CONVERGENCE OF A QUANTUM NORMAL FORM AND AN EXACT QUANTIZATION FORMULA

SANDRO GRAFFI AND THIERRY PAUL

Abstract. Let the quantization of the linear flow of diophantine frequencies ω over the torus $\mathbb{T}^l$, $l > 1$, namely the Schrödinger operator $-i\hbar\omega \cdot \nabla$ on $L^2(\mathbb{T}^l)$, be perturbed by the quantization of a function $\mathcal{V}_\omega : \mathbb{R}^l \times \mathbb{T}^l \to \mathbb{R}$ of the form

$$\mathcal{V}_\omega(\xi, x) = \mathcal{V}(z \circ \mathcal{L}_\omega(\xi), x), \quad \mathcal{L}_\omega(\xi) := \omega_1\xi_1 + \ldots + \omega_l\xi_l$$

where $z \mapsto \mathcal{V}(z, x) : \mathbb{R} \times \mathbb{T}^l \to \mathbb{R}$ is real-holomorphic. We prove that the corresponding quantum normal form converges uniformly with respect to $\hbar \in [0, 1]$. Since the quantum normal form reduces to the classical one for $\hbar = 0$, this result simultaneously yields an exact quantization formula for the quantum spectrum, as well as a convergence criterion for the Birkhoff normal form, valid for a class of perturbations holomorphic away from the origin. The main technical aspect concerns the quantum homological equation $[F(-i\hbar\omega \cdot \nabla), W]/i\hbar + V = N$, $F : \mathbb{R} \to \mathbb{R}$ being a smooth function $\varepsilon-$close to the identity. Its solution is constructed, and estimated uniformly with respect to $\hbar \in [0, 1]$, by solving the equation $\{F(\mathcal{L}_\omega), \mathcal{W}\}_M + \mathcal{V} = \mathcal{N}$ for the corresponding symbols. Here $\{\cdot, \cdot\}_M$ stands for the Moyal bracket. As a consequence, the KAM iteration for the symbols of the quantum operators can be implemented, and its convergence proved, uniformly with respect to $(\xi, \hbar, \varepsilon) \in \mathbb{R}^l \times [0, 1] \times \{\varepsilon \in \mathbb{C} \mid |\varepsilon| < \varepsilon^*\}$, where $\varepsilon^* > 0$ is explicitly estimated in terms only of the diophantine constants. This in turn entails the uniform convergence of the quantum normal form.

Contents

1. Introduction

1.1. **Quantization formulae.** The establishment of a quantization formula (QF) for the eigenvalues of the Schrödinger operators is a classical mathematical problem of quantum mechanics (see e.g.[FM]). To review the notion of QF, consider first a semiclassical pseudodifferential operator H (for this notion, see e.g.[Ro]) acting on $L^2(\mathbb{R}^l)$, $l \geq 1$, of order m, self-adjoint with pure-point spectrum, with (Weyl) symbol $\sigma_H(\xi, x) \in C^\infty(\mathbb{R}^l \times \mathbb{R}^l; \mathbb{R})$.

Definition 1.1. *We say that H admits an M-smooth* exact *QF, $M \geq 2$, if there exists a function $\mu : (A, \hbar) \mapsto \mu(A, \hbar) \in C^M(\mathbb{R}^l \times [0,1]; \mathbb{R})$ such that:*

(1) *$\mu(A, \hbar)$ admits an asymptotic expansion up to order M in $\hbar$ uniformly on compacts with respect to $A \in \mathbb{R}^l$;*
(2) *$\forall \hbar \in]0,1]$, there is a sequence $n_k := (n_{k_1}, \dots, n_{k_l}) \subset \mathbb{Z}^l$ such that all eigenvalues $\lambda_k(\hbar)$ of H admit the representation:*

$$\lambda_k(\hbar) = \mu(n_k\hbar, \hbar). \tag{1.1}$$

Remark 1.2. (Link with the Maslov index) Consider any function $f : \mathbb{R}^l \to \mathbb{R}^l$ with the property $\langle f(A), \nabla\mu(A,0)\rangle = \partial_\hbar \mu(A,0)$. Then we can rewrite the asymptotic expansion of μ at second order as :

$$\mu(n_k\hbar, \hbar) = \mu(n_k\hbar + \hbar f(n_k\hbar)) + O(\hbar^2). \tag{1.2}$$

When $f(m\hbar) = \nu$, $\nu \in \mathbb{Q}^l$, the Maslov index [Ma] is recovered. Moreover, when

$$|\lambda_k(\hbar) - \mu(n_k\hbar, \hbar)| = O(\hbar^M), \quad \hbar \to 0, \quad M \geq 2 \tag{1.3}$$

then we speak of *approximate* QF of order M.

Example 1.3. (Bohr-Sommerfeld-Einstein formula). Let σ_H fulfill the conditions of the Liouville-Arnold theorem (see e.g.[Ar1], §50). Denote $A = (A_1, \dots, A_l) \in \mathbb{R}^l$ the action variables, and

$E(A_1,\dots,A_l)$ the symbol σ_H expressed as a function of the action variables. Then the Bohr-Sommerfeld-Einstein formula (BSE) QF is

$$\lambda_{n,\hbar} = E((n_1+\nu/4)\hbar,\dots,(n_l+\nu/4)\hbar) + O(\hbar^2) \tag{1.4}$$

where $\nu = \nu(l) \in \mathbb{N}\cup\{0\}$ is the Maslov index [Ma]. When H is the Schrödinger operator, and σ_H the corresponding classical Hamiltonian, (1.4) yields the approximate eigenvalues, i.e. the approximate quantum energy levels. In the particular case of a quadratic, positive definite Hamiltonian, which can always be reduced to the harmonic oscillator with frequencies $\omega_1 > 0,\dots,\omega_l > 0$, the BSE is an exact quantization formula in the sense of Definition 1.1 with $\nu = 2$, namely:

$$\mu(A,\hbar) = E(A_1+\hbar/2,\dots,A_l+\hbar/2) = \sum_{k=1}^{l}\omega_k(A_k+\hbar/2)$$

To our knowledge, if $l > 1$ the only known examples of exact QF in the sense of Definition 1.1 correspond to classical systems integrable by separation of variables, such that each separated system admits in turn an exact QF, as in the case of the Coulomb potential (for exact QFs for general one-dimensional Schrödinger operators see [Vo]). For general integrable systems, only the approximate BSE formula is valid. Non-integrable systems admit a formal approximate QF, the so-called Einstein-Brillouin-Keller (EBK), recalled below, provided they possess a normal form to all orders.

In this paper we consider a perturbation of a linear Hamiltonian on $T^*\mathbb{T}^l = \mathbb{R}^l\times\mathbb{T}^l$, and prove that the corresponding quantized operator can be unitarily conjugated to a function of the differentiation operators via the construction of a quantum normal form which converges uniformly with respect to $\hbar\in[0,1]$. This yields immediately an exact, ∞-smooth QF. The uniformity with respect to $\hbar$ yields also an explicit family of classical Hamiltonians admitting a convergent normal form, thus making the system integrable.

1.2. **Statement of the results.** Consider the Hamiltonian family $\mathcal{H}_\varepsilon : \mathbb{R}^l\times\mathbb{T}^l \to \mathbb{R}, (\xi,x)\mapsto \mathcal{H}_\varepsilon(\xi,x)$, indexed by $\varepsilon\in\mathbb{R}$, defined as follows:

$$\mathcal{H}_\varepsilon(\xi,x) := \mathcal{L}_\omega(\xi) + \varepsilon\mathcal{V}(x,\xi); \quad \mathcal{L}_\omega(\xi) := \langle\omega,\xi\rangle, \quad \omega\in\mathbb{R}^l, \quad \mathcal{V}\in C^\infty(\mathbb{R}^l\times\mathbb{T}^l;\mathbb{R}). \tag{1.5}$$

Here $\xi\in\mathbb{R}^l, x\in\mathbb{T}^l$ are canonical coordinates on the phase space $\mathbb{R}^l\times\mathbb{T}^l$, the $2l-$cylinder. $\mathcal{L}_\omega(\xi)$ generates the linear Hamiltonian flow $\xi_i\mapsto\xi_i, x_i\mapsto x_i+\omega_i t$ on $\mathbb{R}^l\times\mathbb{T}^l$. For $l>1$ the dependence of $\mathcal{V}$ on ξ makes non-trivial the integrability of the flow of $\mathcal{H}_\varepsilon$ when $\varepsilon\neq 0$, provided the *frequencies* $\omega := (\omega_1,\dots,\omega_l)$ are independent over $\mathbb{Q}$ and fulfill a diophantine condition such as (1.25) below. Under this assumption it is well known that $\mathcal{H}_\varepsilon$ admits a *normal form* at any order (for this notion, see e.g. [Ar2], [SM]). Namely, $\forall\, N \in \mathbb{N}$ a canonical bijection $\mathcal{C}_{\varepsilon,N} : \mathbb{R}^l \times \mathbb{T}^l \leftrightarrow \mathbb{R}^l \times \mathbb{T}^l$ close to the identity can be constructed in such a way that:

$$(\mathcal{H}_\varepsilon \circ \mathcal{C}_{\varepsilon,N})(\xi, x) = \mathcal{L}_\omega(\xi) + \sum_{k=1}^{N} \mathcal{B}_k(\xi;\omega)\varepsilon^k + \varepsilon^{N+1}\mathcal{R}_{N+1,\varepsilon}(\xi, x) \tag{1.6}$$

This makes the flow of $\mathcal{H}_\varepsilon(\xi, x)$ integrable up to an error of order ε^{N+1}. In turn, $\mathcal{C}_{\varepsilon,N}$ is the Hamiltonian flow at time 1 generated by

$$\mathcal{W}^N_\varepsilon(\xi, x) := \langle \xi, x\rangle + \sum_{k=1}^{N} \mathcal{W}_k(\xi, x)\varepsilon^k, \tag{1.7}$$

where the functions $\mathcal{W}_k(\xi, x) : \mathbb{R}^l \times \mathbb{T}^l \to \mathbb{R}$ are recursively computed by canonical perturbation theory via the standard Lie transform method of Deprit[De] and Hori[Ho] (see also e.g [Ca]).

To describe the quantum counterpart, let $H_\varepsilon = L_\omega + \varepsilon V$ be the operator in $L^2(\mathbb{T}^l)$ of symbol $\mathcal{H}_\varepsilon$, with domain $D(H_\varepsilon) = H^1(\mathbb{T}^l)$ and action specified as follows:

$$\forall u \in D(H_\varepsilon), \quad H_\varepsilon u = L_\omega u + V u, \quad L_\omega u = \sum_{k=1}^{l} \omega_k D_k u, \;\; D_k u := -i\hbar\partial_{x_k} u, \tag{1.8}$$

and V is the Weyl quantization of $\mathcal{V}$ (formula (1.26) below).

Since *uniform* quantum normal forms (see e.g. [Sj],[BGP],[Po1], [Po2]) are not so well known as the classical ones, let us recall here their definition. The construction is reviewed in Appendix.

Definition 1.4. [Quantum normal form (QNF)] *We say that a family of operators H_ε ε-close (in the norm resolvent topology) to $H_0 = L_\omega$ admits a uniform quantum normal form (QNF) at any order if*

(i) *There exists a sequence of continuous self-adjoint operators $W_k(\hbar)$ in $L^2(\mathbb{T}^l)$, $k = 1, \dots$ and a sequence of functions $B_k(\xi_1, \dots, \xi_l, \hbar) \in C^\infty(\mathbb{R}^l \times [0, 1]; \mathbb{R})$, such that, defining $\forall\, N \in \mathbb{N}$ the family of unitary operators:*

$$U_{N,\varepsilon}(\hbar) = e^{iW_{N,\varepsilon}(\hbar)/\hbar}, \quad W_{N,\varepsilon}(\hbar) = \sum_{k=1}^{N} W_k(\hbar)\varepsilon^k \tag{1.9}$$

we have:

$$U_{N,\varepsilon}(\hbar)H_\varepsilon U^*_{N,\varepsilon}(\hbar) = L_\omega + \sum_{k=1}^{N} B_k(D_1, \dots, D_l, \hbar)\varepsilon^k + \varepsilon^{N+1}R_{N+1,\varepsilon}(\hbar). \tag{1.10}$$

(ii) *The operators $B_k(D, \hbar) : k = 1, 2 \dots$, R_{N+1} are continuous in $L^2(\mathbb{T}^l)$; the corresponding symbols $\mathcal{W}_k, \mathcal{B}_k, \mathcal{R}_{N+1}(\varepsilon)$ belong to $C^\infty(\mathbb{R}^l \times \mathbb{T}^l \times [0, 1])$, and reduce to the classical normal form construction (1.6) and (1.7) as $\hbar \to 0$:*

$$\mathcal{B}_k(\xi; 0) = \mathcal{B}_k(\xi); \quad \mathcal{W}_k(\xi, x, 0) = \mathcal{W}_k(\xi, x), \quad \mathcal{R}_{N+1,\varepsilon}(x, \xi; 0) = \mathcal{R}_{N+1,\varepsilon}(x, \xi) \tag{1.11}$$

(1.10) entails that H_ε commutes with H_0 up to an error of order ε^{N+1}; hence the following approximate QF formula holds for the eigenvalues of H_ε:

$$\lambda_{n,\varepsilon}(\hbar) = \hbar\langle n,\omega\rangle + \sum_{k=1}^{N} \mathcal{B}_k(n_1\hbar,\dots,n_l\hbar,\hbar)\varepsilon^k + O(\varepsilon^{N+1}). \tag{1.12}$$

Definition 1.5. (Uniformly convergent quantum normal forms) *We say that the QNF converges M-smoothly, $M > 2l$,* uniformly with respect to the Planck constant $\hbar$, *if there is $\varepsilon^* > 0$ such that*

$$\sum_{k=1}^{\infty} \sup_{\mathbb{R}^l\times\mathbb{T}^l\times[0,1]} \sum_{|\alpha|\le M} |D^\alpha \mathcal{W}_k(\xi,x;\hbar)\varepsilon^k| < +\infty \tag{1.13}$$

$$\sum_{k=1}^{\infty} \sup_{\mathbb{R}^l\times[0,1]} \sum_{|\alpha|\le M} |D^\alpha \mathcal{B}_k(\xi,\hbar)\varepsilon^k| < +\infty, \quad |\varepsilon| < \varepsilon^*. \tag{1.14}$$

Here $D^\alpha = \partial_\xi^{\alpha_1}\partial_x^{\alpha_2}\partial_\hbar^{\alpha_3}$, $|\alpha| = |\alpha_1| + |\alpha_2| + \alpha_3$.

(1.13,1.14) entail that, if $|\varepsilon| < \varepsilon^*$, we can define the symbols

$$\mathcal{W}_\infty(\xi,x;\varepsilon,\hbar) := \langle\xi,x\rangle + \sum_{k=1}^{\infty} \mathcal{W}_k(\xi,x;\hbar)\varepsilon^k \in C^M(\mathbb{R}^l\times\mathbb{T}^l\times[0,\varepsilon^*]\times[0,1];\mathbb{C}), \tag{1.15}$$

$$\mathcal{B}_\infty(\xi;\varepsilon,\hbar) := \mathcal{L}_\omega(\xi) + \sum_{k=1}^{\infty} \mathcal{B}_k(\xi;\hbar)\varepsilon^k \in C^M(\mathbb{R}^l\times[0,\varepsilon^*]\times[0,1];\mathbb{C}) \tag{1.16}$$

By the Calderon-Vaillancourt theorem (see §3 below) their Weyl quantizations $W_\infty(\varepsilon,\hbar)$, $B_\infty(\varepsilon,\hbar)$ are continuous operator in $L^2(\mathbb{T}^l)$. Then:

$$e^{iW_\infty(\varepsilon,\hbar)/\hbar} H_\varepsilon e^{-iW_\infty(\varepsilon,\hbar)/\hbar} = B_\infty(D_1,\dots,D_l;\varepsilon,\hbar). \tag{1.17}$$

Therefore the uniform convergence of the QNF has the following straightforward consequences:

(A1) *The eigenvalues of H_ε are given by the* exact *quantization formula:*

$$\lambda_n(\hbar,\varepsilon) = \mathcal{B}_\infty(n\hbar,\hbar,\varepsilon), \qquad n\in\mathbb{Z}^l, \quad \varepsilon\in\mathfrak{D}^* := \{\varepsilon\in\mathbb{R}\,|\,|\varepsilon|<\varepsilon^*\} \tag{1.18}$$

(A2) *The classical normal form is convergent, uniformly on compacts with respect to $\xi\in\mathbb{R}^l$, and therefore if $\varepsilon\in\mathfrak{D}^*$ the Hamiltonian $\mathcal{H}_\varepsilon(\xi,x)$ is integrable.*

Let us now state explicit conditions on V ensuring the uniform convergence of the QNF. Given $\mathcal{F}(t,x)\in C^\infty(\mathbb{R}\times\mathbb{T}^l;\mathbb{R})$, consider its Fourier expansion

$$\mathcal{F}(t,x) = \sum_{q\in\mathbb{Z}^l} \mathcal{F}_q(t)e^{i\langle q,x\rangle}. \tag{1.19}$$

and define furthermore $\mathcal{F}_\omega : \mathbb{R}^l\times\mathbb{T}^l\to\mathbb{R}; \mathcal{F}_\omega\in C^\infty(\mathbb{R}^l\times\mathbb{T}^l;\mathbb{R})$ in the following way:

$$\mathcal{F}_\omega(\xi,x) := \mathcal{F}(\mathcal{L}_\omega(\xi),x) = \sum_{q\in\mathbb{Z}^l} \mathcal{F}_{\omega,q}(\xi)e^{i\langle q,x\rangle}, \tag{1.20}$$

$$\mathcal{F}_{\omega,q}(\xi) := (\mathcal{F}_q\circ\mathcal{L}_\omega)(\xi) = \frac{1}{(2\pi)^{l/2}}\int_{\mathbb{R}} \widehat{\mathcal{F}}_q(p)e^{-ip\mathcal{L}_\omega(\xi)}\,dp = \tag{1.21}$$

$$= \frac{1}{(2\pi)^{l/2}}\int_{\mathbb{R}} \widehat{\mathcal{F}}_q(p)e^{-i\langle p\omega,\xi\rangle}\,dp, \quad p\omega := (p\omega_1,\dots,p\omega_l). \tag{1.22}$$

Here, as above, $\mathcal{L}_\omega(\xi) = \langle\omega,\xi\rangle$.

Given $\rho>0$, introduce the weighted norms:

$$\|\mathcal{F}_{\omega,q}(\xi)\|_\rho := \int_{\mathbb{R}} |\widehat{\mathcal{F}}_q(p)|e^{\rho|p|}|\,dp \tag{1.23}$$

$$\|\mathcal{F}_\omega(x,\xi)\|_\rho := \sum_{q\in\mathbb{Z}^l} e^{\rho|q|}\|\mathcal{F}_{\omega,q}\|_\rho \tag{1.24}$$

We can now formulate the main result of this paper. Assume:

(H1) There exist $\gamma>1, \tau>l-1$ such that the frequencies ω fulfill the diophantine condition

$$|\langle\omega,q\rangle|^{-1} \le \gamma|q|^\tau, \quad q\in\mathbb{Z}^l,\ q\neq 0. \tag{1.25}$$

(H2) V_ω is the Weyl quantization of $\mathcal{V}_\omega(\xi,x)$ (see Sect.3 below), that is:

$$V_\omega f(x) = \int_{\mathbb{R}}\sum_{q\in\mathbb{Z}^l} \widehat{\mathcal{V}}_q(p)e^{i\langle q,x\rangle+\hbar p\langle\omega,q\rangle/2}f(x+\hbar p\omega)\,dp, \quad f\in L^2(\mathbb{T}^l). \tag{1.26}$$

with $\mathcal{V}(\xi,x;\hbar) = \mathcal{V}(\langle\omega,\xi\rangle,x) = \mathcal{V}_\omega(\xi,x)$ for some function $\mathcal{V}(t;x):\mathbb{R}\times\mathbb{T}^l\to\mathbb{R}$.

(H3)

$$\|\mathcal{V}_\omega\|_\rho < +\infty, \qquad \rho > 1+16\gamma\tau^\tau.$$

Clearly under these conditions the operator family $H_\varepsilon := L_\omega+\varepsilon V_\omega$, $D(H_\varepsilon) = H^1(\mathbb{T}^l)$, $\varepsilon\in\mathbb{R}$, is self-adjoint in $L^2(\mathbb{T}^l)$ and has pure point spectrum. We can then state the main results.

Theorem 1.6. *Under conditions (H1-H3), H_ε admits a uniformly convergent quantum normal form $\mathcal{B}_{\infty,\omega}(\xi,\varepsilon,\hbar)$ in the sense of Definition 1.5, with radius of convergence no smaller than:*

$$\varepsilon^*(\gamma,\tau) := \frac{1}{e^{24(3+2\tau)}2^{2\tau}\|\mathcal{V}\|_\rho}. \tag{1.27}$$

If in addition to (H1-H2) we assume, for any fixed $r\in\mathbb{N}$:

(H4)

$$\rho > \lambda(\gamma,\tau,r) := 1+8\gamma\tau[(2(r+1)^2] \tag{1.28}$$

we can sharpen the above result proving smoothness with respect to $\hbar$:

Theorem 1.7. *Let conditions (H1-H2-H4) be fulfilled. For* $r \in \mathbb{N}$ *define* $\mathfrak{D}_r^* := \{\varepsilon \in \mathbb{C} \,|\, |\varepsilon| < \varepsilon^*(\gamma, \tau, r)\}$*, where:*

$$\varepsilon^*(\gamma, \tau, r) := \frac{1}{e^{24(3+2\tau)}(r+2)^{2\tau}\|\mathcal{V}\|_\rho} \tag{1.29}$$

Then $\hbar \mapsto \mathcal{B}_\infty(t, \varepsilon, \hbar) \in C^\infty([0,1]; C^\omega(\{t \in \mathbb{C} \,|\, |\Im t| < \rho/2 \times \mathfrak{D}_r^*(\rho)\})$*; i.e. there exist* $C_r(\varepsilon^*) > 0$ *such that, for* $\varepsilon \in \mathfrak{D}_r^*$*:*

$$\sum_{\gamma=0}^{r} \max_{\hbar \in [0,1]} \|\partial_\hbar^\gamma \mathcal{B}_{\infty,\omega}(\xi; \varepsilon, \hbar)\|_{\rho/2} \le C_r, \quad r = 0, 1, \dots \tag{1.30}$$

In view of Definition 1.1, the following statement is a straightforward consequence of the above Theorems:

Corollary 1.8 (Quantization formula)**.** $\mathcal{H}_\varepsilon$ *admits an* ∞*-smooth quantization formula in the sense of Definition 1.1. That is,* $\forall\, r \in \mathbb{N}$, $\forall\, |\epsilon| < \varepsilon^*(\gamma, \tau, r)$ *given by (1.29), the eigenvalues of* H_ε *are expressed by the formula:*

$$\lambda(n, \hbar, \varepsilon) = \mathcal{B}_{\infty,\omega}(n\hbar, \varepsilon, \hbar) = \mathcal{L}_\omega(n\hbar) + \sum_{s=1}^{\infty} \mathcal{B}_s(\mathcal{L}_\omega(n\hbar), \hbar)\varepsilon^s \tag{1.31}$$

where $\mathcal{B}_{\infty,\omega}(\xi, \varepsilon, \hbar)$ *belongs to* $C^r(\mathbb{R}^l \times [0, \varepsilon^*(\cdot, r)] \times [0,1])$*, and admits an asymptotic expansion at order* r *in* $\hbar$*, uniformly on compacts with respect to* $(\xi, \varepsilon) \in \mathbb{R}^l \times [0, \varepsilon^*(\cdot, r)]$*.*

Remarks

(i) (1.30) and (1.31) entail also that the Einstein-Brillouin-Keller (EBK) quantization formula:

$$\lambda_{n,\varepsilon}^{EBK}(\hbar) := \mathcal{L}_\omega(n\hbar) + \sum_{s=1}^{\infty} \mathcal{B}_s(\mathcal{L}_\omega(n\hbar))\varepsilon^s = \mathcal{B}_{\infty,\omega}(n\hbar, \varepsilon), \quad n \in \mathbb{Z}^l \tag{1.32}$$

reproduces here Spec(H_ε) up to order $\hbar$.

(ii) Apart the classical Cherry theorem yielding convergence of the Birkhoff normal form for smooth perturbations of the harmonic flow with *complex* frequencies when $l = 2$ (see e.g. [SM], §30; the uniform convergence of the QNF under these conditions is proved in [GV]), no simple convergence criterion seems to be known for the QNF nor for the classical NF as well. (See e.g.[PM], [Zu], [St] for reviews on convergence of normal forms). Assumptions (1) and (2) of Theorem 1.6 entail Assertion (A2) above. Hence they represent, to our knowledge, a first explicit convergence criterion for the NF.

Remark that $\mathcal{L}_\omega(\xi)$ is also the form taken by harmonic-oscillator Hamiltonian in $\mathbb{R}^{2l}$,

$$\mathcal{P}_0(\eta,y;\omega) := \sum_{s=1}^{l}\omega_s(\eta_s^2+y_s^2), \quad (\eta_s,y_s)\in\mathbb{R}^2, \quad s=1,\dots,l$$

if expressed in terms of the action variables $\xi_s>0$, $s=1,\dots,l$, where

$$\xi_s := \eta_s^2+y_s^2 = z_s\overline{z}_s, \quad z_s := y_s+i\eta_s.$$

Assuming (1.25) *and* the property

$$\mathcal{B}_k(\xi) = (\mathcal{F}_k\circ\mathcal{L}_\omega(\xi)) = \mathcal{F}_k(\sum_{s=1}^{l}\omega_s z_s\overline{z}_s), \quad k=0,1,\dots \tag{1.33}$$

Rüssmann [Ru] (see also [Ga]) proved convergence of the Birkhoff NF if the perturbation $\mathcal{V}$, expressed as a function of $(z,\overline{z})$, is *in addition* holomorphic at the origin in $\mathbb{C}^{2l}$. No explicit condition on $\mathcal{V}$ seems to be known ensuring *both* (1.33) and the holomorphy. In this case instead we *prove* that the assumption $\mathcal{V}(\xi,x)=\mathcal{V}(\mathcal{L}_\omega(\xi),x)$ entails (1.33), uniformly in $\hbar\in[0,1]$; namely, we construct $\mathcal{F}_s(t;\hbar):\mathbb{R}\times[0,1]\to\mathbb{R}$ such that:

$$\mathcal{B}_s(\xi;\hbar) = \mathcal{F}_s(\mathcal{L}_\omega(\xi);\hbar) := \mathcal{F}_{\omega,s}(\xi;\hbar), \quad s=0,1,\dots \tag{1.34}$$

The conditions of Theorem 1.6 cannot however be transported to Rüssmann's case: the map

$$\mathcal{T}(\xi,x) = (\eta,y) := \begin{cases}\eta_i = -\sqrt{\xi_i}\sin x_i, \\ y_i = \sqrt{\xi_i}\cos x_i,\end{cases} \qquad i=1,\dots,l,$$

namely, the inverse transformation into action-angle variable, is defined only on $\mathbb{R}_+^l\times\mathbb{T}^l$ and does not preserve the analyticity at the origin. On the other hand, $\mathcal{T}$ is an analytic, canonical map between $\mathbb{R}_+^l\times\mathbb{T}^l$ and $\mathbb{R}^{2l}\setminus\{0,0\}$. Assuming for the sake of simplicity $\mathcal{V}_0=0$ the image of $\mathcal{H}_\varepsilon$ under $\mathcal{T}$ is:

$$(\mathcal{H}_\varepsilon\circ\mathcal{T})(\eta,y) = \sum_{s=1}^{l}\omega_s(\eta_s^2+y_s^2)+\varepsilon(\mathcal{V}\circ\mathcal{T})(\eta,y) := \mathcal{P}_0(\eta,y)+\varepsilon\mathcal{P}_1(\eta,y) \tag{1.35}$$

where

$$\mathcal{P}_1(\eta,y) = (\mathcal{V}\circ\mathcal{T})(\eta,y) = \mathcal{P}_{1,R}(\eta,y)+\mathcal{P}_{1,I}(\eta,y),\ (\eta,y)\in\mathbb{R}^{2l}\setminus\{0,0\}. \tag{1.36}$$

$$\mathcal{P}_{1,R}(\eta,y) = \frac{1}{2}\sum_{k\in\mathbb{Z}^l}(\Re\mathcal{V}_k\circ\mathcal{H}_0)(\eta,y)\prod_{s=1}^{l}\left(\frac{\eta_s-iy_s}{\sqrt{\eta_s^2+y_s^2}}\right)^{k_s}$$

$$\mathcal{P}_{1,I}(\eta,y) = \frac{1}{2}\sum_{k\in\mathbb{Z}^l}(\Im\mathcal{V}_k\circ\mathcal{H}_0)(\eta,y)\prod_{s=1}^{l}\left(\frac{\eta_s-iy_s}{\sqrt{\eta_s^2+y_s^2}}\right)^{k_s}$$

If $\mathcal{V}$ fulfills Assumption (H3) of Theorem 1.6, both these series converge uniformly in any compact of $\mathbb{R}^{2l}$ away from the origin and $\mathcal{P}_1$ is holomorphic on $\mathbb{R}^{2l} \setminus \{0,0\}$. Therefore Theorem 1.6 immediately entails a convergence criterion for the Birkhoff normal form generated by perturbations holomorphic away from the origin. We state it under the form of a corollary:

Corollary 1.9. (A convergence criterion for the Birkhoff normal form) *Under the assumptions of Theorem 1.6 on ω and $\mathcal{V}$, consider on $\mathbb{R}^{2l}\setminus\{0,0\}$ the holomorphic Hamiltonian family $P_\varepsilon(\eta,y) := \mathcal{P}_0(\eta,y)+\varepsilon\mathcal{P}_1(\eta,y)$, $\varepsilon\in\mathbb{R}$, where $\mathcal{P}_0$ and $\mathcal{P}_1$ are defined by (1.35,1.36). Then the Birkhoff normal form of H_ε is uniformly convergent on any compact of $\mathbb{R}^{2l}\setminus\{0,0\}$ if $|\varepsilon|<\varepsilon^*(\gamma,\tau)$.*

1.3. **Strategy of the paper.** The proof of Theorem 1.6 rests on an implementation in the quantum context of Rüssmann's argument[Ru] yielding convergence of the KAM iteration when the complex variables $(z,\overline{z})$ belong to an open neighbourhood of the origin in $\mathbb{C}^{2l}$. Conditions (1.25, 1.34) prevent the occurrence of accidental degeneracies among eigenvalues at any step of the quantum KAM iteration, in the same way as they prevent the formation of resonances at the same step in the classical case. However, the global nature of quantum mechanics prevents phase-space localization; therefore, and this is the main difference, at each step the coefficients of the homological equation for the operator symbols not only have an additional dependence on $\hbar$ but also have to be controlled up to infinity. These difficulties are overcome by exploiting the closeness to the identity of the whole procedure, introducing adapted spaces of symbols i(Section 2), which account also for the properties of differentiability with respect to the Planck constant. The link between quantum and classical settings is provided by a sharp (i.e. without $\hbar^\infty$ approximation) Egorov Theorem established in section 4. Estimates for the solution of the quantum homological equation and their recursive properties are obtained in sections 5.1 (Theorem 5.3) and 5.2 (Theorem 5.5) respectively. Recursive estimates are established in Section 6 (Theorem 6.4) and the proof of our main result is completed in section 7. The link with the usual construction of the quantum normal form described in Appendix.

2. Norms and first estimates

Let $m,l=1,2,\dots$. For $\mathcal{F}\in C^\infty(\mathbb{R}^m\times\mathbb{T}^l\times[0,1];\mathbb{C})$, $(\xi,x,\hbar)\to\mathcal{F}(\xi,x;\hbar)$, and $\mathcal{G}\in C^\infty(\mathbb{R}^m\times[0,1];\mathbb{C})$, $(\xi,\hbar)\to\mathcal{G}(\xi;\hbar)$, consider the Fourier transforms

$$\widehat{\mathcal{G}}(p;\hbar)=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}\mathcal{G}(\xi;\hbar)e^{-i\langle p,\xi\rangle}\,dx \tag{2.1}$$

$$\mathcal{F}(\xi, q; \hbar) := \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{T}^l} \mathcal{F}(\xi, x; \hbar) e^{-i\langle q,x\rangle} \, dx. \tag{2.2}$$

$$\mathcal{F}(\xi, x; \hbar) = \sum_{q\in\mathbb{Z}^l} \mathcal{F}(\xi, q; \hbar) e^{-i\langle q,x\rangle} \tag{2.3}$$

$$\widehat{\mathcal{F}}(p, q; \hbar) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \mathcal{F}(\xi, q; \hbar) e^{-i\langle p,\xi\rangle} \, dx \tag{2.4}$$

It is convenient to rewrite the Fourier representations (2.3, 2.4) under the form a single Lebesgue-Stieltjes integral. Consider the product measure on $\mathbb{R}^m \times \mathbb{R}^l$:

$$d\lambda(t) := dp \, d\nu(s), \quad t := (p, s) \in \mathbb{R}^m \times \mathbb{R}^l; \tag{2.5}$$

$$dp := \prod_{k=1}^{m} dp_k; \quad d\nu(s) := \prod_{h=1}^{l} \sum_{q_h \le s_h} \delta(s_h - q_h), \; q_h \in \mathbb{Z}, h = 1, \dots, l \tag{2.6}$$

Then:

$$\mathcal{F}(\xi, x; \hbar) = \int_{\mathbb{R}^m\times\mathbb{R}^l} \widehat{\mathcal{F}}(p, s; \hbar) e^{i\langle p,\xi\rangle + i\langle s,x\rangle} \, d\lambda(p, s) \tag{2.7}$$

Definition 2.1. *For $\rho \ge 0$, $\sigma \ge 0$, we introduce the weighted norms*

$$|\mathcal{G}|^{\dagger}_{\sigma} \;:=\; \max_{\hbar\in[0,1]} \|\widehat{\mathcal{G}}(.; \hbar)\|_{L^1(\mathbb{R}^m, e^{\sigma|p|}dp)} = \max_{\hbar\in[0,1]} \int_{\mathbb{R}^l} \|\widehat{\mathcal{G}}(.; \hbar)\| \, e^{\sigma|p|} \, dp. \tag{2.8}$$

$$|\mathcal{G}|^{\dagger}_{\sigma,k} \;:=\; \max_{\hbar\in[0,1]} \sum_{j=0}^{k} \|(1 + |p|^2)^{\frac{k-j}{2}} \partial^j_\hbar \widehat{\mathcal{G}}(.; \hbar)\|_{L^1(\mathbb{R}^m, e^{\sigma|p|}dp)}; \quad |\mathcal{G}|^{\dagger}_{\sigma;0} := |\mathcal{G}|^{\dagger}_{\sigma}. \tag{2.9}$$

Remark 2.2. By noticing that $|p| \le |p' - p| + |p'|$ and that, for $x \ge 0$, $x^j e^{-\delta x} \le \frac{1}{e}(\frac{j}{\delta})^j$, we immediately get the inequalities

$$|\mathcal{F}\mathcal{G}|^{\dagger}_{\sigma} \le |\mathcal{F}|_{\sigma} |\mathcal{G}|_{\sigma}, \tag{2.10}$$

$$|(I - \Delta^{j/2})\mathcal{F}|_{\sigma-\delta} \le \frac{1}{e} \left(\frac{j}{\delta}\right)^j |\mathcal{F}|_{\sigma}, \quad k \ge 0. \tag{2.11}$$

Set now for $k \in \mathbb{N} \cup \{0\}$:

$$\mu_k(t) := (1 + |t|^2)^{\frac{k}{2}} = (1 + |p|^2 + |s|^2)^{\frac{k}{2}}. \tag{2.12}$$

and note that

$$\mu_k(t - t') \le 2^{\frac{k}{2}} \mu_k(t) \mu_k(t'). \tag{2.13}$$

because $|x - x'|^2 \le 2(|x|^2 + |x'|^2)$.

Definition 2.3. *Consider $\mathcal{F}(\xi, x; \hbar) \in C^\infty(\mathbb{R}^m \times \mathbb{T}^l \times [0, 1]; \mathbb{C})$, with Fourier expansion*

$$\mathcal{F}(\xi, x; \hbar) = \sum_{q\in\mathbb{Z}^l} \mathcal{F}(\xi, q; \hbar) e^{i\langle q,x\rangle} \tag{2.14}$$

(1) *Set:*

$$\|\mathcal{F}\|^{\dagger}_{\rho,k} := \max_{\hbar\in[0,1]} \sum_{\gamma=0}^{k} \int_{\mathbb{R}^m\times\mathbb{R}^l} |\mu_{k-\gamma}(p,s)\partial_\hbar^\gamma \widehat{\mathcal{F}}(p,s;\hbar)|e^{\rho(|s|+|p|)}\, d\lambda(p,s). \tag{2.15}$$

(2) *Let $\mathcal{O}_\omega$ be the set of functions $\Phi : \mathbb{R}^l\times\mathbb{T}^l\times[0,1]$ such that $\Phi(\xi,x;\hbar) = \mathcal{F}(\mathcal{L}_\omega(\xi),x;\hbar)$ for some $\mathcal{F}:\ \mathbb{R}\times\mathbb{T}^l\times[0,1]\to\mathbb{C}$. Define, for $\Phi\in\mathcal{O}_\omega$:*

$$\|\Phi\|_{\rho,k} := \max_{\hbar\in[0,1]} \sum_{\gamma=0}^{k} \int_{\mathbb{R}} |\mu_{k-\gamma}(p\omega,q)\partial_\hbar^\gamma \widehat{\mathcal{F}}(p,s;\hbar)|e^{\rho(|s|+|p|}\, d\lambda(p,s). \tag{2.16}$$

(3) *Finally we denote $Op^W(\mathcal{F})$ the Weyl quantization of $\mathcal{F}$ recalled in Section 3 and*

$$\begin{aligned}
\mathcal{J}^\dagger_k(\rho) &= \{\mathcal{F}\,|\,\|\mathcal{F}\|^\dagger_{\rho,k}<\infty\}, &(2.17)\\
J^\dagger_k(\rho) &= \{Op^W(\mathcal{F})\,|\,\mathcal{F}\in\mathcal{J}^\dagger(\rho,k)\}, &(2.18)\\
\mathcal{J}_k(\rho) &= \{\mathcal{F}\in\mathcal{O}_\omega\,|\,\|\mathcal{F}\|_{\rho,k}<\infty\}, &(2.19)\\
J_k(\rho) &= \{\mathcal{F}\,|\,\|\mathcal{F}\|_{\rho,k}<\infty\}, &(2.20)
\end{aligned}$$

Finally we denote: $L^1_\sigma(\mathbb{R}^m) := L^1(\mathbb{R}^m, e^{\sigma|p|}dp)$.

Remark 2.4. Note that, if $\mathcal{F}(\xi,q,\hbar)$ is independent of q, i.e. $\mathcal{F}(\xi,q,\hbar)=\mathcal{F}(\xi,\hbar)\delta_{q,0}$, then:

$$\|\mathcal{F}\|^\dagger_{\rho,k} = |\mathcal{F}|^\dagger_{\rho,k};\quad \|\mathcal{F}\|_{\rho,k} = |\mathcal{F}|_{\rho,k} \tag{2.21}$$

while in general

$$\|\mathcal{F}\|_{\rho,k} \le \|\mathcal{F}\|_{\rho',k'}\quad \text{whenever } k\ge k',\ \rho\le\rho'; \tag{2.22}$$

Remark 2.5. (Regularity properties)

Let $\mathcal{F}\in\mathcal{J}^\dagger_k(\rho), k\ge 0$. Then:

(1) There exists $K(\alpha,\rho,k)$ such that

$$\max_{\hbar\in[0,1]} \|\mathcal{F}(\xi,x;\hbar)\|_{C^\alpha(\mathbb{R}^m\times\mathbb{T}^l)} \le K\|\mathcal{F}\|^\dagger_{\rho,k},\quad \alpha\in\mathbb{N} \tag{2.23}$$

and analogous statement for the norm $\|\cdot\|_{\rho,k}$.

(2) Let $\rho>0$, $k\ge 0$. Then $\mathcal{F}(\xi,x;\hbar)\in C^k([0,1];C^\omega(\{|\Im\xi|<\rho\}\times\{|\Im x|<\rho\})$ and

$$\sup_{\{|\Im\xi|<d\}\times\{|\Im x|<d\}} \le \|\mathcal{F}\|^\dagger_{\rho,k}. \tag{2.24}$$

Analogous statements for $\mathcal{F}\in\mathcal{J}_k(\rho)$.

We will show in section 3 that:

$$\|Op^W(F)\|_{\mathcal{B}(L^2)} \le \|\mathcal{F}\|_{\rho,k}\quad \forall k,\ \rho>0. \tag{2.25}$$

In what follows we will often use the notation $\mathcal{F}$ also to denote the function $\mathcal{F}(\mathcal{L}_\omega(\xi))$, because the indication of the belonging to J or $J^\dagger$, respectively, is already sufficient to mark the distinction of the two cases.

Remark 2.6. Without loss of generality we may assume:

$$|\omega| := |\omega_1| + \ldots + |\omega_l| \leq 1 \tag{2.26}$$

Indeed, the general case $|\omega| = \alpha|\omega'|$, $|\omega'| \leq 1$, $\alpha > 0$ arbitrary reduces to the former one just by the rescaling $\varepsilon \to \alpha\varepsilon$.

3. Weyl quantization, matrix elements, commutator estimates

3.1. **Weyl quantization: action and matrix elements.** We sum up here the canonical (Weyl) quantization procedure for functions (classical observables) defined on the phase space $\mathbb{R}^l \times \mathbb{T}^l$. In the present case it seems more convenient to consider the representation (unique up to unitary equivalences) of the natural Heisenberg group on $\mathbb{R}^l \times \mathbb{T}^l$. Of course this procedure yields the same quantization as the standard one via the Brézin-Weil-Zak transform (see e.g. [Fo], §1.10) and has already been employed in [CdV], [Po1],[Po2]).

Let $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R})$ be the Heisenberg group over $\mathbb{R}^{2l+1}$ (see e.g.[Fo], Chapt.1). Since the dual space of $\mathbb{R}^l \times \mathbb{T}^l$ under the Fourier transformation is $\mathbb{R}^l \times \mathbb{Z}^l$, the relevant Heisenberg group here is the subgroup of $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R})$, denoted by $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R})$, defined as follows:

Definition 3.1. *Let $u := (p, q), p \in \mathbb{R}^l, q \in \mathbb{Z}^l$, and let $t \in \mathbb{R}$. Then $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R})$ is the subgroup of $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R})$ topologically equivalent to $\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R}$ with group law*

$$(u, t) \cdot (v, s) = (u + v, t + s + \frac{1}{2}\Omega(u, v)) \tag{3.1}$$

Here $\Omega(u, v)$ is the canonical $2-$form on $\mathbb{R}^l \times \mathbb{Z}^l$:

$$\Omega(u, v) := \langle u_1, v_2\rangle - \langle v_1, u_2\rangle \tag{3.2}$$

$\mathbb{H}_l(\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R})$ is the Lie group generated via the exponential map from the Heisenberg Lie algebra $\mathcal{HL}_l(\mathbb{Z}^l \times \mathbb{R}^l \times \mathbb{R})$ defined as the vector space $\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R}$ with Lie bracket

$$[(u, t) \cdot (v, s)] = (0, 0, \Omega(u, v)) \tag{3.3}$$

The unitary representations of $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R})$ in $L^2(\mathbb{T}^l)$ are defined as follows

$$(U_\hbar(p, q, t)f)(x) := e^{i\hbar t + i\langle q,x\rangle + \hbar\langle p.q\rangle/2} f(x + \hbar p) \tag{3.4}$$

$\forall\,\hbar\neq 0, \forall\,(p,q,t)\in\mathcal{H}_l, \forall\, f\in L^2(\mathbb{T}^l)$. These representations fulfill the Weyl commutation relations

$$U_\hbar(u)^*=U_\hbar(-u),\qquad U_\hbar(u)U_\hbar(v)=e^{i\hbar\Omega(u,v)}U(u+v) \tag{3.5}$$

For any fixed $\hbar>0$ $U_\hbar$ defines the Schrödinger representation of the Weyl commutation relations, which also in this case is unique up to unitary equivalences (see e.g. [Fo], §1.10).

Consider now a family of smooth phase-space functions indexed by $\hbar$, $\mathcal{A}(\xi,x,\hbar):\mathbb{R}^l\times\mathbb{T}^l\times[0,1]\to\mathbb{C}$, written under its Fourier representation

$$\mathcal{A}(\xi,x,\hbar)=\int_{\mathbb{R}^l}\sum_{q\in\mathbb{Z}^l}\widehat{\mathcal{A}}(p,q;\hbar)e^{i(\langle p.\xi\rangle+\langle q,x\rangle)}\,dp=\int_{\mathbb{R}^l\times\mathbb{R}^l}\widehat{\mathcal{A}}(p,s;\hbar)e^{i(\langle p.\xi\rangle+\langle s,x\rangle)}\,d\lambda(p,s) \tag{3.6}$$

Definition 3.2. *The (Weyl) quantization of $\mathcal{A}(\xi,x;\hbar)$ is the operator $A(\hbar)$ definde as*

$$\begin{aligned}(A(\hbar)f)(x)&:=\int_{\mathbb{R}^l}\sum_{q\in\mathbb{Z}^l}\widehat{\mathcal{A}}(p,q;\hbar)U_\hbar(p,q)f(x)\,dp \\ &=\int_{\mathbb{R}^l\times\mathbb{R}^l}\widehat{\mathcal{A}}(p,s;\hbar)U_\hbar(p,s)f(x)\,d\lambda(p,s)\quad f\in L^2(\mathbb{T}^l)\end{aligned} \tag{3.7}$$

Remark 3.3. Formula (3.7) can be also be written as

$$(A(\hbar)f)(x)=\sum_{q\in\mathbb{Z}^l}A(q,\hbar)f,\quad (A(q,\hbar)f)(x)=\int_{\mathbb{R}^l}\widehat{\mathcal{A}}(p,q;\hbar)U_\hbar(p,q)f(x)\,dp \tag{3.8}$$

From this we compute the action of $A(\hbar)$ on the canonical basis in $L^2(\mathbb{T}^l)$:

$$e_m(x):=(2\pi)^{-l/2}e^{i\langle m,x\rangle},\quad x\in\mathbb{T}^l,\ m\in\mathbb{Z}^l.$$

Lemma 3.4.

$$A(\hbar)e_m(x)=\sum_{q\in\mathbb{Z}^l}e^{i\langle(m+q),x\rangle}\mathcal{A}(\hbar(m+q/2),q,\hbar) \tag{3.9}$$

Proof. By (3.8), it is enough to prove that the action of $A(q,\hbar)$ is

$$A(q,\hbar)e_m(x)=e^{i\langle(m+q),x\rangle}\mathcal{A}(\hbar(m+q/2),q,\hbar) \tag{3.10}$$

Applying Definition 3.2 we can indeed write:

$$\begin{aligned}(A(q,\hbar)e_m)(x)&=(2\pi)^{-l/2}\int_{\mathbb{R}^l}\widehat{A}(p,q;\hbar)e^{i\langle q,x\rangle+i\hbar\langle p,q\rangle/2}e^{i\langle m,(x+\hbar p)\rangle}\,dp\\ &=(2\pi)^{-l/2}e^{i\langle(m+q),x\rangle}\int_{\mathbb{R}^l}\widehat{A}(p;q,\hbar)e^{i\hbar\langle p,(m+q/2)\rangle}\,dp=e^{i\langle(m+q),x\rangle}\mathcal{A}(\hbar(m+q/2),q,\hbar).\end{aligned}$$

. ∎

We note for further reference an obvious consequence of (3.10):

$$\langle A(q,\hbar)e_m,A(q,\hbar)e_n\rangle_{L^2(\mathbb{T}^l)}=0,\ m\neq n;\quad \langle A(r,\hbar)e_m,A(q,\hbar)e_n\rangle_{L^2(\mathbb{T}^l)}=0,\ r\neq q. \tag{3.11}$$

As in the case of the usual Weyl quantization, formula (3.7) makes sense for tempered distributions $\mathcal{A}(\xi, x; \hbar)$ [Fo]. Indeed we prove in this context, for the sake of completeness, a simpler, but less general, version of the standard Calderon-Vaillancourt criterion:

Proposition 3.5. *Let $A(\hbar)$ by defined by (3.7). Then*

$$\|A(\hbar)\|_{L^2 \to L^2} \leq \frac{2^{l+1}}{l+2} \cdot \frac{\pi^{(3l-1)/2}}{\Gamma(\frac{l+1}{2})} \sum_{|\alpha| \leq 2k} \|\partial_x^k \mathcal{A}(\xi, x; \hbar)\|_{L^\infty(\mathbb{R}^l \times \mathbb{T}^l)}. \tag{3.12}$$

where

$$k = \begin{cases} \frac{l}{2} + 1, & l \text{ even} \\ \\ \frac{l+1}{2} + 1, & l \text{ odd.} \end{cases}$$

Proof. Consider the Fourier expansion

$$u(x) = \sum_{m \in \mathbb{Z}^l} \widehat{u}_m e_m(x), \quad u \in L^2(\mathbb{T}^l).$$

Since:

$$\|A(q, \hbar)\widehat{u}_m e_m\|^2 = |\mathcal{A}(\hbar(m + q/2), q, \hbar)|^2 \cdot |\widehat{u}_m|^2$$

by Lemma 3.4 and (3.11) we get:

$$\begin{aligned} \|A(\hbar)u\|^2 &\leq \sum_{(q,m) \in \mathbb{Z}^l \times \mathbb{Z}^l} \|A(q, \hbar)\widehat{u}_m e_m\|^2 = \sum_{(q,m) \in \mathbb{Z}^l \times \mathbb{Z}^l} |\mathcal{A}(\hbar(m + q/2), q, \hbar)|^2 \cdot |\widehat{u}_m|^2 \\ &\leq \sum_{q \in \mathbb{Z}^l} \sup_{\xi \in \mathbb{R}^l} |\mathcal{A}(\xi, q, \hbar)|^2 \sum_{m \in \mathbb{Z}^l} |\widehat{u}_m|^2 = \sum_{q \in \mathbb{Z}^l} \sup_{\xi \in \mathbb{R}^l} |\mathcal{A}(\xi, q, \hbar)|^2 \|u\|^2 \\ &\leq \big[\sum_{q \in \mathbb{Z}^l} \sup_{\xi \in \mathbb{R}^l} |\mathcal{A}(\xi, q, \hbar)|\big]^2 \|u\|^2 \end{aligned}$$

Therefore:

$$\|A(\hbar)\|_{L^2 \to L^2} \leq \sum_{q \in \mathbb{Z}^l} \sup_{\xi \in \mathbb{R}^l} |\mathcal{A}(\xi, q, \hbar)|.$$

Integration by parts entails that, for $k \in \mathbb{N}$, and $\forall\, g \in C^\infty(\mathbb{T}^l)$:

$$\begin{aligned} &\left| \int_{\mathbb{T}^l} e^{i\langle q, x\rangle} g(x) dx \right| = \frac{1}{1 + |q|^{2k}} \left| \int_{\mathbb{T}^l} e^{i\langle q, x\rangle} (1 + (-\triangle_x)^k) g(x) dx \right| \\ &\leq \frac{1}{1 + |q|^{2k}} (2\pi)^l \sup_{\mathbb{T}^l} \sum_{|\alpha| \leq 2k} |\partial_x^\alpha g(x)|. \end{aligned}$$

Let us now take:

$$k = \begin{cases} \frac{l}{2} + 1, & l \text{ even} \\ \\ \frac{l+1}{2} + 1, & l \text{ odd} \end{cases} \implies \begin{cases} 2k - l + 1 = 3, & l \text{ even} \\ 2k - l + 1 = 2, & l \text{ odd} \end{cases} \tag{3.13}$$

Then $2k-l+1\geq 2$, and hence:

$$\sum_{q\in\mathbb{Z}^l}\frac{1}{1+|q|^{2k}}\leq 2\int_{\mathbb{R}^l}\frac{du_1\cdots du_l}{1+\|u\|^{2k}}\leq 2\frac{\pi^{(l-1)/2}}{\Gamma(\frac{l+1}{2})}\int_0^\infty\frac{\rho^{l-1}}{1+\rho^{2k}}\,d\rho.$$

Now:

$$\int_0^\infty\frac{\rho^{l-1}}{1+\rho^{2k}}\,d\rho=\frac{1}{2k}\int_0^\infty\frac{u^{l/2k-1}}{1+u}\,du$$
$$\leq\frac{1}{2k}\left(\int_0^1 u^{l/2k-1}\,du+\int_1^\infty u^{l/2k-2}\,du\right)=\frac{1}{(4k-l)(2k-l)}$$

This allows us to conclude:

$$\begin{aligned}\sum_{q\in\mathbb{Z}^l}\sup_\xi|\mathcal{A}(\xi,q,\hbar)| &\leq (2\pi)^l\sum_{|\alpha|\leq 2k}\|\partial_x^\alpha\mathcal{A}(\xi,x;\hbar)\|_{L^\infty(\mathbb{R}^l\times\mathbb{T}^l)}\cdot\sum_{q\in\mathbb{Z}^l}\frac{1}{1+|q|^{2k}}\\ &\leq 2^{l+1}\cdot\frac{\pi^{(3l-1)/2}}{\Gamma(\frac{l+1}{2})}\frac{1}{l+2}\sum_{|\alpha|\leq 2k}\|\partial_x^k\mathcal{A}(\xi,x;\hbar)\|_{L^\infty(\mathbb{R}^l\times\mathbb{T}^l)}.\end{aligned}$$

with k given by (3.13). This proves the assertion. □

Remark 3.6. Thanks to Lemma 3.4 we immediately see that, when $\mathcal{A}(\xi,x,\hbar)=\mathcal{F}(\mathcal{L}_\omega(\xi),x;\hbar)$,

$$\begin{aligned}\mathcal{A}(\hbar)f &=\int_{\mathbb{R}}\sum_{q\in\mathbb{Z}^l}\widehat{\mathcal{F}}(p,q;\hbar)U_h(p\omega,q)f\,dp \\ &=\int_{\mathbb{R}}\sum_{q\in\mathbb{Z}^l}\widehat{\mathcal{F}}(p,q;\hbar)e^{i\langle q,x\rangle+i\hbar p\langle\omega,q\rangle/2}f(x+\hbar p\omega)\,dp\quad f\in L^2(\mathbb{T}^l)\end{aligned}\tag{3.14}$$

where, again, $p\omega:=(p\omega_1,\ldots,p\omega_l)$. Explicitly, (3.10) and (3.9) become:

$$A(\hbar)e_m(x)=\sum_{q\in\mathbb{Z}^l}e^{i\langle(m+q),x\rangle}\mathcal{A}(\hbar\langle\omega,(m+q/2)\rangle,q,\hbar)\tag{3.15}$$

$$A(q,\hbar)e_m(x)=e^{i\langle(m+q),x\rangle}\mathcal{A}(\hbar\langle\omega,(m+q/2)\rangle,q,\hbar)\tag{3.16}$$

Remark 3.7. If $\mathcal{A}$ does not depend on x, then $\mathcal{A}(\xi,q,\hbar)=0, q\neq 0$, and (3.9) reduces to the standard (pseudo) differential action

$$(A(\hbar)u)(x)=\sum_{m\in\mathbb{Z}^l}\overline{\mathcal{A}}(m\hbar,\hbar)\widehat{u}_m e^{i\langle m,x\rangle}=\sum_{m\in\mathbb{Z}^l}\overline{\mathcal{A}}(-i\hbar\nabla,\hbar)\widehat{u}_m e^{i\langle m,x\rangle}\tag{3.17}$$

because $-i\hbar\nabla e_m=m\hbar e_m$. On the other hand, if $\mathcal{F}$ does not depend on ξ (3.9) reduces to the standard multiplicative action

$$(A(\hbar)u)(x)=\sum_{q\in\mathbb{Z}^l}\mathcal{A}(q,\hbar)e^{i\langle q,x\rangle}\sum_{m\in\mathbb{Z}^l}\widehat{u}_m e^{i\langle m,x\rangle}=\mathcal{A}(x,\hbar)u(x)\tag{3.18}$$

Corollary 3.8. *Let $A(\hbar):L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)$ be defined as in 3.2. Then:*

(1) $\forall \rho \geq 0, \forall\, k \geq 0$ *we have:*

$$\|A(\hbar)\|_{L^2 \to L^2} \leq \|\mathcal{A}\|^{\dagger}_{\rho,k} \tag{3.19}$$

and, if $\mathcal{A}(\xi, x, \hbar) = \mathcal{A}(\mathcal{L}_\omega(\xi), x; \hbar)$

$$\|A(\hbar)\|_{L^2 \to L^2} \leq \|\mathcal{A}\|_{\rho,k}. \tag{3.20}$$

(2)

$$\langle e_{m+s}, A(q,\hbar) e_m \rangle = \delta_{q,s} \mathcal{A}((m + q/2)\rangle \hbar, q, \hbar) \tag{3.21}$$

$$\langle e_{m+s}, A(\hbar) e_m \rangle = \mathcal{A}((m + s/2)\hbar, s, \hbar) \tag{3.22}$$

and, if $\mathcal{A}(\xi, x, \hbar) = \mathcal{F}(\mathcal{L}_\omega(\xi), x; \hbar)$

$$\langle e_{m+s}, F(q,\hbar) e_m \rangle = \delta_{q,s} \mathcal{F}(\langle \omega, (m + q/2)\rangle \hbar, q, \hbar) = \delta_{q,s} \mathcal{F}(\mathcal{L}_\omega (m + s/2)\hbar, q, \hbar) \tag{3.23}$$

$$\langle e_{m+s}, F(\hbar) e_m \rangle = \mathcal{F}(\langle \omega, (m\hbar + s\hbar/2)\rangle, s, \hbar) = \mathcal{F}(\mathcal{L}_\omega (m\hbar + s\hbar/2), s, \hbar) \tag{3.24}$$

Equivalently:

$$\langle e_m, A(\hbar) e_n \rangle = \mathcal{A}((m + n)\hbar/2, m - n, \hbar) \tag{3.25}$$

(3) $A(\hbar)$ *is an operator of order* $-\infty$*, namely there exists* $C(k,s) > 0$ *such that*

$$\|A(\hbar)u\|_{H^k(\mathbb{T}^l)} \leq C(k,s) \|u\|_{H^s(\mathbb{T}^l)}, \quad (k,s) \in \mathbb{R},\ k \geq s \tag{3.26}$$

Proof. (1) Formulae (3.19) and (3.20) are straighforward consequences of Formula (2.23).

(2) (3.23) immediately yields (3.24). In turn, (3.23) follows at once by (3.10).

(3) The condition $\mathcal{A} \in \mathcal{J}(\rho)$ entails:

$$\sup_{(\xi;\hbar) \in \mathbb{R}^l \times [0,1]} |\mathcal{A}(\xi; q, \hbar)| e^{\rho|q|} \leq e^{\rho|q|} \max_{\hbar \in [0,1]} \|\widehat{\mathcal{A}}(p; q, \hbar)\|_1 \to 0,\ |q| \to \infty. \tag{3.27}$$

Therefore:

$$\begin{aligned}
\|A(\hbar)u\|^2_{H^k} &\leq \sum_{(q,m) \in \mathbb{Z}^l \times \mathbb{Z}^l} (1 + |q|^2)^k \mathcal{A}((m + q/2)\hbar, q, \hbar)|^2 \cdot |\widehat{u}_m|^2 \\
&\leq \sum_{q \in \mathbb{Z}^l} \sup_{q,m} (1 + |q|^2)^k |\mathcal{A}((m + q/2)\hbar, q, \hbar)|^2 \sum_{m \in \mathbb{Z}^l} (1 + |m|^2)^s |\widehat{u}_m|^2 \\
&= C(k,s) \|u\|^2_{H^s} \\
C(k,s) &:= \sum_{q \in \mathbb{Z}^l} \sup_{q,m} (1 + |q|^2)^k |\mathcal{A}((m + q/2)\hbar, q, \hbar)|^2
\end{aligned}$$

where $0 < C(k,s) < +\infty$ by (3.27) above. The Corollary is proved. □

3.2. **Compositions, Moyal brackets.** We first list the main properties which are straightforward consequences of the definition, as in the case of the standard Weyl quantization in $\mathbb{R}^{2l}$. First introduce the abbreviations

$$t := (p, s); \quad t' = (p', s'); \quad \omega t := (p\omega, s) \tag{3.28}$$

$$\Omega_\omega(t' - t, t') := \langle (p' - p)\omega, s' \rangle - \langle (s' - s), p'\omega \rangle = \langle p'\omega, s \rangle - \langle s', p\omega \rangle. \tag{3.29}$$

Given $\mathcal{F}(\hbar), \mathcal{G}(\hbar) \in \mathcal{J}_k(\rho)$, define their twisted convolutions:

$$(\widehat{\mathcal{F}}(\hbar)\widetilde{*}\widehat{\mathcal{G}}(\hbar))(p, q; \hbar) := \int_{\mathbb{R}\times\mathbb{R}^l} \widehat{\mathcal{F}}(t' - t; \hbar)\widehat{\mathcal{G}}(t'; \hbar)e^{i[\hbar\Omega_\omega(t'-t,t')/2]} \, d\lambda(t') \tag{3.30}$$

$$(\mathcal{F}\sharp\mathcal{G})(x, \xi, \hbar) := \int_{\mathbb{R}\times\mathbb{R}^l} (\widehat{\mathcal{F}}(\hbar)\widetilde{*}\widehat{\mathcal{G}}(\hbar))(t, \hbar)e^{i\langle s,x\rangle + p\mathcal{L}_\omega(\xi)} \, d\lambda(t) \tag{3.31}$$

$$\widehat{\mathcal{C}}(p, q; \hbar) := \frac{1}{\hbar}\int_{\mathbb{R}\times\mathbb{R}^l} \widehat{\mathcal{F}}(t' - t, \hbar)\widehat{\mathcal{G}}(t', \hbar) \sin[\hbar\Omega_\omega(t' - t, t')/2] \, d\lambda(t') \tag{3.32}$$

$$\mathcal{C}(x, \xi; \hbar) := \int_{\mathbb{R}\times\mathbb{R}^l} \widehat{\mathcal{C}}(p, s; \hbar)e^{ip\mathcal{L}_\omega(\xi) + i\langle s,x\rangle} \, d\lambda(t) \tag{3.33}$$

Once more by the same argument valid for the Weyl quantization in $\mathbb{R}^{2l}$:

Proposition 3.9. *The following composition formulas hold:*

$$F(\hbar)G(\hbar) = \int_{\mathbb{R}\times\mathbb{R}^l} (\widehat{\mathcal{F}}(\hbar)\widetilde{*}\widehat{\mathcal{G}}(\hbar))(t; \hbar)U_\hbar(\omega t) \, d\lambda(t). \tag{3.34}$$

$$\frac{[F(\hbar), G(\hbar)]}{i\hbar} = \int_{\mathbb{R}\times\mathbb{R}^l} \widehat{\mathcal{C}}(t; \hbar)U_\hbar(\omega t) \, d\lambda(t) \tag{3.35}$$

Remark 3.10. The symbol of the product $F(\hbar)G(\hbar)$ is then $(\mathcal{F}\sharp\mathcal{G})(\mathcal{L}_\omega(\xi), x, \hbar)$ and the symbol of the commutator $[F(\hbar), G(\hbar)]/i\hbar$ is $\mathcal{C}(\mathcal{L}_\omega(\xi), x; \hbar)$, which is by definition the Moyal bracket of the symbols $\mathcal{F}, \mathcal{G}$. From (3.32) we get the asymptotic expansion:

$$\widehat{\mathcal{C}}(p, q; \omega; \hbar) = \sum_{j=0}^{\infty} \frac{(-1)^j \hbar^{2j}}{(2j + 1)!} D^j(p, q; \omega) \tag{3.36}$$

$$D^j(p, q; \omega) := \int_{\mathbb{R}\times\mathbb{R}^l} \widehat{\mathcal{F}}(t' - t, \hbar)\widehat{\mathcal{G}}(t', \hbar)[\Omega_\omega(t' - t, t')^j \, d\lambda(t') \tag{3.37}$$

whence the asymptotic expansion for the Moyal bracket

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\}_M(\mathcal{L}_\omega(\xi),x;\hbar) &= \{\mathcal{F},\mathcal{G}\}(\mathcal{L}_\omega(\xi),x,\hbar) + \\ &\sum_{|r+j|=0}^{\infty}\frac{(-1)^{|r|}\hbar^{|r+j|}}{r!sj}[\partial_x^r\omega\partial_{\mathcal{L}}^j\mathcal{F}(\mathcal{L}_\omega(\xi),x)]\cdot[\omega\partial_{\mathcal{L}}^j\partial_x^r G(\mathcal{L}_\omega(\xi),x,\hbar)] - \\ &-\sum_{|r+j|=0}^{\infty}\frac{(-1)^{|r|}\hbar^{|r+j|}}{r!j!}[\partial_x^r\omega\partial_{\mathcal{L}}^j\mathcal{G}(\mathcal{L}_\omega(\xi),x)]\cdot[\omega\partial_{\mathcal{L}}^j\partial_x^r F(\mathcal{L}_\omega(\xi),x,\hbar)]\end{aligned} \tag{3.38}$$

Remark that:

$$\{\mathcal{F},\mathcal{G}\}_M(\mathcal{L}_\omega(\xi),x;\hbar) = \{\mathcal{F},\mathcal{G}\}(\mathcal{L}_\omega(\xi),x) + O(\hbar) \tag{3.39}$$

In particular, since $L_\omega(\xi)$ is linear, we have $\forall\,\mathcal{F}(\xi;x;\hbar)\in C^\infty(\mathbb{R}^l\times\mathbb{T}^l\times[0,1])$:

$$\{\mathcal{F},\mathcal{L}_\omega(\xi)\}_M(\mathcal{L}_\omega(\xi),x;\hbar) = \{\mathcal{F},\mathcal{L}_\omega(\xi)\}(\mathcal{L}_\omega(\xi),x;\hbar) \tag{3.40}$$

The observables $\mathcal{F}(\xi,x;\hbar)\in\mathcal{J}(\rho)$ enjoy the crucial property of stability under compositions of their dependence on $\mathcal{L}_\omega(\xi)$ (formulae (3.31) and (3.33) above). As in [BGP], we want to estimate the relevant quantum observables uniformly with respect to $\hbar$, i.e. through the weighted norm (2.16).

3.3. **Uniform estimates.** The following proposition is the heart of the estimates needed for the convergence of the KAM iteration. The proof will be given in the next (sub)section. Even though we could limit ourselves to symbols in $\mathcal{J}(\rho)$, we consider for the sake of generality and further reference also the general case of symbols belonging to $\mathcal{J}^\dagger(\rho)$.

Proposition 3.11. *Let $F,\ G\in J_k^\dagger(\rho)$, $k=0,1,\dots$, $d=d_1+d_2$. Let $\mathcal{F},\mathcal{G}$ be the corresponding symbols, and $0<d+d_1<\rho$. Then:*

($1^\dagger$) *$FG\in J_k^\dagger(\rho)$ and fulfills the estimate*

$$\|FG\|_{\mathcal{B}(L^2)} \le \|\mathcal{F}\sharp\mathcal{G}\|^\dagger_{\rho,k} \le (k+1)4^k\|\mathcal{F}\|^\dagger_{\rho,k}\cdot\|\mathcal{G}\|^\dagger_{\rho,k} \tag{3.41}$$

($2^\dagger$) *$\dfrac{[F,G]}{i\hbar}\in J_k^\dagger(\rho-d)$ and fulfills the estimate*

$$\left\|\frac{[F,G]}{i\hbar}\right\|_{\mathcal{B}(L^2)} \le \|\{\mathcal{F},\mathcal{G}\}_M\|^\dagger_{\rho-d-d_1,k} \le \frac{(k+1)4^k}{e^2d_1(d+d_1)}\|\mathcal{F}\|^\dagger_{\rho,k}\|\mathcal{G}\|^\dagger_{\rho-d,k} \tag{3.42}$$

($3^\dagger$) *$\mathcal{F}\mathcal{G}\in\mathcal{J}_k^\dagger(\rho)$, and*

$$\|\mathcal{F}\mathcal{G}\|^\dagger_{\rho,k} \le (k+1)4^k\|\mathcal{F}\|^\dagger_{\rho,k}\cdot\|\mathcal{G}\|^\dagger_{\rho,k} \tag{3.43}$$

Moreover if $F,\ G\in J_k(\rho)$, $k=0,1,\dots$, and $\mathcal{F},\mathcal{G}\in\mathcal{J}_k(\rho)$, then:

(1) *$FG\in J_k(\rho)$ and fulfills the estimate*

$$\|FG\|_{\mathcal{B}(L^2)} \le \|\mathcal{F}\sharp\mathcal{G}\|_{\rho,k} \le (k+1)4^k\|\mathcal{F}\|_{\rho,k}\cdot\|\mathcal{G}\|_{\rho,k} \tag{3.44}$$

(2) $\dfrac{[F,G]}{i\hbar} \in J_k(\rho - d)$ *and fulfills the estimate*

$$\left\| \frac{[F,G]}{i\hbar} \right\|_{\mathcal{B}(L^2)} \leq \|\{\mathcal{F},\mathcal{G}\}_M\|_{\rho-d-d_1,k} \leq \frac{(k+1)4^k}{e^2 d_1(d+d_1)} \|\mathcal{F}\|_{\rho,k} \cdot \|\mathcal{G}\|_{\rho-d,k} \tag{3.45}$$

(3) $\mathcal{F}\mathcal{G} \in \mathcal{J}_k(\rho)$ *and*

$$\|\mathcal{F}\mathcal{G}\|_{\rho,k} \leq (k+1)4^k \|\mathcal{F}\|_{\rho,k} \cdot \|\mathcal{G}\|_{\rho,k}. \tag{3.46}$$

Remark 3.12. The operators $F(\hbar)$ with the uniform norm $\|F\|_{\rho,k}, k = 0,1,\dots$ form a Banach subalgebra (without unit) of the algebra of the continuous operators in $L^2(\mathbb{T}^l)$.

Before turning to the proof we state and prove two further useful results.

Corollary 3.13. *Let $\mathcal{F},\mathcal{G} \in \mathcal{J}_k(\rho)$, and let $0 < d < \rho$, $r \in \mathbb{N}$. Then:*

$$\frac{1}{r!}\|\{\mathcal{F},\{\mathcal{F},\dots,\{\mathcal{F},\mathcal{G}\}_M\}_M\dots\}_M\|_{\rho-d,k} \leq \frac{\sqrt{2\pi r}(k+1)4^k}{(ed)d^r}\|\mathcal{F}\|^r_{\rho,k}\|\mathcal{G}\|_{\rho,k} \tag{3.47}$$

Proof. We follow the argument of [BGP], Lemma 3.5. If $d = d_1 + d_2$, (3.42) entails:

$$\|\{\mathcal{F},\mathcal{G}\}_M\|_{\rho-d,k} \leq \frac{C_k}{e^2 dd_1}\|\mathcal{F}\|_{\rho,k} \cdot \|\mathcal{G}\|_{\rho-d_2,k}, \quad C_k := (k+1)4^k.$$

because $\|\mathcal{G}\|_{\rho-d,k}\| \leq \|\mathcal{G}\|_{\rho-d_2,k}$ and $d_1(d+d_1) < d_1 d$. Set now $d_2 = \dfrac{r-1}{r}d$ which yields $d_1 = \dfrac{d}{r}$. Then:

$$\|\{\mathcal{F},\mathcal{G}\}_M\|_{\rho-d,k} \leq \frac{C_k}{e^2 d\frac{d}{r}}\|\mathcal{F}\|_{\rho,k} \cdot \|\mathcal{G}\|_{\rho-\frac{r-1}{r}d,k} = \frac{C_k r}{(ed)^2}\|\mathcal{F}\|_{\rho,k} \cdot \|\mathcal{G}\|_{\rho-\frac{r-1}{r}d,k}$$

and

$$\|\{\mathcal{F},\{\mathcal{F},\mathcal{G}\}_M\}_M\|_{\rho-d,k} \leq \frac{C_k}{ed\frac{d}{r},k}\|\mathcal{F}\|_{\rho,k} \cdot \|\{\mathcal{F},\mathcal{G}\}_M\|_{\rho-\frac{r-2}{r}d,k} \leq$$
$$\leq \frac{(C_k r)^2}{(ed)^3}\|\mathcal{F}\|^2_{\rho,k} \cdot \|\mathcal{G}\|_{\rho-\frac{r-1}{r}d,k}$$

Iterating r times we get:

$$\frac{1}{r!}\|\{\mathcal{F},\{\mathcal{F},\cdots,\{\mathcal{F},\mathcal{G}\}_M\}_M,\cdots\}_M\|_{\rho-d,k} \leq \frac{(C_k r)^r}{r!}\frac{1}{(ed)^{r+1}}\|\mathcal{F}\|^r_{\rho,k} \cdot \|\mathcal{G}\|_{\rho-\frac{r-1}{r}d,k}.$$

The Stirling formula and the majorization $\|\mathcal{G}\|_{\rho-\frac{r-1}{r}d,k} \leq \|\mathcal{G}\|_{\rho,k}$ now yield (3.47). □

Proposition 3.14. *Let $\mathcal{F}(\xi;x;\hbar) \in \mathcal{J}_k(\rho)$, $\rho > 0$, $k = 0,1,\dots$. Then $\{\mathcal{F},\mathcal{L}_\omega\}_M \in \mathcal{J}_k(\rho-d)$ $\forall\, 0 < d < \rho$ and the following estimates hold:*

$$\|[F,L_\omega]/i\hbar\|_{\rho-d,k} = \|\{\mathcal{F},\mathcal{L}_\omega\}_M\|_{\rho-d,k} \leq \frac{1}{d}\|\mathcal{F}\|_{\rho,k} \tag{3.48}$$

$$\|[F,[\cdots,[F,L_\omega]\cdots]/(i\hbar)^r\|_{\rho-d,k}=\|\{\mathcal{F},\cdots,\{\mathcal{F},\mathcal{L}_\omega\}_M\cdots,\}_M\|_{\rho-d,k} \tag{3.49}$$

$$\le\frac{\sqrt{2\pi(r-1)}(k+1)4^k}{(ed)d^r}\|\mathcal{F}\|^r_{\rho,k}$$

Proof. By (3.40):

$$\{\mathcal{F},\mathcal{L}_\omega\}_M=\{\mathcal{F},\mathcal{L}_\omega\}=-\langle\omega,\nabla_x\rangle\mathcal{F}(\xi,x;\hbar)=\sum_{q\in\mathbb{Z}^l}\langle\omega,q\rangle e^{i\langle q,x\rangle}\int_{\mathbb{R}}\widehat{\mathcal{F}}_q(p;\hbar)e^{ip\mathcal{L}_\omega(\xi)}\,dp$$

and therefore:

$$\|\{\mathcal{F},\mathcal{L}_\omega\}_M\|_{\rho-d,k}\le\|\{\mathcal{F},\mathcal{L}_\omega\}\|_{\rho-d,k}\le\sum_{q\in\mathbb{Z}^l}|\langle\omega,q\rangle|e^{(\rho-d)|q|}\|\mathcal{F}_q\|_{\rho,k}\le$$

$$\sup_{q\in\mathbb{Z}^l}\langle\omega,q\rangle|e^{-d|q|}\sum_{q\in\mathbb{Z}^l}e^{\rho|q|}\|\mathcal{F}_q\|_{\rho,k}\le\frac{1}{d}\|\mathcal{F}\|_{\rho,k}$$

because $|\omega|\le 1$ by Remark 2.6. This proves (3.48). (3.49) is a direct consequence of Corollary 3.13. □

3.4. **Proof of Proposition 3.11.**

3.4.1. *Three lemmata.* The proof will use the three following Lemmata.

Lemma 3.15. *Let $p,p'\in\mathbb{R}^l$, $s,s'\in\mathbb{R}^l$. Define $t:=(p,s),t':=(p',s')$. Let $\Omega_\omega(\cdot)$ and $\mu_j(\cdot)$ be defined by (3.29) and (2.12), respectively. Then:*

$$|\Omega_\omega(t,t')|^j\le 2^j\mu_j(t)\mu_j(t'). \tag{3.50}$$

The proof is straightforward, because $|\Omega_\omega(t,t')|\le 2|t||t'|$ and $|\omega|\le 1$.

Lemma 3.16.

$$\left|\frac{d^m}{d\hbar^m}\frac{\sin\hbar x/2}{\hbar}\right|\le\frac{|x|^{m+1}}{2^{m+1}}. \tag{3.51}$$

.

Proof. Write:

$$\frac{d^m}{d\hbar^m}\frac{1}{\hbar}\sin\hbar x/2=\frac{d^m}{d\hbar^m}\frac{1}{2}\int_0^x\cos\hbar t/2\,dt=\frac{(-\hbar)^m}{2^{m+1}}\int_0^x t^m\cos^{(m)}(\hbar t/2)\,dt\le\frac{\hbar^m}{2^{m+1}}\int_0^x t^m\,dt.$$

whence

$$\left|\frac{d^m}{d\hbar^m}\frac{\sin\hbar x/2}{\hbar}\right|\le\frac{\hbar^m}{2^{m+1}}\left|\int_0^x t^m\,dt\right|=\frac{\hbar^m|x|^{m+1}}{2^{m+1}(m+1)}\le\frac{|x|^{m+1}}{2^{m+1}}.$$

□

Lemma 3.17. *Let* $(\mathcal{F}, G) \in \mathcal{J}^\dagger_\rho$, $0 < d + d_1 < \rho$, $t = (p, s)$, $t' = (p', s')$, $|t| := |p| + |s|$, $|t'| := |p'| + |s'|$. *Then:*

$$\|\{\mathcal{F}, \mathcal{G}\}_M\|^\dagger_{\rho-d-d_1} \le \frac{1}{e^2 d_1(d+d_1)} \|\mathcal{F}\|^\dagger_\rho \|\mathcal{G}\|^\dagger_{\rho-d} \tag{3.52}$$

Proof. We have by definition

$$\begin{aligned}
&|\{\mathcal{F}, \mathcal{G}\}_M\|^\dagger_{\rho-d-d_1} \le \frac{1}{\hbar} \int_{\mathbb{R}^{2l}} e^{(\rho-d-d_1)|t|} d\lambda(t) \int_{\mathbb{R}^{2l}} |\mathcal{F}(t')\mathcal{G}(t'-t)| \cdot |\sin \hbar(t'-t) \wedge t'/\hbar| \, d\lambda(t') \\
&\le \int_{\mathbb{R}^{2l}} e^{(\rho-d-d_1)|t|} d\lambda(t) \int_{\mathbb{R}^{2l}} |\mathcal{F}(t')| \cdot |\mathcal{G}(t'-t)| \cdot |(t'-t)| \cdot |t'| \, d\lambda(t') \\
&= \int_{\mathbb{R}^{2l}} e^{(\rho-d-d_1)|t|} d\lambda(t) \int_{\mathbb{R}^{2l}} |\mathcal{F}(u+t/2)\mathcal{G}(u-t/2)| \cdot |u-t/2| \cdot |u+t/2| \, d\lambda(u) \\
&= \int_{\mathbb{R}^{2l} \times \mathbb{R}^{2l}} e^{(\rho-d-d_1)(|x|+|y|)} |\mathcal{F}(x)\mathcal{G}(y)| \cdot |x| \cdot |y| \, d\lambda(x) d\lambda(y) \le \\
&\frac{1}{d_1(d+d_1)} \int_{\mathbb{R}^{2l}} |\mathcal{F}(x)| e^{\rho|x|} \, d\lambda(x) \int_{\mathbb{R}^{2l}} |\mathcal{G}(y)| e^{(\rho-d)|y|} \, d\lambda(x) \le \frac{1}{e^2 d_1(d+d_1)} \|\mathcal{F}\|^\dagger_\rho \|\mathcal{G}\|^\dagger_{\rho-d}
\end{aligned}$$

because $\sup_{\alpha \in \mathbb{R}} |\alpha| e^{-\delta\alpha} = \dfrac{1}{e\delta}, \delta > 0.$ □

3.4.2. *Assertion* $(1^\dagger)$. By definition

$$\|\mathcal{F}(\hbar) \sharp \mathcal{G}(\hbar)\|^\dagger_{\rho,k} = \sum_{\gamma=0}^{k} \int_{\mathbb{R}^{2l} \times \mathbb{R}^{2l}} |\partial^\gamma_\hbar [\widehat{\mathcal{F}}(t'-t, \hbar) \widehat{\mathcal{G}}(t', \hbar) e^{i\hbar\Omega_\omega(t', t'-t)}]| \mu_{k-\gamma}(t) e^{\rho|t|} \, d\lambda(t') d\lambda(t)$$

whence

$$\begin{aligned}
&\|\mathcal{F}(\hbar) \sharp \mathcal{G}(\hbar)\|^\dagger_{\rho,k} = \\
&\sum_{\gamma=0}^{k} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \int_{\mathbb{R}^{2l} \times \mathbb{R}^{2l}} |\partial^{\gamma-j}_\hbar [\widehat{\mathcal{F}}(t'-t, \hbar) \widehat{\mathcal{G}}(t', \hbar)]| |\Omega_\omega(t'-t, t')|^j \mu_{k-\gamma}(t) e^{\rho|t|} \, d\lambda(t') d\lambda(t) = \\
&\sum_{\gamma=0}^{k} \sum_{j=0}^{\gamma} \sum_{i=0}^{\gamma-j} \binom{\gamma}{j} \binom{j}{i} \int_{\mathbb{R}^{2l} \times \mathbb{R}^{2l}} |\partial^{\gamma-j-i}_\hbar \widehat{\mathcal{F}}(t'-t, \hbar) \partial^i_\hbar \widehat{\mathcal{G}}(t', \hbar)| |\Omega_\omega(t'-t, t')|^j \mu_{k-\gamma}(t) e^{\rho|t|} \, d\lambda(t') d\lambda(t)
\end{aligned}$$

By Lemma 3.15 and the inequality $\mu_k(t'-t) \le 2^{k/2} \mu_k(t') \mu_k(t)$ we get, with $t = (p, s) : t' = (p', s')$

$$\begin{aligned}
&|\Omega_\omega(t'-t, t')|^j \mu_{k-\gamma}(t) \le 2^j \mu_j(t'-t) \mu_j(t') \mu_{k-\gamma}(t) \\
&\le 2^j \mu_j t'-t) \mu_j(t') \mu_{k-\gamma}(t) 2^{(k-\gamma)/2} \mu_{k-\gamma}(t'-t) \mu_{k-\gamma}(t) \\
&\le 2^{j+(k-\gamma)/2} \mu_{k-\gamma+j}(t'-t) \mu_{k-\gamma+j}(t)
\end{aligned}$$

Denote now $\gamma - j - i = k - \gamma'$, $i = k - \gamma''$ and remark that $j \le \gamma'$, $i \le \gamma - j$. Then:

$$2^{j+(k-\gamma)/2} \mu_{k-\gamma+j}(t'-t) \mu_{k-\gamma+j}(t) \le 2^k \mu_{\gamma'}(t') \mu_{\gamma''}(t)$$

Since $\begin{pmatrix}\gamma\\ j\end{pmatrix}\begin{pmatrix}j\\ i\end{pmatrix} \le 4^k$ and the sum over k has $(k+1)$ terms we get:

$$\|\mathcal{F}(\hbar)\sharp\mathcal{G}(\hbar)\|^{\dagger}_{\rho,k} \le$$
$$(k+1)4^k \sum_{\gamma',\gamma''=0}^{k} \int_{\mathbb{R}^{2l}\times\mathbb{R}^{2l}} |\partial_\hbar^{k-\gamma'}\widehat{\mathcal{F}}(t'-t,\hbar)|\partial_\hbar^{k-\gamma''}\widehat{\mathcal{G}}(t',\hbar)|\mu_{\gamma'}(t'-t)\mu_{\gamma''}(t)e^{\rho|t|}\,d\lambda(t')d\lambda(t)$$

Now we can repeat the argument of Lemma 3.17 to conclude:

$$\|\mathcal{F}(\hbar)\sharp\mathcal{G}(\hbar)\|^{\dagger}_{\rho,k} \le (k+1)4^k\|\mathcal{F}\|^{\dagger}_{\rho,k}\cdot\|\mathcal{G}\|^{\dagger}_{\rho,k}$$

which is (3.41). Assertion $(\mathbf{3}^\dagger)$, formula (3.43) is the particular case of (3.41) obtained for $\Omega_\omega = 0$, and Assertion (**3**), formula (3.46), is in turn particular case of (3.43) .

3.4.3. *Assertion*$(2^\dagger)$. By definition:

$$\|\{\mathcal{F}(\hbar),\mathcal{G}(\hbar)\}_M\|^{\dagger}_{\rho,k} = \sum_{\gamma=0}^{k}\int_{\mathbb{R}^{2l}\times\mathbb{R}^{2l}} |\partial_\hbar^{\gamma}[\widehat{\mathcal{F}}(t'-t,\hbar)\widehat{\mathcal{G}}(t',\hbar)\sin\hbar\Omega(t'-t,t')/\hbar]|\mu_{k-\gamma}(t)e^{\rho|t|}\,d\lambda(t')d\lambda(t).$$

Lemma 3.16 entails:

$$|\partial_\hbar^{j}\sin\hbar\Omega(t'-t,t')/\hbar| \le |\Omega(t'-t,t')|^{j+1}$$

and therefore:

$$\|\{\mathcal{F}(\hbar),\mathcal{G}(\hbar)\}_M\|_{\rho,k} \le$$
$$\sum_{\gamma=0}^{k}\sum_{j=0}^{\gamma}\begin{pmatrix}\gamma\\ j\end{pmatrix}\int_{\mathbb{R}^{2l}\times\mathbb{R}^{2l}} |\partial_\hbar^{\gamma-j}[\widehat{\mathcal{F}}(t'-t,\hbar)\widehat{\mathcal{G}}(t',\hbar)]|\Omega_\omega(t'-t,t')|^{j+1}\mu_{k-\gamma}(t)e^{\rho(|t|}\,d\lambda(t')d\lambda(t) =$$
$$\sum_{\gamma=0}^{k}\sum_{j=0}^{\gamma}\sum_{i=0}^{\gamma-j}\begin{pmatrix}\gamma\\ j\end{pmatrix}\begin{pmatrix}j\\ i\end{pmatrix}\int_{\mathbb{R}^{2l}\times\mathbb{R}^{2l}} |\partial_\hbar^{\gamma-j-i}\widehat{\mathcal{F}}(t'-t,\hbar)\partial_\hbar^{i}\widehat{\mathcal{G}}(t',\hbar)||\Omega_\omega(t'-t,t')|^{j+1}\mu_{k-\gamma}(t)e^{\rho|t|}\,d\lambda(t')d\lambda(t)$$

Let us now absorb a factor $|\Omega_\omega(t'-t,t')|^j$ in exactly the same way as above, and recall that $|\Omega_\omega(t'-t,t')| \le |(t'-t)t'|$. We end up with the inequality:

$$\|\{\mathcal{F}(\hbar),\mathcal{G}(\hbar)\}_M\|^{\dagger}_{\rho,k} \le$$
$$(k+1)4^k \sum_{\gamma',\gamma''=0}^{k}\int_{\mathbb{R}^{2l}\times\mathbb{R}^{2l}} |\partial_\hbar^{k-\gamma'}\widehat{\mathcal{F}}(t'-t,\hbar)|\partial_\hbar^{k-\gamma''}\widehat{\mathcal{G}}(t',\hbar)||t'-t||t'|\mu_{\gamma'}(t'-t)\mu_{\gamma''}(t')e^{\rho(|t|}\,d\lambda(t')d\lambda(t)$$

Repeating once again the argument of Lemma 3.17 we finally get:

$$\|\{\mathcal{F}(\hbar),\mathcal{G}(\hbar)\}_M\|^{\dagger}_{\rho-d-d_1,k} \le \frac{(k+1)4^k}{e^2 d_1(d+d_1)}\|\mathcal{F}\|^{\dagger}_{\rho,k}\cdot\|\mathcal{G}\|^{\dagger}_{\rho-d,k}$$

which is (3.42). Once more, Assertion (**2**) is a particular case of (3.42) and Assertion (**1**) a particular case of (3.41). This completes the proof of Proposition 3.10.

4. A sharper version of the semiclassical Egorov theorem

Let us state and prove in this section a particular variant of the semiclassical Egorov theorem (see e.g.[Ro]) which establishes the relation between the unitary transformation $e^{i\varepsilon W/i\hbar}$ and the canonical transformation $\phi^{\varepsilon}_{\mathcal{W}_0}$ generated by the flow of the symbol $\mathcal{W}(\xi,x;\hbar)|_{\hbar=0} := \mathcal{W}_0(\xi,x)$ (principal symbol) of W at time 1. The present version is sharper in the sense that the usual one allows for a $O(\hbar^\infty)$ error term.

Theorem 4.1. *Let $\rho>0, k=0,1,\dots$ and let $A, W\in J^\dagger_k(\rho)$ with symbols $\mathcal{A}$, $\mathcal{W}$. Then:*

$$S_\varepsilon := e^{i\frac{\varepsilon W}{\hbar}}(L_\omega+A)e^{-i\frac{\varepsilon W}{\hbar}} = L_\omega + B$$

where:

(1) $\forall\, 0<d<\rho,\ B\in J^\dagger_k(\rho-d)$;

(2)

$$\|\mathcal{B}\|^\dagger_{\rho-d,k} \le \frac{(k+1)4^k}{(ed)^2}\left[1-|\varepsilon|\|\mathcal{W}\|^\dagger_{\rho,k}/d\right]^{-1}\left[\|\mathcal{A}\|^\dagger_{\rho,k}+|\varepsilon|\|\mathcal{W}\|^\dagger_{\rho,k}/de\right]$$

(3) *Moreover the symbol $\mathcal{B}$ of B is such that:*

$$\mathcal{L}_\omega+\mathcal{B} = (\mathcal{L}_\omega+\mathcal{A})\circ\Phi^\varepsilon_{\mathcal{W}_0}+O(\hbar)$$

where $\Phi^\varepsilon_{\mathcal{W}_0}$ is the Hamiltonian flow of $\mathcal{W}_0 := \mathcal{W}|_{\hbar=0}$ at time ε.

(4) *Assertions (1), (2), (3) hold true when $(A,B,W)\in J_k(\rho)$ with $\|\mathcal{A}\|^\dagger_{\rho,k}$, $\|\mathcal{B}\|^\dagger_{\rho,k}$, $\|\mathcal{W}\|^\dagger_{\rho,k}$ replaced by $\|\mathcal{A}\|_{\rho,k}$, $\|\mathcal{B}\|_{\rho,k}$, $\|\mathcal{W}\|_{\rho,k}$.*

Proof. The proof is the same in both cases, since it it is based only on Proposition 3.11. Therefore we limit ourselves to the $\mathcal{J}_k(\rho)$ case.

By Corollary 3.8, Assertion (3), under the present assumptions $H^1(\mathbb{T}^l)$, the domain of the self-adjoint operator $\mathcal{F}(L_\omega)+A$, is left invariant by the unitary operator $e^{i\frac{\varepsilon W}{\hbar}}$. Therefore on $H^1(\mathbb{T}^l)$ we can write the commutator expansion

$$S_\varepsilon = L_\omega + \sum_{m=1}^\infty \frac{(i\varepsilon)^m}{\hbar^m m!}[W,[W,\dots,[W,L_\omega]\dots] + \sum_{m=1}^\infty \frac{(i\varepsilon)^m}{\hbar^m m!}[W,[W,\dots,[W,A]\dots]$$

whence the corresponding expansions for the symbols

$$\mathcal{S}(x,\xi;\hbar,\varepsilon) = \mathcal{L}_\omega(\xi) + \sum_{m=1}^\infty \frac{\varepsilon^m}{m!}\{\mathcal{W},\{\mathcal{W},\dots,\{\mathcal{W},\mathcal{L}_\omega\}\dots\}_M$$
$$+\sum_{m=1}^\infty \frac{\varepsilon^m}{m!}\{\mathcal{W},\{\mathcal{W},\dots,\{\mathcal{W},\mathcal{A}\}_M\dots\}_M$$

because $\{\mathcal{W},\mathcal{L}_\omega\}_M=\{\mathcal{W},\mathcal{L}_\omega\}$ by the linearity of $\mathcal{L}_\omega$. Now apply Corollaries 3.13 and 3.14. We get, denoting once again $C_k=(k+1)4^k$:

$$\|\sum_{m=1}^\infty\frac{(i\varepsilon)^m}{\hbar^m m!}[W,[W,\dots,[W,L_\omega]\dots]\|_{L^2\to L^2}\leq\|\sum_{m=1}^\infty\frac{\varepsilon^m}{m!}\{\mathcal{W},\{\mathcal{W},\dots,\{\mathcal{W},\mathcal{L}_\omega\}\dots\}_M\|_{\rho-d,k}$$
$$\leq\sum_{m=1}^\infty\frac{|\varepsilon|^m}{m!}\|\{\mathcal{W},\{\mathcal{W},\dots,\{-i\langle\omega,\nabla_x\rangle\mathcal{W}\}_M\dots\}_M\|_{\rho-d,k}\leq\frac{C_k}{ed}\sum_{m=1}^\infty\sqrt{2\pi m}\left(\frac{|\varepsilon|\|\mathcal{W}\|_{\rho,k}}{d}\right)^m$$

$$\|\sum_{m=1}^\infty\frac{(i\varepsilon)^m}{\hbar^m m!}[W,[W,\dots,[W,A]\dots]\|_{L^2\to L^2}\leq\|\sum_{m=1}^\infty\frac{\varepsilon^m}{m!}\{\mathcal{W},\{\mathcal{W},\dots,\{\mathcal{W},\mathcal{A}\}_M\dots\}_M\|_{\rho-d,k}$$
$$\leq\frac{C_k}{ed}\|\mathcal{A}\|_{\rho,k}\sum_{m=1}^\infty\sqrt{2\pi m}\left(\frac{|\varepsilon|\|\mathcal{W}\|_{\rho,k}}{d}\right)^m$$

Now define:

$$B:=\sum_{m=1}^\infty\frac{(i\varepsilon)^m}{\hbar^m m!}[W,[W,\dots,[W,\mathcal{L}_\omega]\dots]+\sum_{m=1}^\infty\frac{(i\varepsilon)^m}{\hbar^m m!}[W,[W,\dots,[W,A]\dots] \tag{4.53}$$

and remark that $\forall\,\eta>0$ we can always find $0<d'<d-\eta$ such that $\sqrt{2\pi m}d^{-m}\leq(d')^{-m}$. Denoting (abuse of notation) $d'=d$ we can write:

$$\|\mathcal{B}\|_{\rho-d,k}\leq\frac{(k+1)4^k}{(ed)^2}\left[1-|\varepsilon|\|\mathcal{W}\|_{\rho,k}/d\right]^{-1}\left[\|\mathcal{A}\|_{\rho,k}+|\varepsilon|\|\mathcal{W}\|_{\rho,k}/de\right]$$

This proves assertions (1) and (2).

By Remark 2.9, we have:

$$\mathcal{S}^0_\varepsilon(x,\xi;\hbar)|_{\hbar=0}=\mathcal{L}_\omega+\mathcal{B}_\varepsilon(\xi,x;\hbar)|_{\hbar=0}=$$
$$\sum_{k=0}^\infty\frac{(\varepsilon)^k}{k!}\{\mathcal{W}_0,\{\mathcal{W},\dots,\{\mathcal{W}_0,\mathcal{L}+\mathcal{A}\}\dots\}=e^{\varepsilon\mathcal{L}_{\mathcal{W}_0}}(\mathcal{L}_\omega+\mathcal{A})$$

where $\mathcal{L}_{\mathcal{W}_0}\mathcal{F}=\{\mathcal{W},\mathcal{F}\}$ denote the Lie derivative with respect to the Hamiltonian flow generated by $\mathcal{W}_0$. Now, by Taylor's theorem

$$e^{\varepsilon\mathcal{L}_{\mathcal{W}_0}}(\mathcal{L}_\omega+\mathcal{A})=(\mathcal{L}_\omega+\mathcal{A})\circ\phi^\varepsilon_{\mathcal{W}_0}(x,\xi)$$

and this concludes the proof of the Theorem. $\square$

Remark 4.2. Let W be a solution of the homological equation (5.1). Then the explicit expression of $\mathcal{W}_0$ clearly is:

$$\mathcal{W}_0=\frac{1}{\mathcal{F}'(\mathcal{L}_\omega(\xi))}\sum_{q\in\mathbb{Z}^\ell}\frac{\mathcal{V}_q(\xi)}{\langle\omega,q\rangle}e^{i\langle q,x\rangle}$$

and

$$e^{\varepsilon\mathcal{L}_{\mathcal{W}_0}}(\mathcal{F}(\mathcal{L}_\omega)+\varepsilon\mathcal{A})=\mathcal{F}(L_\omega)+\varepsilon\mathcal{N}_{0,\varepsilon}(\mathcal{L}_\omega)+O(\varepsilon^2).$$

Thus $\mathcal{W}_0$ coincides with the expression obtained by first order canonical perturbation theory.

5. Homological equation: solution and estimate

Let us briefly recall the well known KAM iteration in the quantum context.

The first step consists in looking for an $L^2(\mathbb{T}^l)$-unitary map $U_{0,\varepsilon} = e^{i\varepsilon W_0/\hbar}$, $W_0 = W_0^*$, such that

$$S_{0,\varepsilon} := U_{0,\varepsilon}(L_\omega + \varepsilon V_0)U_{0,\varepsilon}^* = \mathcal{F}_{1,\varepsilon}(L_\omega) + \varepsilon^2 V_{1,\varepsilon}, \quad V_0 := V, \quad \mathcal{F}_{1,\varepsilon}(L_\omega) = L_\omega + \varepsilon N_0(L_\omega).$$

Expanding to first order near $\varepsilon = 0$ we get that the two unknowns W_0 and N_0 must solve the equation

$$\frac{[L_\omega, W_0]}{i\hbar} + V = N_0$$

$V_{1,\varepsilon}$ is the second order remainder of the expansion. Iterating the procedure:

$$\begin{aligned}
&U_{\ell,\varepsilon} := e^{i\varepsilon^{2^\ell} W_\ell/\hbar};\\
&S_{\ell,\varepsilon} := U_{\ell.\varepsilon}(\mathcal{F}_{\ell,\varepsilon}(L_\omega) + \varepsilon^{2^\ell} V_{\ell,\varepsilon})U_{\ell,\varepsilon}^* == \mathcal{F}_{\ell+1,\varepsilon}(L_\omega) + \varepsilon^{2^{\ell+1}} V_{\ell+1}(\varepsilon),\\
&\frac{[\mathcal{F}_{\ell,\varepsilon}(L_\omega), W_{\ell,\varepsilon}]}{i\hbar} + V_{\ell,\varepsilon} = N_{\ell,\varepsilon}
\end{aligned}$$

With abuse of notation, we denote by $\mathcal{F}_{\ell,\varepsilon}(\mathcal{L}_\omega,\hbar)$, $\mathcal{N}_{\ell,\varepsilon}(\mathcal{L}_\omega,\hbar)$, $\mathcal{V}_{\ell,\varepsilon}(\mathcal{L}_\omega,\hbar)$ the corresponding symbols.

The KAM iteration procedure requires therefore the solution in $J_k(\rho)$ of the operator homological equation in the two unknowns W and M (here we have dropped the dependence on ℓ and ε, and changed the notation from N to M to avoid confusion with what follows):

$$\frac{[\mathcal{F}(L_\omega), W]}{i\hbar} + V = M(L_\omega) \tag{5.1}$$

with the requirement $M(L_\omega) \in J_k(\rho)$; the solution has to be expressed in terms of the corresponding Weyl symbols $(\mathcal{L}_\omega, \mathcal{W}, \mathcal{V}, \mathcal{M}) \in \mathcal{J}_k(\rho)$ in order to obtain estimates uniform with respect to $\hbar$. Moreover, the remainder has to be estimated in terms of the estimates for W, M.

Equation (5.1), written for the symbols, becomes

$$\{\mathcal{F}(\mathcal{L}_\omega(\xi),\hbar), \mathcal{W}(x,\xi;\hbar)\}_M + \mathcal{V}(x, L_\omega(\xi);\hbar) = \mathcal{M}(\mathcal{L}_\omega(\xi),\hbar) \tag{5.2}$$

5.1. **The homological equation.** We will construct and estimate the solution of (5.1), actually solving (5.2) and estimating its solution, under the following assumptions on $\mathcal{F}$:

Condition (1) $(u,\hbar) \mapsto \mathcal{F}(u;\hbar) \in C^\infty(\mathbb{R}\times[0,1];\mathbb{R})$;

Condition (2)

$$\inf_{(u,\hbar)\in\mathbb{R}\times[0,1]} \partial_u \mathcal{F}(u;\hbar) > 0; \qquad \lim_{|u|\to\infty} \frac{|\mathcal{F}(u,\hbar)|}{|u|} = C > 0$$

uniformly with respect to $\hbar \in [0,1]$.

Condition (3) *Set:*

$$\mathcal{K}_{\mathcal{F}}(u,\eta,\hbar) = \frac{\eta}{\mathcal{F}(u+\eta,\hbar) - \mathcal{F}(u,\hbar)} \tag{5.3}$$

Then there is $0 < \Lambda(\mathcal{F}) < +\infty$ *such that*

$$\sup_{u\in\mathbb{R},\eta\in\mathbb{R},\hbar\in[0,1]} |\mathcal{K}_{\mathcal{F}}(u,\eta,\hbar)| < \Lambda. \tag{5.4}$$

The first result deals with the identification of the operators W and M through the determination of their matrix elements and corresponding symbols $\mathcal{W}$ and $\mathcal{M}$.

Proposition 5.1. *Let* $V \in J(\rho)$*,* $\rho > 0$*, and let* W *and* M *be the minimal closed operators in* $L^2(\mathbb{T}^n)$ *generated by the infinite matrices*

$$\langle e_m, W e_{m+q}\rangle = \frac{i\hbar\langle e_m, V e_{m+q}\rangle}{\mathcal{F}(\langle\omega,m\rangle\hbar,\hbar) - \mathcal{F}(\langle\omega,(m+q)\rangle\hbar,\hbar)}, \quad q \neq 0, \quad \langle e_m, W e_m\rangle = 0 \tag{5.5}$$

$$\langle e_m, M e_m\rangle = \langle e_m, V e_m\rangle, \qquad \langle e_m, M e_{m+q}\rangle = 0, \quad q \neq 0 \tag{5.6}$$

on the eigenvector basis $e_m : m \in \mathbb{Z}^l$ *of* L_ω*. Then:*

(1) W *and* M *are continuous and solve the homological equation (5.1);*
(2) *The symbols* $\mathcal{W}(x,\xi;\hbar)$ *and* $\mathcal{M}(\xi,\hbar)$ *have the expression:*

$$\mathcal{M}(\xi;\hbar) = \overline{\mathcal{V}}(\mathcal{L}_\omega(\xi);\hbar); \quad \mathcal{W}(\mathcal{L}_\omega(\xi),x;\hbar) = \sum_{q\in\mathbb{Z}^l, q\neq 0} \mathcal{W}(\mathcal{L}_\omega(\xi),q;\hbar)e^{i\langle q,x\rangle} \tag{5.7}$$

$$\mathcal{W}(\mathcal{L}_\omega(\xi),q;\hbar) := \frac{i\hbar\mathcal{V}(\mathcal{L}_\omega(\xi);q;\hbar)}{\mathcal{F}(\mathcal{L}_\omega(\xi);\hbar) - \mathcal{F}(\mathcal{L}_\omega(\xi+q),\hbar)}, \; q \neq 0; \quad \overline{\mathcal{W}}(\mathcal{L}_\omega(\xi);\hbar) = 0. \tag{5.8}$$

Here the series in (5.7) is $\|\cdot\|_\rho$ *convergent;* $\overline{\mathcal{V}}(\mathcal{L}_\omega(\xi);\hbar)$ *is the* 0*-th coefficients in the Fourier expansion of* $\mathcal{V}(\mathcal{L}_\omega(\xi),x,\hbar)$*.*

Proof. Writing the homological equation in the eigenvector basis $e_m : m \in \mathbb{Z}^l$ we get

$$\langle e_m, \frac{[\mathcal{F}(L_\omega),W]}{i\hbar} e_n\rangle + \langle e_m, V e_n\rangle = \langle e_m, M(L_\omega)e_n\rangle\delta_{m,n} \tag{5.9}$$

which immediately yields (5.5,5.6) setting $n = m+q$. As far the continuity is concerned, we have:

$$\frac{i\hbar}{\mathcal{F}(\langle\omega,m\rangle\hbar,\hbar) - \mathcal{F}(\langle\omega,(m+q)\rangle\hbar,\hbar)} = \langle\omega,q\rangle^{-1}\frac{\eta}{\mathcal{F}(\langle\omega,m\rangle\hbar,\hbar) - \mathcal{F}(\langle\omega,m\rangle\hbar+\eta,\hbar)}, \quad \eta := \langle q,\omega\rangle\hbar.$$

and therefore, by (5.4) and the diophantine condition:

$$|\langle e_m, W e_{m+q}\rangle| \leq \gamma|q|^\tau\Lambda|\langle e_m, V e_{m+q}\rangle|.$$

The assertion now follows by Corollary 3.8, which also entails the $\|\cdot\|_\rho$ convergence of the series (5.7) because $\mathcal{V}\in\mathcal{J}_\rho$. Finally, again by Corollary 3.8, formulae (3.23), (3.24), we can write

$$\langle e_m, We_{m+q}\rangle = \mathcal{W}(\langle\omega,(m+q/2)\rangle\hbar,q,\hbar);\quad \langle e_m, Me_m\rangle = \mathcal{M}(\omega,m\rangle\hbar,\hbar) = \mathcal{V}(\mathcal{L}_\omega(\omega,m\rangle\hbar,0,\hbar)$$

and this conclude s the proof of the Proposition. □

The basic example of $\mathcal{F}$ is the following one. Let:

- $\mathcal{F}_\ell(u,\varepsilon;\hbar) = u + \Phi_\ell(u,\varepsilon,\hbar), \qquad \ell = 0,1,2,\dots$ (5.10)
- $\Phi_\ell(\varepsilon,\hbar) := \varepsilon\mathcal{N}_0(u;\varepsilon,\hbar) + \varepsilon^2\mathcal{N}_1(u;\varepsilon,\hbar) + \ldots + \varepsilon_\ell\mathcal{N}_\ell(u,\varepsilon,\hbar), \quad \varepsilon_j := \varepsilon^{2^j}.$ (5.11)

where we assume holomorphy of $\varepsilon\mapsto\mathcal{N}_s(u,\varepsilon,\hbar)$ in the unit disk and the existence of $\rho_0>\rho_1>\ldots>\rho_\ell>0$ such that:

$$(N_s)\qquad \max_{|\varepsilon|\le 1}|\mathcal{N}|_{\rho_s}<\infty, \qquad .$$

Denote, for $\zeta\in\mathbb{R}$:

$$g_\ell(u,\zeta;\varepsilon,\hbar) := \frac{\Phi_{\ell-1}(u+\zeta;\varepsilon,\hbar)-\Phi_{\ell-1}(u;\varepsilon,\hbar)}{\zeta} \tag{5.12}$$

Let furthermore:

$$0<d_\ell<\ldots<d_0<\rho_0,\quad 0<\rho_0:=\rho; \tag{5.13}$$

$$\rho_{s+1}=\rho_s-d_s>0,\ s=0,\dots,\ell-1$$

$$\delta_\ell := \sum_{s=0}^{\ell-1} d_\ell < \rho \tag{5.14}$$

and set, for $j=1,2,\ldots$:

$$\theta_{\ell,k}(\mathcal{N},\varepsilon) := \sum_{s=0}^{\ell-1}\frac{|\varepsilon_s|\,|\mathcal{N}_s|_{\rho_s,k}}{ed_s}, \qquad \theta_\ell(\mathcal{N},\varepsilon):=\theta_{\ell,0}(\mathcal{N},\varepsilon). \tag{5.15}$$

By Remark 2.4 we have

$$\theta_{\ell,k}(\mathcal{N},\varepsilon) = \sum_{s=0}^{\ell-1}\frac{|\varepsilon_s|\,\|\mathcal{N}_s\|_{\rho_s,k}}{ed_s} \tag{5.16}$$

Lemma 5.2. *In the above assumptions:*

(1) *For any $R>0$ the function $\zeta\mapsto g_\ell(u,\zeta,\varepsilon,\hbar)$ is holomorphic in $\{\zeta\mid|\zeta|<R\,|\,|\Im\zeta|<\rho\}$, uniformly on compacts with respect to $(u,\varepsilon,\hbar)\in\mathbb{R}\times\mathbb{R}\times[0,1]$;*

(2) *For any $n\in\mathbb{N}\cup\{0\}$:*

$$\sup_{\zeta\in\mathbb{R}}|[g(u,\zeta,\varepsilon,\hbar)]^n|_{\rho_\ell}\le[\theta_\ell(\mathcal{N},\varepsilon)]^n \tag{5.17}$$

(3) *Let:*

$$\max_{|\varepsilon|\leq L}\theta_\ell(\mathcal{N},\varepsilon)<1, \qquad L>0. \tag{5.18}$$

Then:

$$\sup_{\zeta\in\mathbb{R};u\in\mathbb{R}}|\mathcal{K}_{\mathcal{F}}(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell}\leq\frac{1}{|\zeta|}\cdot\frac{1}{1-\theta_\ell(\mathcal{N},\varepsilon)} \tag{5.19}$$

(4)

$$\sup_{\zeta\in\mathbb{R}}|\partial_u^j g(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell}\leq\theta_{\ell,j}(\mathcal{N},\varepsilon) \tag{5.20}$$

$$\sup_{\zeta\in\mathbb{R}}|\partial_\zeta^j g(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell}\leq\theta_{\ell,j}(\mathcal{N},\varepsilon) \tag{5.21}$$

$$\sup_{\zeta\in\mathbb{R}}|\partial_\hbar^j g(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell}\leq\theta_{\ell,j}(\mathcal{N},\varepsilon). \tag{5.22}$$

Proof. The holomorphy is obvious given the holomorphy of $\mathcal{N}_s(u;\varepsilon,\hbar)$. To prove the estimate (5.17), denoting $\widehat{\mathcal{N}}_s(p,\varepsilon,\hbar)$ the Fourier transform of $\mathcal{N}_s(\xi,\varepsilon,\hbar)$ we write

$$g_\ell(u,\zeta,\varepsilon,\hbar)=\frac{1}{\zeta}\sum_{s=0}^{\ell-1}\varepsilon_s\int_{\mathbb{R}}\widehat{\mathcal{N}}_\ell(p,\varepsilon,\hbar)(e^{i\zeta p}-1)e^{iup}\,dp= \tag{5.23}$$
$$\frac{2}{\zeta}\sum_{s=0}^{\ell-1}\varepsilon_s\int_{\mathbb{R}}\widehat{\mathcal{N}}_\ell(p,\varepsilon,\hbar)e^{ip(u+\zeta)/2}\sin\zeta p/2\,dp$$

which entails:

$$\sup_{\zeta\in\mathbb{R}}|g_\ell(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell}=\sup_{\zeta\in\mathbb{R}}\int_{\mathbb{R}}|\widehat{g}_\ell(p,\zeta,\varepsilon,\hbar)|e^{\rho_\ell|p|}\,dp$$
$$\leq\max_{\hbar\in[0,1]}\sum_{s=0}^{\ell-1}|\varepsilon_s|\int_{\mathbb{R}}|\widehat{\mathcal{N}}_s(p,\varepsilon,\hbar)p|e^{(\rho_s-d_s)|p|}\,dp\leq\frac{1}{e}\sum_{s=0}^{\ell-1}|\varepsilon_s|\frac{|\mathcal{N}_s|_{\rho_s}}{d_s}=\theta_\ell(\mathcal{N},\varepsilon,1)\qquad 0<d_s<\rho_s.$$

Hence Assertion (3) of Proposition 3.11, considered for $k=0$, immediately yields (5.17). Finally, if g_ℓ is defined by (5.12), then:

$$\mathcal{K}_{\mathcal{F}}(u,\zeta,\varepsilon,\hbar)=\frac{1}{\zeta}\frac{1}{1+g_\ell(u,\zeta,\varepsilon,\hbar)}$$

and the estimate (5.19) follows from (5.17) which makes possible the expansion into the geometrical series

$$\frac{1}{1+g_\ell(u,\zeta,\varepsilon,\hbar)}=\sum_{n=0}^{\infty}(-1)^n\,g_\ell(u,\zeta,\varepsilon,\hbar)^n \tag{5.24}$$

convergent in the $\theta_\ell(\mathcal{N},\varepsilon)$ norm. To see (5.20), remark that (5.23) yields:

$$\partial_u^j g_\ell(u,\zeta,\varepsilon,\hbar)=\frac{2}{\zeta}\sum_{s=0}^{\ell-1}\varepsilon_s\int_{\mathbb{R}}\widehat{\mathcal{N}}_\ell(p,\varepsilon,\hbar)(ip)^j\,e^{ip(u+\zeta)/2}\sin\zeta p/2\,dp.$$

Therefore:

$$\sup_{\zeta\in\mathbb{R}} |\partial_u^j g_\ell(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell} \le \sup_{\zeta\in\mathbb{R}} \max_{\hbar\in[0,1]} 2\sum_{s=0}^{\ell-1} |\varepsilon_s| \int_{\mathbb{R}} |\widehat{\mathcal{N}}_s(p,\varepsilon,\hbar)||p|^j |\sin\zeta p/2|/\zeta| e^{\rho_\ell |p|}\, dp$$

$$\le \sup_{\zeta\in\mathbb{R}} \max_{\hbar\in[0,1]} 2\sum_{s=}^{\ell-1} |\varepsilon_s| \int_{\mathbb{R}} |\widehat{\mathcal{N}}_s(p,\varepsilon,\hbar)||p|^j |\sin\zeta p/2|/\zeta| e^{(\rho_s-d_s)|p|}\, dp$$

$$\le \sup_{p\in\mathbb{R}} [|p| \sum_{s=0}^{\ell-1} |\varepsilon_s|\, e^{-d_s|p|}] \max_{\hbar\in[0,1]} \int_{\mathbb{R}} |p|^j \widehat{\mathcal{N}}(p,\varepsilon,\hbar) e^{\rho_s|p|}\, dp$$

$$\le \frac{1}{e}\sum_{s=0}^{\ell-1} |\varepsilon_s| \frac{|\mathcal{N}_s|_{\rho_s,j}}{d_s} \le \theta_{\ell,j}(\mathcal{N},\varepsilon)$$

(5.21) is proved by exactly the same argument. Finally, to show (5.22) we write:

$$\sup_{\zeta\in\mathbb{R}} |\partial_\hbar^j g_\ell(u,\zeta,\varepsilon,\hbar)|_{\rho_\ell} \le \sup_{\zeta\in\mathbb{R}} \max_{\hbar\in[0,1]} 2\sum_{s=0}^{\ell-1} |\varepsilon_s| \int_{\mathbb{R}} |\partial_\hbar^j \widehat{\mathcal{N}_s}(p,\varepsilon,\hbar)| \cdot |\sin\zeta p/2|/\zeta| e^{\rho_\ell |p|}\, dp$$

$$\le \max_{\hbar\in[0,1]} \sum_{s=0}^{\ell-1} |\varepsilon_s| \int_{\mathbb{R}} |\partial_\hbar^j \widehat{\mathcal{N}}(p,\varepsilon,\hbar)| e^{(\rho_s-d_s)|p|}\, dp \le \theta_\ell(\mathcal{N},\varepsilon)$$

This proves the Lemma. □

By **Condition (1)** the operator family $\hbar \mapsto \mathcal{F}(L_\omega;\varepsilon,\hbar)$, defined by the spectral theorem, is self-adjoint in $L^2(\mathbb{T}^l)$; by **Condition (2)** $D(\mathcal{F}(L_\omega)) = H^1(\mathbb{T}^l)$. Since L_ω is a first order operator with symbol $\mathcal{L}_\omega$, the symbol of $\mathcal{F}(L_\omega;\varepsilon,\hbar)$ is $\mathcal{F}(\mathcal{L}_\omega(\xi),\varepsilon,\hbar)$. We can now state the main result of this section. Let $\mathcal{F}_\ell(x,\varepsilon,\hbar)$ be as in Lemma 5.2, which entails the validity of **Conditions (1), (2), (3)**.

Theorem 5.3. *Let $V_\ell \in J_k(\rho_\ell)$, $\ell = 0,1\ldots$, $V_1 \equiv V$ for some $\rho_\ell > \rho_{\ell+1} > 0$, $k = 0,1,\ldots$. Let $\mathcal{V}_\ell(\mathcal{L}_\omega(\xi),x;\varepsilon,\hbar) \in \mathcal{J}_k(\rho)$ be its symbol. Then for any $\theta_\ell(\mathcal{N},\varepsilon) < 1$ the homological equation (5.1), rewritten as*

$$\frac{[\mathcal{F}_\ell(L_\omega), W_\ell]}{i\hbar} + V_\ell = N_\ell(L_\omega,\varepsilon) \tag{5.25}$$

$$\{\mathcal{F}_\ell(\mathcal{L}_\omega(\xi),\varepsilon,\hbar), \mathcal{W}_\ell(x,\xi;\varepsilon,\hbar)\}_M + \mathcal{V}_\ell(x,L_\omega(\xi);\varepsilon,\hbar) = \mathcal{N}_\ell(\mathcal{L}_\omega(\xi),\varepsilon,\hbar) \tag{5.26}$$

admits a unique solution (W_ℓ, N_ℓ) of Weyl symbols $\mathcal{W}_\ell(\mathcal{L}_\omega(\xi),x;\varepsilon,\hbar)$, $\mathcal{N}_\ell(\mathcal{L}_\omega(\xi),\varepsilon,\hbar)$ such that

(1) *$W_\ell = W_\ell^* \in J_k(\rho_\ell)$, with:*

$$\|W_\ell\|_{\rho_{\ell+1},k} = \|\mathcal{W}\|_{\rho_{\ell+1},k} \le A(\ell,k,\varepsilon)\|\mathcal{V}_\ell\|_{\rho_\ell,k} \tag{5.27}$$

$$A(\ell,k,\varepsilon) = \gamma\frac{\tau^\tau}{(ed_\ell)^\tau}\left[1 + \frac{2^{k+1}(k+1)^{2(k+1)}k^k}{(e\delta_\ell)^k[1-\theta_\ell(\mathcal{N},\varepsilon)]^{k+1}}\theta_{\ell,k}^{k+1}\right]. \tag{5.28}$$

(2) *$\mathcal{N}_\ell = \overline{\mathcal{V}}_\ell$; therefore $\mathcal{N}_\ell \in J_k(\rho_\ell)$ and $\|\mathcal{N}\|_{\rho_\ell,k} \leq \|\mathcal{V}_\ell\|_{\rho_\ell,k}$.*

Proof. The proof of (2) is obvious and follows from the definition of the norms $\|\cdot\|_\rho$ and $\|\cdot\|_{\rho,k}$. The self-adjointess property $W = W^*$ is implied by the construction itself, which makes W symmetric and bounded.

Consider $\mathcal{W}_\ell$ as defined by (5.7). Under the present assumptions, by Lemma 5.2 we have:

$$\mathcal{W}_\ell(\mathcal{L}_\omega(\xi), q; \varepsilon, \hbar) := \frac{1}{\langle\omega, q\rangle} \frac{i\hbar\mathcal{V}_\ell(\mathcal{L}_\omega(\xi); q; \varepsilon, \hbar)}{1 + g_\ell(\mathcal{L}_\omega(\xi); \langle\omega, q\rangle\hbar, \varepsilon, \hbar)}, \quad q \neq 0; \quad \mathcal{W}_\ell(\cdot, 0; \hbar) = 0.$$

By the $\|\cdot\|_{\rho_\ell}$-convergence of the series (5.24) we can write

$$\partial_\hbar^\gamma \mathcal{W}_\ell(\mathcal{L}_\omega(\xi), q; \varepsilon, \hbar) = \sum_{n=0}^{\infty} (-\varepsilon)^n\, \partial_\hbar^\gamma \mathcal{W}_{\ell,n}(\mathcal{L}_\omega(\xi), q; \varepsilon, \hbar), \tag{5.29}$$

$$\mathcal{W}_{\ell,n}(\mathcal{L}_\omega(\xi), q; \varepsilon, \hbar) = \frac{1}{\langle\omega, q\rangle} \mathcal{V}_\ell(\mathcal{L}_\omega(\xi); q; \varepsilon, \hbar)[g_\ell(\mathcal{L}_\omega(\xi); \langle\omega, q\rangle\hbar, \varepsilon, \hbar)]^n \tag{5.30}$$

$$\partial_\hbar^\gamma \mathcal{W}_{\ell,n}(\mathcal{L}_\omega(\xi), q; \varepsilon, \hbar) = \tag{5.31}$$

$$\sum_{j=0}^{\gamma} \binom{\gamma}{j} \partial_\hbar^{\gamma-j} \mathcal{V}_\ell(\mathcal{L}_\omega(\xi); q; \varepsilon, \hbar) D_\hbar^j [g_\ell(\mathcal{L}_\omega(\xi); \langle\omega, q\rangle\hbar, \varepsilon, \hbar)]^n$$

where $D_\hbar$ denotes the total derivative with respect to $\hbar$. We need the following preliminary result.

Lemma 5.4. *Let $\zeta(\hbar) := \langle\omega, q\rangle\hbar$. Then:*

(1)

$$|D_\hbar^j g_\ell(\mathcal{L}_\omega(\xi), \zeta(\hbar), \varepsilon, \hbar)|_{\rho_\ell} \leq (j+1)(2|q|)^j \theta_{\ell,j}(\mathcal{N}, \varepsilon)^2 \tag{5.32}$$

(2)

$$|D_\hbar^j [g_\ell(\mathcal{L}_\omega(\xi); \zeta(\hbar), \varepsilon, \hbar)]^n|_{\rho_\ell} \leq 2n^j (\theta_\ell(\mathcal{N}, \varepsilon))^{n-j} [2(j+1)|q|]^j \theta_{\ell,j}(\mathcal{N}, \varepsilon)^{2j}. \tag{5.33}$$

Proof. The expression of total derivative $D_\hbar g$ is:

$$D_\hbar g(\cdot; \langle\omega, q\rangle\hbar, \varepsilon, \hbar) = (\langle\omega, q\rangle\ \frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\hbar})\ g_\ell(\cdot; \zeta, \varepsilon, \hbar)|_{\zeta = \langle\omega, q\rangle\hbar} \tag{5.34}$$

By Leibnitz's formula we then have:

$$D_\hbar^j g_\ell(\cdot; \langle\omega, q\rangle\hbar, \varepsilon, \hbar) = \sum_{i=0}^{j} \binom{j}{i} \langle\omega, q\rangle^{j-i} \frac{\partial^{j-i} g_\ell}{\partial\zeta^{j-i}} \frac{\partial^i g_\ell}{\partial\hbar^i} \tag{5.35}$$

Apply now (3.46) with $k = 0$, (5.20) and (5.22). We get:

$$\left| \frac{\partial^{j-i} g_\ell}{\partial\zeta^{j-i}} \frac{\partial^i g_\ell}{\partial\hbar^i} \right|_{\rho_\ell} \leq (j+1)2^j \theta_{\ell,j}(\mathcal{N}, \varepsilon)^2$$

whence, since $|\omega| \le 1$:

$$\left|\frac{D^j g_\ell}{D\hbar^j}\right|_{\rho_\ell} \le (j+1)(2)^j |q|^j \theta_{\ell,j}(\mathcal{N},\varepsilon)^2 \tag{5.36}$$

This proves Assertion (1). To prove Assertion (2), let us first note that

$$D_\hbar^j [g_\ell(\mathcal{L}_\omega(\xi); \langle \omega, q\rangle \hbar, \varepsilon, \hbar)]^n = P_{n,j}\left(g_\ell, \frac{Dg_\ell}{D\hbar}, \dots, \frac{D^j g_\ell}{D\hbar^j}\right). \tag{5.37}$$

where $P_{n,j}(x_1, \dots, x_j)$ is a homogeneous polynomial of degree n with n^j terms. Explicitly:

$$P_{n,j}\left(g_\ell, \frac{Dg_\ell}{D\hbar}, \dots, \frac{D^j g_\ell}{D\hbar^j}\right) = \sum_{j=1}^{n} {g_\ell}^{n-j} \prod_{\substack{k=1 \\ j_1+\dots+j_k=j}}^{j} \frac{D^{j_k} g_\ell}{D\hbar^{j_k}}.$$

Now (5.32), (5.36) and Proposition 3.11 (3) entail:

$$\begin{aligned}
&|D_\hbar^j [g_\ell(\mathcal{L}_\omega(\xi); \langle \omega, q\rangle \hbar, \varepsilon, \hbar)]^n|_{\rho_\ell} \le n^j |g|_{\rho_\ell}^{n-j} \prod_{\substack{k=1 \\ j_1+\dots+j_k=j}}^{j} 2(j_k+1)\,(2|q|)^{j_k}\, \theta_{\ell,j_k}(\mathcal{N},\varepsilon)^2 \\
&\le 2n^j (\theta_\ell(\mathcal{N},\varepsilon))^{n-j} [2(j+1)|q|]^j \theta_{\ell,j}(\mathcal{N},\varepsilon)^{2j}.
\end{aligned}$$

This concludes the proof of the Lemma. $\square$

To conclude the proof of the theorem, we must estimate the $\|\cdot\|_{\rho_{\ell+1},k}$ norm of the derivatives $\partial_\hbar^\gamma \mathcal{W}_{\ell,n}(\mathcal{L}_\omega(\xi), x; \varepsilon, \hbar)$. Obviously:

$$\|\mathcal{W}_\ell(\xi, x; \varepsilon, \hbar)\|_{\rho_\ell+1,k} \le \sum_{n=0}^{\infty} \|\mathcal{W}_{\ell,n}(\xi, x; \varepsilon, \hbar)\|_{\rho_{\ell+1},k}. \tag{5.38}$$

For $n = 0$:

$$\begin{aligned}
&\|\mathcal{W}_{\ell,0}(\xi, x; \varepsilon, \hbar)\|_{\rho_{\ell+1},k} \le \gamma \sum_{\gamma=0}^{k} \int_{\mathbb{R}\times\mathbb{R}^l} |\partial_\hbar^\gamma \widehat{\mathcal{W}}_{\ell,0}(p,s;\cdot)||s|^\tau \mu_{k-\gamma}(p\omega, s)\, e^{\rho_{\ell+1}(|p|+|s|)}\, d\lambda(p,s) \\
&\le \gamma \sum_{\gamma=0}^{k} \int_{\mathbb{R}\times\mathbb{R}^l} |\partial_\hbar^\gamma \widehat{\mathcal{V}}_{\ell,0}(p,s;\cdot)||s|^\tau \mu_{k-\gamma}(p\omega, s)\, e^{\rho_{\ell+1}(|p|+|s|)}\, d\lambda(p,s) \le \gamma \frac{\tau^\tau}{(ed_\ell)^\tau} \|\mathcal{V}_\ell\| \rho_{\ell,k}
\end{aligned}$$

where the inequality follows again by the standard majorization

$$e^{\rho_{\ell+1}(|p|+|s|)} = e^{\rho_\ell(|p|+|s|)} e^{-d_\ell(|p|+|s|)}, \quad \sup_{s\in\mathbb{R}^l} [|s|^\tau e^{-d_\ell |s|}] \le \gamma \frac{\tau^\tau}{(ed_\ell)^\tau}$$

on account of the small denominator estimate (1.25). For $n > 0$ we can write, on account of (2.5,2.6):

$$\|\mathcal{W}_{\ell,n}(\xi,x;\cdot)\|_{\rho_{\ell+1},k} = \sum_{\gamma=0}^{k}\int_{\mathbb{R}\times\mathbb{R}^l} |\partial_\hbar^\gamma \widehat{\mathcal{W}}_{\ell,n}(p,s;\cdot)||s|^\tau \mu_{k-\gamma}(p\omega,s)\, e^{\rho_{\ell+1}(|p|+|s|)}\, d\lambda(p,s) \le$$

$$\le \gamma \frac{\tau^\tau}{(ed_\ell)^\tau}\sum_{\gamma=0}^{k}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\int_{\mathbb{R}^l}\mathcal{Q}(s,\cdot)e^{\rho_\ell|s|}\, d\nu(s)$$

where

$$\mathcal{Q}(s,\cdot) := \int_{\mathbb{R}} |[\partial_\hbar^{\gamma-j}\widehat{\mathcal{V}}_\ell(p;s;\cdot)] * [D^j_\hbar \widehat{g}_\ell^{*n}(p;\langle\omega,s\rangle\hbar,\cdot)]\mu_{k-\gamma}(p\omega,s)\, e^{\rho_\ell|p|}\, dp$$

Here $*$ denotes convolution with respect only to the p variable, and $\widehat{g}_\ell^{*in}(p,\zeta,\cdot)$ denotes the $n-$th convolution of $\widehat{g}_\ell$ with itself, i.e. the p-Fourier transform of g^n_ℓ. Now, by Assertion (3) of Proposition (3.11) and the above Lemma:

$$\begin{aligned}
&\int_{\mathbb{R}^l}\mathcal{Q}(s,\cdot)e^{\rho_\ell|s|}\, d\nu(s) = \\
&= \int_{\mathbb{R}\times\mathbb{R}^l} |[\partial_\hbar^{\gamma-j}\widehat{\mathcal{V}}_\ell(p;s;\cdot)] *_\xi [D^j_\hbar g_\ell^{*_\xi n}(p;\langle\omega,s\rangle\hbar,\cdot)]\mu_{k-\gamma}(p\omega,s)\, e^{\rho_\ell(|p|+|s|)}\, d\lambda(p,s) \\
&\le \int_{\mathbb{R}^l}\left[\int_{\mathbb{R}} |[\partial_\hbar^{\gamma-j}\widehat{\mathcal{V}}_\ell(p;s;\hbar)] * [D^j_\hbar \widehat{g}^{*n}(p;\langle\omega,s\rangle\hbar,\cdot)]|\mu_{k-\gamma}(p\omega,s)\, e^{\rho_\ell|p|}\, dp\right] e^{\rho_\ell|s|}\, d\nu(s) \\
&\le 2A(j)^j\theta_\ell(\mathcal{N},\varepsilon)^{n-j}\int_{\mathbb{R}^l}\int_{\mathbb{R}} |\partial_\hbar^{\gamma-j}\widehat{\mathcal{V}}_\ell(p;s;\cdot)|\mu_{k-\gamma}(p\omega,s)\, e^{\rho_\ell|p|}|s|^j e^{\rho_\ell|s|}\; dpd\nu(s),
\end{aligned}$$

with

$$A(j) := 2n(j+1)\theta_{\ell,j}(\mathcal{N},\varepsilon)^2.$$

This yields, with δ_ℓ defined by (5.13):

$$\begin{aligned}
&\|\mathcal{W}_{\ell,n}(\xi,x;\cdot)\|_{\rho_\ell+1,k} \le \gamma\frac{\tau^\tau}{(ed_\ell)^\tau}\sum_{\gamma=0}^{k}\int_{\mathbb{R}\times\mathbb{R}^l} |\partial_\hbar^\gamma \widehat{\mathcal{W}}_{\ell,n}(p,s;\cdot)\mu_{k-\gamma}(p\omega,s)\, e^{\rho_\ell(|p|+|s|)}\, d\lambda(p,s) \le \\
&\le \frac{\gamma\tau^\tau(k+1)(2A(k))^k}{(ed_\ell)^\tau}\theta_\ell(\mathcal{N},\varepsilon)^{n-j}\sum_{\gamma=0}^{k}\int_{\mathbb{R}\times\mathbb{R}^l} |\partial_\hbar^\gamma\widehat{\mathcal{V}}_\ell(p;s;\cdot)|\cdot\mu_{k-\gamma}(p\omega,s)\, e^{\rho_\ell|p|}|s|^j e^{\rho_\ell|s|}\; d\lambda(p,s) \\
&\le \frac{\gamma\tau^\tau(k+1)(2A(k))^k}{(ed_\ell)^\tau}\frac{k^k}{(e\delta_\ell)^k}\theta_\ell(\mathcal{N},\varepsilon)^{n-j}\sum_{\gamma=0}^{k}\int_{\mathbb{R}^l}\int_{\mathbb{R}} |\partial_\hbar^\gamma\widehat{\mathcal{V}}_\ell(p;s;\cdot)|\mu_{k-\gamma}(p\omega,s)e^{\rho|p|}e^{\rho|s|}\, d\lambda(p,s) \\
&\le \gamma\frac{\tau^\tau}{(ed_\ell)^\tau}\frac{(k+1)k^k}{(e\delta_\ell)^k}2(2n)^k(\theta_\ell(\mathcal{N},\varepsilon))^{n-j}(k+1)^k\theta^{2k}_{\ell,k}\|\mathcal{V}_\ell\|_{\rho,k}.
\end{aligned}$$

Therefore, by (5.38):

$$
\begin{aligned}
&\|\mathcal{W}_\ell(\xi;x;\varepsilon,\hbar)\|_{\rho_{\ell+1},k} \le \sum_{n=0}^{\infty} \mathcal{W}_{\ell,n}(\xi;x;\varepsilon,\hbar)\|_{\rho_{\ell+1},k} \le \\
&\le \gamma\frac{\tau^\tau}{(ed_\ell)^\tau}\|\mathcal{V}_\ell\|_{\rho_\ell,k}\left[1+\frac{2^{k+1}(k+1)^{k+1}k^k}{(e\delta_\ell)^k}\theta_{\ell,k}^{2k}\sum_{n=1}^{\infty} n^k(\theta_\ell(\mathcal{N},\varepsilon))^{n-j}\right] \\
&\le \gamma\frac{\tau^\tau}{(ed_\ell)^\tau}\|\mathcal{V}_\ell\|_{\rho_\ell,k}\left[1+\frac{2^{k+1}(k+1)^{k+1}k^k}{(e\delta_\ell)^k}\theta_{\ell,k}^{2k-j}\sum_{n=1}^{\infty} n^k(\theta_\ell(\mathcal{N},\varepsilon))^{n}\right] \\
&\le \gamma\frac{\tau^\tau}{(ed_\ell)^\tau}\|\mathcal{V}_\ell\|_{\rho_\ell,k}\left[1+\frac{2^{k+1}(k+1)^{2(k+1)}k^k}{(e\delta_\ell)^k[(1-\theta_\ell(\mathcal{N},\varepsilon)^{k+1}]}\theta_{\ell,k}^{k+1}\right].
\end{aligned}
$$

because $j \le k$, and

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^k x^n \le \sum_{n=1}^{\infty}(n+1)\cdots(n+k)x^n = \frac{d^k}{dx^k}\sum_{n=1}^{\infty} x^{n+k} \\
&= \frac{d^k}{dx^k}\frac{x^{k+1}}{1-x} = (k+1)!\sum_{j=0}^{k+1}\binom{k+1-j}{j}\frac{x^{k+1-j}}{(1-x)^j} \le \frac{2^{k+1}(k+1)!}{(1-x)^{k+1}}.
\end{aligned}
$$

By the Stirling formula this concludes the proof of the Theorem. □

5.2. **Towards KAM iteration.** Let us now prove the estimate which represents the starting point of the KAM iteration:

Theorem 5.5. *Let $\mathcal{F}_\ell$ and V_ℓ be as in Theorem 5.3, and let W_ℓ be the solution of the homological equation (5.1) as constructed and estimated in Theorem 5.3. Let (5.18) hold and let furthermore*

$$
|\varepsilon| < \overline{\varepsilon}_\ell, \quad \overline{\varepsilon}_\ell := \left(\frac{d_\ell}{\|\mathcal{W}_\ell\|_{\rho_{\ell+1},k}}\right)^{2^{-\ell}}. \tag{5.39}
$$

Then we have:

$$
e^{i\varepsilon_\ell W_\ell/\hbar}(\mathcal{F}_\ell(L_\omega)+\varepsilon_\ell V_\ell)e^{-i\varepsilon_\ell W_\ell/\hbar} = (\mathcal{F}_\ell+\varepsilon_\ell N_\ell)(L_\omega)+\varepsilon_\ell^2 V_{\ell+1,\varepsilon} \tag{5.40}
$$

where, $\forall\, 0 < 2d_\ell < \rho_\ell$ and $k = 0, 1, \dots$:

$$
\|V_{\ell+1,\varepsilon}\|_{\rho_\ell-2d_\ell,k} \le C(\ell,k,\varepsilon)\frac{\|\mathcal{V}_\ell\|_{\rho_\ell,k}^2}{1-|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}\|_{\rho_\ell,k}/d_\ell} \tag{5.41}
$$

$$
C(\ell,k,\varepsilon) := \frac{(k+1)^2 4^{2k}}{(ed_\ell)^3}A(\ell,k.\varepsilon)\left[2+|\varepsilon_\ell|\frac{(k+1)4^k}{(ed_\ell)^2}A(\ell,k.\varepsilon)\|\mathcal{V}_\ell\|_{\rho_\ell,k}\right] \tag{5.42}
$$

Here $A(\ell,k,\varepsilon)$ is defined by (5.28).

Remark 5.6. We will verify in the next section (Remark 6.26 below) that (5.39) is actually fulfilled for $|\varepsilon| < 1/|\mathcal{V}|_\rho$.

Proof. To prove the theorem we need an auxiliary result, namely:

Lemma 5.7. *For $\ell = 0, 1, \dots$ let $\rho_\ell > 0, \rho_0 := \rho$, $A \in J_k(\rho)$, $W_\ell \in J_k(\rho_\ell)$, $k = 0, 1, \dots$. Let $W_\ell^* = W_\ell$, and define:*

$$A_\varepsilon(\hbar) := e^{i\varepsilon_\ell W_\ell/\hbar} A e^{-i\varepsilon_\ell W/\hbar}. \tag{5.43}$$

Then, for $|\varepsilon| < (d'_\ell/\|\mathcal{W}\|_{\rho_{\ell+1},k})^{2^{-\ell}}$, and $\forall\, 0 < d'_\ell < \rho_\ell$, $k = 0, 1, \dots$:

$$\|A_\varepsilon(\hbar)\|_{\rho - d'_\ell, k} \le \frac{(k+1)4^k}{ed'_\ell} \frac{\|\mathcal{A}\|_{\rho_\ell,k}}{1 - |\varepsilon_\ell| \|\mathcal{W}\|_{\rho_{\ell+1},k}/d'_\ell} \tag{5.44}$$

Proof. Since the operators W_ℓ and A are bounded, there is $\varepsilon_0 > 0$ such that the commutator expansion for $A_\varepsilon(\hbar)$:

$$A_\varepsilon(\hbar) = \sum_{m=0}^\infty \frac{(i\varepsilon_\ell)^m}{\hbar^m m!} [W_\ell, [W_\ell, \dots, [W_\ell, A] \dots]$$

is norm convergent for $|\varepsilon| < \varepsilon_0$ if $\hbar \in]0, 1[$ is fixed. The corresponding expansion for the symbols is

$$\mathcal{A}_\varepsilon(\hbar) = \sum_{m=0}^\infty \frac{(\varepsilon_\ell)^m}{m!} \{\mathcal{W}_\ell, \{\mathcal{W}, \dots, \{\mathcal{W}_\ell, \mathcal{A}\}_M \dots\}_M$$

Now we can apply once again Corollary 3.13. We get, with the same abuse of notation of Theorem 4.1:

$$\frac{1}{m!} \|\{\mathcal{W}_\ell, \{\mathcal{W}_\ell, \dots, \{\mathcal{W}_\ell, \mathcal{A}\}_M \dots\}_M\|_{\rho - d'_\ell, k} \le \frac{(k+1)4^k}{ed_1} \left(\frac{\|\mathcal{W}_\ell\|_{\rho_\ell,k}}{d'_\ell}\right)^m \|\mathcal{A}\|_{\rho_\ell,k} \tag{5.45}$$

Therefore

$$\|A_\varepsilon(\hbar)\|_{\rho_\ell - d'_\ell, k} \le \frac{(k+1)4^k}{ed'_\ell} \|\mathcal{A}\|_{\rho_\ell,k} \sum_{m=0}^\infty |\varepsilon|^m [\|\mathcal{W}\|_{\rho_{\ell+1},k}/d'_\ell]^m = \frac{(k+1)4^k}{ed'_\ell} \frac{\|\mathcal{A}\|_{\rho_\ell,k}}{1 - |\varepsilon_\ell| \|\mathcal{W}\|_{\rho_{\ell+1},k}/d'_\ell}$$

and this concludes the proof. □

W_ℓ solves the homological equation (5.1). Then by Theorem 5.3 $W_\ell = W_\ell^* \in J_k(\rho_\ell - d_\ell)$, $k = 0, 1, \dots$; in turn, by Assertion (3) of Corollary 3.8 the unitary operator $e^{i\varepsilon_\ell W_\ell/\hbar}$ leaves $H^1(\mathbb{T}^l)$ invariant. Therefore the unitary image of H_ε under $e^{i\varepsilon_\ell W/\hbar}$ is the real-holomorphic operator family in $L^2(\mathbb{T}^l)$

$$\varepsilon \mapsto S_\varepsilon := e^{i\varepsilon_\ell W_\ell/\hbar} (\mathcal{F}_\ell(L_\omega) + \varepsilon_\ell V_\ell) e^{-i\varepsilon\_ellW/\hbar}, \quad D(S(\varepsilon)) = H^1(\mathbb{T}^l) \tag{5.46}$$

Computing its Taylor expansion at $\varepsilon_\ell = 0$ with second order remainder we obtain:

$$S_\varepsilon u = \mathcal{F}_\ell(L_\omega)u + \varepsilon_\ell N_\ell(L_\omega)u + \varepsilon_\ell^2 V_{\ell+1,\varepsilon} u, \quad u \in H^1(\mathbb{T}^l) \tag{5.47}$$

$$V_{\ell+1,\varepsilon} = \frac{1}{2} \int_0^{\varepsilon_\ell} (\varepsilon_\ell - t) e^{itW_\ell/\hbar} \left(\frac{[N_\ell, W_\ell]}{i\hbar} + \frac{[W_\ell, V_\ell]}{i\hbar} + t \frac{[W_\ell, [W_\ell, V_\ell]]}{(i\hbar)^2} \right) e^{-itW_\ell/\hbar} \, dt \tag{5.48}$$

To see this, first remark that $S_0 = \mathcal{F}(L_\omega)$. Next, we compute, as equalities between continuous operators in $L^2(\mathbb{T}^l)$:

$$
\begin{aligned}
&S'_\varepsilon = e^{i\varepsilon_\ell W/\hbar}([\mathcal{F}_\ell(L_\omega), W_\ell]/i\hbar + V_\ell + \varepsilon_\ell[V,W]/i\hbar)e^{-i\varepsilon_\ell W/\hbar} = \\
&e^{i\varepsilon_\ell W/\hbar}(N_\ell + \varepsilon_\ell[V_\ell, W_\ell]/i\hbar)e^{i\varepsilon_\ell W_\ell/\hbar}; \qquad S'_0 = N_\ell \\
&S''_\varepsilon = e^{i\varepsilon_\ell W_\ell/\hbar}([N_\ell, W_\ell]/i\hbar + [V_\ell, W_\ell]/i\hbar + \varepsilon_\ell[W_\ell,[W_\ell, V_\ell]]/(i\hbar)^2)e^{-i\varepsilon_\ell W_\ell/\hbar},
\end{aligned}
$$

and this proves (5.47) by the second order Taylor's formula with remainder:

$$
S_\varepsilon = S(0) + \varepsilon S'_0 + \frac{1}{2}\int_0^{\varepsilon_\ell}(\varepsilon - t)S''(t), dt
$$

The above formulae obviously yield

$$
\|V_{l+1,\varepsilon}\| \leq |\varepsilon_\ell|^2 \max_{0\leq|t|\leq|\varepsilon_\ell|}\|S''(t)\| \tag{5.49}
$$

Set now:

$$
R_{\ell+1,\varepsilon} := [N_\ell, W_\ell]/i\hbar + [V_\ell, W_\ell]/i\hbar + \varepsilon_\ell[W_\ell,[W_\ell, V_\ell]]/(i\hbar)^2 \tag{5.50}
$$

$R_{\ell+1,\varepsilon}$ is a continuous operator in L^2, corresponding to the symbol

$$
\mathcal{R}_{\ell+1,\varepsilon}(\mathcal{L}_\omega(\xi), x; \hbar) = \{\mathcal{N}_\ell, \mathcal{W}_\ell\}_M + \{\mathcal{V}_\ell, \mathcal{W}_\ell\}_M + \varepsilon_\ell\{\mathcal{W}_\ell, \{\mathcal{W}_\ell, \mathcal{V}_\ell\}_M\}_M \tag{5.51}
$$

Let us estimate the three terms individually. By Theorems 5.3 and 3.11 we can write, with $A(\ell, k, \varepsilon)$ given by (5.28):

$$
\begin{aligned}
&\|[N_\ell, W_\ell]/i\hbar\|_{\rho_\ell - d_\ell, k} \leq \|\{\mathcal{N}_\ell, \mathcal{W}_\ell\}_M\|_{\rho_\ell - d_\ell, k} \leq \frac{(k+1)4^k}{(ed_\ell)^2}\|\mathcal{W}_\ell\|_{\rho_{\ell+1},k}\|\mathcal{N}_\ell\|_{\rho_\ell,k} \\
&\leq \frac{(k+1)4^k}{(ed)^2}A(\ell,k,\varepsilon)\|\mathcal{V}_\ell\|^2_{\rho_\ell,k} \\
&\|[V_\ell, W_\ell]/i\hbar\|_{\rho_\ell - d_\ell, k} \leq \|\{\mathcal{V}_\ell, \mathcal{W}_\ell\}_M\|_{\rho_\ell - d_\ell, k} \leq \frac{(k+1)4^k}{(ed_\ell)^2}\|\mathcal{V}_\ell\|_{\rho_\ell,k}\|\mathcal{W}_\ell\|_{\rho_{\ell+1},k} \leq \\
&\leq \frac{(k+1)4^k}{(ed_\ell)^2}A(\ell, k.\varepsilon)\|\mathcal{V}_\ell\|^2_{\rho_\ell,k} \\
&\|[W_\ell,[W_\ell, V_\ell]]/(i\hbar)^2\|_{\rho_\ell - d_\ell, k} \leq \|\{\mathcal{W}_\ell, \{\mathcal{W}_\ell, \mathcal{V}_\ell\}_M\}_M\|_{\rho_\ell - d_\ell, k} \leq \frac{(k+1)^2 4^{2k}}{(ed_\ell)^4}\|\mathcal{W}_\ell\|^2_{\rho_{\ell+1},k}\|\mathcal{V}_\ell\|_{\rho_\ell,k} \\
&\leq \frac{(k+1)^2 4^{2k}}{(ed_\ell)^4}A(\ell,k,\varepsilon)^2\|\mathcal{V}_\ell\|^3_{\rho_\ell,k}
\end{aligned}
$$

We can now apply Lemma 5.7, which yields:

$$\|e^{i\varepsilon_\ell W_\ell/\hbar}[N_\ell, W_\ell]e^{-i\varepsilon_\ell W_\ell/\hbar}/i\hbar\|_{\rho_\ell - d_\ell - d'_\ell, k} \leq \frac{(k+1)^2 4^{2k}}{(ed_\ell)^2 ed'_\ell}\Xi(\ell, k)$$

$$\|e^{i\varepsilon_\ell W_\ell/\hbar}[V_\ell, W_\ell]e^{-i\varepsilon_\ell W_\ell/\hbar}/i\hbar\|_{\rho_\ell - d_\ell - d'_\ell, k} \leq \frac{(k+1)^2 4^{2k}}{(ed_\ell)^2 ed'_\ell}\Xi(\ell, k)$$

$$\|e^{i\varepsilon_\ell W_\ell/\hbar}[W_\ell, [W_\ell, V_\ell]]e^{-i\varepsilon_\ell W_\ell/\hbar}/(i\hbar)^2\|_{\rho_\ell - d_\ell - d'_\ell, k} \leq \frac{(k+1)^3 4^{3k}}{(ed_\ell)^4 ed'_\ell}\Xi_1(\ell, k)$$

where

$$\Xi(\ell, k) := A(\ell, k) \cdot \frac{\|\mathcal{V}_\ell\|^2_{\rho_\ell, k}}{1 - |\varepsilon_\ell|\|\mathcal{W}\|_{\rho_{\ell+1}, k}/d'_\ell} \tag{5.52}$$

$$\Xi_1(\ell, k) = A(\ell, k, \varepsilon)^2 \cdot \frac{\|\mathcal{V}\|^3_{\rho_\ell, k}}{1 - |\varepsilon_\ell|\|\mathcal{W}\|_{\rho_{\ell+1}, k}/d'_\ell} \tag{5.53}$$

Therefore, summing the three inequalities we get

$$\|V_{\ell+1,\varepsilon}\|_{\rho_\ell - d_\ell - d'_\ell, k} \leq \frac{(k+1)^2 4^{2k}}{(ed_\ell)^2 ed'_\ell} A(\ell, k, \varepsilon) \times$$
$$\times \frac{\|\mathcal{V}_\ell\|^2_{\rho_\ell, k}}{1 - |\varepsilon_\ell|\|\mathcal{W}_\ell\|_{\rho_{\ell+1}, k}/d'_\ell}\left[2 + |\varepsilon_\ell|\frac{(k+1)4^k}{(ed_\ell)^2} A(\ell, k, \varepsilon)\|\mathcal{V}_\ell\|_{\rho_\ell, k}\right]$$

If we choose $d'_\ell = d_\ell$ this is (5.41) on account of Theorem 5.3. This concludes the proof of Theorem 5.5. □

6. Recursive estimates

Consider the ℓ-th step of the KAM iteration. Summing up the results of the preceding Section we can write:

- $S_{\ell,\varepsilon} := e^{i\varepsilon_\ell W_\ell/\hbar} \cdots e^{i\varepsilon_2 W_1/\hbar} e^{i\varepsilon W_0/\hbar}(\mathcal{F}(L_\omega) + \varepsilon V)e^{-i\varepsilon W_0/\hbar} e^{-i\varepsilon_2 W_1/\hbar} \cdots e^{-i\varepsilon_\ell W_\ell/\hbar}$
  $= e^{i\varepsilon_\ell W_\ell/\hbar}(\mathcal{F}_{\ell,\varepsilon}(L_\omega) + \varepsilon^{2^\ell} V_{\ell,\varepsilon})e^{-i\varepsilon_\ell W_\ell/\hbar} = \mathcal{F}_{\ell+1,\varepsilon}(L_\omega) + \varepsilon_{\ell+1} V_{\ell+1,\varepsilon}$,
- $\mathcal{F}_{\ell,\varepsilon}(L_\omega) = \mathcal{F}(L_\omega) + \displaystyle\sum_{k=1}^{\ell-1} \varepsilon_k N_k(L_\omega), \quad [\mathcal{F}_\ell(L_\omega), W_\ell]/i\hbar + V_{\ell,\varepsilon} = N_\ell(L_\omega, \varepsilon)$
- $V_{\ell+1,\varepsilon} = \dfrac{1}{2}\displaystyle\int_0^{\varepsilon_\ell} (\varepsilon_\ell - t)e^{itW_\ell/\hbar} R_{\ell+1,t} e^{-itW_\ell/\hbar}\, dt$
- $R_{\ell+1,\varepsilon} := [N_\ell, W_\ell]/\hbar + [W_\ell, V_{\ell,\varepsilon}]/\hbar + \varepsilon_\ell [W_\ell, [W_\ell, V_{\ell,\varepsilon}]]/\hbar^2$

We now proceed to obtain recursive estimates for the above quantities in the $\|\cdot\|_{\rho_\ell,k}$ norm. Consider (5.41) and denote:

$$\Psi(\ell,k)=\frac{(k+1)^2 4^k}{(ed_\ell)^3}\Pi(\ell,k);\quad \Pi(\ell,k):=\frac{[2(k+1)^2]^{k+1}k^k}{e^k\delta_\ell^k} \tag{6.1}$$

$$P(\ell,k,\varepsilon):=\frac{\theta_{\ell,k}(\mathcal{N},\varepsilon)^{k+1}}{[1-\theta_\ell(\mathcal{N},\varepsilon)]^{k+1}} \tag{6.2}$$

where $\theta_{\ell,k}(\mathcal{N},\varepsilon)$ is defined by (5.16). (6.1) and (6.2) yield

$$A(\ell,k,\varepsilon)=\gamma\frac{\tau^\tau}{(ed_\ell)^\tau}[1+\Pi(\ell,k)P(\ell,k,\varepsilon)]. \tag{6.3}$$

Set furthermore:

$$E(\ell,k,\varepsilon):=\frac{\Psi(\ell,k)B(\ell,k,\varepsilon)[2+|\varepsilon_\ell|e\Psi(\ell,k)A(\ell,k,\varepsilon)\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k}]}{1-|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k}/d_\ell} \tag{6.4}$$

Then we have:

Lemma 6.1. *Let:*

$$|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k}/d_\ell<1. \tag{6.5}$$

Then:

$$\|V_{\ell+1,\varepsilon}\|_{\rho_{\ell+1},k}\le E(\ell,k,\varepsilon)\|V_{\ell,\varepsilon}\|^2_{\rho_\ell,k} \tag{6.6}$$

Remark 6.2. The validity of the assumption (6.5) is to be verified in Proposition 6.3 below.

Proof. Since $d_\ell<1$, by (5.42), (6.1) and (6.3) we can write:

$$C(\ell,k,\varepsilon)\le\Psi(\ell,k)A(\ell,k,\varepsilon))\,[2+|\varepsilon_\ell|e\Psi(\ell,k)A(\ell,k,\varepsilon)\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k}] \tag{6.7}$$

and therefore, by (5.41):

$$\|V_{\ell+1,\varepsilon}\|_{\rho_\ell-2d_\ell,k}\le C(\ell,k,\varepsilon)\frac{\|\mathcal{V}_\ell\|^2_{\rho_\ell,k}}{1-|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}\|_{\rho_\ell,k}/d_\ell}$$

$$\le\frac{\Psi(\ell,k)A(\ell,k,\varepsilon)\,[2+|\varepsilon_\ell|e\Psi(\ell,k)A(\ell,k,\varepsilon)\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k}]}{1-|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k}/d_\ell}\|\mathcal{V}_\ell\|^2_{\rho_\ell,k}=E(\ell,k,\varepsilon)\|\mathcal{V}_\ell\|^2_{\rho_\ell,k}.$$

This yields (6.6) and proves the Lemma. □

Now recall that the sequence $\{\rho_j\}$ is decreasing. Therefore:

$$\|\mathcal{N}_{j,\varepsilon}\|_{\rho_\ell,k}\le\|\mathcal{N}_{j,\varepsilon}\|_{\rho_j,k}=\|\overline{\mathcal{V}}_{j,\varepsilon}\|_{\rho_j,k}\le\|\mathcal{V}_{j,\varepsilon}\|_{\rho_j,k},\qquad j=0,\dots,\ell-1. \tag{6.8}$$

At this point we can specify the sequence $d_\ell,\ell=1,2,\dots$, setting:

$$d_\ell:=\frac{\rho}{(\ell+1)^2},\qquad \ell=0,1,2,\dots \tag{6.9}$$

Remark that (6.9) yields

$$d-\sum_{\ell=0}^\infty d_\ell=\rho-\frac{\pi^2}{6}>\frac{\rho}{2}.$$

as well as the following estimate

$$\Pi(\ell,k) \le \frac{[2(k+1)^2]^{k+1}}{e^k\rho^k} \tag{6.10}$$

We are now in position to discuss the convergence of the recurrence (6.6).

Proposition 6.3. *Let:*

$$|\varepsilon| < \varepsilon^*(\gamma,\tau,k) := \frac{1}{e^{24(3+2\tau)}(k+2)^{2\tau}\|\mathcal{V}\|_{\rho,k}} \tag{6.11}$$

$$\rho > \lambda(k) := 1 + 8\gamma\tau^\tau[2(k+1)^2]. \tag{6.12}$$

Then the following estimate holds:

$$\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k} \le \left(e^{8(3+2\tau)}\|V_0\|_{\rho,k}\right)^{2^\ell} = \left(e^{8(3+2\tau)}\|\mathcal{V}_0\|_{\rho,k}\right)^{2^\ell}, \quad \ell = 0,1,2,\dots \quad V_0 := V. \tag{6.13}$$

Proof. We proceed by induction. The assertion is true for $\ell = 0$. Now assume inductively:

$$|\varepsilon_j|\|\mathcal{V}_{j,\varepsilon}\|_{\rho_j,k} \le (k+2)^{-2\tau(j+1)}, \qquad 0 \le j \le \ell. \tag{6.14}$$

Out of (6.14) we prove the validity of (6.13) and of (6.5); to complete the induction it will be enough to show that (6.13) implies the validity of (6.14) for $j = \ell+1$.

Let us first estimate $\theta_\ell(\mathcal{N},\varepsilon)$ as defined by (5.15) assuming the validity of (6.14) . We obtain:

$$\theta_\ell(\mathcal{N},\varepsilon) \le \theta_{\ell,k}(\mathcal{N},\varepsilon) \le \sum_{s=0}^{\ell-1}|\varepsilon_s|\|\mathcal{V}\|_{\rho_s,k}/d_s = \frac{1}{\rho}\sum_{s=0}^{\ell-1}(s+1)^2(k+2)^{-2\tau(s+1)} =$$

$$\frac{1}{4\rho}\frac{d^2}{d\tau^2}\sum_{s=0}^{\ell-1}(k+2)^{-2\tau(s+1)} = \frac{1}{4\rho}\frac{d^2}{d\tau^2}[(k+2)^{-2\tau}\frac{1-(k+2)^{-2\tau\ell}}{1-(k+2)^{-2\tau}} \le \frac{1}{\rho}(k+2)^{-2} \le \frac{1}{\rho}$$

because $\tau > l-1 \ge 1$. Now $\rho > 1$ entails that

$$\frac{1}{1-\theta_\ell} < \frac{\rho}{\rho-1}. \tag{6.15}$$

Hence we get, by (6.2) and (5.16), the further $(\ell,\varepsilon)-$independent estimate:

$$P(\ell,k,\varepsilon) \le \frac{\rho^{k+1}}{(\rho-1)^{k+1}}\left((k+2)^2\rho\right)^{-k-1} = \left(\frac{1}{(\rho-1)(k+2)^2}\right)^{k+1}. \tag{6.16}$$

whence, by (6.3):

$$A(\ell,k,\varepsilon) \le \gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}[1+[2(k+1)^2]^{k+1}\left[(\rho-1)(k+2)^2\right]^{-(k+1)}(e\rho^3)^{-k}]$$

$$\le \gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}[1+\frac{2}{(\rho-1)^{k+1}}(e\rho^3)^{-k}]. \tag{6.17}$$

Upon application of the inductive assumption we get:

$$|\varepsilon_\ell|\Psi_{\ell,k}A(\ell,k,\varepsilon)\|\mathcal{V}\|_{\rho_\ell,k}/d_\ell \le \frac{4^k[2(k+1)^2]^{k+3}}{e^{k+3}\rho^{k+4}}(\ell+1)^{2\tau+8}|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}\|_{\rho_\ell,k}$$
$$\le \gamma\frac{\tau^\tau(\ell+1)^{2(\tau+4)}}{(e\rho)^\tau}[1+\frac{2}{(\rho-1)^{k+1}}(e\rho^3)^{-k}]\frac{4^k[2(k+1)^2]^{k+3}}{e^{k+3}\rho^{k+4}}(k+2)^{-2(\ell+1)\tau}$$
$$\le \left(\frac{2(\tau+4)}{2\tau\ln(k+2)}\right)^{2(\tau+4)}(k+2)^{-\frac{4(\tau+4)}{2\tau\ln(k+2)}}\frac{4^k[2(k+1)^2]^{k+3}}{e^{k+3}\rho^{k+4}}\frac{\gamma\tau^\tau}{(e\rho)^\tau}[1+\frac{2}{(\rho-1)^{k+1}}(e\rho^3)^{-k}]$$

because

$$\sup_{\ell\ge 0}(\ell+1)^{2(\tau+4)}(k+2)^{-2(\ell+1)\tau} = \left(\frac{2(\tau+4)}{2\tau\ln(k+2)}\right)^{2(\tau+4)}(k+2)^{-\frac{4(\tau+4)}{2\tau\ln(k+2)}}.$$

Hence:

$$|\varepsilon_\ell|\Psi_{\ell,k}A(\ell,k,\varepsilon)\|\mathcal{V}\|_{\rho_\ell,k}/d_\ell \le \frac{1}{2e} \tag{6.18}$$

provided

$$\rho\ge\lambda(k);\qquad \lambda(k)=1+8\gamma\tau^\tau[2(k+1)^2]. \tag{6.19}$$

Since $\Psi_{\ell,k}\ge 1$, if (6.19) holds, (6.18) a fortiori yields

$$|\varepsilon_\ell|A(\ell,k,\varepsilon)\|\mathcal{V}\|_{\rho_\ell,k}/d_\ell \le \frac{1}{2}.$$

Therefore, by (6.4):

$$E(\ell,k,\varepsilon)\le 3\Psi_{\ell,k}A(\ell,k,\varepsilon)\le 6\gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}\Psi_{\ell,k}$$

and (6.6) in turn entails:

$$\|\mathcal{V}_{\ell+1}\|_{\rho_\ell+1,k}\le\Phi_{\ell,k}\|\mathcal{V}_\ell\|^2_{\rho_\ell,k},\quad \Phi_{\ell,k}:=6\gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}\Psi_{\ell,k}.$$

This last inequality immediately yields

$$\|\mathcal{V}_{\ell+1}\|_{\rho_\ell,k}\le[\|\mathcal{V}\|_{\rho,k}]^{2^{\ell+1}}\prod_{m=0}^{\ell}\Phi^{2m}_{\ell-m,k}. \tag{6.20}$$

Now:

$$\Phi_{\ell,k}=6\gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}\frac{(k+1)^2 4^{2k}}{ed_\ell^3}\frac{[2(k+1)^2]^{k+1}}{e^{k+\tau}d_\ell^\tau\delta_\ell^k}\le\gamma\nu(k,\tau,\rho)(\ell+1)^{6+4\tau}$$

$$\nu(k,\tau,\rho):=6\frac{\tau^\tau 4^{2k}[2(k+1)^2]^{k+2}}{e^{k+\tau+1}\rho^{k+\tau+3}}\le 6\frac{\tau^\tau 4^{2k}[2(k+1)^2]^{k+2}}{e^{k+\tau+1}\lambda(k)^{k+\tau+3}}\le$$
$$\le 6\frac{\tau^\tau 4^{2k}[2(k+1)^2]^{k+2}}{e^{k+\tau+1}[8\gamma\tau^\tau 2(k+1)^2]^{k+\tau+3}}\le 6\left(\frac{2}{e}\right)^k\frac{1}{e^{\tau+1}\gamma^{k+\tau+3}[2(k+1)^2]^{\tau+1}}\le$$
$$\le\frac{6}{\gamma^{\tau+3}\tau^{\tau^2+2}(2e)^{\tau+1}}$$

Therefore

$$\gamma\nu(k,\tau,\rho) \le \frac{6}{\gamma^{\tau+2}\tau^{\tau^2+2}(2e)^{\tau+1}} < 1 \tag{6.21}$$

because $\tau > 1$ and $\gamma > 1$. As a consequence, since $\Phi_{j,k} \le \Phi_{\ell,k}, j = 1,\ldots,$ we get:

$$\prod_{m=1}^{\ell} \Phi^{2m}_{\ell+1-m,k} \le [\Phi_{\ell,k}]^{\ell(\ell+1)} \le [\gamma\nu(k,\tau,\rho)]^{\ell(\ell+1)}(\ell+1)^{(6+4\tau)\ell(\ell+1)} \le (\ell+1)^{(6+4\tau)\ell(\ell+1)}$$

Now $\ell(\ell+1) < 2^{\ell+1}$, $\forall\, \ell \in \mathcal{N}$. Hence we can write:

$$(\ell+1)^{(6+4\tau)\ell(\ell+1)} < [e^{(24+16\tau)}]^{2^{\ell+1}}.$$

The following estimate is thus established

$$\prod_{m=0}^{\ell} \Psi^{2m}_{\ell-m,k} \le [e^{8(3+2\tau)}]^{2^{\ell+1}}. \tag{6.22}$$

If we now define:

$$\mu := e^{8(3+2\tau)}, \qquad \mu_\ell := \mu^{2^\ell} \tag{6.23}$$

then (6.20) and (6.22) yield:

$$\|\mathcal{V}_{\ell+1,\varepsilon}\|_{\ell+1,k} \le [\mu_\ell \|\mathcal{V}_\ell\|_{\rho_\ell,k}]^2 \le [\|\mathcal{V}\|_{\rho,k}\,\mu]^{2^{\ell+1}} \tag{6.24}$$

$$\varepsilon_{\ell+1}\|\mathcal{V}_{\ell+1,\varepsilon}\|_{\ell+1,k} \le [\|\mathcal{V}\|_{\rho_\ell,k}\,\mu_\ell\varepsilon_\ell]^2 \le [\|\mathcal{V}\|_{\rho,k}\,\mu\varepsilon]^{2^{\ell+1}} \tag{6.25}$$

Let us now prove out of (6.24,6.25) that the condition (6.14) preserves its validity also for $j = \ell+1$. We have indeed, by the inductive assumption (6.14) and (6.24):

$$\begin{aligned}|\varepsilon_{\ell+1}|\mathcal{V}_{\ell+1,\varepsilon}\|_{\ell+1,k} &\le [\|\mathcal{V}\|_{\rho_\ell,k}\,\mu_\ell\varepsilon_\ell]^2 \le (k+2)^{-2\tau(\ell+1)}\varepsilon_\ell(\mu_\ell)^2\|\mathcal{V}\|_{\rho_\ell,k} \\ &\le (k+2)^{-2\tau(\ell+1)}\left[\varepsilon\mu^3\|\mathcal{V}\|_{\rho,k}\right]^{2^\ell} \le (k+2)^{-2\tau(\ell+2)}\end{aligned}$$

provided

$$|\varepsilon| < \frac{1}{\mu^3\|\mathcal{V}\|_{\rho,k}(k+2)^{2\tau}} = \frac{1}{e^{24(3+2\tau)}\|\mathcal{V}\|_{\rho,k}(k+2)^{2\tau}} := \varepsilon^*(\gamma,\tau,k) \tag{6.26}$$

where the last expression follows from (6.23). This proves (6.11), and concludes the proof of the Proposition. □

Theorem 6.4. *[Final estimates of W_ℓ, N_ℓ, V_ℓ]*
Let $\mathcal{V}$ fulfill Assumption (H2-H4). Then the following estimates hold, $\forall \ell \in \mathbb{N}$:

$$\varepsilon_\ell\|W_{\ell,\varepsilon}\|_{\rho_{\ell+1},k} \le \gamma\left(\frac{\tau}{e}\right)^\tau (\ell+1)^{2\tau}(1+8\gamma\tau^\tau[2(k+1)^2])^{-\tau}\cdot(\mu\varepsilon\|\mathcal{V}\|_\rho)^{2^\ell}. \tag{6.27}$$

$$\varepsilon_\ell\|N_{\ell,\varepsilon}\|_{\rho_\ell,k} \le \varepsilon_\ell\|\mathcal{V}_{\ell,\varepsilon}\|_{\rho_\ell,k} \le [\|\mathcal{V}\|_\rho\,\varepsilon\mu]^{2^\ell}. \tag{6.28}$$

$$\varepsilon_{\ell+1}\|V_{\ell+1,\varepsilon}\|_{\rho_{\ell+1},k} \leq [\|V\|_\rho\,\varepsilon\mu]^{2^{\ell+1}}. \tag{6.29}$$

Proof. Since $\mathcal{V}$ does not depend on $\hbar$, obviously $|\mathcal{V}\|_{\rho,k} \equiv \|\mathcal{V}\|_\rho$. Then formula (5.27) yields, on account of (6.17), (6.15), (6.19), (6.24), (6.25) and of the obvious inequalities $e\rho^{-3} < 1$, $\rho/(\rho-1) > 1$ when $\rho > \lambda(k)$:

$$\begin{aligned}\varepsilon_\ell\|W_{\ell,\varepsilon}\|_{\rho_\ell,k} &\leq \gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}[1+\frac{2}{(\rho-1)^{k+1}}(e\rho^3)^{-k}](\mu\varepsilon\|\mathcal{V}\|_\rho)^{2^\ell}\\ &\leq 2\gamma\frac{\tau^\tau(\ell+1)^{2\tau}}{(e\rho)^\tau}(\mu\varepsilon\|\mathcal{V}\|_\rho)^{2^\ell} \leq \gamma\left(\frac{\tau}{e}\right)^\tau(\ell+1)^{2\tau}(1+8\gamma\tau^\tau[2(k+1)^2])^{-\tau}\cdot(\mu\varepsilon\|\mathcal{V}\|_\rho)^{2^\ell}.\end{aligned}$$

because of the straightforward inequality

$$[1+\frac{2}{(\rho-1)^{k+1}}(e\rho^3)^{-k}] < 1$$

which in turn follows from $\gamma > 1$. This proves (6.27). Moreover, since $\mathcal{N}_{\ell,\varepsilon} = \overline{\mathcal{V}}_{\ell,\varepsilon}$, again by (6.24), (6.25):

$$\varepsilon_\ell\|\mathcal{N}_{\ell,\varepsilon}\|_{\rho_\ell,k} = \varepsilon_\ell\|\overline{\mathcal{V}}_{\ell,\varepsilon}\|_{\rho_\ell,k} \leq [\|\mathcal{V}\|_\rho\,\varepsilon\mu]^{2^\ell}.$$

The remaining assertion follows once more from (6.25). This concludes the proof of the Theorem. □

Remark 6.5. (6.27) yields, with $K := \gamma\left(\frac{\tau}{e}\right)^\tau(1+8\gamma\tau^\tau[2(k+1)^2])^{-\tau}$:

$$\varepsilon_\ell\frac{\|W_{\ell,\varepsilon}\|_{\rho_{\ell+1},k}}{d_\ell} \leq K\varepsilon^{2^\ell}(\ell+1)^{2(\tau+1)}\|\mathcal{V}\|_\rho^{2^\ell}$$

This yields:

$$|\varepsilon|\left(\frac{\|W_{\ell,\varepsilon}\|_{\rho_{\ell+1},k}}{d_\ell}\right)^{2^{-\ell}} \leq [K(\ell+1)^{2(\tau+1)}]^{2^{-\ell}}\|\mathcal{V}\|_\rho \to \|\mathcal{V}\|_\rho, \quad \ell\to\infty$$

so that (5.39) is actually fulfilled for $|\varepsilon| < \dfrac{1}{\|\mathcal{V}\|_\rho}$.

Corollary 6.6. *In the above assumptions set:*

$$U_{n,\varepsilon}(\hbar) := \prod_{s=0}^n e^{i\varepsilon_{n-s}W_{n-s,\varepsilon}}, \quad n = 0,1,\ldots. \tag{6.30}$$

Then:

(1) $U_{n,\varepsilon}(\hbar)$ *is a unitary operator in* $L^2(\mathbb{T}^l)$, *with*

$$U_{n,\varepsilon}(\hbar)^* = U_{n,\varepsilon}(\hbar)^{-1} = \prod_{s=0}^n e^{-i\varepsilon_s W_{s,\varepsilon}}$$

(2) *Let:*

$$S_{n,\varepsilon}(\hbar) := U_{n,\varepsilon}(\hbar)(\mathcal{L}_\omega + \varepsilon V)U_{n,\varepsilon}(\hbar)^{-1} \tag{6.31}$$

Then:

$$S_n = D_{n,\varepsilon}(\hbar) + \varepsilon_{n+1}V_{n+1,\varepsilon} \tag{6.32}$$

$$D_{n,\varepsilon}(\hbar) = L_\omega + \sum_{s=1}^{n} \varepsilon_s N_{s,\varepsilon} \tag{6.33}$$

The corresponding symbols are:

$$\mathcal{S}_n(\xi, x; \hbar) = \mathcal{D}_{n,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar) + \varepsilon_{n+1}V_{n+1,\varepsilon}(\mathcal{L}_\omega(\xi), x; \hbar) \tag{6.34}$$

$$\mathcal{D}_{n,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar) = \mathcal{L}_\omega(\xi) + \sum_{s=1}^{n} \varepsilon_s \mathcal{N}_{s,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar). \tag{6.35}$$

Here the operators $W_{s,\varepsilon}$, $N_{s,\varepsilon}$, $V_{\ell+1,\varepsilon}$ and their symbols $\mathcal{W}_{s,\varepsilon}$, $\mathcal{N}_{s,\varepsilon}$, $\mathcal{V}_{\ell+1,\varepsilon}$ fulfill the above estimates.

(3) *Let ε^* be defined as in (6.11). Remark that $\varepsilon^*(\cdot, k) > \varepsilon^*(\cdot, k+1)$, $k = 0, 1, \dots$. Then, if $|\varepsilon| < \varepsilon(k, \cdot)$:*

$$\lim_{n\to\infty} \mathcal{D}_{n,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar) = \mathcal{D}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar) \tag{6.36}$$

where in the convergence takes place in the $C^k([0,1]; C^\omega(\rho/2))$ topology, namely

$$\lim_{n\to\infty} \|\mathcal{D}_{n,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar) - \mathcal{D}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi), \hbar)\|_{\rho/2,k} = 0. \tag{6.37}$$

Proof. Since Assertions (1) and (2) are straightforward, we limit ourselves to the simple verification of Assertion (3). If $|\varepsilon| < \varepsilon^*(\cdot, k)$ then $\|V\|_{\rho,k}\mu\varepsilon < \Lambda < 1$. Recalling that $\|\cdot\|_{\rho,,k} \le \|\cdot\|_{\rho',k}$ whenever $\rho \le \rho'$, and that $\rho_\ell < \rho/2$, $\forall\, \ell \in \mathbb{N}$, (6.29) yields:

$$\begin{gathered}\varepsilon_{n+1}\|\mathcal{V}_{n+1,\varepsilon}\|_{\rho/2,k} \le \varepsilon_{n+1}\|\mathcal{V}_{n+1,\varepsilon}\|_{\rho_{n+1},k} \le \\ [\|V\|_{\rho,k}\mu\varepsilon]^{2^{n+1}} \to 0, \quad n \to \infty,\ k \text{ fixed.}\end{gathered}$$

In the same way, by (6.28):

$$\begin{gathered}\|\mathcal{N}_{n,\varepsilon}\|_{\rho/2,k} \le \|\mathcal{N}_{n,\varepsilon}\|_{\rho_n,k} = \|\overline{\mathcal{V}}_{n,\varepsilon}\|_{\rho_n,k} \le \|\mathcal{V}_{n,\varepsilon}\|_{\rho_n,,k} \le \\ [\|V\|_{\rho,k}\mu\varepsilon]^{2^n} \to 0, \quad n \to \infty,\ k \text{ fixed.} \to 0, \quad n \to \infty,\ k \text{ fixed.}\end{gathered}$$

This concludes the proof of the Corollary. $\square$

7. Convergence of the iteration and of the normal form.

Let us first prove the uniform convergence of the unitary transformation sequence as $n \to \infty$. Recall that $\varepsilon^*(\cdot,k) > \varepsilon^*(\cdot,k+1),\ k=0,1,\ldots$, and recall the abbreviation $\|\cdot\|_{\rho,0} := \|\cdot\|_\rho$. Define moreover:

$$\varepsilon^* := \varepsilon_0^* = \varepsilon^*(\gamma,\tau,0). \tag{7.1}$$

where $\varepsilon^*(\gamma,\tau,0)$ is defined by (6.26). Then:

Lemma 7.1. *Let $\hbar$ be fixed, and $|\varepsilon| < \varepsilon_0^*$. Consider the sequence $\{U_{n,\varepsilon}(\hbar)\}$ of unitary operators in $L^2(\mathbb{T}^l)$ defined by (6.30). Then there is a unitary operator $U_{\infty,\varepsilon}(\hbar)$ in $L^2(\mathbb{T}^l)$ such that*

$$\lim_{n\to\infty} \|U_{n,\varepsilon}(\hbar) - U_{\infty,\varepsilon}(\hbar)\|_{L^2\to L^2} = 0$$

Proof. Without loss we can take $\hbar = 1$. We have, for $p = 1,2,\ldots$:

$$\begin{aligned}
&U_{n+p,\varepsilon} - U_{n,\varepsilon} = \Delta_{n+p,\varepsilon} e^{i\varepsilon_n W_n}\cdots e^{i\varepsilon W_1}, \quad \Delta_{n+p,\varepsilon} := (e^{i\varepsilon_{n+p}W_{n+p}}\cdots e^{i\varepsilon_{n+1}W_{n+1}} - I)\\
&\|U_{n+p,\varepsilon} - U_{n,\varepsilon}\|_{L^2\to L^2} \le 2\|\Delta_{n+p,\varepsilon}\|_{L^2\to L^2}
\end{aligned}$$

Now we apply the mean value theorem and obtain

$$e^{i\varepsilon_\ell W_{\ell,\varepsilon}} = 1 + \beta_{\ell,\varepsilon} \quad \beta_{\ell,\varepsilon} := i\varepsilon_\ell W_{\ell,\varepsilon}\int_0^{\varepsilon_\ell} e^{i\varepsilon_\ell' W_{\ell,\varepsilon}}\,d\varepsilon_\ell',$$

whence, by (6.27) in which we make $k = 0$:

$$\|\beta_{\ell,\varepsilon}\| \le \varepsilon_\ell \|W_{\ell,\varepsilon}\|_{\rho_\ell} \le \varepsilon_\ell \|W_{\ell,\varepsilon}\|_{\rho_\ell,k} \le \gamma\tau^\tau(\ell+1)^{2\tau}\frac{(1+8\gamma\tau^\tau[2(k+1)^2])^{2-\tau}}{64\gamma^2\tau^{2\tau}[2(k+1)^2]^4}\cdot(\mu\varepsilon\|\mathcal{V}\|_\rho)^{2^\ell} \le A^\ell \tag{7.2}$$

for some $A < 1$. Now:

$$\begin{aligned}
&\Delta_{n+p,\varepsilon} = [(1+\beta_{n+p,\varepsilon}\varepsilon_{n+p})(1+\beta_{n+p-1,\varepsilon}\varepsilon_{n+p-1})\cdots(1+\beta_{n+1,\varepsilon}\varepsilon_{n+1})] = \sum_{j=1}^p \beta_{n+j,\varepsilon}\varepsilon_{n+j}\\
&+ \sum_{j_1<j_2=1}^p \beta_{n+j_1,\varepsilon}\varepsilon_{n+j_1}\beta_{n+j_2,\varepsilon}\varepsilon_{n+j_2} + \sum_{j_1<j_2<j_3=1}^p \beta_{n+j_1,\varepsilon}\varepsilon_{n+j_1}\beta_{n+j_2,\varepsilon}\varepsilon_{n+j_2}\beta_{n+j_3,\varepsilon}\varepsilon_{n+j_3}\\
&+ \ldots + \beta_{n+1,\varepsilon}\cdots\beta_{n+p,\varepsilon}\varepsilon_{n+1}\cdots\varepsilon_{n+p}
\end{aligned}$$

Therefore, by (7.2):

$$\|\Delta_{n+p,\varepsilon}\|_{L^2\to L^2} \le \sum_{j=1}^{p} A^j + \sum_{j_1<j_2=1}^{p} A^{n+j_1}A^{n+j_2} + \sum_{j_1<j_2<j_3=1}^{p} A^{n+j_1}A^{n+j_2}A^{n+j_3} + \dots$$
$$\le A^n \frac{A}{1-A^n} + A^{2n}\left(\frac{A}{1-A^n}\right)^2 + \dots + A^{pn}\left(\frac{A}{1-A^n}\right)^p =$$
$$\frac{A^n}{1-A^n}\left[1 + A^n\left(\frac{A}{1-A}\right) + \dots + A^{(p-1)n}\left(\frac{A}{1-A^n}\right)^{p-1}\right] =$$
$$\frac{A^n}{1-A^n}\frac{1}{1-A^n\frac{A}{1-A}} \to 0, \quad n\to\infty, \quad \forall\, p>0$$

Hence $\{U_{n,\varepsilon}(\hbar)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the operator norm, uniformly with respect to $|\varepsilon|<\varepsilon_0^*$, and the Lemma is proved. □

We are now in position to prove existence and analyticity of the limit of the KAM iteration, whence the uniform convergence of the QNF.

Proof of Theorems 1.6 and 1.7

The operator family H_ε is self-adjoint in $L^2(T^l)$ with pure point spectrum $\forall\,\varepsilon\in\mathbb{R}$ because V is a continuous operator. By Corollary 6.6, the operator sequence $\{D_{n,\varepsilon}(\hbar)\}_{n\in\mathbb{N}}$ admits for $|\varepsilon|<\varepsilon_0^*$ the uniform norm limit

$$D_{\infty,\hbar}(L_\omega,\hbar) = L_\omega + \sum_{m=0}^{\infty} \varepsilon^{2^m} N_{m,\varepsilon}(L_\omega,\hbar)$$

of symbol $\mathcal{D}_{\infty,\hbar}(\mathcal{L}_\omega(\xi))$. The series is norm-convergent by (6.28). By Lemma (7.1), $D_{\infty,\hbar}(L_\omega,\hbar)$ is unitarily equivalent to H_ε. The operator family $\varepsilon\mapsto D_{\infty,\varepsilon}(\hbar)$ is holomorphic for $|\varepsilon|<\varepsilon_0^*$, uniformly with respect to $\hbar\in[0,1]$. As a consequence, $D_{\infty,\varepsilon}(\hbar)$ admits the norm-convergent expansion:

$$D_{\infty,\varepsilon}(L_\omega,\hbar) = L_\omega + \sum_{s=1}^{\infty} B_s(L_\omega,\hbar)\varepsilon^s, \quad |\varepsilon|<\varepsilon_0^*$$

which is the convergent quantum normal form.

On the other hand, (6.37) entails that the symbol $\mathcal{D}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi),\hbar)$ is a $\mathcal{J}(\rho/2)$-valued holomorphic function of ε, $|\varepsilon|<\varepsilon_0^*$, continuous with respect to $\hbar\in[0,1]$. Therefore it admits the expansion

$$\mathcal{D}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi),\hbar) = \mathcal{L}_\omega(\xi) + \sum_{s=1}^{\infty} \mathcal{B}_s(\mathcal{L}_\omega(\xi),\hbar)\varepsilon^s, \quad |\varepsilon|<\varepsilon^* \tag{7.3}$$

convergent in the $\|\cdot\|_{\rho/2}$-norm, with radius of convergence ε_0^*. Hence, in the notation of Theorem 1.6, $\mathcal{D}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi),\hbar)\equiv\mathcal{B}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi),\hbar)$. By construction, $\mathcal{B}_s(\mathcal{L}_\omega(\xi),\hbar)$ is the symbol of $B_s(L_\omega,\hbar)$. $\mathcal{B}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi),\hbar)$ is the symbol yielding the quantum normal form via Weyl's quantization. Likewise, the symbol $\mathcal{W}_{\infty,\varepsilon}(\xi,x,\hbar)$ is a $J(\rho/2)$-valued holomorphic function of ε, $|\varepsilon|<\varepsilon^*$, continuous with respect to $\hbar\in[0,1]$, and admits the expansion:

$$\mathcal{W}_{\infty,\varepsilon}(\xi,x,\hbar)=\langle\xi,x\rangle+\sum_{s=1}^{\infty}\mathcal{W}_s(\xi,x,\hbar)\varepsilon^s,\quad |\varepsilon|<\varepsilon_0^* \tag{7.4}$$

convergent in the $\|\cdot\|_{\rho/2}$-norm, once more with radius of convergence ε_0^*. Since Since $\|\mathcal{B}_s\|_1\leq\|\mathcal{B}_s\|_{\rho/2}$, $\|\mathcal{W}_s\|_1\leq\|\mathcal{W}_s\|_{\rho/2}$ $\forall\,\rho>0$. By construction, $\mathcal{B}_{\infty,\varepsilon}(\xi,x,\hbar)=\mathcal{B}_{\infty,\varepsilon}(t,x,\hbar)|_{t=\mathcal{L}_\omega(\xi)}$. Theorem 1.6 is proved . Remark that the principal symbol of $\mathcal{B}_{\infty,\varepsilon}(\mathcal{L}_\omega(\xi),\hbar)$ is just the convergent Birkhoff normal form:

$$\mathcal{B}_{\infty,\varepsilon}=\mathcal{L}_\omega(\xi)+\sum_{s=1}^{\infty}\mathcal{B}_s(\mathcal{L}_\omega(\xi))\varepsilon^s,\quad |\varepsilon|<\varepsilon_0^*$$

Theorem (1.7) is a direct consequence of (6.37) on account of the fact that

$$\sum_{\gamma=0}^{r}\max_{\hbar\in[0,1]}\|\partial_\hbar^\gamma\mathcal{B}_\infty(t;\varepsilon,\hbar)\|_{\rho/2}\leq\|\mathcal{B}_\infty\|_{\rho/2,k}$$

Remark indeed that by (6.37) the series (7.3) converges in the $\|\cdot\|_{\rho/2,r}$ norm if $|\varepsilon|<\varepsilon^*(\cdot,r)$. Therefore $\mathcal{B}_s(t,\hbar)\in C^r([0,1];C^\omega(\{t\in\mathbb{C}\,|\,|\Im t|<\rho/2\})$ and the formula (1.31) follows from (7.3) upon Weyl quantization. This concludes the proof of the Theorem.

Appendix A. The quantum normal form

The quantum normal form in the framework of semiclassical analysis has been introduced by Sjöstrand [Sj]. We follow here the presentation of [BGP].

1. The formal construction Given the operator family $\varepsilon\mapsto H_\varepsilon=L_\omega+\varepsilon V$, look for a unitary transformation $U(\omega,\varepsilon,\hbar)=e^{iW(\varepsilon)/\hbar}:L^2(\mathbb{T}^l)\leftrightarrow L^2(\mathbb{T}^l)$, $W(\varepsilon)=W^*(\varepsilon)$, such that:

$$S(\varepsilon):=UH_\varepsilon U^{-1}=L(\omega)+\varepsilon B_1+\varepsilon^2B_2+\ldots+\varepsilon^kR_k(\varepsilon) \tag{A.1}$$

where $[B_p,L_0]=0$, $p=1,\ldots,k-1$. Recall the formal commutator expansion:

$$S(\varepsilon)=e^{itW(\varepsilon)/\hbar}He^{-itW(\varepsilon)/\hbar}=\sum_{l=0}^{\infty}t^lH_l,\quad H_0:=H,\quad H_l:=\frac{[W,H_{l-1}]}{i\hbar l},\ l\geq 1 \tag{A.2}$$

and look for $W(\varepsilon)$ under the form of a power series: $W(\varepsilon)=\varepsilon W_1+\varepsilon^2W_2+\ldots$. Then (A.2) becomes:

$$S(\varepsilon)=\sum_{s=0}^{k-1}\varepsilon^sP_s+\varepsilon^kR^{(k)} \tag{A.3}$$

where

$$P_0=L_\omega;\quad P_s:=\frac{[W_s,H_0]}{i\hbar}+V_s,\quad s\geq 1,\ V_1\equiv V \tag{A.4}$$

$$V_s = \sum_{r=2}^{s} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=s \\ j_l \geq 1}} \frac{[W_{j_1}, [W_{j_2}, \ldots, [W_{j_r}, H_0] \ldots]}{(i\hbar)^r} + \sum_{r=2}^{s-1} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=s-1 \\ j_l \geq 1}} \frac{[W_{j_1}, [W_{j_2}, \ldots, [W_{j_r}, V] \ldots]}{(i\hbar)^r}$$

$$R^{(k)} = \sum_{r=k}^{\infty} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k \\ j_l \geq 1}} \frac{[W_{j_1}, [W_{j_2}, \ldots, [W_{j_r}, L_\omega] \ldots]}{(i\hbar)^r} + \sum_{r=k-1}^{\infty} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k-1 \\ j_l \geq 1}} \frac{[W_{j_1}, [W_{j_2}, \ldots, [W_{j_r}, V] \ldots]}{(i\hbar)^r}$$

Since V_s depends on $W_1, \ldots, W_{s-1}$, (A1) and (A3) yield the recursive homological equations:

$$\frac{[W_s, P_0]}{i\hbar} + V_s = B_s, \qquad [L_0, B_s] = 0 \tag{A.5}$$

To solve for S, W_s, B_s, we can equivalently look for their symbols. The equations (A.2), (A.3), (A.4) become, once written for the symbols:

$$\Sigma(\varepsilon) = \sum_{l=0}^{\infty} \mathcal{H}_l, \quad \mathcal{H}_0 := \mathcal{L}_\omega + \varepsilon\mathcal{V}, \quad \mathcal{H}_l := \frac{\{w, \mathcal{H}_{l-1}\}_M}{l}, \; l \geq 1 \tag{A.6}$$

$$\Sigma(\varepsilon) = \sum_{s=0}^{k} \varepsilon^s \mathcal{P}_s + \varepsilon^{k+1} \mathbb{R}^{(k+1)} \tag{A.7}$$

where

$$\mathcal{P}_0 = \mathcal{L}_\omega; \qquad \mathcal{P}_s := \{\mathcal{W}_s, \mathcal{P}_0\}_M + \mathcal{V}_s, \quad s = 1, \ldots, \qquad \mathcal{V}_1 \equiv \mathcal{V}_0 = \mathcal{V} \tag{A.8}$$

$$\mathcal{V}_s := \sum_{r=2}^{s} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=s \\ j_l \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \ldots, \{\mathcal{W}_{j_r}, \mathcal{L}_\omega\}_M \ldots\}_M +$$

$$+ \sum_{r=1}^{s-1} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=s-1 \\ j_l \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \ldots, \{\mathcal{W}_{j_r}, \mathcal{V}\}_M \ldots\}_M, \quad s > 1$$

$$\mathbb{R}^{(k)} = \sum_{r=k}^{\infty} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k \\ j_l \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \ldots, \{\mathcal{W}_{j_r}, \mathcal{L}_\omega\}_M \ldots\}_M +$$

$$\sum_{r=k-1}^{\infty} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k-1 \\ j_l \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \ldots, \{\mathcal{W}_{j_r}, \mathcal{V}\}_M \ldots\}_M$$

In turn, the recursive homological equations become:

$$\{\mathcal{W}_s, \mathcal{L}_\omega\}_M + \mathcal{V}_s = \mathcal{B}_s, \qquad \{\mathcal{L}_\omega, \mathcal{B}_s\}_M = 0 \tag{A.9}$$

2. Solution of the homological equation and estimates of the solution

The key remark is that $\{\mathcal{A}, \mathcal{L}_\omega\}_M = \{\mathcal{A}, \mathcal{L}_\omega\}$ for any smooth symbol $\mathcal{A}(\xi; x; \hbar)$ because $\mathcal{L}_\omega$ is linear in ξ. The homological equation (A.9) becomes therefore

$$\{\mathcal{W}_s, \mathcal{L}_\omega\} + \mathcal{V}_s = \mathcal{B}_s, \qquad \{\mathcal{L}_\omega, \mathcal{B}_s\} = 0 \tag{A.10}$$

We then have:

Proposition A.1. *Let $\mathcal{V}_s(\xi, x;\hbar) \in \mathcal{J}(\rho_s)$. Then the equation*

$$\{\mathcal{W}_s, \mathcal{L}_\omega\} + \mathcal{V}_s = \mathcal{B}_s, \qquad \{\mathcal{L}_\omega, \mathcal{B}_s\} = 0 \tag{A.11}$$

admits $\forall\, 0 < d_s < \rho_s$ *the solutions* $\mathcal{B}_s(\mathcal{L}_\omega(\xi;)\hbar) \in \mathcal{J}(\rho_s)$, $\mathcal{W} \in \mathcal{J}(\rho - d_s)$ *given by:*

$$\mathcal{B}_s(\xi;\hbar) = \overline{\mathcal{V}_s}; \quad \mathcal{W}_s(\xi, x;\hbar) = \mathcal{L}_\omega^{-1}\mathcal{V}_s, \quad \mathcal{L}_\omega^{-1}\mathcal{V}_s := \sum_{0 \neq \in \mathbb{Z}^l} \frac{\mathcal{V}_{s,q}(\mathcal{L}_\omega(\xi))}{i\langle\omega, q\rangle} e^{i\langle q, x\rangle}. \tag{A.12}$$

Moreover:

$$\|\mathcal{B}_s\|_{\rho_s} \leq \|\mathcal{V}_s\|_{\rho_s}; \qquad \|\mathcal{W}_s\|_{\rho_s - d_s} \leq \gamma \left(\frac{\tau}{d_s}\right)^\tau \|\mathcal{V}_s\|_{\rho_s}. \tag{A.13}$$

Proof. $\mathcal{B}_s$ and $\mathcal{W}_s$ defined by (A.12) clearly solve the homological equation (A.11). The estimate for $\mathcal{B}_s$ is obvious, and the estimate for $\mathcal{W}_s$ follows once more by the small denominator inequality (1.25). □

By definition of $\|\cdot\|_\rho$ norm:

$$\|B_s\|_{L^2 \to L^2} \leq \|B_s\|_\rho \leq \|\mathcal{V}_s\|_{\rho_s}; \quad \|B_s\|_{L^2 \to \mathcal{L}^2} \leq \|B_s\|_\rho \leq \|\mathcal{V}_s\|_{\rho_s} \tag{A.14}$$

Hence all terms of the quantum normal form and the remainder can be recursively estimated in terms of $\|\mathcal{V}\|_\rho$ by Corollary 3.11. Setting now, for $s \geq 1$:

$$\rho_s := \rho - s d_s, \quad d_s < \frac{\rho}{s+1}; \qquad \rho_0 := \rho$$
$$\mu_s := 8\gamma\tau^\tau \frac{E}{d_s^\tau \delta_s^2}, \quad E := \|\mathcal{V}\|_\rho.$$

we actually have, applying without modification the argument of [BGP], Proposition 3.2:

Proposition A.2. *Let $\mu_s < 1/2, s = 1, \ldots, k$. Set:*

$$K := \frac{8 \cdot 2^{\tau+5}\gamma\tau^\tau}{\rho^{2+\tau}}.$$

Then the following estimates hold for the quantum normal form:

$$\sum_{s=1}^k \|B_s\|_{\rho/2}\varepsilon^s \leq \sum_{s=1}^k \|\mathcal{B}_s\|_{\rho/2}\varepsilon^s \leq \sum_{s=1}^k E^s K^s s^{(\tau+2)s}\varepsilon^s$$

$$\|R_{k+1}\|_{\rho/2} \leq \|\mathbb{R}_{k+1}\|_{\rho/2} \leq (EK)^{k+1}(k+1)^{(\tau+2)(k+1)}\varepsilon^{k+1}$$

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Dipartimento di Matematica, Università di Bologna, 40127 Bologna, Italy
E-mail address: `graffi@dm.unibo.it`

Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex, France
E-mail address: `paul@math.polytechnique.fr`