A NOTE ON AFFINE CONNECTION IN $\mathbb{R}^m$

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ABSTRACT. In this paper we consider the space $\mathbb{R}^m$ with a symmetric affine connection. We investigate the sufficient conditions which imply the existence of Euclidian coordinates in the whole space $\mathbb{R}^m$. In these coordinates all the Christoffel symbols are equal to zero identically.

1. Main Theorem

Define a symmetric affine connection in $\mathbb{R}^m = \{(x^1, \ldots, x^m)\}$ by the Christoffel symbols $\{\Gamma^k_{ij}(x)\} \subset C^1(\mathbb{R}^m), \quad \Gamma^k_{ij}(x) = \Gamma^k_{ji}(x)$.

Suppose that the space $\mathbb{R}^m$ is equipped with a norm $\|\cdot\|$. This norm does not have any relation to the connection.

For any vector $z \in \mathbb{R}^m$ introduce a matrix valued function $\Gamma_z(t) = (\Gamma^k_{ij}(tz)z^j), \quad t \geq 0$.

We use the Einstein summation convention. Below the symbol T stands for the matrix transpose operation.

Theorem 1. *Suppose that the Riemann curvature tensor equals zero identically:* $R^i_{skl}(x) = 0$.

Assume that the condition

$$\sup_{\|z\|=1} \int_0^{+\infty} \|\Gamma_z(t) + \Gamma_z^T(t)\| \, dt < \infty \tag{1.1}$$

is fulfilled. Then there exists a C^1*-diffeomorphism* $f : \mathbb{R}^m \to \mathbb{R}^m$ *which generates the change of variables* $y = f(x)$ *such that in these new variables* y *all the Christoffel symbols are equal to zero identically.*

Remark 1. *There is a simple sufficient condition that imply (1.1). This condition is as follows*

$$|\Gamma^k_{ij}(x)| \leq \frac{c}{(1 + \|x\|)^\gamma}.$$

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The constants $c>0$ and $\gamma>1$ are independent on x.

If the curvature tensor is equal to zero identically then by itself it does not guarantee the existence of new global coordinates such as the Theorem gives. The example is as follows.

Introduce in $\mathbb{R}^1$ a connection by the formula $\Gamma^1_{11}(x)=1$. This connection corresponds to a metric tensor $g_{11}(x)=e^{2x}$.

Then to obtain the proper f one must solve the equation $f''(x)=f'(x)$. It is easy to see that there are no diffeomorphisms of $\mathbb{R}^1$ among the solutions to this equation.

2. Proof

In the sequel all the inessential positive constants we denote by the same latter c.

2.1. **Frobenius Theorem for Linear Pfaff System.** Recall the Frobenius theorem.

Let variables $x=(x^1,\ldots,x^l)$ live in the space $\mathbb{R}^l$.

Introduce functions

$$a^k_{ji}(x)\in C^1(\mathbb{R}^l),\quad j=1,\ldots l,\quad i,k=1,\ldots,p.$$

Consider a linear Pfaff problem

$$\frac{\partial u^k}{\partial x^j}(x)=a^k_{ji}(x)u^i(x),\quad u(\hat x)=\hat u. \tag{2.1}$$

Theorem 2 (Frobenius, [1], [2])**.** *Assume that the following identity holds:*

$$\frac{\partial a^k_{ji}}{\partial x^s}-\frac{\partial a^k_{si}}{\partial x^j}+a^k_{jq}a^q_{si}-a^k_{sq}a^q_{ji}=0. \tag{2.2}$$

Then for any initial condition $(\hat x,\hat u)$ problem (2.1) has a unique solution $u^i(x)\in C^1(\mathbb{R}^l)$.

2.2. **Pfaff Problem for Affine Connection.** To proof Theorem 1 one must find a change of variables $y^i=f^i(x)$ such that

$$\frac{\partial y^p}{\partial x^k}=w^p_k,\quad \frac{\partial w^p_r}{\partial x^s}=w^p_k\Gamma^k_{rs}(x). \tag{2.3}$$

By Theorem 2 this problem has a solution for any initial conditions. Identities (2.2) take the form

$$-R^i_{skl}=\frac{\partial\Gamma^i_{sl}}{\partial x^k}-\frac{\partial\Gamma^i_{sk}}{\partial x^l}+\Gamma^i_{rk}\Gamma^r_{sl}-\Gamma^i_{rl}\Gamma^r_{sk}=0.$$

These identities have been assumed by Theorem 1, but they are not sufficient: the solution to system (2.3) must be such that the mapping $x\mapsto y(x)$ to be a diffeomorphism of $\mathbb{R}^m$ to itself.

Let us compliment system (2.3) with initial conditions

$$y^p(0)=0,\quad w^p_r(0)=\delta^p_r. \tag{2.4}$$

To check that the mapping $y(x)$ is a diffemorphism we employ the following proposition.

Proposition 1 ([3]). *Let $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ and for all x one has*

$$\det(df(x)) \neq 0.$$

(Here df is the Jacobi matrix of f.)

Suppose that there exists a constant c such that the estimate

$$\|(df(x))^{-1}\| \leq c$$

holds for all $x \in \mathbb{R}^m$. Then f is a diffeomorphism of $\mathbb{R}^m$ to itself.

Let $w^p_r(x) \in C^1(\mathbb{R}^m)$ be a solution to problem (2.3)-(2.4). This solution exists by the Frobenius theorem.

Introduce a matrix

$$W_z(t) = (w^p_r(tz)), \quad z = \frac{x}{\|x\|}, \quad x \neq 0.$$

Then due to formulas (2.3), (2.4) the matrix W_z satisfy the Cauchy problem

$$\dot{W}_z(t) = W_z(t)\Gamma_z(t), \quad W_z(0) = I, \tag{2.5}$$

Thus the matrix W_z is a fundamental matrix for the linear system

$$\dot{\xi} = \xi\Gamma_z.$$

Particularly, $\det W_z(t) \neq 0$ for all t.

Now the value $w^p_r(x), \quad x \neq 0$ may be obtained in such a way: $(w^p_r(x)) = W_z(\|x\|)$.

By Proposition 1 we must control the norm of the matrix $V_z(t) = W_z^{-1}(t)$. Formulas (2.5) give

$$\dot{V}_z(t) = -\Gamma_z(t)V_z(t), \quad V_z(0) = I.$$

In other words, $V_z(t)$ is a fundamental matrix for the system

$$\dot{\nu} = -\Gamma_z\nu.$$

Let $\lambda_z(t)$ stands for the greatest eigenvalue of the matrix

$$H_z(t) = -\frac{1}{2}\Big(\Gamma_z(t) + \Gamma^T_z(t)\Big).$$

Then by Waźewski's inequality [4] one has

$$\|V_z(t)\| \leq c\exp\Big(\int_0^t \lambda_z(s)\,ds\Big). \tag{2.6}$$

The positive constant c is independent on z, t.

By Proposition 1 the norm $\|V_z(t)\|$ must be bounded. But it is bounded due to inequalities (2.6), (1.1) and virtue of a general fact from the matrix algebra:

$$|\lambda_z(s)| \leq c\|H_z(t)\|.$$

The positive constant c is independent on z, t.

Theorem 1 is proved.

References

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