COMMUTATION METHODS FOR SCHRÖDINGER OPERATORS WITH STRONGLY SINGULAR POTENTIALS

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Abstract. We explore the connections between singular Weyl–Titchmarsh theory and the single and double commutation methods. In particular, we compute the singular Weyl function of the commuted operators in terms of the original operator. We apply the results to spherical Schrödinger operators (also known as Bessel operators). We also investigate the connections with the generalized Bäcklund–Darboux transformation.

1. Introduction

The present paper is concerned with spectral theory for one-dimensional Schrödinger operators

$$H = -\frac{d^2}{dx^2} + q(x), \qquad x \in (a,b), \tag{1.1}$$

on the Hilbert space $L^2(a,b)$ with a real-valued potential $q \in L^1_{\text{loc}}(a,b)$. It has been shown recently by Gesztesy and Zinchenko [19], Fulton and Langer [10], [11], Kurasov and Luger [28] that, for a large class of singularities at a, it is still possible to define a singular Weyl function at the basepoint a. Furthermore, in previous work we have shown that this singular Weyl function shares many properties with the classical Weyl function [24] and established the connection with super singular perturbations for the special case of spherical Schrödinger operators

$$H = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \qquad x \in (0,\infty), \tag{1.2}$$

(also known as Bessel operators) [25].

On the other hand, commutation methods have played an important role in the theory of one-dimensional Schrödinger operators both as a method for inserting eigenvalues as well as for constructing solutions of the (modified) Korteweg–de Vries equation (see, e.g., [14] and the references therein). Historically, these methods of inserting eigenvalues go back to Jacobi [22] and Darboux [6] with decisive later contributions by Crum [5], Krein [26], Schmincke [34], and Deift [7]. Two particular methods turned out to be of special importance: The single commutation method, also called the Crum–Darboux method [5], [6] (actually going back at least to Jacobi [22]) and the double commutation method, to be found, e.g., in the seminal work of Gel'fand and Levitan [12]. For recent extensions of these methods we refer to [13], [16], [17], [35].

1991 *Mathematics Subject Classification.* Primary 34B20, 34L05; Secondary 34B24, 47A10.

Key words and phrases. Schrödinger operators, spectral theory, commutation methods, strongly singular potentials.

Research supported by the Austrian Science Fund (FWF) under Grant No. Y330.

Krein [26] was the first to realize the connection between inverse spectral problems and Crum's results. Namely, in [26], the connection between the spectral measures of the original and transformed operators was established and then exploited to characterize the spectral measures of Bessel operators in the case $l \in \mathbb{N}$ (see also [9]). This idea has been subsequently used by many authors: See, for instance, [2, 4, 7, 9, 20] and references therein.

The purpose of our present paper is to continue the work of Krein and establish the connection between the singular Weyl functions (and hence between the spectral measures) of the original and transformed operators for both the single and double commutation method. In particular, we will obtain an independent proof for the fact that the singular Weyl function of perturbed Bessel operators is a generalized Nevanlinna function. In addition, we investigate the connections with the generalized Bäcklund–Darboux transformation (GBDT) for a particular example. This method is a generalization of the double commutation method which it contains as a special case (cf. Remark 5.2).

2. Singular Weyl–Titchmash theory

We begin by recalling a few facts from [24]. To set the stage, we will consider one-dimensional Schrödinger operators on $L^2(a,b)$ with $-\infty \le a < b \le \infty$ of the form

$$\tau = -\frac{d^2}{dx^2} + q, \tag{2.1}$$

where the potential q is real-valued satisfying

$$q \in L^1_{loc}(a,b). \tag{2.2}$$

We will use τ to denote the formal differential expression and H to denote a corresponding self-adjoint operator given by τ with separated boundary conditions at a and/or b.

If a (resp. b) is finite and q is in addition integrable near a (resp. b), we will say a (resp. b) is a *regular* endpoint. We will say τ, respectively H, is *regular* if both a and b are regular.

We will choose a point $c \in (a,b)$ and also consider the operators $H^D_{(a,c)}$, $H^D_{(c,b)}$ which are obtained by restricting H to (a,c), (c,b) with a Dirichlet boundary condition at c, respectively. The corresponding operators with a Neumann boundary condition will be denoted by $H^N_{(a,c)}$ and $H^N_{(c,b)}$.

Moreover, let $c(z,x)$, $s(z,x)$ be the solutions of $\tau u = z\, u$ corresponding to the initial conditions $c(z,c) = 1$, $c'(z,c) = 0$ and $s(z,c) = 0$, $s'(z,c) = 1$.

Define the Weyl m-functions (corresponding to the base point c) such that

$$\begin{aligned} u_-(z,x) &= c(z,x) - m_-(z) s(z,x), \qquad z \in \mathbb{C} \setminus \sigma(H^D_{(a,c)}),\\ u_+(z,x) &= c(z,x) + m_+(z) s(z,x), \qquad z \in \mathbb{C} \setminus \sigma(H^D_{(c,b)}), \end{aligned} \tag{2.3}$$

are square integrable near a, b and satisfiy the boundary condition of H at a, b (if any), respectively. The solutions $u_\pm(z,x)$ (as well as their multiples) are called Weyl solutions at a, b. For further background we refer to [37, Chap. 9] or [38].

To define an analogous singular Weyl m-function at the, in general singular, endpoint a we will first need the analog of the system of solutions $c(z,x)$ and $s(z,x)$. Hence our first goal is to find a system of entire solutions $\theta(z,x)$ and $\phi(z,x)$ such that $\phi(z,x)$ lies in the domain of H near a and such that the Wronskian

$W(\theta(z), \phi(z)) = 1$. To this end we start with a hypothesis which will turn out necessary and sufficient for such a system of solutions to exist.

Hypothesis 2.1. *Suppose that the spectrum of $H^D_{(a,c)}$ is purely discrete for one (and hence for all) $c \in (a, b)$.*

Note that this hypothesis is for example satisfied if $q(x) \to +\infty$ as $x \to a$ (cf. Problem 9.7 in [37]).

Lemma 2.2 ([24])**.** *Suppose Hypothesis 2.1. Then there exists a fundamental system of solutions $\phi(z, x)$ and $\theta(z, x)$ of $\tau u = zu$ which are real entire with respect to z such that*

$$W(\theta(z), \phi(z)) = 1 \tag{2.4}$$

and $\phi(z, .)$ is in the domain of H near a. Here $W_x(u, v) = u(x)v'(x) - u'(x)v(x)$ is the usual Wronski determinant.

It is important to point out that such a system is not unique and any other such sytem is given by

$$\tilde{\theta}(z, x) = \mathrm{e}^{-g(z)}\theta(z, x) - f(z)\phi(z, x), \qquad \tilde{\phi}(z, x) = \mathrm{e}^{g(z)}\phi(z, x),$$

where $g(z)$, $f(z)$ are real entire functions.

Given a system of real entire solutions $\phi(z, x)$ and $\theta(z, x)$ as in the above lemma we can define the singular Weyl function

$$M(z) = -\frac{W(\theta(z), u_+(z))}{W(\phi(z), u_+(z))} \tag{2.5}$$

such that the solution which is in the domain of H near b (cf. (2.3)) is given by

$$u_+(z, x) = a(z)\big(\theta(z, x) + M(z)\phi(z, x)\big), \tag{2.6}$$

where $a(z) = -W(\phi(z), u_+(z)) = \beta(z) - m_+(z)\alpha(z)$. By construction we obtain that the singular Weyl function $M(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies $M(z) = M(z^*)^*$. Rather than $u_+(z, x)$ we will use

$$\psi(z, x) = \theta(z, x) + M(z)\phi(z, x). \tag{2.7}$$

Recall also from [24, Lem. 3.2] that associated with $M(z)$ is a corresponding spectral measure

$$\frac{1}{2}\left(\rho\big((x_0, x_1)\big) + \rho\big([x_0, x_1]\big)\right) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{x_0}^{x_1} \mathrm{Im}\big(M(x + \mathrm{i}\varepsilon)\big)dx. \tag{2.8}$$

3. Connection with the single commutation method

3.1. **Preliminary basic results.** We begin by recalling a few basic facts from the single commutation method. Let A be a closed operator and recall that $H = A^*A$ is a self-adjoint operator with $\ker(H) = \ker(A)$. Similarly, $\hat{H} = AA^*$ is a self-adjoint operator with $\ker(\hat{H}) = \ker(A^*)$. Then the key observation is the following well-known result (see e.g. [37, Thm. 8.6] for a short proof):

Theorem 3.1 ([7])**.** *Let A be a densely defined closed operator and introduce $H = A^*A$, $\hat{H} = AA^*$. Then the operators $H\big|_{\ker(H)^\perp}$ and $\hat{H}\big|_{\ker(\hat{H})^\perp}$ are unitarily equivalent.*

If $H\psi = E\psi$, $\psi \in \mathfrak{D}(H)$, then $\hat{\psi} = A\psi \in \mathfrak{D}(\hat{H})$ with $\hat{H}\hat{\psi} = E\hat{\psi}$ and $\|\hat{\psi}\| = \sqrt{E}\|\psi\|$. Moreover,

$$R_{\hat{H}}(z) \supseteq \frac{1}{z}\left(AR_H(z)A^* - \mathbb{1}\right), \quad R_H(z) \supseteq \frac{1}{z}\left(A^* R_{\hat{H}}(z)A - \mathbb{1}\right), \tag{3.1}$$

where $R_H(z) = (H - z)^{-1}$ denotes the resolvent of an operator H.

3.2. **Application to Schrödinger operators.** In order to find such a factorization for a given Schrödinger operator H one requires a positive solution of the underlying differential equation. Existence of such a solution is equivalent to semiboundedness of H and we will thus make the following assumption:

Hypothesis 3.2. *In addition to Hypothesis 2.1 assume that H is bounded from below and limit point at b. Let $\lambda \in \mathbb{R}$ be such that $H - \lambda \geq 0$.*

In particular, let $\phi(z,x)$, $\theta(z,x)$ be a fundamental system of solutions as in Lemma 2.2 and recall (cf. [37, Lem. 9.7]) that the Green's function of H is given by

$$G(z,x,y) = \phi(z,x)\psi(z,y), \qquad x \leq y. \tag{3.2}$$

Moreover, if $H - \lambda \geq 0$ we must have $\phi(\lambda,x) > 0$ (as well as $\psi(\lambda,x) > 0$) for $x \in (a,b)$ possibly after flipping signs by [15, Cor. 2.4].

Now consider the operator.

$$\begin{aligned} A_\phi f = a_\phi f, \qquad a_\phi = -\frac{d}{dx} + \frac{\phi'(\lambda,x)}{\phi(\lambda,x)},\\ \mathfrak{D}(A_\phi) = \{f \in L^2(a,b) | f \in AC_{loc}(a,b),\ a_\phi f \in L^2(a,b)\}. \end{aligned} \tag{3.3}$$

Here we use A_ϕ and a_ϕ for the operator and differential expression, respectively. It is straightforward to check (cf. [37, Problem 9.3]) that A_ϕ is closed and that its adjoint is given by

$$\begin{aligned} A_\phi^* f = a_\phi^* f, \qquad a_\phi^* = \frac{d}{dx} + \frac{\phi'(\lambda,x)}{\phi(\lambda,x)},\\ \mathfrak{D}(A_\phi^*) = \{f \in L^2(a,b) | f \in AC_{loc}(a,b),\ a_\phi^* f \in L^2(a,b),\\ \lim_{x\to a,b} f(x)g(x) = 0, \forall g \in \mathfrak{D}(A_\phi)\}. \end{aligned} \tag{3.4}$$

If we also have $\theta(\lambda,x) > 0$ we can also define A_θ by using $\theta(\lambda,x)$ in place of $\phi(\lambda,x)$.

Lemma 3.3. *Assume Hypothesis 3.2. Then $H - \lambda = A_\phi^* A_\phi$. If in addition $\theta(\lambda,x) > 0$ and τ is limit point at a, then we also have $H - \lambda = A_\theta^* A_\theta$.*

Proof. It is simple algebra to check that H and $A_\phi^* A_\phi$ agree on functions with compact support in (a,b) hence $A_\phi^* A_\phi$ is a self-adjoint extension of the minimal operator associated with τ and it remains to identify the boundary conditions. Since by assumption τ is limit point at b we only need to consider a. Moreover, since $\phi(\lambda,x)$ is in the domain of $A_\phi^* A_\phi$ near a by our choice of A, both $A_\phi^* A_\phi$ and H are associated with the boundary condition generated by $\phi(\lambda,x)$ if τ is limit circle at a.

The case of A_θ is even simpler since by our limit point assumption there is no need to identify any boundary conditions. □

The commuted operator $\hat{H} - \lambda = A_\phi A_\phi^*$ is associated with the potential

$$\hat{q}(x) = q(x) - 2\frac{d}{dx}\frac{\phi'(\lambda,x)}{\phi(\lambda,x)}. \tag{3.5}$$

Moreover, it is straightforward to check that if $u(z)$ solves $\tau u = zu$ then

$$\hat{u}(z,x) = a_\phi u(z,x) = -\frac{W_x(\phi(\lambda), u(z))}{\phi(\lambda,x)} \tag{3.6}$$

solves $\hat{\tau}\hat{u} = z\hat{u}$ and given two solutions $u(z)$ and $v(z)$ we have

$$W(\hat{u}(z), \hat{v}(z)) = (z-\lambda) W(u(z), v(z)). \tag{3.7}$$

Similarly, $\check{H} - \lambda = A_\theta A_\theta^*$ is associated with the potential

$$\check{q}(x) = q(x) - 2\frac{d}{dx}\frac{\theta'(\lambda,x)}{\theta(\lambda,x)} \tag{3.8}$$

and

$$\check{u}(z,x) = -a_\theta u(z,x) = \frac{W_x(\theta(\lambda), u(z))}{\theta(\lambda,x)} \tag{3.9}$$

solves $\check{\tau}\check{u} = z\check{u}$ and given two solutions $u(z)$ and $v(z)$ we have

$$W(\check{u}(z), \check{v}(z)) = (z-\lambda) W(u(z), v(z)). \tag{3.10}$$

Theorem 3.4. *Assume Hypothesis 3.2. Then the operator $\hat{H} = A_\phi A_\phi^* + \lambda$ has an entire system of solutions*

$$\hat{\phi}(z,x) := \frac{1}{z-\lambda} a_\phi \phi(z,x) = -\frac{W_x(\phi(\lambda), \phi(z))}{(z-\lambda)\phi(\lambda,x)} = \frac{\int_a^x \phi(\lambda,y)\phi(z,y)dy}{\phi(\lambda,x)}, \tag{3.11}$$

$$\hat{\theta}(z,x) := a_\phi \theta(z,x) = -\frac{W_x(\phi(\lambda), \theta(z))}{\phi(\lambda,x)}, \tag{3.12}$$

which satisfy $W(\hat{\theta}(z), \hat{\phi}(z)) = 1$.

Furthermore, the Weyl solutions of $\hat{H}$ are given by

$$\hat{\phi}(z,x), \qquad \hat{\psi}(z,x) = a_\phi \psi(z,x) = -\frac{W_x(\phi(\lambda), \psi(z))}{\phi(\lambda,x)} = \hat{\theta}(z,x) + \hat{M}(z)\hat{\phi}(z,x), \tag{3.13}$$

where

$$\hat{M}(z) = (z-\lambda) M(z) \tag{3.14}$$

is the singular Weyl function of $\hat{H}$ corresponding the above system of solutions. Moreover, the associated spectral measures are related via

$$d\hat{\rho}(x) = (x-\lambda) d\rho(x). \tag{3.15}$$

Proof. The first part is easy to check using (3.7). To see

$$W_x(\phi(\lambda), \phi(z)) = (z-\lambda)\int_a^x \phi(\lambda,y)\phi(z,y)dy \tag{3.16}$$

note that both sides have the same derivative and are both equal 0 in the limit $x \to a$.

For the second part we set $\lambda = 0$ and fix $z \in \mathbb{C}\backslash\mathbb{R}$ without loss of generality. Moreover we will abbreviate $\phi(\lambda,x) = \phi_0(x)$, $\phi(z,x) = \phi(x)$, and $\psi(z,x) = \psi(x)$.

Now let f be some absolutely continuous function with compact support in (a,b) and $f' \in L^2$. Then invoking (3.1) one computes using (2.4), (3.11), (3.12), and integration by parts that

$$\begin{aligned}
R_{\hat{H}}(z)f(x) =& \frac{1}{z}\left(A_\phi \int_a^b G(z,x,y) A_\phi^* f(y)dy - f(x)\right)\\
=& \frac{1}{z}\left(a_\phi\Big[\psi(x)\int_a^x \phi(y)(a_\phi^* f)(y)dy\Big] + a_\phi\Big[\phi(x)\int_x^b \psi(y)(a_\phi^* f)(y)dy\Big] - f(x)\right)\\
=& \frac{1}{z}\Big((a_\phi\psi)(x)\Big(\phi(x)f(x) + \int_a^x f(y)(a_\phi\phi)(y)dy\Big)\\
&+(a_\phi\phi)(x)\Big(-\psi(x)f(x) + \int_x^b f(y)(a_\phi\psi)(y)dy\Big) - f(x)\Big)\\
=& \frac{1}{z}\Big(z\hat{\psi}(x)\int_a^x f(y)\hat{\phi}(y)dy + z\hat{\phi}(x)\int_x^b f(y)\hat{\psi}(y)dy\\
&+\hat{\psi}(z)\phi(x)f(x) - z\hat{\phi}(x)\psi(x)f(x) - f(x)\Big)\\
=& \hat{\psi}(x)\int_a^x \hat{\phi}(y)f(y)dy + \hat{\phi}(x)\int_x^b \hat{\psi}(y)f(y)dy = \int_a^b \hat{G}(z,x,y)f(y)dy,
\end{aligned}$$

which shows that

$$\hat{G}(z,x,y) = \hat{\phi}(z,x)\hat{\psi}(z,y), \qquad x \le y, \tag{3.17}$$

is the Green's function of $\hat{H}$ since the set of f under consideration is dense. □

Note

$$\hat{\phi}(\lambda,x) = -\frac{W_x(\phi(\lambda),\dot{\phi}(\lambda))}{\phi(\lambda,x)} = \frac{\int_a^x \phi(\lambda,y)^2 dy}{\phi(\lambda,x)}, \qquad \hat{\theta}(\lambda,x) = \frac{1}{\phi(\lambda,x)}, \tag{3.18}$$

where the dot denotes differentiation with respect to z.

Remark 3.5. *A few remarks are in order:*

(i) *If τ is regular at a and $\theta(z,x)$, $\phi(z,x)$ are chosen to satisfy the boundary conditions*

$$\theta(z,a) = \cos(\alpha),\ \theta'(z,a) = -\sin(\alpha),$$
$$\phi(z,a) = \sin(\alpha),\ \phi'(z,a) = \cos(\alpha),$$

then $\hat{\tau}$ will be again regular if and only if $\sin(\alpha) \neq 0$. Moreover, in this case we have

$$\hat{\theta}(z,a) = \sin(\alpha)^{-1},\ \hat{\theta}'(z,a) = \cos(\alpha)\left(z - \lambda - 1 - \cot(\alpha)^2\right),$$
$$\hat{\phi}(z,a) = 0,\ \hat{\phi}'(z,a) = \sin(\alpha).$$

(ii) *If $\phi(\lambda,x)$ is the principal positive solution near a (i.e., if H is the Friedrich's extension of τ, see* [29], [18]*), that is,*

$$\int_a^c \frac{dx}{\phi(\lambda,x)^2} = \infty, \tag{3.19}$$

then $\hat{\tau}$ will be limit point at a.

(iii) *Note that $\hat{H}$ has no eigenvalue at λ: $\sigma_p(\hat{H}) \cap \{\lambda\} = \emptyset$.*

Recall that N_κ^∞ is the subclass of N_κ consisting of generalized Nevanlinna functions with κ negative squares and having no nonreal poles and the only generalized pole at infinity (for further definitions and details we refer e.g. [27], [11, §2.2–2.3], [24, App. C]).

Corollary 3.6. *$M \in N_\kappa^\infty$ for some $\kappa \in \mathbb{N}_0$ if and only if $\hat{M} \in N_{\tilde\kappa}^\infty$ with $\tilde\kappa \in \{\kappa, \kappa+1\}$. Moreover, $\tilde\kappa = \kappa$ if and only if*

$$\lim_{y\uparrow\infty} \frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa}} \in [0,\infty),$$

and $\tilde\kappa = \kappa+1$ if and only if

$$\lim_{y\uparrow\infty} \frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa}} \in [-\infty,0).$$

Proof. The claim is straightforward from the following characterization of N_κ^∞–functions: Given a generalized Nevanlinna function M in N_κ^∞, the corresponding κ is given by the multiplicity of the generalized pole at ∞ which is determined by the facts that the following limits exist and take values as indicated:

$$\lim_{y\uparrow\infty} -\frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa-1}} \in (0,\infty], \qquad \lim_{y\uparrow\infty} \frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa+1}} \in [0,\infty).$$

□

Similarly, we obtain

Theorem 3.7. *Assume Hypothesis 3.2, τ is limit point at a, and let $\theta(\lambda,x)>0$. The operator $\check{H} = A_\theta A_\theta^* - \lambda$ has an entire system of solutions*

$$\check\phi(z,x) := -a_\theta\phi(z,x) = \frac{W_x(\theta(\lambda),\phi(z))}{\theta(\lambda,x)}, \tag{3.20}$$

$$\check\theta(z,x) := -\frac{1}{z-\lambda}a_\theta\theta(z,x) = \frac{W_x(\theta(\lambda),\theta(z))}{(z-\lambda)\theta(\lambda,x)}, \tag{3.21}$$

which satisfy $W(\check\theta(z),\check\phi(z))=1$.

Furthermore, the Weyl solutions of $\check{H}$ are given by

$$\check\phi(z,x), \qquad \check\psi(z,x) = -a_\theta\psi(z,x) = \frac{W_x(\theta(\lambda),\psi(z))}{(z-\lambda)\theta(\lambda,x)} = \check\theta(z,x)+\check{M}(z)\check\phi(z,x), \tag{3.22}$$

where

$$\check{M}(z) = (z-\lambda)^{-1}M(z) \tag{3.23}$$

is the singular Weyl function of $\check{H}$. The associated spectral measures are related via

$$d\check\rho(x) = (x-\lambda)^{-1}d\rho(x) - M(\lambda)d\Theta(x-\lambda), \tag{3.24}$$

where $\Theta(x)=0$ for $x<0$ and $\Theta(x)=1$ for $x\ge 0$ is the usual step function. Here $M(\lambda) = \lim_{\varepsilon\downarrow 0} M(\lambda-\varepsilon)$.

Note

$$\check\phi(\lambda,x) = \frac{1}{\theta(\lambda,x)}, \quad \check\theta(\lambda,x) = \frac{W_x(\theta(\lambda),\dot\theta(\lambda))}{\theta(\lambda,x)}. \tag{3.25}$$

Remark 3.8. *Again a few remarks are in order:*

(i) *If $\theta(\lambda,x)$ is the principal positive solution near b, that is,*

$$\int_c^b \frac{dx}{\theta(\lambda,x)^2} = \infty, \tag{3.26}$$

then $\check{\tau}$ will be limit point at a. (Note that the principal positive solution near a is $\phi(\lambda,x)$ since we assumed the limit point case at a.)

(ii) *Note that $\check{H}$ has an eigenvalue at λ unless $\theta(\lambda,x)$ is the principal positive solution near b.*

(iii) *Note that factorizing $\hat{H}$ using $\hat{\theta}(\lambda,x) = \phi(\lambda,x)^{-1}$ will transform $\hat{H}$ back into H. In particular, $\check{\hat{\phi}}(z,x) = \phi(z,x)$ and $\check{\hat{\theta}}(z,x) = \theta(z,x)$.*

(iv) *Clearly this procedure can be iterated and we refer (e.g.) to Appendix A of* [16] *for the well-known formulas.*

3.3. **Example: The Coulomb Hamiltonian.** We can apply the single commutation method to the Coulomb Hamiltonian

$$H_l = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} - \frac{\gamma}{x}, \qquad l \in \mathbb{N}_0, \tag{3.27}$$

by setting

$$A_l f = -\frac{d}{dx} + \frac{l+1}{x} + \frac{\gamma}{2(l+1)}, \quad A_l^* f = \frac{d}{dx} + \frac{l+1}{x} + \frac{\gamma}{2(l+1)}, \tag{3.28}$$

which gives ([37, Thm. 10.10])

$$H_l = A_l^* A_l - c_l^2, \qquad H_{l+1} = A_l A_l^* - c_l^2, \tag{3.29}$$

where

$$c_l = \frac{\gamma}{2(l+1)}. \tag{3.30}$$

In particular, the singular Weyl function is given by

$$M_l(z) = M_0(z) \prod_{k=0}^{l-1} (z - c_k^2), \tag{3.31}$$

where the singular Weyl function $M_0(z)$ of H_0 is a Herglotz–Nevanlinna function since H_0 is limit circle at $a = 0$.

Remark 3.9. *The singular Weyl function for this case was explicitly computed in* [11, eq. (5.11)]. *Moreover, the fact that it can be reduced to the case $\gamma = 0$ via the above product formula was also first observed in* [11, Lemma 5.1]. *In the special case $\gamma = 0$ we obtain $M_l(z) = \sqrt{-z} z^l$ (cf. Section 5 in* [24]*).*

3.4. **Application to perturbed Bessel operators.** Next we want to apply these results to perturbed spherical Schrödinger equations which have attracted considerable interest in the past [1], [4], [21] [23], [25]. In particular, we want to mention [2], where commutation techniques were used to transfer results for $l = 0$ to $l \in \mathbb{N}_0$.

Lemma 3.10. *Fix $l \geq -\frac{1}{2}$ and $p \in [1, \infty]$. Suppose*

$$H = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \quad x \in (0, b), \tag{3.32}$$

where

$$\begin{cases} xq(x) \in L^p(0,c), & p \in (1,\infty], l \geq -\frac{1}{2}, \\ xq(x) \in L^1(0,c), & p = 1, l > -\frac{1}{2}, \\ x(1-\log(x))q(x) \in L^1(0,c), & p = 1, l = -\frac{1}{2}, \end{cases} \tag{3.33}$$

for some $c \in (0,b)$. *In addition, suppose* H *is bounded from below. If* τ *is limit circle at* $a = 0$ *we impose the usual boundary condition*

$$\lim_{x\to 0} x^l((l+1)f(x) - xf'(x)) = 0, \qquad l \in [-\frac{1}{2}, \frac{1}{2}). \tag{3.34}$$

Then

$$\hat{H} = -\frac{d^2}{dx^2} + \frac{(l+1)(l+2)}{x^2} + \hat{q}(x), \quad x\hat{q} \in L^p(0,c) \tag{3.35}$$

and if $l \geq \frac{1}{2}$ *then*

$$\check{H} = -\frac{d^2}{dx^2} + \frac{(l-1)l}{x^2} + \check{q}(x), \quad x\check{q} \in L^p(0,c). \tag{3.36}$$

Proof. It suffices to observe that by [25, Lem. 3.2, Cor. 3.4]

$$\phi(\lambda, x) = x^{l+1} u_1(\lambda, x),$$

where $u_1(\lambda, x)$, $xu_1'(\lambda, x) \in W^{1,p}(0,c)$, and $\lim_{x\to 0} xu_1'(x,\lambda) = 0$. Noting that $\phi(\lambda, x) > 0$ for all $x \in [0,b)$, by (3.5) we get

$$x\hat{q}(x) = xq(x) - 2x\frac{d}{dx}\frac{u_1'(\lambda,x)}{u_1(\lambda,x)} \in L^p(0,c).$$

Similarly, to prove (3.36) it suffices to note that

$$\theta(\lambda, x) = x^{-l} u_2(\lambda, x),$$

where $u_2(\lambda, x)$, $xu_2'(\lambda, x) \in W^{1,p}(0,c)$ and $\lim_{x\to 0} xu_2'(x,\lambda) = 0$. □

Since H is limit circle for $l \in [-\frac{1}{2}, \frac{1}{2})$ and thus the singular Weyl function is a Herglotz–Nevanlinna function in this case (cf. [24, App. A]), we obtain by induction:

Corollary 3.11. *Suppose*

$$H = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \qquad xq(x) \in L^p, \tag{3.37}$$

where $p \in [1,\infty]$ *if* $l + \frac{1}{2} \notin \mathbb{N}_0$ *and* $p \in (1,\infty]$ *if* $l + \frac{1}{2} \in \mathbb{N}_0$. *Then there is a singular Weyl function of the form*

$$M(z) = (z-\lambda)^{\lfloor l+1/2 \rfloor} M_0(z), \tag{3.38}$$

where $M_0(z)$ *is a Herglotz–Nevanlinna function. The corresponding spectral measure is given by*

$$d\rho(x) = (x-\lambda)^{\lfloor l+1/2 \rfloor} d\rho_0(x), \tag{3.39}$$

where the measure ρ_0 *satisfies* $\int_{\mathbb{R}} d\rho_0(x) = \infty$ *and* $\int_{\mathbb{R}} \frac{d\rho_0(x)}{1+x^2} < \infty$.

Corollary 3.12 ([24, 25])**.** *Assume the conditions of Corollary 3.11. Then there is a singular Weyl function from the class* N_κ^∞ *with* $\kappa = \lfloor \frac{l}{2} + \frac{3}{4} \rfloor$.

Proof. The inequality $\kappa \le \lfloor \frac{l}{2} + \frac{3}{4} \rfloor$ follows after combining Corollary 3.11 with Theorem 4.2 from [24].

It remains to show $\kappa \ge \lfloor \frac{l}{2} + \frac{3}{4} \rfloor$. To this end denote by $M_0(z)$ the Weyl function of the operator H with $l \in [-1/2, 1/2)$. Then, since H is the Friedrichs extension of the minimal operator in this case, $M_0(z)$ satisfies (cf. [8, Proposition 4])

$$M_0(z) \to -\infty, \quad \text{as} \quad z \to -\infty. \tag{3.40}$$

Moreover, by [24, Cor. A.9] and Hypothesis 3.2, $M_0(z)$ admits the following representation

$$M_0(z) = \mathrm{Re}(M_0(\mathrm{i})) + \int_\lambda^{+\infty} \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\rho_0(x), \quad z \notin [\lambda, +\infty),$$

where the measure satisfies $\int_\lambda^{+\infty} d\rho_0(x) = \infty$ and $\int_\lambda^{+\infty} (1+x^2)^{-1} d\rho_0(x) < \infty$. Hence, by (3.40) we conclude that

$$\int_\lambda^{+\infty} \frac{d\rho_0(x)}{1+|x|} = \infty. \tag{3.41}$$

Combining (3.41) with Corollary 3.11 and Theorem 4.2 from [24], we arrive at the desired inequality. □

Remark 3.13. *Corollary 3.12 was first established by Fulton and Langer* [11] *in the case when the potential $q(x)$ is analytic in a neighborhood of $x = 0$ (see also* [28]*). In the general case, it was proven in* [25] *(see also* [24]*). Namely, by using high energy asymptotic of $\phi(z,x)$ it was shown in* [24] *that $\kappa \le \lceil \frac{l+1}{2} \rceil$ (for the details see Section 8 in* [24]*). The equality $\kappa = \lfloor \frac{l}{2} + \frac{3}{4} \rfloor$ was proven in* [25] *with the help of theory of super singular perturbations and detailed analysis of solutions $\phi(z,x)$ and $\theta(z,x)$.*

4. Connection with the double commutation method

In this section we want to look at the effect of the double commutation method. We refer to [13], [17] for further background of this method. We will use the approach from [17] with the only difference that we include the limiting case $\gamma = \infty$ (we omit the necessary minor modifications of the proofs of [17], cf. [36, Sect. 11.6]).

Let H together with a fundamental system of solutions $\phi(z,x)$, $\theta(z,x)$ as in Lemma 2.2 be given.

Hypothesis 4.1. *Assume Hypothesis 2.1. Let $\gamma \in (0, \infty]$ and $\lambda \in \mathbb{R}$ such that $\phi(\lambda, x)$ satisfies the boundary condition at b if τ is limit circle at b (i.e., λ is an eigenvalue with eigenfunction $\phi(\lambda, x)$ if τ is limit circle at b).*

Introduce

$$\tilde{\phi}_\gamma(\lambda, x) = \frac{\phi(\lambda, x)}{\gamma^{-1} + \int_a^x \phi(\lambda, y)^2 dy}. \tag{4.1}$$

Denote by P and P_γ the orthogonal projections onto the one-dimensional spaces spanned by $\phi(\lambda, x)$ and $\tilde{\phi}_\gamma(\lambda, x)$, respectively. Here we set the projection equal to zero if the function is not in $L^2(a,b)$. Note that $\tilde{\phi}_\gamma(\lambda, .) \in L^2(a,b)$ if and only if $\gamma < \infty$ ([17, Lem. 2.1]).

By [17, Lem. 2.1] the transformation

$$U_\gamma f(x) = f(x) - \tilde{\phi}_\gamma(\lambda, x) \int_a^x \phi(\lambda, y) f(y) dy \tag{4.2}$$

is a unitary map from $(1-P)L^2(a,b)$ onto $(1-P_\gamma)L^2(a,b)$. Moreover, by [17, Thm. 3.2]

$$H_\gamma(1-P_\gamma) = U_\gamma H U_\gamma^{-1}(1-P_\gamma), \tag{4.3}$$

where the operator H_γ is associated with the potential

$$q_\gamma(x) = q(x) - 2\frac{d^2}{dx^2} \log\left(\gamma^{-1} + \int_a^x \phi(\lambda, y)^2 dy\right) \tag{4.4}$$

and boundary conditions (if any)

$$W_a(\tilde{\phi}_\gamma(\lambda), f) = W_b(\tilde{\phi}_\gamma(\lambda), f) = 0. \tag{4.5}$$

Note that for $\gamma < \infty$ the operator H_γ is limit circle at a if and only H is and that for $\gamma = \infty$ the operator H_∞ is always limit point at b.

Theorem 4.2. *Assume Hypothesis 4.1 and let $\gamma < \infty$. The operator H_γ has an entire system of solutions*

$$\begin{aligned} \phi_\gamma(z,x) &= \phi(z,x) - \tilde{\phi}_\gamma(\lambda, x) \int_a^x \phi(\lambda, y)\phi(z,y) dy \\ &= \phi(z,x) + \frac{1}{z-\lambda} \tilde{\phi}_\gamma(\lambda, x) W_x(\phi(\lambda), \phi(z)), \end{aligned} \tag{4.6}$$

$$\theta_\gamma(z,x) = \theta(z,x) + \frac{1}{z-\lambda}\left(\tilde{\phi}_\gamma(\lambda,x) W_x(\phi(\lambda), \theta(z)) + \gamma\phi_\gamma(z,x)\right). \tag{4.7}$$

which satisfy $W(\theta_\gamma(z), \phi_\gamma(z)) = 1$.

Furthermore, the Weyl solutions of H_γ are given by

$$\begin{aligned} \phi_\gamma(z,x), \quad \psi_\gamma(z,x) &= \psi(z,x) + \frac{1}{z-\lambda}\tilde{\phi}_\gamma(\lambda,x) W_x(\phi(\lambda), \psi(z)) \\ &= \theta_\gamma(z,x) + M_\gamma(z)\phi_\gamma(z,x), \end{aligned} \tag{4.8}$$

where

$$M_\gamma(z) = M(z) - \frac{\gamma}{z-\lambda}$$

is the singular Weyl function of H_γ. The associated spectral measures are related via

$$d\rho_\gamma(x) = d\rho(x) + \gamma d\Theta(x-\lambda). \tag{4.9}$$

Proof. It is straightforward to check that $\phi_\gamma(z,x)$, $\theta_\gamma(z,x)$ is an entire system of solutions whose Wronskian equals one (cf. [17, (3.14) and (3.16)]). The extra multiple of $\phi_\gamma(z,x)$ has been added to $\theta_\gamma(z,x)$ to remove the pole at $z = \lambda$ (cf. (4.12) below).

That ψ_γ is square integrable follows from (cf. [17, (3.15)])

$$|\psi_\gamma(z,x)|^2 = |\psi(z,x)|^2 - \frac{\gamma}{|z-\lambda|^2}\frac{d}{dx}\left(\frac{|W_x(\phi(\lambda),\psi(z))|^2}{1+\gamma\int_a^x \phi(\lambda,y)^2 dy}\right) \tag{4.10}$$

using the Cauchy–Schwartz inequality since

$$W_x(\phi(\lambda),\psi(z)) = W_c(\phi(\lambda),\psi(z)) + (\lambda - z)\int_c^x \phi(\lambda,y)\psi(z,y)dy.$$

To show that $\psi_\gamma(z,x)$ satisfies the boundary condition at b if τ_γ is limit point at b we use (cf. [17, (3.16)] plus (3.16))

$$W_x(\phi_\gamma(\lambda),\psi_\gamma(z)) = \frac{W_x(\phi(\lambda),\psi(z))}{1+\gamma\int_a^x \phi(\lambda,y)^2 dy}. \tag{4.11}$$

□

Note

$$\phi_\gamma(\lambda,x) = \gamma^{-1}\tilde{\phi}_\gamma(\lambda,x), \qquad \theta_\gamma(\lambda,x) = \theta(\lambda,x) + \tilde{\phi}_\gamma(\lambda,x)W_x(\phi(\lambda),\dot{\theta}(\lambda)). \tag{4.12}$$

Remark 4.3. *Again a few remarks are in order:*

(i) *Clearly this procedure can be iterated and we refer to Section 4 of* [17] *for the corresponding formulas.*
(ii) *If λ is an eigenvalue, one could even admit $\gamma \in [-\|\phi(\lambda)\|^{-2},\infty)$.*
(iii) *This procedure leaves operators of the type* (3.32) *invariant.*

Theorem 4.4. *Assume Hypothesis 4.1 and let $\gamma = \infty$. The operator H_∞ has an entire system of solutions*

$$\phi_\infty(z,x) = \frac{1}{z-\lambda}\left(\phi(z,x) - \tilde{\phi}_\infty(\lambda,x)\int_a^x \phi(\lambda,y)\phi(z,y)dy\right) \tag{4.13}$$

$$\theta_\infty(z,x) = (z-\lambda)\theta(z,x) + \tilde{\phi}_\infty(\lambda,x)W_x(\phi(\lambda),\theta(z)), \tag{4.14}$$

which satisfy $W(\theta_\infty(z),\phi_\infty(z)) = 1$.

Furthermore, the Weyl solutions of H_∞ are given by

$$\begin{aligned}\phi_\infty(z,x), \quad \psi_\infty(z,x) &= (z-\lambda)\psi(z,x) + \tilde{\phi}_\infty(\lambda,x)W_x(\phi(\lambda),\psi(z))\\ &= \theta_\infty(z,x) + M_\infty(z)\phi_\infty(z,x),\end{aligned} \tag{4.15}$$

where

$$M_\infty(z) = (z-\lambda)^2 M(z) \tag{4.16}$$

is the singular Weyl function of H_∞. The associated spectral measures are related via

$$d\rho_\infty(x) = (x-\lambda)^2 d\rho(x). \tag{4.17}$$

Proof. In the limiting case $\gamma \to \infty$ the definition (4.6) from the previous theorem would give $\phi_\infty(\lambda,x) = 0$ and we simply need to remove this zero. The rest follows as in the previous theorem. □

Note

$$\phi_\infty(\lambda,x) = \dot{\phi}(z,x) - \tilde{\phi}_\infty(\lambda,x)\int_a^x \phi(\lambda,y)\dot{\phi}(z,y)dy, \quad \theta_\infty(\lambda,x) = -\tilde{\phi}_\infty(\lambda,x) \tag{4.18}$$

5. Connections with the GBDT

In this section we want to investigate a particular example connected with a class of explicit soliton-type solutions of $\tau u = zu$ in the inteval $(a,b) = (0,\infty)$. We apply a generalized Bäcklund–Darboux transformation (GBDT) approach (see [30, 32] and the references therein). As it was discussed in Remark 4.4 of [32], the GBDT approach is related to the double commutation method. The constructions for the Schrödinger operators below are similar to the ones in Theorem 1.2 and Proposition 1.3 of [20], the main difference being that here we will be interested in solutions with singularities at $x = 0$. The so called Darboux matrix is presented in the GBDT as the transfer matrix function w_A in Lev Sakhnovich form [33], which depends additionally on the variable x. Namely, we put

$$w_A(z,x) = I_2 + J\Lambda(x)^* S(x)^{-1}(zI_2 - A)^{-1}\Lambda(x), \quad J = J^* = J^{-1}, \tag{5.1}$$

where I_2 is the 2×2 identity matrix and the following matrix identity holds

$$AS(x) - S(x)A^* = \Lambda(x)J\Lambda(x)^*. \tag{5.2}$$

Note that the case of scalar A was treated in Appendix B of [24], and here we consider the simplest nondiagonal 2×2 matrix case. (The construction holds for the $n\times n$ matrix A case as well.) We apply the GBDT to the initial trivial Schrödinger equation (i.e., to the equation $-\frac{d^2}{dx^2}u = zu$, where $q \equiv 0$) and transform it into the equation $-\frac{d^2}{dx^2}u + \widetilde{q}u = zu$, where the "transformed" potential $\widetilde{q}$ has the following asymptotics

$$\widetilde{q}(x) = \frac{12}{x^2}\big(1 + O(x)\big), \quad x \geq 0, \quad x \to 0, \tag{5.3}$$

(see a detailed procedure below). Thus we get a perturbed Bessel equation (3.32) corresponding to $l = 3$ whereas a scalar A generates the case $l = 1$ (see (B.16) in [24]). This result should also be compared with Lemma 3.10 for the single commutation method. A fundamental system of entire solutions plus the corresponding singular Weyl function will be constructed explicitly as well.

We start our construction by fixing a 2×2 matrix $J = J^* = J^{-1}$, a constant $d \in \mathbb{R}\backslash 0$, a 2×1 vector υ, and a 2×2 Jordan matrix A:

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \upsilon = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}, \quad \mu \neq 0. \tag{5.4}$$

One could fix an arbitrary vector $\upsilon \in \mathbb{C}^2$ but our choice somewhat simplifies further calculations. The 2×2 matrix function $\Lambda(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}$ is defined as the solution of the linear equations with constant coefficients

$$\begin{cases} \Lambda_1'(x) = A\Lambda_2(x), \\ \Lambda_2'(x) = -\Lambda_1(x), \end{cases} \tag{5.5}$$

which satisfies the following initial condition:

$$\Lambda(0) = \begin{bmatrix} d\upsilon & \upsilon \end{bmatrix}, \quad d = \overline{d}. \tag{5.6}$$

It is immediate that

$$\Lambda(0)J\Lambda(0)^* = d\upsilon\upsilon^* - d\upsilon\upsilon^* = 0. \tag{5.7}$$

The 2×2 matrix function S is defined by the equality

$$S(x) = \int_0^x \Lambda_2(t)\Lambda_2(t)^* dt. \tag{5.8}$$

Using (5.5)–(5.8) we get the identity (5.2). The potential $\widetilde{q}$ corresponding to our triplet $\{\Lambda_1(x), \Lambda_2(x), S(x)\}$ is given, similar to [20], by the formula

$$\widetilde{q}(x) = 2\Big(\big(\Lambda_2(x)^* S(x)^{-1}\Lambda_2(x)\big)^2 + \Lambda_1(x)^* S(x)^{-1}\Lambda_2(x) + \Lambda_2(x)^* S(x)^{-1}\Lambda_1(x)\Big). \tag{5.9}$$

It is straightforward to check that (5.5) and (5.6) hold for

$$\Lambda_1(x) = \frac{\mathrm{i}}{2}\mathcal{A}\left(\mathrm{e}^{-\omega\, x}\begin{bmatrix} c_3 x + c_4 \\ c_1 \end{bmatrix} - \mathrm{e}^{\omega\, x}\begin{bmatrix} c_5 x - c_4 \\ c_2 \end{bmatrix}\right), \tag{5.10}$$

$$\Lambda_2(x) = \frac{1}{2}\left(\mathrm{e}^{-\omega\, x}\begin{bmatrix} c_3 x + c_4 \\ c_1 \end{bmatrix} + \mathrm{e}^{\omega\, x}\begin{bmatrix} c_5 x - c_4 \\ c_2 \end{bmatrix}\right), \tag{5.11}$$

where $\omega = \mathrm{i}\sqrt{\mu}$,

$$\mathcal{A} := -\mathrm{i}\omega I_2 + \frac{i}{2\omega}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}^2 = A, \quad c_1 = 1 + \frac{d}{\omega}, \tag{5.12}$$

$$c_2 = 1 - \frac{d}{\omega}, \quad c_3 = \frac{c_1}{2\omega}, \quad c_4 = \frac{d}{2\omega^3}, \quad c_5 = -\frac{c_2}{2\omega}. \tag{5.13}$$

Now, consider the behavior of the vector functions Λ_k in the neighborhood of $x = 0$. According to (5.4), (5.6), and (5.10)–(5.12) we get

$$\begin{aligned} \Lambda_1(x) =& d\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{x}{2}\begin{bmatrix} \omega(c_5 - c_3) - (1/2)(c_1 + c_2) \\ \omega^2(c_1 + c_2) \end{bmatrix} \\ &+ \frac{x^2}{4}\begin{bmatrix} 2\omega^3 c_4 - 2\omega^2(c_3 + c_5) - (\omega/2)(c_1 - c_2) \\ \omega^3(c_1 - c_2) \end{bmatrix} + O(x^3), \end{aligned} \tag{5.14}$$

$$\Lambda_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{x}{2}\begin{bmatrix} -2\omega c_4 + c_3 + c_5 \\ \omega(c_2 - c_1) \end{bmatrix} + \frac{x^2}{4}\begin{bmatrix} 2\omega(c_5 - c_3) \\ \omega^2(c_1 + c_2) \end{bmatrix} + O(x^3). \tag{5.15}$$

By (5.12) and (5.13) it is immediate that

$$c_1 + c_2 = 2, \ \omega(c_1 - c_2) = 2d, \ 2\omega c_4 = c_3 + c_5 = d/\omega^2, \ \omega(c_3 - c_5) = 1. \tag{5.16}$$

Using (5.16) rewrite (5.14) and (5.15) in the form

$$\Lambda_1(x) = \begin{bmatrix} x - dx^2/2 + O(x^3) \\ d - \omega^2 x + \omega^2 d x^2/2 + O(x^3) \end{bmatrix}, \ \Lambda_2(x) = \begin{bmatrix} -x^2/2 + O(x^3) \\ 1 - dx + \omega^2 x^2/2 + O(x^3) \end{bmatrix}. \tag{5.17}$$

In a similar way, after some standard calculations one gets the behavior of the entries of S. Namely, using (5.8), (5.11), and (5.16) we derive

$$S_{11}(x) = \frac{x^5}{20}(1+O(x)), \quad S_{12}(x) = \overline{S_{21}}(x) = -\frac{x^3}{6}(1+O(x)), \quad S_{22}(x) = x(1+O(x)). \tag{5.18}$$

Equalities (5.18) on the asymptotics of S_{ij} imply:

$$\det S(x) = \frac{x^6}{45}\big(1 + O(x)\big), \quad x \to +0. \tag{5.19}$$

By (5.8) and (5.19), the matrix function $S(x)$ is positive and invertible for all $x > 0$, and so $\widetilde{q}$ is well-defined via (5.9). It follows from (5.18) and (5.19) that

$$S(x)^{-1} = \frac{45}{x^6}\begin{bmatrix} x(1+O(x)) & x^3(\frac{1}{6}+O(x)) \\ x^3(\frac{1}{6}+O(x)) & x^5(\frac{1}{20}+O(x)) \end{bmatrix}. \tag{5.20}$$

In view of (5.9), (5.17), and (5.20) it is straightforward that (5.3) holds true.

To find a fundamental system of entire solutions for our Schrödinger equation, where the potential is given by (5.9), we set

$$T(z,x) := T_1(z)\begin{bmatrix} \mathrm{e}^{\mathrm{i}\sqrt{z}x} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\sqrt{z}x} \end{bmatrix} T_1(z)^{-1}, \quad T_1(z) = \begin{bmatrix} 1 & 1 \\ \mathrm{i}\sqrt{z} & -\mathrm{i}\sqrt{z} \end{bmatrix}. \tag{5.21}$$

It is easy to see that

$$T(z,x) = \cos(\sqrt{z}x)I_2 + \sin(\sqrt{z}x)\begin{bmatrix} 0 & 1/\sqrt{z} \\ -\sqrt{z} & 0 \end{bmatrix} \tag{5.22}$$

is an entire matrix function of z (which has real-valued entries on $\mathbb{R}$). Clearly, the vector $u(z,x) = \begin{bmatrix} 1 & 0 \end{bmatrix} T(z,x)$ satisfies the trivial Schrödinger equation $-\frac{d^2}{dx^2}u = zu$. Using (5.1), (5.2), (5.5), and (5.8) one can show (compare [20]) that the vector function

$$\begin{aligned} y(z,x) &= \begin{bmatrix} 1 & 0 \end{bmatrix} w_A(z,x)T(z,x) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} T(z,x) + \Lambda_2(x)^* S(x)^{-1}(zI_2 - A)^{-1}[\Lambda_1(x)\ \Lambda_2(x)]T(z,x) \end{aligned} \tag{5.23}$$

satisfies $\tau y = zy$, where the potential is given by (5.9). Taking into account that $T(z,x)$ is an entire matrix function, we see that $y(z,x)$ gives a system of linearly independent solutions, which are meromorphic in z with only possible pole at $z = \mu$. Moreover, it is true that

$$y'(z,x) = \begin{bmatrix} -\Lambda_2(x)^* S(x)^{-1}\Lambda_2(x) & 1 \end{bmatrix} w_A(z,x)T(z,x). \tag{5.24}$$

Now, consider

$$\tilde{\phi}(z,x) := y(z,x)\begin{bmatrix} 1 \\ -d \end{bmatrix}. \tag{5.25}$$

Clearly, $\tilde{\phi}$ satisfies $\tau\tilde{\phi} = z\tilde{\phi}$ since $\tau y = zy$. Moreover, we shall show that

$$\tilde{\phi}(z,x) \in L^2(0,c). \tag{5.26}$$

Indeed, it is immediate from (5.22) that

$$T(z,x) = \big(1 - \frac{z}{2}x^2\big)I_2 + x\begin{bmatrix} 0 & 1 \\ -z & 0 \end{bmatrix} + O(x^3). \tag{5.27}$$

By straightforward calculations it follows from (5.17) and (5.27) that

(5.28)

$$\Lambda(x)T(z,x) = \begin{bmatrix} x - dx^2/2 + O(x^3) & x^2/2 + O(x^3) \\ d - (z-\mu)x + (z-\mu)dx^2/2 + O(x^3) & 1 - (z-\mu)x^2/2 + O(x^3) \end{bmatrix}.$$

We also have

$$(zI_2 - A)^{-1} = \frac{1}{(z-\mu)^2}\begin{bmatrix} z-\mu & 1 \\ 0 & z-\mu \end{bmatrix}. \tag{5.29}$$

It follows from (5.28) and (5.29) that

$$(zI_2 - A)^{-1}\Lambda(x)T(z,x) = \frac{1}{(z-\mu)^2}\begin{bmatrix} d + O(x^3) & 1 + O(x^3) \\ d(z-\mu) + O(x) & z - \mu + O(x^2)\end{bmatrix}. \tag{5.30}$$

Finally, relations (5.17) and (5.20) imply

$$\Lambda_2(x)^* S(x)^{-1} = -\begin{bmatrix} 15x^{-3}(1+O(x)) & \frac{3}{2}x^{-1}(1+O(x))\end{bmatrix}. \tag{5.31}$$

Relations (5.23), (5.25), (5.30), and (5.31) yield (5.26).

Next, let us show that $\tilde{\phi}$ is an entire function. For that purpose consider again the resolvent

$$\begin{aligned}(zI_2 - A)^{-1}\Lambda(x)T(z,x)\begin{bmatrix} 1 \\ -d\end{bmatrix} &= \frac{1}{(z-\mu)^2}\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\Lambda(x)T(\mu,x)\begin{bmatrix} 1 \\ -d\end{bmatrix} \\ &+ \frac{1}{(z-\mu)}\left(\Lambda(x)T(\mu,x) + \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\Lambda(x)T_z(\mu,x)\right)\begin{bmatrix} 1 \\ -d\end{bmatrix} + \Omega(z,x),\end{aligned} \tag{5.32}$$

where $T_z(z,x) := \frac{\partial}{\partial z}T(z,x)$, and Ω is an entire vector function of z.

Recall that $\mu = -\omega^2$. According to (5.22) we have

$$\begin{aligned}T_z(-\omega^2,x) = -\frac{1}{4\omega}\Bigg(&(\mathrm{e}^{\omega x} - \mathrm{e}^{-\omega x})\left(xI_2 + \begin{bmatrix} 0 & -\omega^{-2} \\ 1 & 0\end{bmatrix}\right) \\ &+x(\mathrm{e}^{\omega x} + \mathrm{e}^{-\omega x})\begin{bmatrix} 0 & \omega^{-1} \\ \omega & 0\end{bmatrix}\Bigg).\end{aligned} \tag{5.33}$$

Hence, taking into account (5.10), (5.11), and (5.22) we get

$$\begin{bmatrix} 0 & 1\end{bmatrix}\Lambda(x)T(-\omega^2,x) \equiv \frac{1}{2}\begin{bmatrix}\omega(c_1 - c_2) & c_1 + c_2\end{bmatrix}, \tag{5.34}$$

$$\begin{aligned}&\begin{bmatrix} 1 & 0\end{bmatrix}\Lambda(x)T(-\omega^2,x) + \begin{bmatrix} 0 & 1\end{bmatrix}\Lambda(x)T_z(-\omega^2,x) \\ &= (1/4)x\begin{bmatrix} 2\omega(c_3 - c_5) - c_1 - c_2 & 2(c_3 + c_5) - \omega^{-1}(c_1 - c_2)\end{bmatrix} \\ &+ (1/4)\begin{bmatrix} 4\omega c_4 - \omega^{-1}(c_1 - c_2) & 0\end{bmatrix}.\end{aligned} \tag{5.35}$$

By (5.16) it is immediate that

$$\begin{aligned}&\begin{bmatrix}\omega(c_1 - c_2) & c_1 + c_2\end{bmatrix}\begin{bmatrix} 1 \\ -d\end{bmatrix} = 0, \quad 2\omega(c_3 - c_5) - c_1 - c_2 = 0, \\ &2(c_3 + c_5) - \omega^{-1}(c_1 - c_2) = 0, \quad 4\omega c_4 - \omega^{-1}(c_1 - c_2) = 0.\end{aligned} \tag{5.36}$$

Relations (5.32) and (5.34)–(5.36) show that $(zI_2 - A)^{-1}\Lambda(x)T(z,x)\begin{bmatrix} 1 \\ -d\end{bmatrix} = \Omega(z,x)$ is an entire vector function. Therefore, in view of (5.23) and (5.25), one can see that $\tilde{\phi}(z,x)$ is an entire function too.

Let us consider another solution

$$\tilde{\theta}(z,x) := y(z,x)\begin{bmatrix} 0 \\ 1\end{bmatrix}. \tag{5.37}$$

According to (5.21) and (5.23)–(5.25) we have

$$W\big(\tilde{\theta}(z), \tilde{\phi}(z)\big) = W_x\big(\tilde{\theta}(z), \tilde{\phi}(z)\big) = -\det w_A(z,x). \tag{5.38}$$

Thus, $\det w_A(z,x)$ does not depend on x. Moreover, using (5.1) and (5.2) we get

$$w_A(\overline{z},x)^* J w_A(z,x) = J, \tag{5.39}$$

that is, $|\det w_A(z,x)|=1$. Taking into account the fact that $w_A(z,x)$ is a rational function of z with the only possible pole at $z=\mu$ (of order no greater than 2) and that the relations

$$|\det w_A(z,x)|=1, \quad \lim_{z\to\infty}\det w_A(z,x)=1$$

hold, we derive

$$\det w_A(z,x)=\Big(\frac{z-\overline{\mu}}{z-\mu}\Big)^n \quad (0\leq n\leq 4), \quad \det w_A(z,x)=1 \quad \text{for} \quad \mu=\overline{\mu}. \tag{5.40}$$

Furthermore, we assume that $\mu\neq\overline{\mu}$. First, we show that $n=2$ in (5.40) for $\mu\neq\overline{\mu}$. Indeed, use (5.1) and (5.29) to write down $\det w_A$ in the form

$$\begin{aligned}\det w_A&=-(z-\mu)^{-2}\big(Z_{22}+(z-\mu)^{-1}\Lambda_{22}Y_1\big)\big(-Z_{11}+(z-\mu)^{-1}\Lambda_{21}Y_2\big)\\ &+\big(1+(z-\mu)^{-1}Z_{21}+(z-\mu)^{-2}\Lambda_{21}Y_1\big)\big(1-(z-\mu)^{-1}Z_{12}+(z-\mu)^{-2}\Lambda_{22}Y_2\big),\end{aligned} \tag{5.41}$$

where $Z_{ij}:=\Lambda_i^*S^{-1}\Lambda_j$, $\Lambda=:\{\Lambda_{ij}\}_{i,j=1}^2$,

$$Y=\begin{bmatrix}Y_1\\ Y_2\end{bmatrix}:=J\Lambda^*S^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}, \quad S^{-1}=\frac{1}{\det S}\begin{bmatrix}S_{22} & -S_{12}\\ -S_{21} & S_{11}\end{bmatrix}. \tag{5.42}$$

If we write down $\det w_A$ as a series $\det w_A(x,z)=\sum_{k=0}^n C_k(z-\mu)^{-k}$ it is immediate from (5.41) that $C_4=0$. In view of (5.42) one can rewrite $(\det S)Z_{ij}$ and $(\det S)Y_i$ in terms of S_{ij} and Λ_{ij} $(i,j=1,2)$. Then, after standard calculations, (5.41) yields $C_3=0$. Finally, according to (5.41) and (5.42) we have

$$C_2=Z_{11}Z_{22}-Z_{12}Z_{21}+\begin{bmatrix}\Lambda_{21} & \Lambda_{22}\end{bmatrix}Y=\det(\Lambda^*S^{-1}\Lambda)+\begin{bmatrix}0 & 1\end{bmatrix}\Lambda J\Lambda^*S^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}. \tag{5.43}$$

It follows from (5.2) that

$$\Lambda J\Lambda^*S^{-1}=A-\overline{\mu}I_2-S\begin{bmatrix}0 & 1\end{bmatrix}^*\begin{bmatrix}1 & 0\end{bmatrix}S^{-1}. \tag{5.44}$$

Substitute (5.44) into (5.43) and use (5.42) to get

$$C_2=(1/\det S)(|\det\Lambda|^2-S_{22}^2). \tag{5.45}$$

As $\mu\neq\overline{\mu}$ the inequality $\omega=\mathrm{i}\sqrt{\mu}\neq\pm\overline{\omega}$ holds. Hence, without loss of generality we assume $\mathrm{Re}(\omega)>0$. Moreover, relations $\omega\neq\overline{\omega}$ and $d=\overline{d}$ imply $c_2\neq 0$, see the definition of c_2 in (5.12). Then, it is immediate that the asymtotics of $\det\Lambda(x)$ for $x\to+\infty$ is given by $(\mathrm{e}^{2\omega x}/8\omega)c_2^2(1+o(1))$ and the asymtotics of $S_{22}(x)$ is given by the expression $\big(\mathrm{e}^{2\mathrm{Re}(\omega)x}/8\mathrm{Re}(\omega)\big)|c_2|^2(1+o(1))$, that is, $|\det\Lambda|^2-S_{22}^2\not\equiv 0$. As $C_4=C_3=0$ and $C_2\neq 0$, formula (5.40) implies

$$\det w_A(z,x)=\Big(\frac{z-\overline{\mu}}{z-\mu}\Big)^2 \qquad (\mu\neq\overline{\mu}). \tag{5.46}$$

To construct entire real-valued solutions ϕ and θ rewrite (5.39) as $w_A(\overline{z},x)^*=Jw_A(z,x)^{-1}J^*$. In view of (5.46) the last equality yields

$$\begin{bmatrix}\overline{w}_{11}(\overline{z},x) & \overline{w}_{21}(\overline{z},x)\\ \overline{w}_{12}(\overline{z},x) & \overline{w}_{22}(\overline{z},x)\end{bmatrix}=(z-\mu)^2(z-\overline{\mu})^{-2}\begin{bmatrix}w_{11}(z,x) & w_{21}(z,x)\\ w_{12}(z,x) & w_{22}(z,x)\end{bmatrix},$$

where $w_A=:\{w_{ij}\}_{i,j=1}^2$. In other words, we get

$$(z-\overline{\mu})^{-2}w_A(z,x)=(z-\mu)^{-2}\overline{w}_A(\overline{z},x) \qquad (\mu\neq\overline{\mu}). \tag{5.47}$$

Recal that $T(z,x) = \overline{T}(\overline{z},x)$, that is, T is real-valued. Thus, by (5.23) and (5.47) the vector function $(z-\overline{\mu})^{-2}y(z,x)$ is real-valued for $\mu \neq \overline{\mu}$. So, for

$$\phi(z,x) := \frac{\tilde{\phi}(z,x)}{(z-\overline{\mu})^2} = \frac{y(z,x)}{(z-\overline{\mu})^2}\begin{bmatrix} 1 \\ -d \end{bmatrix}, \tag{5.48}$$

$$\theta(z,x) := -(z-\mu)^2\tilde{\theta}(z,x) = -(z-\mu)^2 y(z,x)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\mu \neq \overline{\mu}); \tag{5.49}$$

we derive that ϕ and θ are real-valued. As $\tilde{\phi}$ is an entire function, it follows from (5.48) that the real-valued function ϕ may have only one pole at $z = \overline{\mu}$ $(\mu \neq \overline{\mu})$. Therefore, ϕ is an entire function. It follows from (5.29) and (5.49) that θ is an entire function too. Finally, (5.38), (5.46), (5.48), and (5.49) imply that $W\big(\theta(z), \phi(z)\big) = 1$. Thus, a fundamental system of solutions satisfying conditions of Lemma 2.2 is constructed explicitly.

Now, using these solutions we can construct explicitly a singular Weyl m-function M. Namely, chose the branch cut in $\sqrt{z}$ along the negative real axis and assume that $\mathrm{Im}(\sqrt{z}) > 0$. Observe that (5.21) yields

$$T(z,x)\begin{bmatrix} 1 \\ \mathrm{i}\sqrt{z} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\sqrt{z}x} \\ \mathrm{i}\sqrt{z}\mathrm{e}^{\mathrm{i}\sqrt{z}x} \end{bmatrix}. \tag{5.50}$$

In view of (5.8) and (5.11) we calculate that

$$\det S(x) \sim |c_2c_5|^2(4\mathrm{Re}(\omega))^{-4}\mathrm{e}^{4\mathrm{Re}(\omega)x} \quad (x \to +\infty),$$

and so the transfer function $w_A(z,x)$ given by (5.1) behaves as $O(x^4)$. Hence, it follows from (5.23) and (5.50) that

$$\widetilde{\psi}(z,x) = y(x,z)\begin{bmatrix} 1 \\ \mathrm{i}\sqrt{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} w_A(z,x)\begin{bmatrix} \mathrm{e}^{\mathrm{i}\sqrt{z}x} \\ \mathrm{i}\sqrt{z}\mathrm{e}^{\mathrm{i}\sqrt{z}x} \end{bmatrix} \in L^2(c,\infty). \tag{5.51}$$

Therefore, by (5.48), (5.49), and (5.51) we get

$$\psi(z,x) = \theta(z,x) + M(z)\phi(z,x) \in L^2(c,\infty), \quad M(z) := -\frac{(z-\mu)^2(z-\overline{\mu})^2}{\mathrm{i}\sqrt{z}+d}, \tag{5.52}$$

that is, $M(z)$ above is a singular Weyl m-function of our system. Thus, we proved the following lemma.

Lemma 5.1. *Let the parameters $\mu \neq \overline{\mu}$ and $d \in \mathbb{R}$ be fixed. Then the relations* (5.9)*,* (5.8)*, and* (5.10)*–*(5.12) *explicitly determine a singular potential $\widetilde{q}$ satisfying* (5.3)*. The corresponding entire solutions $\phi(z,x)$ and $\theta(z,x)$, such that $\phi(z,x)$ is nonsingular at $x = 0$ and $W(\theta(z),\phi(z)) = 1$, are given by* (5.48) *and* (5.49)*, where y is constructed via* (5.1)*,* (5.21)*, and* (5.23)*. Moreover, the singular Weyl function corresponding to this problem is given by* (5.52)*.*

Application of the GBDT to non-trivial initial Schrödinger equation was treated in Proposition 2.2 [31], see also [30]. (Assuming $u = \overline{u}$ we put $A_2^* = A_1 = A$, $\Pi_1 = \Lambda$, $\Pi_2 = -\Lambda J$ to switch from notations in [31] to our notations.) It follows from [31] that a set y_0 of two solutions of the initial Schrödinger equation $-\frac{d^2}{dx^2}u + qu = zu$ generates the set

$$y(z,x) = \begin{bmatrix} 1 & 0 \end{bmatrix} w_A(z,x)w(z,x), \quad w(z,x) := \begin{bmatrix} y_0(z,x) \\ y_0'(z,x) \end{bmatrix} \tag{5.53}$$

of solutions of the transformed Schrödinger equation. Note that in the case of the non-trivial initial Schrödinger equation formula (5.5) for Λ is replaced by the equation

$$\begin{cases} \Lambda_1'(x) = A\Lambda_2(x) - \Lambda_2(x)q(x), \\ \Lambda_2'(x) = -\Lambda_1(x). \end{cases} \tag{5.54}$$

Finally, $S(x) = S(x)^*$ is determined by the relations

$$S'(x) = \Lambda_2(x)\Lambda_2(x)^*, \quad AS(0) - S(0)A^* = \Lambda(0)J\Lambda(0)^*, \tag{5.55}$$

which imply (5.2).

Remark 5.2. *Using relations* (5.53) *and* (5.54) *one can easily compare one step double commutation from Section 4 with the GBDT, where A is a scalar. Namely, we can set*

(5.56)
$$A = \lambda \in \mathbb{C}, \quad y_0(z,x) = \begin{bmatrix} \phi(z,x) & \theta(z,x) \end{bmatrix}, \quad \Lambda(x) = \begin{bmatrix} -\phi'(\lambda,x) & \phi(\lambda,x) \end{bmatrix}.$$

Thus, formulas (5.53) *and* (5.56) *imply*

$$\phi_\gamma(z,x) := y(z,x)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} w_A(z,x)\begin{bmatrix} \phi(z,x) \\ \phi'(z,x) \end{bmatrix}. \tag{5.57}$$

As in the case of $A = \lambda$ we have $w_A(z,x) = I_2 + \frac{1}{S(x)(z-\lambda)}J\Lambda(x)^*\Lambda(x)$ and can rewrite (5.57) *in the form*

$$\begin{aligned} \phi_\gamma(z,x) &= \phi(z,x) + \frac{\overline{\phi}(\lambda,x)}{S(x)(z-\lambda)}\Lambda(x)\begin{bmatrix} \phi(z,x) \\ \phi'(z,x) \end{bmatrix} \\ &= \phi(z,x) + \frac{\overline{\phi}(\lambda,x)}{S(x)(z-\lambda)}W_x\big(\phi(\lambda),\phi(z)\big). \end{aligned} \tag{5.58}$$

By (5.55) *we get*

$$S(x) = (\lambda - \overline{\lambda})^{-1}\Lambda(0)J\Lambda(0)^* + \int_0^x |\phi(\lambda,t)|^2 dt \quad \text{for} \quad \lambda \neq \overline{\lambda}. \tag{5.59}$$

In the case of $\lambda = \overline{\lambda}$ and real-valued ϕ, which are treated in Section 4, identity (5.2) *is fulfilled automatically. Therefore, for this case we have*

$$S(x) = \gamma^{-1} + \int_0^x \phi(\lambda,t)^2 dt, \quad \gamma \in (0,\infty]. \tag{5.60}$$

Now, compare (5.60) *with* (4.1) *to see that*

$$\tilde{\phi}_\gamma(\lambda,x) = \frac{\phi(\lambda,x)}{S(x)}. \tag{5.61}$$

Hence, if $\lambda = \overline{\lambda}$ and ϕ are real-valued, expressions for ϕ_γ in (5.58) *and in* (4.6) *coincide.*

Acknowledgments. A.K. acknowledges the hospitality and financial support of the Erwin Schrödinger Institute and financial support from the IRCSET PostDoctoral Fellowship Program.

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