# Vector and Axial anomaly in the Thirring-Wess model

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**Abstract**

We study the 2D Vector Meson model introduced by Thirring and Wess, that is to say the Schwinger model with massive photon and massless fermion. We prove, with a renormalization group approach, that the vector and axial Ward identities are broken by the Adler-Bell-Jackiw anomaly; and we rigorously establish three widely believed consequences: a) the interacting meson-meson correlation equals a free boson propagator, though the mass is additively renormalized by the anomaly; b) the anomaly is quadratic in the charge, in agreement with the Adler-Bardeen formula; c) the fermion-fermion correlation has an anomalous long-distance decay.

## 1 Introduction

Since early days of Quantum Field Theory, (QFT), 1+1 dimensional models have been widely investigated as example of relativistic fields with local interaction: the Thirring and the Schwinger models, [38], [34], are probably the most celebrated cases. Although these systems are so simplified to have an exact solution, they nonetheless suggest ideas and mathematical tools to approach realistic theories in four space-time dimensions.

One of the aspects that are certain relevant also in higher dimensions is the role played by two Ward Identities (WI) related to the invariance of the Lagrangian under vector and axial transformations. In agreement with the general Adler-Bell-Jackiw mechanism, [2], [4], the vector and the axial symmetries are *broken* at quantum level by the WI *anomaly*. Many salient features of QFT are related to such an anomaly; let's consider some of them. In any dimensions, by the Adler-Bardeen (AB) argument, [3], the anomaly is expected to be *linear* in the bare coupling, i.e. not renormalized in loop perturbation theory at any order bigger than one; besides, it is a *topological quantity*, i.e. it doesn't depend on the choice of the cutoff (to some extent). The anomaly is also responsible for the *anomalous dimension* in the distance decay of the fermion correlations, [25]. And, finally, when a fermion field and a gauge photon interact through a *minimal coupling*, as for instance in the Schwinger model, the anomaly also represents a dynamically-generated physical mass for the photon field, [34].

In two dimensions the anomaly is said "mild": although the formal WI are broken, the anomaly has the only consequence of changing the normalization of the vector and axial currents, that remain conserved. Therefore using the WI it was possible to find *formal exact solutions* of the Schwinger-Dyson equation, (SDE), of the Thirring, the Schwinger and related models, in this way computing all the correlations: see [25], [14], [37], [23], [26] and [28]. An alternative approach - still related to the mildness of the anomaly - is the *bosonization*, i.e. the equivalence of the the fermion currents with boson free fields. This fact is behind the solution of the Thirring and the Schwinger model, [21], [19], in the path integral formalism.

The above analysis - and much more, see chapters X and XII of [1] - has been based on formal methods and sound assumptions only. Rigorous results are few; though, at this stage, the reader might have few interest left for them. It is worth explaining, then, why the matter is still very tangled.

In fact, earlier formal solutions of the Thirring and the Schwinger models were incomplete or incorrect, [23], [40], [1]. And, after all, the distinction among *formally correct*, *incorrect* or *incomplete* solutions may be quite faint. Wightman, [40], to put order in the confusion of the results, took the approach of considering any set of correlation functions, no matter how they were obtained, as trial theories; and then of promoting them to QFT if they satisfied certain axioms. This viewpoint has been moderately prolific (see for example [18], [17], [16]). There is at least one clear issue with it: being based of some sort of exact solution of the correlation functions, it is limited to few special models. A massive fermion field, for example, or an additional interactions that, in the renormalization group language, is irrelevant, would represent a severe obstruction to the method.

The recent approach to the Thirring model in [5], [6] is different. We derived the correlations from the Lagrangian, so that in the massless case we obtained the exact solution, while in the massive case, where no exact solution is known, we can still prove the axioms. But, most of all, the major advantage of using the Lagrangian (as opposed to correlations) as starting point, is that it keeps track of the relationship between a special class of statistical mechanical problems - such as the Eight-Vertex models and the XYZ quantum chain - and their scaling limit, that turns out to be the Thirring model; in this way we were able prove some scaling formulas for non solvable models, [7], [12]. The heart of the technique is the control of the vanishing of the "beta function" for the effective coupling: this route, that have been useful also for other statistical models, was opened in [8] by exploiting the exact solution of the Luttinger model; and reached its final form, based on WI and independent from any exact solution, in [10], [11].

The extension of the above techniques to the Schwinger model poses some serious issues because the infrared divergences related with the massless photon; therefore we take up a problem of intermediate difficulty, the theory of the Vector Meson, [39], i.e. a Schwinger model in which a photon mass is added by hand. This model still preserves some interests, because we can prove that the bare photon mass is renormalized into the physical mass just by an additive constant, exactly as expected in the Schwinger model; besides, we can establish the anomalous dimension of the fermion field. Finally, we can prove the AB-formula in a genuine example of QFT, i.e. removing the all the cutoffs of the theory: in this sense this paper is a completion of the objective of [29], [30].

Although the present technique allows us to treat other aspects of the theory, we shall not verify the Osterwalder-Schrader axioms, nor we shall discuss the case with massive fermion or the bosonization of the final result; this is because details would be largely similar to [5], [6]. We shall rather focus on the novelties with respect to those papers.

To conclude, we mention that other rigorous results on the WI anomalies, but in different regimes and with different techniques, were established in [36], [33]. Whereas for the study the true Schwinger model, i.e. the case with null bare photon mass, a change of viewpoint seems needed: boson and fermion propagators should be treated on the same ground along the Renormalization Group (RG) flow; perhaps that is possible in the approach of [31], [15].

# 2 Definitions and main results

Let's begin with the definition of the formal path integral formula in Euclidean formulation; afterwords we will introduce *infrared and ultraviolet cutoffs* to evaluate the correlations. The Vector Meson model is made of one fermion field, $(\bar\psi_{\mathbf{x}}, \psi_{\mathbf{x}})$, for $\mathbf{x} = (x_0, x_1) \in \mathbb{R}^2$, and one vector boson field, $(A^0_{\mathbf{x}}, A^1_{\mathbf{x}})$, interacting through minimal coupling. The free meson with mass $\mu$ (a *gauge field* if $\mu = 0$) is described by the action

$$\frac{1}{4}\int d\mathbf{x}\ (F^{\mu\nu}_{\mathbf{x}})^2 + \frac{\mu^2}{2}\int d\mathbf{x}\ (A^\mu_{\mathbf{x}})^2$$

for $F^{\mu\nu}_{\mathbf{x}} = \partial^\mu A^\nu_{\mathbf{x}} - \partial^\nu A^\mu_{\mathbf{x}}$. The quantization of a gauge theory (namely the case $\mu = 0$) requires a *gauge fixing term* which makes convergent the integration along the orbits of gauge transformation: for $\alpha > 0$

$$\frac{\alpha}{2}\int d\mathbf{x}\ (\partial^\mu_{\mathbf{x}} A^\mu_{\mathbf{x}})^2\ .$$

In our case $\mu \neq 0$ and the gauge fixing would not be required to make sense of the theory. Nevertheless, the Vector Meson model defined by Thirring and Wess had the purpose to be equivalent to the Schwinger model in the limit of vanishing $\mu$; therefore, with them, we shall consider $\alpha > 0$ only, that will make the interaction *superrinormalizable*. For notational convenience, we also introduce a further non-local term for the vector field

$$\frac{\sigma}{2}\int d\mathbf{x}d\mathbf{y}\ (\partial^\mu A^\mu_{\mathbf{x}})\Delta^{-1}(\mathbf{x}-\mathbf{y})(\partial^\mu A^\mu_{\mathbf{y}})$$

where $\Delta^{-1}$ is the inverse of the Laplacian and $\sigma$ is a real parameter with the dimension of the square of a mass. The three terms we have introduced so far are collected together into the following term

$$\frac{1}{2}\int d\mathbf{x}d\mathbf{x}\ A^\mu_{\mathbf{x}} D^{\mu\nu}(\mathbf{x}-\mathbf{y})A^\nu_{\mathbf{y}}$$
$$\stackrel{\text{def.}}{=} \frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^2}\ \widehat{A}^\mu_{\mathbf{k}}\left[(\mathbf{k}^2+\mu^2)\delta^{\mu\nu} - \left(1-\alpha+\frac{\sigma}{\mathbf{k}^2}\right)\mathbf{k}^\mu\mathbf{k}^\nu\right]\widehat{A}^\nu_{-\mathbf{k}}\ .$$

Sometimes we will write $D^{\mu\nu}$ for the operator with kernel $D^{\mu\nu}(\mathbf{x})$. The massless electron, with charge $q$, interacts with the photon through a *minimal coupling*

$$\int d\mathbf{x}\ \bar{\psi}_{\mathbf{x}}\gamma^\mu\left(i\partial^\mu_{\mathbf{x}} + qA^\mu_{\mathbf{x}}\right)\psi_{\mathbf{x}}$$

where $\gamma^0$ and $\gamma^1$ are generators of the Euclidean Clifford algebra. The total Euclidean action of the Vector Meson is

$$\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\ A^\mu_{\mathbf{x}}D^{\mu\nu}(\mathbf{x}-\mathbf{y})A^\nu_{\mathbf{y}} + \int d\mathbf{x}\ \bar{\psi}_{\mathbf{x}}\gamma^\mu\left(i\partial^\mu_{\mathbf{x}}+qA^\mu_{\mathbf{x}}\right)\psi_{\mathbf{x}}\ .$$

Finally, putting together all the above terms, we define the *generating functional* of the truncated correlations of the the Vector Meson model, $\mathcal{K}(J,\eta)$, as follows:

$$e^{\mathcal{K}(J,\eta)} \stackrel{\text{def.}}{=} \int dP(\psi)dP(A)\ \exp\left\{\int d\mathbf{x}\ \left[-qA^\mu_{\mathbf{x}}(\bar{\psi}_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}}) + J^\mu_{\mathbf{x}}A^\mu_{\mathbf{x}} + \bar{\eta}_{\mathbf{x}}\psi_{\mathbf{x}} + \bar{\psi}_{\mathbf{x}}\eta_{\mathbf{x}}\right]\right\} \tag{1}$$

where $J$ is a real external field; $\eta$, $\bar{\eta}$ are Grassmann external fields; $dP(\psi)$ is a Gaussian measure on Grassmann variables $(\psi_{\mathbf{x}}, \bar{\psi}_{\mathbf{x}})_{\mathbf{x}}$ with zero covariances, but for $\int dP(\psi)\ \psi_{\mathbf{x}}\bar{\psi}_{\mathbf{y}}$ that equals

$$S_0(\mathbf{x}-\mathbf{y}) \stackrel{\text{def.}}{=} i\gamma^\mu\int\frac{d\mathbf{k}}{(2\pi)^2}\ \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2}\mathbf{k}^\mu\ ;$$

and $dP(A)$ is a Gaussian measure on real variables $(A^0_{\mathbf{x}}, A^1_{\mathbf{x}})_{\mathbf{x}}$ with covariances $\int dP(A)\ A^\mu_{\mathbf{x}}A^\nu_{\mathbf{y}}$ equal to

$$G^{\mu\nu}_0(\mathbf{x}-\mathbf{y};\mu^2,\sigma) \stackrel{\text{def.}}{=} \int\frac{d\mathbf{k}}{(2\pi)^2}\ e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left[\frac{\delta^{\mu\nu}}{\mathbf{k}^2+\mu^2} - \left(\frac{1}{\mathbf{k}^2+\mu^2} - \frac{1}{\alpha\mathbf{k}^2+\mu^2-\sigma}\right)\frac{\mathbf{k}^\mu\mathbf{k}^\nu}{\mathbf{k}^2}\right]$$

(we will abridge the notation of $G_0^{\mu\nu}(\mathbf{x};\mu^2,\sigma)$ into $G_0^{\mu\nu}(\mathbf{x})$, sometimes). These covariances are also called *free propagators* as

$$i\gamma^\mu(\partial^\mu S)(\mathbf{x}) = \delta(\mathbf{x}) \qquad (D^{\mu\rho}G_0^{\rho\nu})(\mathbf{x};\mu^2,\sigma) = \delta^{\mu\nu}\delta(\mathbf{x}) \ .$$

To make sense of (1) we have to introduce a cutoff function. For a fixed $\gamma > 1$, let $\widehat{\chi}(t)$ be a smooth function, positive in $[0,\gamma)$ and

$$\widehat{\chi}(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t \ge \gamma. \end{cases} \ ; \tag{2}$$

then, given two integers $h, h'$, define

$$\widehat{\chi}_{h,h'}(\mathbf{k}) = \widehat{\chi}\left(\gamma^{-h}|\mathbf{k}|\right) - \widehat{\chi}\left(\gamma^{-h'+1}|\mathbf{k}|\right) \ .$$

and in correspondence, define two Gaussian measure $dP_{h,h'}(\psi)$ and $dP_{h,h'}(A)$, determined by the covariances:

$$S_{0,h,h'}(\mathbf{x}) \stackrel{\text{def.}}{=} \gamma^\mu \int \frac{d\mathbf{k}}{(2\pi)^2}\, \widehat{\chi}_{h,h'}(\mathbf{k}) \frac{e^{-i\mathbf{k}\mathbf{x}}}{\mathbf{k}^2}\mathbf{k}^\mu \ ,$$

$$G_{0,h,h'}^{\mu\nu}(\mathbf{x}) \stackrel{\text{def.}}{=} \int \frac{d\mathbf{k}}{(2\pi)^2}\, \widehat{\chi}_{h,h'}(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}} \left[\frac{\delta^{\mu\nu}}{\mathbf{k}^2+\mu^2} - \left(\frac{1}{\mathbf{k}^2+\mu^2} - \frac{1}{\alpha\mathbf{k}^2+\mu^2-\sigma}\right)\frac{\mathbf{k}^\mu\mathbf{k}^\nu}{\mathbf{k}^2}\right] \ .$$

Given the integers $-l, N > 0$, the *regularized functional integral*, $\mathcal{K}_{l,N}(J,\eta)$, is given by (1), replacing $dP(\psi)$ and $dP(A)$ with $dP_{l,N}(\psi)$ and $dP_{l,N}(A)$. Finally, let $S(\mathbf{x}-\mathbf{y})$ and $G^{\mu\nu}(\mathbf{x}-\mathbf{y})$ be the *interacting propagators*, namely the correlations

$$S(\mathbf{x}-\mathbf{y}) \stackrel{\text{def.}}{=} \lim_{-l,N\to\infty} \frac{\partial^2\mathcal{K}_{l,N}}{\partial\bar{\eta}_{\mathbf{x}}\partial\eta_{\mathbf{y}}}(0,0) \tag{3}$$

$$G^{\mu\nu}(\mathbf{x}-\mathbf{y};\mu^2,\sigma) \stackrel{\text{def.}}{=} \lim_{-l,N\to\infty} \frac{\partial^2\mathcal{K}_{l,N}}{\partial J^\mu_{\mathbf{x}}\partial J^\nu_{\mathbf{y}}}(0,0) \tag{4}$$

where the derivatives in $\eta$ are taken from the right; and the limit in $N$ is taken before the limit in $l$. Define

$$\begin{aligned} F(\mathbf{z};\mu^2,\sigma) &= \int \frac{d\mathbf{k}}{(2\pi)^2}\, \frac{e^{i\mathbf{k}\mathbf{z}}-1}{[\alpha\mathbf{k}^2+\mu^2-\sigma]\mathbf{k}^2} \\ F_5(\mathbf{z};\mu^2) &= \int \frac{d\mathbf{k}}{(2\pi)^2}\, \frac{e^{i\mathbf{k}\mathbf{z}}-1}{[\mathbf{k}^2+\mu^2]\mathbf{k}^2} \ , \end{aligned} \tag{5}$$

and note that, for $\mu^2 > 0$, $\sigma < \mu^2$, $\alpha > 0$, we have the following large $|\mathbf{z}|$ asymptotic:

$$F(\mathbf{z};\mu^2,\sigma) \sim -\frac{1}{2\pi(\mu^2-\sigma)}\ln|\mathbf{z}| \ , \qquad F_5(\mathbf{z};\mu^2,\sigma) \sim -\frac{1}{2\pi\mu^2}\ln|\mathbf{z}| \ .$$

**Theorem 2.1** *Given the meson mass $\mu^2 > 0$, for $\sigma < \mu^2$, $\alpha \geq \alpha_0 > 0$ and $|q|$ small enough, the explicit expression of the interacting propagators are:*

$$S(\mathbf{x}) = e^{q^2\left[F(\mathbf{x};\mu^2-\nu_5,\sigma+\nu-\nu_5)-F_5(\mathbf{x};\mu^2-\nu_5)\right]} S_0(\mathbf{x}) \tag{6}$$

$$G^{\mu\nu}(\mathbf{x};\mu^2,\sigma) = G_0^{\mu\nu}(\mathbf{x};\mu^2-\nu_5,\sigma+\nu-\nu_5) \tag{7}$$

*with $\nu = -\nu_5 = q^2/(2\pi)$.*

This result is uniform in $\alpha > a_0$: Feynman-'t Hooft's and Landau's gauges, for example, are recovered for $\alpha = 1$ and $\alpha \to \infty$, respectively; the two point correlation of $F^{\mu\nu}$ is $\alpha$-independent. Although (7) means that the interaction changes the free meson correlation of an additional mass term only, the theory is not free; indeed, in (6) we can see that the large distance decay of the fermion correlation has an anomalous dimension $\eta$:

$$S(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{1+\eta}} \qquad \eta = \frac{q^2}{2\pi}\left[\frac{1}{\mu^2-\nu^5} - \frac{1}{\mu^2-\sigma-\nu}\right] \tag{8}$$

We shall see that $\nu$ and $\nu_5$ are the anomalies of the vector and axial WI's, respectively. To clarify the relation of this result with the literature, it is worth mentioning the (unproven) *uncertainty principle* of the anomalies, [23],[24], [19], [13], namely the fact that the most general numerical values are

$$\nu = \frac{q^2}{\pi}(1-\xi) \qquad \nu_5 = -\frac{q^2}{\pi}\xi$$

for $\xi$ a real parameter fixed by the kind of regularization of the functional integral. Hence the meson mass, $\nu_5$, is regularization dependent (this is not an issue for the meaning of the model, because $q$ is the *bare*, not the *physical* charge). Our result is in agreement with [39], where $\sigma = 0$, $\xi = 1$ and $\alpha = 1$; and with [19], where $\xi$ is any, $\sigma = 0$, $\alpha = \infty$.

Solutions of the Vector Meson model for $\alpha = 0$ (a case that this paper doesn't cover) are in [37], [14] and [24], for $\xi = 1/2$, $\xi = 1$ and any $\xi$, respectively. Those results are in agreement with our theorem only formally: in that case $F(\mathbf{x},\mu^2,\nu)$ is not defined and from $S(\mathbf{x})$ one has to divide out a divergent factor. This is not a surprise: when $\alpha = 0$ the large momentum asymptotic of the free meson propagator is

$$\widehat{G}_0^{\mu\nu}(\mathbf{k}) \sim \mathbf{k}^\mu\mathbf{k}^\nu/\mathbf{k}^2$$

that makes the interaction *renormalizable* as in the case of the gradient coupling model, [19]. The correct approach for this case would be the one in [5], with vanishing beta function and field renormalization. Anyways that rises a question: the AB-formula is not valid in [5], where radiative corrections do change the numerical value of the anomaly; is this the case also for the $\alpha = 0$ Vector Meson model? We will discuss this issue in a possible forthcoming paper.

As mentioned in the Introduction, were $\mu = 0$, we would read (7) as the *dynamical mass generation* of the Schwinger model; but unfortunately we are not able to cover that case.

## 3 Idea of the proof. Ward Identities. Anomalies

Firstly, we have to prove that there exist the limits (3) and (4). To do that, we use some Lesniewski's ideas, [27]. Define the functional integral

$$\mathcal{W}_{l,N}(J,\eta) = \ln \int dP_{l,N}(\psi)\ e^{\mathcal{V}(\psi,J,\eta)}$$

where $\mathcal{V}(\psi, J, \eta)$ is the interaction (self-interaction plus coupling with external fields) of a non-local version of the Thirring model:

$$\mathcal{V}(\psi,J,\eta) = \frac{\lambda}{4}\int d\mathbf{x}d\mathbf{y}\ (\bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}})\ G_0^{\mu\nu}(\mathbf{x}-\mathbf{y})\ (\bar\psi_{\mathbf{y}}\gamma^\nu\psi_{\mathbf{y}}) + \int d\mathbf{x}\ J^\mu_{\mathbf{x}}\ (\bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}}) + \int d\mathbf{x}\ (\bar\eta_{\mathbf{x}}\psi_{\mathbf{x}} + \bar\psi_{\mathbf{x}}\eta_{\mathbf{x}}) \qquad (9)$$

Now take the integral over the vector field $A^\mu$ in (1), and obtain an identity between the functional integrals of the Thirring and the Vector meson models, for coupling $\lambda = 2q^2$

$$\mathcal{K}(DJ,\eta) = \mathcal{W}(-qJ,\eta) + \frac{1}{2}\int d\mathbf{x}d\mathbf{y}\ J^\mu_{\mathbf{x}} D^{\mu\nu}(\mathbf{x}-\mathbf{y}) J^\nu_{\mathbf{y}}\ . \qquad (10)$$

If we can control the field and the current correlation derived from $\mathcal{W}$, we can also construct the limits (3) and (4). To bound the Feynman graphs, one has to use that, above the scale of the photon mass, the interaction in $\mathcal{K}$ is superrinormalizable; but, to see that in $\mathcal{W}$, we will need some identities among Feynman graphs, as in [27], [29] and [20]. This point is discussed in Section 4.1. Then, with a classical bound for fermion determinant, [27], the convergence of the perturbation theory is proved.

Secondly, in Section 4.2, we shall prove that the correlations generated from $\mathcal{W}$ satisfy two anomalous WI's. From the (formal) invariance under the vector transformation $\psi \to e^{i\vartheta}\psi$, $\bar\psi \to \bar\psi e^{-i\vartheta}$ we obtain the *vector Ward identity*,

$$i\partial^\mu_{\mathbf{x}}\int d\mathbf{z}\ [\delta^{\mu\nu}\delta(\mathbf{x}-\mathbf{z}) - \nu G_0^{\mu\nu}(\mathbf{x}-\mathbf{z})]\,\frac{\partial\mathcal{W}(J,\eta)}{\partial J^\nu_{\mathbf{z}}} - 2\Theta i\partial^\mu_{\mathbf{x}} J^\mu_{\mathbf{x}} = \frac{\partial\mathcal{W}(J,\eta)}{\partial\eta_{\mathbf{x}}}\eta_{\mathbf{x}} - \bar\eta_{\mathbf{x}}\frac{\partial\mathcal{W}(J,\eta)}{\partial\bar\eta_{\mathbf{x}}}\ ; \qquad (11)$$

and from the (formal) invariance under (Euclidean) axial-vector transformation $\psi \to e^{\gamma^5\vartheta}\psi$, $\bar\psi \to \bar\psi e^{\gamma^5\vartheta}$, with $\gamma^5 = -i\gamma^0\gamma^1$, using $\gamma^\mu\gamma^5 = -i\varepsilon^{\mu\nu}\gamma^\nu$, we obtain the *axial Ward identity*

$$\varepsilon^{\mu\rho} i\partial^{\mu}_{\mathbf{x}} \int d\mathbf{z}\ \left[\delta^{\rho\nu}\delta(\mathbf{x}-\mathbf{z}) - \nu_5 G_0^{\rho\nu}(\mathbf{x}-\mathbf{z})\right] \frac{\partial \mathcal{W}(J,\eta)}{\partial J^{\nu}_{\mathbf{z}}} - 2\Theta_5 \varepsilon^{\rho\mu} i\partial^{\rho}_{\mathbf{x}} J^{\mu}_{\mathbf{x}}$$
$$= -\frac{\partial \mathcal{W}(J,\eta)}{\partial \eta_{\mathbf{x}}} i\gamma^5 \eta_{\mathbf{x}} - \bar\eta_{\mathbf{x}} i\gamma^5 \frac{\partial \mathcal{W}(J,\eta)}{\partial \bar\eta_{\mathbf{x}}}\ . \qquad (12)$$

$\nu$, $\nu_5$, $\Theta$ and $\Theta_5$ are the *Adler-Bell-Jackiw anomalies.* Assuming the validity of (11) and (12), the proof of (7) is just a computation in which the AB-formula plays a crucial role. Before showing that, let's pause for some technical comments. At $J = 0$ (i.e. without the terms proportional to $\Theta$ and $\Theta_5$), (11) and (12) were proved in [5] for local self interaction of the fermion field, i.e. $G_0^{\mu\nu}(\mathbf{x}) = \delta^{\mu\nu}\delta(\mathbf{x})$; and later on they were proved in [29] for $G_0^{\mu\nu}(\mathbf{x}) = \delta^{\mu\nu} v(\mathbf{x})$ where $v(\mathbf{x})$ is a short-range, bounded self interaction, i.e. without removing IR and UV cutoffs in $v(\mathbf{x})$. Of course the latter case is technically simpler; nonetheless it is remarkable for, as opposed to the former, it gives an example in which the AB-formula is valid. The main task of the present paper is to extend the proof of the WI's to the case $J \neq 0$ and for the given $G_0^{\mu\nu}$, that is a symmetric matrix with short-ranged but unbounded entries.

To continue the computation, use (10) to turn (11) and (12) into identities for derivatives of $\mathcal{K}$; since

$$\partial^{\mu}_{\mathbf{x}} D^{\mu\nu}_{\mathbf{x}} = \left[-\alpha\Delta_{\mathbf{x}} + \mu^2 - \sigma\right]\partial^{\nu}_{\mathbf{x}}\ , \qquad \varepsilon^{\mu\rho}\partial^{\mu}_{\mathbf{x}} D^{\rho\nu}_{\mathbf{x}} = \left[-\Delta_{\mathbf{x}} + \mu^2\right]\varepsilon^{\rho\nu}\partial^{\rho}_{\mathbf{x}}$$

take a further derivative in $J^{\sigma}_{\mathbf{y}}$, at $\eta = \bar\eta = J = 0$ and obtain:

$$\left[-\alpha\Delta_{\mathbf{x}} + \mu^2 - \sigma - \nu\right] i\partial^{\mu}_{\mathbf{x}} G^{\mu\nu}(\mathbf{x}-\mathbf{y}) = i\partial^{\nu}_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}) +$$
$$+ \left(2q^2\Theta - \nu\right) i\partial^{\mu}_{\mathbf{x}} G_0^{\mu\nu}(\mathbf{x}-\mathbf{y}) \qquad (13)$$

$$\left[-\Delta_{\mathbf{x}} + \mu^2 - \nu_5\right]\varepsilon^{\rho\mu} i\partial^{\rho}_{\mathbf{x}} G^{\mu\nu}(\mathbf{x}-\mathbf{y}) = \varepsilon^{\rho\nu} i\partial^{\rho}_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}) +$$
$$+ \left(2q^2\Theta_5 - \nu_5\right)\varepsilon^{\rho\mu} i\partial^{\rho}_{\mathbf{x}} G_0^{\mu\nu}(\mathbf{x}-\mathbf{y}) \qquad (14)$$

Here comes the crucial point: in establishing the validity of the WI and the presence of the anomaly, we will also verify the AB-formula, i.e. the fact the anomaly is given by the first order perturbation theory, without higher order corrections. Therefore we can explicitly evaluate $\nu = \lambda\Theta = -\nu_5 = -\lambda\Theta_5 = \frac{\lambda}{4\pi}$. Since $\lambda = 2q^2$, the second line in both equations is zero; and the theorem is proved by explicit solution of (13) and (14). We stress that in formal versions of this computation, [32], the quantum anomaly is just added into the classical equations by hand where it is expected, and so the terms proportional to $2q^2\Theta - \nu$ and $2q^2\Theta_5 - \nu_5$ never appear at all.

To prove (6), we need the SDE, i.e. the field equations written in terms of the correlations. In the functional integral approach that is nothing but the Wick theorem for the Gaussian measure. For the fermion fermion correlation we have:

$$i\gamma^\mu\partial^\mu_{\mathbf{x}}\frac{\partial^2\mathcal{W}}{\partial\bar\eta_{\mathbf{x}}\partial\eta_{\mathbf{y}}}(0,0)=\delta(\mathbf{x}-\mathbf{y})+\frac{\lambda}{2}\gamma^\mu\int d\mathbf{z}\; G_0^{\mu\nu}(\mathbf{x}-\mathbf{z})\frac{\partial^3\mathcal{W}}{\partial J^\nu_{\mathbf{z}}\partial\bar\eta_{\mathbf{x}}\partial\eta_{\mathbf{y}}}(0,0) \tag{15}$$

That is not a closed equation, though we can use the WI to close it. Take derivatives in (11) and (12) w.r.t. $\bar\eta_{\mathbf{x}}$ and $\eta_{\mathbf{y}}$ at $J=\bar\eta=\eta=0$: we obtain two equations that are equivalent to

$$\begin{aligned}\int d\mathbf{z}\; G_0^{\mu\nu}(\mathbf{x}-\mathbf{z})\frac{\partial^3\mathcal{W}(0,0)}{\partial J^\nu_{\mathbf{z}}\partial\bar\eta_{\mathbf{w}}\eta_{\mathbf{y}}}&=i\partial^\mu_{\mathbf{x}}\left[F(\mathbf{x}-\mathbf{y})-F(\mathbf{x}-\mathbf{w})\right]\Big|_{\substack{\mu^2-\nu_5\\ \sigma+\nu-\nu_5}}S(\mathbf{w}-\mathbf{y})\\ &+\varepsilon^{\rho\mu}i\partial^\rho_{\mathbf{x}}\left[F_5(\mathbf{x}-\mathbf{y})-F_5(\mathbf{x}-\mathbf{w})\right]\Big|_{\substack{\mu^2-\nu_5\\ \sigma+\nu-\nu_5}}i\gamma^5S(\mathbf{w}-\mathbf{y})\end{aligned} \tag{16}$$

(we have abridged the notation of the mass terms in $F$ and $F_5$). Plug (16) into (15), and obtain the closed equation

$$i\gamma^\mu\partial^\mu_{\mathbf{x}}S(\mathbf{x}-\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})+\frac{\lambda}{2}i\gamma^\mu\left[\partial^\mu_{\mathbf{x}}F(\mathbf{x}-\mathbf{y})-\partial^\mu_{\mathbf{x}}F_5(\mathbf{x}-\mathbf{y})\right]\Big|_{\substack{\mu^2-\nu_5\\ \sigma+\nu-\nu_5}}S(\mathbf{x}-\mathbf{y}) \tag{17}$$

that is solved by (6). There is a subtle point, though. (16) can be plugged into (17) after the limit of removed cutoff has been removed only if one proves that the limit is continuous at $\mathbf{w}=\mathbf{x}$. In Section 4.3 we will prove (17) in a slightly different way: we will plug the WI into the SDE *before* removing the cutoffs, and we will show that the the limit of the remainder is vanishing (as opposed to the case in [5], where the limit of the remainder gives rise to a further anomaly in the closed equation).

Summarizing, in Section 4.1 we will study the limit of removed cutoffs; in Section 4.2 we will prove (11) and (12); in Section (4.3) we will prove (17).

# 4 Renormalization group approach

As mentioned, from the viewpoint of the formal power series in $q$, the RG description of (1) is quite simple. Above the meson mass scale, i.e. in the UV regime, the coupling of a fermion current with a boson field is superrenormalizable; below the meson mass scale, i.e. the IR regime, the interaction is renormalizable, and the RG flow equals, up to irrelevant terms, the flow of the Thirring model. A qualitative explanation is the following. At energy $E>\mu$, boson and fermion propagators have typical sizes $E^0$ and $E^1$, respectively; then, the energy of a graph with $p$ vertexes, $2m$ external fermion legs and $n$ external boson legs is $E^{d(p,n,2m)}$, for $d(p,n,2m)=2-m-p$: the only relevant graphs are $(p,2m)=(1,1)$, the uncontracted vertex, and $(p,2m)=(2,0)$, which is zero by symmetry; so no renormalization of coupling constants is required. At $E < \mu$, the size of the boson propagator becomes $(E/\mu)^2$; then a graph size is $\mu^{n-p} E^{d'(p,n,2m)}$, for $d'(p,n,2m) = 2 - m - n$, that is the same *power counting* of the Thirring model. Still qualitatively, the limit $\mu \to 0$ gives the Schwinger model; the limit $\mu \to \infty$ gives the free boson field; whereas replacing $G_0^{\mu\nu}$ with $\mu^2 G_0^{\mu\nu}$, and taking the limit $\mu \to \infty$ give the Thirring model.

In this paper we consider a fixed $\mu > 0$. The issue with the above argument is that we are not able to prove the convergence of the perturbation theory with both boson and fermion fields. To overcome that we integrate the boson field before the RG analysis so that the fermion-boson interaction is turned into a fermion-fermion quartic interaction. Now the theory looks marginal at any scale. To recover the superrenormalizability of the UV scales we use identities among the Feynman graphs: the identities of this paper are the same as in [20] and [7]; as opposed to approach in [29], [30], they permit to take advantage of $L_p$ inequalities, which is the key to control an unbounded $G_0$. This part is largely inspired to [27].

Before discussing technical details, we set up some more notations. The explicit choice of generators of the Euclidean Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ;$$

then we call $\eta = (\eta^-_{\mathbf{x},-}, \eta^-_{\mathbf{x},+})$, $\bar\eta = (\eta^+_{\mathbf{x},+}, \eta^+_{\mathbf{x},-})$, $\psi_{\mathbf{x}} = (\psi^-_{\mathbf{x},+}, \psi^-_{\mathbf{x},-})$ and $\bar\psi_{\mathbf{x}} = (\psi^+_{\mathbf{x},-}, \psi^+_{\mathbf{x},+})$ - note the opposite notation for the components of the spinors $\bar\eta$, $\eta$ and $\bar\psi$, $\psi$ - so that, for $\partial_{\mathbf{x},\omega} = i\partial^0 - \omega\partial^1$,

$$\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} = \bar\psi_{\mathbf{x}} \frac{\gamma^0 - i\omega\gamma^1}{2} \psi_{\mathbf{x}} , \qquad \sum_\omega \psi^+_{\mathbf{x},\omega} \partial_\omega \psi^-_{\mathbf{x},\omega} = i\bar\psi_{\mathbf{x}} \gamma^\mu \partial^\mu \psi_{\mathbf{x}} .$$

The interaction becomes

$$v_{\omega,\omega'}(\mathbf{x}) = \frac{1}{2} \left[ i\omega G_0^{10}(\mathbf{x}) + i\omega' G_0^{01}(\mathbf{x}) + G_0^{00}(\mathbf{x}) - \omega\omega' G_0^{11}(\mathbf{x}) \right] ;$$

and since $G_0^{01}(\mathbf{x}) = G_0^{10}(\mathbf{x})$, also $v_{\omega,\omega'}(\mathbf{x}) = v_{\omega',\omega}(\mathbf{x})$. Finally, for $J_{\mathbf{x},\omega} = J^0_{\mathbf{x}} + i\omega J^1_{\mathbf{x}}$, the functional integral formula for the non-local Thirring model is

$$e^{\mathcal{W}_{l,N}(J,\eta)} = \int dP_{l,N}(\psi) \ \exp\left\{ \frac{\lambda}{2} \sum_{\omega,\omega'} \int d\mathbf{x} d\mathbf{y} \ \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \ v_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) \ \psi^+_{\mathbf{x},\omega'}\psi^-_{\mathbf{x},\omega'} \right\}$$
$$\exp\left\{ \sum_\omega \int d\mathbf{x} \ J_{\mathbf{x},\omega} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \sum_\omega \int d\mathbf{x} \ (\eta^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega}) \right\} \quad (18)$$

for $dP_{l,N}(\psi)$ determined by the covariance:

$$\int dP_{l,N}(\psi) \ \psi^-_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega} = g^{[l,N]}_\omega(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{e^{-i\mathbf{k}\mathbf{x}}}{D_\omega(\mathbf{k})} \widehat\chi_{l,N}(\mathbf{k}) , \qquad D(\mathbf{k}) = k^0 + i\omega k^1 .$$

Without loss of generality, since $\mu$ is constant, we absorb in the interaction $v_{\omega,\omega'}$ a factor $\mu^2$: as explained, this clarifies the relation with the RG analysis of the Thirring model.

Finally, we stress two points. Firstly, we are assuming that the limit of removed cutoff in the propagator $v_{\omega,\omega'}$ is already taken: this is not an abuse, since, otherwise, the estimates that will follow would be anyways uniform in the $l, N$ of $v_{\omega,\omega'}$. Secondly, all the claims about $\mathcal{W}(J,\eta)$ (and the same for other functional integrals that we are about to define), must actually be understood in terms of the correlations that it generates, i.e. for a finite number of derivatives w.r.t the external fields, at $J = \eta = 0$. (In fact there would be no need to prove that $\mathcal{W}(J,\eta)$ is a convergent power series of the external fields even if we wanted to verify the Osterwalder-Schrader axioms, see [5].)

## 4.1 Correlations.

In evaluating $\mathcal{W}_{l,N}(J,\eta)$, to have bounds that are uniform in $l$ and $N$, we have to slice the range of allowed momenta into scales. We use the decomposition

$$\widehat{\chi}_{h',h}(\mathbf{k}) = \sum_{k=h'+1}^{h} f_k(\mathbf{k})$$

where $f_k(\mathbf{k}) = \widehat{\chi}_{k-1,k}(\mathbf{k})$; in correspondence we have the factorization of the Gaussian measure

$$\begin{aligned} &\psi = \psi^{(h')} + \psi^{(h'+1)} + \cdots + \psi^{(h)} \\ &dP_{h',h}(\psi) = dP_{h'}(\psi^{(h')}) dP_{h'+1}(\psi^{(h'+1)}) \cdots dP_h(\psi^{(h)}) \end{aligned} \tag{19}$$

and $dP_k(\psi^{(k)})$ is determined by the covariance

$$g_\omega^{[k]}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \, \frac{e^{-i\mathbf{k}\mathbf{x}}}{D_\omega(\mathbf{k})} f_k(\mathbf{k}) \; .$$

We integrate iteratively the fields with smaller and smaller momentum. After the integration of $\psi = \psi^{(N)}, \psi^{(N-1)}, \ldots, \psi^{(h+1)}$ we have the effective potential on scale $h$, $\mathcal{V}^{(h)}$, such that

$$e^{\mathcal{W}_{l,N}(J,\eta)} = \int dP_{l,h}(\psi) \; e^{\mathcal{V}^{(h)}(\psi,J,\eta)} \; . \tag{20}$$

Consider the case $\bar\eta = \eta = 0$. Assume by induction that, for any scale $h = k+1$, the effective potential is a polynomial in the fields $(J_{\mathbf{z},\omega})$, $(\psi^+_{\omega,\mathbf{x}})$ and $(\psi^-_{\omega,\mathbf{y}})$. We call kernels on scale $h$ the coefficients of the monomials of $\mathcal{V}^{(h)}$: for $\underline{\mathbf{z}} =$

$(\mathbf{z}_1,\ldots,\mathbf{z}_n)$, $\underline{\mathbf{x}} = (\mathbf{x}_1,\ldots,\mathbf{x}_m)$, $\underline{\mathbf{y}} = (\mathbf{y}_1,\ldots,\mathbf{y}_m)$, $\underline{\omega}' = (\omega_1,\ldots\omega_n)$ and $\underline{\omega} = (\omega_1,\ldots\omega_m)$,

$$W^{(n;2m)(h)}_{\underline{\omega}',\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}}) \stackrel{\text{def.}}{=} \prod_{j=1}^{n}\frac{\partial}{\partial J_{\mathbf{z}_i,\omega_i'}}\prod_{i=1}^{m}\frac{\partial}{\partial\psi^+_{\mathbf{x}_i,\omega_i}}\frac{\partial}{\partial\psi^-_{\mathbf{y}_i,\omega_i}}\mathcal{V}^{(h)}(\psi,J,0)\Big|_{J=\psi=0} \tag{21}$$

(where the derivatives in $\partial\psi^-_{\mathbf{y}_i,\omega_i}$ are taken from the right). To evaluate $\mathcal{V}^{(k)}(\psi,J,0)$, use the formula for the truncated expectations:

$$\begin{aligned}\mathcal{V}^{(k)}(\psi,J,0) &= \ln\int dP_{k+1}(\zeta)\; e^{\mathcal{V}^{(k+1)}(\psi+\zeta,J,0)}\\ &= \sum_{p\geq 0}\frac{1}{p!}E^T_{k+1}\big[\underbrace{\mathcal{V}^{(k+1)}(\psi+\zeta,J,0);\cdots;\mathcal{V}^{(k+1)}(\psi+\zeta,J,0)}_{p\ \text{times}}\big]\end{aligned} \tag{22}$$

where $E^T_{k+1}$ is by definition the truncated expectation w.r.t. the Gaussian random variables $(\zeta^\varepsilon_{\mathbf{x},\omega})$ with covariances $(g^{[k+1]}_\omega(\mathbf{x}))$. Accordingly, (22) gives through (21) the kernels $W^{(n;2m)(k)}_{\underline{\omega}',\underline{\omega}}$. Formula (22) gives also the well known interpretation of each kernels as sum of Feynman graphs belonging to a given class, that is determined by the “external legs”. Later on, we will take advantage of the following two identities on the structure of the graph expansion of the kernels.

**Lemma 4.1** *The derivatives of the effective potential satisfy two identities:*

$$\begin{aligned}&\frac{\partial\mathcal{V}^{(k)}}{\partial\psi^+_{\mathbf{x},\omega}}(\psi,J,0) = J_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + J_{\mathbf{x},\omega}\int d\mathbf{u}\; g_\omega(\mathbf{x}-\mathbf{u})\frac{\partial\mathcal{V}^{(k)}}{\partial\psi^+_{\mathbf{u},\omega}}(\psi,J,0)\\ &+\lambda\sum_{\omega,\omega'}\int d\mathbf{w}d\mathbf{u}\; v_{\omega,\omega'}(\mathbf{x}-\mathbf{w})g_\omega(\mathbf{x}-\mathbf{u})\left[\frac{\partial^2\mathcal{V}^{(k)}}{\partial J_{\mathbf{w},\omega'}\partial\psi^+_{\mathbf{u},\omega}} + \frac{\partial\mathcal{V}^{(k)}}{\partial J_{\mathbf{w},\omega'}}\frac{\partial\mathcal{V}^{(k)}}{\partial\psi^+_{\mathbf{u},\omega}}\right](\psi,J,0)\\ &+\lambda\sum_{\omega,\omega'}\int d\mathbf{w}\; v_{\omega,\omega'}(\mathbf{x}-\mathbf{w})\psi^-_{\mathbf{x},\omega}\frac{\partial\mathcal{V}^{(k)}}{\partial J_{\mathbf{w},\omega'}}(\psi,J,0)\,,\end{aligned} \tag{23}$$

$$\begin{aligned}&\frac{\partial\mathcal{V}^{(k)}}{\partial J_{\mathbf{x},\omega}}(\psi,J,\eta) = \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \int d\mathbf{u}\; g_\omega(\mathbf{x}-\mathbf{u})\left[\psi^+_{\mathbf{x},\omega}\frac{\partial\mathcal{V}^{(k)}}{\partial\psi^+_{\mathbf{u},\omega}} - \frac{\partial\mathcal{V}^{(k)}}{\partial\psi^-_{\mathbf{u},\omega}}\psi^-_{\mathbf{x},\omega}\right](\psi,J,\eta)\\ &+\int d\mathbf{u}d\mathbf{u}'\; g_\omega(\mathbf{x}-\mathbf{u})g_\omega(\mathbf{x}-\mathbf{u}')\left[\frac{\partial^2\mathcal{V}^{(k)}}{\partial\psi^+_{\mathbf{u},\omega}\partial\psi^-_{\mathbf{u}',\omega}} + \frac{\partial\mathcal{V}^{(k)}}{\partial\psi^+_{\mathbf{u},\omega}}\frac{\partial\mathcal{V}^{(k)}}{\partial\psi^-_{\mathbf{u}',\omega}}\right](\psi,J,\eta)\,.\end{aligned} \tag{24}$$

These identities are clear from graphical interpretation of the multiscale integration; in appendix A we will prove them from the definition of $\mathcal{V}^{(k)}$ and $\mathcal{V}$.

We introduce the following $L_1$ norm

$$\|W^{(n;2m)(k)}_{\underline{\omega}',\underline{\omega}}\| = \int d\underline{\mathbf{x}}d\underline{\mathbf{y}}d\underline{\mathbf{z}}_2\;\left|W^{(n;2m)(k)}_{\underline{\omega}',\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})\right| \tag{25}$$

for $d\underline{\mathbf{z}}_2 = (\mathbf{z}_2, \ldots, \mathbf{z}_m)$; namely in (25) we are integrating all but one variable; by translation invariance, the norm does not depend upon $\mathbf{z}_1$. Since $W^{(n;2m)(k)}_{\underline{\omega}',\underline{\omega}}$ may contain delta-distributions, we extend the definition of $L_1$ norm by considering them as positive functions.

Let $\mu^2 = \gamma^{2M}$. We will use the following straightforward bounds, for $c, c_p, c', B, B_p > 1$:

$$
\begin{aligned}
&\|g^{(h)}_\omega\|_{L_\infty} \stackrel{\text{def.}}{=} \sup_{\mathbf{x}} |g^{(h)}_\omega(\mathbf{x})| \le c\gamma^h , \\
&\|g^{(h)}_\omega\|_{L_p} \stackrel{\text{def.}}{=} \left[\int d\mathbf{x}\ |g^{(h)}_\omega(\mathbf{x})|^p\right]^{1/p} \le c_p \gamma^{\left(1-\frac{2}{p}\right)h} \\
&\|g^{(h)}_\omega\|_{L_1(w)} \stackrel{\text{def.}}{=} \int d\mathbf{x}\ |x_j||g^{(h)}_\omega(\mathbf{x})| \le c'\gamma^{-2h}
\end{aligned}
\tag{26}
$$

and, since $\alpha\mathbf{k}^2 + \mu^2 - \sigma \ge \alpha_0[\mathbf{k}^2 + \alpha_0^{-1}(\mu^2 - \sigma)]$, uniformly in $\alpha \ge \alpha_0$

$$
\begin{aligned}
&\|v_{\omega,\omega'}\|_{L_p} \stackrel{\text{def.}}{=} \left[\int d\mathbf{x} |v_{\omega,\omega'}(\mathbf{x})|^p\right]^{1/p} \le B_p \gamma^{2\left(1-\frac{1}{p}\right)M} \\
&\|\partial_j v_{\omega,\omega'}\|_{L_1} \stackrel{\text{def.}}{=} \int d\mathbf{x}\ |(\partial_j v_{\omega,\omega'})(\mathbf{x})| \le B\gamma^M .
\end{aligned}
\tag{27}
$$

Let's consider separately the two different regimes: the UV one, that corresponds to the scales $k : M \le k \le N$, and the IR, for $k : l \le k \le M - 1$.

Define

$$
\begin{aligned}
&w^{(1;2)}_{\omega',\omega}(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \delta(\mathbf{z} - \mathbf{x})\delta(\mathbf{z} - \mathbf{y})\delta_{\omega,\omega'} \\
&w^{(0;4)}_{\omega',\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \delta(\mathbf{x} - \mathbf{y})v_{\omega',\omega}(\mathbf{x} - \mathbf{u})\delta(\mathbf{u} - \mathbf{v})
\end{aligned}
\tag{28}
$$

and note that at $h = N$ we have $W^{(0;2)(N)}_\omega(\mathbf{x}, \mathbf{y}) = 0$, $W^{(1;2)(N)}_{\omega',\omega}(\mathbf{z}, \mathbf{x}, \mathbf{y}) =$

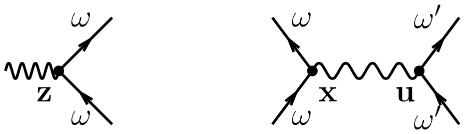


Figure 1: Graphical representation for $w^{(1;2)}_{\omega',\omega}$ and $w^{(0;4)}_{\omega',\omega}$

$w^{(1;2)}_{\omega',\omega}(\mathbf{z}, \mathbf{x}, \mathbf{y})$ and $W^{(0;4)(N)}_{\omega',\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \lambda w^{(0;4)}_{\omega',\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$; all the other $W^{(n;2m)(N)}_{\underline{\omega}',\underline{\omega}}$ are zero.

**Theorem 4.2** *For $|\lambda|$ small enough, there exist constants $C_0, C > 1$ such that, for $M \le h \le N$*

$$
\|W^{(0;2)(h)}_\omega\| \le C|\lambda|\gamma^h ,
$$

$$\|W^{(1;2)(h)}_{\omega',\omega} - w^{(1;2)}_{\omega',\omega}\| \leq C|\lambda| \; ,$$

$$\|W^{(0;4)(h)}_{\omega',\omega} - \lambda w^{(0;4)}_{\omega',\omega}\| \leq C|\lambda| \; ; \tag{29}$$

*and, for any other* $(n;2m)$

$$\|W^{(n;2m)(h)}_{\underline{\omega}',\underline{\omega}}\| \leq C_0^{n+d_{n,2m}}(C|\lambda|)^{d_{n,2m}}\gamma^{h(2-n-m)} \tag{30}$$

*where* $d_{0,2}=1$*,* $d_{n,0}=0$*, otherwise* $d_{n,2m}=m-1$*.*

The point in the bounds is that $C$, $C_0$ are $N-h$ independent. The proof of (30) for $h=k$, assumed iteratively (29) and (30) for $h\geq k+1$, is standard. We shall focus, therefore, on (29) that improves (30) in the cases of *marginal and relevant graphs*, i.e. $(n;2m)=(0;2),(0;4),(1;2)$.

**Proof.** To shorten the notation, in this proof we define $\zeta \stackrel{\text{def.}}{=} \psi^{(k+1)}+\psi^{(k+2)}+\cdots+\psi^{(N)}$ and $g_\omega \stackrel{\text{def.}}{=} g^{[k+1,N]}_\omega$. The derivatives in $\psi^-$, $\eta^-$ and $\zeta^-$ are taken from the right. The proof is for $C$ large enough with respect to $C_0, c, c_p, c', B, B_p$.

*1. Improved bound for* $(0;2)$*.* By symmetry we have $W^{(1;0)(k)}_{-\omega}(\mathbf{w})\equiv 0$; hence from (23) and (24) we expand the two-points kernel as in Fig.2

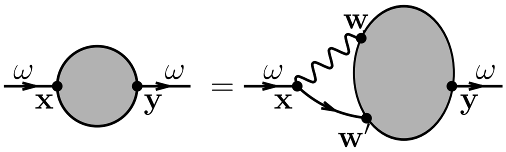


Figure 2: Decomposition of the class of graphs $W^{(0;2)(k)}_\omega$.

$$W^{(0;2)(k)}_\omega(\mathbf{x},\mathbf{y}) = \lambda\sum_{\omega'}\int d\mathbf{w}d\mathbf{w}'\; v_{\omega,\omega'}(\mathbf{x}-\mathbf{w})g_\omega(\mathbf{x}-\mathbf{w}')W^{(1;2)(k)}_{\omega';\omega}(\mathbf{w};\mathbf{w}',\mathbf{y}) \; , \tag{31}$$

so that, from $\|w^{(1;2)}_{\omega';\omega}\|\leq 1$ and from (30) for $(n;2m)=(1;2)$, we obtain, for $C$ large enough,

$$\|W^{(0;2)(k)}_\omega\| \leq |\lambda|(1+C_0)\sum_{\omega'}\|v_{\omega,\omega'}\|_{L_3}\sum_{j=k}^{N}\|g^{(j)}_\omega\|_{L_{3/2}} \leq \frac{C}{r_1}|\lambda|\gamma^k\gamma^{-\frac{4}{3}(k-M)} \; , \tag{32}$$

that proves the first of (29). The factor $r_1>1$ will be useful for later.

*2. Improved bound for* $(1;2)$*.* By (23), the kernel $W^{(1;2)(k)}_{\omega';\omega}$ can be rewritten as in Fig.3. Graph (a) in Fig.3 is given by:

$$W^{(1;2)(k)}_{(a)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) \stackrel{\text{def.}}{=} \lambda\sum_{\omega''}\int d\mathbf{w}d\mathbf{u}\; v_{\omega,\omega''}(\mathbf{x}-\mathbf{w})g_\omega(\mathbf{x}-\mathbf{u})W^{(2;2)(k)}_{\omega',\omega'';\omega}(\mathbf{z},\mathbf{w};\mathbf{u},\mathbf{y}) \tag{33}$$

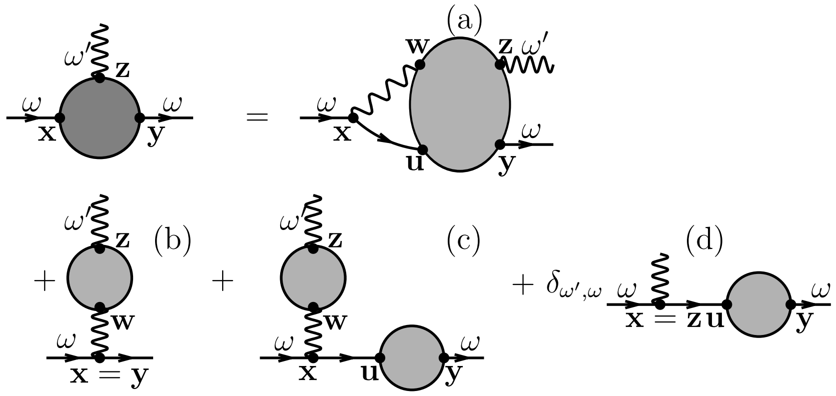


Figure 3: Decomposition of the class of graphs $W^{(1;2)(k)}_{\omega';\omega}$. The darker bubble is for $W^{(1;2)(k)}_{\omega';\omega} - w^{(1;2)}_{\omega';\omega}$

From (30) for $(n; 2m) = (2; 2)$, we obtain

$$\|W^{(1;2)(k)}_{(a);\omega';\omega}\| \leq |\lambda| C_0^2 \gamma^{-k} \sum_{\omega''} \|v_{\omega,\omega''}\|_{L_3} \sum_{j=k}^{N} \|g^{(j)}_\omega\|_{L_{3/2}} \leq \frac{C}{4r_2} |\lambda| \gamma^{-\frac{4}{3}(k-M)} \tag{34}$$

where a large enough constant $r_2 > 1$ will be used later. For graphs (c) and (d) we use the just improved bound for $W^{(0;2)(k)}_\omega$: for instance, graph (d) is given by

$$W^{(1;2)(k)}_{(d)\omega';\omega}(\mathbf{z}; \mathbf{x}, \mathbf{y}) \stackrel{\text{def.}}{=} \delta_{\omega,\omega'} \delta(\mathbf{x} - \mathbf{z}) \int d\mathbf{u}\; g_\omega(\mathbf{x} - \mathbf{u}) W^{(0;2)(k)}_\omega(\mathbf{u}, \mathbf{y}) \tag{35}$$

and using (32) for a $r_1$ large enough to compensate other constants, we get

$$\|W^{(1;2)(k)}_{(d)\omega';\omega}(\mathbf{z}; \mathbf{x}, \mathbf{y})\| \leq \|W^{(0;2)(k)}_\omega\| \sum_{j=k}^{N} \|g^{(j)}_\omega\|_{L_1} \leq \frac{C}{4r_2} |\lambda| \gamma^{-\frac{4}{3}(k-M)} \; . \tag{36}$$

In order to obtain an improved bound also for the graphs (b) of Fig.3, we need to further expand $W^{(2;0)(k)}_{\omega';-\omega}$. Using (24), we find

$$W^{(2;0)(k)}_{\omega',-\omega}(\mathbf{z}, \mathbf{w}) = \int d\mathbf{u}' d\mathbf{u}\; g_\omega(\mathbf{w} - \mathbf{u}) g_\omega(\mathbf{w} - \mathbf{u}') W^{(1;2)(k)}_{\omega';-\omega}(\mathbf{z}; \mathbf{u}', \mathbf{u}) \tag{37}$$

and then, replacing the expansion for $W^{(1;2)(k)}_{\omega';-\omega}(\mathbf{z}; \mathbf{u}', \mathbf{u})$ in the graph (37) we find for (b) what is depicted in Fig.4. Graphs (b4) and (b5) have been obtained also using the expansion (31). A bound for (b2) and (b3) can be found in the same way. Consider, for instance, the expression for (b2):

$$W^{(1;2)(k)}_{(b2)\omega';\omega}(\mathbf{z}; \mathbf{x}, \mathbf{y}) \stackrel{\text{def.}}{=} \lambda \delta(\mathbf{x} - \mathbf{y}) \int d\mathbf{w}\; v_{\omega',\omega}(\mathbf{x} - \mathbf{w}) g^2_{\omega'}(\mathbf{w} - \mathbf{z}) \tag{38}$$

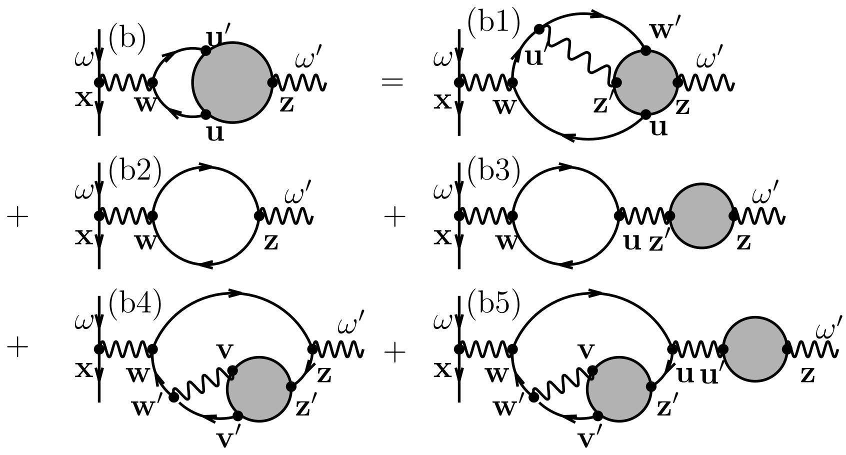


Figure 4: Further decomposition of the class of graphs (b) in Fig.3

We want to use the cancellation $\int d\mathbf{u}\ g_\omega^2(\mathbf{u}) = 0$ that is a consequence of the symmetry under rotation of the model. In order to to that, expand

$$v_{\omega,\omega}(\mathbf{x}-\mathbf{w}) = v_{\omega,\omega'}(\mathbf{x}-\mathbf{z}) + \sum_{j=0,1}(z_j - w_j)\int_0^1 d\tau\ (\partial_j v_{\omega,\omega'})(\mathbf{x}-\mathbf{z}+\tau(\mathbf{z}-\mathbf{w})) \quad (39)$$

and plug (39) into (38): one term in zero; the other can be bounded as follows:

$$\|W^{(1;2)(k)}_{(b2)\omega';\omega}\| \le 4|\lambda|\|\partial_j v_{\omega,\omega'}\|_{L_1}\sum_{i=k}^N\sum_{j=k}^i \|g^{(j)}_{\omega'}\|_{L_\infty}\|g^{(i)}_\omega\|_{L_1(w)} \le |\lambda|\frac{C}{20r_2}\gamma^{-(k-M)} \quad (40)$$

Now consider (b1)

$$W^{(1;2)(k)}_{(b1)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) \stackrel{\text{def.}}{=} \lambda\delta(\mathbf{x}-\mathbf{y})\sum_{\sigma,\sigma'}\int d\mathbf{w}d\mathbf{u}'d\mathbf{z}'\ v_{\omega,\sigma}(\mathbf{x}-\mathbf{w})v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')\cdot$$
$$\cdot\int d\mathbf{u}d\mathbf{w}'\ g_\sigma(\mathbf{w}-\mathbf{u})g_\sigma(\mathbf{w}-\mathbf{u}')g_\sigma(\mathbf{u}'-\mathbf{w}')W^{(2;2)(k)}_{\omega',\sigma';\sigma}(\mathbf{z},\mathbf{z}';\mathbf{w}',\mathbf{u})\ ; \quad (41)$$

therefore

$$\|W^{(1;2)(k)}_{(b1)\omega';\omega}\| \le |\lambda|\|v_{\omega,\sigma}\|_{L_1}\int d\mathbf{w}'\ d\mathbf{u}\ d\mathbf{z}'\ \ |W^{(2;2)(k)}_{\omega',\sigma';\sigma}(0,\mathbf{z}';\mathbf{w}',\mathbf{u})|\cdot$$
$$\cdot\int d\mathbf{u}'\ d\mathbf{w}\ \ |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma(\mathbf{w}-\mathbf{u})g_\sigma(\mathbf{w}-\mathbf{u}')g_\sigma(\mathbf{u}'-\mathbf{w}')|\ . \quad (42)$$

We have to find a bound for the second line that is uniform in $N-k$. For that, it is convenient to decompose the three fermion propagators into scales, $\sum_{j,q,p=k}^N g^{(j)}_\sigma g^{(q)}_\sigma g^{(p)}_\sigma$ and then, for each realization of $j,q,p$, we take the $\|\cdot\|_{L_1}$ on the fermion propagator with lowest scale. This is always possible:
for $p \le q, j$

$$\int d\mathbf{w}\ d\mathbf{u}'\ |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g^{(j)}_\sigma(\mathbf{u}'-\mathbf{w})g^{(q)}_\sigma(\mathbf{u}'-\mathbf{w}')g^{(p)}_\sigma(\mathbf{w}-\mathbf{u})|$$

$$= \int d\mathbf{u}' \, |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma^{(q)}(\mathbf{u}'-\mathbf{w}')| \int d\mathbf{w} \, |g_\sigma^{(j)}(\mathbf{u}'-\mathbf{w})g_\sigma^{(p)}(\mathbf{w}-\mathbf{u})|$$

$$\leq \|v_{\sigma,\sigma'}\|_{L_3}\|g_\omega^{(q)}\|_{L_{3/2}}\|g_\sigma^{(j)}\|_{L_1}\|g_\sigma^{(p)}\|_{L_\infty} \leq C_3\gamma^{\frac{4}{3}M}\gamma^{-\frac{q}{3}}\gamma^{-j}\gamma^{p} \tag{43}$$

for $q \leq p, j$

$$\int d\mathbf{w} \, d\mathbf{u}' \, |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma^{(j)}(\mathbf{u}'-\mathbf{w})g_\sigma^{(q)}(\mathbf{u}'-\mathbf{w}')g_\sigma^{(p)}(\mathbf{w}-\mathbf{u})|$$

$$= \int d\mathbf{w} \, |g_\sigma^{(p)}(\mathbf{w}-\mathbf{u})| \int d\mathbf{u}' \, |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma^{(q)}(\mathbf{u}'-\mathbf{w}')g_\sigma^{(j)}(\mathbf{u}'-\mathbf{w})|$$

$$\leq \|g_\sigma^{(p)}\|_{L_1}\|v_{\sigma,\sigma'}\|_{L_3}\|g_\sigma^{(j)}\|_{L_{3/2}}\|g_\sigma^{(q)}\|_{L_\infty} \leq C_3\gamma^{\frac{4}{3}M}\gamma^{-p}\gamma^{-\frac{j}{3}}\gamma^{q} \tag{44}$$

and finally, for $j \leq p, q$

$$\int d\mathbf{w} \, d\mathbf{u}' \, |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma^{(j)}(\mathbf{u}'-\mathbf{w})g_\sigma^{(q)}(\mathbf{u}'-\mathbf{w}')g_\sigma^{(p)}(\mathbf{w}-\mathbf{u})|$$

$$= \int d\mathbf{w} \, |g_\sigma^{(p)}(\mathbf{w}-\mathbf{u})| \int d\mathbf{u}' \, |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma^{(q)}(\mathbf{u}'-\mathbf{w}')g_\sigma^{(j)}(\mathbf{u}'-\mathbf{w})|$$

$$\leq \|g_\sigma^{(p)}\|_{L_1}\|v_{\sigma,\sigma'}\|_{L_3}\|g_\sigma^{(q)}\|_{L_{3/2}}\|g_\sigma^{(j)}\|_{L_\infty} \leq C_3\gamma^{\frac{4}{3}M}\gamma^{-p}\gamma^{-\frac{q}{3}}\gamma^{j} \tag{45}$$

so that, summing over the scales $q, p, j$, we obtain

$$\int d\mathbf{w} \, d\mathbf{u}' \, |v_{\sigma,\sigma'}(\mathbf{u}'-\mathbf{z}')g_\sigma(\mathbf{u}'-\mathbf{w})g_\sigma(\mathbf{u}'-\mathbf{w}')g_\sigma(\mathbf{w}-\mathbf{u})| \leq C_4\gamma^{\frac{4}{3}M}\gamma^{-\frac{k}{3}} \tag{46}$$

From (46) and (42), we obtain:

$$\|W^{(1;2)(k)}_{(b1)\omega';\omega}\| \leq \frac{C}{20}|\lambda|\gamma^{-\frac{4}{3}(k-M)} \tag{47}$$

Finally, the latter argument applies also to the bounds of (b4) and (b5). For instance, the expression for (b4) is

$$W^{(1;2)(k)}_{(b4)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) \stackrel{\text{def.}}{=} \delta(\mathbf{x}-\mathbf{y})\lambda^2 \int d\mathbf{z}'d\mathbf{w} \; v_{\omega,\omega'}(\mathbf{x}-\mathbf{w})g_{\omega'}(\mathbf{w}-\mathbf{z})\cdot$$
$$\cdot\sum_\sigma \int d\mathbf{w}'d\mathbf{u}'d\mathbf{u} \; g_{\omega'}(\mathbf{w}-\mathbf{w}')g_{\omega'}(\mathbf{w}'-\mathbf{u})v_{\omega',\sigma}(\mathbf{w}'-\mathbf{u}')\cdot$$
$$\cdot W^{(1;2)(k)}_{\sigma;\omega'}(\mathbf{u}';\mathbf{u},\mathbf{z}')g_{\omega'}(\mathbf{z}'-\mathbf{z}) \tag{48}$$

Hence, the bound for such a kernel is:

$$\|W^{(1;2)(k)}_{(b4)\sigma';\sigma}\| \leq 2|\lambda|^2\|v_{\omega,\omega'}\|_{L_1}\int d\mathbf{z}' \, d\mathbf{u}' \, d\mathbf{u} \, |W^{(1;2)(k)}_{-\omega;\omega}(\mathbf{u}';\mathbf{u},\mathbf{z}')g_\omega(\mathbf{z}')|$$
$$\cdot\int d\mathbf{w} \, d\mathbf{w}' \, |g_\omega(\mathbf{w}-\mathbf{z})g_\omega(\mathbf{w}-\mathbf{w}')g_\omega(\mathbf{w}'-\mathbf{u})v_{\omega',\sigma}(\mathbf{w}'-\mathbf{u}')| \tag{49}$$

that by (46) becomes

$$\|W^{(1;2)(k)}_{(b4);\omega';\omega}\| \leq \frac{C}{20r_2}|\lambda|\gamma^{-\frac{4}{3}(k-M)} \tag{50}$$

Therefore we have proved

$$\|W^{(1;2)(k)}_{\omega;\omega'} - w^{(1;2)}_{\omega;\omega'}\| \leq \frac{C}{r_2}|\lambda|\gamma^{-\frac{4}{3}k} \tag{51}$$

that is the second (29).

*3. Improved bound for* (0; 4). By (23) we obtain the identity in Fig.5.

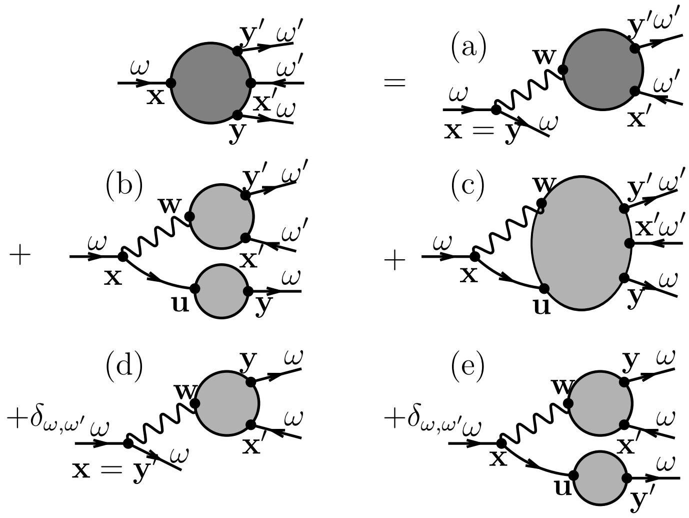


Figure 5: Decomposition of the class of graphs $W^{(0;4)(k)}$. The darker bubbles represent $W^{(0;4)(k)}_{\omega,\omega'} - \lambda w^{(0;4)}_{\omega,\omega'}$ and $W^{(1;2)(k)}_{\omega;\omega'} - w^{(1;2)}_{\omega;\omega'}$.

The bound for the sum of the graphs (a), (b), (d), and (e), all together, can be easily obtained from (51): for $r_2$ large enough

$$|\lambda|\|v_{\omega,\omega'}\|_{L_1}\|W^{(1;2)(k)}_{\omega;\omega'} - w^{(1;2)}_{\omega;\omega'}\| \left(1 + \|g_\omega\|_{L_1}\|W^{(0;2)(k)}_{\omega}\|\right) \leq |\lambda|\frac{C}{2}\gamma^{-\frac{4}{3}(k-M)} \;. \tag{52}$$

To bound (c), use (30) for $(n; 2m) = (1; 4)$, to get, for $|\lambda|$ small enough,

$$\|W^{(0;4)(k)}_{(a),\omega;\omega,\omega'}\| \leq |\lambda|\|v_{\omega,\omega'}\|_{L_3} C_0^2 C|\lambda|\|g_\omega\|_{L_{3/2}} \leq \frac{C}{2}|\lambda|\gamma^{-\frac{4}{3}(k-M)} \tag{53}$$

The proof of theorem 4.2 is complete. ■

The analysis for $\eta^+ = \eta^- \neq 0$ is not different, because the monomials in the effective potential that are proportional to at least one field $\eta^+$ or $\eta^-$ multiply a kernel that doesn't need any power counting improvement. This is important for pointwise estimations on correlations: w.r.t. the $L_1$ bounds of the kernels the pointwise estimates have some missing integrations; but they never involve $W^{(0;2)}$, $W^{(1;2)}$, $W^{(0;4)}$, where, as we showed, missing integrations would spoil the bound.

On scales $k \leq M$ the above arguments do not give a power counting improvement: the factors of type $\gamma^{-\vartheta(k-M)}$ that we have seen in the estimates so far are unbounded. Indeed at IR regime the interaction $v_{\omega,\omega'}$ is effectively local, namely the system is effectively like a Thirring model. The RG approach to use is the one in [5]: with respect to that paper, the UV scale now is replace by $M$; and to the interaction, that there is purely quartic, now has a further term, that is irrelevant, generated by the integrations of the scales $[M, N]$.

We do not repeat all the details of [5]. We just stress that $W^{(0;2)}$, $W^{(1;2)}$, $W^{(0;4)}$ must be *localized* to extract the relevant part of the interaction. In this way, $W^{(0;2)}$ causes the flow of the field renormalization, which is responsible of the anomalous dimension of the large distance decay of fermion correlations. $W^{(1;2)}$ causes the flow of the current renormalization, that is responsible for the anomalous dimension of current correlations. Finally $W^{(0;4)}$ changes scale by scale the effective coupling of the quartic self-interaction: the related flow stays bounded thanks to the vanishing of the beta function, asymptotically for $M-k \to \infty$; this crucial property is not modified by the additional irrelevant interaction on scale $M$, see [5].

The final result of the multiscale integration is that there exists the limit of removed cutoff of the two point correlations; and for large $|\mathbf{x}-\mathbf{y}|$,

$$\frac{\partial^2 \mathcal{W}}{\partial \eta^+_{\mathbf{x},\omega} \partial \eta^-_{\mathbf{y},\omega}}(0,0) \sim \frac{C}{|\mathbf{x}-\mathbf{y}|^{1+\eta}} \ , \qquad \frac{\partial^2 \mathcal{W}}{\partial J_{\mathbf{x},\omega} \partial J_{\mathbf{y},\omega}}(0,0) \sim \frac{C}{|\mathbf{x}-\mathbf{y}|^{2+y}} \ .$$

At this stage we do not know yet the explicit expressions (6), (7) of the two above correlation; nor we know the formula for $\eta$ and the fact that $y = 0$. We only have a convergent power series for them. Explicit evaluations come from WI and SDE.

## 4.2 Ward Identities

By definition of $J_{\mathbf{x},\omega}$, we have $\frac{\partial}{\partial J_{\mathbf{x},\omega}} = \frac{1}{2}\left[\frac{\partial}{\partial J^0_{\mathbf{x}}} - i\omega \frac{\partial}{\partial J^1_{\mathbf{x}}}\right]$. The analysis of this section will be done in Fourier transform: these are the conventions

$$\psi^\varepsilon_{\mathbf{x},\omega} = \int \frac{d\mathbf{k}}{(2\pi)^2}\, e^{i\varepsilon \mathbf{k}\mathbf{x}}\, \widehat{\psi}^\varepsilon_{\mathbf{k},\omega} \ , \qquad \eta^\varepsilon_{\mathbf{x},\omega} = \int \frac{d\mathbf{k}}{(2\pi)^2}\, e^{i\varepsilon \mathbf{k}\mathbf{x}}\, \widehat{\eta}^\varepsilon_{\mathbf{k},\omega} \ ,$$

while $J_{\mathbf{x},\omega}$ and $v_{\omega,\omega'}(\mathbf{x})$ follow the same convention of $\psi^-_{\mathbf{x},\omega}$ and $\psi^+_{\mathbf{x},\omega}$, respectively. Setting

$$\nu_{\omega,\omega'}(\mathbf{p}) = \frac{\lambda}{4\pi}\widehat{v}_{\omega,\omega'}(\mathbf{p}) \ , \qquad B_\omega(\mathbf{p}) = \int \frac{d\mathbf{k}}{(2\pi)^2}\left[\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}\frac{\partial \mathcal{W}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega}} - \frac{\partial \mathcal{W}}{\partial \widehat{\eta}^-_{\mathbf{k}+\mathbf{p},\omega}}\widehat{\eta}^-_{\mathbf{k},\omega}\right] \ ,$$

we want to prove that the correlation functions satisfy, in the limit of removed cutoffs, identities generated by the following equation for $\mathcal{W}$

$$D_\omega(\mathbf{p})\frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\omega}} - D_{-\omega}(\mathbf{p})\sum_{\omega'}\nu_{\omega,\omega'}(\mathbf{p})\frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\omega'}} - \frac{1}{4\pi}D_{-\omega}(\mathbf{p})\widehat{J}_{-\mathbf{p},\omega} = B_{\mathbf{p},\omega}(J,\eta)\ . \quad (54)$$

Since $\nu_{\omega,\omega'}(\mathbf{p}) = \nu_{\omega',\omega}(\mathbf{p})$, summing (54) over $\omega$ we find

$$\sum_{\omega,\omega'} D_\omega(\mathbf{p})\left[\delta_{\omega,\omega'} - \nu_{-\omega,\omega'}(\mathbf{p})\right]\frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\omega'}} - \frac{1}{4\pi}\sum_\omega D_\omega(\mathbf{p})\widehat{J}_{-\mathbf{p},-\omega} = \sum_\omega B_{\mathbf{p},\omega}(J,\eta)\ ,$$

that is (11) for $\nu = \lambda\Theta = \frac{\lambda}{4\pi}$. Whereas multiplying (54) times $\omega$ and summing over $\omega$ we find

$$\sum_{\omega,\omega'} \omega D_\omega(\mathbf{p})\left[\delta_{\omega,\omega'} + \nu_{-\omega,\omega'}(\mathbf{p})\right]\frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\omega'}} + \frac{1}{4\pi}\sum_\omega \omega D_\omega(\mathbf{p})\widehat{J}_{-\mathbf{p},-\omega} = \sum_\omega \omega B_{\mathbf{p},\omega}(J,\eta)\ ,$$

that is (12) for $\nu_5 = \lambda\Theta_5 = -\frac{\lambda}{4\pi}$. In order to prove (54), use a general combination of the vector and axial vector transformations: for a real $\widehat{\alpha}_{\mathbf{p},\omega}$ (with the same Fourier transform convention as $\widehat{J}_{\mathbf{p},\omega}$) and transform the fields in $\mathcal{W}_{l,N}(\eta,J)$ as follows

$$\widehat{\psi}^\varepsilon_{\mathbf{k},\omega} \to \widehat{\psi}^\varepsilon_{\mathbf{k},\omega} + \varepsilon\int\frac{d\mathbf{p}}{(2\pi)^2}\widehat{\alpha}_{\mathbf{p},\omega}\widehat{\psi}^\varepsilon_{\mathbf{k}+\varepsilon\mathbf{p},\omega}\ ;$$

that gives the identity

$$D_\omega(\mathbf{p})\frac{\partial \mathcal{W}_{l,N}}{\partial \widehat{J}_{\mathbf{p},\omega}}(J,\eta) = B_{\mathbf{p},\omega}(J,\eta) + R_{\omega,l,N}(\mathbf{p};J,\eta) \quad (55)$$

where $R_{\omega,l,N}(\mathbf{p};J,\eta)$ is a remainder w.r.t. the formal WI: the presence in the free measure of the cutoff function $\chi_{l,N}(\mathbf{k})$ explicitly breaks the vector and axial-vector symmetries. To study $R_{\omega,l,N}(\mathbf{p};J,\eta)$ we introduce a new functional integral,

$$e^{\mathcal{H}_{l,N}(\alpha,J,\eta)} = \int dP_{l,N}(\psi)\ e^{\mathcal{V}(\psi,\eta,J)+\mathcal{A}_0(\psi,\alpha)-\mathcal{A}_-(\psi,\alpha)}$$

where

$$\mathcal{A}_0(\alpha,\psi) = \sum_{\omega=\pm}\int\frac{d\mathbf{q}\,d\mathbf{p}}{(2\pi)^4}\ C_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{\alpha}_{\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\omega}\widehat{\psi}^-_{\mathbf{q},\omega}\ ,$$
$$\mathcal{A}_-(\alpha,\psi) = \sum_{\omega,\omega'=\pm}\int\frac{d\mathbf{q}\,d\mathbf{p}}{(2\pi)^4}\ D_{-\omega}(\mathbf{p})\nu_{\omega,\omega'}(\mathbf{p})\widehat{\alpha}_{\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\omega'}\widehat{\psi}^-_{\mathbf{q},\omega'}$$

and, for $\mathbf{p},\mathbf{q} \in (\gamma^{l-1},\gamma^{N+1})$,

$$C_\omega(\mathbf{q},\mathbf{p}) = [\chi^{-1}_{l,N}(\mathbf{p}) - 1]D_\omega(\mathbf{p}) - [\chi^{-1}_{l,N}(\mathbf{q}) - 1]D_\omega(\mathbf{q})\ .$$

By explicit computation one can check

$$\frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\omega}}(0,J,\eta) = R_{\omega,l,N}(\mathbf{p};J,\eta) - D_{-\omega}(\mathbf{p})\sum_{\omega'}\nu_{\omega,\omega'}(\mathbf{p})\frac{\partial \mathcal{W}_{l,N}}{\partial \widehat{J}_{\mathbf{p},\omega'}}(J,\eta) \tag{56}$$

The fundamental issue behind the Adler-Bell-Jackiw anomaly corresponds, in the viewpoint of our RG scheme, to the following fact: although $C_\omega(\mathbf{q},\mathbf{p})$ is zero for $\mathbf{p},\mathbf{q}\in[\gamma^l,\gamma^N]$ and point-wise vanishing in $(\gamma^{l-1},\gamma^{N+1})$ in the limit of removed cutoffs, its insertion in the graphs of the perturbation theory, i.e. $R_{\omega,l,N}(\mathbf{p};J,\eta)$, is not vanishing at all. Our result is that the remainder, in the limit of removed cutoffs, can anyways be computed:

$$\lim_{-l,N\to\infty}\frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\omega}}(J,\eta,0) = \frac{1}{4\pi}D_{-\omega}(\mathbf{p})\widehat{J}_{-\mathbf{p},\omega}\ . \tag{57}$$

(Recall that (57) must be understood as generator of identities for correlations). This formula gives (54). In order to prove it, we need a multiscale integration of $\mathcal{H}_{l,N}$. Define the effective potential on scale $h$, $\mathcal{A}^{(h)}$, such that

$$e^{\mathcal{H}_{l,N}(\alpha,J,\eta)} = \int dP_{l,h}(\psi)\ e^{\mathcal{V}^{(h)}(\psi,J,\eta)+\mathcal{A}^{(h)}(\alpha,J,\eta,\psi)}\ , \tag{58}$$

(so that $\mathcal{A}^{(h)}(0,J,\eta,\psi)=0$) and, correspondingly, the kernels of the monomials of $\mathcal{A}^{(h)}$ that are linear in $\alpha$

$$H^{(1;n;2m)(h)}_{\omega;\underline{\omega}';\underline{\omega}}(\mathbf{z};\underline{\mathbf{w}};\underline{\mathbf{x}},\underline{\mathbf{y}}) \stackrel{\text{def.}}{=} \prod_{i=1}^{n}\frac{\partial}{\partial J_{\mathbf{w}_i,\omega'_i}}\prod_{i=1}^{m}\frac{\partial}{\partial\psi^+_{\mathbf{x}_i,\omega_i}}\frac{\partial}{\partial\psi^-_{\mathbf{y}_i,\omega_i}}\frac{\partial\mathcal{A}^{(h)}}{\partial\alpha_{\mathbf{z},\omega}}(0,0,0,0)\ . \tag{59}$$

For the results in this paper, we only need $n=0,1$. Because of the definitions of $\mathcal{A}_0$ and $\mathcal{A}_-$, we can have quite an explicit formula for the Fourier transforms of $H^{(1;n;2m)(h)}_{\omega;\underline{\omega}';\underline{\omega}}$. Consider the identity, at $\alpha=\eta=0$,

$$\frac{\partial\mathcal{A}^{(h)}}{\partial\widehat{\alpha}_{\mathbf{p},\omega}} = \int\frac{d\mathbf{q}}{(2\pi)^2}C_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{\psi}^+_{\mathbf{q},\omega}\widehat{\psi}^-_{\mathbf{p}+\mathbf{q},\omega}$$

$$+\int\frac{d\mathbf{q}}{(2\pi)^2}C_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})\left[\widehat{\psi}^-_{\mathbf{q}+\mathbf{p},\omega}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^-_{\mathbf{q},\omega}}\widehat{g}_\omega(\mathbf{q}) - \widehat{g}_\omega(\mathbf{q}+\mathbf{p})\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^+_{\mathbf{p}+\mathbf{q},\omega}}\widehat{\psi}^+_{\mathbf{q},\omega}\right]$$

$$+\sum_{i,j=h}^{N}\int\frac{d\mathbf{q}}{(2\pi)^2}\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^+_{\mathbf{p}+\mathbf{q},\omega}}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^-_{\mathbf{q},\omega}}$$

$$+\sum_{i,j=h}^{N}\int\frac{d\mathbf{q}}{(2\pi)^2}\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})\frac{\partial^2\mathcal{V}^{(h)}}{\partial\widehat{\psi}^+_{\mathbf{p}+\mathbf{q},\omega}\partial\widehat{\psi}^-_{\mathbf{q},\omega}}$$

$$-\sum_{\omega''} D_{-\omega}(\mathbf{p})\nu_{\omega,\omega''}(\mathbf{p})\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{J}_{\mathbf{p},\omega''}} , \qquad (60)$$

for

$$\widehat{U}^{(i,j)}_{\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \stackrel{\text{def.}}{=} C_{\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{g}^{(i)}_{\omega}(\mathbf{q}+\mathbf{p})\widehat{g}^{(j)}_{\omega}(\mathbf{q}) .$$

That suggests the following decomposition of the kernels:

$$\widehat{H}^{(1;n;2m)(h)}_{\omega;\underline{\omega}';\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}};\underline{\mathbf{q}}) = \widehat{H}^{(1;n;2m)(h)}_{0,\omega;\underline{\omega}';\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}};\underline{\mathbf{q}}) + \sum_{\sigma=\pm} D_{\sigma\omega}(\mathbf{p})\widehat{H}^{(1;n;2m)(h)}_{\sigma,\omega;\underline{\omega}';\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}};\underline{\mathbf{q}})$$
$$+\sum_{\sigma=\pm} D_{\sigma\omega}(\mathbf{p})\widehat{K}^{(1;n;2m)(h)}_{\sigma,\omega;\underline{\omega}';\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}};\underline{\mathbf{q}}) . \quad (61)$$

The first line of (60) corresponds to uncontracted $\mathcal{A}_0$, therefore is not included in the kernels. In $\widehat{H}^{(1;n;2m)(h)}_{0,\omega;\underline{\omega}';\underline{\omega}}$ there are the terms generated by the second line of (60), i.e. graphs in which only one between $\psi^-_{\mathbf{q},\omega}$ and $\psi^+_{\mathbf{q}+\mathbf{p},\omega}$ is contracted; $\widehat{H}^{(1;n;2m)(h)}_{\sigma,\omega;\underline{\omega}';\underline{\omega}}$ comes from the third line of (60), i.e. when $\psi^-_{\mathbf{q},\omega}$ and $\psi^+_{\mathbf{q}+\mathbf{p},\omega}$ are both contracted but to different graphs. Fourth and fifth line of (60) are kept together (because we want to exploit a partial cancellation among them) and generate $\widehat{K}^{(1;n;2m)(h)}_{\sigma,\omega;\underline{\omega}';\underline{\omega}}$. To explain the sum over $\sigma$, we define $\widehat{S}^{(i,j)}_{\omega',\omega}$ such that

$$\chi_{l,N}(\mathbf{p}/2)\widehat{U}^{(i,j)}_{\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) = \sum_{\sigma=\pm} D_{\sigma\omega}(\mathbf{p})\widehat{S}^{(i,j)}_{\sigma\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \qquad (62)$$

(we freely add the factor $\chi_{l,N}(\mathbf{p}/2)$ because we are only interested in the case of fixed $\mathbf{p}\neq 0$), so that, for example, for the Fourier transform of $\widehat{K}^{(1;n;2m)(h)}_{\sigma,\omega;\underline{\omega}';\underline{\omega}}$ we find

$$K^{(1;n;2m)(h)}_{\sigma,\omega;\underline{\omega}';\underline{\omega}}(\mathbf{z};\underline{\mathbf{x}};\underline{\mathbf{y}}) = \sum_{i,j=h}^{N}\int d\mathbf{u}d\mathbf{w}\; S^{(i,j)}_{\sigma\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})W^{(n;2+2m)(h)}_{\underline{\omega}';\omega,\underline{\omega}}(\underline{\mathbf{x}};\mathbf{u},\mathbf{w},\underline{\mathbf{y}})$$
$$-\delta_{\sigma,-1}\sum_{\omega''}\int d\mathbf{w}\;\nu_{\omega,\omega''}(\mathbf{z}-\mathbf{w})W^{(1+n;2m)(h)}_{\omega'',\underline{\omega}';\underline{\omega}}(\mathbf{w};\underline{\mathbf{x}},\underline{\mathbf{y}}) \quad (63)$$

This kernel for $(n,2m)=(0,2)$ and $(1,0)$ is depicted in the the first line of Fig.6 and Fig.7, respectively.

**Theorem 4.3** *For $|\lambda|$ small enough and fixed $\mathbf{p}\neq 0$, there exists a $C>1$ and a $\vartheta : 0<\vartheta<1$ such that, for any $k : M\le k\le N$ and $\sigma=\pm$,*

$$|\widehat{K}^{(1;0;2)(k)}_{\sigma,\omega;\omega'}(\mathbf{p};\mathbf{k})| \le C|\lambda|\gamma^{-\vartheta(N-k)}$$
$$|\widehat{K}^{(1;1;0)(k)}_{\sigma,\omega;\omega'}(\mathbf{p}) - \delta_{\omega,\omega'}\delta_{\sigma,-1}\frac{\delta(\mathbf{p})}{4\pi}| \le C\gamma^{-\vartheta(N-k)} \qquad (64)$$

*and, for any other integer $(n,2m)$,*

$$|\widehat{K}^{(1;n;2m)(k)}_{\sigma,\omega;\underline{\omega}}(\mathbf{p};\underline{\mathbf{q}},\underline{\mathbf{k}})| \le (C|\lambda|)^n\gamma^{(1-m-n)k}\gamma^{-\vartheta(N-k)} \qquad (65)$$

$$|\widehat{H}^{(1;n;2m)(k)}_{\sigma,\omega;\underline{\omega}}(\mathbf{p};\underline{\mathbf{q}},\underline{\mathbf{k}})| \leq (C|\lambda|)^n \gamma^{(1-m-n)k}\gamma^{-\vartheta(N-k)} \tag{66}$$

**Proof.** Note that $\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})$ is zero if none of $i$ and $j$ is $N$ or $l$. Besides, in the appendix of [20] there is the proof of the following bound: for any $q$ positive integer, there exists a constant $C_q > 1$ such that

$$|S^{(i,j)}_{\bar{\omega},\omega}(\mathbf{z};\mathbf{x},\mathbf{y})| \leq C_q \frac{\gamma^i}{1+[\gamma^i|\mathbf{x}-\mathbf{z}|]^q}\frac{\gamma^j}{1+[\gamma^j|\mathbf{y}-\mathbf{z}|]^q}\ . \tag{67}$$

Also, we need the result of [5],

$$\lim_{-l,N\to\infty}\sum_{i,j=l+1}^{N}\int\frac{d\mathbf{p}}{(2\pi)^2}\ \widehat{S}^{(i,j)}_{-\omega,\omega}(\mathbf{p},\mathbf{p}) = \frac{1}{4\pi}\ . \tag{68}$$

We bound $|\widehat{K}^{(1;n;2m)(k)}|$ and $|\widehat{H}^{(1;n;2m)(k)}|$ with the $L_1$ norm of $K^{(1;n;2m)(k)}$ and $H^{(1;n;2m)(k)}$, respectively. The estimate (66) is a straightforward consequence of (29), (30) and (67) for $i$ or $j$ equal to $N$; indeed, in the graphs expansion if $\widehat{H}^{(1;n;2m)(k)}_{\sigma,\omega;\underline{\omega}}$, there is no loop to worry about other than those ones already in the kernels $W^{(n;2m)(k)}$. The estimate (65), for $(n,2m)\neq(0,2),(1,0)$, is simple, because it derives from (64) by standard methods. For the estimate of the relevant and marginal kernels, (64), we have to take advantage of partial cancellations.

Using the expansion for $W^{(0;4)(h)}_{\omega,\omega'}$ in Fig.5, we can expand (63) for $(n,2m) = (0,2)$ according to Fig.6 (we have also used, for the class of graphs (d), the decomposition in Fig.2). Consider the case $\sigma = -$. Fixed the integer $q$ and calling $b_j(\mathbf{x}) \stackrel{\text{def.}}{=} C_q\gamma^j/(1+[\gamma^j|\mathbf{x}|]^q)$, we bound the r.h.s. member in the same spirit as in Section 4.1; though this time we also want to find the exponential small factor $\gamma^{-\vartheta(N-k)}$.

Let's first consider graphs (a) and (b) together:

$$\lambda\sum_{\omega''}\int d\mathbf{u}\ \left[\sum_{i,j=k}^{N} S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) - \frac{\delta(\mathbf{z}-\mathbf{u})}{4\pi}\right]\int d\mathbf{w}\ v_{\omega,\omega''}(\mathbf{u}-\mathbf{w})W^{(1;2)(k)}_{\omega'';\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) \tag{69}$$

Using the identity (39), for graph (a) we have

$$\begin{aligned}
&\lambda\sum_{\omega''}\sum_{i,j=k}^{N}\int d\mathbf{u}d\mathbf{w}\ S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})v_{\omega,\omega''}(\mathbf{u}-\mathbf{w})W^{(1;2),(k)}_{\omega'';\omega'}(\mathbf{w};\mathbf{x},\mathbf{y})\\
&=\lambda\sum_{\omega''}\int d\mathbf{w}\ v_{\omega,\omega''}(\mathbf{z}-\mathbf{w})W^{(1;2),(k)}_{\omega'';\omega'}(\mathbf{w};\mathbf{x},\mathbf{y})\sum_{i,j=k}^{N}\int d\mathbf{u}\ S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})\\
&+\lambda\sum_{p=0,1}\sum_{\omega''}\sum_{i,j=k}^{N}\int d\mathbf{u}\ S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})(u_p-z_p)\cdot\\
&\qquad\cdot\int_0^1 d\tau\ \int d\mathbf{w}\ (\partial_p v_{\omega,\omega''})(\mathbf{z}-\mathbf{w}+\tau(\mathbf{u}-\mathbf{z}))W^{(1;2),(k)}_{\omega'';\omega'}(\mathbf{w};\mathbf{x},\mathbf{y})
\end{aligned} \tag{70}$$

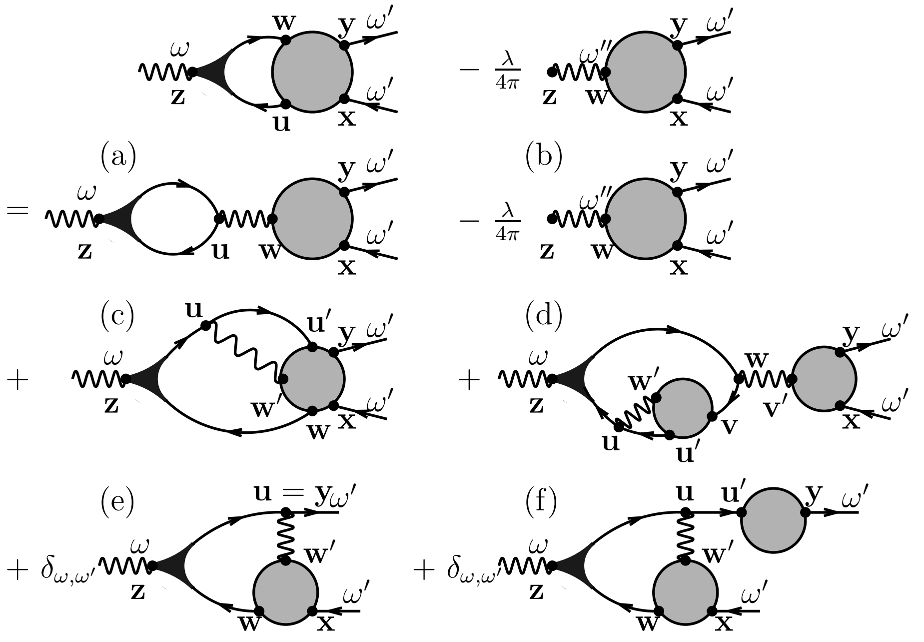


Figure 6: Graphical representation of $K^{(1;0;2)(k)}_{-,\omega,\omega'}$

The latter term of the r.h.s. member of (70) has the wanted estimate: using that one between $i$ and $j$ is on scale $N$, a bound for its norm is

$$8|\lambda|\|W^{(1;2),(k)}_{\omega'';\omega'}\|\|\partial v_{\omega,\omega''}\|_{L_1}\|b_N\|_{L_1(w)}\sum_{i=k}^{N}\|b_i\|_{L_\infty}\leq|\lambda|C_5\gamma^{-(k-M)}\gamma^{-(N-k)}\;. \qquad (71)$$

The former term of the r.h.s. member of (70) - as opposed to what happened for (b3) of Fig4 - is not zero, but is compensated by (b):

$$\sum_{i,j=k}^{N}\int d\mathbf{u}\;S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})-\frac{1}{4\pi}=2\sum_{j\leq k-1}\int d\mathbf{u}\;S^{(N,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) \qquad (72)$$

and hence the bound for such a difference is $C_5\gamma^{-(N-k)}$. The global bound for (a) and (b) together is therefore $|\lambda|C_6\gamma^{-(N-k)}$.

Graph (c) corresponds to

$$\lambda\sum_{i,j=k}^{N}\sum_{\omega''}\int d\mathbf{u}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'\;S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})g_\omega(\mathbf{u}-\mathbf{u}')v_{\omega,\omega''}(\mathbf{u}-\mathbf{w}')\cdot$$
$$\cdot W^{(1;4)(k)}_{\omega'';\omega,\omega'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y})\;. \qquad (73)$$

Since either $i$ or $j$ has to be $N$, and because of the bound (67), the norm of (c) is bounded by

$$|\lambda|\sum_{m=k}^{N}\sum_{i,j=k}^{*N}\int d\mathbf{x}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'\;|W^{(1;4)(k)}_{\omega'';\omega,\omega'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y})|$$

$$\cdot \int d\mathbf{z}d\mathbf{u}\, b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|v_{\omega,\omega''}(\mathbf{u}-\mathbf{w}')g_\omega^{(m)}(\mathbf{u}-\mathbf{u}')| \tag{74}$$

where $*N$ reminds that at least one between $i$ and $j$ has to be $N$. As in the previous section, we bound the second line as follows:

$$\begin{aligned}
&\|b_N\|_{L_1}\ \|b_j\|_{L_{3/2}}\ \|v_{\omega,\omega'}\|_{L_3}\ \|g_\omega^{(m)}\|_{L_\infty} \quad \text{for} \quad i=N,\ m\le j\\
&\|b_N\|_{L_1}\ \|b_j\|_{L_\infty}\ \|v_{\omega,\omega'}\|_{L_3}\ \|g_\omega^{(m)}\|_{L_{3/2}} \quad \text{for} \quad i=N,\ j<m\\
&\|b_i\|_{L_1}\ \|b_N\|_{L_{3/2}}\ \|v_{\omega,\omega'}\|_{L_3}\ \|g_\omega^{(m)}\|_{L_\infty} \quad \text{for} \quad j=N,\ m\le i\\
&\|b_i\|_{L_\infty}\ \|b_N\|_{L_1}\ \|v_{\omega,\omega'}\|_{L_3}\ \|g_\omega^{(m)}\|_{L_{3/2}} \quad \text{for} \quad j=N,\ i<m
\end{aligned} \tag{75}$$

and hence we get the bound, for $0\le\vartheta\le 1/3$, $C_3>1$

$$\int d\mathbf{z}d\mathbf{u}\, b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|v_{\omega,\omega''}(\mathbf{u}-\mathbf{w}')g_\omega^{(m)}(\mathbf{u}-\mathbf{u}')| \le C_3\gamma^{-\frac{4}{3}(k-M)}\gamma^{-\vartheta(N-k)}\gamma^k \tag{76}$$

(with $C_3\to\infty$ if $\vartheta\to 1/3$). Using (30) for $W^{(1;4)(k)}_{\omega'';\omega,\omega'}$ we have for graph (c) the bound $|\lambda|C_6\gamma^{-\frac{4}{3}(k-M)}\gamma^{-\vartheta(N-k)}$. Graph (d) is bounded in the same way as (c) (in fact (d) was distinguished from (c) only for enumeration reasons, whereas topologically they are the same). We find

$$\sum_\sigma \int d\mathbf{x}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'\ |W^{(1;2)(k)}_{\omega'';\omega}(\mathbf{u};\mathbf{w}',\mathbf{u}')g_\omega(\mathbf{v}-\mathbf{w})v_{\omega,\sigma}(\mathbf{w}-\mathbf{v}')W^{(1;2)(k)}_{\sigma;\omega'}(\mathbf{v}';\mathbf{x},\mathbf{y})|$$
$$\cdot|\lambda|^2\sum_{m=k}^N\sum_{i,j=k}^{N*}\int d\mathbf{z}d\mathbf{u}\, b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|v_{\omega,\omega''}(\mathbf{u}-\mathbf{w}')g_\omega^{(m)}(\mathbf{u}-\mathbf{u}')|\ ;$$

then, with the aid of (29) and (76), we find a bound of the type $|\lambda|C_3\gamma^{-\frac{4}{3}(k-M)}\gamma^{-\vartheta(N-k)}$. For (e) and (f), by a simple argument, we have the bound

$$4|\lambda|\|W^{(1;2)(k)}_{\omega'';\omega'}\|\left[1+\|g_\omega\|_{L_1}\|W^{(0;2)(k)}_\omega\|\right]\|v_{\omega,\omega'}\|_{L_3}\sum_{i,j=k}^{*N}\|b_i\|_{L_1}\|b_j\|_{L_{3/2}}\ ,$$

that less than $|\lambda|C_3\gamma^{-\frac{4}{3}(k-M)}\gamma^{-\vartheta(N-k)}$. Now consider the case $\sigma=+$. The graph expansion of $K^{(1;0;2)(k)}_{+,\omega;\omega'}$ is given again by Fig.6; the only differences is that the graph (b) is missing (that because of the $\delta_{\sigma,-1}$ in (63)). Hence a bound can be obtained with the same above argument, with only one important difference: the contribution that in the previous analysis were compensated by (b) now are zero by symmetries. Indeed, calling $\mathbf{k}^*$ the rotation of $\mathbf{k}$ of $\pi/2$ and since $\widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{k}^*,\mathbf{p}^*)=-\omega\bar\omega\widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{k},\mathbf{p})$, in place of (72), in this case we have:

$$\sum_{i,j=k}^N\int d\mathbf{u}\ S^{(i,j)}_{\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})=\sum_{i,j=k}^N\int\frac{d\mathbf{k}}{(2\pi)^2}\ \widehat{S}^{(i,j)}_{\omega,\omega}(\mathbf{k},-\mathbf{k})=0 \tag{77}$$

The proof of the first of (64) is completed. We now consider the second. Expand $W^{(1;2)(h)}_{\omega';\omega}$ as in Fig.3, and obtain the decomposition of Fig.7. In particular, class (e) comes from the kernel $w^{(1;2)}_{\omega,\omega'}$ that is is darker bubble of Fig.3; while for (d) and (f) we also used the identity in Fig.2 to extract a further wiggly line. It is also worth stressing that (e) is not included in (a), because by construction $W^{(2;0)(h)}_{\omega,\omega'}(\mathbf{w},\mathbf{x})$ does not contain $\delta_{\omega,\omega'}\delta(\mathbf{x}-\mathbf{w})$. It is evident that graphs of classes (a), (b), (c)

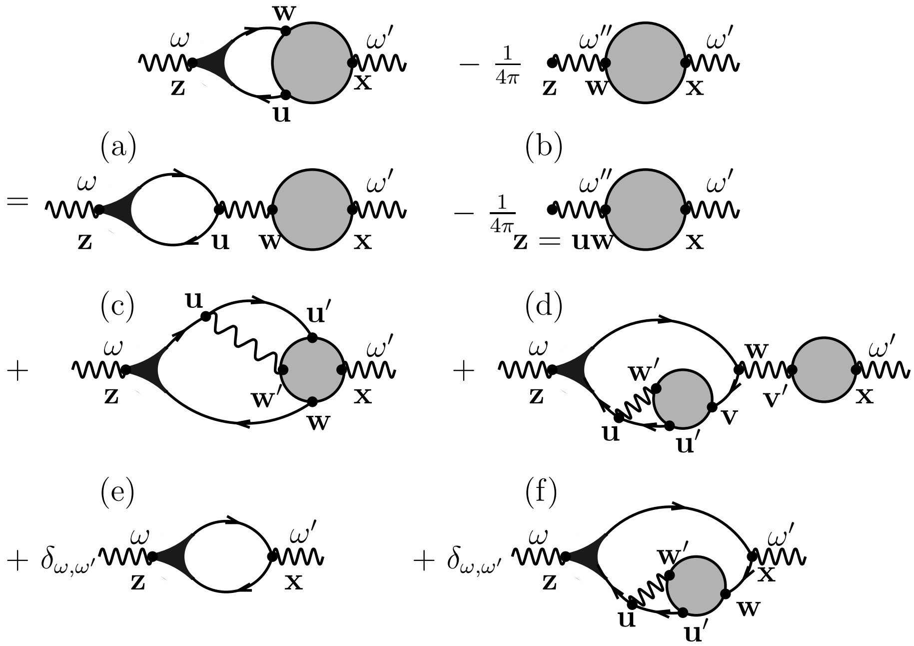


Figure 7: Decomposition of the class $K^{(1;1;0)(h)}_{-,\omega;\omega'}$

and (d) can be bounded as the graphs in homonym classes in Fig.(6): the only difference is one external wiggly line in place of two external fermion lines, but that does not change the power counting, nor the topology of the graph. A bound for graph (f) is:

$$|\lambda|\int d\mathbf{u}'d\mathbf{w}'d\mathbf{w}\;|W^{(1;2)(k)}_{\omega'',\omega}(\mathbf{w}';\mathbf{u}',\mathbf{x})g_\omega(\mathbf{w}-\mathbf{x})|$$

$$\sum_{i,j=k}^{*N}\int d\mathbf{z}d\mathbf{u}\;b_i(\mathbf{z}-\mathbf{u})b_j(\mathbf{z}-\mathbf{x})|v_{\omega,\omega''}(\mathbf{u}-\mathbf{w}')g_\omega(\mathbf{u}-\mathbf{u}')|\;. \tag{78}$$

By (76) we have the bound $|\lambda|C_3\gamma^{-\vartheta(N-k)}$.

The only graph that is not bounded by the exponential small factor is (e). In fact, this kernel is finite; and to cancel it we need the to subtract $\delta_{\sigma,-1}/(4\pi)$, see Fig.8. The difference equals (72) for $\sigma=-1$, and (77) for $\sigma=+1$. Therefore also the second of (30) is proved. ■

We still have to consider a last kind of kernels, $\widehat{H}^{(1;n;2m)(k)}_{0,\omega;\underline{\omega}}$. They can easily bounded, supposing $\mathbf{p}\neq 0$ and finite. Anyways to extract the small factor one

Figure 8: Cancellation of graph (e) in Fig.7 for $\sigma = -1$ with the factor $1/(4\pi)$.

might need the IR integration also: indeed, either the small factor comes from a contraction of $\mathcal{A}_0$ on scale $N$, or on scale $l$; the (simple) details are in [5].

As consequence of the analysis in this section, by the argument in [5], in the limit of removed cutoffs, the correlations generated by $\mathcal{H}$ satisfy the identities generated by (57).

### 4.3 Closed Equation

From the Wick theorem, see (98), we find the SDE equation

$$\frac{\partial^2 \mathcal{W}_{l,N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega} \partial \widehat{\eta}^-_{\mathbf{k},\omega}}(0,0) = \widehat{g}^{[l,N]}_\omega(\mathbf{k}) \left[1 + \lambda \sum_{\omega'} \int \frac{d\mathbf{p}}{(2\pi)^2} \widehat{v}_{\omega,\omega'}(\mathbf{p}) \frac{\partial^3 \mathcal{W}_{l,N}}{\partial \widehat{J}_{\mathbf{p},\omega'} \partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega} \partial \widehat{\eta}^-_{\mathbf{k},\omega}}(0,0)\right] \tag{79}$$

that, in the limit of removed cutoffs, would be equivalent to (15); we do not take the limit now, though. Using (55) and (56), we find

$$\begin{aligned} \sum_{\omega'',\omega'} \left[D_\omega(\mathbf{p})(\widehat{v}^{-1}(\mathbf{p}))_{\omega,\omega''} - D_{-\omega}(\mathbf{p})\frac{\delta_{\omega,\omega''}}{4\pi}\right] \widehat{v}_{\omega'',\omega'}(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}}{\partial \widehat{J}_{\mathbf{p},\omega'}}(0,\eta) \\ = B_{\mathbf{p},\omega}(0,\eta) + \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\omega}}(0,0,\eta) \end{aligned} \tag{80}$$

where the inverse of the matrix $\widehat{v}_{\omega,\omega'(\mathbf{p})}$ is

$$(\widehat{v}^{-1}(\mathbf{p}))_{\omega,\omega'} = (\mathbf{p}^2+\mu^2)\delta_{\omega,-\omega} - \frac{1}{2}\left(1 - \alpha + \frac{\sigma}{\mathbf{p}^2}\right)\left[(\mathbf{p}^0)^2 - \omega\omega'(\mathbf{p}^1)^2 - i(\omega+\omega')\mathbf{p}^0\mathbf{p}^1\right] .$$

As done to prove (13) and (14), use that

$$\sum_\omega D_\omega(\mathbf{p})(\widehat{v}^{-1}(\mathbf{p}))_{\omega,\omega'} = \left(\alpha \mathbf{p}^2 + \mu^2 - \sigma\right) D_{-\omega'}(\mathbf{p})$$

$$\sum_\omega \omega D_\omega(\mathbf{p})(\widehat{v}^{-1}(\mathbf{p}))_{\omega,\omega'} = -\omega' \left(\mathbf{p}^2 + \mu^2\right) D_{-\omega'}(\mathbf{p})$$

to obtain a more explicit form of (80)

$$\sum_{\omega'} \widehat{v}_{\omega,\omega'}(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}}{\partial \widehat{J}_{\mathbf{p},\omega'}}(0,\eta) = \sum_{\omega'} M_{\omega,\omega'}(\mathbf{p}) \left[B_{\mathbf{p},\omega'}(0,\eta) + \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'}}(0,\eta)\right] \tag{81}$$

for $M_{\omega,\omega'}(\mathbf{p})$ the Fourier transform of $-\partial_\omega F(\mathbf{x}) + \omega'\partial_\omega F_5(\mathbf{x})$. Plug (81) into (79) and obtain an equation that, in the limit of removed cutoff, equals (17), but for a remainder term,

$$\sum_{\omega'} \int \frac{d\mathbf{p}}{(2\pi)^2} M_{\omega,\omega'}(\mathbf{p}) \frac{\partial^2 \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'} \partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta) \ . \tag{82}$$

We have to prove that, in the limit of removed cutoffs, (82) is vanishing. To this purpose, as in the previous section, we introduce a new functional integral

$$e^{\mathcal{T}_{\varepsilon,l,N}(\beta,\eta)} = \int dP_{l,N}(\psi) e^{\mathcal{V}(\psi,0,\eta)+\mathcal{B}_{\varepsilon,0}(\psi,\beta)-\mathcal{B}_{\varepsilon,-}(\psi,\beta)} \tag{83}$$

for

$$\mathcal{B}_{\varepsilon,0}(\psi,\beta) = \sum_\omega \int \frac{d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}}{(2\pi)^6}\ M_{\omega,\varepsilon\omega}(\mathbf{p}) C_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\ \widehat{\beta}_{\mathbf{k},\omega} \widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\varepsilon\omega} \widehat{\psi}^-_{\mathbf{q},\varepsilon\omega} \ , \tag{84}$$

$$\mathcal{B}_{\varepsilon,-}(\psi,\beta) = \sum_{\omega,\omega'} \int \frac{d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}}{(2\pi)^6}\ M_{\omega,\omega'}(\mathbf{p}) D_{-\omega'}(\mathbf{p}) \nu_{\omega',\varepsilon\omega}(\mathbf{p})\ \widehat{\beta}_{\mathbf{k},\omega} \widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\varepsilon\omega} \widehat{\psi}^-_{\mathbf{q},\varepsilon\omega} \ , \tag{85}$$

Therefore we find:

$$\frac{\partial \mathcal{T}_{\varepsilon,l,N}}{\partial \beta_{\mathbf{k},\omega}}(0,\eta) = \int \frac{d\mathbf{p}}{(2\pi)^2} M_{\omega,\varepsilon\omega}(\mathbf{p}) \frac{\partial^2 \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\varepsilon\omega} \partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta) \ .$$

We now perform a multiscale integration of $\mathcal{T}_{\varepsilon,l,N}(\beta,0)$. Define $\mathcal{B}^{(h)}(\beta,\eta,\psi)$, the effective potential on scale $h$, to be such that

$$e^{\mathcal{T}_{\varepsilon,l,N}(\beta,\eta)} = \int dP_{l,h}(\psi)\ e^{\mathcal{V}^{(h)}(\psi,0,\eta)+\mathcal{B}^{(h)}_\varepsilon(\beta,\eta,\psi)} \ , \tag{86}$$

and correspondingly, the kernels of the monomials of $\mathcal{T}^{(h)}_\varepsilon$ that are linear in $\beta$:

$$T^{(2m;2)(h)}_{\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{x}},\underline{\mathbf{y}};\mathbf{u},\mathbf{v}) \stackrel{\text{def.}}{=} \prod_{i=1}^m \frac{\partial}{\partial \psi^+_{\mathbf{x}_i,\omega_i}} \frac{\partial}{\partial \psi^-_{\mathbf{y}_i,\omega_i}} \frac{\partial^2 \mathcal{B}^{(h)}_\varepsilon}{\partial \beta_{\mathbf{u},\omega} \partial \psi^-_{\mathbf{v},\omega}}(0,0,0) \ . \tag{87}$$

To make more explicit the kernels $T^{(2m;2)(h)}_{\varepsilon;\underline{\omega},\omega}$ we use the following identity at $\beta = \eta = 0$:

$$\frac{\partial \mathcal{B}^{(h)}_\varepsilon}{\partial \beta_{\mathbf{k},\omega}} = \sum_{i,j=h}^N \int \frac{d\mathbf{p} d\mathbf{q}}{(2\pi)^4} M_{\omega,\varepsilon\omega}(\mathbf{p}) \widehat{U}^{(i,j)}_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \frac{\partial^2 \mathcal{V}^{(h)}}{\partial \widehat{\psi}^+_{\mathbf{q},\varepsilon\omega} \partial \widehat{\psi}^-_{\mathbf{q}+\mathbf{p},\varepsilon\omega}} \frac{\partial \mathcal{V}^{(h)}}{\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}$$

$$+ \sum_{\omega'} \int \frac{d\mathbf{p}}{(2\pi)^2} M_{\omega,\omega'}(\mathbf{p}) D_{-\omega'}(\mathbf{p}) \nu_{\omega',\varepsilon\omega}(\mathbf{p}) \frac{\partial \mathcal{V}^{(h)}}{\partial \widehat{J}_{\mathbf{p},\varepsilon\omega}} \frac{\partial \mathcal{V}^{(h)}}{\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}$$

$$+\sum_{i,j=h}^{N}\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}M_{\omega,\varepsilon\omega}(\mathbf{p})\widehat{U}^{(i,j)}_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{g}_\omega(\mathbf{k}+\mathbf{p})\frac{\partial^3\mathcal{V}^{(h)}}{\partial\widehat{\psi}^+_{\mathbf{k}+\mathbf{p},\omega}\partial\widehat{\psi}^+_{\mathbf{q},\varepsilon\omega}\partial\widehat{\psi}^-_{\mathbf{q}+\mathbf{p},\varepsilon\omega}}$$

$$+\sum_{\omega'}\int\frac{d\mathbf{p}}{(2\pi)^2}M_{\omega,\omega'}(\mathbf{p})D_{-\omega'}(\mathbf{p})\nu_{\omega',\varepsilon\omega}(\mathbf{p})\widehat{g}_\omega(\mathbf{k}+\mathbf{p})\frac{\partial^2\mathcal{V}^{(h)}}{\partial\widehat{\psi}^+_{\mathbf{k}+\mathbf{p},\omega}\partial\widehat{J}_{\mathbf{p},\varepsilon\omega}}$$

$$-\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}M_{\omega,\varepsilon\omega}(\mathbf{p})C_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\;e^{-\mathcal{V}^{(h)}}\frac{\partial}{\partial\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}\left[e^{\mathcal{V}^{(h)}}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\eta}^+_{\mathbf{q},\varepsilon\omega}}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\eta}^-_{\mathbf{q}+\mathbf{p},\varepsilon\omega}}\right]\quad(88)$$

We decompose the kernels as follows:

$$\widehat{T}^{(2m;2)(h)}_{\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{q}};\mathbf{k})=\widehat{T}^{(2m;2)(h)}_{0,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{q}};\mathbf{k})+\widehat{T}^{(2m;2)(h)}_{1,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{q}};\mathbf{k})+\widehat{T}^{(2m;2)(h)}_{2,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{q}};\mathbf{k})\;;$$

in $T^{(2m;2)(h)}_{0,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{x}};\mathbf{u},\mathbf{v})$ we collect the term generated by the last line of (88); in $T^{(2m;2)(h)}_{1,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{x}};\mathbf{u},\mathbf{v})$ we included the terms generated by the third and fourth line of (88); finally, $T^{(2m;2)(h)}_{2,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{x}};\mathbf{u},\mathbf{v})$ is related to the first two lines of (88).

**Theorem 4.4** *For $|\lambda|$ small enough, there exists a $C>1$ and a $\vartheta:0<\vartheta<1$ such that, for any $k:M\le k\le N$, $r=1,2$ and $\varepsilon=\pm$*

$$|\widehat{T}^{(2m;2)(k)}_{r,\varepsilon;\underline{\omega},\omega}(\underline{\mathbf{q}},\mathbf{k})|\le C^m|\lambda|\gamma^{(1-m)k}e^{-\vartheta(N-k)}\qquad(89)$$

**Proof.** Again we use the $L_1$ norm of $T^{(2m;2)(k)}$ as upper bound of $|T^{(2m;2)(k)}|$. In the cases $r=1,2$ the proof of (89) is a direct consequence of (64), (65). Indeed, for $r=2$, there is not new loop in the graph expansion of $\widehat{T}^{(2m;2)(k)}_{r,\varepsilon;\underline{\omega},\omega}$ w.r.t. the graph expansion of $\widehat{K}^{(1;0;2m)(k)}_{\sigma,\omega;\underline{\omega}}$ for $\sigma=\pm$. Whereas for $r=1$ there is only one loop more, that can be easily bounded: for $u^{(\sigma)}_{\omega,\varepsilon\omega}(\mathbf{x})$ the Fourier transform of $M_{\omega,\varepsilon\omega}(\mathbf{p})D_{\sigma\varepsilon\omega}(\mathbf{p})$

$$T^{(2m;2)(k)}_{1,\varepsilon;\omega}(\underline{\mathbf{w}};\mathbf{x},\mathbf{y})=\sum_\sigma\int d\mathbf{z}d\mathbf{u}\;u^{(\sigma)}_{\omega,\varepsilon\omega}(\mathbf{x}-\mathbf{z})g_\omega(\mathbf{x}-\mathbf{u})K^{(1;0;2m+2)(k)}_{\sigma;\varepsilon\omega,\omega'}(\mathbf{z};\underline{\mathbf{w}},\mathbf{u},\mathbf{y})$$

Note that the bound for $\|u^{(\sigma)}_{\omega,\varepsilon\omega}\|_{L_p}$ is essentially the same of $\|v_{\omega,\omega'}\|_{L_p}$; therefore,

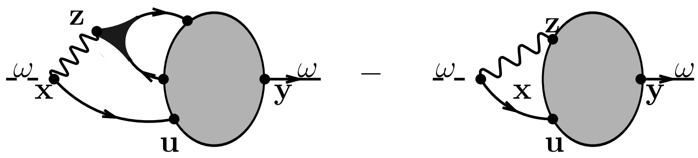


Figure 9: Graphical representation of $T^{(0;2)(k)}_{1,\varepsilon;\omega}$. The dashed line represents the external fermion field $\beta$.

using (64), we obtain the bound

$$2\|u^{(\sigma)}_{\omega,\varepsilon\omega}\|_{L_3}\sum_{j=k}^{N}\|g^{(j)}_{\omega}\|_{L_{3/2}}\|K^{(1;0;2m+2)(k)}_{\sigma;\varepsilon\omega,\omega'}\|\leq C|\lambda|\gamma^{k}\gamma^{-\frac{4}{3}(k-M)}\gamma^{-\vartheta(N-k)}\,. \tag{90}$$

That completes the proof of the theorem. ■

We shall now analyze the last kind of kernel left, $T^{(2m;2)(k)}_{0,\varepsilon;\omega}(\underline{\mathbf{q}};\mathbf{k})$. We further expand the last line of (88)

$$\begin{aligned}
&\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}M_{\omega,\varepsilon\omega}(\mathbf{p})C_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\eta}^{+}_{\mathbf{k}+\mathbf{p},\omega}}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\eta}^{+}_{\mathbf{q},\varepsilon\omega}}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\eta}^{-}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}}\\
&-\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}M_{\omega,\varepsilon\omega}(\mathbf{p})C_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{g}_{\varepsilon\omega}(\mathbf{q})\frac{\partial^2\mathcal{V}^{(h)}}{\partial\widehat{\eta}^{+}_{\mathbf{k}+\mathbf{p},\omega}\partial\widehat{\psi}^{+}_{\mathbf{q},\varepsilon\omega}}\widehat{\psi}^{+}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}\\
&+\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}M_{\omega,\varepsilon\omega}(\mathbf{p})C_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{g}_{\varepsilon\omega}(\mathbf{q}+\mathbf{p})\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\eta}^{+}_{\mathbf{k}+\mathbf{p},\omega}}\widehat{\psi}^{-}_{\mathbf{q},\varepsilon\omega}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^{-}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}}\\
&+\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}M_{\omega,\varepsilon\omega}(\mathbf{p})\widehat{U}_{\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\widehat{g}_{\omega}(\mathbf{k}+\mathbf{p})\frac{\partial}{\partial\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\omega}}\left[\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^{+}_{\mathbf{q},\varepsilon\omega}}\frac{\partial\mathcal{V}^{(h)}}{\partial\widehat{\psi}^{-}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}}\right]
\end{aligned}\tag{91}$$

In the terms generated by the first line each of $\widehat{\psi}^{-}_{\mathbf{k}+\mathbf{p}}$, $\widehat{\psi}^{+}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}$ and $\widehat{\psi}^{-}_{\mathbf{q},\varepsilon\omega}$ is contracted with a different kernel $\widehat{W}^{(0,2m)(k)}$ (if any); therefore this term is bounded with (29), (30); the small factor can be extracted only if at least one between $\widehat{\psi}^{+}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}$ and $\widehat{\psi}^{-}_{\mathbf{q},\varepsilon\omega}$ is contracted; otherwise it comes from the IR integration, as in [5].

In the terms generated by the second and third line, one between $\widehat{\psi}^{+}_{\mathbf{q}+\mathbf{p},\varepsilon\omega}$ and $\widehat{\psi}^{-}_{\mathbf{q},\varepsilon\omega}$ is not contracted; anyways in the graphical representation there is a loop that is not included in the kernels $\widehat{W}^{(0;2m)(k)}$, with momentum $\mathbf{p}$. Its explicit

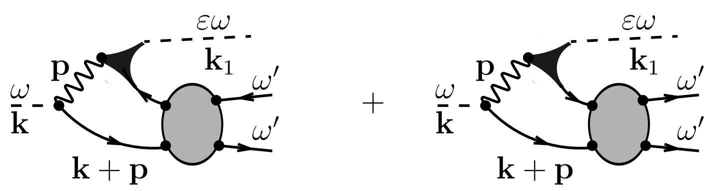


Figure 10: Graphical representation of possible terms generated by the second line of (91)

expression is

$$\int\frac{d\mathbf{p}}{(2\pi)^2}\,M_{\omega,\varepsilon\omega}(\mathbf{p})\Big[(1-f_N(\mathbf{k}_1-\mathbf{p}))-(\chi^{-1}_{l,N}(\mathbf{k}_1)-1)\widehat{g}_{\omega}(\mathbf{k}_1-\mathbf{p})\Big]$$

$$\cdot\widehat{g}_\omega(\mathbf{k}+\mathbf{p})\widehat{W}^{(0;2m+2)(k)}_{\omega,\varepsilon\omega\underline{\omega}}(\mathbf{k}+\mathbf{p},\mathbf{k}_1-\mathbf{p},\underline{\mathbf{k}}) \tag{92}$$

The addend proportional to $(1-f_N(\mathbf{k}_1-\mathbf{p}))$ has the constraint that $|\mathbf{p}|\geq c\gamma^N$; then $|M(\mathbf{p})|\leq c\gamma^{-3N}$, and the bound is $C\gamma^{-N}\|g_\omega\|_{L_1}\|W^{(0;2m+2)(k)}\|\leq C_M\gamma^{-(N-k)}\gamma^{-2(k-M)}\gamma^{-(1-m)k}$; whereas the addend proportional to $(\chi^{-1}_{l,N}(\mathbf{k}_1)-1)$ will be contracted on scale $l$ (otherwise is zero), so obtaining the exponentially small factor, see [5].

Finally, from the fourth line we obtain terms in which $\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}$ is contracted; and also both $\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\varepsilon\omega}$ and $\widehat{\psi}^-_{\mathbf{q},\varepsilon\omega}$ are contracted, but one of them is linked to the same kernel as $\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}$. We find

$$\sum_\sigma\sum_{i,j=k}^N\int d\mathbf{z}d\mathbf{w}d\mathbf{w}'d\mathbf{u}d\mathbf{u}'\; u^\sigma_{\omega,\varepsilon\omega}(\mathbf{z}-\mathbf{w})S^{(i,j)}_{\sigma\omega,\omega}(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u}')g_\omega(\mathbf{z}-\mathbf{w}')$$
$$W^{(0;2m_1)(k)}(\mathbf{u}',\underline{\mathbf{x}})W^{(0;2m_2+2)(k)}(\mathbf{w}',\mathbf{u},\underline{\mathbf{x}}') \tag{93}$$

that is bounded by

$$4\|u^\sigma_{\omega,\varepsilon\omega}\|_{L_3}\|g_\omega\|_{L_{3/2}}\|b_N\|_{L_1}\sum_{j=k}^N\|b_j\|_{L_1}\|W^{(0;2m_1)(k)}\|\|W^{(0;2m_2+2)(k)}\|$$
$$\leq C\gamma^{(2-m_1-m_2)}\gamma^{-(N-k)}\gamma^{-\frac{4}{3}(k-M)}\,. \tag{94}$$

The consequence of the analysis in this section is that, using the argument in [5], the correlations generated by (82) are vanishing when cutoffs are removed.

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## A Proof of the graphical identities

We remind that the derivative in the Grassmann variables $\psi^+$, $\eta^+$ and $\zeta^+$ are taken from the left, while the derivatives in $\psi^-$, $\eta^-$ and $\zeta^-$ are taken from the right. The definition of $\mathcal{V}^{(k)}$ is given by (20). Accordingly, the relation between $\mathcal{V}^{(k)}$ and $\mathcal{V}$ is

$$\mathcal{V}^{(k)}(\psi, J, 0) = \ln \int dP_{k+1,N}(\zeta)\; e^{\mathcal{V}(\psi+\zeta,J,0)} \;. \tag{95}$$

and we have the two identities:

$$\begin{aligned}\frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \psi^+_{\mathbf{x},\omega}}(\psi, J, 0) &= J_{\mathbf{x},\omega}\frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \eta^+_{\mathbf{x},\omega}}(\psi, J, 0) \\ &\qquad + \lambda \sum_{\omega'} \int d\mathbf{w}\; v_{\omega,\omega'}(\mathbf{x}-\mathbf{w}) \frac{\partial^2 e^{\mathcal{V}^{(k)}}}{\partial J_{\mathbf{w},\omega'}\partial \eta^+_{\mathbf{x},\omega}}(\psi, J, 0) \;. \end{aligned}\tag{96}$$

$$\frac{\partial \mathcal{V}^{(k)}}{\partial J_{\mathbf{x},\omega}}(\psi, J, \eta) = -e^{-\mathcal{V}^{(k)}(\psi,J,\eta)} \frac{\partial^2 e^{\mathcal{V}^{(k)}}}{\partial \eta^+_{\mathbf{x},\omega}\partial \eta^-_{\mathbf{x},\omega}}(\psi, J, \eta) \tag{97}$$

Moreover the *Wick theorem* for $dP(\zeta)$ Gaussian mean values gives

$$\int dP_{k+1,N}(\zeta)\; \zeta^{-\varepsilon}_{\mathbf{x},\omega} F(\zeta) = \varepsilon \int d\mathbf{u}\; g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u}) \int dP_{k+1,N}(\zeta)\; \frac{\partial F(\zeta)}{\partial \zeta^{\varepsilon}_{\mathbf{u},\omega}} \tag{98}$$

this identity is straightforward for $F = \exp\{\sum_{\mathbf{x}} \zeta^+_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega} + \sum_{\mathbf{x}} \eta^+_{\mathbf{x},\omega}\zeta^-_{\mathbf{x},\omega}\}$ and so is also for any formal power series with even numbers of fields. Therefore

$$\begin{aligned}\frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \eta^{\varepsilon}_{\mathbf{x},\omega}}(\psi, J, \eta) &= \int dP_{k+1,N}(\zeta)\; \left(\psi^{-\varepsilon}_{\mathbf{x},\omega} + \zeta^{-\varepsilon}_{\mathbf{x},\omega}\right) e^{\mathcal{V}(\psi+\zeta,J,\eta)} \\ &= \psi^{-\varepsilon}_{\mathbf{x},\omega} e^{\mathcal{V}^{(k)}} + \varepsilon \int d\mathbf{u}\; g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u}) \frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \psi^{\varepsilon}_{\mathbf{u},\omega}}(\psi, J, \eta) \;. \end{aligned}\tag{99}$$

We plug the identity for $\varepsilon = +$ into (96) and get (23). Also, since $g^{[k+1,N]}_{\omega}(0) = 0$, from (99) and (97) we obtain:

$$\begin{aligned}\frac{\partial e^{\mathcal{V}^{(k)}}}{\partial J_{\mathbf{x},\omega}}(\psi, J, \eta) &= -\psi^-_{\mathbf{x},\omega}\frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \eta^-_{\mathbf{x},\omega}}(\psi, J, \eta) \\ &\qquad - \int d\mathbf{u}\; g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u}) \frac{\partial^2 e^{\mathcal{V}^{(k)}}}{\partial \psi^+_{\mathbf{u},\omega}\partial \eta^-_{\mathbf{x},\omega}}(\psi, J, \eta) \end{aligned}\tag{100}$$

that gives (24).

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