

QCD on an infinite lattice

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Abstract

We construct a mathematically well-defined framework for the kinematics of Hamiltonian QCD on an infinite lattice in \mathbb{R}^3 , and it is done in a C^* -algebraic context. This is based on the finite lattice model for Hamiltonian QCD developed by Kijowski, Rudolph e.a. (cf. [22]). To extend this model to an infinite lattice, we need to take an infinite tensor product of nonunital C^* -algebras, which is a nonstandard situation. We use a recent construction for such situations, developed in [9]. Once the field C^* -algebra is constructed for the fermions and gauge bosons, we define local and global gauge transformations, and identify the Gauss law constraint. The full field algebra is the crossed product of the previous one with the local gauge transformations. The rest of the paper is concerned with enforcing the Gauss law constraint to obtain the C^* -algebra of quantum observables. For this, we use the method of enforcing quantum constraints developed by Grundling and Hurst (cf. [10]). In particular, the natural inductive limit structure of the field algebra is a central component of the analysis, and the constraint system defined by the Gauss law constraint is a system of local constraints in the sense of [14]. Using the techniques developed in that area, we solve the full constraint system by first solving the finite (local) systems and then combining the results appropriately. We do not consider dynamics.

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1 Introduction

QCD is an important component of the standard model, and the explicit construction of a field C^* -algebra for it is still an unsolved problem in mathematical physics. The construction of a field algebra is a kinematics problem and it precedes the hard problem of dynamics, which involves interactions, so it seems more tractable. There is a deep body of theory developed for the locality properties of the field algebras of quantum field theories in space-time (cf. [15] for a survey), and of course any explicitly constructed field algebra of this system must be consistent with that. There is also extensive work on the Hamiltonian model of a Fermion in a nonabelian *classical* gauge potential in \mathbb{R}^3 , cf. [6, 24, 25], and it leads to interesting proposals for the field algebra of the fully quantized model [16].

Thus far, the best explicit rigorous constructions of appropriate field algebras have been for lattice approximations of Hamiltonian QCD in \mathbb{R}^3 cf. [22, 23, 20, 21]. Unfortunately due to a technical problem explained below, these models have been confined to finite lattices. This is the main problem which we want to address here, i.e. we want to construct the field C^* -algebra for QCD on an infinite lattice in \mathbb{R}^3 . Using this field algebra, we then want to define gauge transformations and solve the Gauss law constraint, hence identifying the physical observables.

More specifically;- for the model of QCD on a finite lattice developed by Kijowski, Rudolph e.a. [22, 23, 20, 21], one finds that the field algebra is isomorphic to the algebra of compact operators $\mathcal{K}(\mathcal{H})$ on a separable infinite dimensional Hilbert space \mathcal{H} . As this has (up to unitary equivalence) only one irreducible representation, one obtains a generalized von Neumann uniqueness theorem for the system. For an infinite lattice, when passing to infinitely many degrees of freedom, one has to expect inequivalent representations. Explicitly, for the gauge part of the algebra, one needs to take an infinite tensor product of the algebras associated to the links of the lattice (these are also isomorphic to $\mathcal{K}(\mathcal{H})$). This means that the standard theory for infinite tensor products does not apply. However, there is a little-known definition for an infinite tensor product of nonunital algebras developed by Blackadar cf. [3], which however has some drawbacks. Recently this approach was further developed by Grundling and Neeb in [9], where an infinite tensor product of nonunital C^* -algebras was constructed which has good representation properties. This is what we use for our construction of the field algebra of our model, and as expected, this new field algebra has many inequivalent representations.

Once we have the field algebra of our model, we can define (local and global) gauge transformations, extend the field algebra to include the implementers of these, and identify the Gauss law constraint. Enforcement of quantum constraints is not a simple matter, in fact compared with Quantum Electrodynamics, the analysis of the Gauss law

is much more complicated. This is due to the fact that in QCD the Gauss law constraint is neither built from gauge invariant operators nor is it linear in the gauge connection fields. Here we use the general method of enforcing quantum constraints developed by Grundling and Hurst (the T-procedure, cf. [10]). It is crucial for this, that the constraint system defined by the Gauss law constraint is a system of local constraints in the sense of Grundling and Lledó [14]. This allows us to solve the full constraint system by first solving the finite (local) systems and then combining the results appropriately.

In this paper, we do not consider boundary effects, and we postpone colour charge analysis to a separate project. Boundary effects were analyzed for finite lattice systems by Kijowski and Rudolph in [20], where it was shown that from the local Gauss equation one can extract a gauge invariant, additive law for operators with eigenvalues in \mathbb{Z}_3 . As in QED, this implies a gauge invariant conservation law:- the global \mathbb{Z}_3 -valued colour charge is equal to a \mathbb{Z}_3 -valued gauge invariant quantity obtained from the color electric flux “at infinity”. The discussion of the boundary data yielding this flux is a subtle task, see [21].

Our paper is organized as follows. We start Sect. 2 with a statement of our initial assumptions, and construction of the Fermion algebra. In Sect. 2.1 we define for each link the field algebra for the gauge connection, recall the method developed in [9], and then use it to construct an infinite tensor product of the link algebras. We then take the tensor algebra of this gauge field algebra with the fermion algebra, and consider a natural inductive structure of it in Sect. 2.2. We call it the kinematic field algebra. In Sect. 3.1 we define the action of the local gauge transformations on the kinematic algebra, and in Sect. 3.2 we do this for global gauge transformations. This requires us to choose a gauge invariant approximate identity in the link algebras, and we analyze this issue. In Sect 3.3 we construct the full field algebra as a (discrete) crossed product of the kinematic algebra with the local gauge transformations. This contains all the relevant information of the system, and in Sect. 3.4 we define the local Gauss law constraint. The rest of the paper is dedicated to the enforcement of this constraint. We first review the heuristic Gupta–Bleuler constraint method in Sect. 4.1, then in Sect. 4.2 we review the T-procedure of enforcing constraints (cf. [10]), and show how the current constraint system fits into it. In Sect. 4.3 we solve the constraint system for a finite lattice in terms of the T-procedure. These results are used in Sect. 4.4 to solve the constraint system for the local algebras in the inductive limit of the full field algebra. Finally, in Sect. 4.5 we show that the full system of constraints is a system of local quantum constraints in the sense of [14]. Using techniques from [14] we then solve the constraint system fully for the local observables, but for global observables the constraining remains unresolved. There are two appendices; one to make contact with physics notation for our system, and the other to state a result on constraint subsystems which we need.

2 The Kinematic Field Algebra

We consider a model for QCD in the Hamiltonian framework on an infinite regular cubic lattice in \mathbb{Z}^3 . For basic notions concerning lattice gauge theories including fermions, we refer to [33] and references therein. For the convenience of the reader, and to make the presentation more self-contained, we will spell out details of the underlying classical gauge connection field.

We list our input assumptions and fix notation. First, for the lattice, define a pair $\Lambda := (\Lambda^0, \Lambda^1)$ as follows:

- $\Lambda^0 := \{(n, m, r) \in \mathbb{R}^3 \mid n, m, r \in \mathbb{Z}\} \cap X$ where X is an open connected set in \mathbb{R}^3 . Thus Λ^0 is a unit cubic lattice (possibly infinite) contained in X , and its elements are called sites.
- Let $\tilde{\Lambda}^1$ be the set of all directed edges (or links) between nearest neighbours, i.e. $\tilde{\Lambda}^1 := \{(x, y) \in \Lambda^0 \times \Lambda^0 \mid y = x \pm \mathbf{e}_i \text{ for some } i\}$ where the $\mathbf{e}_i \in \mathbb{R}^3$ are the standard unit basis vectors. Define a map $\eta : \tilde{\Lambda}^1 \rightarrow \mathcal{P}(\Lambda^0) \equiv$ power set of Λ^0 , by $\eta((x, y)) := \{x, y\}$, i.e. it is the map which “forgets” the orientation of links, then Λ^1 will denote a choice of orientation of $\tilde{\Lambda}^1$, i.e. it is a section of η , i.e. for each $\{x, y\} \in \eta(\tilde{\Lambda}^1)$ it contains either (x, y) or (y, x) but not both. Thus the pair (Λ^0, Λ^1) is a directed graph, and we assume that it is connected.
- Sometimes we need to identify the elements of Λ^i with subsets of \mathbb{R}^3 and we will make the natural identifications, e.g. a link $(x, y) \in \Lambda^1$ is the undirected closed line segment from x to y .

Below, the set X will play no role, so that one may just consider the lattice $\Lambda^0 = \mathbb{Z}^3$. If one wants to analyze surface effects, X will become more important.

Next, to define our model of lattice QCD, we will associate to each lattice site $x \in \Lambda^0$ a fermionic particle (the quarks), and associate to each link $(x, y) \in \Lambda^1$ a bosonic particle (the gluons).

To motivate our definition of lattice QCD on Λ , assume that we have a classical matter field with a gauge connection on \mathbb{R}^3 . To be precise,

- Let G be a connected, compact Lie group (the gauge group, usually $SU(3)$), and let \mathbf{V} be a finite dimensional complex Hilbert space on which G acts smoothly as unitaries (e.g. $\mathbf{V} = \mathbb{C}^3$), so we take $G \subset U(\mathbf{V})$.
- Let $\mathbf{q}_E : E \rightarrow M := \mathbb{R}^3$ be a smooth vector bundle with typical fibre being \mathbf{V} .

The classical matter fields are defined as the smooth sections of E . The space \mathbf{V} represents “internal degrees of freedom” of the matter field.

Consider an atlas \mathcal{T}_E of local trivializations of the bundle $\mathbf{q}_E : E \rightarrow M$, i.e. $\mathcal{T}_E = \{(U_i, \varphi_i) \mid i \in I\}$ for an index set I , where $U_i \subset M$ is open, M is covered by $\{U_i \mid i \in I\}$, and bijections $\varphi_i : \mathbf{q}_E^{-1}(U_i) \rightarrow U_i \times \mathbf{V}$. We denote:

$$\begin{aligned} \varphi_i(p) &= (\mathbf{q}_E(p), \tilde{\varphi}_i(p)), & \tilde{\varphi}_{ix} &:= \tilde{\varphi}_i|_{\mathbf{q}_E^{-1}(x)}, \\ \tilde{\varphi}_{ix} : \mathbf{q}_E^{-1}(x) &\rightarrow \mathbf{V} & \text{and } g_{ij}(x) &:= \tilde{\varphi}_{ix} \circ \tilde{\varphi}_{jx}^{-1} : \mathbf{V} \rightarrow \mathbf{V} \end{aligned}$$

for all $x \in U_i \cap U_j$. The functions $g_{ij} : U_i \cap U_j \rightarrow \text{GL}(\mathbf{V})$ are called the *transition functions*, and these are smooth.

- Let $\mathbf{q}_E : E \rightarrow M$ be a G -bundle, i.e. it has an atlas of local trivializations for which the transition functions g_{ij} take their values in G (such an atlas is called a G -atlas or a G -structure).

This is the structural assumption which is the starting point for a local gauge theory. Using the transition functions, one can construct an associated G -principal fibre bundle $\mathbf{q} : P \rightarrow M$. This principal bundle P encodes the G -structure of E , in the sense that two bundles E and E' with the same base space M , fibre space \mathbf{V} and inclusion $G \subset \text{GL}(\mathbf{V})$ have equivalent G -structures iff their associated G -principal bundles are equivalent (cf. Theorem 8.2 in [34]).

Now we have a radical simplification;- if the base M is σ -compact and contractible to a point, then the principal bundle P is trivial (cf. [17], Sect. 4.12 for the topological category and [26] for the passage to the smooth category). Thus, since $M = \mathbb{R}^3$ satisfies these properties, we conclude that P and hence E must be trivial, i.e. we may assume that $P = \mathbb{R}^3 \times G$ and $E = \mathbb{R}^3 \times \mathbf{V}$. Thus the classical matter fields are $\Gamma(E) = C^\infty(\mathbb{R}^3, \mathbf{V})$. If we restrict $\Gamma(E)$ to the lattice Λ^0 , we obtain $\prod_{x \in \Lambda^0} \mathbf{V}$. For the quantum theory we want to make these into fermions, so

Definition 2.1. Assume the quantum matter field algebra on Λ is:

$$\mathfrak{F}_\Lambda := \text{CAR}(\ell^2(\Lambda^0, \mathbf{V})) = C^*\left(\bigcup_{x \in \Lambda^0} \mathfrak{F}_x\right) \quad (2.1)$$

where $\mathfrak{F}_x := \text{CAR}(V_x)$ and $V_x := \{f \in \ell^2(\Lambda^0, \mathbf{V}) \mid f(y) = 0 \text{ if } y \neq x\} \cong \mathbf{V}$. We interpret $\mathfrak{F}_x \cong \text{CAR}(\mathbf{V})$ as the field algebra for a fermion at x . We denote the generating elements of $\text{CAR}(\ell^2(\Lambda^0, \mathbf{V}))$ by $a(f)$, $f \in \ell^2(\Lambda^0, \mathbf{V})$, and these satisfy the usual CAR-relations:

$$\{a(f), a(g)^*\} = \langle f, g \rangle \mathbf{1} \quad \text{and} \quad \{a(f), a(g)\} = 0 \quad \text{for } f, g \in \ell^2(\Lambda^0, \mathbf{V}) \quad (2.2)$$

where $\{A, B\} := AB + BA$. Note that the odd parts of \mathfrak{F}_x and \mathfrak{F}_y w.r.t. the fields $a(f)$ anticommute if $x \neq y$.

This defines the matter fields on the lattice sites.

2.1 The gauge field algebra.

Next, we want to define the algebra for the quantum gauge fields on Λ . Continue the analysis above of a classical matter field with a gauge connection on \mathbb{R}^3 . There are many equivalent definitions of a connection on $P = \mathbb{R}^3 \times G$. We define a classical gauge connection on P as a $C^\infty(M)$ -linear map

$$\Phi : \mathfrak{X}(M) \rightarrow \mathfrak{aut} P \quad \text{such that} \quad \mathfrak{q}_*(\Phi(X)) = X \quad \text{for} \quad X \in \mathfrak{X}(M),$$

where $\mathfrak{aut} P := \{X \in \mathfrak{X}(P) \mid (R_g)_*(X) = X \ \forall g \in G\} \equiv$ the G -invariant vector fields, and R_g is the principal right action of $g \in G$ on P . Integrable elements of $\mathfrak{aut} P$ generate elements of the principal bundle automorphisms:

$$\text{Aut } P = \left\{ \gamma \in \text{Diff } P \mid \gamma \circ R_g = R_g \circ \gamma \ \forall g \in G \right\}.$$

The parallel transport of an element $p \in \mathfrak{q}^{-1}(x) \subset P$ along a curve $c : \mathbb{R} \rightarrow M$ through $x = c(0)$ is defined as the unique curve $c_\Phi : \mathbb{R} \rightarrow P$ such that $\frac{d}{dt}c_\Phi(t) = \Phi(X(c(t)))$ and $c_\Phi(0) = p$. In fact this defines maps $\tau^t : \mathfrak{q}^{-1}(x) \rightarrow \mathfrak{q}^{-1}(c(t))$ by $\tau^t(p) := c_\Phi(t)$, hence $\tau^t : G \rightarrow G$ as P is trivial. As τ^t commutes with R_g i.e. $\tau^t(hg) = \tau^t(h)g$ for all $h, g \in G$, let $g = h^{-1}$ so $\tau^t(e) = \tau^t(h)h^{-1}$ i.e. $\tau^t(h) = \tau^t(e)h$. Thus τ^t is just left multiplication by the element $\tau^t(e) \in G$.

Returning now to the lattice Λ , it is natural to model the classical connection Φ on it, by associating to each link $(x, y) \in \Lambda^1$ the parallel transport $\tau^1 : \mathfrak{q}^{-1}(x) \rightarrow \mathfrak{q}^{-1}(y)$ along the link, which we have just seen is left multiplication by an element of G . Thus, our model for a classical connection on the lattice Λ , is a map $\Phi_\Lambda : \Lambda^1 \rightarrow G$. Therefore, our classical configuration space for the connections is $\prod_{\ell \in \Lambda^1} G$, hence its phase space is

$$\prod_{\ell \in \Lambda^1} T^*G \cong \prod_{\ell \in \Lambda^1} (G \times \mathfrak{g}^*).$$

To construct the field algebra for the quantum system corresponding to this, we first choose a field algebra for the quantum system corresponding to a single factor $T^*G \cong (G \times \mathfrak{g}^*)$. The classical configuration space G has a distinguished set of motions on it, given by left multiplication by elements of G . Thus it seems reasonable to take the transformation group algebra, which is the crossed product C^* -algebra $C(G) \rtimes_\lambda G$ where $\lambda : G \rightarrow \text{Aut } C(G)$ is the action

$$\lambda_g(f)(h) := f(g^{-1}h), \quad f \in C(G), \quad g, h \in G. \quad (2.3)$$

The algebra $C(G) \rtimes_\lambda G$ is also called the generalised Weyl algebra, and by definition its representations produce covariant representations for the action $\lambda : G \rightarrow \text{Aut } C(G)$. The derived action $d\lambda : \mathfrak{g} \rightarrow \text{Der}(C^\infty(G))$, satisfies the relation:

$$[d\lambda(A), T_f] := T_{X_A(f)} \quad \text{for} \quad A \in \mathfrak{g}, \quad f \in C^\infty(G), \quad (2.4)$$

where $X_A \in \mathfrak{X}(G)$ is the associated right-invariant vector field and $T_f : C^\infty(G) \rightarrow C^\infty(G)$ denotes multiplication by f . These are thought of as generalized canonical commutation relations, especially when represented on $L^2(G)$.

Thus for every link $\ell \in \Lambda^1$ we will assume a generalised Weyl algebra $C(G) \rtimes_\lambda G$ where G is our compact gauge group. It is well-known that $C(G) \rtimes_\lambda G \cong \mathcal{K}(L^2(G))$ cf. [31] and Theorem II.10.4.3 in [2]. Next, we need to combine these. Since for the classical connection field, the phase space is $\prod_{\ell \in \Lambda^1} T^*G$, it seems that for the quantum system we must take a tensor product $\bigotimes_{\ell \in \Lambda^1} (C(G) \rtimes_\lambda G)$. In the case of a finite lattice, this is fine, and the C^* -tensor norms are unique. Moreover, since $C(G) \rtimes_\lambda G \cong \mathcal{K}(L^2(G))$ and $\mathcal{K}(\mathcal{H}_1) \otimes \mathcal{K}(\mathcal{H}_2) \cong \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, it follows that

$$\bigotimes_{\ell \in \Lambda^1} (C(G) \rtimes_\lambda G) \cong \mathcal{K}\left(\bigotimes_{\ell \in \Lambda^1} L^2(G)\right) \cong \mathcal{K}(\mathcal{H})$$

for a finite lattice, where \mathcal{H} is a generic infinite dimensional separable Hilbert space. So the field algebra for a finite lattice is

$$\mathfrak{F}_\Lambda \otimes \bigotimes_{\ell \in \Lambda^1} (C(G) \rtimes_\lambda G) \cong \mathfrak{F}_\Lambda \otimes \mathcal{K}\left(\bigotimes_{\ell \in \Lambda^1} L^2(G)\right) \cong \mathcal{K}(\mathcal{H})$$

as \mathfrak{F}_Λ is a full matrix algebra. This shows that for a finite lattice there will be only one irreducible representation, up to unitary equivalence.

In the case of an infinite lattice, the situation is considerably different, and we will expect inequivalent representations when passing to infinitely many degrees of freedom (as in quantum field theory). First, note that since $C(G) \rtimes_\lambda G \cong \mathcal{K}(L^2(G))$ is nonunital, the standard theory for infinite tensor products breaks down, i.e. an infinite tensor product of these is undefined. The problem of infinite tensor products for nonunital C^* -algebras is still relatively undeveloped, in fact Takesaki states in [39] on p84 that “the infinite tensor product of non-unital C^* -algebras is not defined.” There is however a little-known definition for an infinite tensor product of nonunital algebras developed by Blackadar cf. [3], but this uses a choice of reference projections in the sequence, and representations of the resultant C^* -algebra, depends on the choice of projections. Recently in [9], extending Blackadar’s construction, an infinite tensor product of $\mathcal{K}(\mathcal{H})$ was constructed which has good representation properties w.r.t. a natural Weyl algebra in its multiplier algebra. This is very close to the situation which we have here, so we will choose this method of construction for the full bosonic field algebra. We describe the construction. Further details, and proofs of the rest of the claims in this subsection can be found in [9].

Observe first, that the representation theory of $\mathcal{K}(\mathcal{H})$ (resp. $\bigotimes_{n=1}^k \mathcal{K}(\mathcal{H})$) is precisely the regular representation theory of the Weyl algebra $\text{CCR}(\mathbb{R}^2) \subset M(\mathcal{K}(\mathcal{H}))$ (resp.

$\bigotimes_{n=1}^k \text{CCR}(\mathbb{R}^2) = \text{CCR}(\mathbb{R}^{2k})$ using minimal tensor norm) hence we would expect that the representation theory of “ $\bigotimes_{n=1}^{\infty} \mathcal{K}(\mathcal{H})$ ” (if this object is given a proper meaning) should be the regular representations of the Weyl algebra $\bigotimes_{n=1}^{\infty} \text{CCR}(\mathbb{R}^2)$, where the latter tensor product is well-defined as $\text{CCR}(\mathbb{R}^2)$ is unital. This is precisely what we have for the construction in [9].

We start with Blackadar’s construction [3]. Let $\mathcal{L}_n := \mathcal{K}(\mathcal{H})$, and choose a sequence of “reference projections,” i.e. for each $n \in \mathbb{N}$, choose a nonzero projection $P_n \in \mathcal{L}_n$. Define C^* -embeddings

$$\Psi_{\ell k} : \mathcal{L}^{(k)} \rightarrow \mathcal{L}^{(\ell)} \quad \text{by} \quad \Psi_{\ell k}(A_1 \otimes \cdots \otimes A_k) := A_1 \otimes \cdots \otimes A_k \otimes P_{k+1} \otimes \cdots \otimes P_{\ell},$$

where $k < \ell$ and $\mathcal{L}^{(k)} := \bigotimes_{n=1}^k \mathcal{L}_n$. Then the inductive limit makes sense, so we define

$$\mathcal{L} := \bigotimes_{n=1}^{\infty} \mathcal{L}_n := \varinjlim \{ \mathcal{L}^{(n)}, \Psi_{\ell k} \}$$

and write $\Psi_k : \mathcal{L}^{(k)} \rightarrow \mathcal{L}$ for the corresponding embeddings, satisfying $\Psi_k \circ \Psi_{kj} = \Psi_j$ for $j \leq k$. Since each \mathcal{L}_n is simple, so are the finite tensor products $\mathcal{L}^{(k)}$ ([41], Prop. T.6.25), and as inductive limits of simple C^* -algebras are simple ([19], Prop. 11.4.2), so is \mathcal{L} . It is also clear that \mathcal{L} is separable, and it is nuclear as it is an inductive limit of nuclear algebras.

Since $\Psi_{k+n,k}(L_k) = L_k \otimes P_{k+1} \otimes \cdots \otimes P_{k+n}$, where $L_k \in \mathcal{L}^{(k)}$, this means that we can consider \mathcal{L} to be built up out of elementary tensors of the form

$$\Psi_k(L_1 \otimes \cdots \otimes L_k) = L_1 \otimes L_2 \otimes \cdots \otimes L_k \otimes P_{k+1} \otimes P_{k+2} \otimes \cdots, \quad \text{where } L_i \in \mathcal{L}_i \quad (2.5)$$

i.e. eventually they are of the form $\cdots \otimes P_k \otimes P_{k+1} \otimes \cdots$. We will use this picture below, and generally will not indicate the maps Ψ_k . By componentwise multiplication, we can also identify elementary tensors $\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes P_k \otimes P_{k+1} \otimes \cdots$ in the multiplier algebra $M(\mathcal{L})$. The representations π of \mathcal{L} are well-behaved w.r.t. the reference sequence $\{P_k\}_{k=1}^{\infty}$ in the sense that

$$\text{s-}\lim_{k \rightarrow \infty} \pi(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes P_k \otimes P_{k+1} \otimes \cdots) = \mathbb{1},$$

and this restricts the corresponding regular representations on $\bigotimes_{n=1}^{\infty} \text{CCR}(\mathbb{R}^2) \subset M(\mathcal{L})$. Thus, if we do not want our representations to depend on the choice of the reference sequence of projections, we will need to go beyond a single Blackadar product \mathcal{L} .

As we saw, for every sequence of projections $P_k \in \mathcal{L}_k$ we obtained a Blackadar product \mathcal{L} . We now want to examine a collection of them, where our choices of $P_k \in \mathcal{L}_k$ will “fill out” the full Hilbert space \mathcal{H} .

There is a (countable) approximate identity $(E_n)_{n \in \mathbb{N}}$ in $\mathcal{K}(\mathcal{H})$ consisting of a strictly increasing sequence of projections E_n with $\dim(E_n \mathcal{H}) = n$. For each k , choose such an approximate identity $(E_n^{(k)})_{n \in \mathbb{N}} \subset \mathcal{L}_k = \mathcal{K}(\mathcal{H})$, then for each sequence $\mathbf{n} = (n_1, n_2, \dots) \in \mathbb{N}^\infty := \mathbb{N}^\mathbb{N}$, we have a sequence of projections $(E_{n_1}^{(1)}, E_{n_2}^{(2)}, \dots)$ from which we can construct an infinite tensor product as above, and we will denote it by $\mathcal{L}[\mathbf{n}]$. For the elementary tensors, we streamline the notation to:

$$A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} := A_1 \otimes \cdots \otimes A_k \otimes E_{n_{k+1}}^{(k+1)} \otimes E_{n_{k+2}}^{(k+2)} \otimes \cdots \in \mathcal{L}[\mathbf{n}],$$

where $A_i \in \mathcal{L}_i$, and their closed span is the simple C^* -algebra $\mathcal{L}[\mathbf{n}]$.

Next we define componentwise multiplication between different C^* -algebras $\mathcal{L}[\mathbf{n}]$ and $\mathcal{L}[\mathbf{m}]$. For componentwise multiplication, the sequences give:

$$(E_{n_1}^{(1)}, E_{n_2}^{(2)}, \dots) \cdot (E_{m_1}^{(1)}, E_{m_2}^{(2)}, \dots) = (E_{p_1}^{(1)}, E_{p_2}^{(2)}, \dots)$$

where $p_j := \min(n_j, m_j)$, i.e. multiplication reduces the entries, and hence the sequence $(E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, \dots)$ is invariant under such multiplication. So we define an embedding $\mathcal{L}[\mathbf{n}] \subseteq M(\mathcal{L}[\mathbf{1}])$ for all \mathbf{n} , where $\mathbf{1} := (1, 1, \dots)$ by

$$\begin{aligned} & (A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1}) \cdot (B_1 \otimes \cdots \otimes B_n \otimes E[\mathbf{1}]_{n+1}) \\ & := \begin{cases} A_1 B_1 \otimes \cdots \otimes A_n B_n \otimes A_{n+1} E_1^{(n+1)} \cdots \otimes A_k E_1^{(k)} \otimes E[\mathbf{1}]_{k+1} & \text{if } n \leq k \\ A_1 B_1 \otimes \cdots \otimes A_k B_k \otimes E_{n_{k+1}}^{(k+1)} B_{k+1} \cdots \otimes E_{n_n}^{(n)} B_n \otimes E[\mathbf{1}]_{n+1} & \text{if } n \geq k \end{cases} \end{aligned}$$

for the left action, and similar for the right action on $\mathcal{L}[\mathbf{1}]$. Since multiplication by elements of $\mathcal{L}[\mathbf{1}]$ can separate the elements of $\mathcal{L}[\mathbf{n}]$, the embeddings are faithful. Using these embeddings $\mathcal{L}[\mathbf{n}] \subseteq M(\mathcal{L}[\mathbf{1}])$ we see that

$$\mathcal{L}[\mathbf{n}] \cdot \mathcal{L}[\mathbf{m}] \subseteq \mathcal{L}[\mathbf{p}], \quad (2.6)$$

where $p_j := \min(n_j, m_j)$, and in fact

$$\mathcal{L}[\mathbf{n}] \subset M(\mathcal{L}[\mathbf{p}]) \supset \mathcal{L}[\mathbf{m}]. \quad (2.7)$$

Since $\mathcal{L}[\mathbf{n}] \subseteq M(\mathcal{L}[\mathbf{1}])$ for all \mathbf{n} , we can define the C^* -algebra in $M(\mathcal{L}[\mathbf{1}])$ generated by all $\mathcal{L}[\mathbf{n}]$, and denote it by $\mathcal{L}[E]$. By (2.6), this is just the closed span of all $\mathcal{L}[\mathbf{n}]$ and hence the closure of the dense $*$ -subalgebra $\mathcal{L}_0 \subset \mathcal{L}[E]$, where

$$\mathcal{L}_0 := \sum_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{L}[\mathbf{n}]_0 \quad \text{and} \quad \mathcal{L}[\mathbf{n}]_0 := \bigcup_{k \in \mathbb{N}} \mathcal{L}^{(k)} \otimes E[\mathbf{n}]_{k+1}.$$

Note that if two sequences \mathbf{n} and \mathbf{m} differ only in a finite number of entries, then $\mathcal{L}[\mathbf{n}] = \mathcal{L}[\mathbf{m}]$, and hence we actually have that the correct index set for the algebras $\mathcal{L}[\mathbf{n}]$ is not the sequences \mathbb{N}^∞ , but the set of equivalence classes \mathbb{N}^∞ / \sim where $\mathbf{n} \sim \mathbf{m}$ if they differ only in finitely many entries. We have a partial ordering of equivalence

classes defined by $[\mathbf{n}] \geq [\mathbf{m}]$ if for any representatives \mathbf{n} and \mathbf{m} resp., we have that there is an N (depending on the representatives) such that $n_k \geq m_k$ for all $k > N$. In particular, we note that products reduce sequences, i.e., we have $\mathcal{L}[\mathbf{n}] \cdot \mathcal{L}[\mathbf{p}] \subseteq \mathcal{L}[\mathbf{q}]$ for $q_i = \min(n_i, p_i)$, so $[\mathbf{n}] \geq [\mathbf{q}] \leq [\mathbf{p}]$.

Let $\phi : \mathbb{N}^\infty / \sim \rightarrow \mathbb{N}^\infty$ be a section of the factor map. Then $\mathcal{L}[E]$ is the C^* -algebra generated in $M(\mathcal{L}[\mathbf{1}])$ by $\{\mathcal{L}[\phi(\gamma)] \mid \gamma \in \mathbb{N}^\infty / \sim\}$, and it is the closure of the span of the elementary tensors in this generating set.

From the reducing property of products, we already know that $\mathcal{L}[E]$ has the ideal $\mathcal{L}[\mathbf{1}]$ (we will see that it is proper), hence that it is not simple. However, it has in fact infinitely many proper ideals and each of the generating algebras $\mathcal{L}[\mathbf{n}]$ is contained in such an ideal:

Proposition 2.2. *For the C^* -algebra $\mathcal{L}[E]$, we have the following:*

(i) $\mathcal{L}[E]$ is nonseparable,

(ii) Define $\mathcal{I}[\mathbf{n}_1, \dots, \mathbf{n}_k]$ to be the closed span of

$$\{\mathcal{L}[\mathbf{q}]_0 \mid [\mathbf{q}] \leq [\mathbf{n}_\ell] \text{ for some } \ell = 1, \dots, k\}.$$

Let $[\mathbf{p}] > [\mathbf{n}_\ell]$ strictly for all $\ell \in \{1, \dots, k\}$, then $\mathcal{L}[\mathbf{p}] \cap \mathcal{I}[\mathbf{n}_1, \dots, \mathbf{n}_k] = \{0\}$.

(iii) $\mathcal{I}[\mathbf{n}_1, \dots, \mathbf{n}_k]$ is a proper closed two sided ideal of $\mathcal{L}[E]$.

(iv) Define $\mathcal{L}[\mathbf{n}_1, \dots, \mathbf{n}_k] := C^*(\mathcal{L}[\mathbf{n}_1] \cup \dots \cup \mathcal{L}[\mathbf{n}_k])$.

Then $\mathcal{L}[\mathbf{n}_1, \dots, \mathbf{n}_k] \subset \mathcal{I}[\mathbf{n}_1, \dots, \mathbf{n}_k]$ and

$$C^*(\mathcal{L}[\mathbf{n}_1, \dots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]) \subseteq \mathcal{L}[\mathbf{q}_1, \dots, \mathbf{q}_k], \quad \text{where } (\mathbf{q}_j)_\ell = \min((\mathbf{n}_j)_\ell, (\mathbf{n}_{k+1})_\ell).$$

The main attraction of the C^* -algebra $\mathcal{L}[E]$, is that its representation theory is exactly the regular representations of $\bigotimes_{n=1}^{\infty} \text{CCR}(\mathbb{R}^2)$, which naively is what one would require for the representation theory of “ $\bigotimes_{n=1}^{\infty} \mathcal{K}(\mathcal{H})$ ”.

One of the main costs of using it, is that the finite tensor products $\bigotimes_{n=1}^N \mathcal{K}(\mathcal{H})$ are not contained in $\mathcal{L}[E]$, but are contained in its multiplier algebra $M(\mathcal{L}[E])$. This is not a serious problem because a representation (resp. state) on $\mathcal{L}[E]$ extends uniquely to $M(\mathcal{L}[E])$ on the same representation space (resp. as a state), and hence to subalgebras of $M(\mathcal{L}[E])$.

One could interpret the sequences of projections as specifying the “type” of infinite lattice in which we embed our finite systems. As these sequences restrict the representations, they have physical content, so in the next main section we will try to obtain sequences which are natural from the physical point of view (e.g. being gauge invariant). To conclude:

Definition 2.3. *The field algebra for the quantum connection fields on a lattice is $\mathcal{L}[E]$, where the components $\mathcal{L}_\ell = \mathcal{K}(\mathcal{H}) \cong C(G) \rtimes_\lambda G$ are labelled by links $\ell \in \Lambda^1$.*

2.2 The kinematic field algebra.

From the matter and gauge field algebras, it is now natural to take:

Definition 2.4. *The kinematic field algebra is $\mathfrak{A}_\Lambda := \mathfrak{F}_\Lambda \otimes \mathcal{L}[E]$. It has a unique tensor norm as \mathfrak{F}_Λ is nuclear.*

Note that \mathfrak{A}_Λ is not unital since $\mathcal{L}[E]$ is not unital, and it is not simple since $\mathcal{L}[E]$ is not simple. As mentioned, we will restrict our choice of approximate identities $(E_n^{(k)})_{n \in \mathbb{N}} \subset \mathcal{L}_k = \mathcal{K}(\mathcal{H})$ below when we have defined gauge transformations. In fact \mathfrak{A}_Λ is not yet the full field algebra, since information of important physical transformations is still absent. Below we will extend it to a crossed product of the gauge transformations, to obtain the full field algebra.

We next consider a natural inductive limit structure for this field algebra. Let \mathcal{S} be a directed set of open, bounded convex subsets of \mathbb{R}^3 such that $\bigcup_{S \in \mathcal{S}} S = \mathbb{R}^3$, where the partial ordering is set inclusion. Let $\Lambda_S^i = \{x \in \Lambda^i \mid x \subset S\}$ (using the natural identification of elements of Λ^i with subsets of \mathbb{R}^3), and note that $S_1 \subseteq S_2$ implies $\Lambda_{S_1}^i \subseteq \Lambda_{S_2}^i$ and $\bigcup_{S \in \mathcal{S}} \Lambda_S^i = \Lambda^i$. Define $\mathfrak{F}_S := C^*\left(\bigcup_{x \in \Lambda_S^0} \mathfrak{F}_x\right) \subset \mathfrak{F}_\Lambda$ and then $\mathfrak{F}_\Lambda = \varinjlim \mathfrak{F}_S$ is an inductive limit w.r.t. the partial ordering in \mathcal{S} .

To identify the analogous inductive limit for $\mathcal{L}[E]$, enumerate the links $\{\ell_1, \ell_2, \dots\} = \Lambda^1$ and recall that $\mathcal{L}[E]$ has the dense *-subalgebra

$$\mathcal{L}_0 := \sum_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{L}[\mathbf{n}]_0 \quad \text{and} \quad \mathcal{L}[\mathbf{n}]_0 := \bigcup_{k \in \mathbb{N}} \mathcal{L}^{(k)} \otimes E[\mathbf{n}]_{k+1} \quad \text{where} \quad \mathcal{L}^{(k)} = \mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_k.$$

This suggests that for an $S \in \mathcal{S}$ we should take those elementary tensors in each $\mathcal{L}[\mathbf{n}]_0$ which can only differ from $E[\mathbf{n}]_1 = E_{n_1}^{(1)} \otimes E_{n_2}^{(2)} \otimes \dots$ in entries corresponding to links in Λ_S^1 . Denote the set of these elementary tensors by $\mathcal{E}_S[\mathbf{n}]$, and define

$$\mathcal{L}_S[E] := C^*\left(\bigcup_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{E}_S[\mathbf{n}]\right) \subset \mathcal{L}[E],$$

then again we have the inductive limit structure $\mathcal{L}[E] = \varinjlim \mathcal{L}_S[E]$ w.r.t. set inclusion, since $\mathcal{E}_{S_1}[\mathbf{n}] \subseteq \mathcal{E}_{S_2}[\mathbf{n}]$ if $S_1 \subseteq S_2$, and $\mathcal{L}[\mathbf{n}]_0 = \bigcup_{S \in \mathcal{S}} \mathcal{E}_S[\mathbf{n}]$.

Proposition 2.5. *Given as above, a directed set \mathcal{S} of open, bounded convex subsets of \mathbb{R}^3 such that $\bigcup_{S \in \mathcal{S}} S = \mathbb{R}^3$, partially ordered by inclusion, then*

$$\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_S = \varinjlim (\mathfrak{F}_S \otimes \mathcal{L}_S[E])$$

where $\mathfrak{F}_S := C^*\left(\bigcup_{x \in \Lambda_S^0} \mathfrak{F}_x\right)$ and $\mathcal{L}_S[E] := C^*\left(\bigcup_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{E}_S[\mathbf{n}]\right)$.

Proof: Now the field algebra $\mathfrak{A}_\Lambda = \mathfrak{F}_\Lambda \otimes \mathcal{L}[E] = (\varinjlim \mathfrak{F}_S) \otimes (\varinjlim \mathcal{L}_{S'}[E])$ and we want to show that this is isomorphic to $\varinjlim (\mathfrak{F}_S \otimes \mathcal{L}_S[E])$. Note first that for a fixed $S \in \mathcal{S}$ that $\text{Span}\{F \otimes L \mid F \in \mathfrak{F}_S, L \in \mathcal{L}_S[E]\} \subset \mathfrak{A}_\Lambda$ is the algebraic tensor product of \mathfrak{F}_S with $\mathcal{L}_S[E]$, and that the restriction of the C*-norm of \mathfrak{A}_Λ to this is still a cross-norm (as it is one on the full algebra). Thus the closure of this space in \mathfrak{A}_Λ is precisely $\mathfrak{F}_S \otimes \mathcal{L}_S[E] =: \mathfrak{A}_S$ as this algebra has a unique tensor norm. By construction we have that $\mathfrak{A}_{S_1} \subseteq \mathfrak{A}_{S_2}$ if $S_1 \subseteq S_2$, and the *-algebra $\bigcup_{S \in \mathcal{S}} \mathfrak{A}_S$ contains all of $(\bigcup_{x \in \Lambda^0} \mathfrak{F}_x) \otimes \bigcup_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{L}[\mathbf{n}]_0$, hence it is dense. Thus $\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_S = \varinjlim (\mathfrak{F}_S \otimes \mathcal{L}_S[E])$ as required. \blacksquare

Recall though that the algebras $\mathcal{L}_S[E]$ are not the local algebras $\bigotimes_{\ell_k \in \Lambda_S^1} \mathcal{L}_k \subset M(\mathcal{L}[E])$, since the elementary tensors $A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} \in \mathcal{E}_S[\mathbf{n}]$ generating the $\mathcal{L}_S[E]$ contain the extra parts $E[\mathbf{n}]_{k+1}$. As remarked above, this is not a serious problem because a representation (resp. state) on $\mathcal{L}[E]$ extends uniquely to $M(\mathcal{L}[E])$ on the same representation space (resp. as a state), and hence to subalgebras of $M(\mathcal{L}[E])$. Thus $\mathcal{L}[E]$ determines states and representations on all the local algebras $\bigotimes_{\ell_k \in \Lambda_S^1} \mathcal{L}_k$.

3 Gauge transformations and the local Gauss law

We next consider the gauge transformations. Classically, $\text{Gau } P = C^\infty(M, G)$ because $P = \mathbb{R}^3 \times G$ is trivial. However, $M = \mathbb{R}^3$ is not compact, and in this case it is customary to assume that local gauge transformations are of compact support (cf. [18]). The global gauge transformations are taken to be the constant maps $\gamma : M \rightarrow G$ (for nontrivial P global gauge transformations need not exist).

3.1 Local gauge transformations.

As the local gauge transformations are of compact support, they restrict on the lattice Λ^0 to the group of maps $\gamma : \Lambda^0 \rightarrow G$ of finite support, i.e.

$$\text{Gau } \Lambda := G^{(\Lambda^0)} = \{\gamma : \Lambda^0 \rightarrow G \mid |\text{supp}(\gamma)| < \infty\}, \quad \text{supp}(\gamma) := \{x \in \Lambda^0 \mid \gamma(x) \neq e\}.$$

This is an inductive limit indexed by the finite subsets $S \subset \Lambda^0$, of the subgroups $\text{Gau}_S \Lambda := \{\gamma : \Lambda^0 \rightarrow G \mid \text{supp}(\gamma) \subseteq S\} \cong \prod_{x \in S} G$, and we give it the inductive limit topology. As the groups $\prod_{x \in S} G$ are compact, $\text{Gau } \Lambda$ is amenable, hence any continuous automorphic action of it on a C*-algebra will have an invariant state. Moreover, as G is connected, so is any finite product $\prod_{x \in S} G$, and as every element of $\text{Gau } \Lambda$ is in one of these, $\text{Gau } \Lambda$ is connected (a more general result is in Prop. 4.4 of [8]). By choosing a strictly increasing chain of finite subsets $S \subset \Lambda^0$ with union Λ^0 , we conclude from

[8] that the inductive limit $\text{Gau } \Lambda$ is an infinite dimensional Lie group, with (infinite dimensional) Lie algebra

$$\mathfrak{gau } \Lambda = \mathfrak{g}^{(\Lambda^0)} = \{\nu : \Lambda^0 \rightarrow \mathfrak{g} \mid |\text{supp}(\nu)| < \infty\} = \text{Span}\{Y \cdot \delta_x \mid Y \in \mathfrak{g}, x \in \Lambda^0\}$$

where $\delta_x : \Lambda^0 \rightarrow \mathbb{R}$ is $\delta_x(y) = 1$ if $y = x$ and zero otherwise.

Next, we consider the action of the gauge group on the lattice. The action of $\gamma \in \text{Gau } \Lambda$ on classical configuration space $(\prod_{x \in \Lambda^0} \mathbf{V}) \times (\prod_{\ell \in \Lambda^1} G)$ is by

$$\left(\prod_{x \in \Lambda^0} v_x\right) \times \left(\prod_{\ell \in \Lambda^1} g_\ell\right) \mapsto \left(\prod_{x \in \Lambda^0} \gamma(x) \cdot v_x\right) \times \left(\prod_{\ell \in \Lambda^1} \gamma(x_\ell) g_\ell \gamma(y_\ell)^{-1}\right) \quad \text{where } \ell = (x_\ell, y_\ell).$$

For the quantum case, we define an analogous action $\alpha : \text{Gau } \Lambda \rightarrow \text{Aut } \mathfrak{A}_\Lambda$ as follows. Using the tensor structure $\mathfrak{A}_\Lambda = \mathfrak{F}_\Lambda \otimes \mathcal{L}[E]$, we will define α as a product action:

$$\alpha_\gamma := \alpha_\gamma^1 \otimes \alpha_\gamma^2 \quad \text{where} \quad \alpha^1 : \text{Gau } \Lambda \rightarrow \text{Aut } \mathfrak{F}_\Lambda \quad \text{and} \quad \alpha^2 : \text{Gau } \Lambda \rightarrow \text{Aut } \mathcal{L}[E]$$

for $\gamma \in \text{Gau } \Lambda$. The first component of the action is given by:

$$\alpha_\gamma^1(a(f)) := a(\gamma \cdot f) \quad \text{where} \quad (\gamma \cdot f)(x) := \gamma(x)f(x) \quad \text{for all } x \in \Lambda^0, f \in \ell^2(\Lambda^0, \mathbf{V})$$

since $f \mapsto \gamma \cdot f$ defines a unitary on $\ell^2(\Lambda^0, \mathbf{V})$.

For the second component action α^2 , we first show how to define it on an individual tensor factor $\mathcal{L}_k = C(G) \rtimes_\lambda G$ of $\mathcal{L}[E]$. Fix a pair $x, y \in \Lambda^0$ and let:

$$\tau : \text{Gau } \Lambda \rightarrow \text{Aut } C(G) \quad \text{be} \quad (\tau_\gamma f)(g) := f(\gamma(x)^{-1}g\gamma(y))$$

which corresponds to the classical action on G . Since $\tau_\gamma \circ \lambda_h = \lambda_{\gamma(x)h\gamma(y)^{-1}} \circ \tau_\gamma$, recalling that $C(G) \rtimes_\lambda G$ is generated by $\psi \in L^1(G, C(G))$, we extend τ_γ to an automorphism on $C(G) \rtimes_\lambda G$ by setting $(\theta_\gamma(\psi))(g) := \tau_\gamma(\psi(\gamma(x)^{-1}g\gamma(y)))$. Since the product and adjoint in $L^1(G, C(G)) \subset C(G) \rtimes_\lambda G$ are given by

$$\begin{aligned} (\psi_1 \times \psi_2)(g) &:= \int \psi_1(s) \lambda_s(\psi_2(s^{-1}g)) ds \\ \psi^*(g) &:= \lambda_g(\psi(g^{-1})^*) \end{aligned}$$

it is clear by straightforward verification that θ is an automorphic action. In fact, as it only uses the evaluations of γ at two points, it is a compact action

$$\theta : G \times G \rightarrow \text{Aut}(C(G) \rtimes_\lambda G).$$

This can be simplified by recalling that the crossed product $C(G) \rtimes_\lambda G$ is just the closure of the space spanned by $L^1(G) \cdot C(G)$, using the canonical containments $L^1(G) \subset C^*(G) \subset M(C(G) \rtimes_\lambda G) \supset C(G)$ (cf. Thm 2.6.1 in [43]). Thus, if we consider $\varphi \cdot f \in L^1(G) \cdot C(G)$ for $\varphi \in L^1(G)$, $f \in C(G)$ then

$$\theta_\gamma(\varphi \cdot f) = \sigma_\gamma(\varphi) \cdot \tau_\gamma(f) \quad \text{where} \quad \sigma_\gamma(\varphi)(g) := \varphi(\gamma(x)^{-1}g\gamma(y)) \quad \text{and}$$

$(\tau_\gamma f)(g) := f(\gamma(x)^{-1}g\gamma(y))$ is as above. Thus $d\theta(\nu) = d\sigma(\nu) + d\tau(\nu)$ for $\nu \in \mathfrak{gau} \Lambda$ on the span of $(L^1(G) \cap C^\infty(G)) \cdot C^\infty(G)$. This will be useful below.

Next, to define α^2 , we combine these actions for the full algebra $\mathcal{L}[E]$. Recall that we enumerated the links $\Lambda^1 = \{\ell_n = (x_n, y_n) \mid n \in \mathbb{N}\}$, and that $\mathcal{L}[E]$ is generated by the elements

$$A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} \in \mathcal{L}^{(k)} \otimes E[\mathbf{n}]_{k+1} \quad \text{where} \quad A_i \in \mathcal{L}_i = C(G) \rtimes_\lambda G$$

(note that $\mathcal{L}^{(j)} \otimes E[\mathbf{n}]_{j+1} \subset \mathcal{L}^{(k)} \otimes E[\mathbf{n}]_{k+1}$ if $j < k$, simply by putting some $A_i = E_{n_i}^{(i)}$).

For a given $\gamma \in \text{Gau} \Lambda$ there is always an m large enough so that

$\text{supp}(\gamma) \subset \{x_n, y_n \mid n = 1, \dots, m\}$. Thus

$$\alpha_\gamma^2(A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1}) := \theta_\gamma^1(A_1) \otimes \cdots \otimes \theta_\gamma^k(A_k) \otimes E[\mathbf{n}]_{k+1} \quad \text{for all } k \geq m$$

where $\theta_\gamma^j(A_j)$ is $\theta_\gamma(A_j)$ where the pair (x, y) is replaced by $(x_j, y_j) = \ell_j$ in the definition above. Explicitly, if we let $A_j = \varphi \cdot f \in L^1(G) \cdot C(G)$, then

$$\theta_\gamma^j(A_j)(g) = \sigma_\gamma^j(\varphi) \cdot \tau_\gamma^j(f) \quad \text{where} \quad \sigma_\gamma^j(\varphi)(g) := \varphi(\gamma(x_j)^{-1}g\gamma(x_j)) \quad \text{and} \quad (3.8)$$

$(\tau_\gamma^j f)(g) := f(\gamma(x_j)^{-1}g\gamma(y_j))$. This completely defines $\alpha^2 : \text{Gau} \Lambda \rightarrow \text{Aut} \mathcal{L}[E]$ and hence $\alpha_\gamma := \alpha_\gamma^1 \otimes \alpha_\gamma^2$. Note that α is continuous w.r.t. the inductive limit topology of $\text{Gau} \Lambda$.

Remarks:

1. Note that the orientation of links in Λ^1 was used in the definition of α^2 , because the definition of θ based on a pair (x, y) treated the x and y differently.
2. The use of compact support for the gauge transformations was crucial. If one did not assume this, then it may not be possible to define α_γ^2 because γ may not map elementary tensors of the type $A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1}$ to one of the type $B_1 \otimes \cdots \otimes B_j \otimes E[\mathbf{m}]_{j+1}$ as it may not preserve the approximate identities which they are based on. This means that global gauge transformations cannot be defined, unless one chooses approximate identities $(E_n^{(k)})_{n \in \mathbb{N}}$ which are invariant with respect to the gauge action. This is what we will do in the next subsection.

3.2 Global gauge transformations.

As mentioned in the last remark, the action $\alpha^2 : \text{Gau} \Lambda \rightarrow \text{Aut} \mathcal{L}[E]$ cannot in general be extended to the constant maps, unless the $(E_n^{(k)})_{n \in \mathbb{N}}$ are chosen to be gauge invariant. We examine this issue. Recall that for $\gamma \in \text{Gau} \Lambda$, α_γ^2 is given by θ_γ^k in the k^{th} factor for all k , so we consider the invariance of $(E_n^{(k)})_{n \in \mathbb{N}}$ w.r.t. $\theta^k : \text{Gau} \Lambda \rightarrow \text{Aut} \mathcal{L}_k$. Explicitly this action is $\theta_\gamma^k(L) = \theta_{(\gamma(x_k), \gamma(y_k))}(L)$ where

$$\theta : G \times G \rightarrow \text{Aut}(C(G) \rtimes_\lambda G)$$

is given as follows. Let $\varphi \in L^1(G)$, $f \in C(G)$, then for $L = \varphi \cdot f \in L^1(G) \cdot C(G) \subset C(G) \rtimes_\lambda G$ we have

$$\theta_{(h,s)}(L) = \theta_{(h,s)}(\varphi \cdot f) = \sigma_{(h,s)}(\varphi) \cdot \tau_{(h,s)}(f) \quad \text{where} \quad \sigma_{(h,s)}(\varphi)(g) := \varphi(h^{-1}gh)$$

and $(\tau_{(h,s)}f)(g) := f(h^{-1}gs)$.

Lemma 3.1. (i) Let $\pi_0 : C(G) \rtimes_\lambda G \rightarrow \mathcal{B}(L^2(G))$ be the irreducible representation given by $\pi_0(\varphi \cdot f) = \pi_1(\varphi)\pi_2(f)$ for $\varphi \in L^1(G)$ and $f \in C(G)$ where

$$(\pi_1(\varphi)\psi)(g) := \int \varphi(h)\psi(h^{-1}g) dh \quad \text{and} \quad (\pi_2(f)\psi)(g) := f(g)\psi(g)$$

for all $\psi \in L^2(G)$ (Schrödinger representation). Then π_0 is a covariant representation for θ with unitary implementers $W_{(h,s)} \in U(L^2(G))$, $h, s \in G$, given by $(W_{(h,s)}\psi)(g) := \psi(h^{-1}gs)$. Constant vectors, i.e. $\psi(g) = c \in \mathbb{C}$ for all g are in $L^2(G)$ and invariant w.r.t. W .

(ii) There is an approximate identity of commuting projections $(E_n)_{n \in \mathbb{N}}$ for $C(G) \rtimes_\lambda G$ which is invariant w.r.t. $\theta : G \times G \rightarrow \text{Aut}(C(G) \rtimes_\lambda G)$.

Proof: (i) It is well-known that $\pi_0(C(G) \rtimes_\lambda G) = \mathcal{K}(L^2(G))$ (cf. Theorem II.10.4.3 in [2]), hence that π_0 is irreducible. Direct verification also shows that $W : G \times G \rightarrow U(L^2(G))$ is a continuous unitary representation. We verify implementation of θ :

$$\begin{aligned} (W_{(h,s)}\pi_1(\varphi)W_{(h,s)}^{-1}\psi)(g) &= (\pi_1(\varphi)W_{(h,s)}^{-1}\psi)(h^{-1}gs) \\ &= \int \varphi(t) (W_{(h,s)}^{-1}\psi)(t^{-1}h^{-1}gs) dt = \int \varphi(t) (\psi)(ht^{-1}h^{-1}g) dt \\ &= \int \varphi(h^{-1}th) (\psi)(t^{-1}g) dt = (\pi_1(\sigma_{(h,s)}(\varphi))\psi)(g) \\ (W_{(h,s)}\pi_2(f)W_{(h,s)}^{-1}\psi)(g) &= (\pi_2(f)W_{(h,s)}^{-1}\psi)(h^{-1}gs) \\ &= f(h^{-1}gs) (W_{(h,s)}^{-1}\psi)(h^{-1}gs) = (\pi_2(\tau_{(h,s)}f)\psi)(g) \end{aligned}$$

which produces $W_{(h,s)}\pi(L)W_{(h,s)}^{-1} = \pi(\theta_{(h,s)}(L))$ as required.

(ii) Since $G \times G$ is compact, the representation $W : G \times G \rightarrow U(L^2(G))$ is a direct orthogonal sum of finite dimensional irreducible representations of $G \times G$. The projections onto these finite dimensional subspaces are therefore in $\mathcal{K}(L^2(G)) = \pi_0(C(G) \rtimes_\lambda G)$, and as these projections commute with W they are invariant w.r.t. $\theta : G \times G \rightarrow \text{Aut}(C(G) \rtimes_\lambda G)$. Moreover, they form a commuting set with total sum the identity, hence by taking larger and larger sums of them we obtain the desired approximate identity. ■

Given this Lemma, one may therefore choose approximate identities $(E_n^{(k)})_{n \in \mathbb{N}}$ invariant with respect to θ , and use these to construct $\mathcal{L}[E]$. Henceforth we will assume that such

a choice has been fixed, and we assume that approximate identities $(E_n^{(k)})_{n \in \mathbb{N}}$ have been chosen such that the constant vector $\psi_0 := 1$ is in the range space of each $E_n^{(k)}$ in the Schrödinger representation π_0 .

Given this choice of approximate identities, we now have for $\mathcal{L}[E]$, that the action $\alpha : \text{Gau } \Lambda \rightarrow \text{Aut } \mathfrak{A}_\Lambda$ extends from $\text{Gau } \Lambda = G^{(\Lambda^0)}$ to all of G^{Λ^0} , which includes the constant maps, i.e global gauge transformations. In particular on $\mathfrak{A}_\Lambda = \mathfrak{F}_\Lambda \otimes \mathcal{L}[E]$, we have a product action: $\alpha_\gamma := \alpha_\gamma^1 \otimes \alpha_\gamma^2$, $\gamma \in G^{\Lambda^0}$ where as before

$$\alpha_\gamma^1(a(f)) := a(\gamma \cdot f) \quad \text{where} \quad (\gamma \cdot f)(x) := \gamma(x)f(x) \quad \text{for all } x \in \Lambda^0, f \in \ell^2(\Lambda^0, \mathbf{V})$$

since $f \mapsto \gamma \cdot f$ defines a unitary on $\ell^2(\Lambda^0, \mathbf{V})$. Moreover, by the invariance of $(E_n^{(k)})_{n \in \mathbb{N}}$, the same formula

$$\alpha_\gamma^2(A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1}) := \theta_\gamma^1(A_1) \otimes \cdots \otimes \theta_\gamma^k(A_k) \otimes E[\mathbf{n}]_{k+1}$$

is valid, but now for all $\gamma \in G^{\Lambda^0}$. So global gauge transformations are given by α_γ where $\gamma(x) = g \in G$ for all $x \in \Lambda^0$.

Remarks:

1. Recall from Proposition 2.5 that for a directed set \mathcal{S} of open, bounded convex subsets of \mathbb{R}^3 such that $\bigcup_{S \in \mathcal{S}} S = \mathbb{R}^3$, then

$$\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_S = \varinjlim (\mathfrak{F}_S \otimes \mathcal{L}_S[E])$$

where $\mathfrak{F}_S := C^*(\bigcup_{x \in \Lambda_S^0} \mathfrak{F}_x)$ and $\mathcal{L}_S[E] := C^*(\bigcup_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{E}_S[\mathbf{n}])$. With the choice of invariant approximate identities $(E_n^{(k)})_{n \in \mathbb{N}}$ above, it is clear that the extended action $\alpha : G^{\Lambda^0} \rightarrow \text{Aut } \mathfrak{A}_\Lambda$ preserves each of the “local” algebras $\mathfrak{A}_S = \mathfrak{F}_S \otimes \mathcal{L}_S[E]$. Moreover, a “local” algebra \mathfrak{A}_S cannot tell the global gauge transformations apart from certain local gauge transformations. That is, given any global gauge transformation α_γ where $\gamma(x) = g \in G$ for all $x \in \Lambda^0$ and a “local” algebra \mathfrak{A}_S , then there is a $\gamma_{\text{loc}} \in \text{Gau } \Lambda$ such that $\alpha_\gamma|_{\mathfrak{A}_S} = \alpha_{\gamma_{\text{loc}}}|_{\mathfrak{A}_S}$, for example take $\gamma_{\text{loc}}(x) = g = \gamma(x)$ if $x \in S$ and $\gamma_{\text{loc}}(x) = e$ if $x \notin S$. This is not true for the full algebra \mathfrak{A}_Λ because given a global gauge transformation α_γ , we cannot find a $\gamma_{\text{loc}} \in \text{Gau } \Lambda$ which will work for all $\mathfrak{A}_S \subset \mathfrak{A}_\Lambda$.

2. From Lemma 3.1, we obtain a very natural representation for $\mathcal{L}[E]$ with the choice of approximate identity made here. For $\mathfrak{A}_\Lambda = \mathfrak{F}_\Lambda \otimes \mathcal{L}[E]$ define a product representation $\pi = \pi_{\text{Fock}} \otimes \pi_\infty$ where π_{Fock} is the Fock representation of $\mathfrak{F}_\Lambda = \text{CAR}(\mathcal{H})$, and π_∞ is an infinite tensor product of Schrödinger representations π_0 (one for each factor \mathcal{L}_ℓ of $\mathcal{L}[E]$), but where we choose the reference sequence to be just the sequence (ψ_0, ψ_0, \dots) where $\psi_0 = 1$ is the constant vector. This means we can

consider the representation space \mathcal{H}_∞ of π_∞ to be spanned by elementary tensors of the type

$$\varphi_1 \otimes \cdots \otimes \varphi_k \otimes \psi_0 \otimes \psi_0 \otimes \cdots, \quad \varphi_i \in L^2(G).$$

Then, if we consider the action of $\mathcal{L}[E]$ on it, we see

$$\begin{aligned} \pi_\infty(A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1})(\varphi_1 \otimes \cdots \otimes \varphi_k \otimes \psi_0 \otimes \psi_0 \otimes \cdots) \\ = \pi_0(A_1)\varphi_1 \otimes \cdots \otimes \pi_0(A_k)\varphi_k \otimes \psi_0 \otimes \psi_0 \otimes \cdots \end{aligned}$$

because $\pi_0(E_n^{(j)})\psi_0 = \psi_0$ for all n and j . Hence all of $\mathcal{L}[E]$ can be represented on \mathcal{H}_∞ . In fact, since each factor of the representation is covariant, and ψ_0 is an invariant vector, we also get that π is covariant w.r.t. $\alpha : G^{\Lambda^0} \rightarrow \text{Aut } \mathfrak{A}_\Lambda$, and it has an invariant vector $\Omega \otimes (\psi_0 \otimes \psi_0 \otimes \cdots)$ where Ω is the Fock vacuum vector. Thus the vector state of this vector is a $\alpha(G^{\Lambda^0})$ -invariant state on \mathfrak{A}_Λ . This is interesting as this means that we have an invariant state for the much larger group action $\alpha : G^{\Lambda^0} \rightarrow \text{Aut } \mathfrak{A}_\Lambda$, not just for its restriction to the amenable group $\text{Gau } \Lambda$.

We claim that the representation $\pi = \pi_{\text{Fock}} \otimes \pi_\infty$ is faithful. Since π_{Fock} is already known to be faithful, we only have to show that π_∞ is faithful (since the tensor norm for $\mathfrak{F}_\Lambda \otimes \mathcal{L}[E]$ is unique, using Theorem 4.9(iii), p208 in [38]). Recall that $\mathcal{L}[E]$ is the C*-algebra constructed from all $\mathcal{L}[\mathbf{n}] \subseteq M(\mathcal{L}[\mathbf{1}])$ in $M(\mathcal{L}[\mathbf{1}])$, hence we have a faithful embedding $\mathcal{L}[E] \subset M(\mathcal{L}[\mathbf{1}])$. Now the restriction $\pi_\infty|_{\mathcal{L}[\mathbf{1}]}$ is faithful as $\mathcal{L}[\mathbf{1}]$ is simple and π_∞ is nonzero on it. But then the extension of π_∞ to $M(\mathcal{L}[\mathbf{1}])$ is faithful, hence π_∞ is faithful on $\mathcal{L}[E]$.

3.3 Defining the full Field algebra.

There is physical information contained in the gauge action $\alpha : G^{\Lambda^0} \rightarrow \text{Aut } \mathfrak{A}_\Lambda$ as $\alpha(\text{Gau } \Lambda)$ is the local gauge transformations and $\alpha(G)$ is the global gauge transformations (identifying G with the constant maps in G^{Λ^0}). It is therefore desirable to extend the field algebra \mathfrak{A}_Λ to ensure that in physical representations, the generators of the unitary implementers of α are affiliated to our field algebra. Usually, one takes the crossed product, but in this context e.g. the crossed product “ $\mathfrak{A}_\Lambda \rtimes_\alpha (\text{Gau } \Lambda)$ ” cannot be defined because $\text{Gau } \Lambda$ is not locally compact. In fact for non-locally compact groups, it is a very hard question as to what C*-algebra should play the role of the crossed product. In such a situation, the best one can do at the moment, is to endow the given group with the discrete topology, which makes it locally compact, and then to use the crossed product w.r.t. this discrete group. This has the disadvantage of having too many representations, in particular it allows those covariant representations where the unitary implementers are not continuous w.r.t. the original group topology. In the present context one may argue that as the gauge transformations will be factored out by a constraint procedure, the topology of the gauge group is not physically relevant.

Concretely, our strategy is as follows. Let $\text{Gau}^e \Lambda$ denote our chosen group in G^{Λ^0} of physically relevant transformations (this should at least contain the local gauge transformations $\text{Gau} \Lambda \subset G^{\Lambda^0}$). Let $\text{Gau}_d^e \Lambda$ denote $\text{Gau}^e \Lambda$ equipped with the *discrete* topology. Then take the discrete crossed product $\mathfrak{A}_\Lambda \rtimes_\alpha (\text{Gau}_d^e \Lambda)$. As it is convenient to have an identity in our field algebra, we will take instead

$$\mathcal{F}_e := (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}_d^e \Lambda)$$

where $\mathfrak{A}_\Lambda \oplus \mathbb{C}$ denotes \mathfrak{A}_Λ with an identity adjoined. It is generated as a C^* -algebra by a copy of \mathfrak{A}_Λ as well as by unitaries U_g , $g \in \text{Gau}^e \Lambda$ such that $U_g A U_g^* = \alpha_g(A)$ for all $A \in \mathfrak{A}_\Lambda$, and $U_g U_h = U_{gh}$. Algebraically

$$\begin{aligned} \mathcal{F}_e &= (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}_d^e \Lambda) = C^*(U_{\text{Gau}_d^e \Lambda} \cup \mathfrak{A}_\Lambda) \quad \text{where} \quad \mathfrak{A}_\Lambda := \mathfrak{F}_\Lambda \otimes \mathcal{L}[E] \\ &= [U_{\text{Gau}_d^e \Lambda} \cdot \mathfrak{A}_\Lambda] + [U_{\text{Gau}_d^e \Lambda}] \end{aligned}$$

where we use the notation $[\cdot]$ for the closed linear space generated by its argument. The representations of \mathcal{F}_e consist of *all* covariant representations for $\alpha : \text{Gau}^e \Lambda \rightarrow \text{Aut}(\mathfrak{A}_\Lambda \oplus \mathbb{C})$, whether continuous or not.

The natural choice for our full field algebra, is \mathcal{F}_e where we take $\text{Gau}^e \Lambda$ to be the group generated in G^{Λ^0} by $\text{Gau} \Lambda$ and G (the constant maps in G^{Λ^0}), as this will include both local and global gauge transformations. However, with our eye on the subsequent work below (enforcing constraints) we will make the smaller choice where we take \mathcal{F}_e with $\text{Gau}^e \Lambda = \text{Gau} \Lambda$. The reason why we will not include unitaries corresponding to global gauge transformations, is because locally these implement the same automorphisms as some local gauge transformations (see remark (1) at end of Subsect. 3.2). Thus, if we enforce local gauge invariance through constraints, then the images of these unitaries will commute with all the local algebras, hence with the image of \mathfrak{A}_Λ , and hence will be of no physical relevance. Thus, to conclude, henceforth for our full field algebra we will take

$$\mathcal{F}_e = (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}_d \Lambda) = C^*(U_{\text{Gau}_d \Lambda} \cup \mathfrak{A}_\Lambda) .$$

3.4 The local Gauss law.

We consider the local gauge transformations. Given the action $\alpha : \text{Gau} \Lambda \rightarrow \text{Aut} \mathfrak{A}_\Lambda$ defined above, an (abstract) Gauss law element will be a nonzero element in the range of the derived action $d\alpha : \mathfrak{gau} \Lambda \rightarrow \text{Der}(\mathfrak{A}_\Lambda^\infty)$ where $\mathfrak{A}_\Lambda^\infty$ is the algebra of smooth elements of the action. Since $\alpha_\gamma := \alpha_\gamma^1 \otimes \alpha_\gamma^2$, it is of the form

$$d\alpha(\nu) = d\alpha^1(\nu) \otimes \mathbb{1} + \mathbb{1} \otimes d\alpha^2(\nu), \quad \nu \in \mathfrak{gau} \Lambda, \quad \text{on} \quad \mathfrak{F}_\Lambda^\infty \otimes \mathcal{L}[E]^\infty \subseteq \mathfrak{A}_\Lambda^\infty,$$

i.e. it is a sum of a matter part and a radiation part. The Gauss law condition, is simply the enforcement of of it as a constraint, i.e. setting it to zero in an appropriate way. We

will investigate this below. Concrete Gauss law elements consist of implementers of the derivations $d\alpha(\nu)$ in α -covariant representations by selfadjoint operators, and clearly these will be the generators of the unitaries implementing the one-parameter groups $t \mapsto \alpha(\exp(t\nu))$.

To obtain an explicit form for $d\alpha(\nu)$, recall that $\mathfrak{gau}\Lambda$ consists of finite spans of elements $\nu = Y \cdot \delta_x$ for $Y \in \mathfrak{g}$, $x \in \Lambda^0$, and these are the generators of the one-parameter groups $t \mapsto \exp(tY \cdot \delta_x) \in \text{Gau}\Lambda$. Thus the matter part of the Gauss law, $d\alpha^1(Y \cdot \delta_x) \in \text{Der}(\mathfrak{F}_\Lambda^\infty)$, is given by

$$d\alpha^1(Y \cdot \delta_x)(a(f)) = \frac{d}{dt} a(\exp(tY \cdot \delta_x)f) \Big|_{t=0} = a(\delta_x \cdot Yf) \in \mathfrak{F}_x = \text{CAR}(V_x).$$

In fact, as \mathbf{V} is finite dimensional, this is defined for all $f \in \mathfrak{F}_\Lambda$, hence $\mathfrak{F}_\Lambda^\infty$ contains the dense $*$ -algebra generated in \mathfrak{F}_Λ by the set $\{a(f) \mid f \in \ell^2(\Lambda^0, \mathbf{V})\}$.

Next, we consider the radiation part of the Gauss law, hence α^2 . Recall that we have enumerated the links $\Lambda^1 = \{\ell_n = (x_n, y_n) \mid n \in \mathbb{N}\}$, and that for each link $\ell_k = (x_k, y_k)$ there is an action $\theta^k : \text{Gau}\Lambda \rightarrow \text{Aut}\mathcal{L}_k$ where θ_γ^k only depends on $\gamma(x_k)$ and $\gamma(y_k)$. Thus $\alpha^2(\gamma) = \alpha^2(\exp(tY \cdot \delta_x))$ will only affect the links which contain x . Let $L(x) := \{k \in \mathbb{N} \mid \ell_k = (x, y_k) \text{ or } \ell_k = (x_k, x)\}$. As Λ is a cubic lattice, there are at most 6 links connected to a vertex x so $|L(x)| \leq 6$, hence $L(x) = \{k_1, k_2, \dots, k_j\}$ where $j \leq 6$. Then

$$\alpha_\gamma^2\left(\bigotimes_{i=1}^n A_i \otimes E[\mathbf{n}]_n\right) = A_1 \otimes \dots \otimes A_{k_1-1} \otimes \theta_\gamma^{k_1}(A_{k_1}) \otimes A_{k_1+1} \otimes \dots \otimes A_{k_j-1} \otimes \theta_\gamma^{k_j}(A_{k_j}) \otimes A_{k_j+1} \otimes \dots \otimes A_n \otimes E[\mathbf{n}]_n$$

for $n > k_j$, and so

$$d\alpha^2(Y \cdot \delta_x) = \sum_{k \in L(x)} d\theta^k(Y \cdot \delta_x) = \sum_{k \in L(x)} \left(d\sigma^k(Y \cdot \delta_x) + d\tau^k(Y \cdot \delta_x) \right)$$

since $d\theta^k(\nu) = d\sigma^k(\nu) + d\tau^k(\nu)$ for $\nu \in \mathfrak{gau}\Lambda$ on the span of $(L^1(G) \cap C^\infty(G)) \cdot C^\infty(G)$. In particular, if $k \in L_1(x) := \{k \in L(x) \mid \ell_k = (x, y_k)\}$, then from (3.8) we get for $A_k = \varphi \cdot f \in (L^1(G) \cap C^\infty(G)) \cdot C^\infty(G)$ that

$$\theta_\gamma^k(A_k)(g) = \sigma_\gamma^k(\varphi) \cdot \tau_\gamma^k(f) \quad \text{where} \quad \sigma_\gamma^k(\varphi)(g) := \varphi(e^{-tY} g e^{tY}) \quad \text{and}$$

$(\tau_\gamma^k f)(g) := f(e^{-tY} g)$ as $\gamma(x) = \exp(tY)$. Then clearly

$$\begin{aligned} d\tau^k(Y \cdot \delta_x)(f)(g) &= \frac{d}{dt} f(e^{-tY} g) \Big|_{t=0} = -(\tilde{Y}f)(g) = -df(\tilde{Y})(g) \quad \text{and} \\ d\sigma^k(Y \cdot \delta_x)(\varphi)(e^Z) &= \frac{d}{dt} \varphi(e^{-tY} e^Z e^{tY}) \Big|_{t=0} = \frac{d}{dt} \varphi(\exp(-t \text{ad}_Y(Z))) \Big|_{t=0} \\ &= \text{ad}_Y(Z)(\varphi)(e^Z) \quad \forall Z \in \mathfrak{g} \end{aligned}$$

where \tilde{Y} is the right invariant vector field on G generated by $t \rightarrow e^{tY}g$. On the other hand, if $k \in L_2(x) := L(x) \setminus L_1(x) = \{k \in L(x) \mid \ell_k = (x_k, x)\}$ then σ_γ^k is the identity, so

$$\theta_\gamma^k(A_k)(g) = \varphi \cdot \tau_\gamma^k(f) \quad \text{where} \quad (\tau_\gamma^k f)(g) := f(g e^{tY})$$

hence
$$d\tau^k(Y \cdot \delta_x)(f)(g) = \frac{d}{dt} f(g e^{tY}) \Big|_{t=0} = (Yf)(g) = df(Y)(g).$$

So all components of the Gauss law elements have been made explicit

$$d\alpha(Y \cdot \delta_x) = d\alpha^1(Y \cdot \delta_x) \otimes \mathbb{1} + \mathbb{1} \otimes \sum_{k \in L(x)} \left(d\sigma^k(Y \cdot \delta_x) + d\tau^k(Y \cdot \delta_x) \right) \quad \forall x \in \Lambda^0, Y \in \mathfrak{g}.$$

For the algebra $\mathfrak{F}_\Lambda^\infty \otimes \mathcal{L}[E]^\infty \subseteq \mathfrak{A}_\Lambda^\infty$ on which this acts, the first factor $\mathfrak{F}_\Lambda^\infty$ contains $\ast\text{-alg}\{a(f) \mid f \in \ell^2(\Lambda^0, \mathbf{V})\}$, and the second factor $\mathcal{L}[E]^\infty$ contains the span of all elementary tensors $A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} \in \mathcal{L}[E]$ such that $A_j \in (L^1(G) \cap C^\infty(G)) \cdot C^\infty(G) \subset \mathcal{L}_j$ for all j .

Note that the Gauss law elements will only be represented concretely in representations π for which $t \rightarrow \pi(U_{\exp(t\nu)})$ is continuous for all $\nu \in \mathfrak{gau} \Lambda$. Moreover, in the case that G is abelian, i.e. $G = \mathbb{T}$ (electromagnetism) we see that $d\sigma^k = 0$ for all k , which simplifies the last expression.

4 Enforcement of local Gauss law Constraints.

Here we want to obtain the algebra of physical observables from our chosen field algebra $\mathcal{F}_e = (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}_d \Lambda)$ by enforcing the local Gauss law constraint, and by imposing gauge invariance in an appropriate form. There is a range of methods in the literature for enforcing constraints, but here we will consider two:

- The method developed by Kijowski and Rudolph in [20], and this is summarized below, following Theorem 4.5.
- The **T-procedure** developed by Grundling and Hurst (reviewed in [10]) is based on enforcing the constraints as state conditions in the universal representation. It is based on Dirac's method for enforcement of constraints, and it is summarized below in Subsection 4.2.

For the case of a finite lattice, we will show below in Theorem 4.13 that these two methods produce the same result. Here we want to apply these methods to the system constructed in Section 3.

4.1 Heuristic constraint method.

The best established method in physics for extracting the observables from a gauge theory, is based on the Gupta-Bleuler method for QEM, which we now review. (We will not consider BRST-methods, as there are several non-equivalent methods, and they are hard to cast in C*-algebraic format [5]). It has the following special features.

- (1) The theory is represented on an indefinite inner product Fock–Krein space \mathcal{H} . There is a representation of the Poincare group on \mathcal{H} , and the Krein inner product on \mathcal{H} is invariant w.r.t. the Poincaré transformations, but not the Hilbert inner product. The use of an indefinite inner product space, is required by a number of theorems, e.g. a covariant representation of a vector potential which is weakly local cannot be a normal Hilbert space representation, it must be done w.r.t. an indefinite inner product (cf. [42]).
- (2) Gauge invariance is imposed as a state condition, by the noncausal constraint

$$\chi(x) := \left(\partial^\mu A_\mu \right)^{(+)}(x) = -i(2(2\pi)^3)^{-\frac{1}{2}} \int_{C_+} p^\mu a_\mu(\mathbf{p}) e^{-ip \cdot x} \frac{d^3 p}{p_0}.$$

It is necessary to use this constraint since the canonical commutation relations prohibit nonzero solutions for full constraint $\partial^\mu A_\mu(x)$.

- (3) Then Maxwell’s equations (in terms of the vector potential) are imposed as state conditions instead of as operator identities. This is necessary, because from the work of Strocchi (e.g. [35, 36]), we know that Maxwell’s equations are incompatible with the Lorentz covariance of the vector potential.

The constraint selects the physical subspace

$$\mathcal{H}' := \{ \psi \in \mathcal{H} \mid \chi(h)\psi = 0, h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}) \}.$$

Then \mathcal{H}' is positive semidefinite w.r.t. the Krein inner product $\langle \cdot, \cdot \rangle$, so the heuristic theory constructs the physical Hilbert space $\mathcal{H}_{\text{phys}}$ as the closure of $\mathcal{H}'/\mathcal{H}''$ equipped with inner product $\langle \cdot, \cdot \rangle$ where \mathcal{H}'' is the zero norm part of \mathcal{H}' . At the one particle level, \mathcal{H}' consists of functions satisfying $p_\mu f^\mu(p) = 0$, and \mathcal{H}'' consists of gradients $f_\mu(p) = ip_\mu h(p)$. The physical observables consist of operators which can factor to $\mathcal{H}_{\text{phys}}$, and in particular contains the field operators $F_{\mu\nu}$. These satisfy the Maxwell equations on $\mathcal{H}_{\text{phys}}$, because $F_{\mu\nu}{}^{;\mu}$ maps \mathcal{H}' to \mathcal{H}'' .

The heuristic theory above, has been cast into C*-algebra format cf. [14]. The T-procedure has also been applied to Gupta-Bleuler electromagnetism (cf. [14]), and it was found that one could avoid the use of indefinite metric representations, by allowing the use of nonregular states for the Weyl algebra. This sidestepped the theorems which require indefinite metric for gauge theories, and it produced exactly the same final

algebra of physical observables and representation, than what one obtains from the usual Gupta-Bleuler method.

We also remark that for gauge theory on a lattice, the use of an indefinite metric is again not required, and in e.g. [20], it was sufficient to constrain in ordinary Hilbert space representations.

4.2 Enforcing constraints by T-procedure - method.

In this section we review the T-procedure for the enforcement of constraints, and we show that the system defined in Section 3 satisfies its input assumptions. A convenient review of the T-procedure is in [10]. The starting point is:

Definition 4.1. *A quantum system with constraints is a pair $(\mathcal{F}, \mathcal{C})$ where the field algebra \mathcal{F} is a unital C^* -algebra containing the constraint set $\mathcal{C} = \mathcal{C}^*$. A constraint condition on $(\mathcal{F}, \mathcal{C})$ consists of the selection of the physical state space by:*

$$\mathfrak{S}_D := \left\{ \omega \in \mathfrak{S}(\mathcal{F}) \mid \pi_\omega(C)\Omega_\omega = 0 \quad \forall C \in \mathcal{C} \right\},$$

where $\mathfrak{S}(\mathcal{F})$ denotes the state space of \mathcal{F} , and $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ denotes the GNS-data of ω . The elements of \mathfrak{S}_D are called **Dirac states**. The case of **unitary constraints** means that $\mathcal{C} = \mathcal{U} - \mathbb{1}$ for a set of unitaries $\mathcal{U} \subset \mathcal{F}_u$, and for this we will also use the notation $(\mathcal{F}, \mathcal{U})$.

Thus in the GNS-representation of each Dirac state, the GNS cyclic vector Ω_ω satisfies the physical selection condition for the physical states, e.g. for \mathcal{H}' above. The assumption is that all physical information is contained in the pair $(\mathcal{F}, \mathfrak{S}_D)$.

In our case, of the system defined in Section 3, we will take the field algebra defined above: $\mathcal{F}_e := (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_{\alpha} (\text{Gau}_d \Lambda)$. In representations π for which $t \rightarrow \pi(U_{\exp(t\nu)})$ is continuous for all $\nu \in \mathfrak{gau} \Lambda$, the concrete Gauss law elements $\pi(d\alpha(\nu)) \in \text{Der}(\pi(\mathfrak{F}_\Lambda^\infty \otimes \mathcal{L}[E]^\infty))$ are given by $\pi(d\alpha(\nu))(A) = i[B_\nu, A]$ for $A \in \pi(\mathfrak{F}_\Lambda^\infty \otimes \mathcal{L}[E]^\infty)$ where $\pi(U_{\exp(t\nu)}) = \exp(itB_\nu)$. These are enforced as state constraints by selecting the physical subspace by the condition $B_\nu\psi = 0$. This condition is the same as $\pi(U_{\exp(t\nu)})\psi = \psi$ for all $t \in \mathbb{R}$. As G is a compact connected Lie group, each element in G is an exponential, hence this also holds for any finite product of G 's and hence for $\text{Gau} \Lambda$. Thus the condition $B_\nu\psi = 0$ for all ν is the same as $\pi(U_g)\psi = \psi$ for all $g \in \text{Gau} \Lambda$. This justifies our choice for constraint set as $\mathcal{C} = U_{\text{Gau} \Lambda} - \mathbb{1}$, i.e. we have the case of unitary constraints with $\mathcal{U} = U_{\text{Gau} \Lambda}$. Our system with unitary constraints is the pair $(\mathcal{F}_e, U_{\text{Gau} \Lambda})$.

For the general case of unitary constraints $(\mathcal{F}, \mathcal{U})$, we have the following equivalent characterizations of the Dirac states (cf. [11, Theorem 2.19 (ii)]):

$$\mathfrak{S}_D = \left\{ \omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(U) = 1 \quad \forall U \in \mathcal{U} \right\} \quad (4.9)$$

$$= \left\{ \omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(FU) = \omega(F) = \omega(UF) \quad \forall F \in \mathcal{F}, U \in \mathcal{U} \right\}. \quad (4.10)$$

From these, we note that \mathfrak{S}_D is already selected by any set $\mathcal{U}_0 \subset \mathcal{F}_u$ which generates the same group as \mathcal{U} in \mathcal{F}_u . In particular, in the context of our lattice model, a useful generating subset of $U_{\text{Gau}\Lambda}$ is

$$\mathcal{U}_0 := \{U_{\exp(t\nu)} \mid t \in \mathbb{R}, \nu = Y \cdot \delta_x \text{ for all } Y \in \mathfrak{g}, x \in \Lambda^0\},$$

and in fact the system we will analyze below is $(\mathcal{F}_e, \mathcal{U}_0)$.

Observe that (4.10) shows that $\alpha_{\text{Gau}\Lambda}$ leaves every Dirac state invariant, i.e. we have $\omega \circ \alpha_g = \omega$ for all $\omega \in \mathfrak{S}_D$, $g \in \text{Gau}\Lambda$. Since \mathcal{F}_e is a crossed product, on the kinematic field algebra $\mathfrak{A}_\Lambda \subset \mathcal{F}_e$ we also have the converse:

Proposition 4.2. *For the system above with unitary constraints $(\mathcal{F}_e, U_{\text{Gau}\Lambda})$, we have that $\mathfrak{S}_D \upharpoonright \mathfrak{A}_\Lambda = \mathfrak{S}^{\text{Gau}} \upharpoonright \mathfrak{A}_\Lambda$ where $\mathfrak{S}^{\text{Gau}} = \left\{ \omega \in \mathfrak{S}(\mathcal{F}_e) \mid \omega \circ \alpha_g = \omega \quad \forall g \in \text{Gau}\Lambda \right\}$, and where α_g was extended from \mathfrak{A}_Λ to \mathcal{F}_e by setting it to be $\alpha_g = \text{Ad}(U_g)$.*

Proof: We already know that $\mathfrak{S}_D \subseteq \mathfrak{S}^{\text{Gau}}$, hence that $\mathfrak{S}_D \upharpoonright \mathfrak{A}_\Lambda \subseteq \mathfrak{S}^{\text{Gau}} \upharpoonright \mathfrak{A}_\Lambda$. We only need to prove the inclusion $\mathfrak{S}^{\text{Gau}} \upharpoonright \mathfrak{A}_\Lambda \subseteq \mathfrak{S}_D \upharpoonright \mathfrak{A}_\Lambda$, i.e. that any $\omega \in \mathfrak{S}^{\text{Gau}} \upharpoonright \mathfrak{A}_\Lambda$ has an extension to \mathcal{F}_e as a Dirac state. First recall that

$$\mathcal{F}_e = (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}_d \Lambda) = [U_{\text{Gau}_d \Lambda} \cdot (\mathfrak{A}_\Lambda \oplus \mathbb{C})]$$

where we use the notation $[\cdot]$ for the closed linear space generated by its argument. Now $\text{Gau}\Lambda \subset \text{Gau}_d \Lambda$ where $\text{Gau}_d \Lambda$ is the (discrete) group generated by $\text{Gau}\Lambda \cup G$. Thus

$$\mathcal{F}_e^\circ := (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}\Lambda)_d = [U_{\text{Gau}\Lambda} \cdot (\mathfrak{A}_\Lambda \oplus \mathbb{C})] \subset \mathcal{F}_e$$

where $(\text{Gau}\Lambda)_d$ denotes $\text{Gau}\Lambda$ with the discrete topology. Let $\omega \in \mathfrak{S}^{\text{Gau}}(\mathfrak{A}_\Lambda)$, then by Corr. 2.3.17 [4] we obtain a covariant representation (π_ω, V^ω) of the action $\alpha : \text{Gau}\Lambda \rightarrow \text{Aut}(\mathfrak{A}_\Lambda \oplus \mathbb{C})$ such that $V_g^\omega \Omega_\omega = \Omega_\omega$ for all $g \in \text{Gau}\Lambda$. By Prop. 7.6.4 and Theorem 7.6.6 in [29] we know that this covariant pair defines a representation $\tilde{\pi} : (\mathfrak{A}_\Lambda \oplus \mathbb{C}) \rtimes_\alpha (\text{Gau}\Lambda)_d \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ by $\tilde{\pi}(A) := \pi_\omega(A)$ and $\tilde{\pi}(U_g) := V_g^\omega$ for all $A \in \mathfrak{A}_\Lambda$, $g \in \text{Gau}\Lambda$. It is obvious that this representation extends π_ω , hence we can define an extension of ω to $\mathcal{F}_e^\circ \subset \mathcal{F}_e$ by $\tilde{\omega}(F) := (\Omega_\omega, \tilde{\pi}(F)\Omega_\omega)$ for all $F \in \mathcal{F}_e^\circ$. Since $\tilde{\omega}$ is a state, and as $\tilde{\omega}(U_g) = (\Omega_\omega, V_g^\omega \Omega_\omega) = 1$ it follows that $\tilde{\omega} \in \mathfrak{S}_D$ on \mathcal{F}_e° . Since the unitary constraints $U_{\text{Gau}\Lambda} \subset \mathcal{F}_e^\circ$, any extension of $\tilde{\omega}$ to \mathcal{F}_e is still a Dirac state, and this concludes the proof. \blacksquare

Thus on the kinematical field algebra \mathfrak{A}_Λ , the Dirac states and the $\text{Gau}\Lambda$ -invariant states are the same.

The choice of the Dirac states for a constraint system $(\mathcal{F}, \mathcal{C})$, determines a lot of structure. First, let $N_\omega := \{F \in \mathcal{F} \mid \omega(F^*F) = 0\}$ be the left kernel of a state ω and let $\mathcal{N} := \bigcap \{N_\omega \mid \omega \in \mathfrak{S}_D\}$. Then $\mathcal{N} = [\mathcal{F}\mathcal{C}]$ (where we use the notation $[\cdot]$ for the closed linear space generated by its argument), as every closed left ideal is the intersection of the

left kernels which contains it (cf. 3.13.5 in [29]). Thus \mathcal{N} is the left ideal generated by \mathcal{C} . Since \mathcal{C} is selfadjoint and contained in \mathcal{N} we conclude $\mathcal{C} \subset C^*(\mathcal{C}) \subset \mathcal{N} \cap \mathcal{N}^* = [\mathcal{FC}] \cap [\mathcal{CF}]$, where $C^*(\cdot)$ denotes the C^* -algebra in \mathcal{F} generated by its argument.

By Theorem 5.2.2 in [28], we know that if \mathcal{A} is a Banach algebra with a bounded left approximate identity and $T : \mathcal{A} \rightarrow \mathcal{B}(X)$ is a continuous representation of \mathcal{A} on the Banach space X , then for each $y \in \overline{\text{Span}(T(\mathcal{A})X)}$ there are elements $a \in \mathcal{A}$ and $x \in X$ with $y = T(a)x$, i.e. $[T(\mathcal{A})X] = T(\mathcal{A})X$. Thus, if $X = \mathcal{F}$ and $T : C^*(\mathcal{C}) \rightarrow \mathcal{B}(X)$ is defined by $T(C)F := CF$, then $\mathcal{N}^* = [\mathcal{CF}] = [C^*(\mathcal{C})\mathcal{F}] = C^*(\mathcal{C})\mathcal{F}$, hence $\mathcal{N} = \mathcal{FC}^*(\mathcal{C})$.

Theorem 4.3. *Now for the Dirac states we have [14]:*

- (i) $\mathfrak{S}_D \neq \emptyset$ iff $\mathbb{1} \notin C^*(\mathcal{C})$ iff $\mathbb{1} \notin \mathcal{N} \cap \mathcal{N}^* =: \mathcal{D}$.
- (ii) $\omega \in \mathfrak{S}_D$ iff $\pi_\omega(\mathcal{D})\Omega_\omega = 0$.
- (iii) *An extreme Dirac state is pure.*

We will call a constraint set \mathcal{C} *first class* if $\mathbb{1} \notin C^*(\mathcal{C})$, and this is the nontriviality condition which needs to be checked [12, Section 3].

For our system $(\mathcal{F}_e, U_{\text{Gau}\Lambda})$, we automatically have $\mathfrak{S}_D \neq \emptyset$, since \mathcal{F}_e always has the trivial Dirac state ω_0 given by $\omega_0(U_{\text{Gau}\Lambda}\mathfrak{A}_\Lambda) = 0$, $\omega_0(U_{\text{Gau}\Lambda}) = 1$ on \mathcal{F}_e^o , which extends as a Dirac state to \mathcal{F}_e . However, to verify that constraining will produce physically nontrivial results, we need to check via Proposition 4.2 that there are gauge invariant states on $\mathfrak{A}_\Lambda \subset \mathcal{F}_e$, as these will extend to Dirac states on \mathcal{F}_e for which $N_\omega \cap N_\omega^*$ will not contain \mathfrak{A}_Λ . At the end of Subsection 3.2 we constructed a representation $\pi = \pi_{\text{Fock}} \otimes \pi_\infty$ which was covariant and had a nonzero invariant vector. The vector state of this invariant vector is therefore a gauge invariant state on \mathfrak{A}_Λ , and shows that our constraint system is physically nontrivial.

We recall the rest of the T-procedure before we implement it for the present system. Define

$$\mathcal{O} := \{F \in \mathcal{F} \mid [F, D] := FD - DF \in \mathcal{D} \quad \forall D \in \mathcal{D}\}.$$

Then \mathcal{O} is the C^* -algebraic analogue of Dirac's observables (the weak commutant of the constraints) [7].

Theorem 4.4. *With the preceding notation we have [14]:*

- (i) $\mathcal{D} = \mathcal{N} \cap \mathcal{N}^*$ is the unique maximal C^* -algebra in $\cap \{\text{Ker } \omega \mid \omega \in \mathfrak{S}_D\}$. Moreover \mathcal{D} is a hereditary C^* -subalgebra of \mathcal{F} , and $\mathcal{D} = [\mathcal{CF}\mathcal{C}]$.
- (ii) $\mathcal{O} = M_{\mathcal{F}}(\mathcal{D}) := \{F \in \mathcal{F}_e \mid FD \in \mathcal{D} \ni DF \quad \forall D \in \mathcal{D}\}$, i.e. it is the relative multiplier algebra of \mathcal{D} in \mathcal{F} .
- (iii) $\mathcal{O} = \{F \in \mathcal{F} \mid [F, \mathcal{C}] \subset \mathcal{D}\}$.

(iv) $\mathcal{D} = [\mathcal{O}\mathcal{C}] = [\mathcal{C}\mathcal{O}] = [\mathcal{C}\mathcal{O}\mathcal{C}]$. (Thus $\mathcal{D} = \mathcal{O}C^*(\mathcal{C}) = C^*(\mathcal{C})\mathcal{O}$ by Theorem 5.2.2 in [28] quoted above).

(v) For the present case $\mathcal{C} = U_{\text{Gau}\Lambda} - \mathbb{1}$, we have $U_{\text{Gau}\Lambda} \subset \mathcal{O}$ and $\mathcal{O} = \{F \in \mathcal{F}_e \mid \alpha_g(F) - F \in \mathcal{D} \quad \forall g \in \text{Gau}\Lambda\}$.

Proof: Only the last statement in (i) needs proof, as the rest is in [14]. Clearly $[\mathcal{C}\mathcal{F}\mathcal{C}] \subseteq \mathcal{N} \cap \mathcal{N}^* = \mathcal{D}$ is hereditary by Theorem 3.2.2 in [27]. Since

$$\mathcal{N} = [\mathcal{F}\mathcal{C}] = \{F \in \mathcal{F} \mid F^*F \in [\mathcal{C}\mathcal{F}\mathcal{C}]\}$$

it follows from Theorem 3.2.1 in [27] that $[\mathcal{C}\mathcal{F}\mathcal{C}] = \mathcal{N} \cap \mathcal{N}^* = \mathcal{D}$. ■

Thus \mathcal{D} is a closed two-sided ideal of \mathcal{O} and it is proper when $\mathfrak{S}_D \neq \emptyset$ (which is the case for our current example). From (iii) above, we see that the traditional observables $\mathcal{C}' \subset \mathcal{O}$, where \mathcal{C}' denotes the relative commutant of \mathcal{C} in \mathcal{F}_e . (In our case \mathcal{C}' is just the gauge invariant elements of \mathcal{F}_e .) Note also that two constraint sets $\mathcal{C}_1, \mathcal{C}_2$ which select the same set of Dirac states \mathfrak{S}_D , will produce the same algebras \mathcal{D} and \mathcal{O} , but need not produce the same traditional observables, i.e. $\mathcal{C}'_1 \neq \mathcal{C}'_2$ is possible. In examples, \mathcal{O} is generally much harder to obtain explicitly than $\mathcal{C}' \subset \mathcal{O}$. In our example, $U_{\text{Gau}\Lambda}$ and \mathcal{U}_0 will produce the same \mathcal{D} and \mathcal{O} .

Define the *maximal C^* -algebra of physical observables* as

$$\mathcal{R} := \mathcal{O}/\mathcal{D}.$$

This method of constructing \mathcal{R} from $(\mathcal{F}, \mathcal{C})$ is called the **T-procedure**. We call the factoring map $\xi : \mathcal{O} \rightarrow \mathcal{R}$ the **constraining homomorphism**. We require that after the T-procedure all physical information is contained in the pair $(\mathcal{R}, \mathfrak{S}(\mathcal{R}))$, where $\mathfrak{S}(\mathcal{R})$ denotes the set of states on \mathcal{R} . The following result justifies the choice of \mathcal{R} as the algebra of physical observables (cf. Theorem 2.20 in [11]):

Theorem 4.5. *There exists a w^* -continuous isometric bijection between the Dirac states on \mathcal{O} and the states on \mathcal{R} .*

An established alternative method for enforcing constraints (cf. [20]) in this context, is to take the traditional observables \mathcal{C}' (gauge invariant observables) and then to factor out by the ideal generated by the Gauss law (the state constraint \mathcal{C}). Since \mathcal{C} need not be in \mathcal{C}' (e.g. for nonabelian gauge theory), the term “ideal generated by the Gauss law” needs interpretation. The easiest interpretation of this ideal, is as the intersection of \mathcal{C}' with the ideal which \mathcal{C} generates in $C^*(\mathcal{C}' \cup \mathcal{C}) \subseteq \mathcal{O}$. By Theorem B.1 below, the ideal generated by \mathcal{C} in $C^*(\mathcal{C}' \cup \mathcal{C})$ is just $C^*(\mathcal{C}' \cup \mathcal{C}) \cap \mathcal{D}$, hence the “ideal generated in \mathcal{C}' by \mathcal{C} ” is just $\mathcal{D} \cap \mathcal{C}'$. Thus the physical algebra obtained is $\mathcal{C}'/(\mathcal{D} \cap \mathcal{C}') \subset \mathcal{O}/\mathcal{D} = \mathcal{R}$. For particular field algebras, these algebras can coincide (cf. [14] for the Weyl algebra with

linear constraints), and below in Theorem 4.12 we will show for our model on a finite lattice, that they also do.

We can gain further understanding of the algebras \mathcal{D} , \mathcal{O} , \mathcal{R} through the hereditary property of \mathcal{D} . Denote by π_u the universal representation of \mathcal{F} on the universal Hilbert space \mathcal{H}_u [29, Section 3.7]. \mathcal{F}'' is the strong closure of $\pi_u(\mathcal{F})$ and since π_u is faithful we make the usual identification of \mathcal{F} with a subalgebra of \mathcal{F}'' , i.e. generally omit explicit indication of π_u . If $\omega \in \mathfrak{S}(\mathcal{F})$, we will use the same symbol for the unique normal extension of ω from \mathcal{F} to \mathcal{F}'' . Recall the definition from Pedersen [29]:

Definition 4.6. *For a C*-algebra \mathcal{F} , a projection $P \in \mathcal{F}''$ is **open** if $\mathcal{L} = \mathcal{F} \cap (\mathcal{F}''P)$ is a closed left ideal of \mathcal{F} .*

We then know from Theorem 3.10.7, Proposition 3.11.9 and Remark 3.11.10 in Pedersen [29] that the open projections are in bijection with:

- (i) hereditary C*-subalgebras of \mathcal{F} by $P \rightarrow P\mathcal{F}''P \cap \mathcal{F}$,
- (ii) closed left ideals of \mathcal{F} by $P \rightarrow \mathcal{F}''P \cap \mathcal{F}$,
- (iii) weak *-closed faces containing 0 of the quasi-state space $Q(\mathcal{F})$ by

$$P \rightarrow \{ \omega \in Q(\mathcal{F}) \mid \omega(P) = 0 \} .$$

Theorem 4.7. *For a constrained system $(\mathcal{F}, \mathcal{C})$ there is an open projection $P \in \mathcal{F}''$ such that [14]:*

- (i) $\mathcal{N} = \mathcal{F}''P \cap \mathcal{F}$,
- (ii) $\mathcal{D} = P\mathcal{F}''P \cap \mathcal{F}$ and
- (iii) $\mathfrak{S}_D = \{ \omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(P) = 0 \}$.

Since any hereditary C*-subalgebra of \mathcal{F} can be obtained as the algebra \mathcal{D} of a set of constraints (just take $\mathcal{C} = \mathcal{D}$), this characterises the possible sets of Dirac states \mathfrak{S}_D as the intersections of $\mathfrak{S}(\mathcal{F})$ with weak *-closed faces of $Q(\mathcal{F})$ containing 0.

Theorem 4.8. *Let P be the open projection in Theorem 4.7. Then [14]:*

$$\mathcal{O} = \{ A \in \mathcal{F} \mid PA(\mathbb{1} - P) = 0 = (\mathbb{1} - P)AP \} = P' \cap \mathcal{F}$$

What these two last theorems mean, is that with respect to the decomposition

$$\mathcal{H}_u = P\mathcal{H}_u \oplus (\mathbb{1} - P)\mathcal{H}_u$$

we may rewrite

$$\begin{aligned}\mathcal{D} &= \left\{ F \in \mathcal{F} \mid F = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, D \in P\mathcal{F}''P \right\} \text{ and} \\ \mathcal{O} &= \left\{ F \in \mathcal{F} \mid F = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A \in P\mathcal{F}''P, B \in (\mathbb{1} - P)\mathcal{F}''(\mathbb{1} - P) \right\}.\end{aligned}$$

It is clear that in general $\mathcal{O} = P' \cap \mathcal{F}$ can be much greater than the traditional observables $\mathcal{C}' \cap \mathcal{F}$. To appreciate this difference, consider the example where $\mathcal{F} = \mathcal{B}(\mathcal{H})$ for a separable Hilbert space \mathcal{H} , and let $\mathcal{C} = \mathcal{K}(\mathcal{H}) \equiv$ compact operators. Then $\mathcal{C}' = \mathbb{C}\mathbb{1}$, but $\mathcal{O} = \mathcal{B}(\mathcal{H})$.

We can identify the final algebra of physical observables \mathcal{R} with a subalgebra of \mathcal{F}'' :

Theorem 4.9. *For P as above we have:*

$$\mathcal{R} \cong (\mathbb{1} - P)(P' \cap \mathcal{F}) \subset \mathcal{F}''.$$

Notice that this just means that \mathcal{R} is the restriction of $P' \cap \mathcal{F}$ to the subspace $(\mathbb{1} - P)\mathcal{H}_u$ of the universal representation, and that $(\mathbb{1} - P)\mathcal{H}_u$ is the annihilator of \mathcal{N} , hence of \mathcal{C} . Thus a simplified (equivalent) version of the T-procedure, is to select the space $\{\psi \in \mathcal{H}_u \mid \pi_u(\mathcal{C})\psi = 0\} = (\mathbb{1} - P)\mathcal{H}_u$, select the set of all elements of \mathcal{F} which preserves this space together with their adjoints (this is $P' \cap \mathcal{F}$), and restrict it to $(\mathbb{1} - P)\mathcal{H}_u$ to obtain \mathcal{R} . In other words, it is just the enforcement of the constraints by state condition in the universal representation.

Regarding transformations of the system, consider the automorphisms of \mathcal{F} which factor through to \mathcal{R} , i.e. those which preserve both \mathcal{O} and \mathcal{D} . Define

$$\Upsilon := \left\{ \alpha \in \text{Aut } \mathcal{F} \mid \mathcal{D} = \alpha(\mathcal{D}) \right\},$$

then since $\mathcal{O} = \mathcal{M}_{\mathcal{F}}(\mathcal{D})$, an $\alpha \in \Upsilon$ also preserves \mathcal{O} and so defines canonically an automorphism α' on \mathcal{R} when we factor out by \mathcal{D} . Define the group homomorphism $T: \Upsilon \mapsto \text{Aut } \mathcal{R}$ by $T(\alpha) = \alpha'$, then $\text{Ker } T$ consists of all the transformations which become the identity on the physical algebra \mathcal{R} , i.e. “gauge transformations” (in a different sense than encountered above). In fact, in the case of our assumed constraint system $(\mathcal{F}_e, U_{\text{Gau}\Lambda})$ we obtain from Theorem 4.4(v) and (ii) that $\alpha_{\text{Gau}\Lambda} = \text{Ad } U_{\text{Gau}\Lambda} \subset \text{Ker } T$, so we can indeed claim that \mathcal{R} consists of gauge invariant observables, though not necessarily obtained from the traditional gauge invariant observables $U'_{\text{Gau}\Lambda} \subset \mathcal{F}_e$.

We now return for a more detailed analysis of our assumed constraint system $(\mathcal{F}_e, \mathcal{U}_0)$. Our strategy in the rest of this paper, will be to first analyze the constraint systems for finite lattices, and then to use these to analyze the full system.

Recall from Proposition 2.5 that \mathfrak{A}_Λ has an inductive limit structure over any directed set \mathcal{S} of open, bounded convex subsets of \mathbb{R}^3 such that $\bigcup_{S \in \mathcal{S}} S = \mathbb{R}^3$, partially ordered by inclusion. In particular

$$\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_S = \varinjlim (\mathfrak{F}_S \otimes \mathcal{L}_S[E])$$

where $\mathfrak{F}_S := C^*\left(\bigcup_{x \in \Lambda_S^0} \mathfrak{F}_x\right)$ and $\mathcal{L}_S[E] := C^*\left(\bigcup_{\mathbf{n} \in \mathbb{N}^\infty} \mathcal{E}_S[\mathbf{n}]\right)$. Here we will only be concerned with the particular case where \mathcal{S} is a linear increasing chain $\mathcal{S} = \{S_k \mid k \in \mathbb{N}\}$, $S_1 \subset S_2 \subset S_3 \subset \dots$. Note that as each S_i is open bounded and convex, it only contains finitely many lattice points. We can equip

$$\mathcal{U}_0 := \{U_{\exp(t\nu)} \mid t \in \mathbb{R}, \nu = Y \cdot \delta_x \text{ for all } Y \in \mathfrak{g}, x \in \Lambda^0\},$$

with the same inductive limit structure as follows. Let

$$\mathcal{U}_0^S := \{U \in \mathcal{U}_0 \mid [U, \mathfrak{A}_S] \neq 0\} = \{U_{\exp(tY \cdot \delta_x)} \mid t \in \mathbb{R} \setminus 0, Y \in \mathfrak{g} \setminus 0, x \in S_e\}$$

where the ‘‘lattice envelope’’ S_e of $S \in \mathcal{S}$ is

$$S_e := \{x \in \Lambda^0 \mid \exists \ell = (x_\ell, y_\ell) \in \Lambda^1 \text{ such that } \ell \cap S \neq \emptyset \text{ and } x_\ell = x \text{ or } y_\ell = x\}.$$

If we denote $\mathcal{C}_i := \mathcal{U}_0^{S_i} - \mathbb{1}$, then $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots$, and $\mathcal{C} = \mathcal{U}_0 - \mathbb{1} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. The group generated by $\mathcal{U}_0^{S_i}$ is denoted by $U_{\text{Gau } S_i}$, so these sets of unitaries produce equivalent constraint sets. We also obtain an inductive limit for $\mathcal{F}_e = [U_{\text{Gau } \Lambda} \cdot (\mathfrak{A}_\Lambda \oplus \mathbb{C})]$ by

$$\mathcal{F}_e = \varinjlim \mathcal{F}_S \quad \text{where} \quad \mathcal{F}_S := [U_{\text{Gau } S} \cdot (\mathfrak{A}_S \oplus \mathbb{C})] = [U_{\text{Gau } S} \cdot (\mathfrak{F}_S \otimes \mathcal{L}_S[E] \oplus \mathbb{C})].$$

This suggests that we analyze the ‘‘local constraint systems’’ $(\mathcal{F}_{S_i}, \mathcal{C}_i)$. Below we will see that $i < j$ implies that $\mathcal{U}_0^{S_i} = \mathcal{U}_0^{S_j} \cap \mathcal{F}_{S_i}$, hence the set of constraint systems $(\mathcal{F}_{S_i}, \mathcal{C}_i)$, $i \in \mathbb{N}$, is a system of local quantum constraints in the sense of [14] (Def. 3.3). Such systems were studied in detail in [14], and in Section 4.5 below, we will apply this analysis. However, first we need to solve the constraint system for an individual ‘‘local’’ system $(\mathcal{F}_{S_i}, \mathcal{C}_i)$. To do so, we will solve the corresponding system for a finite lattice in the next subsection.

We start with the constraint system for the *finite* lattice, i.e. the system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$ where

$$\mathcal{F}_{S_i}^F := [U_{\text{Gau } S_i} \cdot (\text{CAR}(\mathcal{H}_{S_i}) \otimes \mathcal{L}^{(S_i)} \oplus \mathbb{C})] \subset M(\mathfrak{A}_\Lambda \rtimes_\alpha (\text{Gau } \Lambda)) \supset \mathcal{F}_e$$

with the same constraints $\mathcal{C}_i := \mathcal{U}_0^{S_i} - \mathbb{1}$. Note that $\mathcal{F}_{S_i}^F$ is not contained in \mathcal{F}_e , though $\mathcal{C}_i \subset \mathcal{F}_e \cap \mathcal{F}_{S_i}^F$. Moreover $\mathcal{F}_{S_i}^F$ only differs from \mathcal{F}_{S_i} by the replacement of \mathcal{L}^{S_i} by $\mathcal{L}_{S_i}[E]$.

4.3 Enforcing the Gauss law constraint for finite lattices.

In this subsection we will obtain a full analysis of the constraint data $(\mathcal{D}_i^F, \mathcal{O}_i^F, \mathcal{R}_i^F)$ for the the finite lattice system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$ in S_i . First observe that

$$\begin{aligned}\mathcal{F}_{S_i}^F &= [U_{\text{Gau } S_i} \cdot (\mathcal{A}_{S_i} \oplus \mathbb{C})] = (\mathcal{A}_{S_i} + \mathbb{C}) \rtimes_{\alpha} (\text{Gau}_d S_i) \\ &= [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] + [U_{\text{Gau } S_i}] \quad \text{where} \quad \mathcal{A}_{S_i} := \mathfrak{F}_{S_i} \otimes \mathcal{L}^{(S_i)}\end{aligned}$$

and $\text{Gau}_d S_i := \{\gamma \in \text{Gau } \Lambda \mid \text{supp}(\gamma) \subset (S_i)_e\} \cong \prod_{x \in (S_i)_e} G$ with the discrete topology.

Lemma 4.10. *With notation as above, we have that $[U_{\text{Gau } S_i} \mathcal{A}_{S_i}]$ is a closed two-sided ideal of $\mathcal{F}_{S_i}^F$. Moreover $[U_{\text{Gau } S_i} \mathcal{A}_{S_i}] \cap [U_{\text{Gau } S_i}] = \{0\}$, i.e. $\mathcal{F}_{S_i}^F / [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] \cong [U_{\text{Gau } S_i}]$.*

Proof: That $[U_{\text{Gau } S_i} \mathcal{A}_{S_i}]$ is a closed two-sided ideal of $\mathcal{F}_{S_i}^F$ is clear by construction, so we prove the second statement. Consider a faithful representation $V : [U_{\text{Gau } S_i}] \rightarrow \mathcal{B}(\mathcal{H})$, and let $\varphi : \mathcal{A}_{S_i} + \mathbb{C} \rightarrow \mathcal{B}(\mathcal{H})$ be the character $\varphi(A + \lambda \mathbb{1}) = \lambda \mathbb{1}$ for all $A \in \mathcal{A}_{S_i}$, $\lambda \in \mathbb{C}$. Then the pair (V, φ) defines a covariant representation, i.e. $\varphi(\alpha_g(B)) = V(U_g)\varphi(B)V(U_g)^* = \varphi(B)$ for all $B \in \mathcal{A}_{S_i} + \mathbb{C}$ and $g \in \text{Gau } S_i$. Thus it defines a representation π of the crossed product $\mathcal{F}_{S_i}^F = (\mathcal{A}_{S_i} + \mathbb{C}) \rtimes_{\alpha} (\text{Gau}_d S_i)$ on \mathcal{H} by $\pi(B) := \varphi(B)$ for all $B \in \mathcal{A}_{S_i} + \mathbb{C}$, and $\pi(U_g) = V(U_g)$ for all $g \in \text{Gau } S_i$. As $\mathcal{F}_{S_i}^F = [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] + [U_{\text{Gau } S_i}]$, it is obvious that $\pi([U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}]) = 0$ and π is faithful on $[U_{\text{Gau } S_i}]$, hence $\pi(\mathcal{F}_{S_i}^F) \cong [U_{\text{Gau } S_i}]$ and $\text{Ker}(\pi) = [U_{\text{Gau } S_i} \mathcal{A}_{S_i}]$. \blacksquare

As \mathcal{H}_{S_i} is finite dimensional, $\mathfrak{F}_{S_i} = \text{CAR}(\mathcal{H}_{S_i})$ is just a (full) matrix algebra, and as $\mathcal{L}^{(S_i)} = \bigotimes \{\mathcal{L}_k \mid \ell_k \cap S_i \neq \emptyset\}$ is a finite tensor product of factors $\mathcal{L}_k \cong \mathcal{K}(\mathcal{H})$, it is isomorphic to $\mathcal{K}(\mathcal{H})$ and hence $\text{CAR}(\mathcal{H}_{S_i}) \otimes \mathcal{L}^{(S_i)} \cong \mathcal{K}(\mathcal{H})$. This has the following consequences:

- The algebra $\mathcal{A}_{S_i} := \text{CAR}(\mathcal{H}_{S_i}) \otimes \mathcal{L}^{(S_i)} \cong \mathcal{K}(\mathcal{H})$ has (up to unitary equivalence) only one irreducible representation $\pi : \mathcal{A}_{S_i} \rightarrow \mathcal{B}(\mathcal{H}_{\pi})$. This representation π is faithful, and $\pi(\mathcal{A}_{S_i}) = \mathcal{K}(\mathcal{H}_{\pi})$.
- Note that the enforcement of the constraints $\mathcal{C}_i := \mathcal{U}_0^{S_i} - \mathbb{1}$ will put $[\mathcal{U}_0^{S_i}] \subset \mathcal{F}_{S_i}^F$ and hence $[U_{\text{Gau } S_i}]$ equal to $\mathbb{C}\mathbb{1}$, hence the only nontrivial part of $\mathcal{F}_{S_i}^F$ which needs to be analyzed w.r.t. constraints is the closed two-sided ideal $[U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] = \mathcal{A}_{S_i} \rtimes_{\alpha} (\text{Gau}_d S_i) \subset \mathcal{F}_{S_i}^F$.
- For the action $\alpha : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{A}_{S_i}$, as $\text{Gau } S_i \cong \prod_{x \in (S_i)_e} G$ is compact, we know from [32] that the invariance algebra is $\mathcal{A}_{S_i}^{\alpha} = p(\mathcal{A}_{S_i} \rtimes_{\alpha} (\text{Gau } S_i))p$ for some projection $p \in M(\mathcal{A}_{S_i} \rtimes_{\alpha} (\text{Gau } S_i))$, where the equality is realised in the multiplier algebra $M(\mathcal{A}_{S_i} \rtimes_{\alpha} (\text{Gau } S_i))$ using the imbedding of \mathcal{A}_{S_i} in it. We will find a similar structure in Theorem 4.12 below.

- All automorphisms of $\mathcal{K}(\mathcal{H}) \cong \mathcal{A}_{S_i}$ are inner, hence there are unitaries $W_g \in M(\mathcal{A}_{S_i})$ implementing $\alpha_g \in \text{Aut } \mathcal{A}_{S_i} = \text{Aut}(\mathcal{A}_{S_i} + \mathbb{C})$, $g \in \text{Gau } S_i$, and the unitaries are unique up to scalar multiples. The map $g \mapsto W_g$ need not be a group homomorphism, and it is well known that the obstruction for this to be the case, is a nontrivial $H^2(\text{Gau } S_i, \mathbb{T})$ (second Moore cohomology group), cf. [30]. Sufficient conditions for a trivial $H^2(\text{Gau } S_i, \mathbb{T})$ are in [40].

Proposition 4.11. *Let G be a connected compact Lie group. Then*

- (i) *there is a representation of $\mathcal{F}_{S_i}^F$ which is irreducible on the subalgebra $\mathcal{A}_{S_i} \subset \mathcal{F}_{S_i}^F$. Hence the irreducible representation π of $\mathcal{A}_{S_i} = \text{CAR}(\mathcal{H}_{S_i}) \otimes \mathcal{L}^{(S_i)}$ is also covariant for the action $\alpha : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{A}_{S_i}$.*
- (ii) *the action $\alpha : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{A}_{S_i}$ is inner, i.e. there is a strictly continuous homomorphism $V : \text{Gau } S_i \rightarrow \text{UM}(\mathcal{A}_{S_i})$ which implements α .*
- (iii) *Let H denote either $\text{Gau } S_i$ or $\text{Gau}_d S_i$. Then we have an isomorphism $\varphi_H : \mathcal{A}_{S_i} \rtimes_{\alpha} H \rightarrow \mathcal{A}_{S_i} \otimes C^*(H)$. Explicitly it is obtained by defining $\varphi_H^{-1}(A \otimes f) \in C_c(H, \mathcal{A}_{S_i})$ by $\varphi_H^{-1}(A \otimes f)(g) = AV_g^{-1}f(g)$ for $A \in \mathcal{A}_{S_i}$, $f \in C_c(H)$ and $g \in H$.*
- (iv) *We have that*

$$\mathcal{F}_{S_i}^F = (\mathcal{A}_{S_i} + \mathbb{C}) \rtimes_{\alpha} (\text{Gau}_d S_i) = \mathcal{A}_{S_i} \rtimes_{\alpha} (\text{Gau}_d S_i) + C^*(\text{Gau}_d S_i)$$

and that $\xi : [\mathcal{A}_{S_i} \otimes C^*(\text{Gau}_d S_i) + \mathbb{1} \otimes C^*(\text{Gau}_d S_i)] \rightarrow \mathcal{F}_{S_i}^F$

is an isomorphism where $\xi(A \otimes f_1 + \mathbb{1} \otimes f_2)(g) = AV_g^{-1}f_1(g) + V_g^{-1}f_2(g)$ for $A \in \mathcal{A}_{S_i}$ and $f_i \in C_c(\text{Gau}_d S_i)$.

Proof: (i) Recall that

$$\alpha_{\gamma} := \alpha_{\gamma}^1 \otimes \alpha_{\gamma}^2 \quad \text{where} \quad \alpha^1 : \text{Gau } S_i \rightarrow \text{Aut } \text{CAR}(\mathcal{H}_{S_i}) \quad \text{and} \quad \alpha^2 : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{L}^{(S_i)}.$$

We show there is a product representation of irreducible covariant representations. For $\alpha^1 : \text{Gau } S_i \rightarrow \text{Aut } \text{CAR}(\mathcal{H}_{S_i})$, we only need to take the Fock representation, which is irreducible and covariant. Regarding $\alpha^2 : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{L}^{(S_i)}$, recall that $\mathcal{L}^{(S_i)} = \bigotimes_{\ell_k \in \Lambda_{S_i}^1} \mathcal{L}_k$ where $\Lambda_{S_i}^1 := \{\ell \in \Lambda^1 \mid \ell \cap S_i \neq \emptyset\}$, and that $\alpha^2 = \bigotimes_{\ell_k \in \Lambda_{S_i}^1} \theta^k$. However, by Lemma 3.1(i), we have for each \mathcal{L}_k an irreducible representation π_0 which is covariant w.r.t. $\theta^k : G^2 \rightarrow \text{Aut } \mathcal{L}_k$. Thus, by taking the (finite) tensor product of all of these covariant representations π_0 for θ^k , $\ell_k \in \Lambda_{S_i}^1$, we obtain an irreducible covariant representation for $\alpha^2 : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{L}^{(S_i)}$, and tensoring it with the Fock representation produces the desired irreducible covariant representation for $\alpha : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{A}_{S_i}$. As $\mathcal{F}_{S_i}^F$ is a crossed product, this defines then a representation of $\mathcal{F}_{S_i}^F$ which is irreducible on \mathcal{A}_{S_i} .

(ii) The representation $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ in (i) is irreducible (hence faithful) on \mathcal{A}_{S_i} . As $\pi(\mathcal{A}_{S_i}) = \mathcal{K}(\mathcal{H}_\pi)$, which is an essential ideal in $\mathcal{B}(\mathcal{H}_\pi) = M(\mathcal{K}(\mathcal{H}_\pi))$, π produces a faithful homomorphism from $\mathcal{B}(\mathcal{H}_\pi)$ into $M(\mathcal{A}_{S_i})$ by Prop. 3.12.8 in [29]. Since π is covariant we have the unitary implementers $U_g^\pi := \pi(U_g) \in \mathcal{B}(\mathcal{H}_\pi)$, which therefore define the unitaries $V_g \in UM(\mathcal{A}_{S_i})$ which also implement α_g . It is clear that $g \rightarrow V_g$ is a homomorphism which is strictly continuous, since the strong operator topology and strict topology w.r.t. the compacts coincide on unitaries. Thus the action $\alpha : \text{Gau } S_i \rightarrow \text{Aut } \mathcal{A}_{S_i}$ is inner.

(iii) By (ii), the action α is inner for H . Thus by Lemma 2.68 and Remark 2.71 of [43], the action $\alpha : H \rightarrow \text{Aut } \mathcal{A}_{S_i}$ is exterior equivalent to the trivial action $\iota : H \rightarrow \text{Aut } \mathcal{A}_{S_i}$, hence the crossed products are isomorphic by $\psi_1 : \mathcal{A}_{S_i} \rtimes_\iota H \rightarrow \mathcal{A}_{S_i} \rtimes_\alpha H$, where the isomorphism is given by $\psi_1(A \cdot f)(g) := Af(g)V_g^{-1}$ for $A \in \mathcal{A}_{S_i}$ and $f \in C_c(H)$. However, by Lemma 2.73 of [43] we know that the crossed product $\mathcal{A}_{S_i} \rtimes_\iota H$ is isomorphic to the tensor product $\mathcal{A}_{S_i} \otimes C^*(H)$ (using nuclearity of \mathcal{A}_{S_i} for the tensor norm). Explicitly this isomorphism $\psi_0 : \mathcal{A}_{S_i} \otimes C^*(H) \rightarrow \mathcal{A}_{S_i} \rtimes_\iota H$ is given by $\psi_0(A \otimes f) = A \cdot f$. Thus we obtain the claimed isomorphism $\varphi_H^{-1} := \psi_1 \circ \psi_0$, which explicitly is given by $\varphi_H^{-1}(A \otimes f)(g) = \psi_1(A \cdot f)(g) = Af(g)V_g^{-1}$.

(iv) This follows from the previous part, keeping in mind that $\mathcal{F}_{S_i}^F = C^*(\mathcal{U}_0^{S_i} \cup \mathcal{A}_{S_i}) = C^*(U_{\text{Gau } S_i} \cup \mathcal{A}_{S_i}) = [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i} + U_{\text{Gau } S_i}]$, and that $[U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] = \mathcal{A}_{S_i} \rtimes_\alpha \text{Gau}_d S_i$. ■

Remark: Observe in part (iv) that if we extend ξ to $M(\mathcal{A}_{S_i}) \otimes C^*(\text{Gau}_d S_i)$, then $\xi(V_g \otimes \delta_g) = \delta_g = U_g$ where $\delta_x(y) = 1$ if $x = y$ and it is zero otherwise. So we may consider U_g to be the product of V_g with some independent part which commutes with \mathcal{A}_{S_i} . Note that $V_g^{-1}U_g$ commutes with all of $M(\mathcal{A}_{S_i})$.

Theorem 4.12. *For the system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$ above, we have that*

(i) *There is a projection $P_\alpha \in M(\mathcal{A}_{S_i})$ such that $\alpha_g(P_\alpha) = P_\alpha$ for all $g \in \text{Gau } S_i$ and*

$$\begin{aligned} \mathcal{O}_i^F \cap \mathcal{A}_{S_i} &= P_\alpha \mathcal{A}_{S_i} P_\alpha \oplus (\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha), \\ \mathcal{D}_i^F \cap \mathcal{A}_{S_i} &= (\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha) \\ \text{and} \quad &(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) / (\mathcal{D}_i^F \cap \mathcal{A}_{S_i}) \cong P_\alpha \mathcal{A}_{S_i} P_\alpha. \end{aligned}$$

Moreover $U_g P_\alpha = P_\alpha = P_\alpha U_g$ for all $g \in \text{Gau } S_i$ and so $\mathcal{C}_i P_\alpha = 0$.

(ii) *Let $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ be a representation which is irreducible (hence faithful) on \mathcal{A}_{S_i} . Then $\pi(P_\alpha)$ is the projection onto $\mathcal{H}_\pi^G := \{\psi \in \mathcal{H}_\pi \mid U_g^\pi \psi = \psi \ \forall g \in \text{Gau}_d S_i\}$, where $U_g^\pi := \pi(U_g)$, and*

$$\begin{aligned} \pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) &= \mathcal{K}(\mathcal{H}_\pi^G) \oplus \mathcal{K}((\mathcal{H}_\pi^G)^\perp), \quad \pi(\mathcal{D}_i^F \cap \mathcal{A}_{S_i}) = \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \\ \text{and} \quad &\tilde{\pi}((\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) / (\mathcal{D}_i^F \cap \mathcal{A}_{S_i})) = \mathcal{K}(\mathcal{H}_\pi^G), \end{aligned}$$

where $\tilde{\pi}$ is the restriction of $\pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})$ to \mathcal{H}_π^G .

(iii) We have $\mathcal{H}_\pi^G \neq \{0\}$.

(iv) $P_\alpha = (\mathbb{1} - P_{S_i})P_J$ where $P_{S_i} \in (\mathcal{F}_{S_i}^F)''$ is the open projection of Theorem 4.7 for the constraint system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$, and $P_J \in (\mathcal{F}_{S_i}^F)''$ is the central projection determined by the ideal $[U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}]$ of $\mathcal{F}_{S_i}^F$.

Proof: (ii) From Proposition 4.11, we have the representation $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ which is irreducible and faithful on \mathcal{A}_{S_i} . As $\pi(\mathcal{A}_{S_i}) = \mathcal{K}(\mathcal{H}_\pi)$, which is an essential ideal in $\mathcal{B}(\mathcal{H}_\pi) = M(\mathcal{K}(\mathcal{H}_\pi))$, π produces a faithful homomorphism from $\mathcal{B}(\mathcal{H}_\pi)$ into $M(\mathcal{A}_{S_i})$ by Prop. 3.12.8 in [29]. Thus there exists a unique projection $P_\alpha \in M(\mathcal{A}_{S_i})$ such that $\pi(P_\alpha)$ is the projection P^G onto \mathcal{H}_π^G . By definition we have that $\pi(U_g)\pi(P_\alpha) = \pi(P_\alpha)$ and hence $U_g P_\alpha = P_\alpha = P_\alpha U_g$ and so $\mathcal{C}_i P_\alpha = 0$.

As π is irreducible, it is cyclic, hence there is a projection $P^\pi \in \pi_u(\mathcal{F}_{S_i}^F)'$ on the Hilbert space \mathcal{H}_u of the universal representation $\pi_u : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_u)$ such that $P^\pi \mathcal{H}_u \rightarrow \mathcal{H}_\pi$ and $P^\pi \circ \pi_u = \pi$.

Let $P_{S_i} \in (\mathcal{F}_{S_i}^F)''$ be the open projection of Theorem 4.7 for the constraint system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$, specified by $\omega(P_{S_i}) = 0$ iff $\pi_\omega(\mathcal{C}_i)\Omega_\omega = 0$ for $\omega \in \mathfrak{S}(\mathcal{F}_{S_i}^F)$. Thus $\mathbb{1} - P_{S_i}$ projects onto the $U_{\text{Gau } S_i}$ -invariant subspace in all representations, and so $P^G := (\mathbb{1} - P_{S_i})P^\pi$ is the projection of \mathcal{H}_π onto \mathcal{H}_π^G .

By Theorem 4.8 we have that $\mathcal{O}_i^F \cap \mathcal{A}_{S_i} = P'_{S_i} \cap \mathcal{A}_{S_i}$, so

$$\begin{aligned} \pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) &:= P^\pi(P'_{S_i} \cap \mathcal{A}_{S_i}) \\ &= \{ \pi(A) \mid A \in \mathcal{A}_{S_i} \text{ such that } [\pi(A), P^\pi(\mathbb{1} - P_{S_i})] = 0 \} \quad \text{as } P^\pi \in (\mathcal{F}_{S_i}^F)' \\ &= (P^G)' \cap \pi(\mathcal{A}_{S_i}) \\ &= \left\{ \pi(A) \in \pi(\mathcal{A}_{S_i}) \mid \pi(A) = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, C \in P^G \pi(\mathcal{A}_{S_i}) P^G, D \in (\mathbb{1} - P^G) \pi(\mathcal{A}_{S_i}) (\mathbb{1} - P^G) \right\}, \end{aligned}$$

where the matrix decomposition corresponds to the decomposition $\mathcal{H}_\pi = \mathcal{H}_\pi^G \oplus (\mathcal{H}_\pi^G)^\perp$. Since $\pi(\mathcal{A}_{S_i}) = \mathcal{K}(\mathcal{H}_\pi)$ we have that $P^G \pi(\mathcal{A}_{S_i}) P^G \in \pi(\mathcal{A}_{S_i})$, hence

$$\pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) = P^G \pi(\mathcal{A}_{S_i}) P^G \oplus (\mathbb{1} - P^G) \pi(\mathcal{A}_{S_i}) (\mathbb{1} - P^G) = \mathcal{K}(\mathcal{H}_\pi^G) \oplus \mathcal{K}((\mathcal{H}_\pi^G)^\perp).$$

Since $\mathcal{D}_i^F \subseteq P_{S_i} \mathcal{O}_i^F$, we conclude that

$$\pi(\mathcal{D}_i^F \cap \mathcal{A}_{S_i}) = (\mathbb{1} - P^G) \pi(\mathcal{A}_{S_i}) (\mathbb{1} - P^G) = \mathcal{K}((\mathcal{H}_\pi^G)^\perp),$$

hence as π is faithful, $\mathcal{O}_i^F \cap \mathcal{A}_{S_i} \cong \mathcal{K}(\mathcal{H}_\pi^G) \oplus \mathcal{K}((\mathcal{H}_\pi^G)^\perp)$ and $\mathcal{D}_i^F \cap \mathcal{A}_{S_i} \cong \mathcal{K}((\mathcal{H}_\pi^G)^\perp)$ and hence the physical algebra in \mathcal{A}_{S_i} is

$$(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) / (\mathcal{D}_i^F \cap \mathcal{A}_{S_i}) \cong \mathcal{K}(\mathcal{H}_\pi^G),$$

where the isomorphism is obtained from restricting $\pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})$ to \mathcal{H}_π^G .

(i) Using the faithful homomorphism from $\mathcal{B}(\mathcal{H}_\pi)$ into $M(\mathcal{A}_{S_i})$, we obtain the corresponding statements in \mathcal{A}_{S_i} for the projection $P_\alpha \in M(\mathcal{A}_{S_i})$ such that $\pi(P_\alpha) = P^G$ from part (ii).

(iii) To see that $\mathcal{H}_\pi^G \neq \{0\}$, recall from the proof of Proposition 4.11(i) that we have a specific representation π as in (ii), and it is $\pi = \pi_1 \otimes \pi_2$ where π_1 is the Fock representation on the CAR-part of $\mathcal{A}_{S_i} = \text{CAR}(\mathcal{H}_{S_i}) \otimes \mathcal{L}^{(S_i)}$, and π_2 is the (finite) tensor product of copies of the usual regular representation of $C(G) \rtimes_\lambda G \cong \mathcal{K}(\mathcal{H}) \cong \mathcal{L}_\ell$ on $L^2(G)$. The implementing unitaries of α^1 for π_1 act on the Fock space of the first factor by second quantized unitaries, hence the vacuum vector is an invariant vector. The implementing unitaries $W_{(h,s)} \in U(L^2(G))$ for the factor actions θ w.r.t. the representation π_0 (cf. Lemma 3.1) are given by $(W_{(h,s)}\psi)(g) := \psi(h^{-1}gs)$. It is clear that if ψ is constant then it is invariant w.r.t. W (as G is compact, the constant functions are in $L^2(G)$). Thus, the tensor product of the Fock vacuum with copies of these constant vectors will produce a nonzero element in \mathcal{H}_π^G . Thus by (ii) the factor algebra $(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})/(\mathcal{D}_i^F \cap \mathcal{A}_{S_i})$ is nontrivial, and as all representations as in (ii) are faithful, we obtain from (ii) that $\mathcal{K}(\mathcal{H}_\pi^G) \neq \{0\}$ hence $\mathcal{H}_\pi^G \neq \{0\}$ for all π as in (ii).

(iv) All representations $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H})$ decompose uniquely into $\pi = \pi_1 \oplus \pi_2$ where $\pi_1(P_J) = \mathbb{1}$ and $\pi_2(P_J) = 0$ as $[U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}]$ is an ideal of $\mathcal{F}_{S_i}^F$. It suffices to verify the claim in all representations of these two types, since then it follows for the universal representation. If $\pi_1(P_J) = \mathbb{1}$, then π_1 is nondegenerate on $[U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}] \cong \mathcal{A}_{S_i} \rtimes_\alpha (\text{Gau}_d S_i)$, hence on \mathcal{A}_{S_i} by this property for crossed products. In particular, for the representation in (ii) we get that $\pi_1((\mathbb{1} - P_{S_i})P_J) = \pi_1(\mathbb{1} - P_{S_i}) = P^G = \pi_1(P_\alpha)$. This determines it in the multiplier of \mathcal{A}_{S_i} , hence for any representation which is nondegenerate on \mathcal{A}_{S_i} , which includes all representations of $\mathcal{F}_{S_i}^F$ such that $\pi(P_J) = \mathbb{1}$. On the other hand, for a representation π_2 with $\pi_2(P_J) = 0$, we have that $\pi_2((\mathbb{1} - P_{S_i})P_J) = 0$. Moreover as $\mathcal{A}_{S_i} \subset [U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}]$ we have (using ideal structure):

$$P_\alpha \in M(\mathcal{A}_{S_i}) \subset \mathcal{A}_{S_i}'' \subset [U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}]'' = P_J(\mathcal{F}_{S_i}^F)'' \subset (\mathcal{F}_{S_i}^F)''$$

and so it is clear that $\pi_2(P_J) = 0$ implies that $\pi_2(P_\alpha) = 0$. Hence we conclude that $P_\alpha = (\mathbb{1} - P_{S_i})P_J$. \blacksquare

Observe that as $\mathcal{K}(\mathcal{H}_1) \cong \mathcal{K}(\mathcal{H}_2)$ iff $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2)$, we conclude from (ii) that $\dim(\mathcal{H}_\pi^G)$ is the same for all representations $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ which are irreducible on \mathcal{A}_{S_i} .

We obtained above precisely the same result for the physical algebra than what the traditional constraint method produces for this system:

Theorem 4.13. *For the system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$ above, the set of traditional observables in \mathcal{A}_{S_i} , i.e. $\mathcal{C}_i' \cap \mathcal{A}_{S_i}$, is the gauge invariant part $\mathcal{A}_{S_i}^G$ of \mathcal{A}_{S_i} . Let $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ be an*

irreducible representation which is irreducible on \mathcal{A}_{S_i} , then

$$\pi(\mathcal{A}_{S_i}^G) = \pi(\mathcal{C}'_i \cap \mathcal{A}_{S_i}) = \mathcal{K}(\mathcal{H}_\pi^G) \oplus \left(\mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)' \right).$$

The “ideal generated in $\mathcal{A}_{S_i}^G$ by the Gauss law” is taken to be $\mathcal{D}_i^F \cap \mathcal{A}_{S_i}^G$, and its image w.r.t. π is

$$\pi(\mathcal{D}_i^F \cap \mathcal{A}_{S_i}^G) = \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)',$$

hence the (traditional) algebra of physical observables is

$$\mathcal{A}_{S_i}^G / (\mathcal{D}_i^F \cap \mathcal{A}_{S_i}^G) \cong \mathcal{K}(\mathcal{H}_\pi^G)$$

where the isomorphism is obtained by restricting $\pi(\mathcal{A}_{S_i}^G)$ to \mathcal{H}_π^G .

Proof: As \mathcal{A}_{S_i} commutes with $\mathcal{U}_0 \setminus \mathcal{U}_0^{S_i}$ we have $\mathcal{A}_{S_i}^G = \mathcal{A}_{S_i}^{G_i} = \mathcal{A}_{S_i} \cap (\mathcal{U}_0^{S_i})' = \mathcal{A}_{S_i} \cap \mathcal{C}'_i \subseteq \mathcal{O}_i^F \cap \mathcal{A}_{S_i}$. Thus by Theorem 4.12(ii):

$$\pi(\mathcal{A}_{S_i}^G) \subseteq \pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) = \mathcal{K}(\mathcal{H}_\pi^G) \oplus \mathcal{K}((\mathcal{H}_\pi^G)^\perp) = \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \mid C \in \mathcal{K}(\mathcal{H}_\pi^G), D \in \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \right\}$$

Note that for $g \in \text{Gau}_d S_i$ we have the decomposition

$$U_g^\pi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & V_g \end{pmatrix} \quad \text{for} \quad V_g = (\mathbb{1} - P^G)U_g^\pi \in U((\mathcal{H}_\pi^G)^\perp),$$

hence

$$\begin{aligned} \pi(\mathcal{A}_{S_i}^G) &\subseteq (U_{\text{Gau}_d S_i}^\pi)' \cap \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \mid C \in \mathcal{K}(\mathcal{H}_\pi^G), D \in \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \right\} \\ &= \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \mid C \in \mathcal{K}(\mathcal{H}_\pi^G), D \in \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap (V_{\text{Gau}_d S_i})' \right\} \\ &= \mathcal{K}(\mathcal{H}_\pi^G) \oplus \left(\mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)' \right). \end{aligned}$$

We prove that the inclusion is an equality. If $A + B \in \mathcal{K}(\mathcal{H}_\pi^G) \oplus \left(\mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)' \right)$, then as $A \in \mathcal{K}(\mathcal{H}_\pi^G) \subset \pi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \supset \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)' \ni B$, and π is faithful on \mathcal{A}_{S_i} , there are unique $A_0, B_0 \in \mathcal{O}_i^F \cap \mathcal{A}_{S_i}$ such that $\pi(A_0) = A$ and $\pi(B_0) = B$. As π is covariant and both A and B commute with all implementers U_g^π , $g \in \text{Gau}_d S_i$, we see that $A_0, B_0 \in \mathcal{O}_i^F \cap \mathcal{A}_{S_i}$ are G -invariant, using faithfulness of π on \mathcal{A}_{S_i} , hence $A_0 + B_0 \in \mathcal{A}_{S_i}^G$. Thus

$$\pi(\mathcal{A}_{S_i}^G) = \mathcal{K}(\mathcal{H}_\pi^G) \oplus \left(\mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)' \right).$$

Then

$$\begin{aligned} \pi(\mathcal{D}_i^F \cap \mathcal{A}_{S_i}^G) &= \pi(\mathcal{D}_i^F \cap \mathcal{A}_{S_i}) \cap \pi(\mathcal{A}_{S_i}^G) \\ &= \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \left(\mathcal{K}(\mathcal{H}_\pi^G) \oplus \left(\mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)' \right) \right) \\ &= \mathcal{K}((\mathcal{H}_\pi^G)^\perp) \cap \pi(\mathcal{C}'_i)', \end{aligned}$$

and clearly $\pi(\mathcal{A}_{S_i}^G) \upharpoonright \mathcal{H}_\pi^G = \mathcal{K}(\mathcal{H}_\pi^G)$, and the claim follows. \blacksquare

Now $\mathcal{F}_{S_i}^F = C^*(\mathcal{U}_0^{S_i} \cup \mathcal{A}_{S_i})$ hence to obtain the full algebras \mathcal{D}_i^F and \mathcal{O}_i^F we need to consider the role of $\mathcal{U}_0^{S_i}$. By construction, $\mathcal{C}_i \subset \mathcal{D}_i^F \subset \mathcal{O}_i^F \ni \mathbb{1}$, hence $\mathcal{U}_0^{S_i} = \mathcal{C}_i + \mathbb{1} \subset \mathcal{O}_i^F$.

Theorem 4.14. *With notation as above, we have that*

$$\mathcal{O}_i^F = [U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i} + \mathbb{C})] = [P_\alpha \mathcal{A}_{S_i} P_\alpha] + [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)] + [U_{\text{Gau } S_i}]$$

$$\text{and } \mathcal{D}_i^F = \mathcal{B}_i + [\mathcal{C}_i U_{\text{Gau } S_i}] \quad \text{where}$$

$$\begin{aligned} \mathcal{B}_i &:= [\mathcal{C}_i U_{\text{Gau } S_i} (\mathcal{D}_i^F \cap \mathcal{A}_{S_i})] = [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] \\ &= [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)] = \mathcal{D}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}]. \end{aligned}$$

$$\text{Moreover:} \quad \mathcal{R}_i^F = \mathcal{O}_i^F / \mathcal{D}_i^F \cong [P_\alpha \mathcal{A}_{S_i} P_\alpha] + \mathbb{C} \cong \mathcal{K}(\mathcal{H}_\pi^G) + \mathbb{C},$$

where the last isomorphism is concretely realized in the representation π of Theorem 4.12(ii).

Proof: From the fact that $\mathcal{U}_0^{S_i}$ generates $U_{\text{Gau } S_i}$ as a group, we have that $C^*(\mathcal{U}_0^{S_i}) = [U_{\text{Gau } S_i}]$ hence via the implementing relations we obtain:

$$\mathcal{F}_{S_i}^F = C^*(\mathcal{U}_0^{S_i} \cup \mathcal{A}_{S_i}) = (\mathcal{A}_{S_i} + \mathbb{C}) \rtimes_\alpha (\text{Gau}_d S_i) = [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] + [U_{\text{Gau } S_i}].$$

As $\mathcal{U}_0^{S_i} \subset \mathcal{O}_i^F$ it is obvious that

$$\mathcal{O}_i^F \supseteq [U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i} + \mathbb{C})],$$

so we show the converse inclusion. By Theorem 4.8 we have that $\mathcal{O}_i^F = P_{S_i}' \cap \mathcal{F}_{S_i}^F$ where $P_{S_i} \in (\mathcal{F}_{S_i}^F)''$ is the open projection of Theorem 4.7 for the constraint system $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$, specified by $\omega(P_{S_i}) = 0$ iff $\pi_\omega(\mathcal{C}_i)\Omega_\omega = 0$ for $\omega \in \mathfrak{S}(\mathcal{F}_{S_i}^F)$. Let $B \in \mathcal{O}_i^F \subset [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] + [U_{\text{Gau } S_i}] = \mathcal{F}_{S_i}^F$, then since we already know that $[U_{\text{Gau } S_i}] \subset \mathcal{O}_i^F$, it suffices to assume that $B \in [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] \cap \mathcal{O}_i^F$, i.e.

$$B = \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} U_j^{(n)} A_j^{(n)} \quad \text{for} \quad U_j^{(n)} \in U_{\text{Gau } S_i} \quad \text{and} \quad A_j^{(n)} \in \mathcal{A}_{S_i}$$

such that $[B, P_{S_i}] = 0$, i.e. $B = P_{S_i} B P_{S_i} + (\mathbb{1} - P_{S_i}) B (\mathbb{1} - P_{S_i})$. Recall from Theorem 4.12(iv) that $P_\alpha = (\mathbb{1} - P_{S_i}) P_J$ where $P_J \in (\mathcal{F}_{S_i}^F)''$ is the central projection determined by the ideal $[U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}]$ of $\mathcal{F}_{S_i}^F$. Thus, since B is in this ideal, we get

$$B = P_J B P_J = (\mathbb{1} - P_\alpha) B (\mathbb{1} - P_\alpha) + P_\alpha B P_\alpha$$

and hence

$$B = \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} U_j^{(n)} ((\mathbb{1} - P_\alpha) A_j^{(n)} (\mathbb{1} - P_\alpha) + P_\alpha A_j^{(n)} P_\alpha). \quad (4.11)$$

However we have that $(\mathbb{1} - P_\alpha)A_j^{(n)}(\mathbb{1} - P_\alpha) + P_\alpha A_j^{(n)}P_\alpha \in \mathcal{O}_i^F \cap \mathcal{A}_{S_i}$ by Theorem 4.12(i), and hence $B \in [U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})]$. We conclude that $\mathcal{O}_i^F = [U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i} + \mathbb{C})]$, as claimed.

By Theorem 4.12(i) we have a projection $P_\alpha \in M(\mathcal{A}_{S_i})$ such that $U_g P_\alpha = P_\alpha = P_\alpha U_g$ for all $g \in \text{Gau } S_i$ and

$$\mathcal{O}_i^F \cap \mathcal{A}_{S_i} = P_\alpha \mathcal{A}_{S_i} P_\alpha \oplus (\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha) \quad (4.12)$$

from which we obtain the second equality for \mathcal{O}_i^F .

Next, recall from Theorem 4.4(iv) that we have

$$\mathcal{D}_i^F = [\mathcal{C}_i \mathcal{O}_i^F] = [\mathcal{C}_i U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] + [\mathcal{C}_i U_{\text{Gau } S_i}].$$

However $DP_{S_i} = D$ for all $D \in \mathcal{D}_i^F \supset \mathcal{C}_i$, hence $[\mathcal{C}_i U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] = [\mathcal{C}_i P_{S_i} U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] = [\mathcal{C}_i (\mathbb{1} - P_\alpha) U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] = [\mathcal{C}_i U_{\text{Gau } S_i}(\mathcal{D}_i^F \cap \mathcal{A}_{S_i})]$ by the decomposition in Equation (4.11) for elements of $[U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})]$. This establishes the first equality $\mathcal{B}_i = [\mathcal{C}_i U_{\text{Gau } S_i}(\mathcal{D}_i^F \cap \mathcal{A}_{S_i})]$ for \mathcal{D}_i^F .

It is clear via Lemma 4.10, that $\mathcal{B}_i = \mathcal{D}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}]$ since $[U_{\text{Gau } S_i} \mathcal{A}_{S_i}]$ is a closed two-sided ideal of $\mathcal{F}_{S_i}^F = [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] + [U_{\text{Gau } S_i}]$ and $[\mathcal{C}_i U_{\text{Gau } S_i}] \subset [U_{\text{Gau } S_i}]$. For the second equality for \mathcal{B}_i use $\mathcal{C}_i P_\alpha = 0$ and

$$\begin{aligned} \mathcal{D}_i^F &= [\mathcal{C}_i \mathcal{O}_i^F \mathcal{C}_i] = [\mathcal{C}_i ([P_\alpha \mathcal{A}_{S_i} P_\alpha] + [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)] + [U_{\text{Gau } S_i}]) \mathcal{C}_i] \\ &= [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] + [U_{\text{Gau } S_i} \mathcal{C}_i] \end{aligned}$$

and so $\mathcal{B}_i = \mathcal{D}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] = [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i]$. For the third equality recall from Theorem 4.12 that $(\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha) = \mathcal{D}_i^F \cap \mathcal{A}_{S_i} \subset \mathcal{D}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] = [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i]$ hence

$$[(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)] \subseteq [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] \quad -(*)$$

To get equality, note that $\mathcal{O}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] = [P_\alpha \mathcal{A}_{S_i} P_\alpha] + [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)]$ is a sum of two ideals with trivial intersection, and that $[\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] = \mathcal{D}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] \subset \mathcal{O}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}]$ is also an ideal, as it is the intersection of two ideals. By (*) it suffices to show that $[\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] \cap [P_\alpha \mathcal{A}_{S_i} P_\alpha] = \{0\}$. If $B \in [P_\alpha \mathcal{A}_{S_i} P_\alpha]$ then $P_\alpha B = B$, however for any element $B \in [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i]$ we have $P_\alpha B = 0$. It follows that $[\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] \cap [P_\alpha \mathcal{A}_{S_i} P_\alpha] = \{0\}$ and hence $\mathcal{B}_i = [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] = [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)]$.

Finally, let $\xi : \mathcal{O}_i^F \rightarrow \mathcal{O}_i^F / \mathcal{D}_i^F$ be the constraining homomorphism. Then $\xi(U_{\text{Gau } S_i}) = \mathbb{1}$ hence

$$\mathcal{R}_i^F = \xi(\mathcal{O}_i^F) = \xi(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) + \mathbb{C} = (\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) / (\mathcal{D}_i^F \cap \mathcal{A}_{S_i}) + \mathbb{C} \cong P_\alpha \mathcal{A}_{S_i} P_\alpha + \mathbb{C}$$

by Theorem 4.12(i), and using the faithful representation π we also get from Theorem 4.12(ii) that $\mathcal{R}_i^F \cong \mathcal{K}(\mathcal{H}_\pi^G) + \mathbb{C}$. \blacksquare

We have now fully specified the constraint data for the finite constraint systems $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$. For the physical algebra, we obtained the same result from two different methods.

4.4 Solving the local constraint systems.

Our aim in this section is to use the results above for the finite lattice constraint systems $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$ to solve the corresponding ‘‘local’’ constraint systems $(\mathcal{F}_{S_i}, \mathcal{C}_i)$ in the infinite lattice. Recall that

$$\begin{aligned}\mathcal{F}_{S_i}^F &= [U_{\text{Gau } S_i} \cdot (\mathcal{A}_{S_i} \oplus \mathbb{C})] = (\mathcal{A}_{S_i} + \mathbb{C}) \rtimes_{\alpha} (\text{Gau}_d S_i) \\ &= [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] + [U_{\text{Gau } S_i}] \subset M(\mathfrak{A}_{\Lambda} \rtimes_{\alpha} (\text{Gau}_d \Lambda)) \supset \mathcal{F}_e \quad \text{and} \\ \mathcal{F}_{S_i} &= [U_{\text{Gau } S_i} \cdot (\mathfrak{A}_{S_i} \oplus \mathbb{C})] = [U_{\text{Gau } S_i} \cdot \mathfrak{A}_{S_i}] + [U_{\text{Gau } S_i}] \subset \mathcal{F}_e\end{aligned}$$

where $\mathcal{A}_{S_i} := \mathfrak{F}_{S_i} \otimes \mathcal{L}^{S_i}$ and $\mathfrak{A}_{S_i} := \mathfrak{F}_{S_i} \otimes \mathcal{L}_{S_i}[E]$ and we have the same constraints $\mathcal{C}_i := \mathcal{U}_0^{S_i} - \mathbb{1}$ for both cases. As $\mathcal{F}_{S_i}^F$ differs from \mathcal{F}_{S_i} only by the replacement of \mathcal{L}^{S_i} by $\mathcal{L}_{S_i}[E]$, we examine the relation between these algebras. Recall that

$$\mathcal{L}_{S_i}[E] := C^*\left(\bigcup_{\mathbf{n} \in \mathbb{N}^{\infty}} \mathcal{E}_{S_i}[\mathbf{n}]\right) \subset \mathcal{L}[E],$$

where $\mathcal{E}_{S_i}[\mathbf{n}]$ denotes those elementary tensors in $\bigcup_{k \in \mathbb{N}} \mathcal{L}^{(k)} \otimes E[\mathbf{n}]_{k+1}$ which can only differ from $E[\mathbf{n}]_1 = E_{n_1}^{(1)} \otimes E_{n_2}^{(2)} \otimes \dots$ in entries corresponding to links in $\Lambda_{S_i}^1$.

Lemma 4.15. *Let $E_{S_i}[\mathbf{n}] \subset M(\mathcal{L}[E])$ consist of $E[\mathbf{n}]_1 = E_{n_1}^{(1)} \otimes E_{n_2}^{(2)} \otimes \dots$ except for entries corresponding to links in $\Lambda_{S_i}^1$, where it is the identity. Let $\mathcal{T}_{S_i}[E] := C^*\left(\bigcup_{\mathbf{n} \in \mathbb{N}^{\infty}} E_{S_i}[\mathbf{n}]\right) = \left[\bigcup_{\mathbf{n} \in \mathbb{N}^{\infty}} E_{S_i}[\mathbf{n}]\right] \subset M(\mathcal{L}[E])$ denote the ‘‘infinite tails’’, then*

$$\begin{aligned}\mathcal{L}_{S_i}[E] &= [\mathcal{L}^{S_i} \cdot \mathcal{T}_{S_i}[E]] \cong \mathcal{L}^{S_i} \otimes \mathcal{T}_{S_i}[E] \\ \text{and} \quad \mathcal{F}_{S_i} &= [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau } S_i}] \otimes \mathbb{1}.\end{aligned}$$

Proof: Identify \mathcal{L}^{S_i} with $\mathcal{L}^{S_i} \otimes \mathbb{1} \subset M(\mathcal{L}[E])$, by which we mean that for the elementary tensors, only the entries corresponding to links in $\Lambda_{S_i}^1$ are not the identity. Observe that $\mathcal{L}_{S_i}[E] = [\mathcal{L}^{S_i} \cdot \mathcal{T}_{S_i}[E]]$. In fact, as both $\mathcal{L}^{S_i} \otimes \mathbb{1}$ and $\mathcal{T}_{S_i}[E]$ are generated by elementary tensors with their only nontrivial parts in complementary factors, it is clear that the algebraic span of the products of these generating tensors is an algebraic tensor product. Recall from the line below equation (2.7) that the C*-norm is taken in $M(\mathcal{L}[\mathbf{1}])$, and $\mathcal{L}[\mathbf{1}]$ is a tensor product over the same index set as the elementary tensors above. It is well-known that for a tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$, its multiplier algebra $M(\mathcal{A} \otimes_{\min} \mathcal{B})$ in general strictly contains $M(\mathcal{A}) \otimes_{\min} M(\mathcal{B})$ (cf. [1] p286–287), and hence on this subalgebra the norm of the multiplier algebra is a cross norm. Thus the norm of $M(\mathcal{L}[\mathbf{1}]) \supset \mathcal{L}^{S_i} \otimes \mathbb{1} \cup \mathcal{T}_{S_i}[E]$ is a cross norm, so if we take the closure of the algebraic tensor product mentioned above, we get that $\mathcal{L}_{S_i}[E] = [\mathcal{L}^{S_i} \cdot \mathcal{T}_{S_i}[E]] = \mathcal{L}^{S_i} \otimes \mathcal{T}_{S_i}[E]$.

Note that $\mathfrak{A}_{S_i} = \mathfrak{F}_{S_i} \otimes \mathcal{L}_{S_i}[E] = \mathfrak{F}_{S_i} \otimes \mathcal{L}^{S_i} \otimes \mathcal{T}_{S_i}[E] = \mathcal{A}_{S_i} \otimes \mathcal{T}_{S_i}[E]$. Moreover $[U_{\text{Gau } S_i} \cdot \mathfrak{A}_{S_i}] = \mathfrak{A}_{S_i} \rtimes_{\alpha} (\text{Gau}_d S_i) = (\mathcal{A}_{S_i} \otimes \mathcal{T}_{S_i}[E]) \rtimes_{\alpha} (\text{Gau}_d S_i)$, and the action α acts trivially on $\mathbb{1} \otimes \mathcal{T}_{S_i}[E]$, hence it is the product action $\alpha = \alpha^F \otimes \iota$ where $\alpha^F : \text{Gau}_d S_i \rightarrow$

Aut \mathcal{A}_{S_i} is the restriction of α to the finite part $\mathcal{A}_{S_i} \otimes \mathbb{1} = \mathfrak{F}_{S_i} \otimes \mathcal{L}^{S_i} \otimes \mathbb{1}$. It follows from Lemma 2.75 in [43] that $(\mathcal{A}_{S_i} \otimes \mathcal{T}_{S_i}[E]) \rtimes_{\alpha} (\text{Gau}_d S_i) = (\mathcal{A}_{S_i} \rtimes_{\alpha^F} (\text{Gau}_d S_i)) \otimes \mathcal{T}_{S_i}[E]$, using the fact that $\mathcal{T}_{S_i}[E]$ is commutative, hence nuclear. Hence the implementing unitaries are of the form $U_g \otimes \mathbb{1}$, and so

$$[U_{\text{Gau}_{S_i}} \cdot \mathfrak{A}_{S_i}] = (\mathcal{A}_{S_i} \rtimes_{\alpha^F} (\text{Gau}_d S_i)) \otimes \mathcal{T}_{S_i}[E] = [U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E]$$

and this proves the last equality, using $\mathcal{F}_{S_i} = [U_{\text{Gau}_{S_i}} \cdot \mathfrak{A}_{S_i}] + [U_{\text{Gau}_{S_i}}]$. \blacksquare

Theorem 4.16. *Given the constraint systems $(\mathcal{F}_{S_i}^F, \mathcal{C}_i)$ and $(\mathcal{F}_{S_i}, \mathcal{C}_i)$ above, and their associated constraint data $(\mathcal{D}_i^F, \mathcal{O}_i^F, \mathcal{R}_i^F, \xi_i^F)$ and $(\mathcal{D}_i, \mathcal{O}_i, \mathcal{R}_i, \xi_i)$ respectively, then*

$$\mathcal{D}_i = \mathcal{B}_i \otimes \mathcal{T}_{S_i}[E] + [\mathcal{C}_i U_{\text{Gau}_{S_i}}] \otimes \mathbb{1}$$

where $\mathcal{B}_i = [\mathcal{C}_i U_{\text{Gau}_{S_i}} (\mathcal{D}_i^F \cap \mathcal{A}_{S_i})] = [\mathcal{C}_i U_{\text{Gau}_{S_i}} \mathcal{A}_{S_i} \mathcal{C}_i] = [(\mathbb{1} - P_{\alpha}) U_{\text{Gau}_{S_i}} \mathcal{A}_{S_i} (\mathbb{1} - P_{\alpha})]$ for a projection $P_{\alpha} \in M(\mathcal{A}_{S_i})$ such that $U P_{\alpha} = P_{\alpha} = P_{\alpha} U$ for all $U \in U_{\text{Gau}_{S_i}} \supset \mathcal{U}_0^{S_i}$ by Theorem 4.12. Moreover

$$\mathcal{O}_i^F = [U_{\text{Gau}_{S_i}} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] + [U_{\text{Gau}_{S_i}}] \quad \text{and} \quad \mathcal{O}_i = [U_{\text{Gau}_{S_i}} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau}_{S_i}}]$$

where $\mathcal{O}_i^F \cap \mathcal{A}_{S_i} = P_{\alpha} \mathcal{A}_{S_i} P_{\alpha} \oplus (\mathbb{1} - P_{\alpha}) \mathcal{A}_{S_i} (\mathbb{1} - P_{\alpha})$. Furthermore

$$\mathcal{R}_i = \xi_i^F (\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E] + \mathbb{C} \cong P_{\alpha} \mathcal{A}_{S_i} P_{\alpha} \otimes \mathcal{T}_{S_i}[E] + \mathbb{C} \cong \mathcal{K}(\mathcal{H}_{\pi}^G) \otimes \mathcal{T}_{S_i}[E] + \mathbb{C}$$

where $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_{\pi})$ is any representation which is irreducible on \mathcal{A}_{S_i} .

Proof: First consider $\mathcal{F}_{S_i} = [U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau}_{S_i}}] \otimes \mathbb{1}$. Note that $[U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E]$ is a closed two-sided ideal of \mathcal{F}_{S_i} . By an analogous proof to Lemma 4.10, we see that $[U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E] \cap [U_{\text{Gau}_{S_i}}] \otimes \mathbb{1} = \{0\}$, and hence decompositions in terms of these two spaces are unique. By Theorem 4.4 we have $\mathcal{D}_i = [\mathcal{C}_i \mathcal{F}_{S_i} \mathcal{C}_i]$ hence

$$\begin{aligned} \mathcal{D}_i &= [\mathcal{C}_i U_{\text{Gau}_{S_i}} \mathcal{A}_{S_i} \mathcal{C}_i] \otimes \mathcal{T}_{S_i}[E] + [\mathcal{C}_i U_{\text{Gau}_{S_i}}] \otimes \mathbb{1} \\ &= \mathcal{B}_i \otimes \mathcal{T}_{S_i}[E] + [\mathcal{C}_i U_{\text{Gau}_{S_i}}] \otimes \mathbb{1} \end{aligned}$$

where $\mathcal{B}_i = [\mathcal{C}_i U_{\text{Gau}_{S_i}} \mathcal{A}_{S_i} \mathcal{C}_i] = [\mathcal{C}_i U_{\text{Gau}_{S_i}} (\mathcal{D}_i^F \cap \mathcal{A}_{S_i})] = [(\mathbb{1} - P_{\alpha}) U_{\text{Gau}_{S_i}} \mathcal{A}_{S_i} (\mathbb{1} - P_{\alpha})]$ by Theorem 4.14, which establishes the first claim.

The equality for \mathcal{O}_i^F follows directly from $\mathcal{O}_i^F = [U_{\text{Gau}_{S_i}} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i} + \mathbb{C})]$, obtained in Theorem 4.14. Recall from equation (4.12) that

$$\mathcal{O}_i^F \cap \mathcal{A}_{S_i} = P_{\alpha} \mathcal{A}_{S_i} P_{\alpha} \oplus (\mathbb{1} - P_{\alpha}) \mathcal{A}_{S_i} (\mathbb{1} - P_{\alpha}). \quad -(*)$$

We now prove the stated equality for \mathcal{O}_i . Let $A \in \mathcal{F}_{S_i}$, then by the decomposition for \mathcal{F}_{S_i} we may write $A = F + C$ where $F \in [U_{\text{Gau}_{S_i}} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E]$ and $C \in [U_{\text{Gau}_{S_i}}] \otimes \mathbb{1}$.

As $C \in \mathcal{O}_i$ already, we only have to consider F . If $F \in [U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$, then

$$F\mathcal{D}_i = F(\mathcal{B}_i \otimes \mathcal{T}_{S_i}[E] + [\mathcal{C}_i U_{\text{Gau } S_i}] \otimes \mathbb{1}) \subseteq \mathcal{B}_i \otimes \mathcal{T}_{S_i}[E] \subset \mathcal{D}_i$$

because $[U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \mathcal{B}_i \subseteq [U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i}(\mathbb{1} - P_\alpha)]$
 $\subseteq [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i}(\mathbb{1} - P_\alpha)] = \mathcal{B}_i \quad (\text{as } [P_\alpha, \mathcal{O}_i^F] = 0)$

and $[U_{\text{Gau } S_i} \cdot (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] [\mathcal{C}_i U_{\text{Gau } S_i}] \subseteq [(\mathbb{1} - P_\alpha) U_{\text{Gau } S_i} \mathcal{A}_{S_i}(\mathbb{1} - P_\alpha)] = \mathcal{B}_i$

using $\mathcal{C}_i P_\alpha = 0$, and the decomposition (*) for $\mathcal{O}_i^F \cap \mathcal{A}_{S_i}$ stated above. Likewise we also get that $\mathcal{D}_i F \subseteq \mathcal{D}_i$, and hence $F \in \mathcal{O}_i$, so we have shown that

$$\mathcal{O}_i \supseteq [U_{\text{Gau } S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau } S_i}].$$

We prove the reverse inclusion. Let $A \in \mathcal{O}_i$, then as above we may write $A = F + C$ where $F \in [U_{\text{Gau } S_i} \cdot \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E]$ and $C \in [U_{\text{Gau } S_i}] \otimes \mathbb{1}$. As $C \in \mathcal{O}_i$ we have $F \in \mathcal{O}_i$, so we need to show that $F \in [U_{\text{Gau } S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$. Let $\widehat{P}_\alpha := P_\alpha \otimes \mathbb{1}$, then $\widehat{P}_\alpha F \widehat{P}_\alpha + (\mathbb{1} - \widehat{P}_\alpha) F (\mathbb{1} - \widehat{P}_\alpha) \in [U_{\text{Gau } S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$ by (*), so it remains to show that the remaining part of F :

$$\widetilde{F} := \widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) + (\mathbb{1} - \widehat{P}_\alpha) F \widehat{P}_\alpha \in [U_{\text{Gau } S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E].$$

Explicitly $F \in [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E]$ has the form

$$F = \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} U_j^{(n)} A_j^{(n)} \otimes T_j^{(n)} \quad \text{for } U_j^{(n)} \in U_{\text{Gau } S_i}, A_j^{(n)} \in \mathcal{A}_{S_i}, T_j^{(n)} \in \mathcal{T}_{S_i}[E]$$

so $\widetilde{F} := \widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) + (\mathbb{1} - \widehat{P}_\alpha) F \widehat{P}_\alpha$

$$= \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} U_j^{(n)} \left(P_\alpha A_j^{(n)} (\mathbb{1} - P_\alpha) + (\mathbb{1} - P_\alpha) A_j^{(n)} P_\alpha \right) \otimes T_j^{(n)} \in \mathcal{O}_i$$

since the other part $\widehat{P}_\alpha F \widehat{P}_\alpha + (\mathbb{1} - \widehat{P}_\alpha) F (\mathbb{1} - \widehat{P}_\alpha)$ is in \mathcal{O}_i . Thus \widetilde{F} is in the relative multiplier of \mathcal{D}_i . Now $(\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha) \otimes \mathcal{T}_{S_i}[E] \subset \mathcal{D}_i$, and so

$$\widetilde{F} \cdot (\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha) \otimes \mathcal{T}_{S_i}[E] = \widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) \cdot \left((\mathbb{1} - P_\alpha) \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha) \otimes \mathcal{T}_{S_i}[E] \right) \subset \mathcal{D}_i,$$

and in fact it is in $\mathcal{D}_i \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E] = \mathcal{B}_i \otimes \mathcal{T}_{S_i}[E]$. However $P_\alpha \in M(\mathcal{A}_{S_i})$ by Theorem 4.12(iv), so if $\{J_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{A}_{S_i}$ is an approximate identity, then $(\mathbb{1} - P_\alpha) J_\lambda (\mathbb{1} - P_\alpha) \rightarrow (\mathbb{1} - P_\alpha)$ in the strict topology of $M(\mathcal{A}_{S_i})$ hence in the strong operator topology of \mathcal{A}_{S_i}'' . Recall that $\mathcal{T}_{S_i}[E] := C^* \left(\bigcup_{\mathbf{n} \in \mathbb{N}^\infty} E_{S_i}[\mathbf{n}] \right) \subset M(\mathcal{L}[E])$, and let $\{K_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{T}_{S_i}[E]$ be an approximate identity of it. Consider

$$\begin{aligned} \widetilde{F} \cdot (\mathbb{1} - P_\alpha) J_\lambda (\mathbb{1} - P_\alpha) \otimes K_\gamma &= \widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) \cdot \left((\mathbb{1} - P_\alpha) J_\lambda (\mathbb{1} - P_\alpha) \otimes K_\gamma \right) \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} U_j^{(n)} P_\alpha A_j^{(n)} (\mathbb{1} - P_\alpha) J_\lambda (\mathbb{1} - P_\alpha) \otimes T_j^{(n)} K_\gamma \end{aligned}$$

which is in $\mathcal{B}_i \otimes \mathcal{T}_{S_i}[E] \subset \mathcal{D}_i$. Construct the faithful representation

$$\pi_1 \otimes \pi_2 : [U_{\text{Gau}S_i} \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E] \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

where $\pi_1 : [U_{\text{Gau}S_i} \mathcal{A}_{S_i}] \rightarrow \mathcal{B}(\mathcal{H}_1)$ is the universal representation of $[U_{\text{Gau}S_i} \mathcal{A}_{S_i}]$ (which restricts on \mathcal{A}_{S_i} to its universal representation on its essential subspace), and $\pi_2 : \mathcal{T}_{S_i}[E] \rightarrow \mathcal{B}(\mathcal{H}_2)$ is the universal representation of $\mathcal{T}_{S_i}[E]$. Then

$$\pi \left(U_j^{(n)} P_\alpha A_j^{(n)} (\mathbb{1} - P_\alpha) J_\lambda (\mathbb{1} - P_\alpha) \otimes T_j^{(n)} K_\gamma \right) \rightarrow \pi \left(U_j^{(n)} P_\alpha A_j^{(n)} (\mathbb{1} - P_\alpha) \otimes T_j^{(n)} \right)$$

as $\lambda, \gamma \rightarrow \infty$ in strong operator topology. As the norm limit w.r.t. n can be interchanged with the strong operator limits w.r.t. λ, γ (since the product is continuous w.r.t. strong operator topology), this implies that

$$\begin{aligned} \pi \left(\tilde{F} \cdot (\mathbb{1} - P_\alpha) J_\lambda (\mathbb{1} - P_\alpha) \otimes K_\gamma \right) &\rightarrow \pi \left(\lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} U_j^{(n)} P_\alpha A_j^{(n)} (\mathbb{1} - P_\alpha) \otimes T_j^{(n)} \right) \\ &= \pi \left(\widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) \right) \in \pi(\mathcal{B}_i \otimes \mathcal{T}_{S_i}[E])^{-\text{s.op}} \end{aligned}$$

as $\lambda, \gamma \rightarrow \infty$ in strong operator topology. As $\mathcal{B}_i = [(\mathbb{1} - P_\alpha) U_{\text{Gau}S_i} \mathcal{A}_{S_i} (\mathbb{1} - P_\alpha)]$, we have $(\mathbb{1} - \widehat{P}_\alpha) B = B$ for $B \in \mathcal{B}_i \otimes \mathcal{T}_{S_i}[E]$, and hence $\pi(\mathbb{1} - \widehat{P}_\alpha) \tilde{B} = \tilde{B}$ for all $\tilde{B} \in \pi(\mathcal{B}_i \otimes \mathcal{T}_{S_i}[E])^{-\text{s.op}}$. Thus

$$\pi \left(\widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) \right) = \pi(\mathbb{1} - \widehat{P}_\alpha) \pi \left(\widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) \right) = 0$$

and as π is faithful, $\widehat{P}_\alpha F (\mathbb{1} - \widehat{P}_\alpha) = 0$. Likewise we get that $(\mathbb{1} - \widehat{P}_\alpha) F \widehat{P}_\alpha = 0$, and hence $F = \widehat{P}_\alpha F \widehat{P}_\alpha + (\mathbb{1} - \widehat{P}_\alpha) F (\mathbb{1} - \widehat{P}_\alpha) = 0 \in [U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$ and so

$$\mathcal{O}_i = [U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau}S_i}].$$

To obtain the claimed equality for \mathcal{R}_i , we consider the factor map $\xi_i : \mathcal{O}_i \rightarrow \mathcal{R}_i$. Since $[U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] \cap [U_{\text{Gau}S_i}] = \{0\}$, we can analyze $\xi_i([U_{\text{Gau}S_i}])$ and $\xi_i([U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E])$ independently. By construction, as $\mathcal{D}_i = [\mathcal{C}_i \mathcal{O}_i \mathcal{C}_i]$ by Theorem 4.4, thus factoring \mathcal{O}_i by \mathcal{D}_i , is a homomorphism which puts $\mathcal{C}_i = \mathcal{U}_0^{S_i} - \mathbb{1}$ to zero, hence $\xi_i(\mathcal{U}_0^{S_i}) = \mathbb{1}$ and as $U_{\text{Gau}S_i}$ is generated as a group by $\mathcal{U}_0^{S_i}$, we have $\xi_i(U_{\text{Gau}S_i}) = \mathbb{1}$ and hence $\xi_i([U_{\text{Gau}S_i}]) = \mathbb{C}\mathbb{1}$.

Next, recall that as $\mathcal{T}_{S_i}[E]$ is commutative (hence nuclear), the tensor norm of $[U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$ is unique. Thus by II.9.6.6 in [2] we have that $\text{Ker}(\check{\xi}_i^F \otimes \iota) = \text{Ker}(\check{\xi}_i^F) \otimes \mathcal{T}_{S_i}[E]$ where ι is the identity map of $\mathcal{T}_{S_i}[E]$ and where $\check{\xi}_i^F : [U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \rightarrow \mathcal{R}_i^F$ is the restriction of ξ_i^F to $[U_{\text{Gau}S_i} (\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] = \mathcal{O}_i^F \cap [U_{\text{Gau}S_i} \mathcal{A}_{S_i}]$. Now $\text{Ker}(\check{\xi}_i^F) = \mathcal{D}_i^F \cap [U_{\text{Gau}S_i} \mathcal{A}_{S_i}] = \mathcal{B}_i$ by Theorem 4.14, hence $\text{Ker}(\check{\xi}_i^F \otimes \iota) = \mathcal{B}_i \otimes \mathcal{T}_{S_i}[E] = \mathcal{D}_i \cap [U_{\text{Gau}S_i} \mathcal{A}_{S_i}] \otimes \mathcal{T}_{S_i}[E]$. As this is precisely the kernel of

ξ_i restricted to $[U_{\text{Gau}S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$, we conclude that ξ_i coincides with $\check{\xi}_i^F \otimes \iota$ on $[U_{\text{Gau}S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]$, thus

$$\xi_i([U_{\text{Gau}S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E]) = \xi_i^F(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E]$$

using $\check{\xi}_i^F([U_{\text{Gau}S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})]) = \xi_i^F(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})$ by $\xi_i(U_{\text{Gau}S_i}) = \mathbf{1}$. Combining this with the previous paragraph, we obtain

$$\mathcal{R}_i = \xi_i(\mathcal{O}_i) = \xi_i([U_{\text{Gau}S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau}S_i}]) = \xi_i^F(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E] + \mathbb{C}$$

as claimed. The remaining equalities are obtained from Theorem 4.14. \blacksquare

4.5 Solving the system of local constraints.

Our aim in this section is to solve the constraint problem for the full system $(\mathcal{F}_e, \mathcal{C})$. As remarked above, the set of local constraint systems $(\mathcal{F}_{S_i}, \mathcal{C}_i)$, $i \in \mathbb{N}$ which it comprises of, is a system of local quantum constraints in the sense of [14] (Def. 3.3). Such systems were studied in detail in [14], and now we will recall and apply the relevant parts of that analysis.

Definition 4.17. *A system of local quantum constraints consists of the following:*

- (1) *A directed set $\tilde{\Gamma}$ of C^* -algebras with a common identity $\mathbf{1}$, partially ordered by inclusion, defining an inductive limit C^* -algebra \mathcal{F}_0 (over $\tilde{\Gamma}$). We will call the elements of $\tilde{\Gamma}$ the **local field algebras** and \mathcal{F}_0 the **quasi-local algebra**. There is directed index set Γ together with a surjection $\mathcal{F}: \Gamma \rightarrow \tilde{\Gamma}$ which is order preserving, i.e. if $\gamma_1 \leq \gamma_2$, then $\mathcal{F}(\gamma_1) \subseteq \mathcal{F}(\gamma_2)$.*
- (2) *(Local Constraints) There is a map \mathcal{U} from Γ to the set of first class subsets of the unitaries in the local field algebras such that*

$$\begin{aligned} \mathcal{U}(\gamma) &\subset \mathcal{F}(\gamma)_u \text{ for all } \gamma \in \Gamma, \text{ and} \\ \text{if } \gamma_1 &\leq \gamma_2, \text{ then } \mathcal{U}(\gamma_1) = \mathcal{U}(\gamma_2) \cap \mathcal{F}(\gamma_1). \end{aligned}$$

This definition was adapted from Definition 3.3 in [14], where Γ and \mathcal{F} had additional structure, which we will not need. We first verify that our current system $(\mathcal{F}_{S_i}, \mathcal{U}_0^{S_i})$, $i \in \mathbb{N}$ satisfies these conditions.

Proposition 4.18. *Let $\Gamma = \mathbb{N}$ with its usual order, and let*

$$\mathcal{F}(i) := \mathcal{F}_{S_i} = [U_{\text{Gau}S_i} \cdot \mathfrak{A}_{S_i}] + [U_{\text{Gau}S_i}] \subset \mathcal{F}_e = \varinjlim \mathcal{F}_{S_i}$$

where we use the notation established above. Define $\mathcal{U}(i) := \mathcal{U}_0^{S_i}$, then the system $\{(\mathcal{F}(i), \mathcal{U}(i)) \mid i \in \mathbb{N}\} = \{(\mathcal{F}_{S_i}, \mathcal{U}_0^{S_i}) \mid i \in \mathbb{N}\}$ is a system of local quantum constraints.

Proof: It is clear that (1) is satisfied. Regarding (2), it is obvious that if $i < j$, then $\mathcal{U}_0^{S_i} \subseteq \mathcal{U}_0^{S_j} \cap \mathcal{F}_{S_i}$, so it suffices to show that no $U \in \mathcal{U}_0^{S_j} \setminus \mathcal{U}_0^{S_i}$ is in \mathcal{F}_{S_i} . Recall that

$$\begin{aligned} \mathcal{U}_0^S &:= \{U \in \mathcal{U}_0 \mid [U, \mathfrak{A}_S] \neq 0\} = \{U_{\exp(tY \cdot \delta_x)} \mid t \in \mathbb{R} \setminus 0, Y \in \mathfrak{g} \setminus 0, x \in S_e\} \quad \text{where} \\ \mathcal{U}_0 &:= \{U_{\exp(t\nu)} \mid t \in \mathbb{R}, \nu = Y \cdot \delta_x \text{ for all } Y \in \mathfrak{g}, x \in \Lambda^0\} \quad \text{and} \\ S_e &:= \{x \in \Lambda^0 \mid \exists \ell = (x_\ell, y_\ell) \in \Lambda^1 \text{ such that } \ell \cap S \neq \emptyset \text{ and } x_\ell = x \text{ or } y_\ell = x\}. \end{aligned}$$

Now $\mathcal{F}_S = [U_{\text{Gau } S} \cdot \mathfrak{A}_S] + [U_{\text{Gau } S}]$ and we have uniqueness for decompositions in terms of these two spaces. If $i < j$, then $[U_{\text{Gau } S_i} \cdot \mathfrak{A}_{S_i}] \subseteq [U_{\text{Gau } S_j} \cdot \mathfrak{A}_{S_j}]$ and $[U_{\text{Gau } S_i}] \subseteq [U_{\text{Gau } S_j}]$ hence $\mathcal{U}_0^{S_j} \cap \mathcal{F}_{S_i} \subseteq [U_{\text{Gau } S_i}] \cap [U_{\text{Gau } S_j}] = [U_{\text{Gau } S_i}]$. Now $[U_{\text{Gau } S_j}] = C^*(\text{Gau}_d S_j)$ so as $\text{Gau } S_j = (\text{Gau } S_i) \times H$ where $H := \{\gamma \in \text{Gau } S_j \mid \text{supp}(\gamma) \cap (S_i)_e = \emptyset\}$, we have $C^*(\text{Gau}_d S_j) = C^*(\text{Gau}_d S_i) \otimes_{\max} C^*(H_d)$. Thus $[U_{\text{Gau } S_i}] = C^*(\text{Gau}_d S_i) \otimes_{\max} \mathbb{C}\mathbb{1}$.

Moreover $U \in \mathcal{U}_0^{S_j} \setminus \mathcal{U}_0^{S_i}$ is of the form $U = U_{\exp(tY \cdot \delta_x)}$ for $t \in \mathbb{R} \setminus 0, Y \in \mathfrak{g} \setminus 0$ and $x \in (S_j)_e \setminus (S_i)_e$ so $\exp(tY \cdot \delta_x) \in H$ hence $U \in \mathbb{C}\mathbb{1} \otimes_{\max} C^*(H_d)$. As U implements a nontrivial automorphism, it cannot be a multiple of the identity, hence $U \notin C^*(\text{Gau}_d S_i) \otimes_{\max} \mathbb{C}\mathbb{1} = \mathcal{U}_0^{S_j} \cap \mathcal{F}_{S_i}$. Thus $\mathcal{U}_0^{S_j} \cap \mathcal{F}_{S_i} = \mathcal{U}_0^{S_i}$ and so (2) is satisfied. ■

Given a system of local quantum constraints, $\gamma \rightarrow (\mathcal{F}(\gamma), \mathcal{U}(\gamma))$, we can apply the T-procedure to each system $(\mathcal{F}(\gamma), \mathcal{U}(\gamma))$, to obtain the “local” objects:

$$\begin{aligned} \mathfrak{S}_D^\gamma &:= \left\{ \omega \in \mathfrak{S}(\mathcal{F}(\gamma)) \mid \omega(U) = 1 \quad \forall U \in \mathcal{U}(\gamma) \right\} = \mathfrak{S}_D(\mathcal{F}(\gamma)), \\ \mathcal{D}(\gamma) &:= [\mathcal{F}(\gamma) \mathcal{C}(\gamma)] \cap [\mathcal{C}(\gamma) \mathcal{F}(\gamma)], \\ \mathcal{O}(\gamma) &:= \{F \in \mathcal{F}(\gamma) \mid FD - DF \in \mathcal{D}(\gamma) \quad \forall D \in \mathcal{D}(\gamma)\} = M_{\mathcal{F}(\gamma)}(\mathcal{D}(\gamma)), \\ \mathcal{R}(\gamma) &:= \mathcal{O}(\gamma) / \mathcal{D}(\gamma) \quad \text{and constraint homomorphism} \quad \xi_\gamma : \mathcal{O}(\gamma) \rightarrow \mathcal{R}(\gamma). \end{aligned}$$

In the case of our system $\{(\mathcal{F}_{S_i}, \mathcal{U}_0^{S_i}) \mid i \in \mathbb{N}\}$, this corresponds to the constraint data $(\mathfrak{S}_D^i, \mathcal{D}_i, \mathcal{O}_i, \mathcal{R}_i, \xi_i)$ analyzed in Theorem 4.16. We need to determine what the inclusions in Definition 4.17 imply for the associated objects $(\mathfrak{S}_D^\gamma, \mathcal{D}(\gamma), \mathcal{O}(\gamma), \mathcal{R}(\gamma), \xi_\gamma)$.

Theorem 4.19. *Let $\Gamma \ni \gamma \rightarrow (\mathcal{F}(\gamma), \mathcal{U}(\gamma))$ be a system of local quantum constraints. Let $\gamma_1 \leq \gamma_2$ imply that $\mathcal{O}(\gamma_1) \subseteq \mathcal{O}(\gamma_2)$ and $\mathcal{D}(\gamma_1) = \mathcal{D}(\gamma_2) \cap \mathcal{O}(\gamma_1)$. Then the constraint homomorphism $\xi_{\gamma_2} : \mathcal{O}(\gamma_2) \rightarrow \mathcal{R}(\gamma_2)$ coincides on $\mathcal{O}(\gamma_1)$ with ξ_{γ_1} , and hence it defines a unital *-monomorphism $\iota_{12} : \mathcal{R}(\gamma_1) \hookrightarrow \mathcal{R}(\gamma_2)$. In this case, the net $\gamma \rightarrow \mathcal{R}(\gamma)$ has an inductive limit, which we denote by $\mathcal{R}_0 := \varinjlim \mathcal{R}(\gamma)$. Now we may consistently write $\mathcal{R}(\gamma_1) \subset \mathcal{R}(\gamma_2)$ if $\gamma_1 \leq \gamma_2$.*

Proof: Let $\gamma_1 \leq \gamma_2$ and $\mathcal{O}(\gamma_1) \subseteq \mathcal{O}(\gamma_2)$ and $\mathcal{D}(\gamma_1) = \mathcal{D}(\gamma_2) \cap \mathcal{O}(\gamma_1)$. From

$$\mathcal{R}(\gamma_2) = \mathcal{O}(\gamma_2) / \mathcal{D}(\gamma_2) = \left(\mathcal{O}(\gamma_1) / \mathcal{D}(\gamma_2) \right) \cup \left((\mathcal{O}(\gamma_2) \setminus \mathcal{O}(\gamma_1)) / \mathcal{D}(\gamma_2) \right),$$

it is enough to show that $\mathcal{O}(\gamma_1) / \mathcal{D}(\gamma_2) \cong \mathcal{R}(\gamma_1) = \mathcal{O}(\gamma_1) / \mathcal{D}(\gamma_1)$. Now, in $\mathcal{O}(\gamma_1)$ a $\mathcal{D}(\gamma_2)$ -equivalence class consists of $A, B \in \mathcal{O}(\gamma_1)$ such that $A - B \in \mathcal{D}(\gamma_2)$ and therefore

$A - B \in \mathcal{D}(\gamma_2) \cap \mathcal{O}(\gamma_1) = \mathcal{D}(\gamma_1)$. This implies $\mathcal{O}(\gamma_1)/\mathcal{D}(\gamma_2) \cong \mathcal{O}(\gamma_1)/\mathcal{D}(\gamma_1) = \mathcal{R}(\gamma_1)$. Moreover, since $\mathbb{1} \in \mathcal{O}(\gamma_1) \subset \mathcal{O}(\gamma_2)$, and the $\mathcal{D}(\gamma_1)$ -equivalence class of $\mathbb{1}$ is contained in the $\mathcal{D}(\gamma_2)$ -equivalence class of $\mathbb{1}$, it follows that the identity maps to the identity. We obtain for $\gamma_1 \leq \gamma_2$ a unital monomorphism $\iota_{12}: \mathcal{R}(\gamma_1) \hookrightarrow \mathcal{R}(\gamma_2)$. Next we have to verify that these monomorphisms satisfy Takeda's criterion: $\iota_{13} = \iota_{23} \circ \iota_{12}$ (cf. [37]), which will ensure the existence of the inductive limit \mathcal{R}_0 , and in which case we can write simply inclusion $\mathcal{R}(\gamma_1) \subset \mathcal{R}(\gamma_2)$ for ι_{12} . Recall that $\iota_{12}(A + \mathcal{D}(\gamma_1)) = A + \mathcal{D}(\gamma_2)$ for $A \in \mathcal{O}(\gamma_1)$. Let $\gamma_1 \leq \gamma_2 \leq \gamma_3$, then by assumption $\mathcal{O}(\gamma_1) \subset \mathcal{O}(\gamma_2) \subset \mathcal{O}(\gamma_3)$, and so for $A \in \mathcal{O}(\gamma_1)$, $\iota_{23}(\iota_{12}(A + \mathcal{D}(\gamma_1))) = \iota_{23}(A + \mathcal{D}(\gamma_2)) = A + \mathcal{D}(\gamma_3) = \iota_{13}(A + \mathcal{D}(\gamma_1))$. This establishes Takeda's criterion. \blacksquare

The pair of conditions $\mathcal{O}(\gamma_1) \subseteq \mathcal{O}(\gamma_2)$ and $\mathcal{D}(\gamma_1) = \mathcal{D}(\gamma_2) \cap \mathcal{O}(\gamma_1)$ were analyzed in Subsections 3.1 and 3.2 of [14] where together they were given the name of reduction isotony.

Theorem 4.20. *With notation established above, the system of local quantum constraints $\{(\mathcal{F}_{S_i}, \mathcal{U}_0^{S_i}) \mid i \in \mathbb{N}\}$ satisfies*

(i) $\mathcal{O}_i \subseteq \mathcal{O}_j$ and $\mathcal{D}_i = \mathcal{D}_j \cap \mathcal{O}_i$ if $i \leq j$. Thus $\mathcal{R}_i \subseteq \mathcal{R}_j$, and there is an inductive limit, which we denote by $\mathcal{R}_0 := \varinjlim \mathcal{R}_i$.

(ii) $\mathcal{O}_i \subseteq \mathcal{O}$ and $\mathcal{D}_i = \mathcal{D} \cap \mathcal{O}_i$ where $(\mathfrak{S}_D, \mathcal{D}, \mathcal{O}, \mathcal{R}, \xi)$ is the constraint data for the full system $\mathcal{F}_e = [U_{\text{Gau}_d \Lambda} \cdot (\mathfrak{A}_\Lambda \oplus \mathbb{C})] = \varinjlim \mathcal{F}_{S_i}$ with constraints $\mathcal{C} = \mathcal{U}_0 - \mathbb{1} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. Thus ξ coincides with ξ_i on \mathcal{O}_i , and hence defines a unital $*$ -monomorphism $\iota_i: \mathcal{R}_i \hookrightarrow \mathcal{R}$ which is compatible with the containments $\mathcal{R}_i \subseteq \mathcal{R}_j$, hence we denote it by $\mathcal{R}_i \subseteq \mathcal{R}$. Thus $\mathcal{R}_0 = \varinjlim \mathcal{R}_i \subseteq \mathcal{R}$.

(iii) $\mathcal{D}_i = \mathcal{D} \cap \mathcal{F}_{S_i}$, $\mathcal{O}_i = \mathcal{O} \cap \mathcal{F}_{S_i}$ and $\mathcal{R}_0 = \xi(C^*(\mathcal{O} \cap \bigcup_{i \in \mathbb{N}} \mathcal{F}_{S_i}))$.

Proof: Let $i \leq j$ and recall

$$\mathcal{F}_{S_i} = [U_{\text{Gau } S_i} \cdot \mathfrak{A}_{S_i}] + [U_{\text{Gau } S_i}] \quad \text{and} \quad \mathcal{U}_0^{S_i} = \{U \in \mathcal{U}_0 \mid [U, \mathfrak{A}_{S_i}] \neq 0\}$$

where $\mathfrak{A}_{S_i} := \mathfrak{F}_{S_i} \otimes \mathcal{L}_{S_i}[E]$. Thus for $U \in \mathcal{U}_0 \setminus \mathcal{U}_0^{S_i}$ we have $[U, \mathfrak{A}_{S_i}] = 0$ and $[U, U_{\text{Gau } S_i}] = 0$, and so $[U, \mathcal{F}_{S_i}] = 0$. Now from Lemma 3.3 in [14] we have that

$$\begin{aligned} \mathcal{O}_i \subseteq \mathcal{O}_j & \quad \text{iff} \quad \mathcal{O}_i \subseteq \{F \in \mathcal{F}_{S_i} \mid UFU^{-1} - F \in \mathcal{D}_j \quad \forall U \in \mathcal{U}_0^{S_j} \setminus \mathcal{U}_0^{S_i}\} \\ \mathcal{O}_i \subseteq \mathcal{O} & \quad \text{iff} \quad \mathcal{O}_i \subseteq \{F \in \mathcal{F}_{S_i} \mid UFU^{-1} - F \in \mathcal{D} \quad \forall U \in \mathcal{U}_0 \setminus \mathcal{U}_0^{S_i}\} \end{aligned}$$

and by the previous lines we have that $UFU^{-1} - F = 0 \in \mathcal{D}_j \cap \mathcal{D}$ for all $F \in \mathcal{F}_{S_i} \supseteq \mathcal{O}_i$ and $U \in \mathcal{U}_0 \setminus \mathcal{U}_0^{S_i}$. So these requirements are always satisfied, hence $\mathcal{O}_i \subseteq \mathcal{O}_j$ and $\mathcal{O}_i \subseteq \mathcal{O}$ as claimed.

Next, to show that $\mathcal{D}_i = \mathcal{D}_j \cap \mathcal{O}_i$ note from the definition of \mathcal{D} that $\mathcal{D}_i \subseteq \mathcal{D}_j$, hence $\mathcal{D}_i \subseteq \mathcal{D}_j \cap \mathcal{O}_i$. Since $\mathcal{O}_i \subseteq \mathcal{O}_j$ we may regard these as new field algebras, so using Lemma 3.2 in [14] we will have that $\mathcal{D}_i = \mathcal{D}_j \cap \mathcal{O}_i$ if we can prove that every Dirac state on \mathcal{O}_i (w.r.t. constraints \mathcal{C}_i) extends to a Dirac state on \mathcal{O}_j (w.r.t. constraints \mathcal{C}_j). In fact, it is enough to prove that every Dirac state on \mathcal{O}_i (w.r.t. constraints \mathcal{C}_i) extends to a Dirac state on $\mathcal{E} := C^*(\mathcal{O}_i \cup U_{\text{Gau}S_j}) \subseteq \mathcal{O}_j$ (w.r.t. constraints \mathcal{C}_j) because any further extension of such a state by Hahn–Banach to \mathcal{O}_j remains a Dirac state w.r.t. \mathcal{C}_j .

Now (as observed in the proof of Proposition 4.18) we have that $\text{Gau}S_j = (\text{Gau}S_i) \times H$ where $H := \{\gamma \in \text{Gau}S_j \mid \text{supp}(\gamma) \cap (S_i)_e = \emptyset\}$, and α_H acts trivially on \mathcal{F}_{S_i} , hence on \mathcal{O}_i . Since $U_{\text{Gau}S_i} \subset \mathcal{O}_i$, we have $\mathcal{E} = C^*(\mathcal{O}_i \cup U_H)$ and recalling that the unitaries $U_{\text{Gau}S_i}$ were the implementing unitaries of $\alpha_{\text{Gau}S_i}$ in the original crossed product, this means that $\mathcal{E} = \mathcal{O}_i \rtimes_{\iota} H_d$ where $\iota : H \rightarrow \text{Aut}\mathcal{O}_i$ is the trivial action. Thus by Lemma 2.73 in [43], we obtain that $\mathcal{E} = \mathcal{O}_i \rtimes_{\iota} H_d = \mathcal{O}_i \otimes_{\max} C^*(H_d)$. Now by Theorem 4.9 in [38], given any two states ω_1 on \mathcal{O}_i and ω_2 on $C^*(H_d)$, we can define a state $\omega_1 \otimes \omega_2$ on $\mathcal{E} = \mathcal{O}_i \otimes_{\max} C^*(H_d)$ by $\omega_1 \otimes \omega_2(A \otimes B) = \omega_1(A)\omega_2(B)$ for all $A \in \mathcal{O}_i$ and $B \in C^*(H_d)$. In particular, as \mathcal{C}_j is first-class, we can choose a state ω_2 on $C^*(H_d) = C^*(U_H)$ such that $\omega_2(U_h) = 1$ for all $h \in H$. Thus if ω_1 is a Dirac state on \mathcal{O}_i (w.r.t. constraints \mathcal{C}_i), then $\omega_1 \otimes \omega_2$ is a Dirac state on \mathcal{E} (w.r.t. constraints \mathcal{C}_j) which extends ω_1 . This concludes the proof that $\mathcal{D}_i = \mathcal{D}_j \cap \mathcal{O}_i$, and thus (i) is proven.

For (ii), the same argument with suitable replacements proves that $\mathcal{D}_i = \mathcal{D} \cap \mathcal{O}_i$. Moreover, if we replace \mathcal{O}_i by \mathcal{F}_{S_i} , this argument also proves that $\mathcal{D}_i = \mathcal{D} \cap \mathcal{F}_{S_i}$. To see that $\iota_i : \mathcal{R}_i \hookrightarrow \mathcal{R}$ is compatible with the containments $\mathcal{R}_i \subseteq \mathcal{R}_j$, i.e. with the monomorphism $\iota_{ij} : \mathcal{R}_i \hookrightarrow \mathcal{R}_j$ obtained from (i), recall that $\iota_{ij}(A + \mathcal{D}_i) = A + \mathcal{D}_j$ for $A \in \mathcal{O}_i$. Then by assumption $\mathcal{O}_i \subset \mathcal{O}_j \subset \mathcal{O}$, and so for $A \in \mathcal{O}_i$, $\iota_j(\iota_{ij}(A + \mathcal{D}_i)) = \iota_j(A + \mathcal{D}_j) = A + \mathcal{D} = \iota_i(A + \mathcal{D}_i)$. Thus $\iota_j \circ \iota_{ij} = \iota_i$, and this also proves that the set of monomorphisms $\{\iota_i \mid i \in \mathbb{N}\}$ defines a monomorphism of $\mathcal{R}_0 = \varinjlim \mathcal{R}_i$ into \mathcal{R} by the universal property of inductive limits (Theorem L.2.1. in [41]).

(iii) We already have above that $\mathcal{D}_i = \mathcal{D} \cap \mathcal{F}_{S_i}$, so we prove that $\mathcal{O}_i = \mathcal{O} \cap \mathcal{F}_{S_i}$. As $\mathcal{O}_i \subseteq \mathcal{O}$ by (ii), we have $\mathcal{O}_i \subseteq \mathcal{O} \cap \mathcal{F}_{S_i}$. Conversely, if $A \in \mathcal{O} \cap \mathcal{F}_{S_i}$, then $A\mathcal{D}_i = A(\mathcal{D} \cap \mathcal{O}_i) \subseteq \mathcal{D} \cap \mathcal{F}_{S_i} = \mathcal{D}_i$. Likewise $\mathcal{D}_i A \subseteq \mathcal{D}_i$ hence $A \in \mathcal{O}_i$. Thus $\mathcal{O}_i = \mathcal{O} \cap \mathcal{F}_{S_i}$. Since $\xi : \mathcal{O} \rightarrow \mathcal{R}$ takes each $\mathcal{O}_i = \mathcal{O} \cap \mathcal{F}_{S_i}$ to $\mathcal{R}_i \subset \mathcal{R}_0$ and it is a homomorphism, it takes $C^*(\mathcal{O} \cap \bigcup_{i \in \mathbb{N}} \mathcal{F}_{S_i}) = C^*(\bigcup_{i \in \mathbb{N}} \mathcal{O}_i)$ to \mathcal{R}_0 . Since all $\mathcal{R}_i \subset \xi(C^*(\mathcal{O} \cap \bigcup_{i \in \mathbb{N}} \mathcal{F}_{S_i}))$, and these generate \mathcal{R}_0 it is clear that we have the claimed equality. \blacksquare

Thus we have shown that the local physical observable algebras \mathcal{R}_i (obtained in Theorem 4.16), combine into an inductive limit \mathcal{R}_0 , and produce a large part of the full observable algebra \mathcal{R} . If $\mathcal{R} \neq \mathcal{R}_0$, then the extra elements must be obtained from $\mathcal{O} \setminus C^*(\mathcal{O} \cap \bigcup_{i \in \mathbb{N}} \mathcal{F}_{S_i})$, i.e. these do not come from “local” observables, so we may regard

the elements of $\mathcal{R} \setminus \mathcal{R}_0$ as global observables. It is not clear if there are any.

We would like to understand the inclusions $\mathcal{R}_i \subseteq \mathcal{R}_j$, in terms of the concrete characterization in Theorem 4.16:

$$\mathcal{R}_i = \xi_i(\mathcal{O}_i) = \xi_i^F(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E] + \mathbb{C} \cong \mathcal{K}(\mathcal{H}_\pi^G) \otimes \mathcal{T}_{S_i}[E] + \mathbb{C}$$

where $\pi : \mathcal{F}_{S_i}^F \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ is any representation which is irreducible on \mathcal{A}_{S_i} . Since for $i < j$ we have $\mathcal{R}_i = \xi_i(\mathcal{O}_i) = \xi_j(\mathcal{O}_i) \subseteq \xi_j(\mathcal{O}_j) = \mathcal{R}_j$, we need to consider the inclusion $\mathcal{O}_i \subseteq \mathcal{O}_j$. By Theorem 4.16

$$\mathcal{O}_i = [U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] + [U_{\text{Gau } S_i}] \quad \text{where} \quad \mathcal{O}_i^F = [U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] + [U_{\text{Gau } S_i}]$$

and $\mathcal{A}_{S_i} := \mathfrak{F}_{S_i} \otimes \mathcal{L}^{S_i}$. As $[U_{\text{Gau } S_i}] \subseteq [U_{\text{Gau } S_j}]$ and $\xi_j([U_{\text{Gau } S_j}]) = \mathbb{C}$, we only need to examine the inclusion $[U_{\text{Gau } S_i}(\mathcal{O}_i^F \cap \mathcal{A}_{S_i})] \otimes \mathcal{T}_{S_i}[E] \subseteq [U_{\text{Gau } S_j}(\mathcal{O}_j^F \cap \mathcal{A}_{S_j})] \otimes \mathcal{T}_{S_j}[E]$.

- We show how $\mathcal{L}_{S_i}[E] \subseteq \mathcal{L}_{S_j}[E]$. Now $\mathcal{L}_{S_i}[E] \cong \mathcal{L}^{S_i} \otimes \mathcal{T}_{S_i}[E]$, and so it is generated by elements of the type $L \otimes E_{S_i}[\mathbf{n}]$ where $L \in \mathcal{L}^{S_i}$ and $E_{S_i}[\mathbf{n}]$ is as in Lemma 4.15. Then $E_{S_i}[\mathbf{n}] = E_{ij} \otimes E_{S_j}[\mathbf{n}]$ where E_{ij} is the finite tensor product consisting of those entries of $E_{n_1}^{(1)} \otimes E_{n_2}^{(2)} \otimes \dots$ corresponding to links in $\Lambda_{S_j}^1 \setminus \Lambda_{S_i}^1$. Now $\mathcal{L}^{S_i} \otimes E_{ij} \subseteq \mathcal{L}^{S_j}$ since $(E_n^{(k)})_{n \in \mathbb{N}} \subset \mathcal{L}_k$, hence $L \otimes E_{S_i}[\mathbf{n}] = L \otimes E_{ij} \otimes E_{S_j}[\mathbf{n}] \in \mathcal{L}^{S_j} \otimes \mathcal{T}_{S_j}[E]$. Thus we have identified $\mathcal{L}^{S_i} \otimes \mathcal{T}_{S_i}[E] \subseteq \mathcal{L}^{S_j} \otimes \mathcal{T}_{S_j}[E]$ and hence $\mathcal{L}_{S_i}[E] \subseteq \mathcal{L}_{S_j}[E]$.
- Since $\mathfrak{F}_{S_i} \subseteq \mathfrak{F}_{S_j}$ it follows that $\mathcal{A}_{S_i} \otimes E_{ij} = \mathfrak{F}_{S_i} \otimes \mathcal{L}^{S_i} \otimes E_{ij} \subseteq \mathfrak{F}_{S_j} \otimes \mathcal{L}^{S_j} = \mathcal{A}_{S_j}$. We claim that $(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes E_{ij} \subseteq (\mathcal{O}_j^F \cap \mathcal{A}_{S_j})$, and hence $(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E] \subseteq (\mathcal{O}_j^F \cap \mathcal{A}_{S_j}) \otimes \mathcal{T}_{S_j}[E]$. That $(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes E_{ij} \subseteq \mathcal{A}_{S_j}$ is obvious. To see that it is in \mathcal{O}_j^F note that $[U_0^{S_j} \setminus U_0^{S_i}, (\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes E_{ij}] = 0$, and that

$$[U_0^{S_i}, (\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes E_{ij}] \subseteq \mathcal{B}_i^F \otimes E_{ij} \subseteq \mathcal{D}_j^F$$

where via Theorem 4.14 we have $\mathcal{B}_i = \mathcal{D}_i^F \cap [U_{\text{Gau } S_i} \mathcal{A}_{S_i}] = [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i]$, and the last inclusion follows from $\mathcal{B}_i^F \otimes E_{ij} = [\mathcal{C}_i U_{\text{Gau } S_i} \mathcal{A}_{S_i} \mathcal{C}_i] \otimes E_{ij} \subseteq [\mathcal{C}_i (U_{\text{Gau } S_i} \mathcal{A}_{S_i} \otimes E_{ij}) \mathcal{C}_i] \subseteq \mathcal{D}_j^F$. This inclusion

$$(\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E] \subseteq (\mathcal{O}_j^F \cap \mathcal{A}_{S_j}) \otimes \mathcal{T}_{S_j}[E]$$

fully specifies the inclusion $\mathcal{R}_i \subseteq \mathcal{R}_j$ because $\mathcal{R}_i = \xi_i((\mathcal{O}_i^F \cap \mathcal{A}_{S_i}) \otimes \mathcal{T}_{S_i}[E]) + \mathbb{C}$.

We conclude that we have concretely characterized the algebra of local physical observables $\mathcal{R}_0 = \varinjlim \mathcal{R}_i \subseteq \mathcal{R}$, but that the existence and nature of the global physical observables $\mathcal{R} \setminus \mathcal{R}_0$ remain an open question.

5 Conclusion.

We have extended the finite QCD lattice model in [22, 23, 20, 21] to an infinite lattice. We defined both local and global gauge transformations on it, and we identified the Gauss law constraint. Using the T-procedure and the local structure of the constraints we solved the constraint system, and identified the algebra of local physical observables.

There are three directions in which this model needs to be developed in future work. First, the open question of the existence and nature of the global physical observables $\mathcal{R} \setminus \mathcal{R}_0$ needs to be settled. Second, we need to analyze boundary effects, i.e do colour charge analysis and connect to the results for the finite lattices in [20, 21]. Third, and more ambitiously, we need to define and analyze the dynamics of the system, and obtain suitable ground states.

A Connecting with physics notation.

Here we will make contact with the formulii appearing in the existing work of constructing a field algebra for QCD on a finite lattice. These formulii are written in a physics viewpoint, i.e. there are explicit choices of bases, and transformations are written infinitesimally, i.e. in a Lie algebra framework.

Explicitly, recall that the quantum matter field algebra on Λ is:

$$\mathfrak{F}_\Lambda := \text{CAR}(\ell^2(\Lambda^0, \mathbf{V})) = C^*\left(\bigcup_{x \in \Lambda^0} \mathfrak{F}_x\right)$$

where $\mathfrak{F}_x := \text{CAR}(V_x)$ and $V_x := \{f \in \ell^2(\Lambda^0, \mathbf{V}) \mid f(y) = 0 \text{ if } y \neq x\} \cong \mathbf{V}$. We interpret $\mathfrak{F}_x \cong \text{CAR}(\mathbf{V})$ as the field algebra for a fermion at x . We denote the generating elements of $\text{CAR}(\ell^2(\Lambda^0, \mathbf{V}))$ by $a(f)$, $f \in \ell^2(\Lambda^0, \mathbf{V})$, and these satisfy the usual CAR-relations:

$$\{a(f), a(g)^*\} = \langle f, g \rangle \mathbf{1} \quad \text{and} \quad \{a(f), a(g)\} = 0 \quad \text{for} \quad f, g \in \ell^2(\Lambda^0, \mathbf{V}).$$

Now $\mathbf{V} = \mathbf{W} \otimes \mathbb{C}^k$ where \mathbf{W} has the non-gauge degrees of freedom, and \mathbb{C}^k has the gauge degrees of freedom. In particular, there is a smooth irreducible unitary action of the structure group G on \mathbb{C}^k (if $G = SU(3)$ we take $k = 3$) which produces a smooth unitary action of G on \mathbf{V} . If $\{w_1, \dots, w_\ell\}$ is an orthonormal basis of \mathbf{W} and $\{e_1, \dots, e_k\}$ is an orthonormal basis of \mathbb{C}^k , then w.r.t. the orthonormal basis $\{w_a \otimes e_A \mid a = 1, \dots, \ell, A = 1, \dots, k\}$ of \mathbf{V} , we obtain the usual physics indices

$$a(w_a \otimes e_A \cdot \delta_x) =: \psi^{aA}(x) \in \mathfrak{F}_x$$

for the quark field generators.

For the gauge connection part, recall that for every link $\ell \in \Lambda^1$ we assumed a generalised Weyl algebra $C(G) \rtimes_\lambda G \cong \mathcal{K}(L^2(G))$ where G is our compact gauge group,

and our full gauge system contains a certain (infinite) tensor product of these. Fix a link $(x, y) \in \Lambda^1$, then using the action of the structure group G on \mathbb{C}^k , define the function $U_B^A(x, y) \in C(G)$ by $U_B^A(x, y)(g) := (e_B, g e_A)$, $g \in G$, using the orthonormal basis $\{e_A \mid A = 1, \dots, k\}$ of \mathbb{C}^k . The algebra generated by these functions (w.r.t. pointwise operations) separate the points in G , hence it is a dense subalgebra of $C(G)$. The action $\lambda : G \rightarrow \text{Aut } C(G)$ by $\lambda_g(f)(h) := f(g^{-1}h)$ $f \in C(G)$, $g, h \in G$ produces

$$\lambda_g(U_B^A(x, y))(h) = (e_B, g^{-1}h e_A) = \sum_C (e_B, g^{-1}e_C)(e_C, h e_A).$$

If $g = \exp(itE)$ for $E \in \mathfrak{g}$, $t \in \mathbb{R}$, then

$$d\lambda(E)(U_B^A(x, y))(h) = i \frac{d}{dt} \sum_C (e_B, e^{-itE} e_C)(e_C, h e_A) \Big|_{t=0} = \sum_C E_B^C(x, y) U_C^A(x, y)(h)$$

gives the derived action, where $E_B^C(x, y) := (e_B, E e_C)$. Since $d\lambda : \mathfrak{g} \rightarrow \text{Der}(C^\infty(G))$, the extension of $d\lambda(E)$ from the $U_B^A(x, y)$ to $C^\infty(G)$ is as a derivation, i.e. via the Leibniz rule for derivations. The physics relations are written in terms of $U_B^A(x, y)$ and $E_B^C(x, y)$.

B More on subsystems of constraints.

Assume that $\mathcal{C} \subset \mathcal{A} \subset \mathcal{F}$ where \mathcal{C} is a first-class constraint set, and \mathcal{A} , \mathcal{F} are unital C^* -algebras. Now there are two constrained systems to consider;- $(\mathcal{A}, \mathcal{C})$ and $(\mathcal{F}, \mathcal{C})$. The first one produces the algebras $\mathcal{D} \subset \mathcal{O} \subseteq \mathcal{A}$, and the second produces $\mathcal{D}_{\mathcal{F}} \subset \mathcal{O}_{\mathcal{F}} \subseteq \mathcal{F}$, where as usual,

$$\begin{aligned} \mathcal{N} &= [\mathcal{A}\mathcal{C}] = \mathcal{A} \cdot C^*(\mathcal{C}), & \mathcal{D} &= \mathcal{N} \cap \mathcal{N}^*, & \mathcal{O} &= M_{\mathcal{A}}(\mathcal{D}) & \text{and} \\ \mathcal{N}_{\mathcal{F}} &= [\mathcal{F}\mathcal{C}] = \mathcal{F} \cdot C^*(\mathcal{C}), & \mathcal{D}_{\mathcal{F}} &= \mathcal{N}_{\mathcal{F}} \cap \mathcal{N}_{\mathcal{F}}^*, & \mathcal{O}_{\mathcal{F}} &= M_{\mathcal{F}}(\mathcal{D}_{\mathcal{F}}) \end{aligned}$$

with constraining homomorphisms $\xi : \mathcal{O} \rightarrow \mathcal{R} = \mathcal{O}/\mathcal{D}$ and $\xi_{\mathcal{F}} : \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{R}_{\mathcal{F}} = \mathcal{O}_{\mathcal{F}}/\mathcal{D}_{\mathcal{F}}$. Then we have (cf. Theorem 3.2 of [13]):

Theorem B.1. *Given as above the constraint systems $\mathcal{C} \subset \mathcal{A} \subset \mathcal{F}$ then*

$$\mathcal{N}_{\mathcal{F}} \cap \mathcal{A} = \mathcal{N}, \quad \mathcal{D}_{\mathcal{F}} \cap \mathcal{A} = \mathcal{D}, \quad \text{and} \quad \mathcal{O}_{\mathcal{F}} \cap \mathcal{A} = \mathcal{O}.$$

Hence $\mathcal{R} = \mathcal{O}/\mathcal{D} = (\mathcal{O}_{\mathcal{F}} \cap \mathcal{A})/(\mathcal{D}_{\mathcal{F}} \cap \mathcal{A})$, thus $\xi_{\mathcal{F}} \upharpoonright \mathcal{O} = \xi$.

Thus we can always enlarge our given algebra to a larger more convenient one, then we only need to intersect our constraint algebras \mathcal{D} , \mathcal{O} , with the original algebra to obtain our required constraint algebras.

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