal{X}^*)$  and  $h : \mathcal{X} \rightarrow (-\infty, \infty]$  be any real functional. Then  $h^{**} = \Gamma_{\mathcal{X}}(h)$  on  $\mathcal{X}$ .

Another important notion related to the Legendre–Fenchel transform is the concept of tangent functionals on real linear spaces:

DEFINITION 12.43 (Tangent functionals).

Let  $h$  be any real functional on a real linear space  $\mathcal{X}$ . A linear functional  $dh : \mathcal{X} \rightarrow (-\infty, \infty]$  is said to be tangent to the function  $h$  at  $x \in \mathcal{X}$  iff, for all  $x' \in \mathcal{X}$ ,  $h(x + x') \geq h(x) + dh(x')$ .

If  $\mathcal{X}$  is a separable real Banach space and  $h$  is convex and continuous then it is well-known that  $h$  has, on each point  $x \in \mathcal{X}$ , at least one continuous tangent functional  $dh \in \mathcal{X}^*$ . This is a crucial result coming from Mazur theorem [29] and Lanford III – Robinson theorem [23, Theorem 1]. Indeed, Mazur theorem describes the set  $\mathcal{Y}$  on which a continuous convex functional  $h$  is Fréchet differentiable, i.e., the set  $\mathcal{Y}$  where  $h$  has exactly one continuous tangent functional  $dh(x) \in \mathcal{X}^*$  at any  $x \in \mathcal{Y}$ :

THEOREM 12.44 (Mazur).

Let  $\mathcal{X}$  be a separable real Banach space and let  $h : \mathcal{X} \rightarrow \mathbb{R}$  be a continuous convex functional. The set  $\mathcal{Y} \subset \mathcal{X}$  of elements where  $h$  is Fréchet differentiable is residual, i.e., a countable intersection of dense open sets.

REMARK 12.45. By Baire category theorem, the set  $\mathcal{Y}$  is dense in  $\mathcal{X}$ .

Lanford III – Robinson theorem [23, Theorem 1] completes Mazur theorem by characterizing the set of continuous tangent functionals  $dh(x) \in \mathcal{X}^*$  for any  $x \in \mathcal{X}$ . In particular, there is at least one continuous tangent functional  $dh(x) \in \mathcal{X}^*$  at any  $x \in \mathcal{X}$ .

THEOREM 12.46 (Lanford III – Robinson).

Let  $\mathcal{X}$  be a separable real Banach space and let  $h : \mathcal{X} \rightarrow \mathbb{R}$  be a continuous convex functional. Then the set of tangent functionals  $dh(x) \in \mathcal{X}^*$  to  $h$ , at any  $x \in \mathcal{X}$ , is the weak\*-closed convex hull of the set  $\mathcal{Z}_x$ . Here, at fixed  $x \in \mathcal{X}$ ,  $\mathcal{Z}_x$  is the set of functionals  $x^* \in \mathcal{X}^*$  such that there is a net  $\{x_i\}_{i \in I}$  in  $\mathcal{Y}$  converging to  $x$  with the property that the unique tangent functional  $dh(x_i) \in \mathcal{X}^*$  to  $h$  at  $x_i$  converges towards  $x^*$  in the weak\*-topology.

The Legendre–Fenchel transform and the tangent functionals are also related to each other via the  $\Gamma$ -regularization of real functionals. Indeed, the  $\Gamma$ -regularization  $\Gamma_{\mathcal{X}}(h)$  of a real functional  $h$  allows to characterize all tangent functionals to  $h^*$  at the point  $x^* \in \mathcal{X}^*$  (see, e.g., [30, Theorem I.6.6]):

THEOREM 12.47 (Tangent functionals as minimizers).

Let  $(\mathcal{X}, \mathcal{X}^*)$  be a dual pair and  $h$  be any real functional from a (non-empty) convex subset  $K \subset \mathcal{X}$  to  $(-\infty, \infty]$ . Then the set  $T \subset \mathcal{X}$  of tangent functionals

to  $h^*$  at the point  $x^* \in \mathcal{X}^*$  is the (non-empty) set  $M$  of minimizers over  $K$  of the map

$$y \mapsto -x^*(y) + \Gamma_K(h)(y)$$

from  $K \subseteq \mathcal{X}$  to  $(-\infty, \infty]$ .

*Proof.* The proof is standard and simple, see, e.g., [30, Theorem I.6.6]. Indeed, by Definition 12.28, any tangent functional  $x \in \mathcal{X}$  to  $h^*$  at  $x^* \in \mathcal{X}$  satisfies the inequality:

$$(12.21) \quad x^*(x) + h^*(y^*) - y^*(x) \geq h^*(x^*)$$

for any  $y^* \in \mathcal{X}^*$ . Since  $\Gamma_K(h) = h^{**}$  and  $h^* = h^{***}$ , we have (12.21) iff

$$x^*(x) + \inf_{y^* \in \mathcal{X}^*} \{h^*(y^*) - y^*(x)\} = x^*(x) - \Gamma_K(h)(x) \geq \sup_{y \in \mathcal{X}} \{x^*(y) - \Gamma_K(h)(y)\}.$$

■

We combine Theorem 12.37 with Theorem 12.47 to characterize the set  $T \subset \mathcal{X}$  of tangent functionals to  $h^*$  at the point  $0 \in \mathcal{X}^*$  as the closed convex hull of the set  $\Omega(h, K)$  of generalized minimizers of  $h$  over a compact convex subset  $K$ , see Definition 12.35.

**COROLLARY 12.48** (Tangent functional and generalized minimizers).

Let  $(\mathcal{X}, \mathcal{X}^*)$  be a dual pair and  $h$  be any functional from a (non-empty) compact convex subset  $K \subseteq \mathcal{X}$  to  $[k, \infty]$  with  $k \in \mathbb{R}$ . Then the set  $T \subset \mathcal{X}$  of tangent functionals to  $h^*$  at the point  $0 \in \mathcal{X}^*$  is the set

$$T = M = \overline{\text{co}(\Omega(h, K))}$$

of minimizers of  $\Gamma_K(h)$  over  $K$ , see Theorem 12.37.

This last result has some similarity with Lanford III – Robinson theorem (Theorem 12.46) which has only been proven for separable real Banach spaces  $\mathcal{X}$  and continuous and convex functionals  $h : \mathcal{X} \rightarrow \mathbb{R}$ .

## 12.7. Two-person zero-sum games

A study of two-person zero-sum games belongs to any elementary book on game theory. These are defined via a map  $(x, y) \mapsto f(x, y)$  from the strategy set  $M \times N$  to  $\mathbb{R}$ . Here,  $M \subset \mathcal{X}$  and  $N \subset \mathcal{Y}$  are subsets of two topological vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . The value  $f(x, y)$  is the loss of the first player making the decision  $x$  and the gain of the second one making the decision  $y$ . Without exchange of information and by minimizing the functional

$$f^\#(x) := \sup_{y \in N} f(x, y)$$



the first player obtain her/his least maximum loss

$$F^\sharp := \inf_{x \in M} f^\sharp(x),$$

whereas the greatest minimum gain of the second player is

$$F^\flat := \sup_{y \in N} f^\flat(y) \quad \text{with} \quad f^\flat(y) := \inf_{x \in M} f(x, y).$$

$F^\flat$  and  $F^\sharp$  are called the *conservative values* of the game. The sets

$$\mathcal{C}^\sharp := \left\{ x \in M : F^\sharp = f^\sharp(x) \right\} \quad \text{and} \quad \mathcal{C}^\flat := \left\{ y \in N : F^\flat = f^\flat(y) \right\}$$

are the so-called set of *conservative strategies* and  $[F^\flat, F^\sharp]$  is the *duality interval*.

*Non-cooperative equilibria* (or Nash equilibria) [48, Definition 7.4.] of two-person zero-sum games are also called *saddle points*. They are defined as follows:

DEFINITION 12.49 (Saddle points).

Let  $M \subset \mathcal{X}$  and  $N \subset \mathcal{Y}$  be two subsets of topological vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . Then the element  $(x_0, y_0) \in M \times N$  is a saddle point of the real functional  $f : M \times N \rightarrow \mathbb{R}$  iff  $x_0 \in \mathcal{C}^\sharp$ ,  $y_0 \in \mathcal{C}^\flat$ , and  $F := F^\flat = F^\sharp$ .

It follows from this definition that a saddle point  $(x_0, y_0) \in M \times N$  satisfies  $F = f(x_0, y_0)$ . In this case  $F := F^\flat = F^\sharp$  is called the *value of the game*. As a sup and a inf do not generally commute we have in general  $F^\flat < F^\sharp$  and so, no saddle point of a two-person zero-sum game. An important criterion for the existence of saddle points is given by the von Neumann min-max theorem [48, Theorem 8.2]:

THEOREM 12.50 (von Neumann).

Let  $M \subset \mathcal{X}$  and  $N \subset \mathcal{Y}$  be two (non-empty) compact convex subsets of topological vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . Assume that  $f : M \times N \rightarrow \mathbb{R}$  is a real functional such that, for all  $y \in N$ , the map  $x \mapsto f(x, y)$  is convex and lower semi-continuous, whereas, for all  $x \in M$ , the map  $y \mapsto f(x, y)$  is concave and upper semi-continuous. Then there exists a saddle point  $(x_0, y_0) \in M \times N$  of  $f$ .

If the game ends up with a maximum loss  $F^\sharp$  for the first player then it means that the second player has full information on the choice of the first one. Indeed, the second player maximizes his gain  $f(x, y)$  knowing always the choice  $x$  of the first player. (Similar interpretations can of course be done if one gets  $F^\flat$  instead of  $F^\sharp$ .)

Another way to highlight this phenomenon can be done by introducing the so-called *decision rule*  $r : M \rightarrow N$ . Indeed, from [48, Proposition 8.7] we

have

$$(12.22) \quad F^\# = \sup_{r \in N^M} f^b(r(x)) = \sup_{r \in N^M} \inf_{x \in M} f(x, r(x))$$

with  $N^M$  being the set of all decision rules (functions from  $M$  to  $N$ ). It means that the second player is informed of the choice  $x$  of the first player and uses a decision rule to maximize his gain. Under stronger assumptions on the sets  $M$ ,  $N$  and on the map  $(x, y) \mapsto f(x, y)$  (cf. [48, Theorem 8.4]), observe that the second player can restrict himself on *continuous decision rules* only:

THEOREM 12.51 (Lasry).

Let  $M \subset \mathcal{X}$  and  $N \subset \mathcal{Y}$  be two subsets of topological vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$  such that  $M$  is compact and  $N$  is convex. Assume that  $f : M \times N \rightarrow \mathbb{R}$  is a real functional such that, for all  $y \in N$ , the map  $x \mapsto f(x, y)$  is lower semi-continuous, whereas, for all  $x \in M$ , the map  $y \mapsto f(x, y)$  is concave. Then

$$\inf_{x \in M} \sup_{y \in N} f(x, y) = \sup_{r \in C(M, N)} \inf_{x \in M} f(x, r(x))$$

with  $C(M, N)$  being the set of continuous mappings from  $M$  to  $N$ .

Equation (12.22) or Theorem 12.51 can be interpreted as an *extension* of the two-person zero-sum game *with exchange of information*. Extension of games are defined for instance in [33, Ch. 7, Section 7.2]. In the special case of two-person zero-sum games, saddle point may not exist, but such a non-cooperative equilibrium may appear by extending the strategy sets  $M$  or  $N$  (or both). This is, in fact, what we prove in Theorem 3.37 for the extended thermodynamic game.

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