Bounds on the Pure Point Spectrum of Lattice Schödinger Operators

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October 18, 2011

Abstract

In dimension $d \geq 3$ a variational principle for the size of the pure point spectrum, thus taking embedded eigenvalues into account, of Schrödinger operators $H(\mathfrak{e}, V)$ on the lattice is proven. The dispersion relations $\mathfrak{e}$ are assumed to be Morse functions and the potentials $V(x)$ to decay faster than $|x|^{-2(d+3)}$, but are not necessarily of definite sign. The proof is based on resolvent estimates for $H(\mathfrak{e}, V')$, for small V', combined with positivity arguments.

1 Introduction

Let $\Gamma = \mathbb{Z}^d$ be the d-dimensional hypercubic lattice. Given a potential $V \in \ell^\infty(\Gamma, \mathbb{R})$, the discrete Schrödinger operator corresponding to V is

$$-\Delta_\Gamma + V(x), \tag{1}$$

where V acts as a multiplication operator and Δ_Γ is the discrete Laplacian defined by

$$[\Delta_\Gamma \varphi](x) \;=\; \sum_{|v|=1} \{\varphi(x+v) - \varphi(x)\}. \tag{2}$$

More generally, we assume to be given a function $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ on the d-dimensional torus (Brillouin zone) $\Gamma^* = (\mathbb{R}/2\pi\mathbb{Z})^d = [-\pi, \pi)^d$, the dual group of Γ. We refer to $\mathfrak{e}$ as a *dispersion relation* or simply a *dispersion*. We then consider the self-adjoint operator

$$H(\mathfrak{e}, V) \;:=\; h(\mathfrak{e}) + V(x), \tag{3}$$

on $\ell^2(\Gamma)$, where $h(\mathfrak{e}) \in \mathcal{B}[\ell^2(\Gamma)]$ is the hopping matrix (convolution operator) corresponding to the dispersion relation $\mathfrak{e}$, i.e.,

$$[\mathcal{F}^*(h(\mathfrak{e})\varphi)](p) \;=\; \mathfrak{e}(p)\,[\mathcal{F}^*(\varphi)](p), \tag{4}$$

for all $\varphi \in \ell^2(\Gamma)$. Here,

$$\mathcal{F}^* :\; \ell^2(\Gamma) \to L^2(\Gamma^*), \quad [\mathcal{F}^*(\varphi)](p) \;:=\; \sum_{x \in \Gamma} e^{-i\langle p, x\rangle} \varphi(x), \tag{5}$$

is the usual discrete Fourier transformation with inverse

$$\mathcal{F} :\; L^2(\Gamma^*) \to \ell^2(\Gamma), \quad [\mathcal{F}(\psi)](x) \;:=\; \int_{\Gamma^*} e^{i\langle p, x\rangle} \psi(p)\, \mathrm{d}\mu^*(p), \tag{6}$$

μ^* is the (normalized) Haar measure on the torus, $\mathrm{d}\mu^*(p) = \frac{\mathrm{d}^d p}{(2\pi)^d}$. Put differently, $h(\mathfrak{e}) = \mathcal{F}\mathfrak{e}\mathcal{F}^*$ is the Fourier multiplier corresponding to $\mathfrak{e}$.

For each $x \in \Gamma$, let $\varphi_x \in \ell^2(\Gamma)$ be the norm-one vector

$$\varphi_x(y) := \delta_{x,y}, \tag{7}$$

where $\delta_{x,y}$ is the Kronecker delta. For a dispersion relation $\mathfrak{e}$ and a pair $(x, y) \in \Gamma^2$, define the hopping amplitude

$$h(\mathfrak{e})_{xy} := \left\langle \varphi_x \middle|\, h(\mathfrak{e})\varphi_y \right\rangle. \tag{8}$$

We say that $h(\mathfrak{e})$ has *finite range* if, for some $R < \infty$ and all $(x, y) \in \Gamma^2$, $|x-y| > R$ implies $h(\mathfrak{e})_{xy} = 0$. The smallest number $R(\mathfrak{e}) \geq 0$ with this property is the *range* of the hopping matrix $h(\mathfrak{e})$.

We assume w.l.o.g. that the minimum of $\mathfrak{e}$ is 0, so

$$\mathfrak{e}(\Gamma^*) \;=\; [0, \mathfrak{e}_{\max}(\mathfrak{e})]. \tag{9}$$

We will further assume that the dispersion relation $\mathfrak{e}$ satisfies the following condition:

$$\textbf{(M)} \quad \mathfrak{e} \text{ and } |\nabla \mathfrak{e}|^2 := \sum_{k=1}^{d} |\partial_{p_k} \mathfrak{e}|^2 \text{ are Morse functions.} \tag{10}$$

Note that $-\Delta_\Gamma = h(\mathfrak{e}_{\mathrm{Lapl}})$ and $\min \mathfrak{e}_{\mathrm{Lapl}}(\Gamma^*) = 0$, with

$$\mathfrak{e}_{\mathrm{Lapl}}(p) := 2 \sum_{i=1}^{d} (1 - \cos(p_i)) \tag{11}$$

fulfilling **(M)**.

To consider more general dispersions than $\mathfrak{e}_{\mathrm{Lapl}}$ is important, for instance, for the analysis of many-body problems on the lattice Γ – even in the situation where the dispersion relation for the one-body sector is chosen to be $\mathfrak{e}_{\mathrm{Lapl}}$: Let $\mathfrak{e}$ be a dispersion relation. For each $K \in \Gamma^*$ define the non-negative function $\mathfrak{e}^{(K)} : \Gamma \to \mathbb{R}_0^+$ by

$$\mathfrak{e}^{(K)}(p) \;=\; \mathfrak{e}\,(p) + \mathfrak{e}\,(K - p) - E_0^{(K)}, \tag{12}$$

where

$$E_0^{(K)} := \min_{p' \in \Gamma^*} \left\{ \mathfrak{e}\,(p') + \mathfrak{e}\,(K - p') \right\}. \tag{13}$$

Dispersions of the form (12) come about in the analysis of systems of two particles on the lattice Γ both having the same dispersion $\mathfrak{e}$ and interacting by a (translation invariant) potential $V(x_1 - x_2)$. Indeed, the two-particle Hamiltonian is unitarily equivalent to the direct integral

$$\int_{\Gamma^*}^{\oplus} [H(\mathfrak{e}^{(K)}, V) + E_0^{(K)}] \, \mathrm{d}\mu^*(K). \tag{14}$$

The function $\mathfrak{e}^{(K)}$ is viewed as the (effective) dispersion of a pair of particles travelling through the lattice with total quasi-momentum $K \in \Gamma^*$. Clearly, $\mathfrak{e}^{(K)}$ fulfills **(M)** – at least in a neighborhood of $K = 0$ –, if $\mathfrak{e}$ does. As soon as $K \neq 0$, however, $\mathfrak{e}_{\mathrm{Lapl}}^{(K)}$ is not proportional to $\mathfrak{e}_{\mathrm{Lapl}}$. Similar facts hold true for the N-body problem, $N > 2$.

Our goal in this paper is to give bounds on the size $N_{pp}[\mathfrak{e}, V]$ of the pure point spectrum of $H(\mathfrak{e}, V)$,

$$N_{pp}[\mathfrak{e}, V] \;:= \dim \operatorname{span} \left\{ x \,|\, x \text{ eigenvector of } H(\mathfrak{e}, V) \right\}, \tag{15}$$

in dimensions $d \geq 3$.

Denote the essential and pure point spectra of $H(\mathfrak{e}, V)$ by $\sigma_{\text{ess}}[\mathfrak{e}, V]$ and $\sigma_{\text{pp}}[\mathfrak{e}, V]$, respectively. Let further

$$\sigma_{\text{emb}}[\mathfrak{e}, V] := \sigma_{\text{ess}}[\mathfrak{e}, V] \cap \sigma_{\text{pp}}[\mathfrak{e}, V] \tag{16}$$

be the set of eigenvalues of $H(\mathfrak{e}, V)$ embedded in its essential spectrum, so $\sigma_{\text{pp}}[\mathfrak{e}, V]$ is the disjoint union of $\sigma_{\text{discr}}[\mathfrak{e}, V] = \sigma_{\text{pp}}[\mathfrak{e}, V] \backslash \sigma_{\text{emb}}[\mathfrak{e}, V]$ and $\sigma_{\text{emb}}[\mathfrak{e}, V]$. Then

$$N_{pp}[\mathfrak{e}, V] = N_{\text{discr}}[\mathfrak{e}, V] + N_{\text{emb}}[\mathfrak{e}, V], \tag{17}$$

where $N_{\text{discr}}[\mathfrak{e}, V]$ and $N_{\text{emb}}[\mathfrak{e}, V]$ denote the size of the discrete and embedded pure point spectrum of $H(\mathfrak{e}, V)$:

$$N_{\text{discr}}[\mathfrak{e}, V] = \dim \operatorname{span} \{x \mid H(\mathfrak{e}, V)x = \lambda x, \ \lambda \in \sigma_{\text{discr}}[\mathfrak{e}, V]\}, \tag{18}$$

$$N_{\text{emb}}[\mathfrak{e}, V] = \dim \operatorname{span} \{x \mid H(\mathfrak{e}, V)x = \lambda x, \ \lambda \in \sigma_{\text{emb}}[\mathfrak{e}, V]\}. \tag{19}$$

Note that for the class of operators $H(\mathfrak{e}, V)$ considered here $\sigma_{\text{emb}}[\mathfrak{e}, V] \neq \emptyset$, in general. In fact, it can be easily shown – through simple explicit examples – that there exist potentials V and dispersion relations $\mathfrak{e}$ fulfilling the assumption **(M)** for which $\sigma_{\text{emb}}[\mathfrak{e}, V] \neq \emptyset$, see (31)–(41) below.

In Theorem 4.4, we show that the following variational principle holds:

$$N_{\text{discr}}[\mathfrak{e}, V], \#\sigma_{\text{pp}}[\mathfrak{e}, V] \leq \min \{\#\text{supp}\, V' \mid \Phi_2(V - V') < c(\mathfrak{e})\}. \tag{20}$$

Here, $c(\mathfrak{e}) > 0$ is a constant depending only on a few derivatives of $\mathfrak{e}$, and, for any $m \geq 0$ and any function $V : \Gamma \to \mathbb{R}$,

$$\Phi_m(V) := \left[\sum_{x \in \Gamma} |V(x)|^{\frac{1}{2}} (|x| + 1)^m \right]^2. \tag{21}$$

The proof of (20) uses resolvent bounds (Theorem 3.2(i)) for $H(\mathfrak{e}, V - V')$, where V' is chosen such that

$$\Phi_2(V - V') < c(\mathfrak{e}), \tag{22}$$

see Theorem 4.4. Note that the variational principle (20) is useful only if $V(x)$ is of sufficiently rapid decay, as $|x| \to \infty$. Indeed, if $\Phi_m(V) < \infty$ then $V(x) = \mathcal{O}((|x| + 1)^{-2(m+d)})$.

In [BdSPL10] we proved a similar (and stronger) variational principle for the size $N_{\text{discr}}[\mathfrak{e}, V]$ of the discrete spectrum of $H(\mathfrak{e}, V)$ in any dimension $d \geq 1$ and for potentials V with a definite sign by using the Birman-Schwinger principle. For the Schrödinger operators considered in this paper, however, that method is not directly applicable – not even for the study of the discrete spectrum. In particular we do not assume V to have a definite sign.

Assume, for simplicity, that the hopping matrix $h(\mathfrak{e})$ has finite range, i.e., $R(\mathfrak{e}) < \infty$. We can also give bounds on the multiplicity of the embedded eigenvalues. Let, for any $\lambda \in \mathbb{R}$,

$$m_\lambda := \dim \{x \mid \ H(\mathfrak{e}, V)x = \lambda x\} \, . \tag{23}$$

Define the symmetric operator $\hat{A} = \hat{A}(\mathfrak{e})$ on $C^\infty(\Gamma^*, \mathbb{C}) \subset L^2(\Gamma^*, \mathbb{C})$ by

$$\hat{A}\hat{\varphi}(p) = i \sum_{i=1}^{d} \left\{ [\partial_{p_i} \mathfrak{e}(p)][\partial_{p_i} \hat{\varphi}(p)] + \frac{1}{2}[\partial_{p_i}^2 \mathfrak{e}(p)]\hat{\varphi}(p) \right\} . \tag{24}$$

We further denote by $A = A(\mathfrak{e})$ the (inverse) Fourier transform of $\hat{A}$, i.e., the operator $A = \mathcal{F}\,\hat{A}\,\mathcal{F}^*$ with $\mathrm{Dom}(A) = \mathcal{F}(C^\infty(\Gamma^*, \mathbb{C}))$. Note that $i[V, A]$ uniquely extends to a bounded self-adjoint operator on $\ell^2(\Gamma)$ (also denoted by $i[V, A]$) whenever V and $\mathfrak{e}$ are of finite range. Densely defined bounded operators will be always identified with their closures in the sequel. We can show that, for any $\lambda \in \sigma_{\mathrm{pp}}[\mathfrak{e}, V]$,

$$m_\lambda \ \leq \ \min \{\dim \mathrm{Ran}(i[V', A]) \mid \Phi_3(V - V') < c'(\mathfrak{e}), V' \text{of finite range}\} \tag{25}$$

$$m_\lambda \ \leq \ \min \{\dim \mathrm{Ran}(V') \mid \Phi_2(V - V') < c(\mathfrak{e}), V' \text{ of finite range}\} \, . \tag{26}$$

Here, $c'(\mathfrak{e})$ is some finite constant depending only on a few derivatives of $\mathfrak{e}$ and $c(\mathfrak{e})$ is the same constant as in (20). Combining this with the bound (20) on $\#\sigma_{\mathrm{pp}}[\mathfrak{e}, V]$, we get a bound on $N_{\mathrm{pp}}[\mathfrak{e}, V]$. The bound (25) follows from resolvent estimates (Theorem 3.2(ii)) for

$$h(|\nabla \mathfrak{e}|^2) + i[V - V', A], \text{ provided } \Phi_3(V - V') < c'(\mathfrak{e}), \tag{27}$$

combined with positivity arguments in the form of a virial theorem (Lemma 2.3) for $H(\mathfrak{e}, V)$, see Corollary 4.1. The bound (26) follows from resolvent estimates (Theorem 3.2(i)) for

$$H(\mathfrak{e}, V - V'), \text{ provided } \Phi_2(V - V') < c(\mathfrak{e}), \tag{28}$$

and basic facts about eigenspaces of self-adjoint operators (Lemma 4.2), see Theorem 4.4.

Observe that the bound (25) on m_λ is tighter than the bound (26), in general. Consider, for instance, the potentials $V_R(x) := \mathbf{1}[|x| \leq R]$, $R \in (0, \infty)$. Then

$$\dim \mathrm{Ran}(V_R) = \mathcal{O}(R^d), \tag{29}$$

whereas

$$\dim \mathrm{Ran}\left(i[V_R, A(\mathfrak{e})]\right) = \mathcal{O}(R^{d-1}). \tag{30}$$

Moreover, note that the bounds in (25)–(26) also hold for more general dispersions $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ including dispersions with infinite range.

Observe that, by a theorem due to von Neumann and Weyl (see, for instance, [Kato, Chapter X, Theorem 2.1]), for any self-adjoint operator H_0 on a separable Hilbert space $\mathcal{H}$ and any prescribed upper bound $\varepsilon > 0$, there is another self-adjoint operator H_1 with a pure point spectrum (i.e., the space of eigenvectors of H_1 is dense in $\mathcal{H}$) and $(H_1 - H_0)$ smaller than ε in the Hilbert-Schmidt norm. Thus, even arbitrarily small perturbations can drastically change the point spectrum of a self-adjoint operator drastically.

It is well-known [NaYa92] that $H(\mathfrak{e}_{\mathrm{Lapl}}, V)$ has no embedded eigenvalues if $d = 1$ and $V(x) = O(|x|^{1+\varepsilon})$ for some $\varepsilon > 0$. This result strongly depends on the particular choice of dimension and dispersion relation and is false, in general, (even in one dimension) for more general C^2-dispersion relations, as can be seen from the following simple examples:

For any $d \geq 1$ define $\psi \in \ell^2(\Gamma)$ and $\widetilde{\mathfrak{e}} \in C^2(\Gamma^*, \mathbb{R})$,

$$\psi(x) = \begin{cases} 1 & , \quad x_k = 0 \text{ for some } k = 1, \ldots, d, \\ \frac{1}{x_1^2 x_2^2 \cdots x_d^2} & , \quad \text{otherwise,} \end{cases} \tag{31}$$

$$\widetilde{\mathfrak{e}}(p) = \prod_{k=1}^{d} \left[1 - 2\cos(p_k) + \cos(2p_k)\right], \tag{32}$$

where $x = (x_1, \ldots x_d) \in \Gamma$ and $p = (p_1, \ldots p_d) \in \Gamma^*$. From straightforward calculations, one easily obtains

$$\left[h(\widetilde{\mathfrak{e}})\psi\right](x) = \mathcal{O}\left(\frac{1}{x_1^4 x_2^4 \cdots x_d^4}\right). \tag{33}$$

Define a potential $\widetilde{V}$ by

$$\widetilde{V}(x) := -\frac{\left[h(\widetilde{\mathfrak{e}})\psi\right](x)}{\psi(x)} \tag{34}$$

and observe that $\widetilde{V}$ is real valued and

$$\widetilde{V}(x) = \mathcal{O}\left(\frac{1}{x_1^2 x_2^2 \cdots x_d^2}\right). \tag{35}$$

By construction,

$$[h(\widetilde{\mathfrak{e}})\psi](x) = -\widetilde{V}(x)\psi(x). \tag{36}$$

In particular, as $\inf \widetilde{\mathfrak{e}}(\Gamma^*) < 0 < \sup \widetilde{\mathfrak{e}}(\Gamma^*)$, we have

$$0 \in \sigma_{\text{ess}}[\widetilde{\mathfrak{e}}, \widetilde{V}] \cap \sigma_{\text{pp}}[\widetilde{\mathfrak{e}}, \widetilde{V}], \tag{37}$$

i.e., in any dimension $d \geq 1$, 0 is an embedded eigenvalue of $H(\widetilde{\mathfrak{e}}, \widetilde{V})$ with $\widetilde{V}(x)$ decaying faster than $1/|x|^{2-\varepsilon}$, for any $\varepsilon > 0$, as $|x| \to \infty$. Observe further that the dispersion relation $\widetilde{\mathfrak{e}}$ defined in (32) satisfies **(M)** if $d = 1, 2$.

In higher dimensions there are even simpler examples of lattice Schrödinger operators having embedded eigenvalues with dispersion relations satisfying **(M)** and even for potentials with finite support, thus decaying arbitrarily fast: Let

$$\widetilde{V}(x) := -\delta_{x,0}\left[\int_{\Gamma^*} \frac{1}{\mathfrak{e}_{\text{Lapl}}(p)} \mathrm{d}\mu^*(p)\right]^{-1}, \tag{38}$$

where $\delta_{x,y}$ is the Kronecker delta. For $d \geq 5$, define $\psi \in \ell^2(\Gamma)$ by

$$\psi(x) = \int_{\Gamma^*} \frac{e^{ip\cdot x}}{\mathfrak{e}_{\text{Lapl}}(p)} \mathrm{d}\mu^*(p), \tag{39}$$

i.e. ψ is the inverse Fourier transform of $\mathfrak{e}_{\text{Lapl}}^{-1}$ (which is an element of $L^2(\Gamma^*)$). Then clearly,

$$[h(\mathfrak{e}_{\text{Lapl}})\psi](x) = -\widetilde{V}(x)\psi(x). \tag{40}$$

In particular,

$$0 = \inf \mathfrak{e}_{\text{Lapl}}(\Gamma^*) \in \sigma_{\text{ess}}[\mathfrak{e}_{\text{Lapl}}, \widetilde{V}] \cap \sigma_{\text{pp}}[\mathfrak{e}_{\text{Lapl}}, \widetilde{V}]. \tag{41}$$

Note that $\mathfrak{e}_{\text{Lapl}}$ satisfies **(M)** in any dimension $d \geq 1$.

This paper is organized as follows:

- In Section 2 we discuss a few general facts about the spectrum of $H(\mathfrak{e}, V)$ and prove a virial theorem for $H(\mathfrak{e}, V)$ as Lemma 2.3.

- In Section 3 we derive resolvent estimates for $H(\mathfrak{e}, V - V')$, assuming $\Phi_2(V - V') < c(\mathfrak{e})$, and for $h(|\nabla\mathfrak{e}|^2) + i[V - V', A]$, assuming $\Phi_3(V - V') < c'(\mathfrak{e})$, respectively. An important technical problem we are facing in these estimates arises from singularities of the type

$$\frac{1}{p_1^2 + \cdots + p_k^2 - p_{k+1}^2 \cdots - p_d^2}$$

  appearing in integrands. Such singularities are called "van Hove singularities" in condensed matter physics and have important physical consequences. They cannot be handled by simple "power counting", and rather sign cancellations have to be exploited in the bounds. This technical aspect is discussed in Appendix 5.3.

- Bound (20) on $\#\sigma_{\mathrm{pp}}[\mathfrak{e}, V]$ is proven in Section 4, formulated in Theorem 4.4.

- Bounds (25) and (26) on the multiplicity m_λ of eigenvalues λ are proven in Section 4 as Corollary 4.1 and Theorem 4.4, respectively.

2 The Spectrum of $H(\mathfrak{e}, V)$ – Generalities

We require that V decays at infinity,

$$V \in \ell_0^\infty(\Gamma, \mathbb{R}) := \Big\{ V : \Gamma \to \mathbb{R} \;\Big|\; \lim_{|x|\to\infty} V(x) = 0 \Big\}, \tag{42}$$

or sometimes even that V has bounded support. Note that $V \in \ell_0^\infty(\Gamma, \mathbb{R})$ is compact as a multiplication operator on $\ell^2(\Gamma)$ and by a theorem of Weyl,

$$\sigma_{\mathrm{ess}}[H(\mathfrak{e}, V)] = \sigma_{\mathrm{ess}}[H(\mathfrak{e}, 0)] = [0, \mathfrak{e}_{\max}], \tag{43}$$

where $\mathfrak{e}_{\max} \equiv \mathfrak{e}_{\max}(\mathfrak{e})$.

Let $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ be a Morse function. As Γ^* is compact, $\mathfrak{e}$ has at most finitely many critical points. We denote the set of all critical points of $\mathfrak{e}$ by

$$\mathrm{Crit}(\mathfrak{e}) := \{p \in \Gamma^* \mid \nabla\mathfrak{e}(p) = 0\}. \tag{44}$$

The critical values of $\mathfrak{e}$, collected in the set

$$\mathrm{Thr}(\mathfrak{e}) := \mathfrak{e}\left(\mathrm{Crit}(\mathfrak{e})\right), \tag{45}$$

are called of *thresholds* of $\mathfrak{e}$.

Observe that, for all $\hat{\varphi} \in C^{\infty}(\Gamma^*, \mathbb{C})$,

$$i[\mathfrak{e}, \hat{A}]\hat{\varphi}(p) = |\nabla \mathfrak{e}(p)|^2 \hat{\varphi}(p) := \sum_{i=1}^{d} [\partial_{p_i} \mathfrak{e}(p)]^2 \hat{\varphi}(p). \tag{46}$$

In particular, $i[\mathfrak{e}, \hat{A}]$ and $i[h(\mathfrak{e}), A]$ extend to positive bounded operators on $L^2(\Gamma^*; \mathbb{C})$ and $\ell^2(\Gamma; \mathbb{C})$, which we also denote a by $i[\mathfrak{e}, \hat{A}]$ and $i[h(\mathfrak{e}), A]$, respectively. See (24) and sentence thereafter for the definition of the operators $\hat{A} = \hat{A}(\mathfrak{e})$ and $A = A(\mathfrak{e})$.

Note that $|\nabla \mathfrak{e}(p)|^2$ is a Morse function provided $\mathfrak{e}$ satisfies Assumption **(M)**. As already mentioned above, one example of a dispersion relation satisfying **(M)** is $\mathfrak{e}_{\mathrm{Lapl}}$ defined by (11). The condition **(M)** is stable under small perturbations in the C^3-sense, i.e., if $\|\mathfrak{e} - \tilde{\mathfrak{e}}\|_{C^3(\Gamma^*)}$ is sufficiently small and $\mathfrak{e}$ satisfies **(M),** then so does $\tilde{\mathfrak{e}}$.

Furthermore, we observe that a dispersion relation $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ has finite range if, and only if, it is a trigonometric polynomial of the form

$$\mathfrak{e}(p) = \sum_{n=1}^{N} c_n e^{ip \cdot x_n} \tag{47}$$

for suitable $c_1, \ldots, c_N \in \mathbb{C}$ and $x_1, \ldots, x_N \in \mathbb{Z}^d$. But then $\nabla \mathfrak{e}$ has finite range, too, and

$$R(\nabla \mathfrak{e}) \leq R(\mathfrak{e}). \tag{48}$$

Using the Fourier transformation one easily sees that if $V(x)|x|$ is summable in Γ then AV and VA define bounded operators. In particular, it follows immediately from this that if $V(x)|x|^p$ is summable for some $p > 1$ then AV and VA define compact operators. For such potentials we have the following estimate for the commutator $i[H(\mathfrak{e}, V), A]$:

Lemma 2.1 (Mourre Estimate for $H(\mathfrak{e}, V)$) *If $\mathfrak{e} \in C^4(\Gamma^*, \mathbb{R})$ is a dispersion relation then $A(\mathfrak{e})$ uniquely extends to a self-adjoint operator (also denoted by $A(\mathfrak{e})$). If $V : \Gamma \to \mathbb{R}$ is such that $i[V, A]$ defines a compact operator, then, for any continuous function $\chi : \mathbb{R} \to \mathbb{R}$ of compact support and satisfying $\mathrm{dist}(\mathrm{Thr}(\mathfrak{e}), \mathrm{supp}\chi) > 0$, there is a compact operator $K_\chi \in \mathfrak{B}[\ell^2(\Gamma, \mathbb{C})]$ and a constant $c_\chi > 0$ such that*

$$\chi\,[H(\mathfrak{e}, V)]\; i[H(\mathfrak{e}, V), A]\, \chi\,[H(\mathfrak{e}, V)] \geq c_\chi \chi^2\,[H(\mathfrak{e}, V)] + K_\chi. \tag{49}$$

Observe that if $\Delta \subset \mathbb{R}$ is a compact subset with $\mathrm{dist}(\mathrm{Thr}(\mathfrak{e}), \Delta) > 0$, then there is a continuous function $\chi : \mathbb{R} \to \mathbb{R}$ with compact support such that $\mathrm{dist}(\mathrm{Thr}(\mathfrak{e}), \mathrm{supp}\chi) > 0$, and $\chi(\Delta) = \{1\}$. Let E_Δ be the spectral projection of $H(\mathfrak{e}, V)$ associated with Δ. Then $\chi E_\Delta = \chi E_\Delta = E_\Delta$ and by multiplying equation (49) with E_Δ from the left and from the right it follows that, for some $c_\Delta > 0$ and some compact operator K_Δ,

$$E_\Delta i[H(\mathfrak{e}, V), A]E_\Delta \geq c_\Delta E_\Delta + K_\Delta. \tag{50}$$

Again using the Fourier transformation, one easily checks that if $\Phi_2(V) < \infty$ then AAV, AVA and VAA define bounded operators. By (46), if $\mathfrak{e} \in C^3(\Gamma^*, \mathbb{R})$, $[[h(\mathfrak{e}), A], A]$ extends to a bounded operator. In particular, if $\Phi_2(V) < \infty$ and $\mathfrak{e} \in C^3(\Gamma^*, \mathbb{R})$, then $[[H(\mathfrak{e}, V), A], A]$ extends to a bounded operator on $\ell^2(\Gamma)$.

The following corollary is a consequence of the last two remarks and the fact that $H(\mathfrak{e}, V)$ is bounded, see [CFKS, Theorems 4.7 and 4.9].

Corollary 2.2 *Let $\mathfrak{e} \in C^4(\Gamma^*, \mathbb{R})$ be a dispersion relation and let V be a potential with $\Phi_2(V) < \infty$. Then $H(\mathfrak{e}, V)$ has no singular continuous spectrum and its eigenvalues can only accumulate in points of* $\mathrm{Thr}(\mathfrak{e})$.

Lemma 2.3 (Virial Theorem for $H(\mathfrak{e}, V)$) *Let $\mathfrak{e} \in C^4(\Gamma^*, \mathbb{R})$ be a dispersion relation and let V_1, V_2 be potentials such that $i[V_1, A(\mathfrak{e})]$ and $i[V_2, A(\mathfrak{e})]$ define bounded operators. If φ is an eigenvector of $H(\mathfrak{e}, V_1 + V_2)$, then*

$$\langle \varphi \,|\, i[V_2, A]\varphi \rangle = -\langle \varphi \,|\, i[H(\mathfrak{e}, V_1), A]\varphi \rangle. \tag{51}$$

Note that the restriction $\mathfrak{e} \in C^4(\Gamma^*, \mathbb{R})$ is only relevant for Corollary 2.2 and Lemma 2.3 above.

The proofs of Lemmata 2.1 and 2.3 use adaptations for the lattice case of known methods used for the continuum and are given in the Appendix 5.1– 5.2 for completeness. See also [CFKS, Chapter 4] and [GSch97].

The following upper bound on the multiplicity of embedded eigenvalues of $H(\mathfrak{e}, V)$, in the case that V and $h(\mathfrak{e})$ have finite range, is an immediate consequence of the virial theorem (Lemma 2.3) above:

Corollary 2.4 (Upper Bound on m_λ, Finite Range Case) *Let $d \geq 1$, $\lambda \in \mathbb{R}$, V be a potential of finite range, and $\mathfrak{e}$ be a dispersion relation of finite range. Let*

$$\Sigma(\mathfrak{e}, V) := \dim \mathrm{Ran}\,(i[V, A(\mathfrak{e})]) \,. \tag{52}$$

Then $m_\lambda \leq \Sigma(\mathfrak{e}, V)$.

Proof. $\dim \mathrm{Ran}\,(i[V,A])$ is finite and $\mathfrak{e} \in C^\infty(\Gamma^*, \mathbb{R})$. We assume w.l.o.g. that λ is an eigenvalue of $H(\mathfrak{e}, V)$. Otherwise, the bound $m_\lambda \leq \Sigma(\mathfrak{e}, V)$ is trivial. Observe that by (46) $i[h(\mathfrak{e}), A]$ is positive and has purely absolutely continuous spectrum. Thus, for all $\varphi \in \ell^2(\Gamma)\backslash\{0\}$, $\langle \varphi \,|\, i[h(\mathfrak{e}), A]\varphi\rangle > 0$. Setting $V_1 = 0$ and $V_2 = V$ it follows from Lemma 2.3 that, for any normalized eigenvector φ of $H(\mathfrak{e}, V)$, we have

$$\langle \varphi \,|\, i[V,A]\varphi\rangle = -\langle \mathcal{F}^*(\varphi) \,|\, |\nabla \mathfrak{e}|^2 \mathcal{F}^*(\varphi)\rangle < 0. \tag{53}$$

Hence, denoting by E any finite dimensional subspace of $\{x \mid H(\mathfrak{e}, V)x = \lambda x\}$ we obtain, by compactness of the n-sphere, $n = \dim E - 1$, the estimate

$$\max\{\langle \varphi \,|\, i[V,A]\varphi\rangle \mid \varphi \in E,\ \|\varphi\|_2 = 1\} < 0. \tag{54}$$

By the min-max principle, the dimension of E cannot exceed the number of negative eigenvalues of the self-adjoint operator $i[V,A]$ which, in turn, cannot exceed the rank of $i[V,A]$. Hence,

$$m_\lambda \leq \Sigma(\mathfrak{e}, V). \tag{55}$$

□

In dimension $d \geq 3$ the upper bound on m_λ given in the corollary above can be improved in the following sense:

- If $\Phi_3(V)$ is small enough then $H(\mathfrak{e}, V)$ has no bound states cf. Corollary 3.3(i), i.e., $m_\lambda = 0$ for all $\lambda \in \mathbb{R}$.
- If $V = V_1 + V_2$ with V_1 of finite range and $\Phi_3(V_2)$ small enough (but V not necessarily of finite range), then $m_\lambda \leq \Sigma(\mathfrak{e}, V_1)$, i.e., the bound on the multiplicities m_λ of eigenvalues λ of $H(\mathfrak{e}, V)$ in the corollary above is true with V replaced by V_1 cf. Corollary 4.1.

3 Resolvent Estimates

Let $\mathfrak{e}''(p)$ be the Hessian matrix of the dispersion relation $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ at $p \in \mathrm{Crit}(\mathfrak{e})$. Define the *minimal curvature of* $\mathfrak{e}$ *at* $p \in \mathrm{Crit}(\mathfrak{e})$ by:

$$K(\mathfrak{e}, p) := \min\left\{ |\lambda|^{\frac{1}{2}} \ :\ \lambda \text{ eigenvalue of } \mathfrak{e}''(p) \right\}. \tag{56}$$

Define also the *minimal (critical) curvature of* $\mathfrak{e}$ by

$$K(\mathfrak{e}) := \min\{K(\mathfrak{e}, p) \mid p \in \mathrm{Crit}(\mathfrak{e})\}. \tag{57}$$

Note that $K(\mathfrak{e}) > 0$ and $K(|\nabla \mathfrak{e}|^2) > 0$ under Assumption **(M).**

For any function $\mathfrak{e} \in C^m(\Gamma^*, \mathbb{C})$ and $m \in \mathbb{N}_0$, define the C^m-norms as usual by

$$\|\mathfrak{e}\|_{C^m} := \max_{\substack{\underline{n} \in \mathbb{N}_0^d \\ |\underline{n}|=m}} \max_{p \in \Gamma^*} |\partial_p^{\underline{n}} \mathfrak{e}(p)|. \tag{58}$$

Lemma 3.1 *Let $\mathfrak{e}$ be any dispersion relation from $C^3(\Gamma^*, \mathbb{R})$. Let $K > 0$ and $C < \infty$ be constants with $K(\mathfrak{e}) \geq K$, and $\|\mathfrak{e}\|_{C^3} \leq C$. Then there is a constant $c_{3.1} < \infty$ depending only on K and C such that*

$$\left\| V^{\frac{1}{2}} (z - h(\mathfrak{e}))^{-1} V^{\frac{1}{2}} \right\|_{\mathfrak{B}[\ell^2(\Gamma)]} \leq c_{3.1} \, \Phi_2(V), \tag{59}$$

$$\left. \begin{array}{l} \left\| V^{\frac{1}{2}} (z - h(\mathfrak{e}))^{-1} A V^{\frac{1}{2}} \right\|_{\mathfrak{B}[\ell^2(\Gamma)]} \\[2ex] \left\| V^{\frac{1}{2}} A (z - h(\mathfrak{e}))^{-1} V^{\frac{1}{2}} \right\|_{\mathfrak{B}[\ell^2(\Gamma)]} \\[2ex] \left\| V^{\frac{1}{2}} A (z - h(\mathfrak{e}))^{-1} A V^{\frac{1}{2}} \right\|_{\mathfrak{B}[\ell^2(\Gamma)]} \end{array} \right\} \leq c_{3.1} \, \Phi_3(V), \tag{60}$$

$$|\langle \varphi_x | (z - h(\mathfrak{e}))^{-1} \varphi_y \rangle| \leq c_{3.1}^2 (1 + |x|)^2 (1 + |y|)^2, \tag{61}$$

$$\left\| V^{\frac{1}{2}} (z - h(\mathfrak{e}))^{-1} \varphi_x \right\|_2 \leq c_{3.1} \, (1 + |x|)^2 \Phi_2(V)^{\frac{1}{2}}, \tag{62}$$

$$\left\| V^{\frac{1}{2}} A (z - h(\mathfrak{e}))^{-1} \varphi_x \right\|_2 \leq c_{3.1} \, (1 + |x|)^2 \Phi_3(V)^{\frac{1}{2}}, \tag{63}$$

for all potentials $V : \Gamma \to \mathbb{R}$, all $z \in \mathbb{C} \backslash \mathbb{R}$, and all $x, y \in \Gamma$. Here, $V^{\frac{1}{2}}$ denotes an arbitrary function $V : \Gamma \to \mathbb{C}$ with $(V^{\frac{1}{2}}(x))^2 = V(x)$.

Proof. We freely use the equality $((V^{\frac{1}{2}})^*)^2 = (V^{\frac{1}{2}})^2 = V$ in the sequel without further mentioning. We write

$$\langle \varphi_x | (z - h(\mathfrak{e}))^{-1} \varphi_y \rangle = (1 + |x|)^2 (1 + |y|)^2 \int_{\Gamma^*} \frac{F_{xy}(p)}{z - e(p)} \mathrm{d}\mu^*(p), \tag{64}$$

with

$$F_{xy}(p) := \frac{e^{ip \cdot (x-y)}}{(1 + |x|)^2 (1 + |y|)^2}, \tag{65}$$

and note that $\sup\{\|F_{xy}\|_{C^2} \mid x, y \in \Gamma\} < \infty$. Hence, it follows from Lemma 5.1 that there is a constant $\mathrm{const} < \infty$, such that

$$|\langle \varphi_x | (z - h(\mathfrak{e}))^{-1} \varphi_y \rangle| \leq \mathrm{const}(1 + |x|)^2 (1 + |y|)^2$$

for all $z \in \mathbb{C} \backslash \mathbb{R}$ and all $x, y \in \Gamma$.

Let $V : \Gamma \to \mathbb{R}$ be a potential with $\mathrm{Ran}(V^{\frac{1}{2}}) \subset \mathrm{dom}(A)$. For all $\varphi \in \ell^2(\Gamma)$, we define the following functions on Γ^*,

$$F_V^{\varphi}(p) := \mathcal{F}^* \circ V^{\frac{1}{2}}(\varphi)(p) = \sum_{x \in \Gamma} \mathrm{e}^{-ip \cdot x} V^{\frac{1}{2}}(x) \varphi(x), \tag{66}$$

$$F_{AV}^{\varphi}(p) := \sum_{i=1}^{d} i[\partial_{p_i} \mathfrak{e}(p)][\partial_{p_i} F_V^{\varphi}](p) + \frac{i}{2} |\nabla \mathfrak{e}(p)|^2 F_V^{\varphi}(p). \tag{67}$$

Then, for all $x \in \Gamma$,

$$\langle (\bar{z} - h(\mathfrak{e}))^{-1} V^{\frac{1}{2}} \varphi \,|\, \varphi_x \rangle = \int_{\Gamma^*} \frac{\overline{F_V^{\varphi}(p)} e^{-ip \cdot x}}{z - \mathfrak{e}(p)} \mathrm{d}\mu^*(p), \tag{68}$$

$$\langle (\bar{z} - h(\mathfrak{e}))^{-1} A V^{\frac{1}{2}} \varphi \,|\, \varphi_x \rangle = \int_{\Gamma^*} \frac{\overline{F_{AV}^{\varphi}(p)} e^{-ip \cdot x}}{z - \mathfrak{e}(p)} \mathrm{d}\mu^*(p). \tag{69}$$

We note

$$\overline{F_{\#}^{\varphi}(p)} e^{-ip \cdot x} = (1 + |x|)^2 \left[(1 + |x|)^{-2} \overline{F_{\#}^{\varphi}(p)} e^{-ip \cdot x} \right], \tag{70}$$

where $\#$ denotes V or AV, and observe that the C^2-norms of the functions

$$p \mapsto (1 + |x|)^{-2} \overline{F_V^{\varphi}(p)} e^{-ip \cdot x}, \quad p \mapsto (1 + |x|)^{-2} \overline{F_{AV}^{\varphi}(p)} e^{-ip \cdot x}$$

are bounded by $\mathrm{const}\, \Phi_2(V)^{\frac{1}{2}}$ and $\mathrm{const}\, \Phi_3(V)^{\frac{1}{2}}$, $\mathrm{const} < \infty$, respectively, uniformly in $x \in \Gamma$ and $\varphi \in \ell^2(\Gamma)$, $\|\varphi\|_2 \leq 1$. It follows from Lemma 5.1 that, for some constant $\mathrm{const} < \infty$, all $x \in \Gamma$, all $z \in \mathbb{C} \backslash \mathbb{R}$, and all $\varphi \in \ell^2(\Gamma)$, $\|\varphi\|_2 \leq 1$,

$$\|V^{\frac{1}{2}} (z - h(\mathfrak{e}))^{-1} \varphi_x\|_2 \leq \mathrm{const}\, (1 + |x|)^2 \Phi_2(V)^{\frac{1}{2}}, \tag{71}$$

$$\|V^{\frac{1}{2}} A (z - h(\mathfrak{e}))^{-1} \varphi_x\|_2 \leq \mathrm{const}\, (1 + |x|)^2 \Phi_3(V)^{\frac{1}{2}}, \tag{72}$$

$$\sum_{x \in \Gamma} |\langle (z - h(\mathfrak{e}))^{-1} V^{\frac{1}{2}} \varphi \,|\, V^{\frac{1}{2}} \varphi_x \rangle|^2 \leq \mathrm{const}^2\, \Phi_2(V)^2, \tag{73}$$

$$\sum_{x \in \Gamma} |\langle (z - h(\mathfrak{e}))^{-1} A V^{\frac{1}{2}} \varphi \,|\, V^{\frac{1}{2}} \varphi_x \rangle|^2 \leq \mathrm{const}^2\, \Phi_3(V) \Phi_2(V) \tag{74}$$

$$\leq \mathrm{const}^2\, \Phi_3(V)^2. \tag{75}$$

Thus,

$$\|V^{\frac{1}{2}}(z-h(\mathfrak{e}))^{-1}V^{\frac{1}{2}}\|_{\mathfrak{B}[\ell^2(\Gamma)]} \leq \text{const}\,\Phi_2(V), \tag{76}$$
$$\|V^{\frac{1}{2}}(z-h(\mathfrak{e}))^{-1}AV^{\frac{1}{2}}\|_{\mathfrak{B}[\ell^2(\Gamma)]} \leq \text{const}\,\Phi_3(V), \tag{77}$$

for some $\text{const} < \infty$, all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $x \in \Gamma$. By taking adjoints, we further obtain

$$\|V^{\frac{1}{2}}A(z-h(\mathfrak{e}))^{-1}V^{\frac{1}{2}}\|_{\mathfrak{B}[\ell^2(\Gamma)]} \leq \text{const}\,\Phi_3(V). \tag{78}$$

Similarly, it follows, for a suitable constant $\text{const} < \infty$, all $z \in \mathbb{C}\backslash\mathbb{R}$, all $x \in \Gamma$, and all $\varphi \in \ell^2(\Gamma)$, $\|\varphi\|_2 \leq 1$, that

$$\sum_{x\in\Gamma} |\langle (z-h(\mathfrak{e}))^{-1}AV^{\frac{1}{2}}\varphi \,|\, AV^{\frac{1}{2}}\varphi_x\rangle|^2 \leq \text{const}^2\,\Phi_3(V)^2. \tag{79}$$

Thus,

$$\|V^{\frac{1}{2}}A(z-h(\mathfrak{e}))^{-1}AV^{\frac{1}{2}}\|_{\mathfrak{B}[\ell^2(\Gamma)]} \leq \text{const}\,\Phi_3(V) \tag{80}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $x \in \Gamma$. □

Theorem 3.2 (Resolvent Estimates in Dimension $d \geq 3$) *Let $d \geq 3$ and $\mathfrak{e} \in C^3(\Gamma^*, \mathbb{R})$ be a dispersion relation satisfying* ***(M)***.

(i) If $c_{3.1}\,\Phi_2(V) < 1$ then there exists a constant $c_{3.2} < \infty$ such that, for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $x, y \in \Gamma$,

$$|\langle\varphi_x|(z-H(\mathfrak{e},V))^{-1}\varphi_y\rangle| \leq c_{3.2}(1+|x|)^2(1+|y|)^2. \tag{81}$$

(ii) If $2\,c_{3.1}\,\Phi_3(V) < 1$ then there exists a constant $c_{3.2}^{\text{pc}} < \infty$ such that, for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $x \in \Gamma$,

$$|\langle\varphi_x|(z-i[H(\mathfrak{e},V),A])^{-1}\varphi_y\rangle| \leq c_{3.2}^{\text{pc}}(1+|x|)^2(1+|y|)^2. \tag{82}$$

Proof. For $n \in \mathbb{N}$ and $z \in \mathbb{C}\backslash\mathbb{R}$ let

$$O_n(z) := [V(z-h(\mathfrak{e}))^{-1}]^n = V^{\frac{1}{2}}\widetilde{O}_n(z)V^{\frac{1}{2}}(z-h(\mathfrak{e}))^{-1}, \tag{83}$$

where

$$\widetilde{O}_n(z) := \left[V^{\frac{1}{2}}(z-h(\mathfrak{e}))^{-1}V^{\frac{1}{2}}\right]^{n-1}. \tag{84}$$

Assume that $c_{3.1}\,\Phi_2(V) < 1$. Then, by (59), $\|\widetilde{O}_n(z)\| < a^{n-1}$, for some $0 < a < 1$. Thus we can define the operators

$$\widetilde{O}(z) := \sum_{n=1}^{\infty} \widetilde{O}_n(z).$$

It follows that, for each $z \in \mathbb{C}\backslash\mathbb{R}$, $\|\tilde{O}(z)\|_{B(\ell^2(\Gamma))} \le (1-a)^{-1}$ and that $(z - h(\mathfrak{e}) - V)$ has a bounded inverse given by

$$(z - h(\mathfrak{e}) - V)^{-1} = (z - h(\mathfrak{e}))^{-1} \left[1 + V^{\frac{1}{2}} \widetilde{O}(z) V^{\frac{1}{2}} (z - h(\mathfrak{e}))^{-1}\right]. \tag{85}$$

This, (61), and (62) imply (81).

To prove (ii), we temporarily ignore questions of convergence and write

$$(z - i[H(\mathfrak{e}, V), A])^{-1} = \sum_{n=0}^{\infty} R_0 (i[V, A] R_0)^n, \tag{86}$$

where $R_0 := (z - i[h(\mathfrak{e}), A])^{-1}$. Observe that

$$\begin{aligned} i[V, A] &= iV^{\frac{1}{2}}(V^{\frac{1}{2}}A) + i(-AV^{\frac{1}{2}})V^{\frac{1}{2}} \qquad (87) \\ &= i\sum_{\sigma=0}^{1}(-1)^{\sigma}(A^{\sigma}V^{\frac{1}{2}})(V^{\frac{1}{2}}A^{1-\sigma}). \end{aligned}$$

Hence,

$$\begin{aligned} &(z - i[H(\mathfrak{e}, V), A])^{-1} - R_0 \qquad (88) \\ = &\sum_{n=1}^{\infty} \sum_{\sigma_1,\ldots,\sigma_n=0}^{1} i^n (-1)^{|\underline{\sigma}|} R_0 A^{\sigma_1} V^{\frac{1}{2}} \left(\prod_{j=1}^{n-1} V^{\frac{1}{2}} A^{1-\sigma_j} R_0 A^{\sigma_{j+1}} V^{\frac{1}{2}} \right) V^{\frac{1}{2}} A^{1-\sigma_n} R_0. \end{aligned}$$

Now, due to Lemma 3.1, we have that

$$\begin{aligned} \left\| V^{\frac{1}{2}} A^{\sigma} R_0 \varphi_x \right\|_2 &\le c_{3.1}(1 + |x|^2) \max\{\Phi_2(V)^{\frac{1}{2}}, \Phi_3(V)^{\frac{1}{2}}\} \qquad (89) \\ &= c_{3.1}(1 + |x|^2)\Phi_3(V)^{\frac{1}{2}}, \qquad (90) \end{aligned}$$

$$\begin{aligned} \left\| V^{\frac{1}{2}} A^{\sigma} R_0 A^{\eta} V^{\frac{1}{2}} \right\|_{\mathfrak{B}[\ell^2(\Gamma)]} &\le c_{3.1} \max\{\Phi_2(V), \Phi_3(V)\} \qquad (91) \\ &= c_{3.1}\Phi_3(V). \qquad (92) \end{aligned}$$

for all $x \in \Gamma$ and $\sigma, \eta \in \{0,1\}$. By assumption, $2c_{3.1}\Phi_3(V) < 1$, and the Neumann series evaluated on the vectors φ_x and φ_y, converges. Namely,

$$
\begin{aligned}
& |\langle \varphi_x | (z - i[H(\mathfrak{e}, V), A])^{-1} \varphi_y \rangle| \\
\leq \; & c_{3.1}^2 (1+|x|^2)(1+|y|^2) \sum_{n=0}^{\infty} (2c_{3.1}\Phi_3(V))^n \qquad (93) \\
= \; & \frac{c_{3.1}^2 (1+|x|^2)(1+|y|^2)}{1 - 2c_{3.1}\Phi_3(V)}. \qquad (94)
\end{aligned}
$$

□

Corollary 3.3 *Let $d \geq 3$ and $\mathfrak{e} \in C^3(\Gamma^*, \mathbb{R})$ be a dispersion relation satisfying* ***(M)****.*

(i) If $c_{3.1}\,\Phi_2(V) < 1$ then $H(\mathfrak{e}, V)$ has purely absolutely continuous spectrum and

$$\sigma_{\mathrm{ac}}(H(\mathfrak{e}, V)) = [0, \mathfrak{e}_{\max}]. \qquad (95)$$

(ii) If $2\,c_{3.1}\,\Phi_3(V) < 1$ then $i[H(\mathfrak{e}, V), A]$ is positive and has purely absolutely continuous spectrum.

Proof. Assume that $c_{3.1}\,\Phi_2(V) < 1$. From Theorem 3.2(i), for all $z \in \mathbb{C} \backslash \mathbb{R}$ and all vectors $\varphi \in \mathrm{span}\{\varphi_x \,|\, x \in \Gamma\}$, i.e. φ of finite support, we have that

$$|\langle \varphi | (z - H(\mathfrak{e}, V))^{-1} \varphi \rangle| \leq c(\varphi) < \infty \qquad (96)$$

with $c(\varphi)$ depending only on φ. As $\mathrm{span}\{\varphi_x \,|\, x \in \Gamma\}$ is dense in $\ell^2(\Gamma, \mathbb{C})$, (96) implies the absolute continuity of the spectrum of $H(\mathfrak{e}, V)$; see, for instance, [CFKS, Proposition 4.1]. Analogously, by Theorem 3.2(ii), $i[H(\mathfrak{e}, V), A]$ has only absolutely continuous spectrum whenever $2\,c_{3.1}\Phi_3(V) < 1$. If $\Phi_2(V) < \infty$ then V and $i[V, A]$ define trace class operators. By the Kato-Rosenblum theorem,

$$\sigma_{\mathrm{ac}}(H(\mathfrak{e}, V)) = \sigma_{\mathrm{ac}}(\,h(\mathfrak{e})\,) = [0, \mathfrak{e}_{\max}] \quad \text{and} \quad \sigma_{\mathrm{ac}}(i[H(\mathfrak{e}, V), A]) \subset \mathbb{R}_0^+. \qquad (97)$$

□

4 Bounds on the Size of $\sigma_{\rm pp}[\mathfrak{e}, V]$

Corollary 3.3 yields useful upper bounds on the multiplicity m_λ of eigenvalues of $H(\mathfrak{e}, V)$ for $d \geq 3$ without assuming V to be of finite range:

Corollary 4.1 (Upper Bound on m_λ, Infinite Range Case, $d \geq 3$) *Let $d \geq 3$, $\lambda \in \mathbb{R}$, and let $\mathfrak{e}$ be a dispersion relation from $C^4(\Gamma^*, \mathbb{R})$ satisfying **(M)**. Let $V : \Gamma \to \mathbb{R}$ be a potential with $\Phi_3(V) < \infty$ and choose $V_1, V_2 : \Gamma \to \mathbb{R}$ such that $2\, c_{3.1}\, \Phi_3(V_1) < 1$ and V_2 has finite support. Then $m_\lambda \leq \Sigma(\mathfrak{e}, V_2)$, where m_λ and $\Sigma(\mathfrak{e}, V_2)$ are defined in (23) and (52), respectively.*

Proof. If $2\, c_{3.1}\, \Phi_3(V_1) < 1$, then by Corollary 3.3$(ii)$, $i[H(\mathfrak{e}, V_1), A] \geq 0$ and has purely absolutely continuous spectrum. Thus, by Lemma 2.3, if φ is an eigenvector of $H(\mathfrak{e}, V)$ then $\langle \varphi \,|\, i[V_2, A]\varphi \rangle < 0$ and hence $m_\lambda \leq \dim \mathrm{Ran}(i[V_2, A])$. See the proof of Corollary 2.4 for more details. □

Lemma 4.2 (Bound on the Point Spectrum at Finite Rank Perturbations) *Let H_0 be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ for which $\sigma_{\rm pp}[H_0] = \emptyset$, i.e. H_0 has no eigenvalue. Let W be another bounded self-adjoint operator on $\mathcal{H}$. Then*

$$\# \sigma_{\rm pp}[H_0 + W] \leq \dim \mathrm{Ran}\,(W). \tag{98}$$

Moreover, if $\lambda \in \sigma_{\rm pp}[H_0 + W]$ then

$$\dim \ker(H_0 + W - \lambda) \leq \dim \mathrm{Ran}(W). \tag{99}$$

Proof. We assume w.l.o.g. that W has finite rank. Let $\lambda \in \sigma_{\rm pp}[H_0 + W]$. Then, for some $\psi \in \mathcal{H} \backslash \{0\}$,

$$(H_0 - \lambda)\psi = -W\psi. \tag{100}$$

By assumption $\sigma_{\rm pp}[H_0] = \emptyset$, hence

$$W\psi = -(H_0 - \lambda)\psi \neq 0. \tag{101}$$

This implies that $P_W \psi \neq 0$, where P_W is the orthogonal projection onto the range of W. Let $H_0^W := P_W H_0$. Then, as W has finite rank,

$$H_0^W \varphi = \sum_{k=1}^{N} \langle H_0 \psi_k | \varphi \rangle \, \psi_k, \tag{102}$$

where $\{\psi_k\}_{k=1}^N$ is an ONB of $\mathrm{Ran}(W)$. Let P be the orthogonal projection on the finite dimensional subspace

$$\mathrm{span}\,\{\mathrm{Ran}(W) \cup \mathrm{Ran}(H_0 W)\} \subset \mathcal{H}. \tag{103}$$

From the eigenvalue equation it follows that

$$P_W[H_0 - \lambda + W]\psi = 0. \tag{104}$$

Hence, if $\lambda \in \sigma_{\mathrm{pp}}[H_0 + W]$ then, for some $\varphi \in \mathrm{ran}(P)\backslash\{0\}$,

$$[H_0^W - \lambda P_W + W]\varphi = 0, \tag{105}$$

since $W = WP_W$. This implies that

$$D(\lambda) := \det\big(P[H_0^W - \lambda P_W + W]P\big) = 0 \tag{106}$$

whenever $\lambda \in \sigma_{\mathrm{pp}}[H_0 + W]$. Observe that $D(\lambda)$ is a polynomial of degree at most $\dim\mathrm{Ran}\,(W)$ and has thus at most $\dim\mathrm{Ran}\,(W)$ zeros. This proves the first assertion.

Let $\psi_1, \psi_2 \in \ker(H_0 + W - \lambda)$ for some $\lambda \in \sigma_{\mathrm{pp}}[H_0 + W]$. Then

$$(H_0 - \lambda)(\psi_1 - \psi_2) = -W(\psi_1 - \psi_2). \tag{107}$$

Thus, as $\sigma_{\mathrm{pp}}[H_0] = \emptyset$, $\psi_1 = \psi_2$ iff $W(\psi_1 - \psi_2) = 0$. In other words, $W : \ker(H_0 + W - \lambda) \to \mathrm{Ran}\,(W)$ is injective and thus

$$m_\lambda = \dim\ker(H_0 + W - \lambda) \le \dim\mathrm{Ran}\,(W). \tag{108}$$

□

Observe that the bound (98) above takes embedded eigenvalues into account but disregards multiplicities. If we only consider the discrete spectrum of $H(\mathfrak{e}, V)$ it is easy to see, by using the min-max principle, that a similar bound, but counting multiplicities instead, holds true. A proof of this fact is given below for completeness:

Lemma 4.3 (Bound on $N_{\mathrm{discr}}[\mathfrak{e}, V]$) *Let $V_1 \in \ell_0^\infty(\Gamma)$ be any potential such that $\sigma_{\mathrm{discr}}[H(\mathfrak{e}, V_1)] = \emptyset$. For any potential V_2,*

$$N_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2] \le \#\mathrm{supp}\, V_2. \tag{109}$$

Proof. We assume that $\#\mathrm{supp}\, V_2 < \infty$, otherwise there is nothing to prove. Let $N^+_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2]$ and $N^-_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2]$ be the size of the discrete spectrum above and below zero, respectively. Then

$$N^-_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2] \leq \#\mathrm{supp}\, V_2^-, \tag{110}$$

where $V_2^- := -V_2\, \mathbf{1}[V_2 < 0]$ is the negative part of V_2: As

$$H(\mathfrak{e}, V_1 + V_2) \geq H(\mathfrak{e}, V_1 - V_2^-), \tag{111}$$

it suffices to prove that $H(\mathfrak{e}, V_1 - V_2^-)$ has at most $M := \#\mathrm{supp}\, V_2^-$ eigenvalues below zero (counting multiplicities). Assume thus that $H(\mathfrak{e}, V_1 - V_2^-)$ has at least $M+1$ eigenvalues below zero. Then, by the min-max principle, there is a subspace $X \subset \ell^2(\Gamma)$, $\dim X = M + 1$, for which

$$\sup_{\psi \in X,\, \|\psi\|_2 = 1} \langle \psi \mid H(\mathfrak{e}, V_1 - V_2^-)\psi \rangle < 0. \tag{112}$$

Hence

$$\sup_{\substack{\psi \in X \cap \ker(V_2^-),\\ \|\psi\|_2 = 1}} \langle \psi \mid H(\mathfrak{e}, V_1 - V_2^-)\psi \rangle = \sup_{\substack{\psi \in X \cap \ker(V_2^-),\\ \|\psi\|_2 = 1}} \langle \psi \mid H(\mathfrak{e}, V_1)\psi \rangle < 0. \tag{113}$$

As $\sigma_{\mathrm{ess}}[H(\mathfrak{e}, V_1)] = [0, \mathfrak{e}_{\max}]$, again by the min-max principle, this would then imply that $\sigma_{\mathrm{discr}}[H(\mathfrak{e}, V_1)]$ is not empty. But, by assumption, $\sigma_{\mathrm{discr}}[H(\mathfrak{e}, V_1)] = \emptyset$.

Let $V_2^+ := V_2\, \mathbf{1}[V_2 > 0]$. Note that

$$N^+_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2] = N^-_{\mathrm{discr}}[\mathfrak{e}_{\max} - \mathfrak{e}, -V_1 - V_2] \leq N^-_{\mathrm{discr}}[\mathfrak{e}_{\max} - \mathfrak{e}, -V_1 - V_2^+]. \tag{114}$$

and that $\sigma_{\mathrm{discr}}[H(\mathfrak{e}, V_1)] = \emptyset$ implies $\sigma_{\mathrm{discr}}[H(\mathfrak{e}_{\max} - \mathfrak{e}, -V_1] = \emptyset$. Thus, similarly,

$$N^+_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2^+] \leq \#\mathrm{supp}\, V_2^+ \tag{115}$$

and hence

$$\begin{aligned} N_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2] &= N^-_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2] + N^+_{\mathrm{discr}}[\mathfrak{e}, V_1 + V_2^+] && (116)\\ &\leq \#\mathrm{supp}\, V_2^- + \#\mathrm{supp}\, V_2^+ && (117)\\ &= \#\mathrm{supp}\, V_2. && (118) \end{aligned}$$

□

Combining Corollary 3.3 with Lemmata 4.2 and 4.3 we finally arrive at Bounds (20) and (26):

Theorem 4.4 (Bound on the Size of $\sigma_{\mathrm{pp}}(\mathfrak{e}, V)$) *Let $d \geq 3$ and assume that the dispersion relation $\mathfrak{e}$ satisfies **(M)**. Then there exists a constant $c(\mathfrak{e}) > 0$ depending only on three derivatives of the dispersion relation $\mathfrak{e}$ such that, for all potentials $V : \Gamma \to \mathbb{R}$ with $\Phi_2(V) < \infty$, we have*

$$N_{\mathrm{discr}}[\mathfrak{e}, V], \#\sigma_{\mathrm{pp}}[\mathfrak{e}, V] \leq \min\{\#\mathrm{supp}\, V' \mid \Phi_2(V - V') < c(\mathfrak{e})\}, \tag{119}$$

$$m_\lambda \leq \min\{\dim \mathrm{Ran}(V') \mid \Phi_2(V - V') < c(\mathfrak{e}),\ V' \textit{ of finite range}\} \tag{120}$$

and, for all potentials $V : \Gamma \to \mathbb{R}$, with $\Phi_3(V) < \infty$, we further have

$$m_\lambda \leq \min\{\dim \mathrm{Ran}(i[V', A]) \mid \Phi_3(V - V') < c(\mathfrak{e}), V' \textit{ of finite range}\}. \tag{121}$$

5 Appendix

5.1 Proof of Lemma 2.1

Let N be the unique self-adjoint extension of the operator $\tilde{N}$ defined on $C^\infty(\Gamma^*, \mathbb{C}) \subset L^2(\Gamma^*)$ by

$$\tilde{N}\varphi(p) = \sum_{i=1}^{d} (1 - \partial_{p_i}^2)\varphi(p). \tag{122}$$

Observe that for some $\mathrm{const} < \infty$ and all $\varphi, \varphi' \in C^\infty(\Gamma^*, \mathbb{C})$,

$$|\langle \varphi' \mid A\varphi \rangle| \leq \mathrm{const}\ \| \varphi'_2 \|_2 \| N^{\frac{1}{2}}\varphi_2 \|_2 \leq \mathrm{const}\ \| N^{\frac{1}{2}}\varphi' \|_2 \| N^{\frac{1}{2}}\varphi_2 \|_2 . \tag{123}$$

For all $\varphi \in C^\infty(\Gamma^*, \mathbb{C})$,

$$\begin{aligned}(NA - AN)\varphi(p) \;=\; -i \sum_{k,k'=1}^{d} \Big\{ &(2[\partial_{p_k}^2 \partial_{p_{k'}} \mathfrak{e}(p)][\partial_{p_{k'}} \varphi(p)] + \frac{1}{2}[\partial_{p_k}^2 \partial_{p_{k'}}^2 \mathfrak{e}(p)]\varphi(p) \\ &+ 2[\partial_{p_k} \partial_{p_{k'}} \mathfrak{e}(p)][\partial_{p_k} \partial_{p_{k'}} \varphi(p)] \Big\}.\end{aligned} \tag{124}$$

An integration of the terms with second derivatives of φ by parts yields, for some $0 < \mathrm{const} < \infty$ and all $\varphi, \varphi' \in C^\infty(\Gamma^*, \mathbb{C})$, that

$$|\langle N\varphi' \mid A\varphi \rangle - \langle A\varphi' \mid N\varphi \rangle| \leq \mathrm{const}\ \| N^{\frac{1}{2}}\varphi'_2 \|_2 \| N^{\frac{1}{2}}\varphi_2 \|_2 . \tag{125}$$

Thus, by Nelson's commutator theorem (see [RS2, Theorem X.36]), A is essentially self-adjoint on $C^\infty(\Gamma^*, \mathbb{C})$.

Clearly, as χ is continuous and has compact support,

$$\chi(H(\mathfrak{e}, V)) - \chi(H(\mathfrak{e}, 0)) \tag{126}$$
$$= \lim_{\eta\downarrow 0} \frac{1}{\sqrt{\pi\eta}} \int_0^\infty \chi(t)\Big[\exp\Big(\frac{-(H(\mathfrak{e}, V) - t)^2}{\eta}\Big) - \exp\Big(\frac{-(H(\mathfrak{e}, 0) - t)^2}{\eta}\Big)\Big]\mathrm{d}t$$

in norm sense. Observe that

$$-\eta \int_0^\infty \chi(t)\left[\exp\left(\frac{-(H(\mathfrak{e}, V) - t)^2}{\eta}\right) - \exp\left(\frac{-(H(\mathfrak{e}, 0) - t)^2}{\eta}\right)\right]\mathrm{d}t \tag{127}$$
$$= \int_0^1 \left[\int_0^\infty \chi(t)\exp\left(\frac{-s(H(\mathfrak{e}, V) - t)^2}{\eta}\right)(Vh(\mathfrak{e}) + h(\mathfrak{e})V + V^2 - 2tV)\right.$$
$$\left.\exp\left(\frac{-(1-s)(H(\mathfrak{e}, 0) - t)^2}{\eta}\right)\mathrm{d}t\right]\mathrm{d}s.$$

As V is a compact operator, it follows from (127) that $\chi(H(\mathfrak{e}, V)) - \chi(H(\mathfrak{e}, 0))$ is compact.

The difference

$$i[H(\mathfrak{e}, V), A] - i[H(\mathfrak{e}, 0), A] = i[V, A] \tag{128}$$

is also a compact operator, by assumption. To finish the proof observe that, as $i[H(\mathfrak{e}, 0), A]$ is unitarily equivalent to the multiplication operator $|\nabla\mathfrak{e}|^2$, there is a constant $c_\chi^0 > 0$ such that

$$\chi(H(\mathfrak{e}, 0))i[H(\mathfrak{e}, 0), A]\chi(H(\mathfrak{e}, 0)) \geq c_\chi^0\chi^2(H(\mathfrak{e}, 0)). \tag{129}$$

□

5.2 Proof of Lemma 2.3

Let φ be an eigenvector of $H(\mathfrak{e}, V_1 + V_2)$ and define, for each $n \in \mathbb{Z}\backslash\{0\}$, the vector

$$\varphi_n := \frac{i\,n}{i\,n + A}\varphi. \tag{130}$$

Since $i[H(\mathfrak{e}, V_1), A]$ and $i[V_2, A]$ are bounded operators, by assumption, we have that

$$\lim_{n\to\infty}\langle\varphi_{-n}\,|\,i[H(\mathfrak{e}, V_1), A]\varphi_n\rangle = \langle\varphi\,|\,i[H(\mathfrak{e}, V_1), A]\varphi\rangle, \tag{131}$$
$$\lim_{n\to\infty}\langle\varphi_{-n}\,|\,i[V_2, A]\varphi_n\rangle = \langle\varphi\,|\,i[V_2, A]\varphi\rangle. \tag{132}$$

Note that

$$\begin{aligned} &\langle \varphi_{-n} \,|\, i[H(\mathfrak{e}, V_1 + V_2), A]\varphi_n \rangle \\ &= \ \langle \varphi_{-n} \,|\, i[H(\mathfrak{e}, V_1), A]\varphi_n \rangle + \langle \varphi_{-n} \,|\, i[V_2, A]\varphi_n \rangle. \end{aligned} \tag{133}$$

Hence it suffices to prove, for all $n \in \mathbb{N}$, that

$$\langle \varphi_{-n} \,|\, i[H(\mathfrak{e}, V_1 + V_2), A]\varphi_n \rangle = 0. \tag{134}$$

For all $n \in \mathbb{N}$,

$$\begin{aligned} &\langle \varphi_{-n} \,|\, i[H(\mathfrak{e}, V_1 + V_2), A]\varphi_n \rangle \\ &= \ \left\langle \varphi \,\middle|\, i \left[H(\mathfrak{e}, V_1 + V_2), \frac{i\,nA}{i\,n + A} \right] \varphi \right\rangle = 0. \end{aligned} \tag{135}$$

□

5.3 Proof of Lemma 3.1

In order to prove Lemma 3.1 we need the following estimate:

Lemma 5.1 *Assume that $d \geq 3$ and let $\mathfrak{e}$ be a dispersion relation with $K(\mathfrak{e}) > 0$ and $\|\mathfrak{e}\|_{C^3} < \infty$. Suppose that $\chi \in C^2(\Gamma^*, \mathbb{R})$. Then there exists a constant $\mathrm{c}_{5.1} < \infty$ depending only on $K(\mathfrak{e})$, $\|\mathfrak{e}\|_{C^3}$ and $\|\chi\|_{C^2}$ such that*

$$\left| \int_{\Gamma^*} \frac{\chi(p)}{z - \mathfrak{e}(p)} \mathrm{d}\mu^*(p) \right| \leq \mathrm{c}_{5.1} \tag{136}$$

for all $z \in \mathbb{C} \backslash \mathbb{R}$.

Proof. We assume w.l.o.g. that z is bounded by $|z| \leq \mathfrak{e}_{\max} + 1$, say. We further note that $\mathfrak{e}$ has only finitely many critical points, $\#Q < \infty$, abbreviating $Q := \mathrm{Crit}(\mathfrak{e})$, since Γ^* is compact and $\mathfrak{e}$ is a Morse function. The latter is also the reason that, for each $q \in Q$, there exist an index $m_q \in \{0, \ldots, d\}$ and a C^2–coordinate chart $\xi_q \in C^2(B_{d-m} \times B_m; \mathcal{U}_q)$, for

$$B_n := B_{\mathbb{R}^n}(0, r) = \{x \in \mathbb{R}^n \ : \ |x| < r\}, \ r > 0, \tag{137}$$

denoting the Euclidean open ball in $\mathbb{R}^n$ of radius r and $\mathcal{U}_q \subset \Gamma^*$ being an open neighborhood of q such that, for all $x \in B_{d-m}$, $y \in B_m$,

$$c_1 \leq |\det \mathrm{Jac}\, \xi_q(x, y)| \leq c_2, \tag{138}$$

$$\mathfrak{e}\circ\xi_q(x,y) = \mathfrak{e}(q) + x^2 - y^2, \tag{139}$$

$$\mathcal{U}_q \supseteq B_{\Gamma^*}(q,\delta), \tag{140}$$

for suitable constants $c_1, \delta > 0$, $r \in (0,1)$ and $c_2 < \infty$. $\delta > 0$ can be chosen such that away from the critical points we can find a finite set

$$\widetilde{Q} \subseteq \mathcal{N} := \{q \in \Gamma^* \mid \mathfrak{e}(q) = \mathrm{Re}\{z\}\} \tag{141}$$

and, for each $q \in \widetilde{Q}$, a C^2–coordinate chart $\tilde{\xi}_q \in C^2((-r,r) \times B_{d-1}; \widetilde{\mathcal{U}}_q)$, with $\widetilde{\mathcal{U}}_q \subset \Gamma^*$ being an open neighborhood of q, such that, for all $x \in (-r,r)$, $y \in B_{d-1}$,

$$c_1 \leq |\det \mathrm{Jac}\, \tilde{\xi}_q(x,y)| \leq c_2, \tag{142}$$

$$\mathfrak{e}\circ\tilde{\xi}_q(x,y) = \mathfrak{e}(q) + x, \tag{143}$$

$$\bigcup_{q\in\widetilde{Q}} \widetilde{\mathcal{U}}_q \supseteq \{p \in \Gamma^* \ :\ |\mathfrak{e}(q) - z| < \delta,\ \mathrm{dist}(p,Q) \geq \delta\}. \tag{144}$$

Let

$$\widehat{\mathcal{N}} := \left\{p \in \Gamma^* \ :\ |\mathfrak{e}(p) - z| > \frac{\delta}{2}\right\}. \tag{145}$$

Then $\{\widehat{\mathcal{N}}\} \cup \{\mathcal{U}_q\}_{q\in Q} \cup \{\widetilde{\mathcal{U}}_q\}_{q\in\widetilde{Q}}$ is a finite open covering of Γ^* and there exists a subordinate partition of unity,

$$\{\hat{\eta}\} \cup \{\eta_q\}_{q\in Q} \cup \{\widetilde{\eta}_q\}_{q\in\widetilde{Q}} \subseteq C^\infty(\Gamma^*; [0,1]), \tag{146}$$

such that

$$\mathrm{supp}\hat{\eta} \subset \widehat{\mathcal{N}},\ \mathrm{supp}\eta_q \subset \mathcal{U}_q,\ \mathrm{supp}\widetilde{\eta}_q \subset \widetilde{\mathcal{U}}_q, \tag{147}$$

for $q \in Q \cup \widetilde{Q}$, and

$$\hat{\eta} + \sum_{q\in Q} \eta_q + \sum_{q\in\widetilde{Q}} \widetilde{\eta}_q \equiv 1. \tag{148}$$

It follows that

$$\int_{\Gamma^*} \frac{\chi(p)}{z - \mathfrak{e}(p)} \mathrm{d}\mu^*(p) = \hat{I} + \sum_{q\in Q} I_q + \sum_{q\in\widetilde{Q}} \widetilde{I}_q, \tag{149}$$

where

$$\hat{I} \ := \ \int_{\Gamma^*} \frac{\hat{\eta}(p)\chi(p)}{z - \mathfrak{e}(p)} \mathrm{d}\mu^*(p), \tag{150}$$

$$\widetilde{I}_q \ := \ \int_{B_{d-1}} \mathrm{d}^{d-1}y \int_{-r}^{r} \mathrm{d}x \frac{\widetilde{f}_q(x,y)}{ib - x}, \tag{151}$$

$$I_q \ := \ \int_{B_{d-m_q}} \mathrm{d}^{d-m_q}x \int_{B_{m_q}} \mathrm{d}^{m_q}y \frac{f_q(x,y)}{a_q + ib - x^2 + y^2}, \tag{152}$$

where $b := \mathrm{Im}\{z\}$, $a_q := \mathrm{Re}\{z\} - \mathfrak{e}(q)$, and

$$\begin{aligned} \widetilde{f}_q &:= (\widetilde{\eta}_q \circ \tilde{\xi}_q)(\chi \circ \tilde{\xi}_q) |\det \mathrm{Jac}\, \tilde{\xi}_q|, & (153) \\ f_q &:= (\eta_q \circ \xi_q)(\chi \circ \xi_q) |\det \mathrm{Jac}\, \xi_q|. & (154) \end{aligned}$$

Note that $\widetilde{f}_q \in C_0^2((-r,r) \times B_{d-1}; \mathbb{R})$ and $f_q \in C_0^2(B_{d-m_q} \times B_{m_q}; \mathbb{R})$, due to (147). Moreover, $\|\widetilde{f}_q\|_{C^2}, \|f_q\|_{C^2} < \infty$. The asserted estimate (136) now follows from Lemmata 5.2–5.4 and the trivial estimate

$$|\hat{I}| \leq \frac{2}{\delta} \int_{\Gamma^*} |\chi(p)| \mathrm{d}\mu^*(p). \qquad (155)$$

Observe that the constants r, δ, c_1, c_2 and $\#Q$, $\#\widetilde{Q}$ only depend on $K(\mathfrak{e})$, $\|\mathfrak{e}\|_{C^3}$ and $\|\chi\|_{C^2}$. ☐

Lemma 5.2 *Assume that* $d \geq 1$ *and* $0 < r < 1$. *There is a constant* $\widehat{C}_1 < \infty$ *such that, for all* $f \in C^1((-r,r) \times B_{d-1}; \mathbb{R})$ *and all* $b \in \mathbb{R} \backslash \{0\}$,

$$\left| \int_{B_{d-1}} \mathrm{d}^{d-1}y \int_{-r}^{r} \mathrm{d}x \frac{f(x,y)}{ib - x} \right| \leq \widehat{C}_1 \|f\|_{C^1}. \qquad (156)$$

Proof. For all $x \in (-r,r)$ and all $y \in B_{d-1}$, the fundamental theorem of calculus gives

$$\left| \frac{f(x,y) - f(0,y)}{ib - x} \right| \leq \left| \frac{x}{ib - x} \right| \|\partial_x f\|_\infty \leq \|f\|_{C^1}, \qquad (157)$$

and thus

$$\left| \int_{B_{d-1}} \mathrm{d}^{d-1}y \int_{-r}^{r} \mathrm{d}x \frac{f(x,y)}{ib - x} \right| \leq 2|B_{d-1}| \, \|f\|_{C^1} \left(1 + \left| \int_{-r}^{r} \frac{\mathrm{d}x}{ib - x} \right| \right). \qquad (158)$$

The assertion follows then from

$$\left| \int_{-r}^{r} \frac{\mathrm{d}x}{ib - x} \right| \leq \left| \int_{-1}^{1} \frac{b\,\mathrm{d}x}{b^2 + x^2} \right| = 2\arctan(|b|) \leq \pi. \qquad (159)$$

☐

Lemma 5.3 *Assume that* $d \geq 3$ *and* $0 < r < 1$. *There is a constant* $\widehat{C}_2 < \infty$ *such that, for all* $f \in C_0^1(B_d; \mathbb{R})$, *all* $a \in \mathbb{R}$ *and all* $b \in \mathbb{R} \backslash \{0\}$,

$$\left| \int_{B_d} \frac{f(x)}{a + ib - x^2} \mathrm{d}^d x \right| \leq \widehat{C}_2 \|f\|_{C^1} (1 + a^2 + b^2). \qquad (160)$$

Proof. Introducing spherical coordinates, we observe that

$$\mathcal{J} := \int_{B_d} \frac{f(x)}{a+ib-x^2} \mathrm{d}^d x = \int_0^r \frac{g(s)s^{d-1}}{a+ib-s^2} \mathrm{d}s, \tag{161}$$

where $g \in C^1([0,1];\mathbb{R})$ is the spherical average of f, $\|g\|_{C^1} \leq \|f\|_{C^1}$, defined by

$$g(s) := \int_{\mathbb{S}^{d-1}} f(s\vartheta)\, \mathrm{d}^{d-1}\sigma(\vartheta). \tag{162}$$

An integration by parts gives

$$\begin{aligned}
&\mathrm{Re}\{\mathcal{J}\} \\
&= \int_0^r g(s)s^{d-1} \frac{a-s^2}{(a-s^2)^2+b^2} \mathrm{d}s && (163)\\
&= -\frac{1}{4}\int_0^r g(s)s^{d-2} \left(\frac{\mathrm{d}}{\mathrm{d}s} \ln\left[(a-s^2)^2+b^2\right]\right) \mathrm{d}s && (164)\\
&= \frac{1}{4}\int_0^r \left(g'(s)s^{d-2} + (d-2)g(s)s^{d-3}\right) \ln\left[(a-s^2)^2+b^2\right] \mathrm{d}s. && (165)
\end{aligned}$$

We used above that $g(r) = 0$. Now, use the elementary estimate

$$|\ln(\lambda)| \leq \frac{1}{2\alpha}(\lambda^\alpha + \lambda^{-\alpha}), \tag{166}$$

which holds true for all $\lambda, \alpha > 0$. Choosing $\alpha := \frac{1}{8}$, (166) yields

$$\begin{aligned}
|\mathrm{Re}\{\mathcal{J}\}| &\leq \frac{d-1}{4} \|g\|_{C^1} \int_0^1 \left|\ln\left[(a-s^2)^2+b^2\right]\right| \mathrm{d}s && (167)\\
&\leq C\,\|g\|_{C^1} \left[(1+a^2+b^2)^{\frac{1}{8}} + \int_0^1 \frac{\mathrm{d}s}{|a-s^2|^{\frac{1}{4}}}\right] && (168)\\
&\leq C'\,\|g\|_{C^1}\,(1+a^2+b^2), && (169)
\end{aligned}$$

for some universal constants $C, C' < \infty$. Similarly,

$$\begin{aligned}
&|\mathrm{Im}\{\mathcal{J}\}| \\
&= \frac{1}{|b|}\left|\int_0^r \frac{g(s)s^{d-1}}{1+b^{-2}(a-s^2)^2} \mathrm{d}s\right| && (170)
\end{aligned}$$

$$= \frac{1}{2}\left|\int_0^r g(s)s^{d-2}\left(\frac{\mathrm{d}}{\mathrm{d}s}\arctan\left[\frac{a-s^2}{|b|}\right]\right)\mathrm{d}s\right| \tag{171}$$

$$= \frac{1}{2}\left|\int_0^r \left(g'(s)s^{d-2}+(d-2)g(s)s^{d-3}\right)\arctan\left[\frac{a-s^2}{|b|}\right]\mathrm{d}s\right| \tag{172}$$

$$\leq C\,\|g\|_{C^1}\,. \tag{173}$$

□

Lemma 5.4 *Assume that $d \geq 3$, $0 < r < 1$, and $1 \leq m \leq d-1$. There is a constant $\widehat{C}_3 < \infty$ such that, for all $f \in C_0^1(B_{d-m}\times B_m;\mathbb{R})$, all $a \in \mathbb{R}$ and all $b \in \mathbb{R}\backslash\{0\}$,*

$$\left|\int_{B_{d-m}}\int_{B_m}\frac{f(x,y)}{a+ib-x^2+y^2}\mathrm{d}^{d-m}x\,\mathrm{d}^m y\right| \leq \widehat{C}_3\,\|f\|_{C^1}\,(1+a^2+b^2). \tag{174}$$

Proof. As in Lemma 5.3, we introduce spherical coordinates on B_{d-m} and B_m and define $g \in C^1([0,1]\times[0,1];\mathbb{R})$, with $\|g\|_{C^1} \leq \|f\|_{C^1}$, by

$$g(x,y) := \int_{\mathbb{S}^{d-m-1}}\int_{\mathbb{S}^{m-1}} f(x\vartheta, y\kappa)\,\mathrm{d}^{d-m-1}\sigma(\vartheta)\,\mathrm{d}^{m-1}\sigma(\kappa), \tag{175}$$

so that

$$\mathcal{K} := \int_{B_{d-m}}\int_{B_m}\frac{f(x,y)}{a+ib-x^2+y^2}\,\mathrm{d}^{d-m}x\,\mathrm{d}^m y \tag{176}$$

$$= \int_0^r\int_0^r\frac{g(x,y)x^{d-m-1}y^{m-1}}{a+ib-x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{177}$$

We perform yet another smooth coordinate change by $\phi \in C^\infty((0,r)\times(-1,1);(0,2r)\times(0,2r))$,

$$x = \phi_1(s,u) := s(1+u), \quad y = \phi_2(s,u) := s(1-u), \tag{178}$$

$$|\det \mathrm{Jac}\,\phi(s,u)| = 2s, \tag{179}$$

$$\widetilde{g}(s,u) := (1+u)^{d-m-1}(1-u)^{m-1}\,g\left(s(1+u),s(1-u)\right), \tag{180}$$

from which we obtain

$$\mathcal{K} = 2\int_0^r\int_{-1}^1\frac{s^{d-1}\widetilde{g}(s,u)}{a+ib-(2s)^2u}\mathrm{d}u\,\mathrm{d}r. \tag{181}$$

Note that $\widetilde{g}(s,u) = 0$ whenever $s(1 \pm u) \geq r$. Following a similar strategy as in the proof of Lemma 5.3, we first observe that

$$\begin{aligned}
&\mathrm{Re}\{\mathcal{K}\} \\
&= 2\int_0^r \int_{-1}^1 s^{d-1}\widetilde{g}(s,u)\frac{a-(2s)^2u}{(a-(2s)^2u)^2+b^2}\,\mathrm{d}u\,\mathrm{d}s && (182)\\
&= -\frac{1}{4}\int_0^r s^{d-3}\left[\int_{-1}^1 \widetilde{g}(s,u)\left(\frac{\mathrm{d}}{\mathrm{d}u}\ln\left[(a-(2s)^2u)^2+b^2\right]\right)\,\mathrm{d}u\right]\mathrm{d}s && (183)\\
&= -\frac{1}{4}\int_0^r s^{d-3}\widetilde{g}(s,1)\ln\left[(a-(2s)^2)^2+b^2\right]\mathrm{d}s && (184)\\
&\quad +\frac{1}{4}\int_0^r s^{d-3}\widetilde{g}(s,-1)\ln\left[(a+(2s)^2)^2+b^2\right]\mathrm{d}s && (185)\\
&\quad +\frac{1}{4}\int_0^r s^{d-3}\left[\int_{-1}^1 (\partial_u\widetilde{g})(s,u)\ln\left[(a-(2s)^2u)^2+b^2\right]\,\mathrm{d}u\right]\mathrm{d}s. && (186)
\end{aligned}$$

We use (166) again to bound

$$\ln\left[(a-(2s)^2u)^2+b^2\right] \leq 8(8+2a^2+b^2)^{\frac{1}{8}} + 8|(2s)^2|u|-|a||^{-\frac{1}{4}} \tag{187}$$

for $u = \pm 1$ and $u \in [-1,1]$, respectively, and hence

$$\begin{aligned}
|\mathrm{Re}\{\mathcal{K}\}| \;\leq\; & C\,\|f\|_{C^1}\Big(1+a^2+b^2+\int_0^2\frac{\mathrm{d}s}{|s^2-|a||^{\frac{1}{4}}} && (188)\\
&+\int_0^2\left[\int_0^1\frac{\mathrm{d}u}{|s^2u-|a||^{\frac{1}{4}}}\right]\mathrm{d}s\Big)
\end{aligned}$$

for a suitable constant $C < \infty$. Since

$$\int_0^2 \frac{\mathrm{d}s}{|s^2-|a||^{\frac{1}{4}}} = \int_0^2 \frac{\mathrm{d}s}{(s+|a|^{\frac{1}{2}})^{\frac{1}{4}}|s-|a|^{\frac{1}{2}}|^{\frac{1}{4}}} \leq \int_0^2 \frac{\mathrm{d}s}{|s-|a|^{\frac{1}{2}}|^{\frac{1}{2}}} \leq 4 \tag{189}$$

and

$$\int_0^2\int_0^2 \frac{\mathrm{d}s\,\mathrm{d}u}{|s^2u-|a||^{\frac{1}{4}}} = \int_0^2 \frac{1}{s^{\frac{1}{2}}}\left(\int_0^2\frac{\mathrm{d}u}{|u-\frac{|a|}{s^2}|^{\frac{1}{4}}}\right)\mathrm{d}s \leq 8, \tag{190}$$

we obtain that

$$|\mathrm{Re}\{\mathcal{K}\}| \leq 2^4 C\,\|f\|_{C^1}\left(1+a^2+b^2\right). \tag{191}$$

Similarly,

$$\begin{aligned}
&\mathrm{Im}\{\mathcal{K}\} \\
&= -2\int_0^r\int_{-1}^1 s^{d-1}\widetilde{g}(s,u)\frac{bs^{d-1}\widetilde{g}(s,u)}{(a-(2s)^2u)^2+b^2}\,\mathrm{d}u\,\mathrm{d}s && (192)\\
&= \frac{1}{2}\int_0^r s^{d-3}\left[\int_{-1}^1 \widetilde{g}(s,u)\left(\frac{\mathrm{d}}{\mathrm{d}u}\arctan\left[\frac{a-(2s)^2u}{|b|}\right]\right)\mathrm{d}u\right]\mathrm{d}s && (193)\\
&= \frac{1}{2}\int_0^r s^{d-3}\widetilde{g}(s,1)\arctan\left[\frac{a-(2s)^2}{|b|}\right]\mathrm{d}s && (194)\\
&\quad -\frac{1}{2}\int_0^r s^{d-3}\widetilde{g}(s,-1)\arctan\left[\frac{a+(2s)^2}{|b|}\right]\mathrm{d}s && (195)\\
&\quad -\frac{1}{2}\int_0^r s^{d-3}\left[\int_{-1}^1(\partial_u\widetilde{g})(s,u)\arctan\left[\frac{a-u(2s)^2}{|b|}\right]\mathrm{d}u\right]\mathrm{d}s && (196)
\end{aligned}$$

and $|\arctan(\varphi)| \leq \frac{\pi}{2}$ immediately implies that

$$|\mathrm{Im}\{\mathcal{K}\}| \leq C\,\|f\|_{C^1} \tag{197}$$

for a suitable constant $C < \infty$.

□

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