Rate of convergence towards the Hartree-von Neumann limit in the mean-field regime *

I. Anapolitanos †

March 31, 2010

Abstract

In the mean-field regime we prove convergence, with explicit bounds, of N-particle density matrices satisfying the time-dependent von Neumann equation with factorized initial data to a product of one particle density matrices satisfying the Hartree-von Neumann equation. To prove explicit bounds we generalize techniques developed by Pickl [Pi] and Knowles-Pickl [KP].

1 Introduction

We consider a quantum system of N particles in d dimensions described by a density matrix, i.e a positive trace class operator ρ_N of trace 1, acting on $L^2(\mathbb{R}^d)^{\otimes N} := L^2(\mathbb{R}^d) \otimes ... \otimes L^2(\mathbb{R}^d)$, whose evolution is described by the time-dependent von Neumann equation with factorized initial data:

$$\begin{cases} i\frac{\partial \rho_N}{\partial t} = [H_N, \rho_N] \\ \rho_N|_{t=0} = \rho_0^{\otimes N}, \end{cases} \tag{1}$$

where ρ_0 is a positive trace class operator on $L^2(\mathbb{R}^d)$ of trace 1. Here $H_N = H_N^0 + V_N$, with $H_N^0 = \sum_{i=1}^N h_{x_i}$, $h_x \equiv h$ is a self-adjoint operator in a d-dimensional variable x, e.g. $h_x := -\Delta_x + W(x)$, and

$$V_N = \frac{g}{2} \sum_{i \neq j}^{N} v(x_i - x_j). \tag{2}$$

We can assume without loss of generality that the two-body potential v is even: $v(x) = v(-x)$. We consider the mean-field regime: $N \to \infty$ and $g \to 0$ with $gN \to c$. By changing v, if necessary, we can assume that

$$g = \frac{1}{N-1}. \tag{3}$$

Equation (3) means that the particles are weakly interacting and that the potential and kinetic energy of the system are formally of the same order. The initial condition in (1) has the meaning that the particles are initially uncorrelated and on the same one-particle state ρ_0. The initial value problem (1) is a generalization of the N-body Schrödinger equation for a quantum system of weakly interacting Bosons, which is described by a wave function $\Psi_N \in L^2(\mathbb{R}^d)^{\otimes N}$ evolving according to the initial value problem

$$\begin{cases} i\frac{\partial \Psi_N}{\partial t} = H_N \Psi_N \\ \Psi_N|_{t=0} = \psi_0^{\otimes N}, \end{cases} \tag{4}$$

where $\psi_0 \in L^2(\mathbb{R}^d)$. In the case that ρ_0 is the orthogonal projection onto the wave function ψ_0, the problem (1) reduces to (4). It is well known that density matrices can describe open quantum systems which in general can not be described by wave functions.

In the case that $v = 0$, the solution ρ_N of (1) remains factorized for all times so the particles remain uncorrelated. This is not the case for arbitrary potential v. However, in the case of weakly interacting particles it turns out that $\rho_N \approx \rho^{\otimes N}$ for all times in an appropriate sense of convergence. Here ρ is the solution of the Hartree-von Neumann equation

$$\begin{cases} i\frac{\partial \rho}{\partial t} = [h + (v * n_\rho), \rho] \\ \rho_t|_{t=0} = \rho_0, \end{cases} \tag{5}$$

*This paper is a part of the author's Ph.D thesis.

†Department of Mathematics, University of Toronto, Toronto, Canada.

where $n_\rho(x,t) := \rho(x;x,t)$, the probability or charge density, with $\rho(x;y,t)$ the integral kernel of ρ. If v is even and bounded, then the initial value problem (5) is globally well-posed on the space of trace class operators on $L^2(\mathbb{R}^d)$ in the sense of mild solutions. For more details we refer to Appendix A. In this paper we will deal with mild solutions of initial value problems unless we emphasize that they are strong. To explain in what sense ρ_N converges to $\rho^{\otimes N}$ we need to define the well known notion of the partial trace:

Definition 1. *Consider a trace class operator R on $L^2(\mathbb{R}^d)^{\otimes m}$. For any $j < m$ we define the partial trace of R over the coordinates $j+1,...,m$ and we denote it by $Tr_{j+1,m}(R)$, the trace class operator on $L^2(\mathbb{R}^d)^{\otimes j}$ for which*

$$Tr(R(a \otimes I^{\otimes m-j})) = Tr(Tr_{j+1,m}(R)a), \tag{6}$$

for all linear bounded operators $a : L^2(\mathbb{R}^d)^{\otimes j} \to L^2(\mathbb{R}^d)^{\otimes j}$, where I is the identity on $L^2(\mathbb{R}^d)$.

It is well known that the partial trace exists and it is unique. We are now ready to state our main theorem:

Theorem 1.1. *Suppose that ρ_0 is a positive trace class operator with $Tr(\rho_0) = 1$, $v \in L^\infty$, v is even and (3) holds. If ρ_N is the solution of (1) and ρ is the solution of (5), then we have that*

$$Tr|Tr_{p+1,N}(\rho_N) - \rho^{\otimes p}| \le p\sqrt{\frac{8(e^{12\|v\|_\infty t} - 1)}{3N}}. \tag{7}$$

Remark 1. *The inequality (7) has a physical interpretation in terms of measurements of expectations of quantum observables: if $A = a \otimes I^{\otimes N-p}$ where $a : L^2(\mathbb{R}^d)^{\otimes p} \to L^2(\mathbb{R}^d)^{\otimes p}$ is a linear bounded operator, then using (6) and the standard inequality $|Tr(ar)| \le \|a\| Tr|r|$ (see for example [ReS]), we obtain that*

$$|Tr(A\rho_N) - Tr(a\rho^{\otimes p})| \le p\sqrt{\frac{8(e^{12\|v\|_\infty t} - 1)}{3N}}\|a\|. \tag{8}$$

$Tr(A\rho_N)$ is the expectation of the observable A in the quantum system with state ρ_N and $Tr(a\rho^{\otimes p})$ is the expectation that is predicted by (5). Theorem 1.1 gives an error estimate of this prediction.

The convergence result was first proven in [S]. The method could give estimates on the left hand side of (7) but they are very weak. The result of [S] was extended in [ES] for the case of Coulomb potentials and semi-relativistic kinetic energy in three dimensions. The proof uses compactness arguments and does not give any explicit estimates on the left hand side of (7). There is a considerable literature in the mean-field limit of (4) for both bounded and singular interaction potentials v and for closely related problems as well (see for example [AN], [AN1], [BGGM], [CP], [ErSY], [ErSYa], [EESY], [KSS], [AW], [LS], [FGS], [FKP], [FKS], [GiV], [GMM], [He], [RS], [ErS], [AS], [KF], [Sc], [P], [Pi] and [KP]).

To prove theorem 1.1 we rely heavily on ideas of [Pi] and [KP]. We also note that initial value problems similar to (10) and (12) where introduced in [KF] to write a Hamiltonian formulation for the Hartree-Fock equation.

The paper is organized as follows: In Section 2 we prove our main theorem given a rather technical estimate (proposition 2.3). In Section 3 we prove proposition 2.3.

Acknowledgements

The author is grateful to Israel Michael Sigal for suggesting this problem and for suggestions and numerous stimulating discussions. The author would also like to thank Antti Knowles for discussing his work [KP].

2 Proof of Theorem 1.1

In this Section we prove Theorem 1.1 given the estimate (27) which is proven in the next Section. In Subsection 2.1 we express ρ_N and ρ in terms of two Hilbert-Schmidt operators K_N, k that solve initial value problems which we discuss. In Subsection 2.2, using K_N and k, we introduce a functional a_N counting for the N-body state ρ_N the relative number of particles that are not in the one-body state ρ. This generalizes the functional counting bad particles introduced in [Pi]. In Subsection 2.3 we control the left hand side of (7) in terms of the functional a_N. In Subsection 2.4 we bound a_N.

2.1 Reformulation of the initial value problems (1) and (5)

Denote the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)^{\otimes n}$ by $\mathcal{L}_2^n$. If $n = 1$ the super index will be omitted. It is well known that $\mathcal{L}_2^n$ is a Hilbert space equipped with the inner product that induces the norm

$$\|K\|_{\mathcal{L}_2^n} := \sqrt{Tr(K^*K)}. \tag{9}$$

Let K_N be the solution of

$$\begin{cases} i\frac{\partial K_N}{\partial t} = [H_N, K_N] \\ K_N|_{t=0} = \sqrt{\rho_0^{\otimes N}}. \end{cases} \tag{10}$$

Since $K_N = e^{-iH_Nt}\sqrt{\rho_0^{\otimes N}}e^{iH_Nt}$ and $\rho_N = e^{-iH_Nt}\rho_0^{\otimes N}e^{iH_Nt}$, where ρ_N is the solution of (1), we have that $K_N \in \mathcal{L}_2^N$ and that

$$K_N \geq 0, \ \rho_N = K_N^2, \text{ and } Tr(\rho_N) = Tr(K_N^2) = 1, \tag{11}$$

for all times. Consider the following initial value problem in $\mathcal{L}_2$:

$$\begin{cases} i\frac{\partial k}{\partial t} = [h + (v * n_{k^*k}), k] \\ k|_{t=0} = k_0, \end{cases} \tag{12}$$

In Appendix A it is proven that, under the assumptions of theorem 1.1, the initial value problem (12) is globally well-posed on $\mathcal{L}_2$ in the sense of mild solutions. In Appendix A we also prove the following lemma:

Lemma 2.1. *If k is the solution of* (12) *with $k_0 = \sqrt{\rho_0}$ then we have that*

$$k \geq 0, \tag{13}$$

$$Tr(k^2) = 1, \tag{14}$$

and

$$\rho = k^2, \tag{15}$$

for all times, where ρ is the solution of (5).

We note that (13),(14) and (15) imply immediately that $\rho \geq 0$ and that $Tr(\rho) = 1$ for all times, which is well known. A similar idea to introducing equations (10) and (12) has been used in [KF] to write a Hamiltonian formulation for the Hartree-Fock equation. We note that if K_N solves (10) then K_N is symmetric with respect to the particle coordinates in the sense that

$$K_N = \pi K_N \pi^{-1} \text{ for all permutations } \pi \in S_N. \tag{16}$$

2.2 Counting functional a_N

The following definition generalizes the functional counting bad particles introduced in [Pi]. Let

$$a_N(K_N, k) := 1 - Tr((k \otimes K_N)(K_N \otimes k)). \tag{17}$$

The quantity $a_N(K_N, k)$ measures for a system of N particles in the state $K_N^2 = \rho_N$ the relative number of particles that are not in the one-body state $k^2 = \rho$. If for example $K_N = \frac{1}{\sqrt{N}}\sum_{m=1}^N k^{\otimes m-1} \otimes k^{\perp} \otimes k^{\otimes(N-m)}$, where $k^{\perp} \in \mathcal{L}_2$ is a positive operator that has Hilbert-Schmidt norm 1 and is orthogonal to k with respect to the inner product that induces the Hilbert-Schmidt norm, then $Tr(K_N^2) = 1$, K_N satisfies (16) and

$$a_N(K_N, k) = \frac{1}{N}.$$

The definition (17) generalizes the corresponding definition of [Pi]. Indeed, if $\rho_N = P_{\Psi_N}$ and $\rho = P_\psi$, where P_ϕ denotes the orthogonal projection onto the function ϕ and $\|\Psi_N\|_{L^2(\mathbb{R}^d)^{\otimes N}} = \|\psi\|_{L^2(\mathbb{R}^d)} = 1$, then by (11), (13) and (15) we obtain that $K_N = P_{\Psi_N}$ and $k = P_\psi$. Therefore,

$$a_N(K_N, k) \overset{(17)}{=} 1 - Tr(P_{\psi_N \otimes \psi} P_{\psi \otimes \Psi_N}) = 1 - Tr(P_{\Psi_N \otimes \psi} P_{\psi \otimes \Psi_N} P_{\Psi_N \otimes \psi})$$

$$= 1 - |(\Psi_N \otimes \psi, \psi \otimes \Psi_N)|^2 = 1 - (\langle \psi, Tr_{2,N}(P_{\Psi_N})\psi \rangle)^2,$$

where the last equality follows by extending ψ to an orthonormal basis of $L^2(\mathbb{R}^d)$, and expanding Ψ_N in terms of tensor products of the elements of the orthonormal basis. As a consequence, in this case we have that

$$a_N(K_N, k) \leq 2(1 - \langle \psi, Tr_{2,N}(P_{\Psi_N})\psi \rangle) = 2a_N(\Psi_N, \psi),$$

where $a_N(\Psi_N, \psi)$ is the functional counting bad particles introduced in [Pi].

2.3 Control of left hand side of (7) by a_N

In the following proposition we prove that as long as the relative number of particles that are not in the state ρ is small we can control the left hand side of (7).

Proposition 2.2. *If ρ_N, ρ solve (1) and (5), respectively, then under the assumptions of theorem 1.1*

$$Tr|Tr_{p+1,N}(\rho_N) - \rho^{\otimes p}| \leq p\sqrt{8a_N(K_N, k)}, \tag{18}$$

where K_N solves (10), and k solves (12) with initial condition $k_0 = \sqrt{\rho_0}$.

Proof. First we prove the following inequality:

$$Tr|Tr_{2,N}(\rho_N) - \rho| \leq \sqrt{8a_N(K_N, k)}. \tag{19}$$

For a separable Hilbert space X let $B(X) := \{F : X \to X, F \text{ linear and bounded}\}$. It is well known that $B(X)$ is the dual of the trace class operators on X (see for example [ReS]). Therefore, if r is a trace class operator acting on $L^2(\mathbb{R}^d)^{\otimes m}$ and $q \leq m$ then we have that

$$Tr|Tr_{q+1,m}(r)| = \sup_{\|a\|_{B(L^2(\mathbb{R}^d)^{\otimes q})}=1} |Tr(aTr_{q+1,m}(r))|$$

$$\overset{(6)}{=} \sup_{\|a\|_{B(L^2(\mathbb{R}^d)^{\otimes q})}=1} |Tr((a \otimes I^{\otimes m-q})r)| \leq \sup_{\|A\|_{B(L^2(\mathbb{R}^d)^{\otimes m})}=1} |Tr(Ar)| = Tr|r|. \tag{20}$$

Since $Tr(\rho_N) = Tr(\rho) = 1$, a simple computation gives that

$$Tr_{p+1,N+p}(\rho^{\otimes p} \otimes \rho_N) = \rho^{\otimes p} \text{ and } Tr_{p+1,N+p}(\rho_N \otimes \rho^{\otimes p}) = Tr_{p+1,N}(\rho_N),\ \forall p < N. \tag{21}$$

Therefore,

$$\begin{aligned} &Tr|Tr_{2,N}(\rho_N) - \rho| \\ &\overset{(20),(21)}{\leq} Tr|\rho_N \otimes \rho - \rho \otimes \rho_N| \\ &\overset{(11),(15)}{=} Tr|K_N^2 \otimes k^2 - k^2 \otimes K_N^2| \\ &\leq Tr|K_N \otimes k(K_N \otimes k - k \otimes K_N)| + Tr|(K_N \otimes k - k \otimes K_N)k \otimes K_N|. \end{aligned} \tag{22}$$

We will now prove that

$$Tr|K_N \otimes k(K_N \otimes k - k \otimes K_N)| \leq \sqrt{2a_N(K_N, k)}. \tag{23}$$

Indeed, we prove in Appendix B that if $L, M \in \mathcal{L}_2^N$ then

$$(Tr|LM|)^2 \leq Tr(LL^*)Tr(M^*M). \tag{24}$$

Therefore,

$$\begin{aligned} &Tr|K_N \otimes k(K_N \otimes k - k \otimes K_N)| \\ &\overset{(11),(13),(24)}{\leq} \sqrt{Tr(K_N^2 \otimes k^2)}\sqrt{Tr((K_N \otimes k - k \otimes K_N)^2)} \\ &\overset{(11),(14)}{=} \sqrt{2 - 2Tr((K_N \otimes k)(k \otimes K_N))} \overset{(17)}{=} \sqrt{2a_N(K_N, k)}. \end{aligned}$$

Similarly,

$$Tr|(K_N \otimes k - k \otimes K_N)k \otimes K_N| \leq \sqrt{2a_N(K_N, k)}. \tag{25}$$

Equations (22), (23) and (25) yield (19). We now finish the proof of proposition 2.2:

$$\begin{aligned} &Tr|Tr_{p+1,N}(\rho_N) - \rho^{\otimes p}| \\ &\overset{(20),(21)}{\leq} Tr|\rho_N \otimes \rho^{\otimes p} - \rho^{\otimes p} \otimes \rho_N| \\ &\leq \sum_{j=1}^{p} Tr|\rho^{\otimes j-1} \otimes \rho_N \otimes \rho^{\otimes p+1-j} - \rho^{\otimes j} \otimes \rho_N \otimes \rho^{\otimes p-j}| \\ &= pTr|\rho_N \otimes \rho - \rho \otimes \rho_N| \leq p\sqrt{8a_N(K_N, k)}, \end{aligned} \tag{26}$$

where the last inequality was proven in the proof of (19). □

2.4 Bound on a_N

In the next section we prove the following proposition:

Proposition 2.3. *Under the assumptions of proposition 2.2 and the additional assumption that h is bounded we have that*

$$\frac{d}{dt}a_N(K_N,k) \leq 4\|v\|_\infty(3a_N(K_N,k)+\frac{1}{N}). \tag{27}$$

Using this inequality and Gronwall's inequality to the function $a_N(K_N,k)+\frac{1}{3N}$ we obtain that

$$a_N(K_N,k) \leq \frac{(e^{12\|v\|_\infty t}-1)}{3N}, \tag{28}$$

in the case that h is bounded. Using lemma A.3 we extend (28) to h unbounded. Equations (18) and (28) imply the theorem.

3 Proof of proposition 2.3

To prove proposition 2.3 we use heavily ideas of [Pi]. We first start with some useful notation. For any $k \in \mathcal{L}_2$ self-adjoint and any $1 \leq j \leq N$ we let $k_j := I^{\otimes j-1} \otimes k \otimes I^{\otimes N-j}$. Let also $Tr_j(k_jK_N)$ denote the operator with integral kernel

$$\int k(y_j,x_j)K_N(x_1,...,x_N,y_1,...,y_N)dx_jdy_j,$$

where $k(x_j,y_j)$ and $K_N(x_1,...,x_N,y_1,...,y_N)$ are the integral kernels of k and K_N, respectively. Also let $p_j^k, q_j^k, \hat{n}^k : \mathcal{L}_2^N \to \mathcal{L}_2^N$ be defined by the relations

$$p_j^kK_N := k_jTr_j(k_jK_N), \tag{29}$$

$$q_j^k := 1-p_j^k,$$

$$\hat{n}^k := N^{-1}\sum_{j=1}^N q_j^k. \tag{30}$$

We mention some elementary relations that will be used throughout this section and they are proven in Appendix B:

Lemma 3.1. *Let $l \in \mathcal{L}_2$, $L, M \in \mathcal{L}_2^N$ where N is any natural number. Assume also that l is self-adjoint. Then the following are true*
(i)

$$Tr(LM) = Tr(ML). \tag{31}$$

The last equation also holds when one of the operators is a trace class operator and the other is a bounded operator.
(ii) $Tr_j(l_jL)$ is equal to $Tr_j(Ll_j)$ and it is Hilbert-Schmidt. Therefore p_j^lL and q_j^lL are also Hilbert-Schmidt. Also

$$s_j^l(L^*) = (s_j^l(L))^*, \tag{32}$$

where $s_j^l = p_j^l$ or $s_j^l = q_j^l$. In particular, if L is self-adjoint then p_j^lL, q_j^lL are also self-adjoint.
(iii) If s_j^l is as above then

$$Tr(Ls_j^lM) = Tr((s_j^lL)M). \tag{33}$$

(iv) The operators p_j^l, q_j^l are self-adjoint and therefore orthogonal projections.
(v) The operators $p_i^l, p_j^l, q_i^l, q_j^l$ all commute pairwise.
(vi) If k and K_N solve (12) and (10), respectively, then for all $j = 1,2,...,N$ we have that

$$a_N(K_N,k) = Tr(K_Nq_j^kK_N) = Tr(K_N\hat{n}^kK_N). \tag{34}$$

Lemma 3.2. *Under the assumptions of proposition 2.3 we have that*

$$\frac{d}{dt}a_N(K_N,k)$$

$$= -4ImTr(K_Np_1^kp_2^kB^kq_1^kq_2^kK_N) - 4ImTr(K_Np_1^kq_2^kB^kq_1^kq_2^kK_N), \tag{35}$$

where $B^k = v(x_1-x_2) - v * n_{k^2}$.

Proof. In what follows the projections p_j^k, q_j^k apply to products of all operators standing on their right. First we show the following relation:

$$\frac{d}{dt}a_N(K_N,k) = 2iTr(K_N p_1^k B^k K_N) - 2iTr(K_N B^k p_1^k K_N), \tag{36}$$

Indeed, due to equations (29) and (34) we have that

$$a_N(K_N,k) = 1 - Tr(K_N k_1 Tr_1(k_1 K_N)) = 1 - Tr(Tr_1(K_N k_1)Tr_1(k_1 K_N)). \tag{37}$$

Differentiating the last relation (which we can do because in this lemma h is assumed to be bounded), using equations (10) and (12) and using that $Tr_1(K_N k_1) = Tr_1(k_1 K_N)$ we obtain that

$$\begin{aligned}\frac{d}{dt}a_N(K_N,k) &= -2Tr((\frac{d}{dt}Tr_1(K_N k_1))Tr_1(k_1 K_N)) \\ &= 2iTr(Tr_1(([H_N,K_N]k_1 + K_N[H^k,k_1]))Tr_1(k_1K_N)).\end{aligned} \tag{38}$$

If we write $H_N = H_N^1 + H_N^2$, where H_N^1 is the sum of the terms of H_N that involve the first particle coordinate and $H_N^2 = H_N - H_N^1$, then a simple calculation gives that

$$Tr(Tr_1([H_N^2,K_N]k_1)Tr_1(k_1K_N)) = Tr([H_N^2, Tr_1(K_N k_1)]Tr_1(K_N k_1)) \overset{(31)}{=} 0.$$

The last relation together with (38) gives that

$$\begin{aligned}&\frac{d}{dt}a_N(K_N,k) \\ &= 2iTr(Tr_1(([H_N^1,K_N]k_1 + K_N[H^k,k_1]))Tr_1(k_1K_N)) \\ &= 2iTr(([H_N^1,K_N]k_1 + K_N[H^k,k_1])Tr_1(k_1K_N)) \\ &\overset{(31)}{=} 2iTr([H_N^1 - H^k, K_N]p_1^k K_N) \\ &\overset{(3),(16)}{=} 2iTr([B^k,K_N]p_1^kK_N) \\ &\overset{(31),(33)}{=} 2iTr(K_Np_1^kB^kK_N) - 2iTr(K_NB^kp_1^kK_N),\end{aligned}$$

so (36) is proven. Using (36) and that $p_1^k + q_1^k = 1$ we obtain that

$$\begin{aligned}&\frac{d}{dt}a_N(K_N,k) \\ &= 2iTr(K_Np_1^kB^kq_1^kK_N) - 2iTr(K_Nq_1^kB^kp_1^kK_N).\end{aligned} \tag{39}$$

Equations (33) and (39) together with the self-adjointness of B^k, $p_1^kK_N$ and $q_1^kK_N$ (see lemma 3.1 (ii)) give that

$$\begin{aligned}\frac{d}{dt}a_N(K_N,k) &= -4ImTr(K_Np_1^kB^kq_1^kK_N) \\ &\overset{p_2^k+q_2^k=1}{=} -4ImTr(K_Np_1^kp_2^kB^kq_1^kp_2^kK_N) \\ &\quad -4ImTr(K_Np_1^kq_2^kB^kq_1^kp_2^kK_N) \\ &\quad -4ImTr(K_Np_1^kp_2^kB^kq_1^kq_2^kK_N) \\ &\quad -4ImTr(K_Np_1^kq_2^kB^kq_1^kq_2^kK_N).\end{aligned} \tag{40}$$

To prove lemma 3.2 it suffices to prove that the first two terms on the righthand side of (40) are zero. Indeed, due to lemma 3.1 (v) we have that $Tr(K_Np_1^kp_2^kB^kq_1^kp_2^kK_N) = Tr(K_Np_1^kp_2^kB^kp_2^kq_1^kK_N)$. But

$$p_2^kB^kp_2^k = k_2Tr_2(k_2B^kk_2Tr_2(k_2.)),$$

and $Tr_2(k_2B^kk_2) = Tr_2(k_2^2B^k) = Tr_2(\rho_2 v(x_1-x_2) - \rho_2 v * n_\rho) = v*n_\rho - v*n_\rho = 0$. This shows that the first term on the righthand side of (40) is zero. On the other hand since $q_2^kp_2^k = 0$, we have that

$$Tr(K_Np_1^kq_2^kB^kq_1^kp_2^kK_N) = Tr(K_Np_1^kq_2^kv(x_1-x_2)q_1^kp_2^kK_N). \tag{41}$$

In addition,

$$\begin{aligned}&Tr(K_Np_1^kq_2^kv(x_1-x_2)q_1^kp_2^kK_N) \\ &\overset{v \text{ even},(16)}{=} Tr(K_Np_2^kq_1^kv(x_1-x_2)q_2^kp_1^kK_N) \\ &\overset{(33)}{=} Tr((p_2^kq_1^kK_N)v(x_1-x_2)q_2^kp_1^kK_N) \\ &\overset{\text{lemma 3.1 (ii)}}{=} \overline{Tr((q_2^kp_1^kK_N)v(x_1-x_2)p_2^kq_1^kK_N)} \\ &\overset{(33)}{=} \overline{Tr(K_Np_1^kq_2^kv(x_1-x_2)q_1^kp_2^kK_N)},\end{aligned} \tag{42}$$

which together with (41) implies that the second term of the righthand side of equation (40) is also zero. □

We will now proceed by estimating the terms on the righthand side of (35). We have that

$$|Im(Tr(K_N p_1^k q_2^k B^k q_1^k q_2^k K_N))| \leq 2\|v\|_\infty a_N(K_N,k). \tag{43}$$

Indeed,

$$\begin{aligned}
&|Im(Tr(K_N p_1^k q_2^k B^k q_1^k q_2^k K_N))| \\
&\overset{(33)}{\leq} |Tr((p_1^k q_2^k K_N) B^k q_1^k q_2^k K_N)| \\
&\overset{(31)}{\leq} |Tr(B^k (q_1^k q_2^k K_N) p_1^k q_2^k K_N)| \\
&\leq \|B^k\| Tr|(q_1^k q_2^k K_N) p_1^k q_2^k K_N| \\
&\overset{(24),\ \text{lemma 3.1 (ii), (iv)}}{\leq} 2\|v\|_\infty Tr((q_2^k K_N)^2) \\
&\overset{(33),(34)}{=} 2\|v\|_\infty a_N(K_N,k).
\end{aligned}$$

We next prove that

$$|ImTr(K_N p_1^k p_2^k B^k q_1^k q_2^k K_N)| \leq \|v\|_\infty (\frac{1}{N} + a_N(K_N,k)). \tag{44}$$

Indeed,

$$\begin{aligned}
&|ImTr(K_N p_1^k p_2^k B^k q_1^k q_2^k K_N)| \\
&\overset{\text{lemma 3.1 (v)},\ p_2^k q_2^k = 0}{=} |ImTr(K_N p_1^k p_2^k v(x_1 - x_2) q_1^k q_2^k K_N)| \\
&\overset{(33)}{\leq} |Tr((p_1^k p_2^k K_N) v(x_1 - x_2) q_1^k q_2^k K_N)|.
\end{aligned} \tag{45}$$

Now we will use some notation defined in [Pi]. Let

$$P_{N,l}^k := \sum_{a \in \mathcal{A}_l} \prod_{m=1}^{N} (p_m^k)^{1-a_m} q_m^{a_m}, \tag{46}$$

where

$$\mathcal{A}_l := \{(a_1, ..., a_N) \in \{0,1\}^N : \sum_{j=1}^{N} a_j = l\}. \tag{47}$$

Since $p_j^k + q_j^k = 1$ we obtain that

$$\sum_{l=0}^{N} P_{N,l}^k = I. \tag{48}$$

Repeating arguments of [Pi] we can prove that

$$\hat{n}^k = \sum_{l=1}^{N} \frac{l}{N} P_{N,l}^k, \tag{49}$$

where $\hat{n}^k$ was defined in (30). Following [Pi] we define

$$(\hat{n}^k)^j := \sum_{l=1}^{N} (\frac{l}{N})^j P_{N,l}^k. \tag{50}$$

where j is any real number. Since $P_{N,l}^k P_{N,b}^k = \delta_{bl} P_{N,l}^k$, one can see that

$$(\hat{n}^k)^{j_1} (\hat{n}^k)^{j_2} = (\hat{n}^k)^{j_1+j_2},\ \forall j_1, j_2 \in \mathbb{R}. \tag{51}$$

Equations (48), (50) and (51) give that

$$(\hat{n}^k)^j (\hat{n}^k)^{-j} + P_{N,0}^k = I, \tag{52}$$

which implies that for any $m \in \{1, 2, ..., N\}$, we have that

$$(\hat{n}^k)^j (\hat{n}^k)^{-j} q_m^k = q_m^k. \tag{53}$$

In addition by (33),(46) and (50) we obtain that

$$Tr(K(\hat{n}^k)^j L) = Tr(((\hat{n}^k)^j K) L),\ \forall j \in \mathbb{R},\ \forall K, L \in \mathcal{L}_2^N. \tag{54}$$

Therefore,

$$
\begin{aligned}
&|Tr((p_1^k p_2^k K_N)v(x_1-x_2)q_1^k q_2^k K_N)| \\
&\stackrel{(53)}{=} |Tr((p_1^k p_2^k K_N)v(x_1-x_2)(\hat{n}^k)^{\frac{1}{2}}(\hat{n}^k)^{-\frac{1}{2}}q_1^k q_2^k K_N)| \\
&\stackrel{(54)}{=} |Tr(\left[(\hat{n}^k)^{\frac{1}{2}}(p_1^k p_2^k K_N)v(x_1-x_2)\right](\hat{n}^k)^{-\frac{1}{2}}q_1^k q_2^k K_N)| \\
&\stackrel{(32),(46),(50)}{\leq} \sqrt{Tr(\left[(\hat{n}^k)^{\frac{1}{2}}(p_1^k p_2^k K_N)v(x_1-x_2)\right](\hat{n}^k)^{\frac{1}{2}}v(x_1-x_2)p_1^k p_2^k K_N)}\sqrt{Tr(\left[(\hat{n}^k)^{-\frac{1}{2}}q_1^k q_2^k K_N\right](\hat{n}^k)^{-\frac{1}{2}}q_1^k q_2^k K_N)} \\
&\stackrel{(51),(54)}{=} \sqrt{Tr((p_1^k p_2^k K_N)v(x_1-x_2)(\hat{n}^k)v(x_1-x_2)p_1^k p_2^k K_N)}\sqrt{Tr(K_N q_1^k q_2^k(\hat{n}^k)^{-1}q_1^k q_2^k K_N)}. \qquad (55)
\end{aligned}
$$

But

$$
\begin{aligned}
&|Tr((p_1^k p_2^k K_N)v(x_1-x_2)\hat{n}^k v(x_1-x_2)p_1^k p_2^k K_N)| \\
&\stackrel{(30)}{=} |\frac{1}{N}\sum_{j=1}^{N} Tr((p_1^k p_2^k K_N)v(x_1-x_2)q_j^k v(x_1-x_2)p_1^k p_2^k K_N)| \\
&\stackrel{v\text{ even},(16)}{\leq} |\frac{2}{N}Tr((p_1^k p_2^k K_N)v(x_1-x_2)q_1^k v(x_1-x_2)p_1^k p_2^k K_N)| \\
&+|\frac{N-2}{N}Tr((p_1^k p_2^k K_N)v(x_1-x_2)q_3^k v(x_1-x_2)p_1^k p_2^k K_N)| \\
&\stackrel{\text{lemma3.1}(iv),(v)}{\leq} \frac{2}{N}\|v\|_\infty^2 + \frac{N-2}{N}|Tr((p_1^k p_2^k q_3^k K_N)v(x_1-x_2)v(x_1-x_2)p_1^k p_2^k q_3^k K_N)| \\
&= \frac{2}{N}\|v\|_\infty^2 + \frac{N-2}{N}|Tr(v(x_1-x_2)v(x_1-x_2)(p_1^k p_2^k q_3^k K_N)(p_1^k p_2^k q_3^k K_N))| \\
&\leq \frac{2}{N}\|v\|_\infty^2 + \|v\|_\infty^2\frac{N-2}{N}a_N(K_N,k), \qquad (56)
\end{aligned}
$$

where the last inequality is proven similarly to (43). On the other hand, since q_i^k commutes with $(n^k)^j$, we have that

$$
\begin{aligned}
&N(N-1)Tr(K_N q_1^k q_2^k(\hat{n}^k)^{-1}q_1^k q_2^k K_N) \\
&= N(N-1)Tr(K_N(\hat{n}^k)^{-1}q_1^k q_2^k K_N), \qquad (57)
\end{aligned}
$$

and

$$
\begin{aligned}
&Tr(K_N(\hat{n}^k)^{-1}q_j^k q_j^k K_N) \\
&\stackrel{(33)}{=} Tr((q_j^k K_N)(\hat{n}^k)^{-1}q_j^k K_N) \\
&\stackrel{(51),(54)}{=} Tr(((\hat{n}^k)^{-\frac{1}{2}}q_j^k K_N)^2) \geq 0, \qquad (58)
\end{aligned}
$$

where the last inequality holds because lemma 3.1 (ii) together with (46) and (50) imply that $(\hat{n}^k)^{-\frac{1}{2}}q_j^k K_N$ is self-adjoint. Using (16), (57) and (58) we obtain that

$$
\begin{aligned}
&N(N-1)Tr(K_N q_1^k q_2^k(\hat{n}^k)^{-1}q_1^k q_2^k K_N) \\
&\leq \sum_{j,l=1}^{N} Tr(K_N(\hat{n}^k)^{-1}q_j^k q_l^k K_N) \\
&\stackrel{(30),(51)}{=} N^2 Tr(K_N\hat{n}^k K_N) \leq N^2 a_N(K_N,k), \qquad (59)
\end{aligned}
$$

Equations (45), (55), (56) and (59) together with the arithmetic mean-geometric mean inequality imply equation (44). Equations (35), (43) and (44) imply equation (27). This completes the proof of proposition 2.3.

A Global well-posedness of (5) and (12) and proof of lemma 2.1

For convenience of the reader we prove in this Appendix that (12) is globally well-posed on $\mathcal{L}_2$. The global well-posedness of (5) on the space of trace class operators on $L^2(\mathbb{R}^d)$ is proven in a similar way. For closely related results we refer to [BDF], [C], [CG].

If k solves (12) then by the Duhamel principle we obtain that

$$k(t) = e^{-iht}k_0 e^{iht} - i\int_0^t e^{-ih(t-s)}[v * n_{k(s)^*k(s)}, k(s)]e^{ih(t-s)}ds. \qquad (60)$$

A solution of (60) is called a mild solution of (12).

Theorem A.1. *If $v \in L^\infty$ and v is even, then the initial value problem (12) is globally well-posed on $\mathcal{L}_2$ in the sense of mild solutions. Moreover, the Hilbert-Schmidt norm of its solution is conserved.*

Proof. For any $T > 0$ the space $C([0,T],\mathcal{L}_2)$ equipped with the norm

$$||k||_C := \sup_{t\in[0,T]} \|k(t)\|_{\mathcal{L}_2}$$

is a Banach space. We first prove that

$$k \in C([0,T],\mathcal{L}_2) \implies e^{-ih\cdot}k(.)e^{ih\cdot} \in C([0,T],\mathcal{L}_2). \tag{61}$$

Indeed, we have that

$$\begin{aligned} &\|e^{-iht}k(t)e^{iht} - e^{-ihs}k(s)e^{ihs}\|_{\mathcal{L}_2} \\ \le &\|e^{-iht}(k(t)-k(s))e^{iht}\|_{\mathcal{L}_2} + \|e^{-iht}k(s)e^{iht} - e^{-ihs}k(s)e^{ihs}\|_{\mathcal{L}_2} \\ \le &\|k(t)-k(s)\|_{\mathcal{L}_2} + \|e^{-ih(t-s)}k(s)e^{ih(t-s)} - k(s)\|_{\mathcal{L}_2}. \end{aligned}$$

To finish the proof of (61) it suffices to show that

$$\lim_{t\to 0} e^{-iht}ke^{iht} = k, \forall k \in \mathcal{L}_2. \tag{62}$$

Indeed, since finite rank operators are dense to the Hilbert-Schmidt operators we can assume without loss of generality that k is a rank one operator. But in this case since k is bounded by the Rietz representation theorem there exist $\psi_1,\psi_2 \in L^2(\mathbb{R}^d)$ such that $k\phi = (\psi_1,\phi)\psi_2$ for all $\phi \in L^2(\mathbb{R}^d)$. Using the fact that $e^{-iht}\psi_j \to \psi_j$ as $t \to 0$ and that for a two rank operator its Hilbert-Schmidt norm can be controlled by its operator norm we obtain (62). A similar proof has been given in [BDF] for trace class operators. We now prove that

$$k \in C([0,T],\mathcal{L}_2) \implies -i[v * n_{k^*k}, k] \in C([0,T],\mathcal{L}_2). \tag{63}$$

To prove (63) it suffices to prove that for all $k_1,k_2 \in \mathcal{L}_2$ we have that

$$\begin{aligned} &\|[v*n_{k_1^*k_1},k_1] - [v*n_{k_2^*k_2},k_2]\|_{\mathcal{L}_2} \\ &\le 2\|v\|_\infty(\|k_1\|^2_{\mathcal{L}_2} + \|k_2\|^2_{\mathcal{L}_2} + \|k_1\|_{\mathcal{L}_2}\|k_2\|_{\mathcal{L}_2})\|k_1-k_2\|_{\mathcal{L}_2}. \end{aligned} \tag{64}$$

Indeed, we have

$$\begin{aligned} &\|[v*n_{k_1^*k_1},k_1] - [v*n_{k_2^*k_2},k_2]\|_{\mathcal{L}_2} \\ = &\|[v*n_{k_1^*k_1},k_1-k_2]\|_{\mathcal{L}_2} + \|[v*(n_{k_1^*k_1} - n_{k_2^*k_2}),k_2]\|_{\mathcal{L}_2} \\ \le &2\|k_1\|^2_{\mathcal{L}_2}\|v\|_\infty\|k_1-k_2\|_{\mathcal{L}_2} + 2\|k_2\|_{\mathcal{L}_2}\|v\|_{L^\infty}\|n_{k_1^*k_1} - n_{k_2^*k_2}\|_{L^1}. \end{aligned} \tag{65}$$

To complete the proof of (64) is suffices to observe that if we use the integral kernels of k_1,k_2 together with the Cauchy-Schwartz inequality we obtain that

$$\|n_{k_1^*k_1} - n_{k_2^*k_2}\|_{L^1} \le (\|k_1\|_{\mathcal{L}_2} + \|k_2\|_{\mathcal{L}_2})\|k_1-k_2\|_{\mathcal{L}_2}. \tag{66}$$

We know proceed to prove local well-posedness of (12). By (60) a solution of (12) is a fixed point of the map

$$H(k)(t) := e^{-iht}k_0e^{iht} - i\int_0^t e^{-ih(t-s)}[v*n_{k(s)^*k(s)},k(s)]e^{ih(t-s)}ds. \tag{67}$$

Using the fact that $\|n_{k^*k}\|_{L^1} = \|k\|^2_{\mathcal{L}_2}$, it is easy to show that

$$||H(k)||_C \le \|k_0\|_{\mathcal{L}_2} + 2T\|v\|_\infty||k||^3_C. \tag{68}$$

From equations (61),(63) and (68) it follows that $H(k)$ maps any ball in $C([0,T],\mathcal{L}_2)$ of radius $R \ge 2\|k_0\|_{\mathcal{L}_2}$ to itself provided that $T \le \frac{1}{4\|v\|_\infty R^2}$. On the other hand in this ball we have that

$$\begin{aligned} &||H(k_1) - H(k_2)||_C \\ &\le T||[v*n_{k_1^*k_1},k_1] - [v*n_{k_2^*k_2},k_2]||_C \\ &\le 6TR^2\|v\|_\infty||k_1-k_2||_C, \end{aligned} \tag{69}$$

where the last inequality follows from (64). Therefore choosing $T < \frac{1}{6\|v\|_\infty R^2}$ we can apply the Banach fixed point theorem to obtain local well-posedness of (12).

We prove now that if k is a solution of (12) then its Hilbert-Schmidt norm is conserved. Thus, we can iterate the previous argument to obtain global well-posedness of (12).

Lemma A.2. *Suppose that k is a solution of (12) on $C([0,T],\mathcal{L}_2)$ where T is a positive real number. Then the Hilbert-Schmidt norm of k is conserved.*

Proof. Assume first that h is bounded. Then k is differentiable. Therefore k is a strong solution of (12). As a consequence k is the unique solution of the linear problem

$$\begin{cases} i\frac{\partial r}{\partial t} = [h + (v * n_{k^*k}), r] \\ r|_{t=0} = k_0. \end{cases} \tag{70}$$

Since it is easy to verify that

$$r = e^{-i\int_0^t h+v*n_{k*(s)k(s)}ds} k_0 e^{i\int_0^t h+v*n_{k^*(s)k(s)}ds}, \tag{71}$$

and since the operator $\int_0^t (h + v * n_{k^*(s)k(s)})ds$ is self-adjoint (because it is bounded and symmetric), we obtain the lemma and therefore global well-posedness of (12) for the case that h is bounded. A similar proof has been given in [BDF] for trace class operators. To prove the lemma for arbitrary self-adjoint operator h it suffices to prove the following lemma:

Lemma A.3. *Assume that k is a solution of (12) on $C([0,T],\mathcal{L}_2)$. Then there exists a family h_λ of bounded self-adjoint operators, such that*

$$\lim_{\lambda\to\infty} \|k_\lambda(t) - k(t)\|_{\mathcal{L}_2} \to 0, \forall t \in [0,T], \tag{72}$$

where k_λ denotes the solution of (12) with h replaced by h_λ. (We also note that if we define ρ_λ and $K_{N,\lambda}$ in the same way, then by a similar argument we have that $Tr|\rho_\lambda - \rho| \to 0$ and $\|K_{N,\lambda} - K_N\|_{\mathcal{L}_2} \to 0$.)

Proof. Let $h_\lambda := \frac{1}{2}\lambda^2[(h+i\lambda)^{-1} + (h-i\lambda)^{-1}]$. Then h_λ is bounded and $e^{-ih_\lambda t}\psi \to e^{-iht}\psi$, as $\lambda \to \infty$ for all $\psi \in L^2$. For a proof of the last relation we refer, for example, to [GS]. Since by Duhamel principle

$$k(t) = e^{-iht}k_0e^{iht} - i\int_0^t e^{-ih(t-s)}[v * n_{k(s)^*k(s)}, k(s)]e^{ih(t-s)}ds, \tag{73}$$

and

$$k_\lambda(t) = e^{-ih_\lambda t}k_0e^{ih_\lambda t} - i\int_0^t e^{-ih_\lambda(t-s)}[v * n_{k^*_{\lambda(s)}k_\lambda(s)}, k_\lambda(s)]e^{ih_\lambda(t-s)}ds, \tag{74}$$

we have that

$$\begin{aligned} &\|k_\lambda(t) - k(t)\|_{\mathcal{L}_2} \le \epsilon_\lambda \\ &+\|\int_0^t dse^{-ih(t-s)}\left([v * n_{k_\lambda(s)^*k_\lambda(s)}, k_\lambda(s)] - [v * n_{k(s)^*k(s)}, k(s)]\right)e^{ih(t-s)}\|_{\mathcal{L}_2}, \end{aligned} \tag{75}$$

where

$$\begin{aligned} &\epsilon_\lambda := \|e^{-ih_\lambda t}k_0e^{ih_\lambda t} - e^{-iht}k_0e^{iht}\|_{\mathcal{L}_2} \\ &+\int_0^t ds\|e^{-ih_\lambda(t-s)}[v * n_{k_\lambda(s)^*k_\lambda(s)}, k_\lambda(s)]e^{ih_\lambda(t-s)} - e^{-ih(t-s)}[v * n_{k_\lambda(s)^*k_\lambda(s)}, k_\lambda(s)]e^{ih(t-s)}\|_{\mathcal{L}_2}. \end{aligned} \tag{76}$$

Similarly to showing equation (62) one can show that

$$\lim_{\lambda\to\infty} \|e^{-ih_\lambda t}k_0e^{ih_\lambda t} - e^{-iht}k_0e^{iht}\|_{\mathcal{L}_2} \to 0,$$

and that for any $s \in [0,t]$

$$\lim_{\lambda\to\infty} \|e^{-ih_\lambda(t-s)}[v * n_{k_\lambda(s)^*k_\lambda(s)}, k_\lambda(s)]e^{ih_\lambda(t-s)} - e^{-ih(t-s)}[v * n_{k_\lambda(s)^*k_\lambda(s)}, k_\lambda(s)]e^{ih(t-s)}\|_{\mathcal{L}_2} = 0.$$

This together with the dominated convergence theorem for the integral on the righthand side of (76) gives that

$$\epsilon_\lambda \to 0, \text{ when } \lambda \to \infty. \tag{77}$$

On the other hand equations (64) and (75) together with the conservation of the Hilbert-Schmidt norm of k_λ give us that

$$\|k_\lambda(t) - k(t)\|_{\mathcal{L}_2} \le \epsilon_\lambda + D\|v\|_\infty \int_0^t ds\|k_\lambda(s) - k(s)\|_{\mathcal{L}_2}, \tag{78}$$

where $D = \sup_{t\in[0,T]} 2(\|k(t)\|^2_{\mathcal{L}_2} + \|k(t)\|_{\mathcal{L}_2}\|k_0\|_{\mathcal{L}_2} + \|k_0\|^2_{\mathcal{L}_2})$. This together with Gronwall's inequality gives that

$$\|k_\lambda(t) - k(t)\|_{\mathcal{L}_2} \le \epsilon_\lambda e^{D\|v\|_\infty t}. \tag{79}$$

The last equation together with equation (77) imply equation (72). □

This completes the proof of lemma A.2. □

This completes the proof of theorem A.1. □

Proof of lemma 2.1

Proof. Assume first that h is bounded. Then by equation (71) and the self-adjointness of $\int_0^t (h + v * n_{k^*(s)k(s)})ds$ it follows immediately that the spectrum and therefore the Hilbert-Schmidt norm and the positivity of k are preserved. Therefore, (13) and (14) follow immediately. To prove (15) it suffices to observe that k^2 solves (5). Using lemma A.3 we extend the result to the case that h is unbounded. □

B Proof of equation (24) and of lemma 3.1

The equation (24) is well known (see for example [KP]). For convenience of the reader we outline its proof. By polar decomposition, there exists a partial isometry U such that $|LM| = ULM$. Therefore, by the Cauchy-Schwartz inequality we obtain that

$$(Tr(|LM|))^2 = (Tr(ULM))^2 \leq Tr(ULL^*U^*)Tr(M^*M) \leq Tr(LL^*)Tr(M^*M).$$

We will now prove lemma 3.1.

(i) For a proof we refer to [ReS].

(ii) We can assume without loss of generality that $j = 1$. We have that

$$\begin{aligned}
&\|Tr_1(l_1 L)\|_{\mathcal{L}_2^{N-1}} \\
&\leq \left(\int \left(\int |l_1(y_1, x_1)||L(x_1, ..., x_N, y_1, ..., y_N)|dx_1 dy_1\right)^2 dx_2...dx_N dy_2...dy_N\right)^{\frac{1}{2}} \\
&\leq \int |l_1(y_1, x_1)|[\int |L(x_1, ..., x_N, y_1, ..., y_N)|^2 dx_2...dx_N dy_2...dy_N]^{\frac{1}{2}} dx_1 dy_1 \\
&\leq (\int |l_1(y_1, x_1)|^2 dx_1 dy_1)^{\frac{1}{2}} (\int |L(x_1, x_2, ...x_N, y_1, ..., y_N)|^2 dx_1...dx_N dy_1...dy_N)^{\frac{1}{2}} \\
&= \|l_1\|_{\mathcal{L}_2} \|L\|_{\mathcal{L}_2^N},
\end{aligned}$$

where in the third last line we used Minkowski inequality for integrals. If we compute the integral kernel of $Tr_1(Ll_1)$ and we apply Fubbini's theorem, which can be applied because we have just shown that

$$\int |l(y_1, x_1||L(x_1, ..., x_N, y_1, ..., y_N)|dx_1 dy_1 < \infty, \text{ for almost all } (x_2, ..., x_N, y_2, ..., y_N),$$

we obtain that $Tr_1(l_1 L)$ has the same integral kernel as $Tr_1(Ll_1)$ almost everywhere. Therefore, the operators are equal. To prove equation (32) we will do computations with the integral kernels. The integral kernel of $p_1^l(L^*)$ is

$$\begin{aligned}
&\overline{l_1(x_1, y_1) \int l_1(y_{N+1}, x_{N+1}) L^*(x_{N+1}, x_2, ..., x_N, y_{N+1}, y_2, ..., y_N) dx_{N+1} dy_{N+1}} \\
&= l_1(y_1, x_1) \int l_1(x_{N+1}, y_{N+1}) L(y_{N+1}, y_2, ..., y_N, x_{N+1}, x_2, ..., x_N) dx_{N+1} dy_{N+1}.
\end{aligned}$$

In a similar way one can compute the integral kernel of $(p_1^l(L))^*$.Using Fubbini's theorem we obtain as above that the operators are equal. This implies equation (32) immediately.

(iii) It similarly follows by using integral kernels and applying Fubbini's theorem.

(iv) It is easy to verify that p_j^l and therefore q_j^l is a projection. From equations (32) and (33) it follows that p_j^l is self-adjoint. Therefore, p_j^l is an orthogonal projection and as a consequence so is q_j^l.

(v) and (vi) also follow from Fubbini's theorem. For (vi) we also need to use (16).

This completes the proof of lemma 3.1.

References

[AS] Abou Salem W.K.: *A Remark on the mean-field dynamics of many-body bosonic systems with random interactions and in a random potential*. Lett. Math. Phys. 84, 231-243 (2008).

[AW] Abou Salem W.K.: *Mean-field dynamics of rotating bosons in confining traps.* Available online at http://www.math.toronto.edu/walid/RotMF.pdf. Preprint (2009).

[AN1] Ammari Z., Nier F.: *Mean-field limit for bosons and infinite dimensional phase space analysis*. Annales Henri Poincaré, vol. 9, issue 8, 1503-1574 (2008).

[AN] Ammari Z., Nier F.: *Mean-field propagation of Wigner measures and BBGKY hierarchies for general bosonic states.* ArXiv:1003.2054 (2010).

[BGGM] Bardos C., Golse F., Gottlieb A.D., Mauser N.: *Mean-field dynamics of fermions and the time-dependent Hartree-Fock equation*. J. Math. Pures Appl. 82, 665-683 (2003).

[BDF] Bove A., Da Prato G., Fano G.: *An existence proof for the Hartree-Fock time-dependent problem with bounded two-body interaction.* Comm. Math. Phys. 37, 183-191 (1974).

[C] Chadam J. M.: *The time-dependent Hartree-Fock equations with Coulomb two-body interaction*. Comm. Math. Phys. 46, 99-104 (1976).

[CG] Chadam J. M., Glassey R. T.: *Global existence of solutions to the Cauchy problem for time-dependent Hartree equations*. J. Math. Phys. 16, 1122-1130 (1975).

[CP] Chen T., Pavlovic N.: *The quintic NLS as the mean-field limit of a Boson gas with three-body interactions*. ArXiv:0812.2740 (2009).

[CPT] Chen T., Pavlovic N., Tzirakis N.: *Energy conservation and blowup of solutions for focusing Gross-Pitaevskii hierarchies* . ArXiv:0905.2704 (2009).

[ESY] Elgart A., Erdös L., Schlein B., Yau H-T.: *Gross-Pitaevskii equation as the mean-field limit of weakly coupled Bosons*. Arch. Rat. Mech. Anal. 179, no. 2, 265-283 (2006).

[EESY] Elgart A., Erdös L., Schlein B., Yau H-T.: *Nonlinear Hartree equation as the mean-field limit of weakly coupled fermions*. J. Math. Pure Appl. 83, issue 10, 1241-1273 (2004).

[ES] Elgart A., Schlein B.: *Mean-field dynamics of boson stars*. Comm. Pure and Applied Math. 60, no. 4, 500-545 (2007).

[ErS] Erdös L., Schlein B.: *Quantum dynamics with mean-field interactions: a New Approach*. J. Stat. Phys. 134, 859-870 (2009).

[ErSY] Erdös L., Schlein B., Yau H-T.: *Rigorous Derivation of the Gross-Pitaevskii Equation with a Large Interaction Potential.* J. Amer. Math. Soc. 22, 1099-1156 (2009).

[ErSYa] Erdös L., Schlein B., Yau H-T.: *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Eistein condensate.* To appear in Annals Math.

[FGS] Fröhlich J., Graffi S., Schwarz S.: *Mean-field and classical limit of many-body Schrödinger dynamics for Bosons*. Comm. Math. Phys. 271, no.3, 681-697 (2007).

[FKP] Fröhlich J., Knowles A., Pizzo A.: *Atomism and quantization*. J. Phys. A: Math. Theor. 40 3033-3045 (2007).

[FKS] Fröhlich J., Knowles A., Schwarz S.: *On the mean-field limit of bosons with Coulomb two body interaction*. Comm. Math. Phys. 288, no.3, 1023-1059 (2009).

[GiV] Ginibre J., Velo G.: *The classical field limit of scattering theory for non relativistic many-boson systems, I and II.* Comm. Math. Phys. 66, 37-76 (1979) and 68, 45-68 (1979).

[GMM] Grillakis M.G, Machedon M., Margetis D.: *Second-order corrections to mean-field evolution of weakly interacting Bosons, I and II.* . Comm. Math. Phys. 294, 273-301 (2010) and ArXiv:1003.4713 (2010).

[GS] Gustafson S., Sigal I.M: *Mathematical concepts of quantum mechanics.* Springer (2006).

[He] Hepp K.: *The classical limit for quantum mechanical correlation functions*. Commun. Math. Phys. 35, 265-277 (1974).

[KSS] Kirkpatrick K., Schlein B., Staffilani G.: *Derivation of the two dimensional nonlinear Schödinger equation from many-body quantum dynamics*. ArXiv:0808.0505 (2008).

[KF] Knowles A., Fröhlich J.: *A microscopic derivation of the time-dependent Hartree-Fock equation with Coulomb two-body interaction*. ArXiv:0810.4282 (2008).

[KP] Knowles A., Pickl P.: *Mean-Field Dynamics: Singular Potentials and Rate of Convergence.* ArXiv:0907.4313 (2009).

[LS] Lieb E., Seiringer R.: *Proof of Bose-Einstein condensation for dilute trapped gases*. Phys. Rev. Lett. 88, no.17, 170409 (2002).

[Pi] Pickl P.: *A simple derivation of mean-field limits for quantum systems*. ArXiv:0907.4464 (2009).

[P] Pickl P.: *Derivation of the Time Dependent Gross Pitaevskii Equation with External Fields.* ArXiv:1001.4894 (2010).

[ReS] Reed M., Simon B.: *Methods of modern Mathematical Physics I: Functional Analysis*, Academic Press (1980).

[RS] Rodnianski I., Schlein B.: *Quantum fluctuations and rate of convergence towards mean-field dynamics.* Comm. Math. Phys. 291, no. 1, 31-61 (2009).

[Sc] Schlein B.: *Derivation of effective evolution equations from many-body quantum dynamics.* ArXiv:0910.3969 (2009).

[S] Spohn H.: *Kinetic equations from Hamiltonian dynamics*. Rev. Mod. Phys. 52, no. 3, 569-615 (1980).