LANDAU'S PROBLEM OF DEGENERATE PLASMA OSCILLATIONS WITH SPECULAR – ACCOMMODATIVE BOUNDARY CONDITIONS

N.V. Gritsienko, A.V. Latyshev, A.A. Yushkanov

(Moscow State Regional University,
Radio st., 10a, Moscow, 105005)
e-mail: natafmf@yandex.ru,
avlatyshev@mail.ru,
yushkanov@inbox.ru

In the present paper the linearized problem of plasma oscillations in external longitudinal alternating electric field is solved analytically. Specular – accommodative boundary conditions of electron reflection from the plasma boundary are considered. Coefficients of continuous and discrete spectra of the problem are found, and electron distribution function on the plasma boundary and electric field are expressed in explicit form.

Refs. 38. Figs. 2.

Keywords: degenerate plasma, half-space, normal momentum accommodation coefficient, specular – accommodative boundary condition, plasma mode, expansion by eigenfunctions, singular integral equation.

PACS numbers: 52.35.-g, 52.90.+z

1. INTRODUCTION

The present paper is devoted to degenerate electron plasma behaviour research. Analysis of processes taking place in plasma under effect of external electric field, plasma waves oscillations with various types of conditions of electron reflection from the boundary has important significance today in connection with problems of such intensively developing fields as microelectronics and nanotechnologies [1] – [6].

The concept of "plasma" appeared in the works of Tonks and Langmuir for the first time (see [7]–[9]), the concept of "plasma frequency" was introduced in the same works and first questions of plasma oscillations were considered there. However, in these works equation for the electric field was considered separately from the kinetic equation.

A.A. Vlasov [10] for the first time introduced the concept of "self-consistent electric field" and added the corresponding item to the kinetic equation. Now equations describing plasma behaviour consist of anchor system of equations of Maxwell and Boltzmann. The problem of electron plasma oscillations was considered by A.A. Vlasov [10] by means of solution of the kinetic equation which included self-consistent electric field.

L.D. Landau [11] had supposed that outside of the half-space containing degenerate plasma external electromagnetic field causing oscillations in plasma is situated. By this Landau has formulated a boundary condition on the plasma boundary. After that the problem of plasma oscillation turns out to be formulated correctly as a boundary problem of mathematical physics.

In [11] L.D. Landau has solved analytically by Fourier series the problem of collisionless plasma behaviour in a half-space, situated in external longitudinal (perpendicular to the surface) electric field, in conditions of specular refl ection of electrons from the boundary.

Further the problem of electron plasma oscillations was considered by many authors. Full analytical solution of the problem is given in the works [12] and [13].

This problem has important significance in the theory of plasma (see, for instance, [2], [14] and the references in these works, and also [15], [16]).

The problem of plasma oscillations with diff use boundary condition was considered in the works [17], [18] by method of integral transformations. In the works [19], [20] general asymptotic analysis of electric field behaviour at the large distance from the surface was carried out. In the work [19] particular significance of plasma behaviour analysis close to plasma resonance was shown. And in the same work [19] it was stated that plasma behaviour in this case for conditions of specular and diff use electron scattering on the surface differs substantially.

In the works [21] and [22] general questions of this problem solvability were considered, but diffuse boundary conditions were taken into account. In the work [22] structure of discrete spectrum in dependence of parameters of the problem was analyzed. The detailed analysis of the solution in general case in the works mentioned above hasn't been carried out considering the complex character of this solution.

The present work is a continuation of electron plasma behaviour in external longitudinal alternating electric field research [21] – [26].

In the present paper the linearized problem of plasma oscillations in external alternating electric field is solved analytically. Specular – accommodative boundary conditions for electron reflection from the boundary are considered. In [24]–[26] diffuse boundary conditions were considered.

The coefficients of continuous and discrete spectra of the problem are obtained in the present work, which allows us to derive expressions for electron distribution function at the boundary of conductive medium and electric field in explicit form, to reveal the dependence of this expressions on normal momentum accommodation coefficient and to show that in the case when normal electron momentum accommodation coefficient equals to zero electron distribution function and electric field are expressed by known formulas obtained earlier in [12], [13].

The present work is a continuation of our work [27], in which questions of plasma waves reflection from the plane boundary bounding degenerate plasma were considered.

Let us note, that questions of plasma oscillations are also considered in nonlinear statement (see, for instance, the work [28], [29]).

2. PROBLEM STATEMENT

Let degenerate plasma occupy a half-space $x > 0$.

We take system of equations describing plasma behaviour. As a kinetic equation we take Boltzmann —Vlasov τ–model kinetic equation:

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}}=\frac{f_{eq}(\mathbf{r},t)-f(\mathbf{r},\mathbf{v},t)}{\tau}. \qquad (1.1)$$

Here $f=f(\mathbf{r},\mathbf{v},t)$ is the electron distribution function, e is the electron charge, $\mathbf{p}=m\mathbf{v}$ is the momentum of an electron, m is the electron mass, τ is the character time between two collisions, $\mathbf{E}=\mathbf{E}(\mathbf{r},t)$ is the self-consistent electric field inside plasma, $f_{eq}=f_{eq}(\mathbf{r},t)$ is the local equilibrium Fermi —Dirac distribution function, $f_{eq}=\Theta(\mathcal{E}_F(t,x)-\mathcal{E})$, where $\Theta(x)$ is the function of Heaviside,

$$\Theta(x)=\begin{cases}1, & x>0,\\ 0, & x<0,\end{cases}$$

$\mathcal{E}_F(t,x)=\frac{1}{2}mv_F^2(t,x)$ is the disturbed kinetic energy of Fermi, $\mathcal{E}=\frac{1}{2}mv^2$ is the kinetic energy of electron.

Let us take the Maxwell equation for electric field

$$\operatorname{div}\mathbf{E}(\mathbf{r},t)=4\pi\rho(\mathbf{r},t). \qquad (1.2)$$

Here $\rho(\mathbf{r},t)$ is the charge density,

$$\rho(\mathbf{r},t)=e\int(f(\mathbf{r},\mathbf{v},t)-f_0(\mathbf{v}))\,d\Omega_F, \qquad (1.3)$$

where

$$d\Omega_F=\frac{2d^3p}{(2\pi\hbar)^3},\qquad d^3p=dp_xdp_ydp_z.$$

Here f_0 is the undisturbed Fermi —Dirac electron distribution function,

$$f_0(\mathcal{E})=\Theta(\mathcal{E}_F-\mathcal{E}),$$

$\hbar$ is the Planck's constant, ν is the eff ective frequency of electron collisions, $\nu=1/\tau$, $\mathcal{E}_F=\frac{1}{2}mv_F^2$ is the undisturbed kinetic energy of Fermi, v_F is the electron velocity at the Fermi surface, which is supposed to be spherical.

We assume that external electric field outside the plasma is perpendicular to the plasma boundary and changes according to the following law:$E_0\exp(-i\omega t)$.

Then one can consider that self-consistent electric field $\mathbf{E}(\mathbf{r}, t)$ inside plasma has one x–component and changes only lengthwise the axis x:

$$\mathbf{E} = \{E_x(x,t), 0, 0\}.$$

Under this configuration the electric field is perpendicular to the boundary of plasma, which is situated in the plane $x = 0$.

We will linearize the local equilibrium Fermi —Dirac distribution f_{eq} in regard to the undisturbed distribution $f_0(\mathcal{E})$:

$$f_{eq} = f_0(\mathcal{E}) + [\mathcal{E}_F(x,t) - \mathcal{E}]\delta(\mathcal{E}_F - \mathcal{E}),$$

where $\delta(x)$ is the delta – function of Dirac.

We also linearize the electron distribution function f in terms of absolute Fermi —Dirac distribution. $f_0(\mathcal{E})$:

$$f = f_0(\mathcal{E}) + f_1(x, \mathbf{v}, t) \tag{1.4}$$

After the linearization of the equations (1.1)–(1.3) with the help of (1.4) we obtain the following system of equations:

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} + \nu f_1(x, \mathbf{v}, t) =$$

$$= \delta(\mathcal{E}_F - \mathcal{E})\big[eE_x(x,t)v_x + \nu[\mathcal{E}_F(x,t) - \mathcal{E}_F]\big], \tag{1.5}$$

$$\frac{\partial E_x(x,t)}{\partial x} = \frac{8\pi e}{(2\pi\hbar)^3}\int f_1(x, \mathbf{v}', t)d^3p' \tag{1.6}$$

From the law of preservation of number of particles

$$\int f_{eq} d\Omega_F = \int f d\Omega_F$$

we find:

$$[\mathcal{E}_F(x,t) - \mathcal{E}_F]\int \delta(\mathcal{E}_F - \mathcal{E})d^3p = \int f_1 d^3p. \tag{1.7}$$

From the equation (1.5) it is seen that we should search for the function f_1 in the form proportional to the delta – function:

$$f_1 = \mathcal{E}_F \delta(\mathcal{E}_F - \mathcal{E})H(x, \mu, t), \qquad \mu = \frac{v_x}{v}. \tag{1.8}$$

The system of equations (1.5) and (1.6) with the help of (1.7) and (1.8) can be transformed to the following form:

$$\frac{\partial H}{\partial t}+v_F\mu\frac{\partial H}{\partial x}+\nu H(x,\mu,t)=$$

$$=\frac{ev_F\mu}{\mathcal{E}_F}E_x(x,t)+\frac{\nu}{2}\int\limits_{-1}^{1}H(x,\mu',t)d\mu',$$

$$\frac{\partial E_x(x,t)}{\partial x}=\frac{16\pi^2e\mathcal{E}_Fm^2v_F}{(2\pi\hbar)^3}\int\limits_{-1}^{1}H(x,\mu',t)d\mu'.$$

Further we introduce dimensionless function

$$e(x,t)=\frac{ev_F}{\nu\mathcal{E}_F}E_x(x,t)$$

and pass to dimensionless coordinate $x_1=x/l$, where $l=v_F\tau$ is the mean free path of electrons, and we introduce dimensionless time $t_1=\nu t$. We obtain the following system of equations:

$$\frac{\partial H}{\partial t_1}+\mu\frac{\partial H}{\partial x_1}+\nu H(x_1,\mu,t_1)=$$

$$=\mu e(x_1,t_1)+\frac{1}{2}\int\limits_{-1}^{1}H(x_1,\mu',t_1)d\mu', \qquad (1.9)$$

$$\frac{\partial e(x_1,t_1)}{\partial x_1}=\frac{3\omega_p^2}{2\nu^2}\int\limits_{-1}^{1}H(x_1,\mu',t_1)d\mu'. \qquad (1.10)$$

Here ω_p is the electron (Langmuir) frequency of plasma oscillations,

$$\omega_p^2=\frac{4\pi e^2N}{m},$$

N is the numerical density (concentration), m is the electron mass.

We used the following well-known relation for degenerate plasma for the derivation of the equations (1.9) and (1.10)

$$\Big(\frac{v_Fm}{\hbar}\Big)^3=3\pi^2N.$$

Let k to be a dimensional wave number, and let us introduce dimensionless wave number $k_1 = k\dfrac{v_F}{\omega_p}$, then $kx = \dfrac{k_1 x_1}{\varepsilon}$, where $\varepsilon = \dfrac{\nu}{\omega_p}$. We introduce the quantity $\omega_1 = \omega\tau = \dfrac{\omega}{\nu}$.

3. BOUNDARY CONDITIONS STATEMENT

Let us outline the time variable of the functions $H(x_1, \mu, t_1)$ and $e(x_1, t_1)$, assuming

$$H(x_1, \mu, t_1) = e^{-i\omega_1 t_1} h(x_1, \mu), \tag{2.1}$$

$$e(x_1, t_1) = e^{-i\omega_1 t_1} e(x_1). \tag{2.2}$$

The system of equations (1.9) and (1.10) in this case will be transformed to the following form:

$$\mu \frac{\partial h}{\partial x_1} + (1 - i\omega_1) h(x_1, \mu) = \mu e(x_1) + \frac{1}{2}\int\limits_{-1}^{1} h(x_1, \mu') d\mu', \tag{2.3}$$

$$\frac{de(x_1)}{dx_1} = \frac{3\omega_p^2}{2\nu^2}\int\limits_{-1}^{1} h(x_1, \mu') d\mu'. \tag{2.4}$$

Further instead of x_1, t_1 we write x, t. We rewrite the system of equations (2.3) and (2.4) in the form:

$$\mu \frac{\partial h}{\partial x} + z_0 h(x, \mu) = \mu e(x) + \frac{1}{2}\int\limits_{-1}^{1} h(x, \mu') d\mu'. \tag{2.5}$$

$$\frac{de(x)}{dx} = \frac{3}{2\varepsilon^2}\int\limits_{-1}^{1} h(x, \mu') d\mu'. \tag{2.6}$$

Here

$$z_0 = 1 - i\omega_1 = 1 - \frac{\omega}{\nu} = 1 - i\omega\tau.$$

We consider the external electric field outside the plasma limit is absent. This means that for the field inside plasma on the plasma boundary the following condition is satisfied:

$$e(0) = 0. \tag{2.7}$$

The non-flowing condition for the particle (electric current) flow through the plasma boundary means that

$$\int\limits_{-1}^{1} \mu\, h(0,\mu)\, d\mu = 0. \tag{2.8}$$

In the kinetic theory for the description of the surface properties the accommodation coefficients are used often. Tangential momentum and energy accommodation coefficients are the most–used. For the problem considered the normal electron momentum accommodation under the scattering on the surface has the most important significance.

The normal momentum accommodation coefficient is defined by the following relation:

$$\alpha_p = \frac{P_i - P_r}{P_i - P_s}, \quad 0 \leqslant \alpha_p \leqslant 1, \tag{2.9}$$

where P_i and P_r are the flows of normal to the surface momentum of incoming to the boundary and reflected from it electrons,

$$P_i = \int\limits_{-1}^{0} \mu^2 h(0,\mu) d\mu, \tag{2.10}$$

$$P_r = \int\limits_{0}^{1} \mu^2 h(0,\mu) d\mu, \tag{2.11}$$

quantity P_s is the normal momentum flow for electrons reflected from the surface which are in thermodynamic equilibrium with the wall,

$$P_s = \int\limits_{0}^{1} \mu^2 h_s(\mu) d\mu, \tag{2.12}$$

where the function

$$h_s(\mu) = A_s, \quad 0 < \mu < 1,$$

is the equilibrium distribution function of the corresponding electrons. This function is to satisfy the condition similar to the non-flowing condition:

$$\int\limits_{-1}^{0}\mu h(0,\mu)d\mu+\int\limits_{0}^{1}\mu h_s(\mu)d\mu=0. \qquad (2.13)$$

We are going to consider the relation between the normal momentum accommodation coefficient α_p and the diffuseness coefficient q for the case of specular and diffuse boundary conditions which are written in the following form:

$$h(0,\mu)=(1-q)h(0,-\mu)+a_s, \quad 0<\mu<1. \qquad (2.14)$$

Here q is the diffuseness coefficient ($0 \leqslant q \leqslant 1$), a_s is the quantity determined from the non-flowing condition.

From the non-flowing condition we derive

$$\int\limits_{-1}^{1}\mu h(0,\mu)d\mu=\int\limits_{-1}^{0}\mu h(0,\mu)d\mu+\int\limits_{0}^{1}\mu h(0,\mu)d\mu=0.$$

In the second integral we replace the integrand according to the right-hand side of the specular–diffuse boundary condition (2.14). After that, using the obvious change of integration variable, we obtain that

$$a_s=-2q\int\limits_{-1}^{0}\mu h(0,\mu)d\mu.$$

Let us use the boundary condition (2.13). Using the analogous to the preceded line of reasoning we get:

$$A_s=-2\int\limits_{-1}^{0}\mu h(0,\mu)d\mu.$$

From the two last equations we find that

$$a_s=qA_s. \qquad (2.15)$$

Further we find the difference between two flows

$$P_i - P_r = \int\limits_{-1}^{0} \mu^2 h(0,\mu)d\mu - \int\limits_{0}^{1} \mu^2 h(0,\mu)d\mu.$$

In the second integral we use the boundary condition (2.14) again. With the help of (2.15) we obtain that

$$P_i - P_r = q\int\limits_{-1}^{0} \mu^2 h(0,\mu)d\mu - \int\limits_{0}^{1} \mu^2 a_s d\mu =$$

$$= q\int\limits_{-1}^{0} \mu^2 h(0,\mu)d\mu - q\int\limits_{0}^{1} \mu^2 A_s d\mu = qP_i - qP_s.$$

Substituting the expressions obtained to the definition of the normal momentum accommodation coefficient, we have:

$$\alpha_p = \frac{P_i - P_r}{P_i - P_s} = \frac{qP_i - qP_s}{P_i - P_s} = q.$$

. Thus, for specular – diffuse boundary conditions normal momentum accommodation coefficient α_p coincides with the diffuseness coefficient q.

Equally with the specular – diffuse boundary conditions another variants of boundary conditions are used in kinetic theory as well.

In particular, accommodation boundary conditions are used widely. They are divided into two types: diffuse – accommodative and specular – accommodative boundary conditions (see [30]).

We consider specular – accommodative boundary conditions. For the function h this conditions will be written in the following form:

$$h(0,\mu) = h(0,-\mu) + A_0 + A_1\mu, \quad 0 < \mu < 1. \qquad (2.16)$$

If in (2.16) we assume $A_0 = A_1 = 0$, then specular – accommodative boundary conditions pass into pure specular boundary conditions.

Coefficients A_1 and A_2 can be derived from the non-flowing condition and the definition of the normal electron momentum accommodation coefficient.

The problem statement is completed. Now the problem consists in finding of such solution of the system of equations (2.5) and (2.6), which satisfies the boundary conditions (2.7)–(2.13). Further, with the use of the solution of the problem, it is required to built the profiles of the distribution function of the electrons moving to the plasma surface, and profile of the electric field.

4. THE RELATION BETWEEN FLOWS AND BOUNDARY CONDITIONS

First of all let us find expression which relates the constants $A_0,\ A_1$ from the boundary condition (2.16). To carry this out we will use the condition of non-fl owing (2.12) of the particle fl ow through the plasma boundary, which we will write as a sum of two fl ows:

$$N_0 \equiv \int\limits_0^1 \mu h(0,\mu)d\mu + \int\limits_{-1}^0 \mu h(0,\mu)d\mu = 0.$$

After evident substitution of the variable in the second integral we obtain:

$$N_0 \equiv \int\limits_0^1 \mu\Big[h(0,\mu) - h(0,-\mu)\Big]d\mu = 0.$$

Taking into account the relation (2.16), we obtain that

$$A_0 = -\frac{2}{3}A_1.$$

With the help of this relation we can rewrite the condition (2.16) in the following form:

$$h(0,\mu) = h(0,-\mu) + A_1(\mu - \frac{2}{3}), \qquad 0 < \mu < 1. \tag{3.1}$$

We consider the momentum fl ow of the electrons which are moving to the boundary. According to (3.1) we have:

$$P_i = P_r - \frac{1}{36}A_1. \tag{3.2}$$

It is easy to see further that

$$P_s = \frac{A_s}{3}. \tag{3.3}$$

With the help of the formulas (3.2) and (3.3) we will rewrite the definition of the accommodation coeffi cient (2.9) in the form:

$$\alpha_p P_r - \alpha_p \frac{A_s}{3} + \frac{A_1}{36}(1-\alpha_p) = 0. \tag{3.4}$$

Let us consider the condition (2.13). We rewrite it in the following form:

$$\frac{A_s}{2} + \int\limits_{-1}^{0} \mu h(0,\mu)d\mu = 0.$$

From this condition we obtain

$$A_s = -2\int\limits_{-1}^{0} \mu h(0,\mu)d\mu = 2\int\limits_{0}^{1} \mu h(0,-\mu)d\mu.$$

Using the condition (3.1), we then get

$$A_s = 2\int\limits_{0}^{1} \mu h(0,\mu)d\mu. \tag{3.5}$$

Now with the help of the second equality from (2.16) and (3.5) we rewrite the relation (3.4) in the integral form:

$$\alpha_p \int\limits_{0}^{1} \Big(\mu^2 - \frac{2}{3}\mu\Big) h(0,\mu)d\mu = -\frac{1}{36}(1-\alpha_p)A_1. \tag{3.6}$$

Now the boundary problem consists of the equations (2.5) and (2.6) and boundary conditions (2.7), (3.1) и(3.6).

5. SEPARATION OF VARIABLES AND CHARACTERISTIC SYSTEM

Application of the general Fourier method of the separation of variables in several steps results in the following substitution [31]:

$$h_\eta(x,\mu)=\exp(-\frac{z_0x}{\eta})\Phi(\eta,\mu), \tag{4.1}$$

$$e_\eta(x)=\exp(-\frac{z_0x}{\eta})E(\eta), \tag{4.2}$$

where η is the spectrum parameter or the parameter of separation, which is complex in general.

We substitute the equalities (4.1) and (4.2) into the equations (2.5) and (2.6). We obtain the following characteristic system of equations:

$$z_0(\eta-\mu)\Phi(\eta,\mu)=\eta\mu E(\eta)+\frac{\eta}{2}\int\limits_{-1}^{1}\Phi(\eta,\mu')d\mu', \tag{4.3}$$

$$E(\eta)=-\frac{3}{\varepsilon^2 z_0}\cdot\frac{\eta}{2}\int\limits_{-1}^{1}\Phi(\eta,\mu')d\mu'. \tag{4.4}$$

Let us introduce the designations:

$$\gamma=\frac{\omega}{\omega_p}-1,\qquad \gamma\geq -1,\qquad \eta_1^2=\frac{\varepsilon^2 z_0}{3},$$

$$c=z_0\eta_1^2=\frac{\varepsilon^2z_0^2}{3}=\frac{1}{3}[\varepsilon-i(1+\varepsilon)]^2,$$

$$z_0=1-i\frac{1+\gamma}{\varepsilon}.$$

Substituting the integral from the equation (4.4) into (4.3), we come to the following system of equations:

$$(\eta-\mu)\Phi(\eta,\mu)=\frac{E(\eta)}{z_0}(\eta\mu-\eta_1^2), \tag{4.5}$$

$$-\eta_1^2E(\eta)=\frac{\eta}{2}\int\limits_{-1}^{1}\Phi(\eta,\mu')d\mu'. \tag{4.6}$$

Here

$$\eta_1^2 = \frac{\varepsilon^2}{3} - i\frac{\varepsilon(1+\gamma)}{3}.$$

Solution of the system (4.5) and (4.6) depends essentially on the condition whether the spectrum parameter η belongs to the interval $-1 < \eta < 1$. In connection with this the interval $-1 < \eta < 1$ we will call as continuous spectrum of the characteristic system.

Let the parameter $\eta \in (-1, 1)$. Then from the equations (4.5) and (4.6) in the class of general functions we will find eigenfunction corresponding to the continuous spectrum:

$$\Phi(\eta, \mu) = F(\eta, \mu)\frac{E(\eta)}{z_0}, \tag{4.7}$$

where

$$F(\eta, \mu) = P\frac{\mu\eta - \eta_1^2}{\eta - \mu} - c\frac{\lambda(\eta)}{\eta}\delta(\eta - \mu). \tag{4.8}$$

In the equation (4.8) $\delta(x)$ is the delta–function of Dirac, the symbol Px^{-1} means the principal value of the integral under integrating of the expression x^{-1}, the function $\lambda(z)$ is called as dispersion function of the problem,

$$\lambda(z) = 1 + \frac{z}{c}\int\limits_{-1}^{1} \frac{\eta_1^2 - z\mu}{\mu - z} d\mu. \tag{4.9}$$

Functions (4.8) are called eigenfunction of the continuous spectrum, since the spectrum parameter η fills out the continuum $(-1, +1)$ compactly. The eigensolutions of the given problem can be found from the equalities (4.7).

The dispersion function $\lambda(z)$ we express in the terms of the Case dispersion function [31]:

$$\lambda(z) = 1 - \frac{1}{z_0} + \frac{1}{z_0}\Big(1 - \frac{z^2}{\eta_1^2}\Big)\lambda_c(z),$$

where

$$\lambda_c(z) = 1 + \frac{z}{2}\int\limits_{-1}^{1} \frac{d\tau}{\tau - z} = \frac{1}{2}\int\limits_{-1}^{1} \frac{\tau\, d\tau}{\tau - z}$$

is the Case dispersion function [31].

The boundary values of the dispersion function from above and below the cut (interval $(-1,1)$) we define in the following way:

$$\lambda^{\pm}(\mu)=\lim_{\varepsilon\to 0,\varepsilon>0}\lambda(\mu\pm i\varepsilon),\qquad \mu\in(-1,1).$$

The boundary values of the dispersion function from above and below the cut are calculated according to the Sokhotzky formulas

$$\lambda^{\pm}(\mu)=\lambda(\mu)\pm\frac{i\pi\mu}{2\eta_1^2z_0}(\eta_1^2-\mu^2),\quad -1<\mu<1,$$

from where

$$\lambda^{+}(\mu)-\lambda^{-}(\mu)=\frac{i\pi}{\eta_1^2z_0}\,\mu(\eta_1^2-\mu^2),$$

$$\frac{\lambda^{+}(\mu)+\lambda^{-}(\mu)}{2}=\lambda(\mu),\quad -1<\mu<1,$$

where

$$\lambda(\mu)=1+\frac{\mu}{2\eta_1^2z_0}\int\limits_{-1}^{1}\frac{\eta_1^2-\eta^2}{\eta-\mu}\,d\eta,$$

and the integral in this equality is understood as singular in terms of the principal value by Cauchy. Besides that, the function $\lambda(\mu)$ can be represented in the following form:

$$\lambda(\mu)=1-\frac{1}{z_0}+\frac{1}{z_0}\Big(1-\frac{\mu^2}{\eta_1^2}\Big)\lambda_c(\mu),$$

$$\lambda_c(\mu)=1+\frac{\mu}{2}\ln\frac{1-\mu}{1+\mu}.$$

6. EIGENFUNCTIONS OF THE DISCRETE SPECTRUM

According to the definition, the discrete spectrum of the characteristic equation is a set of zeroes of the dispersion equation

$$\frac{\lambda(z)}{z}=0. \tag{5.1}$$

We start to search zeroes of the equation (5.1). Let us take Laurent series of the dispersion function:

$$\lambda(z)=\lambda_\infty+\frac{\lambda_2}{z^2}+\frac{\lambda_4}{z^4}+\cdots, \qquad |z|>1. \tag{5.2}$$

Here

$$\lambda_\infty\equiv\lambda(\infty)=1-\frac{1}{z_0}+\frac{1}{3z_0\eta_1^2},$$

$$\lambda_2=-\frac{1}{z_0}\Big(\frac{1}{3}-\frac{1}{5\eta_1^2}\Big),$$

$$\lambda_4=-\frac{1}{z_0}\Big(\frac{1}{5}-\frac{1}{7\eta_1^2}\Big).$$

We express these parameters through the parameters γ and ε:

$$\lambda_\infty\equiv\lambda(\infty)=\frac{2\gamma+i\varepsilon+\gamma(\gamma+i\varepsilon)}{(1+\gamma+i\varepsilon)^2},$$

$$\lambda_2=-\frac{9+5i\varepsilon(1+\gamma+i\varepsilon)}{15(1+\gamma+i\varepsilon)^2},$$

$$\lambda_4=-\frac{15+7i\varepsilon(1+\gamma+i\varepsilon)}{35(1+\gamma+i\varepsilon)^2}.$$

It is easy seen that the dispersion function (4.9) in collisional plasma (i.e. when $\varepsilon>0$) in the infinity has the value which doesn't equal to zero: $\lambda_\infty=\lambda(\infty)\neq 0$.

Hence, the dispersion equation has infinity as a zero $\eta_i=\infty$, to which the discrete eigensolutions of the given system correspond:

$$h_\infty(x,\mu)=\frac{\mu}{z_0}, \qquad e_\infty(x)=1.$$

This solution is naturally called as mode of Drude. It describes the volume conductivity of metal, considered by Drude (see, for example, [32]).

Let us consider the question of the plasma mode existence in details. We find finite complex zeroes of the dispersion function. We use the principle of argument. We take the contour (see Fig. 1)

$$\Gamma_\varepsilon^+=\Gamma_R\cup\gamma_\varepsilon,$$

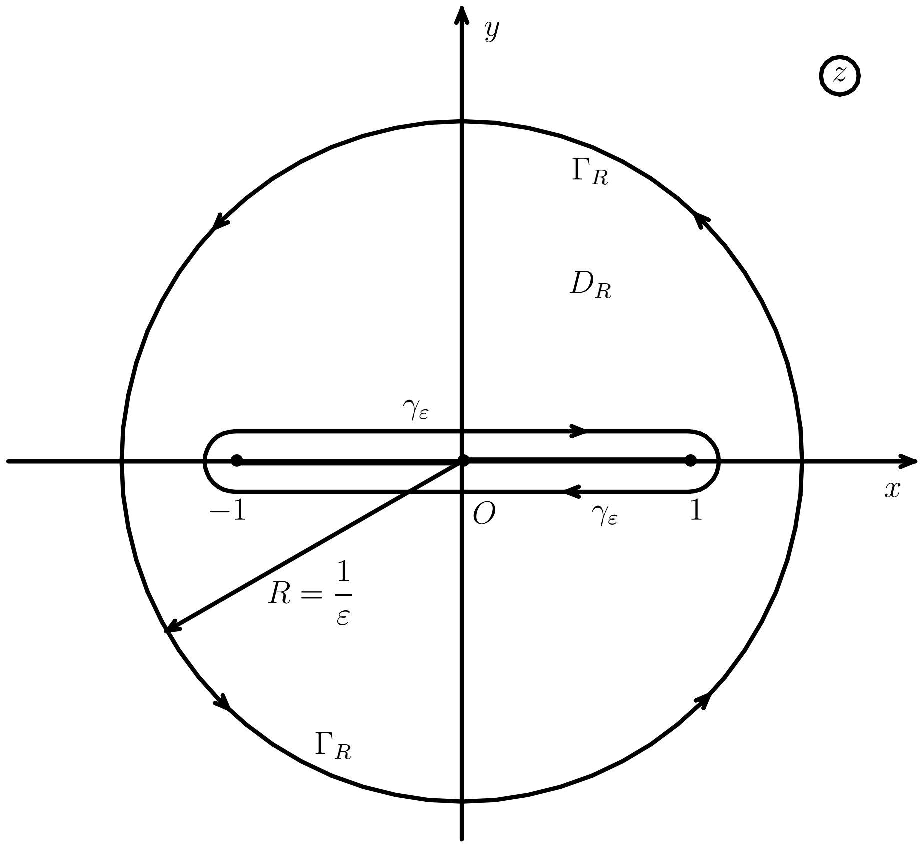

Fig. 1.

which is passed in the positive direction and which bounds the biconnected domain D_R. This contour consists of the circumference

$$\Gamma_R = \{z: \quad |z| = R, \quad R = \frac{1}{\varepsilon}, \quad \varepsilon > 0\},$$

and the contour γ_ε, which includes the cut $[-1, +1]$, and stands at the distance of ε from it.

Let us note that the dispersion function has not any poles in the domain D_R. Then according to the principle of argument the number [33] of zeroes N in the domain D_R equals to:

$$N = \frac{1}{2\pi i} \oint\limits_{\Gamma_\varepsilon} d \ln \lambda(z).$$

Considering the limit in this equality when $\varepsilon \to 0$ and taking into account that the dispersion function is analytic in the neighbourhood of the infinity, we obtain that

$$N=\frac{1}{2\pi i}\int\limits_{-1}^{1}d\,\ln\lambda^+(\tau)-\frac{1}{2\pi i}\int\limits_{-1}^{1}d\,\ln\lambda^-(\tau)$$

$$=\frac{1}{2\pi i}\int\limits_{-1}^{1}d\,\ln\frac{\lambda^+(\mu)}{\lambda^-(\mu)}.$$

So, we obtained that

$$N=\frac{1}{2\pi i}\int\limits_{-1}^{1}d\,\ln\frac{\lambda^+(\tau)}{\lambda^-(\tau)}.$$

We divide this integral into two integrals by segments $[-1,0]$ and $[0,1]$. In the first integral by the segment $[-1,0]$ we carry out replacement of variable $\tau\to-\tau$. Taking into account that $\lambda^+(-\tau)=\lambda^-(\tau)$, we obtain that

$$N=\frac{1}{2\pi i}\int\limits_{-1}^{1}d\,\ln\frac{\lambda^+(\tau)}{\lambda^-(\tau)}=\frac{1}{\pi i}\int\limits_{0}^{1}d\,\ln\frac{\lambda^+(\tau)}{\lambda^-(\tau)}=\frac{1}{\pi}\arg\frac{\lambda^+(\tau)}{\lambda^-(\tau)}\Bigg|_0^1. \qquad (5.3)$$

Here under symbol $\arg G(\tau)=\arg\dfrac{\lambda^+(\tau)}{\lambda^-(\tau)}$ we understand the regular branch of the argument, fixed in zero with the condition: $\arg G(0)=0$.

We consider the curve

$$\gamma=\{z:\ z=G(\tau),\quad 0\leqslant\tau\leqslant+1\},$$

where

$$G(\tau)=\frac{\lambda^+(\tau)}{\lambda^-(\tau)}.$$

It is obvious that

$$G(0)=1,\qquad \lim_{\tau\to+1}G(\tau)=1.$$

Consequently, according to (5.3), the number of values N equals to doubled number of turns of the curve γ around the point of origin, i.e.

$$N = 2\varkappa(G),$$

where

$$\varkappa(G) = \operatorname{Ind}_{[0,+1]} G(\tau)$$

is the index of the function $G(\tau)$.

Thus, the number of zeroes of the dispersion function, which are situated in complex plane outside of the segment $[-1, 1]$ of the real axis, equals to doubled index of the function $G(\tau)$, calculated on the "semi-segment" $[0, +1]$.

Let us single real and imaginary parts of the function $G(\mu)$ out. At first, we represent the function $G(\mu)$ in the form:

$$G(\mu) = \frac{(z_0-1)\eta_1^2 + (\eta_1^2-\mu^2)\lambda_0(\mu) + is(\mu)(\eta_1^2-\mu^2)}{(z_0-1)\eta_1^2 + (\eta_1^2-\mu^2)\lambda_0(\mu) - is(\mu)(\eta_1^2-\mu^2)}.$$

where

$$s(\mu) = \frac{\pi}{2}\mu,$$

$$\lambda(\mu) = 1 - \frac{1}{z_0} + \frac{1}{z_0}\Big(1 - \frac{\mu^2}{\eta_1^2}\Big)\lambda_c(\mu),$$

and

$$\lambda_c(\mu) = 1 + \frac{\mu}{2}\ln\frac{1-\mu}{1+\mu}$$

is the dispersion function of Case, calculated on the cut (i.e., in the interval $(-1, 1)$).

Taking into account that

$$z_0 - 1 = -i\frac{\omega}{\nu} = -i\frac{1+\gamma}{\varepsilon}, \qquad \eta_1^2 = \frac{\varepsilon c}{3} = \frac{\varepsilon^2}{3} - i\frac{\varepsilon(1+\gamma)}{3},$$

$$(z_0-1)\eta_1^2 = -\frac{(1+\gamma)^2}{3} - i\frac{\varepsilon(1+\gamma)}{3},$$

we obtain

$$G(\mu) = \frac{P^-(\mu) + iQ^-(\mu)}{P^+(\mu) + iQ^+(\mu)},$$

where

$$P^{\pm}(\mu) = (1+\gamma)^2 - \lambda_0(\mu)(\varepsilon^2 - 3\mu^2) \pm \varepsilon(1+\gamma)s(\mu),$$

$$Q^{\pm}(\mu) = \varepsilon(1+\gamma)(1+\lambda_0(\mu)) \pm s(\mu)(\varepsilon^2 - 3\mu^2).$$

Now we can easily single real and imaginary parts of the function $G(\mu)$ out:

$$G(\mu) = \frac{g_1(\mu)}{g(\mu)} + i\frac{g_2(\mu)}{g(\mu)}.$$

Here

$$g(\mu) = [P^+(\mu)]^2 + [Q^+(\mu)]^2 = [(1+\gamma)^2 + \lambda_0(3\mu^2 - \varepsilon^2)-$$

$$-\varepsilon(1+\gamma)s]^2 + [\varepsilon(1+\gamma)(1+\lambda_0) - s(3\mu^2 - \varepsilon^2)]^2,$$

$$g_1(\mu) = P^+(\mu)P^-(\mu) + Q^+(\mu)Q^-(\mu) = [(1+\gamma)^2 + \lambda_0(3\mu^2-$$

$$-\varepsilon^2)]^2 - \varepsilon^2(1+\gamma)^2[s^2 - (1+\lambda_0)^2] - (3\mu^2 - \varepsilon^2)^2 s^2,$$

$$g_2(\mu) = P^+(\mu)Q^-(\mu) - P^-(\mu)Q^+(\mu) = 2s[(1+\gamma)^2(3\mu^2 - \varepsilon^2)+$$

$$+\lambda_0(3\mu^2 - \varepsilon^2)^2 + \varepsilon^2(1+\gamma)^2(1+\lambda_0)],$$

We consider (see Fig. 2) the curve L, which is defined in implicit form by the following parametric equations:

$$L = \{(\gamma,\varepsilon): \qquad g_1(\mu;\gamma,\varepsilon) = 0, \quad g_2(\mu;\gamma,\varepsilon) = 0, \quad 0 \le \mu \le 1\},$$

and which lays in the plane of the parameters of the problem (γ,ε), and when passing through this curve the index of the function $G(\mu)$ at the positive "semi-segment" $[0,1]$ changes stepwise.

From the equation $g_2 = 0$ we find:

$$(1+\gamma)^2 = -\frac{\lambda_0(\mu)(3\mu^2 - \varepsilon^2)}{3\mu^2 + \varepsilon^2\lambda_0(\mu)}. \tag{5.4}$$

Now from the equation $g_1 = 0$ with the help of (5.4) we find that

$$\varepsilon = \sqrt{L_2(\mu)}, \tag{5.5}$$

where

$$L_2(\mu) = -\frac{3\mu^2 s^2(\mu)}{\lambda_0(\mu)[s^2(\mu) + (1+\lambda_0(\mu))^2]}.$$

Substituting (5.5) into (5.4), we obtain:

$$\gamma = -1 + \sqrt{L_1(\mu)}, \tag{5.6}$$

where

$$L_1(\mu) = -\frac{3\mu^2[s^2(\mu) + \lambda_0(\mu)(1+\lambda_0(\mu))]^2}{\lambda_0(\mu)[s^2(\mu) + (1+\lambda_0(\mu))^2]}.$$

Functions (5.5) and (5.6) determine the curve L which is the border if the domain D^+ (we designate the external area to the domain as D^-) in explicit parametrical form (see Fig. 2). As in the work [34] we can prove that if $(\gamma, \varepsilon) \in D^+$, then $\varkappa(G) = \operatorname{Ind}_{[0,+1]} G(\mu) = 1$ (the curve L encircles the point of origin once), and if $(\gamma, \varepsilon) \in D^-$, then $\varkappa(G) = \operatorname{Ind}_{[0,+1]} G(\mu) = 0$ (the curve L doesn't encircle the point of origin).

We note, that in the work [34] the method of analysis of boundary regime when $(\gamma, \varepsilon) \in L$ was developed.

From the expression (3.2) one can see that the number of zeroes of the dispersion function equals to two if $(\gamma, \varepsilon) \in D^+$, and equals to zero if $(\gamma, \varepsilon) \in D^-$.

Since the dispersion function is even its zeroes diff er from each other by sign. We designate these zeroes as following $\pm\eta_0$, by η_0 we take the zero which satisfies the condition $\operatorname{Re}\eta_0 > 0$. The following solution corresponds to the zero η_0

$$h_{\eta_0}(x,\mu) = \exp\Big(-\frac{z_0 x}{\eta_0}\Big)\frac{E_2}{z_0}\frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu}, \tag{5.7}$$

$$e_{\eta_0}(x) = \exp\Big(-\frac{z_0 x}{\eta_0}\Big)E_2. \tag{5.8}$$

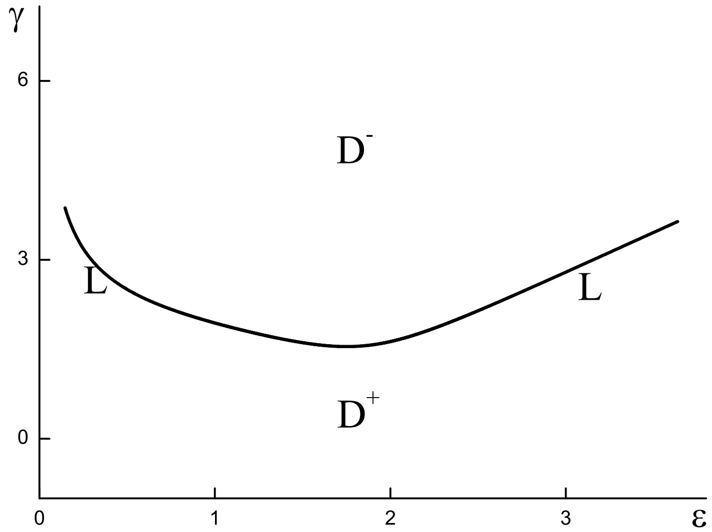

Fig. 2.

From the equalities (5.7) and (5.8) it follows that the following solution corresponds to zero $-\eta_0$:

$$h_{-\eta_0}(x,\mu)=\exp(-\frac{z_0x}{\eta_0})\frac{E_1}{z_0}\frac{\eta_0\mu+\eta_1^2}{\eta_0+\mu}, \tag{5.9}$$

$$e_{-\eta_0}(x)=\exp(\frac{z_0x}{\eta_0})E_1. \tag{5.10}$$

This solution is naturally called as mode of Debay (this is plasma mode). In the case of low frequencies it describes well-known screening of Debay [3]. The external field penetrates into plasma on the depth of r_D, r_D is the raduis of Debay. When the external field frequencies are close to Langmuir frequencies, the mode of Debay describes plasma oscillations (see, for instance, [3, 32]).

7. EXPANSION IN THE TERMS OF EIGENFUNCTIONS

We will seek for the solution of the system of equations (2.5) and (2.6) with boundary conditions (3.1), (3.6) and (2.7) in the form of linear combination of discrete eigensolutions of the characteristic system and integral taken over continuous spectrum of the system. Let us prove that the following theorem is true.

Theorem 7.1. *System of equations (2.5) and (2.6) with boundary conditions (3.1), (3.6) and (2.7) has a unique solution, which can be presented as an expansion by eigenfunctions of the characteristic system:*

$$h(x,\mu)=\frac{E_\infty}{z_0}\mu+\frac{E_0}{z_0}\frac{\eta_0\mu-\eta_1^2}{\eta_0-\mu}\exp\Big(-\frac{z_0x}{\eta_0}\Big)+$$

$$+\frac{1}{z_0}\int\limits_0^1\exp\Big(-z_0\frac{x}{\eta}\Big)F(\eta,\mu)E(\eta)\,d\eta, \qquad (6.1)$$

$$e(x)=E_\infty+E_0\exp\Big(-z_0\frac{x}{\eta}\Big)+\int\limits_0^1\exp\Big(-z_0\frac{x}{\eta}\Big)E(\eta)\,d\eta. \qquad (6.2)$$

Here E_0 and E_∞ is unknown coefficients corresponding to the discrete spectrum (E_0 is the amplitude of Debay, E_1 is the amplitude of Drude), $E(\eta)$ is unknown function, which is called as coefficient of discrete spectrum.

When $(\gamma,\varepsilon)\in D^-$ in expansions (6.1) and (6.2) we should take $E_0=0$. Then from (6.1) and (6.2) we obtain:

$$h(x,\mu)=\frac{E_\infty}{z_0}\mu+\frac{1}{z_0}\int\limits_0^1\exp\Big(-z_0\frac{x}{\eta}\Big)F(\eta,\mu)E(\eta)\,d\eta, \qquad (6.1')$$

$$e(x)=E_\infty+\int\limits_0^1\exp\Big(-z_0\frac{x}{\eta}\Big)E(\eta)\,d\eta. \qquad (6.2')$$

Further we will consider the following case $(\gamma,\varepsilon)\in D^+$.

Our purpose is to find the coefficient of the continuous spectrum, coefficients of the discrete spectrum and to built expressions for electron distribution function at the plasma surface and electric field.

Proof. We substitute the expression (6.1) into the boundary condition (3.1). We obtain the following equation in the interval $0 < \mu < 1$:

$$2\frac{E_\infty}{z_0}\mu + \frac{E_0}{z_0}\varphi_0(\mu)+$$

$$+\frac{1}{z_0}\int\limits_0^1 \Big[F(\eta,\mu) - F(\eta,-\mu)\Big]E(\eta)d\eta = A_1\Big(\mu - \frac{2}{3}\Big), \qquad (6.3)$$

where the following designation is introduced

$$\varphi_0(\mu) = \frac{\eta_1^2 - \eta_0\mu}{\mu - \eta_0} + \frac{\eta_1^2 + \eta_0\mu}{\mu + \eta_0}.$$

Extending the function $E(\eta)$ into the interval $(-1, 0)$ evenly, so that $E(\eta) = E(-\eta)$, and extending the equation into the interval $(-1; 1)$ unevenly, we transform the equation (6.3) to the form:

$$2\frac{E_\infty}{z_0}\mu + \frac{E_0}{z_0}\varphi_0(\mu) + \frac{1}{z_0}\int\limits_{-1}^1 F(\eta,\mu)E(\eta)d\eta$$

$$= A_1\Big(\mu - \frac{2}{3}\operatorname{sign}\mu\Big). \qquad (6.4)$$

We substitute the eigenfunctions of the continuous spectrum into the equation (6.4). We obtain the singular integral equation with the Cauchy kernel in the interval $(-1, 1)$:

$$2E_\infty\mu + E_0\varphi_0(\mu) + \int\limits_{-1}^1 \frac{\eta\mu - \eta_1^2}{\eta - \mu}E(\eta)d\eta -$$

$$-2\eta_1^2 z_0\frac{\lambda(\mu)}{\mu}E(\mu) - z_0A_1\mu = -\frac{2}{3}z_0A_1\operatorname{sign}\mu. \qquad (6.5)$$

8. SOLUTION OF THE SINGULAR EQUATION

We introduce the auxiliary function

$$M(z) = \int\limits_{-1}^1 \frac{\eta z - \eta_1^2}{\eta - z}E(\eta)d\eta, \qquad (7.1)$$

the boundary values of which on the real axis above and below it are related by the Sohotzky formulas:

$$M^+(\mu) - M^-(\mu) = 2\pi i(\mu^2 - \eta_1^2)E(\mu). \tag{7.2}$$

$$\frac{M^+(\mu) + M^-(\mu)}{2} = M(\mu), \qquad -1 < \mu < +1, \tag{7.3}$$

where

$$M(\mu) = \int\limits_{-1}^{1} \frac{\eta\mu - \eta_1^2}{\eta - \mu} E(\eta) d\eta,$$

and the singular integral in this equality is understood as singular in the sense of principal value of Cauchy.

With the help of the equalities (7.2) and (7.3) and the similar equalities for the dispersion and auxiliary functions we reduce the equation (6.5) to the boundary condition of the problem of determination of analytic function by its jump on the contour:

$$\lambda^+(\mu)[M^+(\mu) + 2E_\infty\mu + E_0\varphi_0(\mu) - z_0A_1\mu] -$$

$$-\lambda^-(\mu)[M^-(\mu) + 2E_\infty\mu + E_0\varphi_0(\mu) -$$

$$-z_0A_1\mu] = -\frac{2i\pi}{3\eta_1^2 z_0} z_0A_1\mu(\eta_1^2 - \mu^2)\operatorname{sign}\mu, \quad -1 < \mu < 1.$$

This equation has general solution (see [33]):

$$\lambda(z)[M(z) + E_0\varphi(z) + 2E_\infty z_0 - z_0A_1 z] =$$

$$= \frac{2}{3}\frac{z_0A_1}{2\eta_1^2 z_0} \int\limits_{-1}^{1} \frac{\mu(\mu^2 - \eta_1^2)\operatorname{sign}\mu}{\mu - z} d\mu + C_1 z,$$

where C_1 is an arbitrary constant.

Let us introduce auxiliary function

$$T(z) = \frac{1}{2\eta_1^2 z_0} \int\limits_{-1}^{1} \frac{\mu(\mu^2 - \eta_1^2)\operatorname{sign}\mu}{\mu - z} d\mu.$$

Then the general solution will be written in the form:

$$\lambda(z)[M(z)+E_0\varphi_0(z)+2E_\infty z_0-z_0A_1z]$$
$$=\frac{2}{3}z_0A_1T(z)+C_1z,$$

From the general solution we can easy find the function $M(z)$:

$$M(z)=-2E_\infty z-E_0\varphi_0(z)+z_0A_1z$$
$$+\frac{2}{3}z_0A_1\frac{T(z)}{\lambda(z)}+\frac{C_1z}{\lambda(z)}. \tag{7.4}$$

Let us eliminate the pole of the solution (7.4) in the infinity. We get that

$$C_1=(2E_\infty-z_0A_1)\lambda_\infty.$$

Now we eliminate the poles of the solution (7.4) in the points $\pm\eta_0$. We single out items which contain polar singularity in the points $z=\pm\eta_0$ in the right-hand side of the solution (7.4):

$$E_0\frac{\eta_0z-\eta_1^2}{\eta_0-z}=\frac{C_1z+\frac{2}{3}z_0A_1T(z)}{\lambda(z)}$$

and

$$E_0\frac{\eta_0z+\eta_1^2}{\eta_0+z}=\frac{C_1z+\frac{2}{3}z_0A_1T(z)}{\lambda(z)}.$$

From the last two relations one can see that poles in the points $z=\pm\eta_0$ can be eliminated by one equality since the functions which compose the general solution are uneven:

$$z_0A_1=\frac{E_0\lambda'(\eta_0)(\eta_1^2-\eta_0^2)-2E_\infty\lambda_\infty\eta_0}{(2/3)T(\eta_0)-\lambda_\infty\eta_0}. \tag{7.5}$$

We find the coeffi cient of the continuous spectra from the Sohotzky formula (7.2):

$$E(\eta)=\frac{1}{2\pi i(\mu^2-\eta_1^2)}\Big[M^+(\mu)-M^-(\mu)\Big]. \tag{7.6}$$

We substitute the expansion (6.1) for the function $h(x,\mu)$ into the integral boundary condition (3.6). We derive the following equation:

$$\frac{1}{36}E_\infty+E_0m(\eta_0)+\int\limits_0^1 m(\eta)E(\eta)d\eta=$$

$$=-\frac{1}{36}z_0A_1\frac{1-\alpha_p}{\alpha_p}. \qquad (7.7)$$

We introduced the following designations in (7.7):

$$m(\pm\eta_0)=\int\limits_0^1\Big(\mu^2-\frac{2}{3}\mu\Big)F(\pm\eta_0,\mu)d\mu,$$

and

$$m(\eta)=\int\limits_0^1\Big(\mu^2-\frac{2}{3}\mu\Big)F(\eta,\mu)d\mu.$$

The difference between boundary values of the auxiliary function $M(z)$ from the expression (7.6) can be found with the help of the general solution (7.4):

$$E(\eta)=\frac{1}{2\pi i(\eta^2-\eta_1^2)}\Bigg\{\frac{2}{3}z_0A_1\Big[\frac{T^+(\eta)}{\lambda^+(\eta)}-\frac{T^-(\eta)}{\lambda^-(\eta)}\Big]+$$

$$+(2E_\infty-z_0A_1)\lambda_\infty\eta\Big[\frac{1}{\lambda^+(\eta)}-\frac{1}{\lambda^-(\eta)}\Big]\Bigg\}. \qquad (7.8)$$

Let us note, that when passing the positive part of the cut $(0,1)$ the functions $T(z)$ and $\lambda(z)$ make jumps, which differs from each other only by sign. Indeed, let us represent the formula for $T(z)$ in the following form:

$$T(z)=\frac{1}{2\eta_1^2z_0}\int\limits_0^1\mu(\mu^2-\eta_1^2)\left[\frac{1}{\mu-z}-\frac{1}{\mu+z}\right]d\mu=$$

$$=\frac{z}{\eta_1^2z_0}\int\limits_0^1\frac{\mu(\mu^2-\eta_1^2)}{\mu^2-z^2}d\mu,$$

or

$$T(z)=\frac{z}{2\eta_1^2z_0}\int\limits_0^1(\mu^2-\eta_1^2)\left[\frac{1}{\mu-z}+\frac{1}{\mu+z}\right]d\mu=\frac{z}{\eta_1^2z_0}\int\limits_0^1\frac{\mu(\mu^2-\eta_1^2)}{\mu^2-z^2}d\mu.$$

This integral can be easily calculated in explicit form. On the cut it is calculated by the following formula:

$$T(\eta)=\frac{\eta}{2\eta_1^2z_0}\Big[1+(\eta^2-\eta_1^2)\ln\Big(\frac{1}{\eta^2}-1\Big)\Big],\qquad -1<\eta<+1.$$

Now from the Sokhotzky formula for difference of boundary values we obtain that when $0<\eta<1$:

$$\lambda^+(\eta)-\lambda^-(\eta)=\lambda(\eta)\pm\frac{i\pi\eta(\eta_1^2-\eta^2)}{2\eta_1^2z_0},$$

$$T^+(\eta)-T^-(\eta)=T(\eta)\pm\frac{i\pi\eta(\eta^2-\eta_1^2)}{2\eta_1^2z_0}.$$

Now one can find that

$$T^+(\eta)\lambda^-(\eta)-T^-(\eta)\lambda^+(\eta)=2(T(\eta)+\lambda(\eta))\cdot\frac{i\pi\eta(\eta^2-\eta_1^2)}{2\eta_1^2z_0},$$

$$\lambda^-(\eta)-\lambda^+(\eta)=2\frac{i\pi\eta(\eta^2-\eta_1^2)}{2\eta_1^2z_0}.$$

Let us introduce an integral:

$$T_0(z)=\frac{1}{2\eta_1^2z_0}\int\limits_0^1\frac{\eta^2-\eta_1^2}{\eta-z}d\eta.$$

It is obvious, that in the comlex plane, this integral is calculated by the formula:

$$T_0(z)=\frac{z}{2\eta_1^2z_0}\Big[\frac{1}{2}+z+(z^2-\eta_1^2)\ln\Big(\frac{1}{z}-1\Big)\Big].$$

With the help of this function we present the dispersion function in the following form:

$$\lambda(z)=1-zT_0(z)+zT_0(-z),$$

we also express the function $T(z)$ in terms of this integral:

$$T(z)=zT_0(z)+zT_0(-z).$$

Sum of the two last expressions equals to:

$$\lambda(z)+T(z)=1+2zT_0(-z).$$

We note, that the integral $T(-z)$ is not singular on the cut $0<\eta<1$. The sum $\lambda(\eta)+T(\eta)$ on the cut $0<\eta<1$ is calculated in explicit form without applying to integrals:

$$\lambda(\eta)+T(\eta)=1+\frac{1}{2\eta_1^2z_0}\Big[\eta-2\eta^2+2\eta(\eta^2-\eta_1^2)\ln\Big(\frac{1}{\eta}+1\Big)\Big].$$

From the relations (7.6) and (7.8) we find the coeffi cient of continuous spectra in explicit form:

$$E(\eta)=\frac{1}{2\eta_1^2z_0\lambda^+(\eta)\lambda^-(\eta)}\Big[2E_\infty\lambda_\infty\eta^2+$$
$$+z_0A_1\frac{2}{3}\eta(1+2zT_0(-z))-\lambda_\infty\eta^2\Big],$$

or

$$E(\eta)=\frac{1}{2\eta_1^2z_0\lambda^+(\eta)\lambda^-(\eta)}\Big[(2E_\infty-1)\lambda_\infty\eta^2+$$
$$+z_0A_1\Big(\frac{2}{3}\eta(1+2zT_0(-\eta))\Big)\Big]. \qquad (7.9)$$

Let us calculate the integrals $m(\pm\eta_0)$ and $m(\eta)$ in explicit form. The integrals $m(\pm\eta_0)$ can be calculated easily:

$$m(\eta_0)=(\eta_1^2-\eta_0^2)\Bigg[\Big(-\frac{1}{6}+\eta_0\Big)+\Big(\eta_0^2-\frac{2}{3}\eta_0\Big)\ln\Big(1-\frac{1}{\eta_0}\Big)\Bigg],$$

$$m(-\eta_0)=(\eta_1^2-\eta_0^2)\Bigg[\Big(-\frac{1}{6}-\eta_0\Big)+\Big(\eta_0^2+\frac{2}{3}\eta_0\Big)\ln\Big(1+\frac{1}{\eta_0}\Big)\Bigg].$$

Let us calculate the integral $m(\eta)$. We have:

$$m(\eta)=\int\limits_0^1\Big(\mu^2-\dfrac{2}{3}\mu\Big)\Bigg[P\dfrac{\mu\eta-\eta_1^2}{\eta-\mu}-2\eta_1^2z_0\dfrac{\lambda(\eta)}{\eta}\delta(\eta-\mu)\Bigg]d\mu=$$

$$=\int\limits_0^1\Big(\mu^2-\dfrac{2}{3}\mu\Big)(\eta_1^2-\eta\mu)\dfrac{d\mu}{\mu-\eta}-2\eta_1^2z_0(\eta-\dfrac{2}{3})\lambda(\eta)\theta_+(\eta).$$

Here $\theta_+(\eta)$ is the characteristic function for the interval $0<\eta<1$, i.e.

$$\theta_+(\eta)=\left\{\begin{array}{ll} 1, & 0<\eta<1,\\ 0, & -1<\eta<0.\end{array}\right.$$

From here we get that

$$m(\eta)=(\eta^2-\eta_1^2)\Big(\dfrac{1}{6}-\eta\Big)-(\eta^2-\eta_1^2)\Big(\eta^2-\dfrac{2}{3}\eta\Big)\ln\Big(1+\dfrac{1}{\eta}\Big)+$$

$$+2(\eta^2-\eta_1^2z_0)\Big(\eta-\dfrac{2}{3}\Big),$$

and when $-1<\eta<0$

$$m(\eta)=(\eta^2-\eta_1^2)\Big[\dfrac{1}{6}-\eta-\Big(\eta^2-\dfrac{2}{3}\eta\Big)\ln\Big(1-\dfrac{1}{\eta}\Big)\Big].$$

The last two formulas can be written as one:

$$m(\eta)=(\eta^2-\eta_1^2)\Big[\dfrac{1}{6}-\eta-\Big(\eta^2-\dfrac{2}{3}\eta\Big)f_+(\eta)\Big]+$$

$$+2(\eta^2-\eta_1^2z_0)\Big(\eta-\dfrac{2}{3}\Big)\theta_+(\eta),$$

where

$$f_+(\eta)=\left\{\begin{array}{ll} \ln\dfrac{1+\eta}{\eta}, & 0<\eta<1,\\ \ln\dfrac{1-\eta}{\eta}, & -1<\eta<0.\end{array}\right.$$

We substitute the expansion (6.2) to the condition for the field (2.7), then this condition can be transformed to the following form:

$$E_\infty+E_0+\int\limits_0^1 E(\eta)d\eta=1. \qquad (7.10)$$

Singular integral equation (6.5) is solved completely. The unknown function $E(\eta)$ is found unambiguously and is determined by the equality (7.9). The unknown constants can be found from the equations (7.5), (7.7) and (7.10).

9. COEFFICIENTS OF THE CONTINUOUS AND DISCRETE SPECTRA

Now for finding of the coeffi cients of continuous and discrete spectra we construct system of equations (7.5), (7.10), (7.9) and (7.7):

$$z_0 A_1 = \frac{E_0\lambda'(\eta_0)(\eta_1^2-\eta_0^2)-2E_\infty\lambda_\infty\eta_0}{(2/3)T(\eta_0)-\lambda_\infty\eta_0}, \tag{8.1}$$

$$E_\infty + E_0 + \int\limits_0^1 E(\eta)d\eta = 1, \tag{8.2}$$

$$E(\eta) = \frac{1}{2\eta_1^2 z_0\lambda^+(\eta)\lambda^-(\eta)}\Big[(2E_\infty-1)\lambda_\infty\eta^2+$$

$$+z_0A_1\Big(\frac{2}{3}\eta(1+2zT_0(-\eta))\Big)], \tag{8.3}$$

$$\frac{1}{36}E_\infty + E_0 m(\eta_0) + \int\limits_0^1 m(\eta)E(\eta)d\eta =$$

$$= -\frac{1}{36}z_0A_1\frac{1-\alpha_p}{\alpha_p}. \tag{8.4}$$

From (7.5) we find the coeffi cient corresponding to the mode of Debay:

$$E_0 = \frac{(2E_\infty - z_0A_1)\lambda_\infty\eta_0+\frac{2}{3}z_0A_1T(\eta_0)}{\lambda'(\eta_0)(\eta_1^2-\eta_0^2)} \qquad (8.5).$$

We substitute the expression (8.3) into (8.2). From here we derive:

$$E_\infty + E_0 + (2E_\infty-1)\frac{\lambda_\infty J_1}{2\eta_1^2 z_0}+\frac{z_0A_1}{3\eta_1^2 z_0}J_2 = 1, \tag{8.6}$$

where the following designations are introduced:

$$J_1 = \int\limits_0^1 \frac{\eta^2 d\eta}{\lambda^+(\eta)\lambda^-(\eta)}, \quad J_2 = \int\limits_0^1 \frac{\eta T_2(\eta) d\eta}{\lambda^+(\eta)\lambda^-(\eta)},$$

$$T_2(z) = 1 + 2zT_0(-z).$$

The integrals J_1 and J_2 can be calculated analytically with the help of methods of contour integration.

Substituting the equality (8.1) into the expression obtained (8.6) , we find the relation connecting E_0 and E_∞:

$$E_0 = \frac{1}{S_1}\Big(1 - E_\infty S_2 + \frac{\lambda_\infty J_1}{2\eta_1^2 z_0}\Big), \qquad (8.7)$$

where

$$S_1 = S_1(\eta, \eta_0) = 1 + \frac{\lambda'(\eta_0)(\eta_1^2 - \eta_0^2) J_2}{\eta_1^2 z_0 [2T(\eta_0) - 3\lambda_\infty \eta_0]},$$

$$S_2 = S_2(\eta, \eta_0) = 1 + \frac{\lambda_\infty}{\eta_1^2 z_0}\Big[J_1 - \frac{2\eta_0 J_2}{2T(\eta_0) - 3\lambda_\infty \eta_0}\Big].$$

We consider the integral from the left-hand side of the expression (8.4). We present it in the following form:

$$\int\limits_0^1 m(\eta) E(\eta) d\eta = \frac{1}{2\eta_1^2 z_0}(2E_\infty - 1)\lambda_\infty M_1 + \frac{A_1}{3\eta_1^2} M_2, \qquad (8.8)$$

where

$$M_1 = \int\limits_0^1 \frac{\eta^2 m(\eta) d\eta}{\lambda^+(\eta)\lambda^-(\eta)}, \quad M_2 = \int\limits_0^1 \frac{\eta m(\eta) T_2(\eta)}{\lambda^+(\eta)\lambda^-(\eta)} d\eta.$$

Let us rewrite (8.4) with taking into account the designations introduced:

$$\frac{E_\infty}{36} + E_0 m(\eta_0) + \frac{1}{2\eta_1^2 z_0}(2E_\infty - 1)\lambda_\infty M_1 = -A_1\Big[\frac{M_2}{3\eta_1^2} + z_0 \frac{1-\alpha_p}{36\alpha_p}\Big]. \qquad (8.9)$$

With use of the relations (8.7) and (8.1), we express A_1 in terms of E_∞:

$$A_1 = \frac{3\lambda'(\eta_0)(\eta_1^2 - \eta_0^2)(1 - E_\infty S_2 + \dfrac{\lambda_\infty J_1}{2\eta_1^2 z_0}) - 6E_\infty \lambda_\infty \eta_0}{z_0 S_1 \Big(2T(\eta_0) - 3\lambda_\infty \eta_0\Big)}. \tag{8.10}$$

Let us substitute (8.7) and (8.10) into (8.9), then we obtain the expression for the coeffi cient of the discrete spectrum E_∞:

$$E_\infty = \frac{V_1 - Q'(\alpha_p)K}{S_3 - Q'(\alpha_p)N + \frac{1}{36}}, \tag{8.11}$$

where the following designations were introduced:

$$Q'(\alpha_p) = \frac{Q(\alpha_p)}{W_1},$$

$$Q(\alpha_p) = \frac{M_2}{3\eta_1^2} + z_0 \frac{1 - \alpha_p}{36\alpha_p},$$

$$W_1 = z_0 S_1 (2T(\eta_0) - 3\lambda_\infty \eta_0),$$

$$V_1 = \frac{\lambda_\infty M_1}{2\eta_1^2 z_0} - \frac{m(\eta_0)}{S_1}\Big(1 + \frac{\lambda_\infty J_1}{2\eta_1^2 z_0}\Big),$$

$$K = 3\lambda'(\eta_0)(\eta_1^2 - \eta_0^2)\Big(1 + \frac{\lambda_\infty J_1}{2\eta_1^2 z_0}\Big),$$

$$S_3 = \frac{\lambda_\infty M_1}{2\eta_1^2 z_0} - m(\eta_0)\frac{S_2}{S_1},$$

$$N = 3\lambda'(\eta_0)(\eta_1^2 - \eta_0^2)S_2 + 6\lambda_\infty \eta_0.$$

With the help of the obtained expression for E_∞ we find the rest of unknown variables: E_0 and A_1:

$$E_0 = \frac{1}{S_1}\Big(1 - S_2 \frac{V_1 - Q'(\alpha_p)K}{S_3 - Q'(\alpha_p)N + \frac{1}{36}} + \frac{\lambda_\infty J_1}{2\eta_1^2 z_0}\Big), \tag{8.12}$$

$$A_1 = \frac{1}{W_1}[K - E_\infty(S_2 + 6\lambda_\infty \eta_0)]. \qquad (8.13)$$

Thus, all the coefficients of the expansions (6.1) and (6.2) are found unambiguously: the coefficient E_∞ is determined according to (8.11), the coefficient E_0 is determined according to (8.12), the coefficient of the continuous spectrum $E(\eta)$ is determined according to (8.3). Finding of the coefficients of discrete and continuous spectra of the expansions (6.1) and (6.2) completes the proving of existence of this expansions. Uniqueness of the solution in the form of expansions (6.1) and (6.2) can be proved easily with the use of contraposition method.

10. CONCLUSION

In the present work new boundary conditions for the questions of plasma oscillations in degenerate plasma were proposed. These boundary conditions are naturally called as specular accommodative conditions. Such boundary conditions are most adequate for the problems of **normal** propagation of plasma waves (perpendicular to the boundary), since accommodation coefficient under such boundary conditions is **normal** electron momentum accommodation coefficient.

In the present paper the analytical solution of the problem of plasma oscillations in degenerate plasma with accommodation of electron normal momentum, the coefficients of the discrete and continuous spectra of the problem are found in explicit form.

REFERENCES

1. *Fortov V. E.* (ed.) The encyclopaedia of low temperature plasmas// 1997–2009. Volums I–VII. Moscow, Nauka (in Russian).

2. *Vedenyapin V. V.* Boltzmann and Vlasov kinetic equations. Moscow, Fizmatlit, 2001 (in Russian).

3. *A. A. Abrikosov* Fundamentals of the Theory of Metals. Nauka, Moscow, 1977; North Holland, Amsterdam, 1988.

4. Dressel M., Grüner G. *Electrodynamics of Solids. Optical Properties of Electrons in Matter*. Cambridge. Univ. Press. 2003. 487 p.

5. *Boyd T.J.N., Sanderson J.J.* The Physics of Plasmas// Cambridge Univ. Press. 2003. 546 pp.

6. *Liboff R.L.* Kinetic theory: classical, quantum, and relativistic description// 2003. Springer Verlag. New York, Inc. 587 pp.

7. *Tonks L., Langmuir I.* Oscillations in ionized gases// Phys. Rev. 1929. V. 33 P. 195–210; *Langmuir I.* Phys. Rev., 1929. V. 33, p. 954; *Tonks L., Langmuir I.* Phys. Rev., 1929. V. 34, p. 876.

8. *Tonks L.* The high frequency behavior of a plasma // Phys. Rev. V. 37, 1931, pp. 1458–1483.

9. *Tonks L.* Plasma–electron resonance, plasma resonance and plasma shape// Phys. Rev. 1931. V. 38. P. 1219–1223.

10. *Vlasov A.A.* On high-frequency properties of electron gas// Journal of Experimental and Theoretical Physics.1938. Vol. 8. No. 3. p. 291–318.

11. *Landau L.D.* On electron plasma oscillations // Collection of works. Moscow, Nauka, 1969. Vol. 2. p. 7–25. (in Russian) (See also the Journal of Experimental and Theoretical Physics. 1946. Vol. 26. No. 7. p. 547–586.)

12. *Latyshev A.V., Yushkanov A.A.* Boundary value problems for degenerate electronic plasmas// Monograph. Moscow State Regional University, 2006, 274 p. (in Russian).

13. *Latyshev A.V., Yushkanov A.A.* Analytical solution of the problem on behaviour the degenerate electronic plasmas// – Chapter 10 in "Encyclopaedia of low temperature plasma". Vol. VII-I. P. 159–177. Moscow, 2008 (in Russian).

14. *Vedenyapin V. V.* Kinetic theory by Maxwell, Boltzmann and Vlasov. Moscow State Regional University, 2005. (in Russian).

15. *Morozov A.I., Savelyev V.V.* Structure of the stationary Debay layers in rarefied plasma near dielectric surface// Plasma Physics. 2004. Vol. 30. №4. Pp. (in Russian). 330–338.

16. *Morozov I.V., Norman G.E.* Collisions and plasma waves in collisional plasma// Journal of Experimental and Theoretical Physics. 2005. Vol. 127. №2. Pp. 412–430. (in Russian).

17. *Keller J.M., Fuchs R., Kliewer K.L.* *p*–polarized optical properties of a metal with a diff usely scattering surface// Physical Review B. 1975. V. 12. №6. P. 2012–2029.

18. *Kliewer K.L., Fuchs R.* *s*–polarized optical properties of metals// Phys. Rev. B. 1970. V. 2. №8. P. 2923–2936.

19. *Gohfeld V.M., Gulyanskiy M.A., Kaganov M.I., Plyavenek A.G.* Non-exponential damping of electromagnetic field in normal metals// Journal of Experimental and Theoretical Physics. 1985. Vol. 89. №3(9). P. 985–1001.

20. *Gohfeld V.M., Gulyanskiy M.A., Kaganov M.I.* Anomalous penetration of longitudinal alternating electric field into degenerate plasma under arbitrary parameter of refl ectivity // Journal of Experimental and Theoretical Physics. 1987. Vol. 92. №2. P. 523–530.

21. *Latyshev A.V., Yushkanov A.A.* Analytic solution to the problem of the behavior of an electron plasma in a half–space of a metal in an al-

ternating electric field// Surface. Physics. Chemistry. Mechanics. 1993. №2 P.25–32. (in Russian)

22. *Latyshev A.V., Yushkanov A.A.* Electron Plasma in a Metal Half–Space in an Alternating Electric Field. – Comp. Math. and Math. Phys. 2001. Vol. 41. №8. Pp. 1169–1181.

23. *Latyshev A.V., Yushkanov A.A.* Analytic solution of the problem of the behavior of a collisional plasma in a half–space in an external alternating electric field . – Teor. Math. Phys. 1995. V. 103. №2, May, p.p. 573–582.

24. *Latyshev A.V., Yushkanov A.A.* Reflection and Transmission of Plasma Waves at the Interface of Crystallities. – Computational Mathematics and Mathematical Physics, 2007, Vol. 47, No 7, pp. 1179–1196.

25. *Latyshev A.V., Yushkanov A.A.* Reflection of plasma waves from a plane boundary. – Theoretical and Mathematical Physics. V. 150 (3), 425 – 435 (2007).

26. *Latyshev A.V., Yushkanov A.A.* Reflection of a Plasma Wave from the Flat Boundary of a Degenerate Plasma. – Technical Physics. 2007. Vol. 52. No. 3, pp. 306 – 312.

27. *Gritsienko N.V., Latyshev A.V. and Yushkanov A.A.* On The Theory of Plasma Waves Reflection from the Boundary with Specular Accommodative Boundary Conditions// Proc. 3rd Intern. Conf. on Appl. Maths, Simulation, Modelling (ASM'09), Proc. 3rd Intern. Conf. on Circuits, Systems and Signals (CSS'09) Vouliagmeni, Athens, Greece. December 29-31, 2009. P. 68 – 75.

28. *Stenflo L. and Tsytovich V.N.* Effect of Collisions on the Nonlinear Scattering of Waves in Plasmas// Comments on Plasma Physics and Controlled Fusion. 1994. Vol. 16, No. 2, pp. 105–112.

29. *Bingham R., De Angelis U., Shukla P.K., Stenfl o L.* Largeamplitude waves and fields in plasmas.// Proceedings, Workshop, Spring College on Plasma Physics, Trieste, Italy, May 22-26, 1989 (1990).

30. *Latyshev A.V., Yushkanov A.A.* Moment Boundary Condition in Rarefied Gas Slip–Flow Problems. – Fluid Dynamics. 2004. V. 39. No. 2, pp. 339–353.

31. *Case K. M. and Zweifel P. F.* Linear Transport Theory. Addison–Wesley Publ. Comp. Reading, 1967.

32. *Platzman P. M. and Wolf P. A.* Waves and interactions in solid state plasmas. Academic Press. New York and London. 1973.

33. *Gakhov F. D.*, Boundary Value Problems [in Russian], Nauka, Moscow (1977); English transl., Dover, New York (1990).

34. *Latyshev A. V. and Yushkanov A. A.* Nonstationary boundary problem for model kinetic equations at critical parameters. – Teor. and Math. Phys. 1998. V. 116. №2. P. 978 – 989.

35. *Latyshev A. V. and Yushkanov A. A.* Analytic solution of boundary – value problems for nonstationary model kinetic equations . – Theor. Math. Phys. 1992. V. 92. №; p.p. 782–790.

36. *A. Arnold, J. A. Carrillo, I. Gamba, C.–W. Shu* Low and high field scaling limits for the Vlasov- and Wigner-Poisson-Fokker-Planck systems//Transport theory and statistical physics, 2001. Vol. 30, pp. 121—153.