# Renormalization Group and Problem of Radiation
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**Abstract**

The standard model of non-relativistic quantum electrodynamics describes non-relativistic quantum matter, such as atoms and molecules, coupled to the quantized electromagnetic field. Within this model, we review basic notions, results and techniques in theory radiation. We describe the key technique in this area - the spectral renormalization group. Our review is based on joint works with Volker Bach and Jürg Fröhlich and with Walid Abou Salem, Thomas Chen, Jérémy Faupin and Marcel Griesemer. Brief discussion of related contributions is given at the end of these lectures. This review will appear in "Quantum Theory from Small to Large Scales", Lecture Notes of the Les Houches Summer Schools, volume 95, Oxford University Press, 2011.

*Key words*: quantum electrodynamics, photons and electrons, renormalization group, quantum resonances, spectral theory, Schrödinger operators, ground state, quantum dynamics, non-relativistic theory.

## 1 Overview

We will describe some key results in theory of radiation for the standard model of non-relativistic electrodynamics (QED). The non-relativistic QED was proposed in early days of Quantum Mechanics[1] and it describes quantum-mechanical particle systems coupled to quantized electromagnetic field. It arises from a standard quantization of the corresponding classical systems (with possible addition of internal - spin - degrees of freedom)[2] and it gives a complete and consistent account of electrons and nuclei interacting with electro-magnetic radiation at low energies. In fact, it accounts for all physical phenomena in QED, apart from vacuum polarization. Sample of issues it addresses are

- Stability;
- Radiation;
- Renormalization of mass;
- Anomalous magnetic moment;
- One-particle states;
- Scattering theory.

There was a remarkable progress in the last 10 or so years in rigorous understanding of the corresponding phenomena. In this brief review we will deal with results concerning the first two items. We translate them into mathematical terms:

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[1] It was used, as already known, by Fermi ([37]) in 1932 in his review of theory of radiation.

[2] In fact, it is the most general quantum theory obtained as a quantization of a classical system.

- Stability $\Longleftrightarrow$ Existence of the *ground state*;
- Radiation $\Longleftrightarrow$ Formation of *resonances* out of the excited states of particle systems, scattering theory.

One of the key notions here is that of the *resonance*. It gives a clear-cut mathematical description of processes of emission and absorption of the electro-magnetic radiation.

The key and unifying technique we will concentrate on is the spectral *renormalization group*. It is easily combined with other techniques, e.g. complex deformations (for resonances), the Mourre estimate (for dynamics), analyticity, fiber integral decompositions and Ward identities (used so far for translationally invariant systems). It was also extended to analysis of existence and stability of thermal states.

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# 2 Non-relativistic QED

## 2.1 Schrödinger equation

We consider a system consisting of $n$ charged particles interacting between themselves and with external fields, which are coupled to quantized electromagnetic field. The starting point of the non-relativistic QED is the state Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, which is the tensor product of the state spaces of the particles, $\mathcal{H}_p$, say, $\mathcal{H}_p = L^2(\mathbb{R}^{3n})$, and of Bosonic Fock space $\mathcal{H}_f$ of the quantized electromagnetic field, and the standard quantum Hamiltonian $H \equiv H_{g\chi}$ on $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, given (in the units in which the Planck constant divided by $2\pi$ and the speed of light are equal to $1: \ \hbar = 1$ and $c = 1$) by

$$H = \sum_{j=1}^{n} \frac{1}{2m_j}(i\nabla_{x_j} - gA_\chi(x_j))^2 + V(x) + H_f \tag{1}$$

(see [37] and [96]). Here, $m_j$ and $x_j$, $j = 1, ..., n$, are the ('bare') particle masses and the particle positions, $x = (x_1, \dots, x_n)$, $V(x)$ is the total potential affecting particles and $g > 0$ is a coupling constant related to the particle charges, $A_\chi := \check{\chi} * A$, where $A(y)$ is the *quantized vector potential*, in the Coulomb gauge ($\mathrm{div} A(y) = 0$), describing the quantized electromagnetic field, and $\chi$ is an *ultraviolet cut-off*,

$$A_\chi(y) = \int (e^{iky}a(k) + e^{-iky}a^*(k))\chi(k)\frac{d^3k}{\sqrt{|k|}} \tag{2}$$

($a(k)$ and $a^*(k)$ are annihilation and creation operators acting on the Fock space $\mathcal{H}_f \equiv \mathcal{F}$, see Supplement H for the definitions), and $H_f$ is the quantum Hamiltonian of the quantized electromagnetic field, describing the dynamics of the latter, it is given by

$$H_f \ = \int d^3k \ \omega(k) a^*(k) \cdot a(k), \tag{3}$$

where $\omega(k) \ = \ |k|$ is the dispersion law connecting the energy of the field quantum with its wave vector $k$. For simplicity we omitted the interaction of the spin with magnetic field. (For a discussion of this Hamiltonian including units, the removal of the center-of-mass motion of the particle system and taking into account the spin of the particles, see Appendix A. Note that our units are not dimensionless. We use this units since we want to keep track of the particle masses. To pass to the dimensionless units we would have to set $m_{\mathrm{el}} = 1$ also.) The Hamiltonian $H$ determines the dynamics via the time-dependent Schrödinger equation

$$i\partial_t\psi = H\psi,$$

where $\psi$ is a differentiable path in $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$.

The ultraviolet cut-off, $\chi$, satisfies $\chi(k) = 1$ in a neighborhood of $k = 0$ and is decaying at infinity on the scale $\kappa$ and sufficiently fast. We assume that $V(x)$ is a generalized $n$-body potential, i.e. it satisfies the assumptions:

(V) $V(x) = \sum_i W_i(\pi_i x)$, where $\pi_i$ are a linear maps from $\mathbb{R}^{3n}$ to $\mathbb{R}^{m_i}$, $m_i \leq 3n$ and $W_i$ are Kato-Rellich potentials (i.e. $W_i(\pi_i x) \in L^{p_i}(\mathbb{R}^{m_i}) + (L^\infty(\mathbb{R}^{3n}))_\varepsilon$ with $p_i = 2$ for $m_i \leq 3,\ p_i > 2$ for $m_i = 4$ and $p_i \geq m_i/2$ for $m_i > 4$).

Under the assumption (V), the operator $H$ is self-adjoint and bounded below.

We assume for simplicity that our matter consists of electrons and the nuclei and that the nuclei are infinitely heavy and therefore are manifested through the interactions only (put differently, the molecules are treated in the Born - Oppenheimer approximation). In this case, the coupling constant $g$ is related to the electron charge $-e$ as $g := \alpha^{3/2}$, where $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$, the fine-structure constant, and $m_j = m$. It is shown (see Section 12 and a review in [6]) for references and discussion) that the physical electron mass, $m_{\rm el}$, is not the same as the parameter $m \equiv m_j$ (the 'bare' electron mass) entering (1), but depends on $m$ and $\kappa$. Inverting this relation, we can think of $m$ as a function of $m_{\rm el}$ and $\kappa$. If we fix the particle potential $V(x)$ (e.g. taking it to be the total Coulomb potential), and $m_{\rm el}$ and $e$, then the Hamiltonian (1) depends on one free parameter, the bare electron mass $m$ (or the ultraviolet cut-off scale, $\kappa$).

## 2.2 Stability and radiation

We begin with considering the matter system alone. As was mentioned above, its state space, $\mathcal{H}_p$, is either $L^2(\mathbb{R}^{3n})$ or a subspace of this space determined by a symmetry group of the particle system, and its Hamiltonian operator, $H_p$, acting on $\mathcal{H}_p$, is given by

$$H_p := \sum_{j=1}^{n} \frac{-1}{2m_j}\Delta_{x_j} + V(x), \tag{4}$$

where $\Delta_{x_j}$ is the Laplacian in the variable $x_j$ and, recall, $V(x)$ is the total potential of the particle system. Under the conditions on the potentials $V(x)$, given above, the operator $H_p$ is self-adjoint and bounded below. Typically, according to the HVZ theorem, its spectrum consists of isolated eigenvalues, $\epsilon_0^{(p)} < \epsilon_1^{(p)} < \ldots < \Sigma^{(p)}$, and continuum $[\Sigma^{(p)}, \infty)$, starting at the ionization threshold $\Sigma^{(p)}$, as shown in the figure below.

**Hidden Picture**

The eigenfunctions corresponding to the isolated eigenvalues are exponentially localized. Thus left on its own the particle system, either in its ground state or in one of the excited states, is stable and well localized in space. We expect that this picture changes dramatically when the total system (the universe) also includes the electromagnetic field, which at this level must be considered to be quantum. As was already indicated above what we expect is the following

- The stability of the system under consideration is equivalent to the statement of existence of the ground state of $H$, i.e. an eigenfunction with the smallest possible energy.

- The physical phenomenon of radiation is expressed mathematically as emergence of resonances out of excited states of a particle system due to coupling of this system to the quantum electro-magnetic field.

Our goal is to develop the spectral theory of the Hamiltonian $H$ and relate to the properties of the relevant evolution. Namely, we would like to show that

1) The *ground state* of the particle system is *stable* when the coupling is turned on, while

2) The excited states, generically, are not. They turn into *resonances*.

### 2.3 Ultra-violet cut-off

We reintroduce the Planck constant, $\hbar$, speed of light, $c$, and electron mass, $m_{\rm el}$, for a moment. Assuming the ultra-violet cut-off $\chi(k)$ decays on the scale $\kappa$, in order to correctly describe the phenomena of interest, such as emission and absorption of electromagnetic radiation, i.e. for optical and rf modes, we have to assume that the cut-off energy,

$$\hbar c\kappa \gg \alpha^2 m_{\rm el}c^2, \text{ ionization energy, characteristic energy of the particle motion.}$$

On the other hand, we should exclude the energies where the relativistic effects, such as electron-positron pair creation, vacuum polarization and relativistic recoil, take place, and therefore we assume

$$\hbar c\kappa \ll m_{\rm el}c^2, \text{ the rest energy of the the electron.}$$

Combining the last two conditions we arrive at $\alpha^2 m_{\rm el}c/\hbar \ll \kappa \ll m_{\rm el}c/\hbar$, or in our units,

$$\alpha^2 m_{\rm el} \ll \kappa \ll m_{\rm el} \quad .$$

The Hamiltonian (1) is obtained by the rescaling $x \to \alpha^{-1}x$ and $k \to \alpha^2 k$ of the original QED Hamiltonian (see Appendix A). After this rescaling, the new cut-off momentum scale, $\kappa' = \alpha^{-2}\kappa$, satisfies

$$m_{\rm el} \ll \kappa' \ll \alpha^{-2} m_{\rm el},$$

which is easily accommodated by our estimates (e.g. we can have $\kappa' = O(\alpha^{-1/3} m_{\rm el})$).

## 3 Resonances

As was mentioned above, the mathematical language which describes the physical phenomenon of radiation is that of *quantum resonances*. We expect that the latter emerge out of excited states of a particle system due to coupling of this system to the quantum electro-magnetic field.

Quantum resonances manifest themselves in three different ways:
1) Eigenvalues of complexly deformed Hamiltonian;
2) Poles of the meromorphic continuation of the resolvent across the continuous spectrum;
3) Metastable states.

### 3.1 Complex deformation

To define resonances we use complex deformation method. In order to be able to apply this method we choose the ultraviolet cut-off, $\chi(k)$, so that

> The function $\theta \to \chi(e^{-\theta}k)$ has an analytic continuation from the real axis, $\mathbb{R}$, to the strip $\{\theta \in \mathbb{C} | |\mathrm{Im}\ \theta| < \pi/4\}$ as a $L^2 \bigcap L^\infty(\mathbb{R}^3)$ function,

e.g. $\chi(k) = e^{-|k|^2/\kappa^2}$. For the same purpose, we assume that the potential, $V(x)$, satisfies the condition:

(DA) The the particle potential $V(x)$ is dilation analytic in the sense that the operator-function $\theta \rightarrow V(e^\theta x)$ $(-\Delta+1)^{-1}$ has an analytic continuation from the real axis, $\mathbb{R}$, to the strip $\{\theta \in \mathbb{C} | |\mathrm{Im}\ \theta| < \theta_0\}$ for some $\theta_0 > 0$.

To define the resonances for the Hamiltonian $H$ we pass to the one-parameter (deformation) family

$$H_\theta := U_\theta H U_\theta^{-1}, \tag{5}$$

where $\theta$ is a real parameter and $U_\theta$, on the total Hilbert space $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$, is the one-parameter group of unitary operators, whose action is rescaling particle positions and of photon momenta:

$$x_j \rightarrow e^\theta x_j \text{ and } k \rightarrow e^{-\theta} k.$$

One can show show that:

1) Under a certain analyticity condition on coupling functions, the family $H_\theta$ has an analytic continuation in $\theta$ to the disc $D(0, \theta_0)$, as a type A family in the sense of Kato;

2) The real eigenvalues of $H_\theta,\ \mathrm{Im}\,\theta > 0$, coincide with eigenvalues of $H$ and that complex eigenvalues of $H_\theta,\ \mathrm{Im}\,\theta > 0$, lie in the complex half-plane $\mathbb{C}^-$;

3) The complex eigenvalues of $H_\theta,\ \mathrm{Im}\,\theta > 0$, are locally independent of $\theta$. The typical spectrum of $[H_\theta \equiv H_\theta^{SM}]|_{g=0},\ \mathrm{Im}\,\theta > 0$ (here the superindex SM stands for the standard model) is shown in the figure below.

**Hidden Picture**

We call complex eigenvalues of $H_\theta,\ \mathrm{Im}\,\theta > 0$ the *resonances* of $H$.

As an example of the above procedure we consider the complex deformation of the hydrogen atom and photon Hamiltonians $H_{hydr} := -\frac{1}{2m}\Delta - \frac{\alpha}{|x|}$ and $H_f$:

$$H_{hydr\theta} = e^{-2\theta}(-\frac{1}{2m}\Delta) - e^{-\theta}\frac{\alpha}{|x|},\ H_{f\theta} = e^{-\theta} H_f.$$

Let $e_j^{hydr}$ be the eigenvalues of the hydrogen atom. Then the spectra of these deformations are

$$\sigma(H_{hydr\theta}) = \{e_j^{hydr}\} \cup e^{-2\mathrm{Im}\theta}[0,\infty),\ \sigma(H_{f\theta}) = \{0\} \cup e^{-\mathrm{Im}\theta}[0,\infty).$$

## 3.2 Resonances as poles

Similarly to eigenvalues, we would like to characterize the resonances in terms of poles of matrix elements of the resolvent $(H-z)^{-1}$ of the Hamiltonian $H$. To this end we have to go beyond the spectral analysis of $H$. Let $\Psi_\theta = U_\theta \Psi$, etc., for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^+$. Use the unitarity of $U_\theta$ for real $\theta$, to obtain (the Combes argument)

$$\langle \Psi,\ (H-z)^{-1}\Phi\rangle = \langle \Psi_{\bar\theta}, (H_\theta - z)^{-1}\Phi_\theta\rangle. \tag{6}$$

Assume now that for a dense set of $\Psi$'s and $\Phi$'s (say, $\mathcal{D}$, defined below), $\Psi_\theta$ and $\Phi_\theta$ have analytic continuations into a complex neighbourhood of $\theta = 0$ and continue the r.h.s of (6) analytically first in $\theta$ into the upper half-plane and then in $z$ across the continuous spectrum. This meromorphic continuation has the following properties:

- The real eigenvalues of $H_\theta$ give real poles of the r.h.s. of (6) and therefore they are the eigenvalues of $H$.

- The complex eigenvalues of $H_\theta$ are poles of the meromorphic continuation of the l.h.s. of (6) across the spectrum of $H$ onto the second Riemann sheet.

The poles manifest themselves physically as bumps in the scattering cross-section or poles in the scattering matrix.

The r.h.s. of (6) has an analytic continuation into a complex neighbourhood of $\theta = 0$, if $\Psi, \Phi \in \mathcal{D}$, where

$$\mathcal{D} := \bigcup_{n>0,a>0} \mathrm{Ran}\big(\chi_{N\leq n}\chi_{|T|\leq a}\big). \tag{7}$$

Here $N = \int d^3k a^*(k)a(k)$ be the photon number operator and $T$ be the self-adjoint generator of the one-parameter group $U_\theta,\ \theta \in \mathbb{R}$. (It is dense, since $N$ and $T$ commute.)

## 3.3 Resonance states as metastable states

While bound states are stationary solutions, one expects that resonances to lead to almost stationary, long-living solutions. Let $z_*,\ \mathrm{Im}\, z_* \leq 0$, be the ground state or resonance eigenvalue. One expects that for an initial condition, $\psi_0$, localized in a small energy interval around the ground state or resonance energy, $\mathrm{Re}\, z_*$, the solution, $\psi = e^{-iHt}\psi_0$, of the time-dependent Schrödinger equation, $i\partial_t\psi = H\psi$, is of the form

$$\psi = e^{-iz_*t}\phi_* + O_{\text{loc}}(t^{-\alpha}) + O_{\text{res}}(g^\beta), \tag{8}$$

for some $\alpha,\ \beta > 0$ (depending on $\psi_0$), where

- $\phi_*$ is either the ground state or an excited state of the unperturbed system, depending on whether $z_*$ is the ground state energy or a resonance eigenvalue;
- The error term $O_{\text{loc}}(t^{-\alpha})$ satisfies $\|(\mathbf{1} + |T|)^{-\nu}O_{\text{loc}}(t^{-\alpha})\| \leq Ct^{-\alpha}$, where $T$ is the generator of the group $U_\theta$, with an appropriate $\nu > 0$.

For the *ground state*, (8), without the error term $O_{\text{res}}(g^\beta)$, is called the local decay property (see Section 10). One way to prove it is to use the formula connecting the propagator and the resolvent:

$$e^{-iHt}f(H) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda f(\lambda)e^{-i\lambda t}\mathrm{Im}\,(H - \lambda - i0)^{-1}. \tag{9}$$

Then one controls the boundary values of the resolvent on the spectrum (the corresponding result is called the limiting absorption principle, see Appendix D.2) and uses properties of the Fourier transform.

For the *resonances*, (8) implies that $-\mathrm{Im}\, z_*$ has the meaning of the decay probability per unit time, and $(-\mathrm{Im}\, z_*)^{-1}$, as the life-time of the resonance. To prove it, one uses (6) and the analyticity of its r.h.s. in $z$ and performs in (9) a suitable deformation of the contour of integration to the second Riemann sheet to pick up the contribution of poles there. This works when the resonances are isolated. In the present case, they are not. This is a consequence of the infrared problem. Hence, determining the long-time behaviour of $e^{-iHt}\psi_0$ is a subtle problem in this case.

## 3.4 Comparison with Quantum Mechanics

This situation is quite different from the one in Quantum Mechanics (e.g. Stark effect or tunneling decay) where the resonances are isolated eigenvalues of complexly deformed Hamiltonians. This makes the proof of their existence and establishing their properties, e.g. independence of $\theta$ (and, in fact, of the transformation group $U_\theta$), relatively easy. In the non-relativistic QED (and other massless theories), giving meaning of the resonance poles and proving independence of their location of $\theta$ is a rather involved matter.

### 3.5 Infrared problem

The resonances arise from the eigenvalues of the non-interacting Hamiltonian $H_{g=0}$. The latter is of the form

$$H_0 = H_{\rm part} \otimes \mathbf{1}_f + \mathbf{1}_{\rm part} \otimes H_f \,. \tag{10}$$

The low energy spectrum of the operator $H_0$ consists of branches $[\epsilon_i^{(p)}, \infty)$ of absolutely continuous spectrum and of the eigenvalues $\epsilon_i^{(p)}$'s, sitting at the continuous spectrum 'thresholds' $\epsilon_i^{(p)}$'s. Here, recall, $\epsilon_0^{(p)} < \epsilon_1^{(p)} < \ldots < \Sigma^{(p)}$ are the isolated eigenvalues of the particle Hamiltonian $H_p$. Let $\phi_i^{(p)}$ be the eigenfunctions of the particle system, while $\Omega$ be the photon vacuum. The eigenvalues $\epsilon_i^{(p)}$'s correspond to the eigenfunctions $\phi_i^{(p)} \otimes \Omega$ of $H_0$. The branches $[\epsilon_i^{(p)}, \infty)$ of absolutely continuous spectrum are associated with generalized eigenfunctions of the form $\phi_i^{(p)} \otimes g_\lambda$, where $g_\lambda$ are the generalized eigenfunctions of $H_f$ : $H_f g_\lambda = \lambda g_\lambda$, $0 < \lambda < \infty$.

The absence of gaps between the eigenvalues and thresholds is a consequence of the fact that the photons are massless. To address this problem we use the spectral renormalization group (RG). The problem here is that the leading part of the perturbation in $H$ is marginal.

# 4 Existence of the Ground and Resonance States

### 4.1 Bifurcation of eigenvalues and resonances

Stated informally what we show is

- The ground state of $H|_{g=0} \Rightarrow$ the ground state $H$ ($\epsilon_0 = \epsilon_0^{(p)} + O(g^2)$ and $\epsilon_0 < \epsilon_0^{(p)}$);
- The excited states of $H|_{g=0} \Rightarrow$ (generically) the resonances of $H$ ($\epsilon_{j,k} = \epsilon_j^{(p)} + O(g^2)$);
- There is $\Sigma > \inf \sigma(H)$ (the ionization threshold, $\Sigma + \Sigma^{(p)} + O(g^2)$) s.t. for energies $< \Sigma$ that particles are exponentially localized around the common center of mass.

For energies $> \Sigma$ the system either sheds off locally the excess of energy and descends into a localized state or breaks apart with some of the particles flying off to infinity.

To formulate this result more precisely, denote

$$\epsilon_{gap}^{(p)}(\nu) := \min\{|\epsilon_i^{(p)} - \epsilon_j^{(p)}| \mid i \neq j,\ \epsilon_i^{(p)}, \epsilon_j^{(p)} \leq \nu\}.$$

**Theorem 4.1** *(Fate of particle bound states)*. Fix $e_0^{(p)} < \nu < \inf \sigma_{ess}(H_p)$ and let $g \ll \epsilon_{gap}^{(p)}(\nu)$. Then for $g \neq 0$,

- $H$ has a ground state, originating from a ground state of $H|_{g=0}$ ($\epsilon_0 = \epsilon_0^{(p)} + O(g^2),\ \epsilon_0 < \epsilon_0^{(p)}$);
- Generically, $H$ has no other bound state (besides the ground state);
- Eigenvalues, $\epsilon_j^{(p)} < \nu,\ j \neq 0$, of $H|_{g=0} \implies$ resonance eigenvalues, $\epsilon_{j,k}$, of $H$;
- $\epsilon_{j,k} = \epsilon_j^{(p)} + O(g^2)$ and the total multiplicity of $\epsilon_{j,k}$ equals the multiplicity of $\epsilon_j^{(p)}$;
- The ground and resonance states are exponentially localized in the physical space:

$$\|e^{\delta|x|}\psi\| < \infty,\ \forall \psi \in {\rm Ran} E_\Delta(H),\ \delta < \Sigma^{(p)} - \sup \Delta.$$

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**Remark.** The relation $\epsilon_0 < \epsilon_0^{(p)}$ is due to the fact that the electron surrounded by clouds of photons become heavier.

## 4.2 Meromorphic continuation across spectrum

**Theorem 4.2**. (*Meromorphic continuation of the matrix elements of the resolvent*) Assume $g \ll \epsilon_{gap}^{(p)}(\nu)$ and let $\epsilon_0 := \inf \sigma(H)$ be the ground state energy of $H$. Then

- For a dense set (defined in (7) below) of vectors $\Psi$ and $\Phi$, the matrix elements

  $$F(z, \Psi, \Phi) := \langle \Psi, (H - z)^{-1} \Phi \rangle$$

  have meromorphic continuations from $\mathbb{C}^+$ across the interval $(\epsilon_0, \nu) \subset \sigma_{ess}(H)$ into

  $$\{z \in \mathbb{C}^- \mid \epsilon_0 < \mathrm{Re}\, z < \nu\} / \bigcup_{0 \le j \le j(\nu)} S_{j,k},$$

  where $S_{j,k}$ are the wedges starting at the resonances

  $$S_{j,k} := \{z \in \mathbb{C} \mid \frac{1}{2} \mathrm{Re}\, (e^\theta (z - \epsilon_{j,k})) \ge |\mathrm{Im}\, (e^\theta (z - \epsilon_{j,k}))|\}; \tag{11}$$

- This continuation has poles at $\epsilon_{j,k}$: $\lim_{z \to \epsilon_{j,k}} (\epsilon_{j,k} - z) F(z, \Psi, \Phi)$ is finite and $\neq 0$.

## 4.3 Discussion

- Generically, excited states turn into the resonances, not bound states.
- The second theorem implies the absolute continuity of the spectrum and its proof gives also the limiting absorption principle for $H$ (see Appendix D.2 for the definitions).
- The proof of first theorem gives fast convergent expressions in the coupling constant $g$ for the ground state energy and resonances.
- One can show analyticity of $\epsilon_{j,k}$ in the coupling constant $g$ (see Appendix 10 for a result on the ground state energies).
- The meromorphic continuation in question is constructed in terms of matrix elements of the resolvent of a complex deformation, $H_\theta,\ \mathrm{Im}\, \theta > 0$, of the Hamiltonian $H$.
- A description of resonance poles is given in Section 10.

### 4.4 Approach

The main steps in our analysis of the spectral structure of the quantum Hamiltonian $H$ are:

- Perform a new canonical transformation (a generalized Pauli-Fierz transform)

$$H \to H^{PF} := e^{-igF} H e^{igF},$$

  in order to bring $H$ to a more convenient for our analysis form;
- Apply the spectral renormalization group (RG) on new – momentum anisotropic – Banach spaces.

The main ideas of the spectral RG are as follows:

- Pass from a single operator $H_\theta^{PF}$ to a Banach space $\mathcal{B}$ of Hamiltonian-type operators;
- Construct a map, $\mathcal{R}_\rho$, (RG transformation) on $\mathcal{B}$, with the following properties:
  (a) $\mathcal{R}_\rho$ is 'isospectral';
  (b) $\mathcal{R}_\rho$ removes the photon degrees of freedom related to energies $\geq \rho$.
- Relate the dynamics of semi-flow, $\mathcal{R}_\rho^n, n \geq 1$, (called renormalization group) to spectral properties of individual operators in $\mathcal{B}$.

## 5 Generalized Pauli-Fierz transformation

We perform a canonical transformation (generalized Pauli-Fierz transform) of $H$ in order to bring it to a form which is accessible to spectral renormalization group (it removes the marginal operators). For simplicity, consider one particle of mass 1. We define the generalized Pauli-Fierz transformation as:

$$H^{PF} := e^{-igF} H e^{igF}, \tag{12}$$

where $F(x)$ is the self-adjoint operator given by

$$F(x) = \sum_\lambda \int (\bar{f}_{x,\lambda}(k) a_\lambda(k) + f_{x,\lambda}(k) a_\lambda^*(k)) \frac{\chi(k) d^3k}{\sqrt{|k|}}, \tag{13}$$

with the coupling function $f_{x,\lambda}(k)$ chosen as

$$f_{x,\lambda}(k) := e^{-ikx} \frac{\varphi(|k|^{\frac{1}{2}} e_\lambda(k) \cdot x)}{\sqrt{|k|}}, \tag{14}$$

with $\varphi \in C^2$ bounded, with bounded derivatives and satisfying $\varphi'(0) = 1$. For the *standard* Pauli-Fierz transformation, we have $\varphi(s) = s$.

The Hamiltonian $H^{PF}$ is of the same form as $H$. Indeed, using the commutator expansion $e^{-igF(x)} H_f e^{igF(x)} = -ig[F, H_f] - g^2[F,[F,H_f]]$, we compute

$$H^{PF} = \frac{1}{2m}(p + gA_{\chi\varphi}(x))^2 + V_g(x) + H_f + gG(x), \tag{15}$$

where

$$A_{\chi\varphi}(x) = \sum_\lambda \int (\bar{\varphi}_{x,\lambda}(k) a_\lambda(k) + \varphi_{x,\lambda}(k) a^*_\lambda(k)) \frac{\chi(k) d^3k}{\sqrt{|k|}},$$

with the new coupling function $\varphi_{\lambda,x}(k) := e_\lambda(k) e^{-ikx} - \nabla_x f_{x,\lambda}(k)$ and

$$V_g(x) := V(x) + 2g^2 \sum_\lambda \int |k||f_{x,\lambda}(k)|^2 d^3k,$$

$$G(x) := -i \sum_\lambda \int |k|(\bar{f}_{x,\lambda}(k) a_\lambda(k) - f_{x,\lambda}(k) a^*_\lambda(k)) \frac{\chi(k) d^3k}{\sqrt{|k|}}.$$

The potential $V_g(x)$ is a small perturbation of $V(x)$ and the operator $G(x)$ is easy to control. The new coupling function has better infrared behaviour for bounded $|x|$:

$$|\varphi_{\lambda,x}(k)| \leq \text{const} \min(1, \sqrt{|k|}\langle x \rangle). \tag{16}$$

To prove the results above we first establish the spectral properties of the generalized Pauli-Fierz Hamiltonian $H^{PF}$ and then transfer the obtained information to the original Hamiltonian $H$.

# 6 Renormalization Group Map

The *renormalization map* is defined on Hamiltonians acting on $\mathcal{H}_f$ which as follows

$$\mathcal{R}_\rho = \rho^{-1} S_\rho \circ F_\rho, \tag{17}$$

where $\rho > 0$, $S_\rho : \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]$ is the *scaling transformation*:

$$S_\rho(\mathbf{1}) := \mathbf{1}, \qquad S_\rho(a^\#(k)) := \rho^{-3/2} a^\#(\rho^{-1}k), \tag{18}$$

and $F_\rho$ is the *Feshbach-Schur map*, or decimation, map,

$$F_\rho(H) := \chi_\rho(H - H\overline{\chi}_\rho(\overline{\chi}_\rho H \overline{\chi}_\rho)^{-1} \overline{\chi}_\rho H)\chi_\rho, \tag{19}$$

where $\chi_\rho$ and $\overline{\chi}_\rho$ is a pair of orthogonal projections, defined as

$$\chi_\rho = \chi_{H_{p\theta}=e_j} \otimes \chi_{H_f \leq \rho} \quad \text{and} \quad \overline{\chi}_\rho := \mathbf{1} - \chi_\rho.$$

**Remark.** For simplicity we defined the decimation map as the Feshbach-Schur map. Technically it is more convenient to use the *smooth Feshbach-Schur map*, which we defined and discussed in Appendix D. The smooth Feshbach-Schur map uses 'smooth' projections which form a partition of unity $\chi_\rho^2 + \overline{\chi}_\rho^2 = \mathbf{1}$, instead of true projections as defined above.

The map $F_\rho$ is *isospectral* in the sense of the following theorem:

**Theorem 6.1.** (i) $\lambda \in \rho(H) \Leftrightarrow 0 \in \rho(F_\rho(H - \lambda))$;

(ii) $H\psi = \lambda\psi \iff F_\rho(H - \lambda)\varphi = 0$;

(iii) $\dim \text{Null}(H - \lambda) = \dim \text{Null} F_\rho(H - \lambda)$;

(iv) $(H - \lambda)^{-1}$ exists $\iff F_\rho(H - \lambda)^{-1}$ exists.

For the proof of this theorem as well as for the relation between $\psi$ and $\varphi$ in (ii) and between $(H - \lambda)^{-1}$ and $F_\rho(H - \lambda)^{-1}$ in (iv) see Appendix C.

# 7 A Banach Space of Hamiltonians

We will study operators on the subspace $\mathrm{Ran}\,\chi_1$ of the Fock space $\mathcal{F}$. Such operators are said to be in the *generalized normal form* if they can be written as:

$$H = \sum_{m+n\geq 0} W_{m+n}, \tag{20}$$
$$W_{m+n} = \int_{B_1^{m+n}} \prod_{i=1}^{m+n} d^3k_i \prod_{i=1}^{m} a^*(k_i)\, w_{m,n}\big(H_f; k_{(m+n)}\big) \prod_{i=m+1}^{m+n} a(k_i),$$

where $B_1^r$ denotes the Cartesian product of $r$ unit balls in $\mathbb{R}^3$, $k_{(m)} := (k_1, \ldots, k_m)$ and $w_{m,n} : I \times B_1^{m+n} \to \mathbb{C}$, $I := [0,1]$. We sometimes we display the dependence of $H$ and $W_{m,n}$ on the coupling functions $\underline{w} := (w_{m,n},\ m+n \geq 0)$ by writing $H[\underline{w}]$ and $W_{m,n}[\underline{w}]$.

We assume that the functions $w_{m,n}(r, , k_{(m+n)})$ are continuously differentiable in $r \in I$, symmetric w. r. t. the variables $(k_1, \ldots, k_m)$ and $(k_{m+1}, \ldots, k_{m+n})$ and obey $\|w_{m,n}\|_{\mu,1} := \sum_{n=0}^{1} \|\partial_r^n w_{m,n}\|_\mu < \infty$, where $\mu \geq 0$ and

$$\|w_{m,n}\|_\mu := \max_j \sup_{r\in I, k_{(m+n)} \in B_1^{m+n}} \big| |k_j|^{-\mu} \prod_{i=1}^{m+n} |k_i|^{1/2} w_{m,n}(r; k_{(m+n)})\big|. \tag{21}$$

Here $k_j \in \mathbb{R}^3$ is the $j-$th $3-$vector in $k_{(m,n)}$ over which we take the supremum. Note that these norms are anisotropic in the total momentum space.

For $\mu \geq 0$ and $0 < \xi < 1$ we define the Banach space

$$\mathcal{B}^{\mu\xi} := \{H \ :\ \big\|H\big\|_{\mu,\xi} := \sum_{m+n\geq 0} \xi^{-(m+n)}\, \|w_{m,n}\|_{\mu,1} < \infty\}. \tag{22}$$

We mention some properties of these spaces

- For any $\mu \geq 0$ and $0 < \xi < 1$, the map $H : \underline{w} \to H[\underline{w}]$, given in (20), is one-to-one.
- If $H$ is self-adjoint, then so are $W_{0,0}$ and $\sum_{m+n\geq 1} \chi_1 W_{m,n} \chi_1$ (see (20)).

**Remarks.** 1) Unlike the Banach spaces defined in [13, 14, 7], the Banach spaces are anisotropic in the momentum space. This is needed to overcome the problem of marginal operators which arise in the renormalization group approach.

2) The self-adjointness statement follows from [7], Eq. (3.33).

## 7.1 Basic bound

The following bound shows that our Banach space norm control the operator norm and the terms with higher numbers of creation and annihilation operators make progressively smaller contributions:

**Theorem 7.1.** Let $\chi_\rho \equiv \chi_{H_f \leq \rho}$. Then for all $\rho > 0$ and $m + n \geq 1$

$$\big\|\chi_\rho\, H_{m,n}\, \chi_\rho\big\| \leq \frac{\rho^{m+n+\mu}}{\sqrt{m!\, n!}} \|w_{m,n}\|_\mu. \tag{23}$$

*Sketch of proof.* For simplicity we prove this inequality for $m = n = 1$. Let $\phi \in \mathcal{F}$ and $\Phi_k = a(k)\chi_\rho\phi$. We have

$$\langle \chi_\rho \phi, H_{1,1}\, \chi_\rho \phi\rangle = \int_{B_1^2} \prod_{i=1}^{2} d^3k_i\ \langle \Phi_{k_1},\, w_{1,1}\big(H_f; k_1, k_2\big)\, \Phi_{k_2}\rangle.$$

Now we write $\chi_\rho = \chi_\rho \chi_{2\rho}$ and pull $\chi_{2\rho}$ toward $w_{1,1}(H_f; k_1, k_2)$ using the pull-trough formulae

$$a(k)\, f(H_f) \;=\; f(H_f + |k|)\, a(k), \;\; f(H_f)\, a^*(k) \;=\; a^*(k)\, f(H_f + |k|)$$

(see Appendix G). This gives

$$\begin{aligned}
& |\langle \phi, \chi_\rho\, H_{1,1}\, \chi_\rho \phi \rangle| \\
= \;& |\int_{|k_i| \le 2\rho, i=1,2} \prod_{i=1}^{2} d^3 k_i \; \langle \Phi_{k_1}, \chi_{2\rho - |k_1|} w_{1,1}\big(H_f; k_1, k_2\big) \chi_{2\rho - |k_2|} \Phi_{k_2} \rangle| \\
\le \;& \int_{|k_i| \le 2\rho, i=1,2} \prod_{i=1}^{2} d^3 k_i \; \|\Phi_{k_1}\| \; \|w_{1,1}\big(H_f; k_1, k_2\big)\| \; \|\Phi_{k_2}\| \\
\le \;& \Big( \int_{|k_i| \le 2\rho, i=1,2} \prod_{i=1}^{2} d^3 k_i \frac{\|w_{1,1}\big(H_f; k_1, k_2\big)\|^2}{|k_1||k_2|} \Big)^{1/2} \int d^3 k \| \sqrt{|k|} \Phi_k \|^2 .
\end{aligned}$$

Now, using $\|w_{1,1}\big(H_f; k_1, k_2\big)\| \le \|w_{1,1}\|_\mu \frac{|k_1|^\mu + |k_2|^\mu}{|k_1|^{\frac{1}{2}} |k_2|^{\frac{1}{2}}}$ and $\int d^3 k \| \sqrt{|k|} \Phi_k \|^2 = \| \sqrt{H_f} \chi_\rho \phi \|^2$, we find

$$|\langle \phi, \chi_\rho\, H_{1,1}\, \chi_\rho \phi \rangle| \lesssim \rho^{2+\mu} \|w_{1,1}\|_\mu .$$

### 7.2 Unstable, neutral and stable components

We decompose $H \in \mathcal{B}^{\mu\xi}$ into the components $E := \langle \Omega, H\Omega \rangle$, $T := H_{0,0} - \langle \Omega, H\Omega \rangle$, $W := \sum_{m+n \ge 1} W_{m,n}$, so that

$$H = E\mathbf{1} + T + W. \tag{24}$$

If we assume $\sup_{r \in [0,\infty)} |T'(r) - 1| \ll 1$, then we have $T \sim H_f$. These Hamiltonian components scale as follows

- $\rho^{-1} S_\rho\big(H_f\big) \;=\; H_f$ ($H_f$ is a *fixed point* of $\rho^{-1} S_\rho$);
- $\rho^{-1} S_\rho(E \cdot \mathbf{1}) = \rho^{-1} E \cdot \mathbf{1}$ ($E \cdot \mathbf{1}$ *expand* under $\rho^{-1} S_\rho$ at a rate $\rho^{-1}$);
- $\|S_\rho(W_{m,n})\|_\mu \le \rho^\alpha \, \|w_{m,n}\|_\mu, \; \alpha := m + n - 1 + \mu \delta_{m+n=1}$ ($W_{mn}$ *contract* under $\rho^{-1} S_\rho$, if $\mu > 0$).

Thus for $\mu > 0$, $E$, $T$, $W$ behave, in the terminology of the renormalization group approach, as relevant, marginal, and irrelevant operators, respectively. For $\mu = 0$, the operators $W_{mn}, \; m + n = 1$, become marginal.

## 8 Action of Renormalization Map

To control the components $E, T, W$ of $H$ we introduce, for $\alpha, \beta, \gamma > 0$, the following polydisc:

$$\begin{aligned}
\mathcal{D}^\mu(\alpha, \beta, \gamma) \quad & := \Big\{ H = E + T + W \in \mathcal{B}^{\mu\xi} \mid |E| \le \alpha, \\
& \sup_{r \in [0,\infty)} |T'(r) - 1| \le \beta, \; \|W\|_{\mu,\xi} \le \gamma \Big\}.
\end{aligned}$$

(Strictly speaking we should write $\chi_{H_{p\theta} = e_j} \otimes \mathcal{D}^\mu(\alpha, \beta, \gamma)$ instead of $\mathcal{D}^\mu(\alpha, \beta, \gamma)$ (for various sets of parameters $\alpha, \beta, \gamma$) in the statement below.)

**Theorem 8.1.** Let $0 < \rho < 1/2,\ \alpha,\ \beta,\ \gamma \le \rho/8$ and $\mu_0 = 1/2$. Then there is $c > 0$, s.t.

- $\mathcal{R}_\rho(H_\theta^{PF}) \in \mathcal{D}^{\mu_0}(\alpha_0, \beta_0, \gamma_0),\ \alpha_0 = cg^2\rho^{\mu_0-2}, \beta_0 = cg^2\rho^{\mu_0-1}, \gamma_0 = cg\rho_0^{\mu}$, provided $g \ll 1$;
- $\mathcal{D}^\mu(\alpha, \beta, \gamma) \subset D(\mathcal{R}_\rho)$, provided $\mu > 0$;
- $\mathcal{R}_\rho : \mathcal{D}^\mu(\alpha, \beta, \gamma) \to \mathcal{D}^\mu(\alpha', \beta', \gamma')$, continuously, with $\alpha' = \rho^{-1}\alpha + c\left(\gamma^2/2\rho\right), \beta' = \beta + c\left(\gamma^2/2\rho\right),\ \gamma' = c\rho^\mu\gamma$.

*Sketch of proof of the second and third properties.* 1) $\mathcal{D}^\mu(\rho/8, 1/8, \rho/8) \subset D(\mathcal{R}_\rho)$. Since $W := H - E - T$ defines a bounded operator on $\mathcal{F}$, we only need to check the invertibility of $H_{\tau\chi_\rho}$ on $\mathrm{Ran}\,\overline{\chi}_\rho$. The operator $E + T$ is invertible on $\mathrm{Ran}\,\overline{\chi}_\rho$: for all $r \in [3\rho/4, \infty)$

$$\begin{aligned}
\mathrm{Re}\,T(r) + \mathrm{Re}\,E \quad &\ge \quad r\ -\ |T(r) - r|\ -\ |E| \\
&\ge \quad r\big(1\ -\ \sup_r |T'(r) - 1|\big)\ -\ |E| \\
&\ge \quad \frac{3\,\rho}{4}(1 - 1/8)\ -\ \frac{\rho}{8}\ \ge\ \frac{\rho}{2} \\
&\Rightarrow \quad E + T \text{ is invertible and } \|(E + T)^{-1}\| \le 2/\rho.
\end{aligned}$$

Now, by the basic estimate, $\|W\| \le \rho/8$ and therefore,

$$\begin{aligned}
&\|\overline{\chi}_\rho W \overline{\chi}_\rho (E + T)^{-1}\| \le 1/4 \\
\Rightarrow\quad &E + T + \overline{\chi}_\rho W \overline{\chi}_\rho \text{ is invertible on } \mathrm{Ran}\,\overline{\chi}_\rho \\
\Rightarrow\quad &\mathcal{D}^\mu(\rho/8, 1/8, \rho/8) \subset D(F_\rho) = D(\mathcal{R}_\rho).
\end{aligned}$$

2) $\mathcal{R}_\rho : \mathcal{D}^\mu(\alpha, \beta, \gamma) \to \mathcal{D}^\mu(\alpha', \beta', \gamma')$ (normal form of $\mathcal{R}_\rho(H)$). Recall that $\chi_\rho \equiv \chi_{H_f \le \rho}$ and $\overline{\chi}_\rho := 1 - \chi_\rho$. Let $H_0 := E + T$, so that $H = H_0 + W$. We have shown above

$$\|H_0^{-1}\overline{\chi}_\rho\|\ \le\ \frac{2}{\rho} \qquad \text{and} \qquad \|W\|\ \le\ \frac{\rho}{8}.$$

In the Feshbach-Schur map, $F_\rho$,

$$F_\rho\big(H\big)\ =\ \chi_\rho\big(H_0 + W - W\,\overline{\chi}_\rho\big(\overline{\chi}_\rho(H_0 + W)\overline{\chi}_\rho\big)^{-1}\overline{\chi}_\rho\,W\,\big)\chi_\rho,$$

we expand the resolvent $(\overline{\chi}_\rho(H_0 + W)\overline{\chi}_\rho\big)^{-1}$ in the norm convergent Neumann series

$$F_\rho\big(H\big)\ =\ \chi_\rho\big[H_0 + \sum_{s=0}^{\infty}(-1)^s\,W\big(H_0^{-1}\overline{\chi}_\rho^2\,W\big)^s\big]\chi_\rho.$$

Next, we transform the right side to the generalized normal form using generalized Wick's theorem.

**Generalized Wick's theorem.** To write the product $W\big(H_0^{-1}\overline{\chi}_\rho^2\ W\big)^s$ in the generalized normal form we pull the annihilation operators, $a$, to the right and the creation operators, $a^*$, to the left, apart from those which enter $H_f$. We use the rules (see Appendix G):

$$a(k)a^*(k') = a^*(k')a(k) + \delta(k - k'),$$

$$a(k)\,f(H_f)\ =\ f(H_f + |k|)\,a(k),\ f(H_f)\,a^*(k)\ =\ a^*(k)\,f(H_f + |k|).$$

Some of the creation and annihilation operators reach the extreme left and right positions, while the remaining ones contract (see the figure below). The terms with $m$ creation operators on the left and $n$ annihilation operators on the right contribute to the $(m,n)-$ formfactor, $w^{(s)}_{m,n}$, of the operator $W\big(H_0^{-1}\overline{\chi}_\rho^2\, W\big)^s$. As the result we obtain the generalized normal form of $F_\rho(H)$:

$$F_\rho(H) = \sum_{m+n\geq 0} W^{'}_{m,n}.$$

The term $W^{'}_{0,0} = \langle W^{(s)}_{0,0}\rangle_\Omega + (W^{(s)}_{0,0} - \langle W^{(s)}_{0,0}\rangle_\Omega)$ contributes the corrections to $E+T$.

**Hidden Picture**

This is the standard way for proving the Wick theorem, taking into account the presence of $H_f-$ dependent factors. (See [13] for a different, more formal proof.)

**Estimating formfactors.** The problem here is that the number of terms generated by various contractions is $O(s!)$. Therefore a simple majoration of the series for the $(m,n)-$ formfactor, $w^{(s)}_{m,n}$, of the operator $W\big(H_0^{-1}\overline{\chi}_\rho^2\, W\big)^s$ will diverge badly.

To overcome this we re-sum the series by, roughly, representing the sum over all contractions, for a given $m$ and $n$, as

$$w^{(s)}_{m,n} \sim \langle \Omega, [W\big(H_0^{-1}\overline{\chi}_\rho^2\, W\big)^s]_{m,n}\Omega\rangle,$$

where $[W\big(H_0^{-1}\overline{\chi}_\rho^2\, W\big)^s]_{m,n}$ is $W\big(H_0^{-1}\overline{\chi}_\rho^2\, W\big)^s$, with $m$ escaping creation operators and $n$ escaping annihilation operators deleted. Now the estimate of $w^{(s)}_{m,n}$ is straightforward and can be written (symbolically) as

$$\|w^{(s)}_{m,n}\| \lesssim \|\chi_{\rho'}\, W'\chi_{\rho'}\|^{s+1},$$

( for the operator norm, and similarly for $\mathcal{B}^{\mu,\xi}-$norm.

# 9 Renormalization Group

To analyze spectral properties of individual operators in $\mathcal{B}^{\mu\xi}$ we use the discrete semi-flow, $\mathcal{R}^n_\rho$, $n\geq 1$ (called renormalization group) , generated by the renormalization transformation, $\mathcal{R}_\rho$. By Theorem 8.1, in order to iterate $\mathcal{R}_\rho$ we have to control the expanding direction: $\mathcal{R}_\rho(\zeta\mathbf{1}) = \rho^{-1}\zeta\mathbf{1}$. To control this direction, we adjust, inductively, at each step the constant component $\langle H\rangle_\Omega := \langle \Omega, H\Omega\rangle$ of the initial Hamiltonian, $H$:

$$|\langle H\rangle_\Omega - e_{n-1}| \leq \frac{1}{12}\rho^{n+1},$$

$$e_{n-1} \text{ is the unique zero of the function } \lambda \to \big\langle \mathcal{R}^{n-1}_\rho(H - \langle H\rangle_\Omega + \lambda)\big\rangle_\Omega,$$

so that

$$H \in \quad \text{the domain of} \quad \mathcal{R}^n_\rho.$$

This way one adjusts the initial conditions closer and closer to the stable manifold $\mathcal{M}_s = \cap_n D(\mathcal{R}^n_\rho)$.

**Hidden Picture**

The procedure above leads to the following results:

- $\mathcal{R}_\rho^n$ has the fixed-point manifold $\mathcal{M}_{fp} := \mathbb{C}H_f$;
- $\mathcal{R}_\rho^n$ has an unstable manifold $\mathcal{M}_u := \mathbb{C}\mathbf{1}$;
- $\mathcal{R}_\rho^n$ has a (complex) co-dimension 1 stable manifold $\mathcal{M}_s$ for $\mathcal{M}_{fp}$;
- $\mathcal{M}_s$ is foliated by (complex) co-dimension 2 stable manifolds for each fixed point.

**Hidden Picture**

Define the polydisc $\mathcal{D}_s := \mathcal{D}^\mu(0, \beta_0, \gamma_0)$, with $\mu > 0,\ \beta_0 = cg^2\rho^{\mu_0 - 1}$ and $\gamma_0 = cg\rho_0^\mu$, the same as in Theorem 8.1. Using the above results, we draw the following conclusions about the spectrum of an operator $H \in \mathcal{D}_s$. Find $\lambda \in$ a neighbourhood of 0, s.t. $H(\lambda) := H - \lambda \in \mathcal{D}(\mathcal{R}_\rho^n)$ and define $H^{(n)}(\lambda) := \mathcal{R}_\rho^n(H(\lambda))$. We illustrate this as putting $H(\lambda)$ through the RG (black) box:

$$H(\lambda) \Longrightarrow \textbf{RG BOX} \Longrightarrow H^{(n)}(\lambda).$$

Use that for $n$ sufficiently large, $H^{(n)}(\lambda) \approx \zeta H_f$, for some $\zeta \in \mathbb{C},\ \mathrm{Re}\ \zeta > 0$, to find wanted spectral information about $H^{(n)}(\lambda)$ in a neighbourhood of 0. Then, use the isospectrality of $\mathcal{R}_\rho$ to pass this information up the chain. These steps are described schematically as follows:

Spectral information about $H^{(n)}(\lambda) \Longrightarrow$ (by 'isospectrality' of $\mathcal{R}_\rho$)
Spectral information about $H^{(n-1)}(\lambda)$
...
$\Longrightarrow$ Spectral information about $H(\lambda)$.
We sum up this as

$$\textbf{Spec info}\ (H) \Longleftarrow \textbf{RG BOX} \Longleftarrow \textbf{Spec info}\ (H^{(n)}(\lambda)).$$

If we start with the Hamiltonian $H_\theta$, then we are interested in, we derive this way the required spectral information about it.

## 10 Related Results

**Existence of the ionization threshold**. There is $\Sigma = \Sigma^{(p)} + O(g^2) > \inf \sigma(H)$ (the ionization threshold) s.t. for any energy interval in $\Delta \subset (\inf \sigma(H), \Sigma)$,

$$\|e^{\delta|x|}\psi\| < \infty,\ \forall \psi \in \mathrm{Ran} E_\Delta(H),\ \delta < \Sigma - \sup \Delta.$$

**Analyticity.** If the ground state energy $\epsilon_g$ of $H$ is non-degenerate, then it is analytic in $g$ and has the following expansion

$$\epsilon_g = \sum_{j=1}^{\infty} \epsilon_\alpha^{2j} \alpha^{3j}, \tag{25}$$

where $\epsilon_\alpha^{2j}$ are smooth function of $\alpha$ (recall that $g := \alpha^{3/2}$).

The proof relies on the renormalization group analysis together with the analyticity and the form-factor rotation symmetry transfer by the Feshbach-Schur map (see Appendix D.3) and on treating two sources of the dependence of $H$ on the coupling constant $g$ - in the prefactor of $A_\chi(y)$ and in its argument (see the line after (31)), differently, i.e. by considering the following family of Hamiltonians

$$H := \sum_{j=1}^{n} \frac{1}{2m_j}(i\nabla_{x_j} - gA_\chi(\alpha x_j))^2 + V(x) + H_f, \tag{26}$$

and consider $g$ and $\alpha$ as independent variables. The powers of $g$ in (25) come from $g$ in (26).

**Analyticity of all parts of $H$.** Suppose that $\lambda \mapsto H_\lambda \equiv H(\underline{w}^\lambda)$ is of the form (20) and is analytic in $\lambda \in S \subset \mathbb{C}$ and that $H(\underline{w}^\lambda)$ belongs to some polydisc $\mathcal{D}(\alpha, \beta, \gamma)$ for all $\lambda \in S$. Then $\lambda \mapsto E_\lambda := w_{0,0}^\lambda(0)$, $T_\lambda := w_{0,0}^\lambda(H_f) - w_{0,0}^\lambda(0)$, $W_\lambda := H_\lambda - E_\lambda - T_\lambda$ are analytic in $\lambda \in S$.

**Resonance poles.** Can we make sense of the resonance poles in the present context? Let

$$Q := \{z \in \mathbb{C}^- \mid \epsilon_0 < \operatorname{Re} z < \nu\} / \bigcup_{j \le j(\nu), k} S_{j,k}.$$

**Theorem 10.1.** For each $\Psi$ and $\Phi$ from a dense set of vectors (say, (7)), the meromorphic continuation, $F(z, \Psi, \Phi)$, of the matrix element $\langle \Psi, (H - z)^{-1}\Phi\rangle$ is of the following form near the resonance $\epsilon_j$ of $H$:

$$F(z, \Psi, \Phi) = (\epsilon_{j,k} - z)^{-1} p(\Psi, \Phi) + r(z, \Psi, \Phi). \tag{27}$$

Here $p$ and $r(z)$ are sesquilinear forms in $\Psi$ and $\Phi$, s.t.

- $r(z)$ is analytic in $Q$ and bounded on the intersection of a neighbourhood of $\epsilon_{j,k}$ with $Q$ as

$$|r(z, \Psi, \Phi)| \le C_{\Psi,\Phi} |\epsilon_{j,k} - z|^{-\gamma} \text{ for some } \gamma < 1;$$

- $p \neq 0$ at least for one pair of vectors $\Psi$ and $\Phi$ and $p = 0$ for a dense set of vectors $\Psi$ and $\Phi$ in a finite co-dimension subspace.

**Local decay.** For any compactly supported function $f(\lambda)$ with $\operatorname{supp} f \subseteq (\inf H, \infty)/$(a neighbourhood of $\Sigma$), and for $\theta > \frac{1}{2}$, $\nu < \theta - \frac{1}{2}$, we have that

$$\|\langle Y\rangle^{-\theta} e^{-iHt} f(H) \langle Y\rangle^{-\theta}\| \le Ct^{-\nu}. \tag{28}$$

Here $\langle Y\rangle := (\mathbf{1} + Y^2)^{1/2}$, and $Y$ denotes the self-adjoint operator on Fock space $\mathcal{F}$ of photon co-ordinate,

$$Y := \int d^3k \, a^*(k) \, i\nabla_k \, a(k), \tag{29}$$

extended to the Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}$. (A self-adjoint operator $H$ obeying (28) is said to have the $(\Delta, \nu, Y, \theta)$ - *local decay* (LD) property.) (28) shows that for well-prepared initial conditions $\psi_0$, the probability of finding photons within a ball of an arbitrary radius $R < \infty$, centered say at the center-of-mass of the particle system, tends to 0, as time $t$ tends to $\infty$.

Note that one can also show that as $t \to \infty$, the photon coordinate and wave vectors in the support of the solution $e^{-iHt}\psi_0$ of the Schrödinger equation become more and more parallel. This follows from the local decay for the self-adjoint generator of dilatations on Fock space $\mathcal{F}$,

$$B := \frac{i}{2} \int d^3k \, a^*(k) \left\{k \cdot \nabla_k + \nabla_k \cdot k\right\} a(k). \tag{30}$$

(In fact, one first proves the local decay property for $B$ and then transfers it to the photon co-ordinate operator $Y$.)

# 11 Conclusion

Apart from the vacuum polarization, the non-relativistic QED provides a good qualitative description of the physical phenomena related to the interaction of quantized electrons and nuclei and the electro-magnetic field. (Though construction of the scattering theory is not yet completed and the correction to the gyromagnetic ratio is not established, it is fair to conjecture that while both are difficult problems, they should go through without a hitch.)

The quantitative results though are still missing. Does the free parameter, $m$ (or $\kappa$), suffice to give a good fit with the experimental data say on the radiative corrections? Another important open question is the behaviour of the theory in the ultra-violet cut off.

# 12 Comments on Literature

These lectures follow the papers [103, 45, 1], which in turn extend [13, 14, 7]. The papers [103, 45, 1] use the smooth Feshbah-Schur map ([7, 51]), which is much more powerful (see Appendix D), while in these lectures we use, for simplicity, the the original, Feshbach-Schur map (see [13, 14]), which is simpler to formulate.

The self-adjointness of $H$ is not difficult and was proven in [15] for sufficiently small coupling constant ($g$) and in [72] (see also [70]), for an arbitrary one.

Theorems 4.1 and 4.2 were proven in [13, 14, 15] for 'confined particles' (the exact conditions are somewhat technical) and in the present form in [103].

The results of [15] on existence of the ground state were considerably improved in [68, 69, 71, 77, 4, 89] (by compactness techniques) and [9] (by multiscale techniques), with the sharpest result given in [53, 87]. (The papers [15, 68, 53, 85, 86, 87, 77, 89] include the interaction of the spin with magnetic field in the Hamiltonian.)

Related results:

The asymptotic stability of the ground state (local decay, see Section 10): [17, 44, 46].

The survival probabilities of excited states (see (8)): [15, 62, 1].

Atoms with dynamic nuclei: [35, 2, 88].

Analyticity of the ground state eigenvalues in parameters and asymptotic expansions (see Section 10): [15, 9, 11, 52, 63, 64, 65].

Existence of the ionization threshold (see Section 10): [50].

Resonance poles (see Section 10): [1] (see also [15]).

Self-energy and binding energy: [78, 85, 86, 87, 56, 19, 32, 60, 57, 59, 58, 25, 22, 20].

Electron mass renormalization: [59, 79, 8, 27, 47].

One particle states: [39, 40, 30, 31, 47].

Scattering amplitudes: [10].

Semi-relativistic Hamiltonians: [21, 93, 90, 84, 105, 76].

Other aspects of non-relativistic QED:

Photo-electric effect: [18, 54].

Scattering theory: [41, 42, 43].

Stability of matter: [24, 36, 85].

There is an extensive literature on related models, which we do not mention here: Nelson model describing a particle linearly coupled to a free massless scalar field (phonons), semi-relativistic models, based on Dirac equation, and quantum statistics (open systems, positive temperature) models. (In the latter case, one deals with Liouvillians, rather than Hamiltonians, on positive temperature Hilbert spaces. The main results above were proved simultaneously for the QED and Nelson models and extended, at least partially, to positive temperatures.)

# A Hamiltonian of the Standard Model

In this appendix we demonstrate the origin the quantum Hamiltonian $H_g$ given in (1). To be specific consider an atom or molecule with $n$ electrons interacting with radiation field. In this case the Hamiltonian of the system in our units is given by

$$H(\alpha) = \sum_{j=1}^{n} \frac{1}{2m}(i\nabla_{x_j} - \sqrt{\alpha} A_{\chi'}(x_j))^2 + \alpha V(x) + H_f, \tag{31}$$

where $\alpha V(x)$ is the total Coulomb potential of the particle system, $m$ is the electron bare mass, $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$ (the fine-structure constant) and $A_{\chi'}(y)$ is the original vector potential with the ultraviolet cut-off $\chi'$. Rescaling $x \to \alpha^{-1}x$ and $k \to \alpha^2 k$, we arrive at the Hamiltonian (1), where $g := \alpha^{3/2}$ and $A(y) = A_\chi(\alpha y)$, with $\chi(k) := \chi'(\alpha^2 k)$. After that we relax the restriction on $V(x)$ by allowing it to be a standard generalized $n$-body potential (see Subsection 2.1). Note that though this is not displayed, $A(x)$ does depend on $g$. This however does not effect the analysis of the Hamiltonian $H$. (If anything, this makes certain parts of it simpler, as derivatives of $A(x)$ bring down $g$.)

$* * **$

In order not to deal with the problem of center-of-mass motion which is not essential in the present context, we assume that either some of the particles (nuclei) are infinitely heavy (a molecule in the Born-Oppenheimer approximation), or the system is placed in a binding, external potential field. (In the case of the Born-Oppenheimer molecule, the resulting $V(x)$ also depends on the rescaled coordinates of the nuclei, but this does not effect our analysis except of making the complex deformation of the particle system more complicated (see [82]).) This means that the operator $H_p$ has isolated eigenvalues below its essential spectrum. The general case is considered below.

In order to take into account the particle spin we change the state space the particle system to $\mathcal{H}_p = \otimes_1^n L^2(\mathbb{R}^3, \mathbb{C}^2)$ (or the antisymmetric in identical particles subspace thereof), and the standard quantum Hamiltonian on $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, is taken to be (see e.g. [33, 34, 100])

$$H_{spin} = \sum_{j=1}^{n} \frac{1}{2m}[\sigma_j \cdot (i\nabla_{x_j} - gA_\chi(x_j))]^2 + V(x) + H_f, \tag{32}$$

where $\sigma_j := (\sigma_{j1}, \sigma_{j2}, \sigma_{j3}),\ \sigma_{ji}$ are the Pauli matrices of the $j-$th particle and the identity operator on $\mathbb{C}^{2n}$ is omitted in the last two terms. It is easy to show that

$$[\sigma \cdot (i\nabla_x - gA_\chi(x))]^2 = (i\nabla_x - gA_\chi(x))^2 + g\sigma \cdot B(x). \tag{33}$$

where $B(x) := \mathrm{curl} A_\chi(x)$ is the magnetic field. As a result the operator (34) can be rewritten as

$$H_{spin} = H \otimes \mathbf{1} + \mathbf{1} \otimes g \sum_{j=1}^{n} \frac{1}{2m} \sigma_j \cdot B(x_j). \tag{34}$$

For the semi-relativistic Hamiltonian, the non-relativistic kinetic energy $\frac{1}{2m}|p|^2$ is replaced by the relativistic one, $\sqrt{|p|^2 + m^2}$ or $\sqrt{(\sigma \cdot p)^2 + m^2}$.

# B Translationally Invariant Hamiltonians

If we do not assume that the nuclei are infinitely and there are no external forces acting on the system, then the Hamiltonian (1) is translationally symmetric. This leads to conservation of the total momentum (a quantum version of the classical Noether theorem). Indeed, the system of particles interacting with the quantized electromagnetic fields is invariant under translations of the particle coordinates, $\underline{x} \to \underline{x} + \underline{y}$, where $\underline{y} = (y, \ldots, y)$ ($n-$ tuple) and the fields, $A(x) \to A(x-y)$, i.e. $H$ commutes with the translations $T_y$ : $\Psi(\underline{x}) \to e^{iy\cdot P_{\mathrm{f}}}\Psi(\underline{x} + \underline{y})$, where $P_{\mathrm{f}}$ is the momentum operator associated to the quantized radiation field,

$$P_f = \sum_\lambda \int dk\, k\, a^*_\lambda(k) a_\lambda(k).$$

It is straightforward to show that $T_y$ are unitary operators and that they satisfy the relations $T_{x+y} = T_x T_y$, and therefore $y \to T_y$ is a unitary Abelian representation of $\mathbb{R}^3$. Finally, we observe that the group $T_y$ is generated by the total momentum operator, $P_{tot}$, of the electrons and the photon field: $T_y = e^{iy\cdot P_{tot}}$. Here $P_{tot}$ is the selfadjoint operator on $\mathcal{H}$, given by

$$P_{tot} := \sum_i p_i \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes P_f \tag{35}$$

where, as above, $p_j := -i\nabla_{x_j}$, the momentum of the $j-$th electron and $P_f$ is the field momentum given above. Hence $[H, P_{tot}] = 0$.

Let $\mathcal{H}$ be the direct integral $\mathcal{H} = \int_{\mathbb{R}^3}^\oplus \mathcal{H}_P dP$, with the fibers $\mathcal{H}_P := L^2(X) \otimes \mathcal{F}$, where $X := \{x \in \mathbb{R}^{3n} \mid \sum_i m_i x_i = 0\} \simeq \mathbb{R}^{3(n-1)}$, (this means that $\mathcal{H} = L^2(\mathbb{R}^3, dP; L^2(X)\otimes\mathcal{F})$) and define $U : \mathcal{H}_{\mathrm{el}}\otimes\mathcal{H}_{\mathrm{f}} \to \mathcal{H}$ on smooth functions with compact domain by the formula

$$(U\Psi)(\underline{x}', P) = \int_{\mathbb{R}^3} e^{i(P-P_{\mathrm{f}})\cdot x_{cm}} \Psi(\underline{x}' + \underline{x}_{cm}) \mathrm{d}y, \tag{36}$$

where $\underline{x}'$ are the coordinates of the $N$ particles in the center-of-mass frame and $\underline{x}_{cm} = (x_{cm}, \ldots, x_{cm})$ ($n-$ tuple), with $x_{cm} = \frac{1}{\sum_i m_i}\sum_i m_i x_i$, the center-of-mass coordinate, so that $\underline{x} = \underline{x}' + \underline{x}_{cm}$. Then $U$ extends uniquely to a unitary operator (see below). Its converse is written, for $\Phi(\underline{x}', P) \in L^2(X) \otimes \mathcal{F}$, as

$$(U^{-1}\Phi)(\underline{x}) = \int_{\mathbb{R}^3} e^{-\imath x_{cm}\cdot(P-P_{\mathrm{f}})} \Phi(\underline{x}', P) \mathrm{d}P. \tag{37}$$

The functions $\Phi(\underline{x}', P) = (U\Psi)(\underline{x}', P)$ are called fibers of $\Psi$. One can easiely prove the following

**Lemma B.1.** The operations (36) and (37) define unitary maps $L^2(\mathbb{R}^{3n}) \otimes \mathcal{F} \to \mathcal{H}$ and $\mathcal{H} \to L^2(\mathbb{R}^{3n}) \otimes \mathcal{F}$, and are mutual inverses.

Since $H$ commutes with $P_{tot}$, it follows that it admits the fiber decomposition

$$UHU^{-1} = \int_{\mathbb{R}^3}^\oplus H(P) \mathrm{d}P, \tag{38}$$

where the fiber operators $H(P)$, $P \in \mathbb{R}^3$, are self-adjoint operators on $\mathcal{F}$. Using $a(k)e^{-\imath y\cdot P_{\mathrm{f}}} = e^{-\imath y\cdot(P_{\mathrm{f}}+k)}a(k)$ and $a^*(k)e^{-iy\cdot P_{\mathrm{f}}} = e^{-iy\cdot(P_{\mathrm{f}}-k)}a^*(k)$, we find $\nabla_y e^{iy\cdot(P-P_{\mathrm{f}})} A_\chi(x'+y) e^{\imath y\cdot(P-P_{\mathrm{f}})} = 0$ and therefore

$$A_\chi(x) e^{iy\cdot(P-P_{\mathrm{f}})} = e^{iy\cdot(P-P_{\mathrm{f}})} A_\chi(x-y). \tag{39}$$

Using this and (37), we compute $H(U^{-1}\Phi)(x) = \int_{\mathbb{R}^3} e^{ix\cdot(P-P_{\mathrm{f}})} H(P)\Phi(P) dP$, where $H(P)$ are Hamiltonians on the space fibers $\mathcal{H}_P := \mathcal{F}$ given explicitly by

$$H(P) = \sum_j \frac{1}{2m_i}\big(P - P_{\mathrm{f}} - i\nabla_{x'_j} - e_i A_\chi(x'_j)\big)^2 + V_{\mathrm{coul}}(\underline{x}') + H_{\mathrm{f}} \tag{40}$$

where $x'_i = x_i - x_{cm}$ is the coordinate of the $i-$th particle in the center-of-mass frame. Now, this hamiltonian can be investigated similarly to the one in (1).

# C Proof of Theorem 6.1

In this appendix we *omit the subindex* $\rho$ at $\chi_\rho$ and $\overline{\chi}_\rho$, and *replace the subindex* $\rho$ in other operators by the subindex $\chi$. Moreover, we replace $H - \lambda$ by $H$. Though $\chi$ and $\overline{\chi}$ we deal with are projections, we often keep the powers $\chi^2$ and $\overline{\chi}^2$, which occur often below, having in mind showing possible generalization to $\chi$ and $\overline{\chi}$ which are 'almost (or smooth) projections' satisfying $\chi^2 + \overline{\chi}^2 = 1$ (see Appendix D).

First we note that the relation between $\psi$ and $\varphi$ in Theorem 6.1 (ii) is $\varphi = \chi\psi,\ \psi = Q_\chi(H)\varphi$, and between $H^{-1}$ and $F_\chi(H)^{-1}$ in (iv) is

$$H^{-1} = Q_\chi(H)\, F_\chi H^{-1}\, Q_\chi(H)^\# +\ \overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi}, \tag{41}$$

where $H_{\overline{\chi}} := \overline{\chi}_\rho H \overline{\chi}_\chi$ and $Q_\chi(H)$ and $Q_\chi(H)^\#$ are the operators, given by

$$Q_\chi(H) := \chi\ -\ \overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi} H\chi,$$

$$Q_\chi^\#(H) := \chi\ -\ \chi H\overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi}.$$

*Proof of Theorem 6.1.* Throughout the proof we use the notation $F := F_\chi(H)$, $Q := Q_\chi(H)$, and $Q^\# := Q_\chi^\#(H)$. Note that (i) ($0 \in \rho(H) \Leftrightarrow 0 \in \rho(F_\chi(H))$) follows from (iv) ($H^{-1}$ exists $\Leftrightarrow F_\chi(H)^{-1}$ exists) and (iv) follows from (41), so we start with the latter.

**Proof of** (41)**.** The next two identities,

$$H\, Q\ =\ \chi\, F \qquad \text{and} \qquad Q^\#\, H\ =\ F\, \chi, \tag{42}$$

are of key importance in the proof. They both derive from a simple computation, which we give only for the first equality in (42). We observe the relations

$$H\, \chi\ =\ \chi\, H_\chi\ +\ \overline{\chi}^2\, H\chi, \qquad \text{and} \qquad H\, \overline{\chi}\ =\ \overline{\chi}\, H_{\overline{\chi}}\ +\ \chi^2\, H\overline{\chi}, \tag{43}$$

which follow from $\chi^2 + \overline{\chi}^2 = \mathbf{1}$. Now, using the definition of the operator $Q$ and the relations (43), we obtain

$$H\, Q\ \ =\ \ \chi H_\chi\ +\ \overline{\chi}^2 H\chi\ -\ \left(\overline{\chi} H_{\overline{\chi}} + \chi^2 H\overline{\chi}\right) H_{\overline{\chi}}^{-1}\overline{\chi} H\chi. \tag{44}$$

Canceling the second term on the r.h.s. with the first term in the parentheses in the third term, we see that the r.h.s is equal $\chi\, F$, which gives the first equality in (42).

Now, suppose first that the operator $F$ has bounded invertible and define

$$R\ :=\ Q\, F^{-1}\, Q^\#\ +\ \overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi}. \tag{45}$$

Using (42) and (43), we obtain

$$\begin{aligned} H\, R\ &=\ H\, Q\, F^{-1}\, Q^\#\ +\ \left(\overline{\chi} H_{\overline{\chi}} + \chi^2 H\overline{\chi}\right) H_{\overline{\chi}}^{-1}\overline{\chi} \\ &=\ \chi\, Q^\#\ +\ \overline{\chi}^2\ +\ \chi^2 H\overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi} \\ &=\ \chi^2\ +\ \overline{\chi}^2\ =\ \mathbf{1}, \end{aligned} \tag{46}$$

and, similarly, $RH = \mathbf{1}$. Thus $R = H^{-1}$, and (41) holds true.

Conversely, suppose that $H$ is bounded invertible. Then, using the definition of $F$ and the relation $\chi^2 + \overline{\chi}^2 = \mathbf{1}$, we obtain

$$\begin{aligned} F\,\chi\, H^{-1}\,\chi\ &=\ \chi H\,\chi^2\ H^{-1}\,\chi -\ \chi H\overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi} H\chi^2\, H^{-1}\chi \\ &=\ \chi H\,\chi^2\ H^{-1}\,\chi -\ \chi H\overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi} H\, H^{-1}\chi +\ \chi H\overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi} H\overline{\chi}^2\, H^{-1}\chi \\ &=\ \chi H\,\chi^2\ H^{-1}\,\chi +\ \chi H\overline{\chi}^2\, H^{-1}\chi = \chi^2. \end{aligned} \tag{47}$$

Similarly, one checks that $\chi\, H^{-1}\, \chi F = \mathbf{1}$. Thus $F$ is invertible on $\operatorname{Ran}\chi$ with inverse $F^{-1} = \chi\, H^{-1}\, \chi$.
**Proof of (ii) ($H\psi = \lambda\psi \iff F_\rho(H-\lambda)\,\varphi = 0$).** If $\psi \in \mathcal{H}\setminus\{0\}$ solves $H\psi = 0$ then (42) implies that

$$F\chi\psi \;=\; Q^{\#}\, H\,\psi \;=\; 0. \tag{48}$$

Furthermore, by (43), $0 \;=\; \overline{\chi}\, H\,\psi \;=\; H_{\overline{\chi}}\overline{\chi}\psi \;+\; \overline{\chi}H\chi^2\psi$, and hence

$$Q\,\chi\psi \;=\; \chi^2\psi \;-\; \overline{\chi}H_{\overline{\chi}}^{-1}\overline{\chi}H\chi^2\psi \;=\; \chi^2\psi \;+\; \overline{\chi}^2\psi \;=\; \psi. \tag{49}$$

Therefore, $\psi \neq 0$ implies $\chi\psi \neq 0$.

If $\varphi \in \operatorname{Ran}\chi\setminus\{0\}$ solves $F\varphi = 0$ then the definition of $Q$ implies that

$$\chi Q\varphi = \chi\varphi = \varphi, \tag{50}$$

which implies that $Q\varphi \neq 0$ provided $\varphi \neq 0$.
**Proof of (iii) ($\dim\operatorname{Null}(H-\lambda) = \dim\operatorname{Null}F_\rho(H-\lambda)$).** By (i), $\dim\operatorname{Null}H = 0$ is equivalent to $\dim\operatorname{Null}F = 0$, assuming that $H \in D(F)$. We may therefore assume that $\operatorname{Null}H \neq 0$ and $\operatorname{Null}F \neq 0$ are both nontrivial. Eq. (49) shows that $\chi : \operatorname{Null}H \to \operatorname{Null}F$ is injective, hence $\dim\operatorname{Null}H \leq \dim\operatorname{Null}F$, and Eq. (50) shows that $Q : \operatorname{Null}F \to \operatorname{Null}H$ is injective, hence $\dim\operatorname{Null}H \geq \dim\operatorname{Null}F$. This establishes (iv) and moreover that $\chi : \operatorname{Null}H \to \operatorname{Null}F$ and $Q : \operatorname{Null}F \to \operatorname{Null}H$ are actually bijections. □

# D Smooth Feshbach-Schur Map

## D.1 Definition and isospectrality

We define the smooth Feshbach-Schur map and formulate its important isospectral property Let $\chi$, $\overline{\chi}$ be a partition of unity on a separable Hilbert space $\mathcal{H}$, i.e. $\chi$ and $\overline{\chi}$ are positive operators on $\mathcal{H}$ whose norms are bounded by one, $0 \leq \chi, \overline{\chi} \leq \mathbf{1}$, and $\chi^2 + \overline{\chi}^2 = \mathbf{1}$. We assume that $\chi$ and $\overline{\chi}$ are nonzero. Let $\tau$ be a (linear) projection acting on closed operators on $\mathcal{H}$ with the property that operators in its image commute with $\chi$ an $\tau(\mathbf{1}) = \mathbf{1}$. Let $\overline{\tau} := \mathbf{1} - \tau$ and define

$$H_{\tau,\chi^{\#}} \;\; := \tau(H) \;+\; \chi^{\#}\overline{\tau}(H)\chi^{\#}, \tag{51}$$

where $\chi^{\#}$ stands for either $\chi$ or $\overline{\chi}$.

Given $\chi$ and $\tau$ as above, we denote by $D_{\tau,\chi}$ the space of closed operators, $H$, on $\mathcal{H}$ which belong to the domain of $\tau$ and satisfy the following three conditions:

(i) $\tau$ and $\chi$ (and therefore also $\overline{\tau}$ and $\overline{\chi}$) leave the domain $D(H)$ of $H$ invariant:

$$D(\tau(H)) = D(H) \text{ and } \chi D(H) \subset D(H), \tag{52}$$

(ii)

$$H_{\tau,\overline{\chi}} \text{ is (bounded) invertible on } \operatorname{Ran}\overline{\chi}, \tag{53}$$

(iii)

$$\overline{\tau}(H)\chi \text{ and } \chi\overline{\tau}(H) \text{ extend to bounded operators on } \mathcal{H}. \tag{54}$$

(For more general conditions see [7, 51].)

The *smooth Feshbach-Schur map (SFM)* maps operators from $D_{\tau,\chi}$ into operators on $\mathcal{H}$ by

$$F_{\tau,\chi}(H) \;:=\; H_0 \;+\; \chi W\chi \;-\; \chi W\overline{\chi}H_{\tau,\overline{\chi}}^{-1}\overline{\chi}W\chi, \tag{55}$$

where $H_0 := \tau(H)$ and $W := \overline{\tau}(H)$. Note that $H_0$ and $W$ are closed operators on $\mathcal{H}$ with coinciding domains, $D(H_0) = D(W) = D(H)$, and $H = H_0 + W$. We remark that the domains of $\chi W\chi$, $\overline{\chi}W\overline{\chi}$, $H_{\tau,\chi}$, and $H_{\tau,\overline{\chi}}$ all contain $D(H)$.

Define operators $Q_{\tau,\chi}(H) := \chi \;-\; \overline{\chi}\,H_{\tau,\overline{\chi}}^{-1}\overline{\chi}W\chi$ and $Q_{\tau,\chi}^{\#}(H) := \chi \;-\; \chi W\overline{\chi}\,H_{\tau,\overline{\chi}}^{-1}\overline{\chi}$. The following result ([7]) generalizes Theorem 6.1 above; its proof is similar to the one of that theorem:

**Theorem D.1** (Isospectrality of SFM)**.** Let $0 \leq \chi \leq \mathbf{1}$ and $H \in D_{\tau,\chi}$ be an operator on a separable Hilbert space $\mathcal{H}$. Then we have the following results:

(i) $H$ is bounded invertible on $\mathcal{H}$ if and only if $F_{\tau,\chi}(H)$ is bounded invertible on $\operatorname{Ran}\chi$. In this case

$$H^{-1} = Q_{\tau,\chi}(H)\, F_{\tau,\chi}(H)^{-1}\, Q_{\tau,\chi}(H)^{\#} + \overline{\chi}\, H_{\overline{\chi}}^{-1}\overline{\chi}, \tag{56}$$

$$F_{\tau,\chi}(H)^{-1} = \chi\, H^{-1}\,\chi + \overline{\chi}\,\tau(H)^{-1}\overline{\chi}. \tag{57}$$

(ii) If $\psi \in \mathcal{H}\setminus\{0\}$ solves $H\psi = 0$ then $\varphi := \chi\psi \in \operatorname{Ran}\chi\setminus\{0\}$ solves $F_{\tau,\chi}(H)\,\varphi = 0$.

(iii) If $\varphi \in \operatorname{Ran}\chi\setminus\{0\}$ solves $F_{\tau,\chi}(H)\,\varphi = 0$ then $\psi := Q_{\tau,\chi}(H)\varphi \in \mathcal{H}\setminus\{0\}$ solves $H\psi = 0$.

(iv) The multiplicity of the spectral value $\{0\}$ is conserved in the sense that $\dim\operatorname{Null} H = \dim\operatorname{Null} F_{\tau,\chi}(H)$.

We also mention the following useful property of $F_{\tau,\chi}$:

$$H \text{ is self-adjoint} \quad\Rightarrow\quad F_{\tau,\chi}(H) \text{ is self-adjoint.} \tag{58}$$

## D.2 Transfer of local decay

We have shown above that the smooth Feshbach-Schur map is isospectral. In fact, under certain additional conditions it preserves (or transfers) much stronger spectral property - the limiting absorption principle (LAP) ([46]), which is defined as follows. Let $\Delta \subset \mathbb{R}$ be an interval, $\nu > 0$ and $B$, a self-adjoint operator. We say that a $C^1$ family of self-adjoint operators, s.t. $H(\lambda) \in D_{\tau,\chi}$ has the $(\Delta, \nu, B, \theta)$ *limiting absorption principle* (LAP) property iff

$$\lim_{\varepsilon\to 0+} \langle B\rangle^{-\theta}\big(H(\lambda) - i\varepsilon\big)^{-1}\langle B\rangle^{-\theta} \text{ exists and } \in C^{\nu}(\Delta). \tag{59}$$

Usually LAP holds for $\nu < \theta - \frac{1}{2}$. One can show that the LAP implies the local decay property (see, e.g. [99], vol III; recall that the definition of the local decay property is given in Section 10).

**Theorem D.2.** Let $\Delta \subset \mathbb{R}$ and , $\forall\lambda \in \Delta,\ H(\lambda)$ be a $C^1$ family of self-adjoint operators, s.t. $H(\lambda) \in D_{\tau,\chi}$. Assume that there is a self-adjoint operator $B$ s.t.

$$\mathrm{ad}_B^j(A) \text{ is bounded and differentiable in } \lambda,\ \forall j \leq 2, \tag{60}$$

where $A$ is one of the operators $\chi,\ \overline{\chi},\ \chi\bar{\tau}(H(\lambda)),\ \bar{\tau}(H(\lambda))\chi,\ \partial_\lambda^k(\overline{\chi}H_{\tau,\overline{\chi}}(\lambda)^{-1}\overline{\chi}),\ k = 0, 1$. Then, for any $0 \leq \nu \leq 1$ and $0 < \theta \leq 1$ and in the operator norm, we have

$$\lim_{\varepsilon\to 0+} \langle B\rangle^{-\theta}\left(F_{\tau,\chi}(H(\lambda)) - i\varepsilon\right)^{-1}\langle B\rangle^{-\theta} \text{ exists and } \in C^{\nu}(\Delta) \tag{61}$$

$$\Rightarrow \lim_{\varepsilon\to 0+} \langle B\rangle^{-\theta}\big(H(\lambda) - i\varepsilon\big)^{-1}\langle B\rangle^{-\theta} \text{ exists and } \in C^{\nu}(\Delta). \tag{62}$$

This allows one to reduce the proof of the LAP for the original operator, $H - \lambda$, to the proof of this property for a much simpler one, $\mathcal{R}_\rho^n(H - \lambda)$.

### D.3 Transfer of analyticity

**Theorem D.3.** Let $\Lambda$ be an open set in $\mathbb{C}$ and $H(\lambda)$, $\lambda \in \Lambda$, a family of operators with a fixed domain, which belong to the domain of $F_{\tau\chi}$. Assume $H(\lambda)$ and $\tau(H(\lambda))$, with the same domain, are analytic in the sense of Kato (see e.g. [99], vol IV). Then we have that

- $F_{\tau,\chi}(H(\lambda))$ is an analytic in $\lambda \in \Lambda$ family of operators.

*Proof.* Since that $H(\lambda)$, $H_0(\lambda) := \tau(H(\lambda))$ and $W(\lambda)\chi$ and $\chi W(\lambda)$, where $W(\lambda) := \overline{\tau}(H(\lambda))$, are analytic in $\lambda \in \Lambda$, we see from the definition of the smooth Feshbach-Schur map $F_{\tau\chi}$ in (55) that $F_{\tau\chi}(H(\lambda))$ is analytic in $\lambda \in \Lambda$, provided $\overline{\chi}_\rho H(\lambda)^{-1}_{\tau,\overline{\chi}_\rho}\,\overline{\chi}_\rho$ is analytic in $\Lambda$. The analyticity of the latter family follows by the Neumann series argument. □

One can generalize the above result to $\Lambda$'s which are open sets in a complex Banach space. Recall that a complex vector-function $f$ in an open set $\Lambda$ in a complex Banach space $\mathcal{W}$ is said to be *analytic* iff it is locally bounded and Gâteaux-differentiable. One can show that $f$ is analytic iff $\forall \xi \in \mathcal{W},\ f(H + \tau\xi)$ is analytic in the complex variable $\tau$ for $|\tau|$ sufficiently small (see [23, 66]). Furthermore if $f$ is analytic in $\Lambda$ and $g$ is an analytic vector-function from an open set $\Omega$ in $\mathbb{C}$ into $\Lambda$, then the composite function $f \circ g$ is analytic on $\Omega$.

# E Mass Renormalization

As the free electron is surrounded by virtual 'soft' photons its effective (inertial0 mass is greater than the value ('bare' mass) entering its Hamiltonian. One calls this electron *mass renormalization*. We begin with analyzing the definition of (inertial) mass in Classical Mechanics. Consider a classical particle with the Hamiltonian $h(x, k) := K(k) + V(x)$, where $K(k)$ is some function describing the kinetic energy of the particle. To find the particle mass in this case we have to determine the relation between the force and acceleration at very low velocities. The Hamilton equations give $\dot{x} = \partial_k K$ and $\dot{k} = F$, where $F = -\partial_x V$ is the force acting on the particle. Assuming that $K$ has a minimum at $k = 0$ and expanding $\partial_k K(k)$ around 0, differentiating the resulting relation $\dot{x} = K''(0)k$, where $K''(0)$ is the hessian of $K$ at $k = 0$, w.r. to time and using the second Hamilton equation, we obtain $\ddot{x} = K''(0)F(x)$. This suggests to define the mass of the particle as $m = K''(0)^{-1}$, i.e. as the inverse of the Hessian of the energy, in the absence of external forces, as a function of of momentum at 0. ($K(k)$ is called the dispersion relation.) We adopt this as a general definition: *the (effective) mass of a particle interacting with fields is the inverse of the Hessian of the energy of the total system as a function of of the total momentum at* 0.

Now, we consider a single non-relativistic electron coupled to quantized electromagnetic field. Recall that the charge of electron is denoted by $-e$ and its *bare* mass in our units is $m$. The corresponding Hamiltonian is

$$H := \frac{1}{2m}(i\nabla_x \otimes 1_f - eA_\chi(x))^2 + \mathbf{1}_{el} \otimes H_f, \tag{63}$$

acting on the space $L^2(\mathbb{R}^3) \otimes \mathcal{F} \equiv \mathcal{H}_{\rm part} \otimes \mathcal{H}_f$. It is the generator for the dynamics of a single non-relativistic electron, and of the electromagnetic radiation field, which interact via minimal coupling. Here recall $A_\chi(x)$ and $H_f$ are the quantized electromagnetic vector potential with ultraviolet cutoff and the field Hamiltonian and are defined in (2) and (3)

The system considered is *translationally invariant* in the sense that $H$ commutes with the translations, $T_y$,

$$T_y H = H T_y,$$

which in the present case take the form

$$T_y : \Psi(\underline{x}) \to e^{iy\cdot P_{\rm f}}\Psi(\underline{x} + \underline{y}), \tag{64}$$

This as before leads to $H$ commuting with the total momentum operator,

$$P_{tot} := P_{\mathrm{el}} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes P_f, \tag{65}$$

of the electron and the photon field: $[H, P_{tot}] = 0$. Here $P_{\mathrm{el}} := -i\nabla_x$ and $P_f = \sum_\lambda \int dk\, k\, a^*_\lambda(k) a_\lambda(k)$ are electron and field momenta. Again as in Appendix B, this leads to the fiber decomposition

$$UHU^{-1} = \int_{\mathbb{R}^3}^{\oplus} H(P)\mathrm{d}P, \tag{66}$$

where the fiber operators $H(P)$, $P \in \mathbb{R}^3$, are self-adjoint operators on $\mathcal{F}$. Using $a(k)e^{-\imath x\cdot P_{\mathrm{f}}} = e^{-\imath x\cdot(P_{\mathrm{f}}+k)}a(k)$ and $a^*(k)e^{-\imath x\cdot P_{\mathrm{f}}} = e^{-\imath x\cdot(P_{\mathrm{f}}-k)}a^*(k)$, we find $\nabla_x e^{\imath x\cdot(P-P_{\mathrm{f}})}A_\chi(x)e^{\imath x\cdot(P-P_{\mathrm{f}})} = 0$ and therefore

$$A_\chi(x)e^{\imath x\cdot(P-P_{\mathrm{f}})} = e^{\imath x\cdot(P-P_{\mathrm{f}})}A_\chi(0). \tag{67}$$

Using this and (37), we compute $H(U^{-1}\Phi)(x) = \int_{\mathbb{R}^3} e^{\imath x\cdot(P-P_{\mathrm{f}})}H(P)\Phi(P)\mathrm{d}P$, where $H(P)$ are Hamiltonians on the fibers $\mathcal{H}_P := \mathcal{F}$ given explicitly by

$$H(P) = \frac{1}{2m}\big(P - P_{\mathrm{f}} - eA_\chi\big)^2 + H_{\mathrm{f}} \tag{68}$$

where $A_\chi := A_\chi(0)$. Explicitly, $A_\chi$ is given by

$$A_\chi = \sum_\lambda \int \mathrm{d}k\, \frac{\chi(|k|)}{|k|^{1/2}}\, \epsilon_\lambda(k)\,\{\, a_\lambda(k) + a^*_\lambda(k)\,\}. \tag{69}$$

Consider the infimum $E(P) := \inf \sigma(H(P))$ of the spectrum of the fiber Hamiltonian $H(P)$. Note that for $e = 0$, $E(P)|_{e=0} =: E_0(P)$ is the ground state energy of $H_0(P) := H(P)|_{e=0} = \frac{1}{2m}\big(P - P_{\mathrm{f}}\big)^2 + H_{\mathrm{f}}$ with the ground state $\Omega$ and is $E_0(P) = \frac{|P|^2}{2m}$. The renormalized electron mass is defined as

$$E(P) = \frac{|P|^2}{2m_{ren}} + O(|P|^3)$$

where the left hand side is computed perturbatively up the second order in the coupling constant (charge). Provided that $E(P)$ is spherically symmetric and $C^2$ at $P = 0$, and therefore, in particular, $\partial_{|P|}E(0) = 0$, we define the renormalized electron mass at zero total momentum as

$$m_{ren} := \frac{1}{\partial^2_{|P|}E(0)}\ .$$

The kinematic meaning of this expression is as follows. The ground state energy $E(P)$ can be considered as an effective Hamiltonian of the electron in the ground state. (The propagator $\exp(-itE(P))$ determines the propagation properties of a wave packet formed of dressed one-particle states with a wave function supported near $p = 0$ – which exist as long as there is an infrared regularization.) The first Hamilton equation gives the expression for the electron velocity as

$$v = \partial_P E(P)\ .$$

Expanding the right hand side in $P$ we find $v = \mathrm{Hess}\, E(0)P + O(P^2)$, where

$$\big(\mathrm{Hess}\, E(P)\big)_{ij} = \Big(\delta_{ij} - \frac{P_iP_j}{|P|^2}\Big)\frac{\partial_{|P|}E(P)}{|P|} + \frac{P_iP_j}{|P|^2}\partial^2_{|P|}E(P) \tag{70}$$

is the Hessian of $E(P)$ at $P \in \mathbb{R}^3$ (given that $E(P)$ is spherically symmetric, and $C^2$ in $|P|$ near $P = 0$). It follows from (70) and the fact $\partial_{|P|} E(0) = 0$ that Hess $E(0) = \partial^2_{|P|} E(0)\, \mathbf{1}$, so that

$$v \;=\; \partial^2_{|P|} E(0,\sigma)\, P \;+\; O(P^2)\;.$$

This suggests taking $(\partial^2_{|P|} E(0))^{-1}$ as the renormalized electron mass at $P = 0$.

The following result is proven in [8, 27, 30]:

**Theorem E.1.** For any $P$, s.t. $|P| < \frac{1}{3}$, the infimum of the spectrum $E(P) = \inf \mathrm{spec}(H(P))$ is twice differentiable and satisfies $1 \;\leq\; m_{ren} \;\leq\; 1 + c\, g^2$ for some $c > 0$.

A presentation of the leading order calculations can be found in [79].

**Remark 1.** The estimate $1 \;\leq\; m_{ren} \;\leq\; 1 + c\, g^2$ reflects the fact that the mass of the electron is increased by interactions with the photon field.

# F One-particle States

First we note that for $e = 0$, the one-particle states of the Hamiltonian $H_0 := H|_{e=0} = |P_{\mathrm{el}}|^2 + H_{\mathrm{f}}$ are the generalized eigenfunctions

$$e^{-iP\cdot x} \otimes \Omega, \tag{71}$$

corresponding to the spectral points $E(P) = \frac{|P|^2}{2m}$. This corresponds to the true ground state $\Omega$ of the fiber Hamiltonians $H_0(P) \;:=\; H(P)|_{e=0} \;=\; \frac{1}{2m}(P - P_{\mathrm{f}})^2 + H_{\mathrm{f}}$. The generalization of such a state for the interacting system would be the ground state of $H(P)$, if it existed. However, we have

**Theorem 1.** *$H(P)$ has a ground state if and only if $P = 0$.*

To define one particle states for the interacting model, we first introduce IR regularization, $\kappa_\sigma \in C_0^\infty([0,\kappa];\mathbb{R}_+)$ is assumed to be a smooth cutoff function obeying $\lim_{x\to 0} \frac{\kappa_\sigma(x)}{x^\sigma} \;=\; 1$ . The corresponding Hamiltonian is

$$H_\sigma \;:=\; \frac{1}{2m}\Big( -i\nabla_x \otimes 1_f \;+\; eA_\sigma(x)\Big)^2 \;+\; \mathbf{1}_{el} \otimes H_f \tag{72}$$

with the quantized electromagnetic vector potential subjected, besides the ultra-violet cutoff, also infrared regularization,

$$A_\sigma(x) \;=\; \sum_\lambda \int dk\, \frac{\kappa_\sigma(|k|)}{|k|^{1/2}}\, \{\, \epsilon_\lambda(k)\, e^{-ikx} \otimes a_\lambda(k) \;+\; h.c.\,\}. \tag{73}$$

Let $\mathcal{S} := \{P \in \mathbb{R}^3 \mid |P| < 1/3\}$.

**Theorem 2.** *For $P \in \mathcal{S}$ and for any $\sigma > 0$, the infimum of the spectrum $E_\sigma(P) = \inf \mathrm{spec}(H_\sigma(P))$ is a simple eigenvalue.*

**Remark 2.** *The upper bound on $|P|$ of $\frac{1}{3}$ is not optimal, but we note that, For $E(P)$ to be an eigenvalue, $|P|$ cannot exceed a critical value $P_c < 1$ (corresponding to the speed of light). As $|P| \to P_c$, it is expected that the eigenvalue at $E(P)$ dissolves in the continuous spectrum, while a resonance appears. This is a manifestation of a phenomenon analogous to Cherenkov radiation.*

Let $\Psi_\sigma(P) \in \mathcal{F}$ denote the associated normalized fiber ground state, $\|\Psi_\sigma(P)\|_\mathcal{F} = 1$, for $P \in \mathcal{S}$,

$$H_\sigma(P)\Psi_\sigma(P) \,=\, E_\sigma(P)\,\Psi_\sigma(P)\,. \tag{74}$$

The vector $\Psi(P,\sigma)$ is an *infraparticle state*, describing a compound particle comprising the electron together with a cloud of low-energy (soft) photons whose expected number diverges as $\sigma \to 0$, unless $p = 0$. For $P \in \mathcal{S}$, we introduce the Weyl operators

$$W_{\nabla E_\sigma(P)}(x) := e^{D(x)-D^*(x)} \tag{75}$$

where

$$D(x) \,:=\, \sum_\lambda \int dk\, G_\lambda(k,p)\, e^{-ikx}\, a_\lambda(k)\,, \tag{76}$$

with

$$G_\lambda(k,p) \,:=\, \alpha^{\frac{1}{2}}\, \kappa_\sigma(|k|)\, \frac{\nabla E_\sigma(p)\cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p)\cdot k)}\,. \tag{77}$$

(Here and in the sequel, we will use the abbreviated notation $\nabla E_\sigma(p) \,\equiv\, \nabla_p E(p)\,$.) We observe that they commute with the total momentum operator,

$$[P_{tot}\,,\, W_{\nabla E_\sigma(p)}(x)] \,=\, 0\,. \tag{78}$$

To see this, we note that $[P_{tot}, D(x)] = 0 = [P_{tot}, D^*(x)]$. Indeed, we have that

$$\begin{aligned} & P_{tot} \sum_\lambda \int dk\, G_\lambda(k,p)\, e^{-ikx}\, a_\lambda(k)\, \psi(x) \\ & = \sum_\lambda \int dk\, G_\lambda(k,p)\, (-i\nabla_x + P_f)\, e^{-ikx}\, a_\lambda(k)\, \psi(x) \\ & = \sum_\lambda \int dk\, G_\lambda(k,p)\, e^{-ikx}\, a_\lambda(k)\, (-i\nabla_x + k + P_f - k)\, \psi(x) \\ & = \sum_\lambda \int dk\, G_\lambda(k,p)\, e^{-ikx}\, a_\lambda(k)\, P_{tot}\, \psi(x)\,. \end{aligned} \tag{79}$$

Accordingly, we infer that $W_{\nabla E_\sigma(p)}(x) = \exp[D(x) - D^*(x)]$ commutes with $P_{tot}$.

Furthermore, we observe that

$$W_{\nabla E_\sigma(p)}(x)\, e^{i(p-P_f)x} \,=\, e^{i(p-P_f)x}\, W_{\nabla E_\sigma(p)} \tag{80}$$

holds. Here and in what follows, we will use the abbreviated notation

$$W_{\nabla E_\sigma(p)} \,\equiv\, W_{\nabla E_\sigma(p)}(x=0)\,. \tag{81}$$

We define the maps

$$\begin{aligned} (\mathcal{W}\phi)(x) \quad &:= \quad \int dp\, W_{\nabla E_\sigma(p)}(x)\, e^{i(p-P_f)x}\, \widehat{\phi}(p) \\ &= \quad \int dp\, e^{i(p-P_f)x}\, W_{\nabla E_\sigma(p)}\, \widehat{\phi}(p)\,. \end{aligned} \tag{82}$$

Likewise,

$$(\mathcal{W}^*\phi)(x) \quad := \quad \int dp\, W^*_{\nabla E_\sigma(p)}(x)\, e^{i(p-P_f)x}\, \widehat{\phi}(p)\,. \tag{83}$$

The associated Bogoliubov-transformed Hamiltonian is given by

$$K_\sigma \;:=\; (\mathcal{W} H_\sigma \mathcal{W}^*)\,. \tag{84}$$

We also introduce the Bogoliubov-transformed fiber Hamiltonians

$$K_\sigma(p) \;:=\; W_{\nabla E_\sigma(p)}\, H_\sigma(p)\, W^*_{\nabla E_\sigma(p)}\,. \tag{85}$$

Then,we observe that

$$\begin{aligned} K_\sigma &= (\mathcal{W} H_\sigma \mathcal{W}^*)(x) \\ &= \int W_{\nabla E_\sigma(p)}(x)\, e^{i(p-P_f)x} H(p) e^{-i(p-P_f)x}\, dP_{P_{tot}}(p)\, W^*_{\nabla E_\sigma(p)}(x) \\ &= \int W_{\nabla E_\sigma(p)}(x)\, e^{i(p-P_f)x} H(p) e^{-i(p-P_f)x}\, W^*_{\nabla E_\sigma(p)}(x)\, dP_{P_{tot}}(p) \\ &= \int e^{i(p-P_f)x} K_\chi(p) e^{-i(p-P_f)x} dP_{P_{tot}}(p). \end{aligned} \tag{86}$$

In particular, we have that

$$\mathcal{W}(H_\sigma \psi) \;=\; K_\sigma(\mathcal{W}\psi)\,, \tag{87}$$

as can be readily verified. Defining

$$\Phi_\sigma(p) \;:=\; W_{\nabla E_\sigma(p)}\, \Psi_\sigma(p)\,, \tag{88}$$

we obtain

$$K_\sigma(p)\, \Phi_\sigma(p) \;=\; E_\sigma(p)\, \Phi_\sigma(p)\,. \tag{89}$$

The following result is proven in [30].

**Theorem 3.** *For any $P \in \mathcal{S}$, the ground state eigenvector $\Phi_\sigma(P)$ of $K_\sigma(P)$ converges strongly in $\mathcal{F}$: $\Phi(P) := \lim_{\sigma\to 0} \Phi_\sigma(P)$ exists in $\mathcal{F}$.*

# G Pull-through formulae

In this appendix we prove the very useful "pull-through" formulae (see [13])

$$a(k)f(H_f) = f(H_f + \omega(k))a(k) \tag{90}$$

and

$$f(H_f)a^*(k) = a^*(k)f(H_f + \omega(k)), \tag{91}$$

valid for any piecewise continuous, bounded function, $f$, on $\mathbb{R}$. First, using the commutation relations for $a(k),\ a^*(k)$, one proves relations (90)- (91) for $f(H) = (H_f - z)^{-1}$, $z \in \mathbb{C}/\bar{\mathbb{R}}^+$. Then using the Stone-Weierstrass theorem, one can extend (90)- (91) from functions of the form $f(\lambda) = (\lambda - z)^{-1}$, $z \in \mathbb{C}\backslash\bar{\mathbb{R}}^+$, to the class of functions mentioned above.

## H Supplement: Creation and Annihilation Operators

Let $\mathfrak{h}$ be either $L^2(\mathbb{R}^3, \mathbb{C}, d^3k)$ or $L^2(\mathbb{R}^3, \mathbb{C}^2, d^3k)$. In the first case we consider $\mathfrak{h}$ to be the Hilbert space of one-particle states of a scalar boson or phonon, and in the second case, of a photon. The variable $k \in \mathbb{R}^3$ is the wave vector or momentum of the particle. (Recall that throughout these lectures, the propagation speed $c$, of photon or photons and Planck's constant, $\hbar$, are set equal to 1.) The Bosonic Fock space, $\mathcal{F}$, over $\mathfrak{h}$ is defined by

$$\mathcal{F} \;:=\; \bigoplus_{n=0}^{\infty} \mathcal{S}_n \, \mathfrak{h}^{\otimes n}, \tag{92}$$

where $\mathcal{S}_n$ is the orthogonal projection onto the subspace of totally symmetric $n$-particle wave functions contained in the $n$-fold tensor product $\mathfrak{h}^{\otimes n}$ of $\mathfrak{h}$; and $\mathcal{S}_0 \mathfrak{h}^{\otimes 0} := \mathbb{C}$. The vector $\Omega := (1, 0, ...)$ is called the *vacuum vector* in $\mathcal{F}$. Vectors $\Psi \in \mathcal{F}$ can be identified with sequences $(\psi_n)_{n=0}^{\infty}$ of $n$-particle wave functions, which are totally symmetric in their $n$ arguments, and $\psi_0 \in \mathbb{C}$. In the first case these functions are of the form, $\psi_n(k_1, \dots, k_n)$, while in the second case, of the form $\psi_n(k_1, \lambda_1, \dots, k_n, \lambda_n)$, where $\lambda_j \in \{-1, 1\}$ are the polarization variables.

In what follows we present some key definitions in the first case, limiting ourselves to remarks at the end of this appendix on how these definitions have to be modified for the second case. The scalar product of two vectors $\Psi$ and $\Phi$ is given by

$$\langle \Psi \,,\, \Phi \rangle \;:=\; \sum_{n=0}^{\infty} \int \prod_{j=1}^{n} d^3k_j \; \overline{\psi_n(k_1, \dots, k_n)} \; \varphi_n(k_1, \dots, k_n). \tag{93}$$

Given a one particle dispersion relation $\omega(k)$, the energy of a configuration of $n$ *non-interacting* field particles with wave vectors $k_1, \dots, k_n$ is given by $\sum_{j=1}^{n} \omega(k_j)$. We define the *free-field Hamiltonian*, $H_f$, giving the field dynamics, by

$$(H_f \Psi)_n(k_1, \dots, k_n) \;=\; \Big( \sum_{j=1}^{n} \omega(k_j) \Big) \, \psi_n(k_1, \dots, k_n), \tag{94}$$

for $n \geq 1$ and $(H_f \Psi)_n = 0$ for $n = 0$. Here $\Psi = (\psi_n)_{n=0}^{\infty}$ (to be sure that the r.h.s. makes sense we can assume that $\psi_n = 0$, except for finitely many $n$, for which $\psi_n(k_1, \dots, k_n)$ decrease rapidly at infinity). Clearly that the operator $H_f$ has the single eigenvalue 0 with the eigenvector $\Omega$ and the rest of the spectrum absolutely continuous.

With each function $\varphi \in L^2(\mathbb{R}^3, \mathbb{C}, d^3k)$ one associates an *annihilation operator* $a(\varphi)$ defined as follows. For $\Psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F}$ with the property that $\psi_n = 0$, for all but finitely many $n$, the vector $a(\varphi)\Psi$ is defined by

$$(a(\varphi)\Psi)_n(k_1, \dots, k_n) \;:=\; \sqrt{n+1} \; \int d^3k \; \overline{\varphi(k)} \; \psi_{n+1}(k, k_1, \dots, k_n). \tag{95}$$

These equations define a closable operator $a(\varphi)$ whose closure is also denoted by $a(\varphi)$. Eqn (**??**) implies the relation

$$a(\varphi)\Omega \;=\; 0, \tag{96}$$

The creation operator $a^*(\varphi)$ is defined to be the adjoint of $a(\varphi)$ with respect to the scalar product defined in Eq. (93). Since $a(\varphi)$ is anti-linear, and $a^*(\varphi)$ is linear in $\varphi$, we write formally

$$a(\varphi) \;=\; \int d^3k \; \overline{\varphi(k)} \, a(k), \qquad a^*(\varphi) \;=\; \int d^3k \; \varphi(k) \, a^*(k), \tag{97}$$

where $a(k)$ and $a^*(k)$ are unbounded, operator-valued distributions. The latter are well-known to obey the *canonical commutation relations* (CCR):

$$\big[ a^{\#}(k) \,,\, a^{\#}(k') \big] \;=\; 0, \qquad \big[ a(k) \,,\, a^*(k') \big] \;=\; \delta^3(k - k'), \tag{98}$$

where $a^{\#} = a$ or $a^*$.

Now, using this one can rewrite the quantum Hamiltonian $H_f$ in terms of the creation and annihilation operators, $a$ and $a^*$, as

$$H_f \;=\; \int d^3k\; a^*(k)\; \omega(k)\; a(k), \tag{99}$$

acting on the Fock space $\mathcal{F}$.

More generally, for any operator, $t$, on the one-particle space $L^2(\mathbb{R}^3, \mathbb{C}, d^3k)$ we define the operator $T$ on the Fock space $\mathcal{F}$ by the following formal expression $T := \int a^*(k)ta(k)dk$, where the operator $t$ acts on the $k-$variable ($T$ is the second quantization of $t$). The precise meaning of the latter expression can obtained by using a basis $\{\phi_j\}$ in the space $L^2(\mathbb{R}^3, \mathbb{C}, d^3k)$ to rewrite it as $T := \sum_j \int a^*(\phi_j)a(t^*\phi_j)dk$.

To modify the above definitions to the case of photons, one replaces the variable $k$ by the pair $(k, \lambda)$ and adds to the integrals in $k$ sums over $\lambda$. In particular, the creation- and annihilation operators have now two variables: $a^{\#}_{\lambda}(k) \equiv a^{\#}(k, \lambda)$; they satisfy the commutation relations

$$\left[a^{\#}_{\lambda}(k)\, ,\, a^{\#}_{\lambda'}(k')\right] \;=\; 0, \qquad \left[a_{\lambda}(k)\, ,\, a^*_{\lambda'}(k')\right] \;=\; \delta_{\lambda,\lambda'}\delta^3(k-k'). \tag{100}$$

One can introduce the operator-valued transverse vector fields by

$$a^{\#}(k) := \sum_{\lambda\in\{-1,1\}} e_{\lambda}(k)a^{\#}_{\lambda}(k),$$

where $e_{\lambda}(k) \equiv e(k, \lambda)$ are polarization vectors, i.e., orthonormal vectors in $\mathbb{R}^3$ satisfying $k \cdot e_{\lambda}(k) = 0$. Then, in order to reinterpret the expressions in this paper for photons, one either adds the variable $\lambda$, as was mentioned above, or replaces, in appropriate places, the usual product of scalar functions or scalar functions and scalar operators by the dot product of vector-functions or vector-functions and operator-valued vector-functions.

# References


[1] W. Abou Salem, J. Faupin, J. Fröhlich, I.M.Sigal, On theory of resonances in non-relativisitc QED, *Adv. Appl. Math.*, 2009, to appear; arXiv, 2007.

[2] L. Amour, B. Grébert, J.-C. Guillot, The dressed mobile atoms and ions, *J. Math. Pures Appl.* (9) 86, no. 3, 177–200, 2006.

[3] A. Arai, Mathematical analysis of a model in relativistic quantum electrodynamics. Applications of renormalization group methods in mathematical sciences (in Japanese) Kyoto, 1999.

[4] A. Arai, M. Hirokawa, Ground states of a general class of quantum field Hamiltonians, *Rev. Math. Phys.* 12 , no. 8, 1085–1135, 2000.

[5] A. Arai, M. Hirokawa and Fumio Hiroshima, Regularities of ground states of quantum field models, arXiv.

[6] V. Bach, Mass renormalization in nonrelativisitic quantum electrodynamocs, in Quantum Theory from Small to Large Scales: Lecture Notes of the Les Houches Summer Schools, volume 95. Oxford University Press, 2011.

[7] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, Smooth Feshbach map and operator-theoretic renormalization group methods, *Journal of Functional Analysis*, 203, 44-92, 2003.

[8] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, The renormalized electron mass in non-relativistic quantum electrodynamics, *Journal of Functional Analysis*, 243, 426–535, 2007.

[9] V. Bach, J. Fröhlich, and A. Pizzo, Infrared-Finite Algorithms in QED: The Groundstate of an atom interacting with the quantized radiation field, *Communications in Mathematical Physics* 264, Issue: 1, 145 - 165, 2006.

[10] V. Bach, J. Fröhlich, and A. Pizzo, An infrared-finite algorithm for Rayleigh scattering amplitudes, and Bohr's frequency condition, *Comm. Math. Phys.* 274, no. 2, 457–486, 2007.

[11] V. Bach, J. Fröhlich, and A. Pizzo, Infrared-finite algorithms in QED II. The expansion of the groundstate of an atom interacting with the quantized radiation field, *Adv. in Math.* , 220 (4), 1023-1074, 2009. mp_arc.

[12] V. Bach, J. Fröhlich, and I. M. Sigal, Mathematical theory of non-relativistic matter and radiation, *Letters in Math. Physics*, 34:183–201, 1995.

[13] V. Bach, J. Fröhlich, and I. M. Sigal, Quantum electrodynamics of confined non-relativistic particles, *Adv. in Math.* , 137:299–395, 1998.

[14] V. Bach, J. Fröhlich, and I. M. Sigal, Renormalization group analysis of spectral problems in quantum field theory, *Adv. in Math.* , 137:205–298, 1998.

[15] V. Bach, J. Fröhlich, and I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.*, 207(2):249–290, 1999.

[16] V. Bach, J. Fröhlich, and I. M. Sigal, Return to equilibrium, *J. Math. Phys.*, 41(6):3985–4060, 2000.

[17] V. Bach, J. Fröhlich, I. M. Sigal, and A. Soffer, Positive commutators and spectrum of Pauli-Fierz Hamiltonian of atoms and molecules, *Commun. Math. Phys.*, 207(3):557–587, 1999.

[18] V. Bach, F. Klopp, and H. Zenk, Mathematical analysis of the photoelectric effect. *Adv. Theor. Math. Phys.*, 5(6):969999, 2001.

[19] J.-M. Barbaroux, T. Chen, and S. A. Vugalter, Binding conditions for atomic N-electron systems in non-relativistic QED, *Ann. H. Poinc.*, 4 (6), 1101 - 1136, 2003.

[20] J.-M. Barbaroux, T. Chen, V. Vougalter and S. A. Vugalter, Quantitative estimates on the hydrogen ground state energy in non-relativistic QED, *Ann. H. Poincare*, 11 (8), 1487-1544, 2010.

[21] J.-M. Barbaroux, M. Dimassi, J.C. Guillot, Quantum electrodynamics of relativistic bound states with cutoffs, *J. Hyperbolic Differ. Equ.* **1**, (2004) 271–314.

[22] J.-M. Barbaroux and S. A. Vugalter, Non analyticity of the ground state energy of the Hamiltonian for Hydrogen atom in nonrelativistic QED (2010), Preprint.

[23] M. Berger, *Nonlinearity and functional analysis. Lectures on nonlinear problems in mathematical analysis.* Pure and Applied Mathematics. Academic Press, New York-London, 1977.

[24] L. Bugliaro, J. Fröhlich and G.M. Graf, Stability of quantum electrodynamics with nonrelativistic matter,*Phys.Rev. Lett.* 77 (1996), 3494-3497.

[25] I. Catto, P. Exner, Ch. Hainzl, Enhanced binding revisited for a spinless particle in nonrelativistic QED, *J. Math. Phys.* 45, no. 11, 4174–4185, 2004.

[26] I. Catto, Ch. Hainzl, Self-energy of one electron in non-relativistic QED, *J. Funct. Anal.* 207, no. 1, 68–110, 2004.

[27] T. Chen, Infrared renormalization in non-relativistic QED and scaling criticality, *J. Funct. Anal.* 254, no. 10, 2555–2647, 2008.

[28] T. Chen, J. Faupin, J. Fröhlich, I.M. Sigal, Local decay in non-relativistic QED, *arXiv*.

[29] T. Chen, J. Fröhlich, Coherent infrared representations in non-relativistic QED, *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, Proc. Symp. Pure Math.*, AMS, 2007.

[30] T. Chen, J. Fröhlich, A. Pizzo, Infraparticle scattering states in non-relativistic QED - I. The Bloch-Nordsieck paradigm, *Commun. Math. Phys.*, **294** (3), 761-825 (2010).

[31] T. Chen, J. Fröhlich, A. Pizzo, Infraparticle scattering states in non-relativistic QED - II. Mass shell properties, *J. Math. Phys.*, **50** (1), 012103 (2009). arXiv:0709.2812

[32] T. Chen, V. Vougalter and S. A. Vugalter, The increase of binding energy and enhanced binding in non-relativistic QED, *J. Math. Phys.*, 44 (5), 1961-1970, 2003.

[33] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms – Introduction to Quantum Electrodynamics*. John Wiley, New York, 1991.

[34] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions – Basic Processes and Applications*. John Wiley, New York, 1992.

[35] J. Faupin, Resonances of the confined hydrogen atom and the Lamb-Dicke effect in non-relativisitc quantum electrodynamics, *Ann. Henri Poincaré* 9, no. 4, 743–773, 2008.

[36] C. Fefferman, J. Fröhlich and G.M. Graf, Stability of ultraviolet cutoff quantum electrodynamics with non-relativistic matter,*Commun. Math. Phys.* 190 (1997), 309330.

[37] E. Fermi, Quantum theory of radiation, *Rev. Mod. Phys.*, 4:87–132, 1932.

[38] H. Feshbach, Unified theory of nuclear reactions, *Ann. Phys.*, 5:357–390, 1958. .

[39] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. Henri Poincaré*, Section Physique Théorique, **19** (1), 1-103 (1973).

[40] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22**, 159-198 (1974).

[41] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field, *Advances in Mathematics* 164, Issue: 2, 349-398, 2001.

[42] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic completeness for Rayleigh scattering, *Ann. Henri Poincaré* 3, no. 1, 107–170, 2002.

[43] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic completeness for Compton scattering, *Comm. Math. Phys.* 252, no. 1-3, 415–476, 2004.

[44] J. Fröhlich, M. Griesemer and I.M. Sigal, Spectral theory for the standard model of non-relativisitc QED, *Comm. Math. Phys.*, 283, no 3, 613-646, 2008.

[45] J. Fröhlich, M. Griesemer and I.M. Sigal, Spectral renormalization group analysis, *Rev. Math. Phys.* 21(4) (2009) 511-548, e-print, ArXiv, 2008.

[46] J. Fröhlich, M. Griesemer and I.M. Sigal, Spectral renormalization group and limiting absorption principle for the standard model of non-relativisitc QED, *Rev. Math. Phys.* 23(2) (2011), e-print, ArXiv, 2010.

[47] J. Fröhlich, A. Pizzo, The renormalized electron mass in nonrelativistic QED, *Commun. Math. Phys.*

[48] C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki, Infrared problem for the Nelson model on static space-times, *Commun. Math. Phys.*, 2011, To appear.

[49] V. Gergescu, C. Gérard, and J.S. Møller, Spectral Theory of massless Pauli-Fierz models, *Commun. Math. Phys.*, 249:29–78, 2004.

[50] M. Griesemer, Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics, *J. Funct. Anal.*, 210(2):321–340, 2004.

[51] M. Griesemer and D. Hasler, On the smooth Feshbach-Schur map, *J. Funct. Anal.*, 254(9):2329–2335, 2008.

[52] M. Griesemer and D. Hasler, Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation, arXiv:0801.4458.

[53] M. Griesemer, E.H. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* 145, no. 3, 557–595, 2001.

[54] M. Griesemer and H. Zenk, On the Atomic Photoeffect in Non-relativistic QED, Comm. Math. Phys.

[55] S. Gustafson and I.M. Sigal, *Mathematical Concepts of Quantum Mechanics.* 2nd edition. Springer 2006.

[56] Ch. Hainzl, Enhanced binding through coupling to a photon field. Mathematical results in quantum mechanics, Contemp, Math., 307, Amer. Math. Soc., Providence, RI (2002), 149154.

[57] Ch. Hainzl, One non-relativistic particle coupled to a photon field, *Ann. Henri Poincaré* 4, no. 2, 217 – 237, 2003.

[58] Ch. Hainzl, M. Hirokawa, H. Spohn, Binding energy for hydrogen-like atoms in the Nelson model without cutoffs, *J. Funct. Anal.* 220, no. 2, 424459 (2005).

[59] Ch. Hainzl, R. Seiringer, Mass renormalization and energy level shift in non-relativistic QED, *Adv. Theor. Math. Phys.* 6 (5) 847 – 871, 2003.

[60] Ch. Hainzl, V. Vougalter, S. Vugalter, Enhanced binding in non-relativistic QED, *Commun. Math. Phys.* 233, no. 1, 13–26, 2003.

[61] D. Hasler and I. Herbst, Absence of ground states for a class of translation invariant models of non-relativistic QED, *Commun. Math. Phys.* 279, no. 3, 769–787, 2008. ArXiv

[62] D. Hasler, I. Herbst and M.Huber, On the lifetime of quasi-stationary states in non-relativistic QED, *Ann. Henri Poincaré* 9, no. 5, 1005–1028, 2008, ArXiv:0709.3856.

[63] D. Hasler and I. Herbst, Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation, 2008, ArXiv:0801.4458v1.

[64] D. Hasler and I. Herbst, Smoothness and analyticity of perturbation expansions in QED, 2010, ArXiv:1007.0969v1.

[65] D. Hasler and I. Herbst, Convergent expansions in non-relativistic QED, 2010, ArXiv:1005.3522v1.

[66] E. Hille and R.S.Phillips, *Functional Analalysis and Semi-groups*. AMS 1957.

[67] M. Hirokawa, Recent developments in mathematical methods for models in non-relativistic quantum electrodynamics, *A garden of quanta,* 209–242, World Sci. Publishing, River Edge, NJ, 2003.

[68] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics. I, *J. Math. Phys.* 40 (1999), no. 12, 6209–6222.

[69] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics. II, J. Math. Phys. 41 (2000), no. 2, 661–674.

[70] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, Commun. Math. Phys. 211 (2000), 585613.

[71] F. Hiroshima, Ground states and spectrum of quantum electrodynamics of nonrelativistic particles, *Trans. Amer. Math.* Soc. 353 (2001), no. 11, 4497–4528 (electronic).

[72] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré* 3 (2002), no. 1, 171–201.

[73] F. Hiroshima, Nonrelativistic QED at large momentum of photons,*A garden of quanta*, 167–196, World Sci. Publishing, River Edge, NJ, 2003.

[74] F. Hiroshima, Localization of the number of photons of ground states in nonrelativistic QED, *Rev. Math. Phys*. 15 (2003), no. 3, 271–312.

[75] F. Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation field, *Topics in the theory of Schrödinger operators*, World Sci. Publishing, River Edge, NJ, 2004, 145–272.

[76] F. Hiroshima and I. Sasaki, On the ionization energy of the semi-relativistic Pauli-Fierz model for a single particle, Preprint, arXiv:1003.1661v4 (2010)

[77] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz Hamiltonian with spin. *Adv. Theor. Math. Phys.* 5 (2001), no. 6, 1091–1104.

[78] F. Hiroshima and H. Spohn,. Enhanced binding through coupling to a quantum field, *Ann. Inst. Henri Poincaré* 2(6) (2001), no. 6, 1159–1187.

[79] F. Hiroshima, H. Spohn, Mass renormalization in nonrelativistic quantum electrodynamics, *J. Math. Phys.* 46 (4), 2005.

[80] M. Hübner and H. Spohn, Radiative decay: nonperturbative approaches, *Rev. Math. Phys.* 7, no. 3, 363–387, 1995.

[81] W. Hunziker, Resonances, metastable states and exponential decay laws in perturbation theory, *Comm. Math. Phys.* 132, no. 1, 177–188, 1990.

[82] W. Hunziker and I.M. Sigal, The quantum $N$-body problem, *J. Math. Phys.* 41, no. 6, 3448–3510, 2000.

[83] Ch. King, Resonant decay of a two state atom interacting with a massless non-relativistic quantized scalar field, Comm. Math. Phys. ,165(3), 569-594, (1994).

[84] M. Könenberg, O. Matte, E. Stockmeyer, Existence of ground states of hydrogen-like atoms in relativistic QED I: The semi-relativistic Pauli-Fierz operator, arXiv:0912.4223.

[85] E.H. Lieb and M. Loss, Self-energy of electrons in non-perturbative QED, in: Conference Moshe Flato 1999, vol. I, Dijon, in: *Math. Phys. Stud.* vol. 21, Kluwer Acad. Publ., Dordrecht, 2000, pp. 327-344.

[86] E.H. Lieb and M. Loss, A bound on binding energies and mass renormalization in models of quantum electrodynamics, *J. Statist. Phys.* 108, 2002.

[87] E.H. Lieb and M. Loss, Existence of atoms and molecules in non-relativistic quantum electrodynamics, *Adv. Theor. Math. Phys.* **7**, 667-710 (2003). arXiv math-ph/0307046.

[88] M. Loss, T. Miyao and H. Spohn, Lowest energy states in nonrelativistic QED: Atoms and ions in motion, *J. Funct. Anal.* 243, Issue 2, 353-393, 2007.

[89] M. Loss, T. Miyao and H. Spohn, Kramers degeneracy theorem in nonrelativistic QED, 2008, arXiv:0809.4471.

[90] O. Matte, E. Stockmeyer, Exponential localization for a hydrogen-like atom in relativistic quantum electrodynamics, Comm. Math. Phys. **295**, 551–583 (2010)

[91] M. Merkli, M. Mück, I.M. Sigal, Instability of Equilibrium States for Coupled Heat Reservoirs at Different Temperatures, *J. Funct. Anal.*, 243, no. 1, 87-120 (2007).

[92] M. Merkli, M. Mück, I.M. Sigal, Nonequilibrium stationary states and their stability, *Ann. Inst. H. Poincaré*, 243, no. 1, 87-120 (2008).

[93] T. Miyao, H. Spohn, Spectral analysis of the semi-relativistic Pauli-Fierz Hamiltonian, *J. Funct. Anal.*, 256, 2123–2156 (2009), arxiv, 2008.

[94] M. Mück, *Thermal Relaxation for Particle Systems in Interaction with Several Bosonic Heat Reservoirs*. Ph.D. Dissertation, Department of Mathematics, Johannes-Gutenberg University, Mainz, July 2004, ISBN 3-8334-1866-4

[95] M. Mück, Construction of metastable states in quantum electrodynamics, *Rev. Math. Phys.* 16, no. 1, 1–28, 2004.

[96] W. Pauli and M. Fierz, Zur Theorie der Emission langwelliger Lichtquanten, *Il Nuovo Cimento* 15 (1938) 167-188.

[97] A. Pizzo, One-particle (improper) States in Nelson's massless model, *Annales Henri Poincaré* 4, Issue: 3, 439 - 486, June, 2003.

[98] A. Pizzo, Scattering of an infraparticle: The one particle sector in Nelson's massless model, *Annales Henri Poincaré* 6, Issue: 3, 553 - 606 , 2005.

[99] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, volumes III, IV.* Academic Press, 1978.

[100] J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, 1987.

[101] I. Sasaki, Ground state of a model in relativistic quantum electrodynamics with a fixed total momentum, Preprint, arXiv:math-ph/0606029v4 (2006)

[102] J. Schur, Über Potenzreihen die im Inneren des Einheitskreises beschränkt sind, *J. reine u. angewandte Mathematik*, 205–232, 1917.

[103] I.M. Sigal, Ground state and resonances in the standard model of the non-relativistic quantum electrodynamics, *J. Stat. Physics*, 134 (2009), no. 5-6, 899–939. arXiv 0806.3297.

[104] H. Spohn, *Dynamics of Charged Particles and their Radiation Field*, Cambridge University Press, Cambridge, 2004.

[105] E. Stockmeyer, On the non-relativistic limit of a model in quantum electrodynamics, Preprint, arXiv:0905.1006v1 (2009)