SPATIAL ANALYTICITY OF SOLUTIONS TO INTEGRABLE SYSTEMS. I. THE KDV CASE

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ABSTRACT. We are concerned with the Cauchy problem for the KdV equation for nonsmooth locally integrable initial profiles q's which are, in a certain sense, essentially bounded from below and $q(x) = O\left(e^{-cx^{\varepsilon}}\right), x \to +\infty$, with some positive c and ε. Using the inverse scattering transform, we show that the KdV flow turns such initial data into a function which is (1) meromorphic (in the space variable) on the whole complex plane if $\varepsilon > 1/2$, (2) meromorphic on a strip around the real line if $\varepsilon = 1/2$, and (3) Gevrey regular if $\varepsilon < 1/2$. Note that q's need not have any decay or pattern of behavior at $-\infty$.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

The gain and persistence of regularity effects are important features of many dispersive (linear and nonlinear) partial differential equations (PDEs). The literature on the subject is truly enormous and we make no attempt to give a comprehensive review here. We only mention two recent relevant papers by Himonas et al [18], [19] where the interested reader can find further references on analytic and Gevrey regularity properties for KdV-type equations. In fact, we are interested in a much stronger effect of formation of meromorphic solution out of nonsmooth data. More specifically, in the current paper, we are concerned with the following problem.

Problem 1. *Given the Cauchy problem for the KdV equation*[1]

$$\begin{cases} \partial_t u - 6u\partial_z u + \partial_z^3 u = 0 \\ u|_{t=0} = q \end{cases}, \tag{1.1}$$

describe the largest possible class of (non-smooth) initial data q which evolve into functions $u(z,t)$ meromorphic with respect to z for any $t > 0$.

Meromorphic (or, more generally, analytic) solutions have of course been intensively studied since the boom around integrable systems started in the late 60s. A pure soliton (reflectionless) solution, historically the first explicit solution, is meromorphic on the whole complex plane having infinitely many double poles. This fact is of course a trivial observation immediately following from the explicit formula for multisoliton solutions. We emphasize that how those poles interact is not obvious at all. This question was raised back in earlier 70s by Kruskal and has been followed up by many. We refer the interested reader to a particularly influential 1977

Date: September, 2011.

1991 *Mathematics Subject Classification.* 37K15, 47B35, 35B65.

Key words and phrases. Korteweg-de Vries equation, inverse scattering transform, Schrödinger operator, Hankel operator, Gevrey regularity.

Based on research supported in part by the NSF under grant DMS 1009673.

[1]We use z instead of x for the spatial variable as it will frequently be complex.

paper [2] by Airault-McKean-Moser and recent Bona-Weissler [5] and the literature cited therein. More complicated examples of explicit solutions include algebraric, rational, meromorphic simply periodic, elliptic, etc. (see, e.g. [1], [4], [14], [3] and the literature cited therein). All these examples are of course very specific and in addition those q's are already meromorphic (i.e. smooth on the real line). Although Problem 1 is not addressed in those papers but they demonstrate the importance of meromorphic solutions.

Through the paper we deal with initial data subject to

Hypothesis 1.1. *q is real and L^1_{loc} such that*

(1) *(semiboundedness from below)*

$$\inf \operatorname{Spec}\left(-\partial_x^2 + q(x)\right) = -h_0^2 \tag{1.2}$$

with some $h_0 \geq 0$.

(2) *(subexponential decay at $+\infty$) For x large enough*

$$\int_x^\infty |q| \leq C_q e^{-cx^\varepsilon} \tag{1.3}$$

with some positive C_q, c, ε.

We assume that the constants c, ε in (1.3) are chosen optimal.

Note that the set of such functions is very large. Indeed, in terms of q itself, Condition (1.2) is satisfied if

$$\operatorname*{Sup}_x \int_{x-1}^x \max(-q, 0) < \infty, \tag{1.4}$$

i.e. q is essentially bounded from below [15]. The condition (1.4) cannot be improved since (1.4) becomes also necessary for (1.2) if q's are negative. Therefore, any q subject to Hypothesis 1.1 is essentially bounded from below, has subexponential decay at $+\infty$ and arbitrary otherwise. Such functions can grow (arbitrarily fast) at $-\infty$ or look like a stock market (Gaussian white noise on a left half line) but still satisfy our hypothesis as long as they exhibit rapid decay (1.3) at $+\infty$. In spectral terms (1.3) implies that $(0, \infty)$ belongs to the absolutely continuous spectrum of $-\partial_x^2 + q(x)$.

We now state our main results.

Theorem 1.2. *Under Hypothesis 1.1 with $\varepsilon \geq 1/2$ on the initial data q in* (1.1) *, the problem* (1.1) *has an analytic in z solution $u(z,t)$ given by*

$$u(z,t) = -2\partial_z^2 \log \det\left(1 + \mathbb{M}(z,t)\right), \tag{1.5}$$

where $\mathbb{M}(z,t)$ is a trace class operator-valued function constructed in Proposition 4.1 below for any $t > 0$. Moreover, for any $t > 0$

(1) *If $\varepsilon > 1/2$ then $u(z,t)$ is meromorphic on $\mathbb{C}$.*

(2) *If $\varepsilon = 1/2$ then $u(z,t)$ is meromorphic in the strip*

$$|\operatorname{Im} z| < \frac{9\sqrt{2}}{8} c\sqrt{t} \tag{1.6}$$

where c is as in (1.3).

Theorem 1.3. *Under Hypothesis 1.1 with $0 < \varepsilon < 1/2$ on the initial data q in* (1.1)*, the operator-valued function $\mathbb{M}(x,t)$ given in Proposition 4.1 is trace class for any real x and $t > 0$ and*

$$\mathbb{M}(x,t) = \mathbb{M}^{(1)}(x,t) + \mathbb{M}^{(2)}(x,t),$$

where $\mathbb{M}^{(1)}(x,t)$ is meromorphic in x and $\mathbb{M}^{(2)}(x,t)$ is Gevrey $G^{\frac{1}{2\varepsilon}-1}$ regular. If in addition $1 + \mathbb{M}(x,t)$ is invertible for any real x and $t > 0$ then the problem (1.1) *has a solution $u(x,t)$ given by*

$$u(x,t) = -2\partial_x^2 \log \det (1 + \mathbb{M}(x,t)), \tag{1.7}$$

belonging to the Gevrey class $G_{\mathrm{loc}}^{\frac{1}{2\varepsilon}-1}$.

Theorems 1.2 and 1.3 significantly improve our results in [29] which in turn improve Tarama [31]. Theorems 1.2 and 1.3 have also some important corollaries. We will come back to the relevant discussions in the last section when we have the necessary background. We only mention here that our approach is based on the Inverse Scattering Transform (IST) combined with pseudo-analytic continuation techniques developed by E.M. Dyn'kin (see e.g. [10], [6]) and we do not believe that any of the statements of Theorem 1.2 can be obtained by purely PDE techniques.

The paper is organized as follows. In Section 2, for the reader's convenience we list our main notation and give the relevant preliminaries. In Section 3 we define a suitable reflection coefficient and investigate its properties which will play a central role in our consideration. The results of this section may have some independent interest. In Section 4 we give a brief review of the classical IST stated in terms of Hankel operators and further prepare to prove our main results in Section 5. Section 6, the last one, is devoted to discussions of our results and some corollaries which directly follow from them. It also contains some open problems.

2. Notation and Preliminaries

We adhere to standard terminology accepted in Analysis. Namely, $\mathbb{R}_\pm \equiv [0, \pm\infty)$, $\mathbb{C}$ is the complex plane,

$$\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}.$$

Through the paper the subscript $\pm$ indicates objects (functions, operators, spaces, etc.) somehow related to $\mathbb{R}_\pm$ or $\mathbb{C}_\pm$. The bar $\overline{z}$ denotes the complex conjugate of z.

When appropriate, we write

$$y \approx x \text{ in place of } \; y = \text{const} \cdot x$$

and similarly whenever convenient

$$y \lesssim_a x \text{ in place of } y \leq C_a x$$

with some $C_a > 0$ dependent on a parameter a but independent of x. If C_a is an absolute constant we then write $y \lesssim x$. This will help us keep bulky formulas under control.

We use $\|\cdot\|_X$ to denote the norm in a Banach (Hilbert) space X.

We will need the Gevrey classes $G^\alpha, \alpha > 0$, on $\mathbb{R}$ of all functions f:

$$\|\partial_x^n f\|_{L^\infty} \lesssim_f Q_f^n \, (n!)^{1+\alpha}, n = 0, 1, 2, ...$$

with some $Q_f > 0$.

By [6], Theorem 3, the statement $f \in G^\alpha$ is equivalent to the statement that f admits a pseudo analytic extension to the whole complex plane such that

$$|\partial_{\overline{z}} f| \lesssim_f \exp\left\{-Q\left|\operatorname{Im} z\right|^{-\frac{1}{\alpha}}\right\} \tag{2.1}$$

with some $Q > 0$.

In a similar manner one introduces local Gevrey classes G^α_{loc}.

Next, $\mathfrak{S}_2$ denotes the Hilbert-Schmidt class

$$\mathfrak{S}_2 = \left\{A : \|A\|^2_{\mathfrak{S}_2} \equiv \operatorname{tr}\left(A^* A\right) < \infty\right\}$$

and $\mathfrak{S}_1$ is the trace class:

$$\mathfrak{S}_1 = \left\{A : \|A\|_{\mathfrak{S}_1} \equiv \operatorname{tr}\left(A^* A\right)^{1/2} < \infty\right\}.$$

Note that $A \in \mathfrak{S}_1$ if and only if $A = A_1 A_2$ with some $A_1, A_2 \in \mathfrak{S}_2$.

Some other miscellaneous notation: $\chi_S(x)$ is the characteristic function of a set S, i.e.

$$\chi_S(x) \equiv \begin{cases} 1, x \in S \\ 0, x \notin S \end{cases}.$$

In particular $\chi_\pm \equiv \chi_{\mathbb{R}_\pm}$ is the Heaviside function of $\mathbb{R}_\pm$. We also write

$$f|_S = \chi_S f.$$

The notation $H_q \equiv -\partial_x^2 + q(x)$ for the Schrödinger operator on $L^2(\mathbb{R})$ will be frequently used.

3. The reflection coefficient and its analytic structure

In this section we define a suitable reflection coefficient and investigate its properties which will play a central role in our consideration. The results of this section may have some independent interest.

In the short-range scattering for the full line Schrödinger operator, one typically introduces the right and left reflection coefficients $R(\lambda), L(\lambda)$ and the transmission coefficient $T(\lambda)$ as functions of the momentum λ (see e.g. [9]). These quantities (also called transition coefficients) can also be properly defined in much larger spectral situations through Wronskians and/or Titchmarsh-Weyl m-functions (see e.g. [12, 13]). Such extensions need not be unique. However, in our setting of step-like potentials decaying at $+\infty$, there is a natural candidate for the right reflection coefficient $R(\lambda)$.

Definition 3.1 ([29])**.** *Let $q(x)$ be real, locally integrable such that $q \in L^1(\mathbb{R}_+)$ and $-\partial_x^2 + q(x)$ is in the limit point case at $-\infty$. Denoting by $R_n(\lambda)$ the right reflection coefficient (which is necessarily well defined) from the potential $q_n = q|_{(-n,\infty)}$, we call the weak limit (if it exists)*

$$R(\lambda) \equiv w\text{-}\lim R_n(\lambda),\ n \to \infty, \tag{3.1}$$

the right reflection coefficient from the potential q.

Note that one should not expect in (3.1) pointwise convergence as an explicit counterexample $q = \chi_-$ readily shows. Uniform convergence in (3.1) is not available in general even in the short-range setting [9].

As shown in [29], Lemma 5.4, the reflection coefficient introduced this way is well defined. The following statement will play a crucial role in our consideration.

Proposition 3.2 (the analytic structure of the reflection coefficient)**.** *Under Hypothesis 1.1, the right reflection coefficient given by* (3.1) *exists and admits the representation*

$$R(\lambda) = A(\lambda) + \frac{S(\lambda)G(\lambda)}{\lambda B(\lambda)} \tag{3.2}$$

where functions A, B, S, G have the properties

(1) *A is an analytic on $\mathbb{C}^+ \setminus [0, ih_0]$ function such that $|A| \leq 2$ on $\mathbb{R}$ and $A(\lambda) = o(1/\lambda)$, $\lambda \to \infty$ along any ray in $\mathbb{C}^+$*

(2) *B is the Blaschke product*

$$B(\lambda) = \prod_{k=1}^{N} \frac{\lambda - i\varkappa_k}{\lambda + i\varkappa_k}$$

where real $\varkappa_k$'s are such that $\left\{-\varkappa_k^2\right\}_{k=1}^N$ is the negative discrete spectrum of H_{q_+}, $q_+ \equiv q|_{\mathbb{R}_+}$

(3) *$|S(\lambda)| \leq 1$, $\lambda \in \mathbb{C}^+$*

(4) *$G \in G^{\frac{1}{\varepsilon}-1}$*

(5) *$|S(\lambda)G(\lambda)/\lambda| \leq 1$ a.e. on $\mathbb{R}$*

(6) *If R_n is as in Definition 3.1 then*

$$R_n(\lambda) = A_n(\lambda) + \frac{S(\lambda)G(\lambda)}{\lambda B(\lambda)}$$

and

$$A_n \to A, \; n \to \infty$$

uniformly on any compact in $\mathbb{C}^+ \setminus [0, ih_0]$.

Proof. Most of statements in Proposition 3.2 (save (4)) are proven in [29] and we restrict ourselves to some comments only. Note first that Condition 1 of Hypothesis 1.1 implies that $-\partial_x^2 + q(x)$ is in the limit point case at $-\infty$ (see, e.g. [7] for complete results on this matter). Splitting

$$q = q_- + q_+, \quad q_\pm = q|_{\mathbb{R}_\pm} \tag{3.3}$$

induces the representation

$$R = \frac{T_+^2 R_-}{1 - R_- L_+} + R_+$$

where $\pm$ label scattering quantities associated with $q_\pm$. The functions T_+, L_+, R_- can be analytically continued into $\mathbb{C}^+$ and

$$A \equiv \frac{T_+^2 R_-}{1 - R_- L_+}$$

has properties (1), (6). For R_+, which is independent of n, we use the representation [9], Theorem 2,

$$R_+(\lambda) = \frac{T_+(\lambda)}{\lambda} G(\lambda)$$

where

$$G(\lambda) = \frac{1}{2i} \int_{-\infty}^{\infty} e^{-2i\lambda x} g(x) dx \tag{3.4}$$

with some g obeying

$$|g(x)| \leq |q(x)| + \text{const} \int_x^\infty |q| \,. \tag{3.5}$$

Since $R_+(\lambda)$ is a reflection coefficient we have (5). Since T_+ is a transmission coefficient,

$$T_+(\lambda) = \prod_{k=1}^N \frac{\lambda + i\varkappa_k}{\lambda - i\varkappa_k} \cdot S(\lambda) = B(\lambda)^{-1} S(\lambda)$$

where S is an outer function of $\mathbb{C}^+$: $|S(\lambda)| \leq 1$, $\lambda \in \mathbb{C}^+$. This proves (2) and (3).

The proposition is proven if we show (4). Due to (2.1) we should demonstrate that G admits a pseudo analytic extension the whole complex plane such that

$$\left|\partial_{\overline{\lambda}} G\right| \lesssim \exp\left\{-Q\,|\operatorname{Im}\lambda|^{-\frac{\varepsilon}{1-\varepsilon}}\right\} \tag{3.6}$$

with some $Q > 0$. There are a few explicit ways to construct pseudo analytic continuations (see e.g. [10], [6], [31]) producing different extensions. We modify the one used in [31] to obtain a better Q in 3.6. Note that

$$G(\lambda) \eqsim \widehat{g}(2\lambda) \tag{3.7}$$

where $\widehat{g}$ is the Fourier transform of g which due to (3.5) satisfies Condition 2 of Hypothesis 1.1 with some $\widetilde{c} < c$. I.e.

$$\int_x^\infty |g| \lesssim_g e^{-\widetilde{c}x^\varepsilon}. \tag{3.8}$$

For any $\lambda \in \mathbb{C}^+$ define

$$\widetilde{G}(\lambda) = \sum_{n\geq 1} \theta\left(r^{\varepsilon+2} x_n^{1-\varepsilon} \frac{\operatorname{Im}\lambda}{\widetilde{c}}\right) G_n(\lambda), \tag{3.9}$$

where θ is a smooth on $\mathbb{R}_+$ function such that:

$$\theta(x) = 1,\ x \in [0,1],$$
$$\theta(x) = 0,\ x \geq r,$$

$r > 1$, $x_n = r^n$ and

$$G_n(\lambda) = \int_{x_{n-1}}^{x_n} e^{-i\lambda x} g(x)dx.$$

The formula (3.9) clearly defines an extension of $\widehat{g}(\lambda)$ to complex λ. We next show that $\widetilde{G}$ is uniformly bounded on $\mathbb{C}^+$. Bound G_n first. By (3.8)

$$|G_n(\lambda)| \lesssim e^{|\operatorname{Im}\lambda|\cdot x_n} \int_{x_{n-1}}^{x_n} |g| \lesssim_g \exp\left\{|\operatorname{Im}\lambda| \cdot x_n - \widetilde{c}x_{n-1}^\varepsilon\right\}$$

and one has

$$\left|\widetilde{G}(\lambda)\right| \lesssim_g \sum_{n\geq 1}\sum_{n\geq 1} \theta\left(r^{\varepsilon+2} x_n^{1-\varepsilon} \frac{\operatorname{Im}\lambda}{\widetilde{c}}\right) \exp\left\{|\operatorname{Im}\lambda| \cdot x_n - \widetilde{c}x_{n-1}^\varepsilon\right\}. \tag{3.10}$$

In (3.10) many terms are in fact zero and nontrivial ones are subject to

$$r^{\varepsilon+2} x_n^{1-\varepsilon} \frac{|\operatorname{Im}\lambda|}{\widetilde{c}} \leq r.$$

I.e. only nonzero terms in (3.10) are the ones obeying

$$x_n^{1-\varepsilon} \leq \frac{\widetilde{c}}{r^{\varepsilon+1}} \cdot \frac{1}{|\operatorname{Im}\lambda|}. \tag{3.11}$$

Under the condition (3.11), for the argument of the exponential in (3.10), we have $(1/r < \delta < 1)$

$$\begin{aligned} |\operatorname{Im}\lambda| \cdot x_n - \widetilde{c}r^{-\varepsilon}x_n^\varepsilon &= \left(|\operatorname{Im}\lambda| \cdot x_n - \delta\widetilde{c}r^{-\varepsilon}x_n^\varepsilon\right) - (1-\delta)r^{-\varepsilon}x_n^\varepsilon \\ &= |\operatorname{Im}\lambda|\, x_n^\varepsilon \left(x_n^{1-\varepsilon} - \delta\frac{\widetilde{c}}{|\operatorname{Im}\lambda|}\right) - (1-\delta)r^{-\varepsilon}x_n^\varepsilon. \end{aligned} \tag{3.12}$$

By (3.11) the right hand side of (3.12) doesn't exceed

$$\begin{aligned} & |\operatorname{Im}\lambda|\, x_n^\varepsilon \left(\frac{\widetilde{c}}{|\operatorname{Im}\lambda|}\frac{1}{r^{\varepsilon+1}} - \frac{\widetilde{c}}{|\operatorname{Im}\lambda|}\frac{\delta}{r^\varepsilon}\right) - (1-\delta)r^{-\varepsilon}x_n^\varepsilon \\ = \quad & -\widetilde{c}\left(\delta - \frac{1}{r}\right)x_{n-1}^\varepsilon - (1-\delta)x_{n-1}^\varepsilon \\ < \quad & -\operatorname{const} x_{n-1}^\varepsilon. \end{aligned}$$

It follows now from this estimate and (3.10) that

$$\left|\widetilde{G}(\lambda)\right| \lesssim_g \sum_{n\geq 0} \exp\{-\operatorname{const} x_n^\varepsilon\} < \infty. \tag{3.13}$$

Similarly one proves that all derivatives of G are also bounded on $\mathbb{C}^+$.

It remains now to show (3.9). One has

$$\begin{aligned} \left|\partial_{\overline{\lambda}}\widetilde{G}\right| &\leq \sum_{n\geq 1} \theta'\left(r^{\varepsilon+2}x_n^{1-\varepsilon}\frac{|\operatorname{Im}\lambda|}{\widetilde{c}}\right)\frac{r^{\varepsilon+1}x_n^{1-\varepsilon}}{2\widetilde{c}}\,|G_n| \\ &\lesssim_g \sum_{n\geq 1} x_n^{1-\varepsilon}\exp\{|\operatorname{Im}\lambda| \cdot x_n - \widetilde{c}r^{-\varepsilon}x_n^\varepsilon\}. \end{aligned} \tag{3.14}$$

Only terms subject to

$$\frac{\widetilde{c}r^{-\varepsilon-2}}{|\operatorname{Im}\lambda|} \leq x_n^{1-\varepsilon} \leq \frac{\widetilde{c}r^{-\varepsilon-1}}{|\operatorname{Im}\lambda|} \tag{3.15}$$

make a non trivial contribution to the series in (3.14). The inequality (3.15) implies

$$x_n \geq \left(\frac{\widetilde{c}r^{-\varepsilon-2}}{|\operatorname{Im}\lambda|}\right)^{\frac{1}{1-\varepsilon}},$$

or

$$x_n^\varepsilon \geq \left(\frac{\widetilde{c}r^{-\varepsilon-2}}{|\operatorname{Im}\lambda|}\right)^{\frac{\varepsilon}{1-\varepsilon}}. \tag{3.16}$$

Splitting the argument of the exponential in (3.14) same way as (3.12) and using (3.16), we have

$$
\begin{aligned}
|\operatorname{Im}\lambda|\cdot x_n^{\varepsilon}-\frac{\widetilde{c}}{r^{\alpha}}x_n^{\alpha} &\leq |\operatorname{Im}\lambda|\cdot x_n^{\varepsilon}\left(\frac{\widetilde{c}r^{-\varepsilon-1}}{|\operatorname{Im}\lambda|}-\frac{r^{-\varepsilon}\delta\widetilde{c}}{|\operatorname{Im}\lambda|}\right)-(1-\delta)\frac{\widetilde{c}x_n^{\varepsilon}}{r^{\varepsilon}}\\
&= -x_n^{\varepsilon}\widetilde{c}r^{-\varepsilon-1}(r\delta-1)-(1-\delta)\frac{\widetilde{c}x_n^{\varepsilon}}{r^{\varepsilon}}\\
&\leq -\left(\frac{\widetilde{c}r^{-\varepsilon-2}}{|\operatorname{Im}\lambda|}\right)^{\frac{\varepsilon}{1+\varepsilon}}\widetilde{c}r^{-\varepsilon-1}(r\delta-1)-(1-\delta)\frac{\widetilde{c}x_n^{\varepsilon}}{r^{\varepsilon}}\\
&\quad -\frac{\widetilde{c}^{\frac{1}{1-\varepsilon}}}{r^{\frac{2\varepsilon+1}{1-\varepsilon}}}\frac{r\delta-1}{|\operatorname{Im}\lambda|^{\frac{\varepsilon}{1-\varepsilon}}}-(1-\delta)\frac{\widetilde{c}x_n^{\varepsilon}}{r^{\varepsilon}}.
\end{aligned}
$$

Inserting this into (3.14) we obtain

$$
\left|\partial_{\overline{\lambda}}\widetilde{G}\right| \lesssim_g \left(\sum_{n\geq 0}x_n^{1-\varepsilon}\exp\{-\operatorname{const}x_n^{\varepsilon}\}\right)\cdot\exp\left\{-\widetilde{Q}\,|\operatorname{Im}\lambda|^{-\frac{\varepsilon}{1-\varepsilon}}\right\} \tag{3.17}
$$

$$
\widetilde{Q}\equiv(r\delta-1)\frac{\widetilde{c}^{\frac{1}{1-\varepsilon}}}{r^{\frac{2\varepsilon+1}{1-\varepsilon}}}<(r\delta-1)\frac{c^{\frac{1}{1-\varepsilon}}}{r^{\frac{2\varepsilon+1}{1-\varepsilon}}}. \tag{3.18}
$$

The series in (3.17) is convergent and $\widetilde{G}(\lambda)$ is an pseudo analytic extension of $\widehat{g}(\lambda)$ from the real line to the upper half plane. Due to (3.7) we have found a pseudo analytic extension of G subject to (3.6) with $Q=2\widetilde{Q}$. This completes our proof. □

Remark 3.3. *The representation* (3.2) *is not unique. It depends on the reference point in the splitting of* (3.3)*. This flexibility will be used later.*

Remark 3.4. *We have also had some flexibility in choosing r and δ in* (3.18) *subject to $r>1$, $1/r<\delta<1$. The range for $Q=2\widetilde{Q}$ given by* (3.18) *is*

$$
0<Q<\frac{2(\varepsilon-1)(3\varepsilon)^{\frac{2\varepsilon+1}{1-\varepsilon}}}{(2\varepsilon+1)^{\frac{1+\varepsilon}{1-\varepsilon}}}\cdot c^{\frac{1}{1-\varepsilon}}
$$

which is inessential to what follows but the borderline case $\varepsilon=1/2$. For this case,

$$
0<Q<\frac{3^3}{2^7}c^2. \tag{3.19}
$$

4. The Marchenko integral operator and the inverse scattering transform

The integral operator we are concerned with in this section appears to have been introduced by Marchenko and received a comprehensive treatment in his classical book [23]. To acknowledge Marchenko's profound contribution to the subject, we denote this operator by $\mathbb{M}$ (Marchenko used $\mathbb{F}$) but otherwise try to retain as much of his original notation as possible.

We call the integral operator $\mathbb{M}: L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$ Marchenko if

$$
\begin{aligned}
(\mathbb{M}f)(x) &= \int_0^{\infty}M(x+y)f(y)dy,\quad f\in L^2(\mathbb{R}_+),\\
M(\cdot) &= \int_0^{\infty}e^{-(\cdot)\lambda}d\rho(\lambda)+\int_{-\infty}^{\infty}e^{2i(\cdot)\lambda}R(\lambda)\frac{d\lambda}{2\pi},
\end{aligned} \tag{4.1}
$$

where ρ is a finite nonnegative measure and R is such that for a.e. $\lambda \in \mathbb{R}$

$$R(-\lambda) = \overline{R(\lambda)} \quad , \quad |R(\lambda)| \leq 1.$$

The operator $\mathbb{M}$ introduced this way is clearly a Hankel operator (the kernel depends on the sum of the arguments). We say that $\mathbb{M}$ is associated with a potential q if R is a reflection coefficient from q and ρ characterizes the negative spectrum of H_q. For short-range q's, the measure ρ is purely discrete and (δ is the Dirac δ-function)

$$d\rho(\lambda) = \sum_{n=1}^{N} c_n^2 \delta(\lambda - \varkappa_n) d\lambda$$

where $\varkappa_n$'s are such that $\left\{-\varkappa_n^2\right\}_{n=1}^N$ are the negative bound states of H_q and $\{c_n\}_{n=1}^N$ are the associated norming constants. Of course, the kernel M (which we also call Marchenko) is nothing but the sum of the Laplace transform of the (finite, positive) measure ρ and the Fourier transform of the (symmetric, bounded) function R. This is the main feature of the Marchenko operator resulting in the decomposition

$$\mathbb{M} = \mathbb{M}_1 + \mathbb{M}_2, \tag{4.2}$$

where $\mathbb{M}_1 \geq 0$ and ($\chi = \chi_+$, $\mathcal{F}$ is the Fourier transform)

$$\mathbb{M}_2 = \chi \mathcal{F}^{-1} R \mathcal{F}^{-1}$$

and is selfadjoint.

Note that the Marchenko operator is not typically studied in the context of Hankel operators. We have found in [29] that the language of Hankel operators is very convenient in the IST formalism. In this language, ρ, R are called the measure and symbol of $\mathbb{M}_1, \mathbb{M}_2$ respectively (see e.g. [24]). In the language of inverse scattering, (ρ, R) are commonly referred to as the (right) scattering data[2].

In the context of the Cauchy problem for the KdV equation, we have a two parametric family of Marchenko operators $\mathbb{M}(z,t)$, where (z,t) are the (real) variables in (1.1). Namely, the measure $\rho_{z,t}$ and the symbol $R_{z,t}$ of $\mathbb{M}(z,t)$ are given by

$$\begin{aligned} d\rho_{z,t}(\lambda) &= \zeta_{z,t}(i\lambda) d\rho(\lambda), \\ R_{z,t}(\lambda) &= \zeta_{z,t}(\lambda) R(\lambda), \\ \zeta_{z,t}(\lambda) &:\equiv e^{8i\lambda^3 t + 2i\lambda z} \end{aligned}$$

and (ρ, R) are the scattering data for the profile q. Clearly for $\lambda \in \mathbb{R}$

$$\begin{aligned} \left|\zeta_{z,t}(\lambda)\right| &= 1, \\ \left|\zeta_{z,t}(i\lambda)\right| &= e^{8\lambda^3 t - 2\lambda z}. \end{aligned}$$

The whole point of the IST is that $(\rho_{0,t}, R_{0,t})$ are the scattering data for $H_{u(z,t)}$ where $u(z,t)$ solves (1.1). The actual mechanism to recover $u(z,t)$ amounts to solving the Marchenko integral equation[3] or equivalently through the Riemann-Hilbert problem. For our purposes the explicit formula (1.5) is convenient. Note that (1.5) is nothing but Cramer's rule for linear integral equations. For $R \equiv 0$ (reflectionless initial profile), assuming that q is short-range, $\mathbb{M}(z,t)$ turns into a

[2]For short-range q's, the so-called left scattering data are also considered, which need not be well-defined in our setting.

[3]also known as Gelfand-Levitan-Marchenko.

finite rank operator of a very explicit structure. The formula (1.5) in this particular case was discovered in the 1960s. In the general short-range case ($R \neq 0$), we are not sure whom it should be attributed to but it was systemically studied by Pöppe in the 1980s (see, e.g. [26] and also [27] for the Kadomtsev-Petviashvili and [25] for the sine-Gordon equations. In the context of nondecaying initial data, it appears first in [29]. Note that the sense in what the determinant in (1.5) is understood is an issue which doesn't seem to be fully addressed in the literature. It is typically defined by the Fredholm formula through an absolutely convergent (Fredholm) series. We actually show that $\mathbb{M}(z,t)$ is trace class for any z (even complex) and $t>0$ under Hypothesis 1.1. This means that $\det(1+\mathbb{M}(z,t))$ is an invariant, i.e. it produces the same value in any basis in $L^2(\mathbb{R}_+)$.

In the setting of step-like potentials, the Marchenko operator has been intensively studied in the Kharkov mathematical school by Hruslov, Kotlyarov and their students[4] (see, e.g. [20], [22]).

We also refer to Cohen [8], Kappeler [21], Venakides [32] (and the literature cited therein), and recent Egorova-Teschl [11]. In all the above papers save [11], q's are assumed to have a specific type of behavior at $-\infty$ (approaching either a constant or a periodic function) and fall off at $+\infty$. In [11], the interesting case of two finite gap potentials fused together is considered.

We summarize important properties of the Marchenko operator in the following (see [29] for details).

Proposition 4.1 (The structure of the Marchenko operator)**.** *Assuming Hypothesis 1.1, let $\mathbb{M}(z,t)$ be the Marchenko operator associated with q and let A be as in Proposition 3.2. Then for any $z\in\mathbb{R}$, $t>0$,*

$$\mathbb{M}(z,t)=\mathbb{M}_+(z,t)+\mathbb{A}(z,t), \tag{4.3}$$

where $\mathbb{M}_+(z,t)$ is the Marchenko operator associated with $q_+=q|_{\mathbb{R}_+}$ and $\mathbb{A}(z,t)$ is a Hankel integral operator with the kernel

$$\frac{1}{2\pi}\int_{\mathbb{R}+ih} e^{2i\lambda(\cdot)}\zeta_{z,t}(\lambda)A(\lambda)d\lambda, \quad h>h_0.$$

Furthermore, $\mathbb{A}(z,t)$ is an entire operator-valued function of trace class for any complex z and $t>0$, continuous with respect to q in the following sense: If q_1,q_2 are two functions subject to Hypothesis 1.1 then

$$\left\|\mathbb{A}_1(z,t)-\mathbb{A}_2(z,t)\right\|_{\mathfrak{S}_1}\leq\frac{1}{4\pi h}\left\|\zeta_{z,t}(A_1-A_2)\right\|_{L^1(\mathbb{R}+ih)}$$

for any $z\in\mathbb{C}$, $t>0$.

Note that $\mathbb{M}(z,t)$ depends on (z,t) through $\zeta_{z,t}$.

5. Proof of the main results

With all the preparations done in the previous sections, the actual proofs will be quite short. It is convenient to conduct both proofs at a time. Note first that, by

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[4] Remark that this school has been greatly infuenced by Marchenko himself and he remains to be its part.

a trivial shifting, we may assume without loss of generality that H_{q_+} has at most one bound state $-\varkappa^2$. Consider the problem (1.1) with

$$q_n(x) = \begin{cases} q(x) & , \quad x \geq -n \\ 0 & , \quad x < -n \end{cases}.$$

It is well-known that for such initial profiles[5]

$$u_n(z,t) = -2\partial_z^2 \log\det\left(1 + \mathbb{M}_n(z,t)\right). \tag{5.1}$$

By Proposition 4.1

$$\mathbb{M}(z,t) = \mathbb{M}_+(z,t) + \mathbb{A}(z,t) + \delta\mathbb{A}(z,t)$$

where $\delta\mathbb{A} \equiv \mathbb{A}_n - \mathbb{A}$ is meromorphic in z for any $t > 0$ and small in the $\mathfrak{S}_1$-norm for n large enough. I.e.

$$\left\|\mathbb{M}_n(z,t) - \mathbb{M}(z,t)\right\|_{\mathfrak{S}_1} \to 0, \quad n \to \infty. \tag{5.2}$$

Therefore, $\mathbb{M}(z,t) \in \mathfrak{S}_1$ is proven if we show that $\mathbb{M}_+(z,t) \in \mathfrak{S}_1$.

Split

$$\mathbb{M}_+(z,t) = \mathbb{M}_1^+(z,t) + \mathbb{M}_2^+(z,t)$$

where $\mathbb{M}_1^+(z,t), \mathbb{M}_2^+(z,t)$ are the Hankel operators with the kernels

$$c_0^2 \zeta_{z,t}(i\varkappa) e^{-\varkappa(x+y)},$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x+y)} \zeta_{z,t}(\lambda) R_+(\lambda) d\lambda$$

respectively. Here c_0 stands for the norming constant associated with the bound state $-\varkappa^2$.

The operator $\mathbb{M}_1(z,t)$ is rank 1 and clearly entire in z. Thus we only need to properly control $\partial_z^n \mathbb{M}_2^+(z,t)$ in the $\mathfrak{S}_1$-norm. Evaluate (so far formally) the kernel of $\partial_z^n \mathbb{M}_2^+(z,t), n = 0,1,2,...$, by the Green formula applied to the strip $\mathbb{R} \times (0, \varkappa/2)$ and by Proposition 3.2 ($\lambda = \alpha + i\beta, \partial_{\overline{\lambda}} = \frac{1}{2}(\partial_\alpha + i\partial_\beta)$)

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x+y)} (2i\lambda)^n \zeta_{z,t}(\lambda) R_+(\lambda) d\lambda &= \int_{\mathbb{R}+i\varkappa/2} e^{i\lambda(x+y)} (2i\lambda)^n \left(B^{-1}SG\right)(\lambda) \frac{d\lambda}{2\pi} \\ &\quad + 2i\int_0^{\varkappa/2} d\beta \int d\alpha\; e^{i\lambda(x+y)} F(\alpha,\beta) \\ &\equiv H_1(x+y) + H_2(x+y), \end{aligned} \tag{5.3}$$

where

$$F(\alpha,\beta) \equiv \frac{1}{2\pi}\zeta_{z,t}(\lambda)(2i\lambda)^{n-1}\frac{S(\lambda)}{B(\lambda)}\partial_{\overline{\lambda}} G(\alpha,\beta).$$

Due to the rapid decay of $e^{8i\lambda^3 t}$ as $\lambda \to \infty$ along $\mathbb{R} + ih$, the function $F(\alpha,\beta)$ is subject to the conditions of Proposition 4.1 and hence the integral operator with kernel H_1 is trace class. Our analysis of the integral operator with kernel H_2 is based upon the following lemma.

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[5] So far we only know that the determinant exists in the Fredholm sense.

Lemma 5.1. *Let $F(\alpha,\beta)$ be such that for some $h>0$*

$$\int_0^h \left(\int_{-\infty}^{\infty} |F(\alpha,\beta)|\, d\alpha\right) \frac{d\beta}{\beta} < \infty. \tag{5.4}$$

Then the integral Hankel operator $\mathbb{H}$ with the kernel ($\lambda = \alpha + i\beta$)

$$H(x) = \int_0^h d\beta \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{i\lambda x} F(\alpha,\beta)$$

is trace class and

$$\|\mathbb{H}\|_{\mathfrak{S}_1} \le \frac{1}{2}\int_0^h \frac{d\beta}{\beta}\int_{-\infty}^{\infty} d\alpha\, |F(\alpha,\beta)|\,.$$

Proof. We have

$$\begin{aligned} H(x+y) &\eqsim \int_0^h d\beta e^{-\beta x}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x} F(\alpha,\beta) \\ &= \int_0^h e^{-\beta(x+y)} \widehat{F_\beta}(x+y) dx, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \widehat{F_\beta}(x+y) &\equiv \int_{-\infty}^{\infty} e^{i\alpha(x+y)} F(\alpha+\beta) d\alpha \\ &\eqsim \widehat{F_\beta^{1/2}} * \widehat{F_\beta^{1/2}}(x+y) \\ &= \int_{-\infty}^{\infty} F_\beta^{1/2}(x+s) F_\beta^{1/2}(y-s) ds. \end{aligned} \tag{5.6}$$

Here we have used the convolution theorem. Inserting (5.6) into (5.5) implies that

$$\mathbb{H} = \int_0^h \mathbb{H}_{\beta,1}\mathbb{H}_{\beta,2} d\beta, \tag{5.7}$$

where $\mathbb{H}_{\beta,1}$ and $\mathbb{H}_{\beta,2}$ and integral (but not Hankel) operators on $L^2(\mathbb{R})$ with the kernels

$$\begin{aligned} H_{\beta,1}(x,s) &= \chi(x) e^{-\beta x}\widehat{F_\beta^{1/2}}(x+s), \\ H_{\beta,2}(s,y) &= \chi(y) e^{-\beta y}\widehat{F_\beta^{1/2}}(y-s) \end{aligned}$$

respectively.

It follows from (5.7) that

$$\|\mathbb{H}\|_{\mathfrak{S}_1} \le \int_0^h \|\mathbb{H}_{\beta,1}\|_{\mathfrak{S}_2}\cdot\|\mathbb{H}_{\beta,2}\|_{\mathfrak{S}_2} d\beta. \tag{5.8}$$

Evaluate now the Hilbert-Schmidt norms of $\mathbb{H}_{\beta,1}$ and $\mathbb{H}_{\beta,2}$. By the Plancherel equation we have

$$\begin{aligned} \|\mathbb{H}_{\beta,1}\|^2_{\mathfrak{S}_2} &= \int_0^\infty \int_{-\infty}^{\infty} |H_{\beta,1}(x,s)|^2\, ds\, dx \\ &= \int_0^\infty dx e^{-2\beta x}\int_{-\infty}^{\infty} ds \left|\widehat{F_\beta^{1/2}}(x+s)\right|^2 \\ &= \frac{\left\|F_\beta^{1/2}\right\|^2_{L^2(\mathbb{R})}}{2\beta} = \frac{1}{2\beta}\|F_\beta\|_{L^1(\mathbb{R})}\,. \end{aligned}$$

That is

$$\|H_{\beta,1}\|_{\mathfrak{S}_2} \le \frac{1}{\sqrt{2\beta}} \|F_\beta\|^{1/2}_{L^1(\mathbb{R})} .$$

Similarly,

$$\|H_{\beta,2}\|_{\mathfrak{S}_2} \le \frac{1}{\sqrt{2\beta}} \|F_\beta\|^{1/2}_{L^1(\mathbb{R})}$$

and (5.8) yields

$$\|\mathbb{H}\|_{\mathfrak{S}_1} \le \frac{1}{2}\int_0^h \|F_\beta\|_{L^1} \frac{d\beta}{\beta}.$$

The lemma is proven. □

Let us find suitable bounds on $\mathbb{R} \times [0, \varkappa/2]$ for the functions involved in F:

$$\begin{aligned} \left|\zeta_{z,t}(\lambda)\right| &= \left|e^{8i\lambda^3 t + 2i\lambda z}\right| = e^{8\beta^3 t - 2\beta \operatorname{Re} z} \cdot e^{-24\beta t\alpha^2 - 2\alpha \operatorname{Im} z} \\ &\le e^{\varkappa(\varkappa^2 t + |z|)} \cdot \exp\left\{-\left(\sqrt{24\beta t}\alpha + \frac{\operatorname{Im} z}{\sqrt{24\beta t}}\right)^2 + \frac{\operatorname{Im}^2 z}{24\beta t}\right\}, \\ \left|\lambda^{n-1}B^{-1}(\lambda)S(\lambda)\right| &\lesssim (|\alpha| + \beta)^{n-1}, \\ \left|\partial_{\overline{\lambda}} G\right| &\lesssim_{q_+} \exp\left\{-Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}\right\}. \end{aligned}$$

Thus

$$|F(\alpha,\beta)| \lesssim_{q_+} e^{\varkappa\left(\varkappa^2 t + |z|\right)} (|\alpha| + \beta)^{n-1} e^{-\left(\sqrt{24\beta t}\alpha + \frac{\operatorname{Im} z}{\sqrt{24\beta t}}\right)^2} \exp\left\{\frac{\operatorname{Im}^2 z}{24t}\beta^{-1} - Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}\right\}. \tag{5.9}$$

To prove Theorem 1.2 we only need to consider the case $n = 1$. We have

$$\int_0^{\varkappa/2} \frac{d\beta}{\beta} \int_{-\infty}^{\infty} |F(\alpha,\beta)|\, d\alpha \lesssim_{z,t,q_+} \int_0^{\varkappa/2} \beta^{-3/2} \exp\left\{\frac{\operatorname{Im}^2 z}{24t}\beta^{-1} - Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}\right\} d\beta. \tag{5.10}$$

So F is subject to the condition of Lemma 5.1 if the integral in (5.10) converges, which depends on ε and $\operatorname{Im} z$.

Case 1. $\varepsilon > 1/2$. Then[6] $\frac{\varepsilon}{1-\varepsilon} > 1$ and the right hand side of (5.10) is finite for any $z \in \mathbb{C}$. This means that $\mathbb{M}_+(z,t)$ is an entire $\mathfrak{S}_1$-valued function for any $t > 0$ and due to (5.2), we can pass to the limit in (5.1) as $n \to \infty$ by standard properties of infinite determinants (see e.g. [16]). This proves (1) in Theorem 1.2.

Case 2. $\varepsilon = 1/2$. Then $\frac{\varepsilon}{1-\varepsilon} = 1$ and the right hand side of (5.10) converges if and only if

$$\frac{\operatorname{Im}^2 z}{24t} - Q < 0$$

or when

$$|\operatorname{Im} z| < \sqrt{12Q} \cdot \sqrt{t}.$$

Choosing the maximum possible value of Q in (3.19) we get

$$|\operatorname{Im} z| < \frac{9\sqrt{2}}{8} c\sqrt{t}$$

[6] We assume $\varepsilon < 1$. If $\varepsilon \ge 1$ then Theorem 1.2 is trivial.

and (2) of Theorem 1.2 follows. Thus, Theorem 1.2 is proven.

Case 3. $0 < \varepsilon < 1/2$. Then $\frac{\varepsilon}{1-\varepsilon} < 1$ and (5.10) clearly diverges for any $\operatorname{Im} z \neq 0$ and our method fails to establish analyticity and we have to go back to (5.9) and analyze it for any natural n. Expanding $(|\alpha| + \beta)^{n-1}$ in (5.9) by the binomial formula we have

$$\int_0^{\varkappa/2} \frac{d\beta}{\beta} \int_{-\infty}^{\infty} F(\alpha,\beta) d\alpha \lesssim_{z,t,q_+} \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^{\varkappa/2} d\beta \beta^{k-1} e^{-Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}} \int_0^{\infty} \alpha^{n-k-1} e^{-24\beta t \alpha^2} d\alpha. \quad (5.11)$$

Reducing the inner integral in (5.11) to the gamma function[7],

$$(5.11) = \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^{\varkappa/2} d\beta \beta^{k-1} e^{-Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}} \cdot \frac{1}{(3\beta t)^{\frac{n-k}{2}}} \Gamma\left(\frac{n-k}{2}\right)$$
$$\lesssim \sum_{k=0}^{n} \binom{n}{k} (3t)^{-\frac{n-k}{2}} \Gamma\left(\frac{n-k}{2}\right) \int_0^1 d\beta \beta^{\frac{3k-n}{2}-1} e^{-Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}}. \quad (5.12)$$

Introducing in the last integral the new variable $s = \beta^{-\frac{\varepsilon}{1-\varepsilon}}$ and setting $\gamma \equiv \frac{1-\varepsilon}{2\varepsilon} > 1/2$ we get

$$\int_0^1 d\beta \beta^{\frac{3k-n}{2}-1} e^{-Q\beta^{-\frac{\varepsilon}{1-\varepsilon}}} = \frac{1-\varepsilon}{\varepsilon} \int_1^{\infty} s^{\frac{\varepsilon-1}{\varepsilon}\left(\frac{3k-n}{2}-1\right)-1} e^{-Qs} ds$$
$$\lesssim \int_1^{\infty} s^{\gamma(n-3k)-1} e^{-Qs} ds$$
$$\lesssim Q^{-\gamma(n-3k)+1} \int_Q^{\infty} s^{\gamma(n-3k)} e^{-s} ds. \quad (5.13)$$

The behavior of the last integral depends on the sign of $\omega_k \equiv \gamma(n-3k)$. If $\omega_k \geq 0$, i.e. $3k \leq n$, then

$$J_k \equiv \int_Q^{\infty} s^{\omega_k} e^{-s} ds$$
$$\leq \int_0^{\infty} s^{\omega_k - 1} e^{-s} ds = \Gamma(\omega_k)$$
$$= \Gamma(\gamma(n-3k)).$$

If $\omega_k < 0$, i.e. $3k > n$, then

$$J_k \leq Q^{\omega_k - 1} \int_Q^{\infty} e^{-s} ds \leq Q^{\omega_k - 1}.$$

Splitting the sum in (5.12) accordingly, we see that the right hand side of (5.12) is dominated by

$$\sum_{0 \leq 3k \leq n} \binom{n}{k} (3t)^{-\frac{n-k}{2}} Q^{-\omega_k} \Gamma\left(\frac{n-k}{2}\right) \Gamma(\omega_k) + \sum_{n < 3k \leq 3n} \binom{n}{k} (3t)^{-\frac{n-k}{2}} \Gamma\left(\frac{n-k}{2}\right) \equiv S_1 + S_2. \quad (5.14)$$

[7]Recall $\Gamma(z) = \int_0^{\infty} \alpha^{z-1} e^{-\alpha} d\alpha$.

Analyze now S_1 and S_2. For S_1 we have

$$S_1 \leq \Gamma\left(\frac{n}{2}\right)\Gamma(\gamma n)\sum_{0\leq 3k\leq n}\binom{n}{k}(3t)^{-\frac{n-k}{2}}Q^{-\omega_k}$$
$$\leq \left(Q^{2\gamma}+\frac{1}{\sqrt{3t}Q^{\gamma}}\right)^n \Gamma\left(\frac{n}{2}\right)\Gamma(\gamma n). \tag{5.15}$$

For S_2 we obtain

$$S_2 \leq \left(1+\frac{1}{\sqrt{3t}}\right)^n \Gamma\left(\frac{n}{3}\right)$$

and hence the contribution from S_2 to (5.13) produces a real analytic function. On the other hand, as it easily follows from (5.15), the contribution from S_1 produces a function from $G^{\gamma-1/2} = G^{\frac{1}{2\varepsilon}-1}$. Thus we have proven that if $0<\varepsilon<1/2$ then

$$\mathbb{M}_+(x,t) = \mathbb{M}_+^{(1)}(x,t) + \mathbb{M}_+^{(2)}(x,t)$$

where $\mathbb{M}_+^{(1)}(x,t)$ is a real analytic $\mathfrak{S}_1$-valued function and $\mathbb{M}_+^{(2)}(x,t)$ is a $\mathfrak{S}_1$-valued function from the Gevrey class $G^{\frac{1}{2\varepsilon}-1}$. Thus, we can pass to the limit as before. The limiting function is from the Gevrey class $G^{\frac{1}{2\varepsilon}-1}$ if $\det(1+\mathbb{M}(x,t))$ doesn't vanish for all $x\in\mathbb{R}$. The latter occurs if $1+\mathbb{M}(x,t)$ is invertible on $\mathbb{R}$ for any $t>0$.

Theorem 1.3 is proven.

6. Discussions, corollaries, and open problems

6.1. **Discussions.**

Remark 6.1. *Theorem 1.2 improves our main result from* [29] *where* $\mathbb{M}(x,t)\in\mathfrak{S}_1$ *was not proven and only real analyticity of* $u(x,t)$ *was obtained. The main idea of* [29] *is to put together the analytic continuation arguments of* [28] *to treat initial data on* $\mathbb{R}_-$ *and Tarama's approach from* [31] *to handle the data on* $\mathbb{R}_+$. *As far as we know the solution to Problem 1 given in* [31] *was best known back then. The main result of* [31] *says that* $u(x,t)$ *is real analytic under the following conditions:* q *is real and* L^2_{loc} *such that*

$$\int_{-\infty}^{\infty}(1+|x|)\,|q(x)|\,dx<\infty$$

and for x *large enough there are positive* C_q, c *so that*

$$\int_x^{\infty}|q|^2 \leq C_q e^{-cx^{1/2}}.$$

Note that these conditions are much stronger than Hypothesis 1.1. The techniques used in [31] *are also based upon the (classical) IST but his analysis relies on the properties of the Airy function as opposed to ours which is based on analytic and pseudo-analyitc continuations. The latter appears particularly well-suited for addressing Problem 1 and consequently significantly less involved.*

Remark 6.2. *It is proven in* [9]*, Theorem 7.2 that if q is analytic in the strip $|\operatorname{Im} z| < a$ and has Schwartz decay there, then $u(z,t)$ is meromorphic in a strip with at most N poles (where N is the number of bound states of H_q) off the real line. By Theorem 7.1 from the same* [9]*, for the reflection coefficient one then have $R(\lambda) = O\left(e^{-2a|\lambda|}\right)$ as $\lambda \to \infty$ which of course need not occur in our case. This implies that our real meromorphic solution $u(z,t)$ in Theorem 1.2 has, in general, infinitely many poles for any $t > 0$ in any strip around the real line accumulating only to infinity. By general theorems* [30] *on families of compact meromorphic operators these poles continuously depend on t and hence may appear or disappear only on the boundary of analyticity of $u(x,t)$ (including infinity).*

6.2. **Corollaries.** The following statement is a direct consequence of the analyticity of $u(z,t)$ for $t > 0$.

Corollary 6.3. *Under conditions of Theorem 1.2 the solution $u(z,t)$ can not vanish on an open set for any $t > 0$ unless q is identically zero.*

This quickly recovers and improves a number of unique continuation results due to Zhang [33]. E.g., one of the main results of [33] says that $u(x,t)$ cannot have compact support at two different moments unless it vanishes identically. The techniques of [33] rely upon the classical IST (coupled with some Hardy space arguments) and are valid under certain decay and regularity conditions on q.

Corollary 6.4. *The class of (nonsmooth) initial data q such that*

$$\int_{-\infty}^{\infty} e^{c|x|^{\varepsilon}} |q(x)|\, dx < \infty \text{ for some } c, \varepsilon > 0 \tag{6.1}$$

is not preserved under the KdV flow.

Proof. Assume that for some $t = t_0$ the function $u(x,t_0)$ is subject to (6.1). Since the KdV equation is invariant under the transformation $(x,t) \to (-x,-t)$, the solution $u_0(x,t)$ to the problem (1.1) with the initial data $q_0(x) = u(-x,t_0)$, by Theorems 1.2, 1.3, will be at least smooth for any $t > 0$. But $u_0(x,t_0) = q(x)$ forcing original q to be smooth too. □

Corollary 6.4, in turn, implies that under the KdV flow neither an exponential decay at $-\infty$ nor smoothness persist in general. Note in this connection that issues related to persistence of regularity are also very important and have been extensively studied but we don't touch on this here.

The explicit formula (1.5), which was used to derive our analyticity results, does have some practical value. E.g. it implies that the large time asymptotic behavior of $u(x,t)$ is completely determined by the measure $\rho(\lambda)$ in (4.1) alone. This fact is so far rigorously proven for q's tending to a negative constant or a periodic function at $-\infty$ and was used to obtain explicit expressions for the so-called asymptotic solitons (see, e.g. [20], [32], and [22]). We plan to return to this important issue elsewhere.

6.3. **Open problems.**

(1) We believe that under Hypothesis 1.1 our solutions $u(x,t)$ have no singularities on the real line for any $t > 0$. If this held then the problem (1.1) would be globally well-posed under Hypothesis 1.1 only and no blow-up solution could develop. That is to say that $1 + \mathbb{M}(x,t)$ is automatically invertible

for any real x and $t > 0$ under Hypothesis 1.1 alone. This fact is quite easy if in (4.1) the support of $\rho(\lambda)$ is rich enough (a set of uniqueness of an analytic function) or $|R(\lambda)| < 1$ on any set of positive Lebesgue measure (see [29]). The situation is much less trivial if $R(\lambda)$ in (4.1) is unimodular for a.e. real λ (i.e. q is completely reflecting). An affirmative answer is given in [17] for the case of q such that $q|_{\mathbb{R}_+} = 0$ and $H_q \geq 0$ (absence of negative spectrum). To address the problem as stated one needs to show that $1 + \mathbb{M}(x,t)$ is invertible in the case when in (4.1) $\rho(\lambda)$ is supported on a set $\{\lambda_n\} \subset \mathbb{R}_+$ such that $\sum \lambda_n < \infty$ and $|R(\lambda)| = 1$ a.e. on the real line. In term of the Schrodinger operator H_q itself this means that the absolutely continuous spectrum of H_q is simple and supported on $\mathbb{R}_+$ but there is a rich embedded positive singular spectrum. Physically relevant examples can be constructed from the Gaussian white noise, Pearson sparse blocks, Kotani potentials, etc.

(2) We do not know much about the Banach (or Hilbert) space of meromorphic function to which $u(z,t)$ from 1.2 belongs. It would be very interesting to find such spaces as this would give, among others, important norm estimates for $u(z,t)$ which our paper lacks.

(3) We (cautiously) conjecture that in Theorem 1.3 $u(x,t)$ could be represented for any $t > 0$ as a meromorphic function plus a small Gevrey regular function. We can in fact show that the trace norm of $\mathbb{M}^{(2)}(x,t)$ from Theorem 1.3 can be made small but it is not clear if after taking the det and then log the analytic and small Gevrey parts will still be separated. Of course, this question will immediately have an affirmative answer if under conditions of Theorem 1.3 the solution $u(x,t)$ happens to be real analytic. Our methods however fail to yield such results.

Acknowledgement

We are grateful to Fritz Gesztesy for valuable discussions.

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