Stability and symmetry-breaking bifurcation for the ground states of a NLS with a δ' interaction

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Abstract

We determine and study the ground states of a focusing Schrödinger equation in dimension one with a power nonlinearity $|\psi|^{2\mu}\psi$ and a strong inhomogeneity represented by a singular point perturbation, the so-called (attractive) δ' interaction, located at the origin.
The time-dependent problem turns out to be globally well posed in the subcritical regime, and locally well posed in the supercritical and critical regime in the appropriate energy space. The set of the (nonlinear) ground states is completely determined. For any value of the nonlinearity power, it exhibits a symmetry breaking bifurcation structure as a function of the frequency (i.e., the nonlinear eigenvalue) ω. More precisely, there exists a critical value ω^* of the nonlinear eigenvalue ω, such that: if $\omega_0 < \omega < \omega^*$, then there is a single ground state and it is an odd function; if $\omega > \omega^*$ then there exist two non-symmetric ground states.
We prove that before bifurcation (i.e., for $\omega < \omega^*$) and for any subcritical power, every ground state is orbitally stable. After bifurcation ($\omega = \omega^* + 0$), ground states are stable if μ does not exceed a value $\mu^\star$ that lies between 2 and 2.5, and become unstable for $\mu > \mu^*$. Finally, for $\mu > 2$ and $\omega \gg \omega^*$, all ground states are unstable. The branch of odd ground states for $\omega < \omega^*$ can be continued at any $\omega > \omega^*$, obtaining a family of orbitally unstable stationary states.
Existence of ground states is proved by variational techniques, and the stability properties of stationary states are investigated by means of the Grillakis-Shatah-Strauss framework, where some non standard techniques have to be used to establish the needed properties of linearization operators.

1 Introduction

The present paper is devoted to the analysis of existence and stability of the ground states of a nonlinear Schrödinger equation with a point defect in dimension one. The Schrödinger equation bears an attractive power nonlinearity, and the defect is described by a particular point interaction in dimension one, the so-called attractive δ' interaction. In a formal way, the time dependent equation to be studied is given by

$$\begin{cases} i\partial_t\psi(t) & = - \quad \partial_{xx}\psi(t) - \gamma\delta_0'\psi(t) - \lambda|\psi(t)|^{2\mu}\psi(t) \\ \psi(0) & = \qquad\qquad\qquad \psi_0 \end{cases} \tag{1.1}$$

where ψ_0 represents the initial data, and $\lambda > 0$, $\gamma > 0$, $\mu > 0$. The non rigorous character of the expression (1.1) is due to the fact that the combination $\frac{1}{2}\partial_{xx}\psi(t) + \gamma\delta_0'\psi(t)$ is meaningless if literally interpreted as an operator sum or as a form sum, due to the exceedingly singular character of the perturbation given by δ_0'. Nevertheless, it is well known ([5]) that it is possible to define a singular perturbation H_γ of the one-dimensional laplacian $-\frac{d^2}{dx^2}$ which is a self-adjoint operator in $L^2(\mathbb{R})$ and has the expected properties of the stated formal expression. The self-adjointness is implemented through the singular boundary condition which defines the domain of the operator, i.e

$$D(H_\gamma) = \{\psi \in H^2(\mathbb{R}\backslash\{0\}), \quad | \quad \psi'(0+) = \psi'(0-)\ , \quad \psi(0+) - \psi(0-) = -\gamma\psi'(0+)\ \}$$

and the action is $H_\gamma\psi = -\psi''$ out of the origin.

The δ' is called *repulsive* when $\gamma < 0$ and *attractive* when $\gamma > 0$. Note that the generic element of the domain of δ' interactions is not continuous at the origin (but its left and right derivative exist and coincide), at variance with the milder and better known case of the δ interaction (which is defined by $\psi \in H^1(\mathbb{R})$ and boundary condition $\psi'(0^+) - \psi'(0^-) = \alpha\psi(0)$, where the real parameter α is interpreted as the strength of the interaction). The quadratic form computed on a domain element ψ turns out to be

$$(H_\gamma\psi, \psi) = \|\psi'\|^2_{L^2} - \gamma|\psi'(0)|^2$$

which justifies the name of δ' interaction given to H_γ. Concerning the physical meaning of the δ' interaction, perhaps its best known use is in the analysis of Wannier-Stark effect (see [6]). It is known that the spectrum of Wannier-Stark hamiltonians in the presence of a periodic array of δ' interactions shows a remarkably different behaviour with respect to the case of regular periodic potential; in particular, the spectrum has no absolutely continuous part and it is typically pure point (by the way, the corresponding properties of Wannier-Stark for the δ array are not known). In the present paper, the core interpretation is that of a strongly singular and non trivial scatterer. It is known that the δ' interaction cannot be obtained as the limit of Schrödinger operators in which the potential is a derivative of a δ–like regular function, as the name could erroneously suggest (see [11, 14] for a thorough analysis of this problem). An approximation through three scaled δ potentials exists, but the scaling is nonlinear as the distance of the centres vanishes. Nevertheless, the δ' interaction has a high energy scattering behaviour that can be reproduced, up to a phase factor, through scaling limits of scatterers with internal structure, the so-called spiked-onion graphs (see [6]). These are obtained joining two halflines by N edges of length L and letting $L \to 0$, $N \to \infty$ while keeping the product NL fixed. An analogous behaviour is obtained considering a sphere with two halflines attached. These results enforce the interpretation of the δ' interaction as an effective model of a scatterer with non elementary structure. By the way, in this respect the results of the present paper give support to this view through the analysis of the bifurcation of nonlinear bound states.

From a more abstract point of view, both δ and δ' interactions are members of a 4-parameter family of self-adjoint perturbations of the one dimensional laplacian, the so-called 1-dimensional point interactions (see [4, 5, 13]). As explained above, we interpret the presence of a point interaction in the equation (1.1) as a model of strongly singular interaction between nonlinear waves and an

inhomogeneity. When the inhomogeneity is described by a δ interaction, a fairly extended literature exists of both physical and numerical character; more recently, there has been a growing interest in this model from the mathematical side, in the attempt to establish rigorous results concerning the existence of stationary states ([9, 34]), the asymptotic behaviour in time ([25]), and the reduced dynamics on the stable soliton manifold ([20]). To the knowledge of the authors the only rigorous result about the NLS with δ' interaction concerns the well-posedness of the dynamics, and is contained in [1], where the whole family of point perturbation is treated in the presence of a cubic nonlinearity.
We give here a brief description of the results of the present paper.
We first extend the result given in [1] to cover global well-posedness for the problem 1.1 in the energy space ($Q = H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-)$, coinciding with the form domain of H_γ) in the subcritical case $\mu < 2$ and local well-posedness for $\mu \geq 2$. In particular, the equation has two conserved quantities, the energy E and the charge (i.e. the L^2-norm) M, as in the unperturbed NLS equation. They are associated to symmetries of the space of solutions, respectively time translation and phase invariance. The free NLS equation has one more conserved quantity, linear momentum, associated to space translation; this symmetry is broken by the defect, and correspondingly there is not momentum conservation.
In the main part of the paper we are concerned with the identification of the ground states for equation 1.1 and the analysis of their stability in the case of attractive δ' interaction.
A nonlinear standing wave, or a nonlinear bound state in physical terminology, as in the linear case is a solution of the form

$$\psi(t) \;=\; e^{i\omega t}\phi_\omega \;.$$

Correspondingly, ϕ_ω fulfils the stationary equation

$$H_\gamma\phi_\omega - \lambda|\phi_\omega|^{2\mu}\phi_\omega = -\omega\phi_\omega. \tag{1.2}$$

We call $\mathcal{A}$ the set of stationary states of the equation 1.1.
Every member of $\mathcal{A}$ has to be a classical ($C^2(\mathbb{R}^\pm)$) and square integrable solution of the standard NLS to the left and to the right of the singularity. This gives, for the spatial part of the standing wave, the only possible forms

$$\phi_\omega(x) = \begin{cases} \pm\lambda^{-\frac{1}{2\mu}}(\mu+1)^{\frac{1}{2\mu}}\omega^{\frac{1}{2\mu}}\cosh^{-\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_1)], & x<0 \\ \lambda^{-\frac{1}{2\mu}}(\mu+1)^{\frac{1}{2\mu}}\omega^{\frac{1}{2\mu}}\cosh^{-\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_2)], & x>0. \end{cases}$$

Note that in the introduction we omit, to simplify notation, the dependence of ϕ from every parameter other than the frequency ω. The standing wave solution is represented as a solitary wave of the NLS centered at x_1 on the left of the origin and a solitary wave of the same NLS centered at x_2 on the right of the origin; the parameters x_1, x_2 defining the solution are to be chosen in such a way that the function ϕ_ω satisfies the boundary conditions embodied in domain of H_γ. Eventually, they depend on the parameter λ, μ, γ, ω which enter the equation. It turns out that there exist two families of stationary states, a family $\mathcal{F}_1$ whose members respect the symmetry $x_1 = -x_2$ and a family $\mathcal{F}_2$ whose members do not enjoy this symmetry.

It is an important point that the analysis of standing waves by ODE methods has a variational counterpart in the fact that the standing waves turn out to be critical points of an action functional. The action functional for our problem is defined as

$$S_\omega(\phi) = \frac{1}{2}F_\gamma(\phi) - \frac{\lambda}{2\mu+2}\|\phi\|_{2\mu+2}^{2\mu+2} + \frac{\omega}{2}\|\phi\|_2^2 = E(\phi) + \frac{\omega}{2}M(\phi), \qquad \phi \in Q, \tag{1.3}$$

where we indicate by $F_\gamma(\phi)$ the quadratic form associated to the δ' point interaction H_γ.
It is easy to see that solutions of (1.2) are critical point of (1.3). The above action is easily seen to be unbounded from below for focusing nonlinearities. We show however that the action S_ω attains a minimum when constrained on the natural constraint $\langle S'_\omega(\phi), \phi\rangle = 0$ (the so-called Nehari manifold) which obviously contains the set of solutions to Euler-Lagrange equations for S_ω. In the present paper we adhere to the customary mathematical use to call *ground states* the minimizers of the action on the natural constraint. To prove that the constrained minimum problem has a solution we exploit a) the boundedness from below of the action on the associated Nehari manifold; b) the fact that the Nehari manifold is bounded away from zero; c) classical Brezis-Lieb inequalities showing that the limit of a minimizing sequence exists and it is an element of the minimization domain; d) finally, minimizers turn out to be elements of $D(H_\gamma)$ and satisfy equation (1.2).
For an analysis of the analogous problem in the simpler case of NLS with a δ interaction see [17].
It turns out that ground states do not exist for $\omega \leq \frac{4}{\gamma^2}$ and that for every $\omega > \frac{4}{\gamma^2}$ there is at least one ground state. More precisely, for every $\omega \in (\frac{4}{\gamma^2}, \frac{4(\mu+1)}{\gamma^2\mu})$, there exists a unique (up to a phase factor) stationary state. Furthermore, it belongs to the family $\mathcal{F}_1$, so it is symmetric with respect to the defect. From such a family, a couple of non symmetric stationary states, i.e., belonging to the family $\mathcal{F}_2$, bifurcates in correspondence of the value $\omega^* = \frac{4(\mu+1)}{\gamma^2\mu}$.

Before focusing on the issue of determining the stability of the stationary states, we recall that, to the $U(1)$-invariance of equation (1.1), the appropriate notion of stability for our problem corresponds to the so-called *orbital stability*, which is Lyapunov stability up to symmetries. The stationary state ϕ_ω is orbitally stable if given a tubular neighbourhood $U(\phi_\omega)$ of the orbit of the ground state (i.e., a neighbourhood modulo symmetries), the evolution $\psi(t)$ is in $U(\phi_\omega)$ for all times if the $\psi(0)$ datum is near as it needs to the orbit of ϕ_ω. Precise definitions are given in section 6.3.

There are two main approaches to determine orbital stability or instability of stationary states: the variational or direct method, pioneered in the classic paper [8], and linearization. In our analysis we employ the linearization method, studied in a rigorous way first by Weinstein ([32] [33]) and then developed as a general theory for hamiltonian systems by Grillakis, Shatah, and Strauss ([21], [22]). "Linearization" in this context is a conventional denomination because the theory includes in fact the control of the nonlinear remainder.

A summary of the essential steps of the method can be sketched as follows. The equation (1.1) can be written as a hamiltonian system on a real Hilbert space after separating real and imaginary part of $\psi = u + iv$. Given a stationary state, the linearization of the so constructed hamiltonian system at the point ϕ_ω is described by the second derivative of the action $S''(\phi_\omega)$. Such a quantity is identified with a linear operator, whose action is suitably represented with the aid of operators $L_1^{\gamma,\omega}$ and $L_2^{\gamma,\omega}$ defined through

$$S''(\phi_\omega)(u + iv) \;=\; L_1^{\gamma,\omega}u + iL_2^{\gamma,\omega}v$$

(we refer to 6.2 and subsequent comments for the explicit definition). $L_1^{\gamma,\omega}$ and $L_2^{\gamma,\omega}$ are easily seen to be self-adjoint on their natural domains.
Then, denote by $n(L_1^{\gamma,\omega})$ is the number of negative eigenvalues of $L_1^{\gamma,\omega}$, and define the function

$$p(\omega) = \begin{cases} 1 & \text{if} \quad \frac{d^2}{d\omega^2}S_\omega(\phi_\omega) > 0 \\ 0 & \text{if} \quad \frac{d^2}{d\omega^2}S_\omega(\phi_\omega) < 0 \end{cases} .$$

Now, provided that
a) the essential spectrum of $S''_\omega(\phi_\omega)$ is bounded away from zero,
b) $\mathrm{Ker}(L_2^{\gamma,\omega}) = \mathrm{Span}\{\phi_\omega\}$,

c) $n(L_1^{\gamma,\omega}) = n$,
the stationary state ϕ_ω is stable if $n-p=0$, and unstable if $n-p$ is odd.

We accurately compute $\frac{d^2}{d\omega^2}S_\omega(\phi_\omega)$ for the ground states of our model, in critical and supercritical regime, and prove the occurrence of an exchange of stability between the two subfamilies of $\mathcal{F}_1$. For subcritical and critical nonlinearity power $\mu \leq 2$ and low frequency $\omega \in (\frac{4}{\gamma^2}, \frac{4(\mu+1)}{\gamma^2\mu})$, the symmetric (odd) states are stable, while crossing the critical frequency ω^* the symmetric branch of stationary states becomes unstable and the two newborn asymmetric states prove stable. After bifurcation ($\omega = \omega^* + 0$), ground states are stable for not too strong nonlinearities $2 < \mu < \mu^\star$, where $\mu^\star$ lies between 2 and 2.5, and they become unstable for $\mu > \mu^\star$. Finally, in the supercritical regime $\mu > 2$ and large frequency $\omega \gg \omega^*$, all ground states are unstable.

We are then in the presence of a pitchfork bifurcation, accompanied by a spontaneous symmetry breaking on the stable branches of the bifurcation. The phenomenon of bifurcation of asymmetric ground states from a branch of symmetric ones was discovered by Akhmedeev ([3]) in a model of propagation of electromagnetic pulses in nonlinear layered media, and then studied in a rigorous way in several mathematical papers (see [29] and references therein) with various generalizations of the original result, which was concerned with an exactly solvable example. In these works the bifurcation is induced by a change of sign in the nonlinearity (i.e., a transition to a defocusing regime) in a localized region. Here we have an analogous effect in a different model, in which the bifurcation is induced by a strong point defect.
We remark that no such a phenomenon shows up in the case of a single δ perturbation of NLS. Nevertheless, a situation similar in some respects appears in the case of the NLS with *two* attractive δ interactions separated by a distance, a *double δ - well.* This model is studied in [26], where an analysis of the model is given by means of dynamical systems techniques, and [18], where the bifurcation is explored in the semiclassical regime. See also [27] for the analogous phenomenon with a regular potential of double well type and [30] for the introduction of a related normal form. The analogy is a reminiscence of the the fact that the δ' interaction can be approximated in the norm resolvent sense by a suitable scaled set of three δ interactions (see [11] and [14]), so it may be considered as a singular model of a multiple well. We finally recall that a different definition of ground state for a semilinear Schrödinger equation requires that they are minimizers of the total energy constrained to have constant mass. The two definitions are related, but not in an obvious way, because the constrained action is bounded from below irrespective of the power nonlinearity μ and the constrained energy on the contrary is bounded from below only in the subcritical case $\mu < 2$. On the other hand, the general GSS theory guarantees that the ground states are *local* minima of the constrained energy if and only if they are stable, and this gives a connection between two definitions.

The paper is organized as follows: after giving the main notation, we introduce the model by recalling few elementary properties of the NLS and of the point interaction (sections 2.1. and 2.2). Then we state the variational problem that embodies the search for the ground states and two variants of it (section 2.3). In section 3 we establish the well-posedness of the dynamical problem, that turns out to be only local in time for strong nonlinearity ($\mu \geq 2$) and global in time for weak nonlinearity ($\mu < 2$). In section 4 we prove, by variational techniques, the existence of a ground state (theorem 4.1), while in section 5 we explicitly compute the ground state (proposition 5.1) and show the occurrence of a bifurcation with symmetry breaking (theorem 5.3). Section 6 is devoted to the issue of determining the stability and instability of the ground states: first, in section 6.1, we prove the spectral properties a), b), and c) for S''_ω evaluated at the ground states (propositions 6.1, 6.3, and 6.4); then, in section 6.2, we compute $\frac{d^2}{d\omega^2}S_\omega(\psi_\omega)$ and study its sign (proposition 6.5); finally, in section 6.3 we collect all information and prove stability, instability and the occurrence of a pitchfork bifurcation (proposition 6.9 and theorem 6.11).

1.1 Notation

For the convenience of the reader, we collect here some notation that will be used through the paper. Most symbols will be defined again, when introduced.
- $\omega_0 \ := \ \frac{4}{\gamma^2}$ is the frequency of the linear ground state.
- $\omega^* \ := \ \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$ is the frequency of bifurcation.
- The symbol f' denotes the derivative of the function f with respect to the space variable x. Time derivative is explicited through the symbol ∂_t.
- We denote the L^p-norm by $\|\cdot\|_p$. When $p = 2$ we omit the subscript. The squared H^1-norm of f is defined as the sum of the squared L^2-norms of f and of f'.
- The following abuse of notation is repeatedly used:

$$\|\psi'\|_p^p \ := \ \lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{-\varepsilon}|\psi'(x)|^p\,dx + \int_{\varepsilon}^{\infty}|\psi'(x)|^p\,dx, \qquad 1 \le p < \infty\right)$$

- Our framework is the energy space

$$\begin{aligned} Q \ &:= H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-) \\ \|\psi\|_Q^2 \ &:= \|\psi\|^2 + \|\psi'\|^2 \end{aligned}$$

- The ordinary scalar product in L^2 is denoted by $(\cdot,\cdot)$ and is antilinear on the left factor. We often use the duality product in the space Q, denoted by $\langle f, g\rangle$, where $f \in Q^\star$ and $g \in Q$. Again, it is antilinear in f. For simplicity, in the bracket $\langle\cdot,\cdot\rangle$ we often exchange in the bracket the place of the element of Q with the place of the element of Q^*.
- The symbol ϵ denotes the sign function.

2 Preliminaries

Here we recall some well-known facts.

2.1 Free stationary NLS

The set of the solutions to the free (i.e. without point perturbation) stationary Schrödinger equation with focusing power nonlinearity

$$-\psi'' - \lambda|\psi|^{2\mu}\psi = -\omega\psi, \qquad \omega \ > \ 0, \ \ \lambda \ > \ 0, \ \ \mu \ > \ 0$$

is given by $\{\phi_\omega^{x_0}, x_0 \in \mathbb{R}, \omega > 0\}$, where

$$\phi_\omega^{x_0}(x) \ = \ \lambda^{-\frac{1}{2\mu}}(\mu+1)^{\frac{1}{2\mu}}\omega^{\frac{1}{2\mu}}\operatorname{sech}^{\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_0)] \tag{2.1}$$

Note that they are smooth, exponentially decaying functions.

2.2 δ' interaction: hamiltonian operator

The hamiltonian operator describing the so-called δ_0' interaction is defined on the domain

$$D(H_\gamma) \;=\; \{\psi \in H^2(\mathbb{R}\backslash\{0\}),\, \psi'(0^+) = \psi'(0^-)\;,\; \psi(0^+) - \psi(0^-) = -\gamma\psi'(0^+)\} \tag{2.2}$$

and its action is given by

$$H_\gamma\psi = -\psi'' \;, \qquad x \neq 0 \;.$$

Note that in the current literature the opposite sign of γ appears in the definition of H_γ. This is a matter of convention, and we prefer to adopt the present one because we consider the case of attractive δ' interaction only.

An equivalent description of the elements of the operators domain $D(H_\gamma)$ is

$$\psi = \chi_+\psi_+ + \chi_-\psi_-, \tag{2.3}$$

where

$$\psi_\pm \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R})$$

are *even* functions which satisfy the boundary conditions

$$\psi_+(0) - \psi_-(0) = -\gamma\psi_+'(0) = -\gamma\psi_-'(0).$$

In this representation, the action of H_γ is

$$H_\gamma\psi(x) := -\chi_+(x)\psi_+''(x) - \chi_-(x)\psi_-''(x).$$

Let us recall the main spectral properties of H_γ. The essential spectrum is purely absolutely continuous: $\sigma_{ess}(H_\gamma) = [0, +\infty)$.

Concerning the discrete spectrum, if $\gamma \leq 0$, then $\sigma_p(H_\gamma) = \emptyset$; if $\gamma > 0$, then there exists a unique eigenvalue, given by $\sigma_p(H_\gamma) = \{-\frac{4}{\gamma^2}\}$. The corresponding normalized eigenfunction is given, $\forall\, \gamma \in (0, +\infty)$, by

$$\psi_\gamma(x) = \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} \epsilon(x) e^{-\frac{2}{\gamma}|x|} \;, \quad \epsilon(x) \equiv \frac{x}{|x|} \;.$$

Finally, the singular continuous spectrum is empty: $\sigma_{sc}(H_\gamma) = \emptyset$.

In the following we use the notation $\omega_0 = \frac{4}{\gamma^2}$.

2.3 δ'-interaction: quadratic form

It is known (see [1] that the description of the dynamics generated by H_γ can be equivalently accomplished by using the quadratic form

$$F_\gamma(\psi) \;:=\; \|\psi'\|^2 - \gamma^{-1}|\psi(0^+) - \psi(0^-)|^2, \tag{2.4}$$

where, with an abuse of notation, we denoted $\psi' := \chi_+\psi_+' + \chi_-\psi_-'$, according to the notation introduced in Section 1.1. The domain of F_γ is given by (see also Section 2.2 in [1])

$$D(F_\gamma) \;:=\; H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-). \tag{2.5}$$

We will consider the form domain as the natural energy space for the δ' interaction; it is independent of γ and from now on we will indicate it by Q. If ψ belongs to the operator domain of a δ'-interaction with strength γ, then one has

$$F_\gamma(\psi) \;:=\; \|\psi'\|^2 - \gamma|\psi'(0)|^2 \;.$$

which explains the name given to the operator.

Due to (2.5), it is natural to endow the space Q with the norm

$$\|\psi\|_Q^2 \;:=\; \frac{1}{2}\|\psi_+\|_{H^1(\mathbb{R})}^2 + \frac{1}{2}\|\psi_-\|_{H^1(\mathbb{R})}^2 \;=\; \|\psi\|^2 + \|\psi'\|^2.$$

It is immediately seen that the following proposition holds

Proposition 2.1. *The form F_γ is continuous in the topology induced by the norm $\|\cdot\|_Q$.*

Proof. In order to estimate the pointwise term in F_γ, notice that

$$|\psi(0+) - \psi(0-)|^2 \;\leq 2|\psi(0+)|^2 + 2|\psi(0-)|^2 \;\leq\; 2\|\psi_+\|\|\psi_+'\| + 2\|\psi_-\|\|\psi_-'\| \;\leq\; 2\|\psi\|_Q^2, \tag{2.6}$$

where we used the well-known fact that $\|\phi\|_\infty^2 \leq \|\phi\|_2\|\phi'\|_2$ for any $\phi \in H^1(\mathbb{R})$. The first term in (2.4) is trivially estimated by the squared Q-norm, so the proof is complete. $\square$

Finally, we notice that, for any $m \geq -\omega_0$ one has

$$F_\gamma(\psi) \;=\; ((H_\gamma + m)^{\frac{1}{2}}\psi, (H_\gamma + m)^{\frac{1}{2}}\psi) - m(\psi, \psi). \tag{2.7}$$

This is immediate for $\psi \in D(H_\gamma)$, then a density argument shows that (2.7) holds for any $\psi \in Q$.

2.4 Functionals and variational problems

We first define the functionals we use through the paper. All of them act on the energy space Q, defined in (2.5).

We define the Hamiltonian of the NLS with a point defect as the sum of the linear Hamiltonian of the corresponding point interaction and of the nonlinear self-interacting part.

So, in the particular case of the H_γ interaction, or δ' defect, we have for the total energy

$$E(\psi) \;=\; \frac{1}{2}\|\psi'\|_2^2 - \frac{1}{2\gamma}|\psi(0+) - \psi(0-)|^2 - \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2} \;. \tag{2.8}$$

Standard results in the calculus of variations (see for example [10] show that $E \in C^1(Q, \mathbb{R})$ and the Fréchet derivative

$$E'(\psi) = H_\gamma\psi - \lambda|\psi|^{2\mu}\psi \in Q^* \qquad \forall\psi \in Q \;.$$

Moreover we define the *mass functional* (sometimes called charge)

$$M(\psi) \;:=\; \|\psi\|^2.$$

The mass obviously belongs to $C^1(Q, \mathbb{R})$. Both mass and energy are conserved by the flow (see Proposition 3.4).

A prominent role in the variational characterization of stationary states is played by the *action functional*

$$S_\omega := \; E + \frac{\omega}{2}M \;, \tag{2.9}$$

or explicitly

$$S_\omega(\psi) = \frac{1}{2}\|\psi'\|^2 + \frac{\omega}{2}\|\psi\|^2 - \frac{1}{2\gamma}|\psi(0+) - \psi(0-)|^2 - \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2} . \tag{2.10}$$

Again, $S_\omega \in C^1(Q, \mathbb{R})$ and for every $\psi \in Q$

$$S'_\omega(\psi) = H_\gamma\psi + \omega\psi - \lambda|\psi|^{2\mu}\psi \in Q^* .$$

Stationary states ψ_ω satisfy $S'_\omega(\psi_\omega) = 0$.

Moreover, it is useful to define the so-called *Nehari functional*

$$\begin{aligned} I_\omega(\psi) \ &:= \langle S'_\omega(\psi), \psi\rangle \ = \|\psi'\|^2 + \omega\|\psi\|^2 - \frac{1}{\gamma}|\psi(0+) - \psi(0-)|^2 - \lambda\|\psi\|_{2\mu+2}^{2\mu+2} \\ &= \ 2S_\omega(\psi) - \frac{\lambda\mu}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}. \end{aligned} \tag{2.11}$$

It is immediately seen by its definition that the zero-level set of the Nehari functional contains the stationary states associated to the action S_ω.
The action S_ω restricted to the Nehari manifold gives the last auxiliary functional

$$\widetilde{S}(\psi) := \ \frac{\lambda\mu}{2(\mu+1)}\|\psi\|_{2\mu+2}^{2\mu+2} \ = \ S_\omega(\psi) - \frac{1}{2}I_\omega(\psi) . \tag{2.12}$$

In many physical contexts, the search for the ground states can be formulated as follows:

Problem 2.2. *Given $m > 0$, find the minimum and the minimizers of the functional E in the energy space Q under the constraint $M = m$.*

Nevertheless, in the investigation of orbital stability of stationary states it proves useful to study another variational problem, namely

Problem 2.3. *Find the minimum and the non vanishing minimizers of the functional S_ω in the energy space Q under the constraint $I_\omega = 0$.*

This is the problem studied in the present paper. It can be rephrased to the issue of characterizing, as explicitly as possible, the function

$$d(\omega) \ := \ \inf\{S_\omega(\psi), \psi \in Q\backslash\{0\}, \ I_\omega(\psi) = 0\}. \tag{2.13}$$

Finally, studying problem 2.3 is equivalent (see Section 4) to the

Problem 2.4. *Find the minima and the non vanishing minimizers of the functional $\widetilde{S}$ in the energy space Q under the constraint $I_\omega \leq 0$.*

Problems 2.2 and 2.3 are related, but not in an obvious way. Of course, when 2.2 has a solution ψ, via Lagrange multiplier theory it turns out that there exists a real multiplier ω such that $E'(\psi) + \omega M'(\psi) = 0$, which coincides with $S'_\omega(\psi) = 0$, meaning that ψ is a stationary point of S_ω and by definition it belongs to I_ω, so it is a solution of 2.3. Nevertheless the two problems are not equivalent and a complete analysis will be given elsewhere. See however the remarks at the end of Section 4 and Section 6.

3 Well-posedness and conservation laws

Here we treat the problem of the existence and uniqueness of the solution to equation (1.1). In the present paper we are mainly interested in solution lying in the energy space Q and in the operator domain $D(H_\gamma)$, i.e. weak or strong solutions respectively. Let us stress that it is possible to obtain local well-posedness in L^2 by proving suitable Strichartz estimates and then following the traditional line (see e.g. [10]); this route is followed for a general point interaction and a cubic nonlinearity in [1], to which we refer.

We begin with weak solutions and to this end, instead of equation (1.1) we study its integral form, i.e.

$$\psi(t) \; = \; e^{-iH_\gamma t}\psi_0 + i\lambda \int_0^t e^{-iH_\gamma(t-s)}|\psi(s)|^{2\mu}\psi(s)\,ds, \tag{3.1}$$

where the initial data ψ_0 belong to Q.

For an exhaustive treatment of the problem given by a general point interaction at the origin and a cubic nonlinearity, see [1]. In particular, we recall that the dual $Q^\star$ of the energy space Q, i.e. the space of the bounded linear functionals on Q, can be represented as

$$Q^\star \; = \; H^{-1}(\mathbb{R}) \oplus \mathrm{Span}(\delta(0+), \delta(0-)), \tag{3.2}$$

where the action of the functionals $\delta(0\pm)$ on a function $\varphi \in Q$ reads

$$\langle \delta(0\pm), \varphi \rangle \; = \; \varphi(0\pm).$$

As usual, exploiting formula (2.7), one can extend the action of H_γ to the space Q, with values in $Q^\star$, by

$$\langle H_\gamma \psi_1, \psi_2 \rangle \; := \; ((H_\gamma + m)^{\frac{1}{2}}\psi_1, (H_\gamma + m)^{\frac{1}{2}}\psi_2) - m(\psi_1, \psi_2), \tag{3.3}$$

where $m > -\omega_0$. The continuity of $H_\gamma \psi_1$ as a functional of the space Q is immediately proved by Cauchy-Schwarz inequality, (3.3) and (2.7), that together give

$$|\langle H_\gamma \psi_1, \psi_2 \rangle| \; \leq \; C\|\psi_1\|_Q \|\psi_2\|_Q. \tag{3.4}$$

Now we are ready to state the following lemma.

Lemma 3.1. *For any $\psi \in Q$ the identity*

$$\frac{d}{dt} e^{-iH_\gamma t}\psi \; = \; -iH_\gamma e^{-iH_\gamma t}\psi \tag{3.5}$$

holds in $Q^\star$.

Proof. The time derivative of the functional $e^{-iH_\gamma t}\psi$ is defined in the weak sense, namely

$$\left\langle \frac{d}{dt} e^{-iH_\gamma t}\psi, \cdot \right\rangle := \lim_{h \to 0} \frac{1}{h}\left[(e^{-iH_\gamma(t+h)}\psi, \cdot) - (e^{-iH_\gamma t}\psi, \cdot) \right].$$

Now, fix ξ in the operator domain $D(H_\gamma)$ defined in (2.2). Then,

$$\left\langle \frac{d}{dt} e^{-iH_\gamma t}\psi, \xi \right\rangle \; = \; \lim_{h \to 0} \left(\psi, \frac{e^{iH_\gamma(t+h)}\xi - e^{iHt}\xi}{h} \right) \; = \; (\psi, iH_\gamma e^{iH_\gamma t}\xi) \; = \; \langle -iH_\gamma e^{-iH_\gamma t}\psi, \xi \rangle,$$

where we used (3.3).

Then, the result can be extended to $\xi \in Q$ by a density argument, and by (3.4) we get the continuity of the functional $\frac{d}{dt} e^{-iH_\gamma t}\psi$ on Q, so the result is proven. ∎

Corollary 3.2. *By* (3.5)*, one immediately has that the formulation* (1.1) *of the Schrödinger equation holds in* $Q^\star$*.*

In order to prove a well-posedness result we recall from Section 4 in [1] that the one-dimensional Gagliardo-Nirenberg estimates holds for any ψ in Q, i.e.

$$\|\psi\|_p \;\leqslant\; C\|\psi'\|_{L^2}^{\frac{1}{2}-\frac{1}{p}}\|\psi\|_{L^2}^{\frac{1}{2}+\frac{1}{p}}, \qquad \psi\in Q \tag{3.6}$$

where the $C>0$ is a positive constant which depends on the index p only.
Notice that from inequality (3.6) one immediately obtains the Sobolev-type estimate

$$\|\psi\|_{2\mu+2} \;\leq\; C\|\psi\|_Q. \tag{3.7}$$

Proposition 3.3 (Local well-posedness in Q)**.** *Fixed* $\psi_0\in Q$*, there exists* $T>0$ *such that equation* (3.1) *has a unique solution* $\psi\in C^0([0,T),Q)\cap C^1([0,T),Q^\star)$*.*
Moreover, eq. (3.1) *has a maximal solution* $\psi^{\max}$ *defined on an interval of the form* $[0,T^\star)$*, and the following "blow-up alternative" holds: either* $T^\star=\infty$ *or*

$$\lim_{t\to T^\star}\|\psi^{\max}(t)\|_Q \;=\; +\infty.$$

Proof. We denote by $\mathcal{X}$ the space $L^\infty([0,T),Q)$, endowed with the norm $\|\psi\|_{\mathcal{X}} \;:=\; \sup_{t\in[0,T)}\|\psi(t)\|_Q$. Given $\psi_0\in Q$, we define the map $G:\mathcal{X}\longrightarrow\mathcal{X}$ as

$$G\phi := e^{-iH_\gamma\cdot}\psi_0 + i\lambda\int_0^{\cdot} e^{-iH_\gamma(\cdot-s)}|\phi(s)|^{2\mu}\phi(s)\,ds.$$

Notice first that the nonlinearity preserves the space Q. Indeed, referring to the decomposition (2.3), since both $\psi_\pm$ belong to $H^1(\mathbb{R})$, the functions $|\psi_\pm|^2\psi_\pm$ belong to $H^1(\mathbb{R})$ too, and so the energy space is preserved.

Now, estimate (3.6) with $p=\infty$ yields

$$\||\phi(s)|^{2\mu}\phi(s)\|_Q \;\leq\; C\|\phi(s)\|_Q^{2\mu+1},$$

so

$$\|G\phi\|_{\mathcal{X}} \;\leq \|\psi_0\|_Q + C\int_0^T \|\phi(s)\|_Q^{2\mu+1}\,ds \;\leq\; \|\psi_0\|_Q + CT\|\phi\|_{\mathcal{X}}^{2\mu+1}. \tag{3.8}$$

Analogously, given $\phi,\xi\in Q$,

$$\|G\phi - G\xi\|_{\mathcal{X}} \;\leqslant CT\left(\|\phi\|_{\mathcal{X}}^{2\mu}+\|\xi\|_{\mathcal{X}}^{2\mu}\right)\|\phi-\xi\|_{\mathcal{X}}. \tag{3.9}$$

We point out that the constant C appearing in (3.8) and (3.9) is independent of ψ_0, ϕ, and ξ. Now let us restrict the map G to elements ϕ such that $\|\phi\|_{\mathcal{X}}\leqslant 2\|\psi_0\|_Q$. From (3.8) and (3.9), if T is chosen to be strictly less than $(8C\|\psi_0\|_Q^2)^{-1}$, then G is a contraction of the ball in $\mathcal{X}$ of centre zero and radius $2\|\psi_0\|_Q$, and so, by the contraction lemma, there exists a unique solution to (3.1) in the time interval $[0,T)$. By a standard one-step bootstrap argument one has that the solution actually belongs to $C^0([0,T),Q)$ and, due to the validity of (1.1) in the space $Q^\star$ (see Corollary 3.2), we immediately have that the solution ψ actually belongs to $C^0([0,T),Q)\cap C^1([0,T),Q^\star)$.

The proof of the existence of a maximal solution is standard, while the blow-up alternative is a consequence of the fact that, whenever the Q-norm of the solution is finite, it is possible to extend it for a further time by the same contraction argument. □

The next step consists in the proof of the conservation laws.

Proposition 3.4. *For any solution $\psi \in C^0([0,T), Q)$ to the equation* (3.1)*, the following conservation laws hold at any time t:*

$$\|\psi(t)\| \;=\; \|\psi_0\|, \qquad E(\psi(t)) \;=\; E(\psi_0),$$

where the symbol E denotes the energy functional introduced in (2.8)*.*

Proof. The conservation of the L^2-norm can be immediately obtained using Lemma 3.1 and Corollary 3.2. So,

$$\frac{d}{dt}\|\psi(t)\|^2 \;=\; 2\,\mathrm{Re}\left\langle \psi(t), \frac{d}{dt}\psi(t)\right\rangle \;=\; 2\,\mathrm{Im}\,\langle \psi(t), H_\gamma \psi(t)\rangle \;=\; 0$$

by the self-adjointness of H_γ. In order to prove the conservation of the energy, notice that $\langle \psi(t), H_\gamma\psi(t)\rangle$ is differentiable as a function of time. Indeed,

$$\frac{1}{h}\left[\langle \psi(t+h), H_\gamma\psi(t+h)\rangle - \langle\psi(t), H_\gamma\psi(t)\rangle\right] \;=\; \left\langle \frac{\psi(t+h)-\psi(t)}{h}, H_\gamma\psi(t+h)\right\rangle + \left\langle H_\gamma\psi(t), \frac{\psi(t+h)-\psi(t)}{h}\right\rangle$$

and then, passing to the limit $h \to 0$,

$$\frac{d}{dt}(\psi(t), H_\gamma\psi(t)) \;=\; 2\,\mathrm{Re}\left\langle \frac{d}{dt}\psi(t), H_\gamma\psi(t)\right\rangle \;=\; 2\,\mathrm{Im}\,\langle |\psi(t)|^2\psi(t), H_\gamma\psi(t)\rangle, \tag{3.10}$$

where we used the self-adjointness of H_γ and corollary 3.2. Furthermore,

$$\frac{d}{dt}(\psi(t), |\psi(t)|^{2\mu}\psi(t)) \;=\; \frac{d}{dt}(\overline{\psi^\mu(t)}\psi(t), \overline{\psi^\mu(t)}\psi(t)) \;=\; (2\mu+2)\,\mathrm{Im}\,\langle |\psi(t)|^{2\mu}\psi(t), H_\gamma\psi(t)\rangle. \tag{3.11}$$

From (3.10) and (3.11) one then obtains

$$\frac{d}{dt}E(\psi(t)) \;=\; \frac{1}{2}\frac{d}{dt}\langle\psi(t), H_\gamma\psi(t)\rangle - \frac{1}{2\mu+2}\frac{d}{dt}(\psi(t), |\psi(t)|^{2\mu}\psi(t)) \;=\; 0$$

and the proposition is proved. □

Corollary 3.5. *For $\mu < 2$, all solutions to* (3.1) *are globally defined in time.*

Proof. By estimate (3.6) with $p = \infty$ and conservation of the L^2-norm, there exists a constant M, that depends on ψ_0 only, such that

$$E(\psi_0) \;=\; E(\psi(t)) \;\geq\; \frac{1}{2}\|\psi'(t)\|^2 - M\|\psi'(t)\|$$

Therefore a uniform (in t) bound on $\|\psi'(t)\|^2$ is obtained. As a consequence, one has that no blow-up in finite time can occur, and therefore, by the blow-up alternative proved in theorem (3.3), the solution is global in time. □

Using a well established analysis which we will not repeat here (see again [1] for a detailed study in the case $\mu = 1$)), one can get well posedness for strong solutions of (3.1), i.e. in the operator domain.

Theorem 3.6 (Local and global well posedness in the operator domain)**.** *For any $\mu > 0$ and initial datum $\psi_0 \in D(H_\gamma)$ there exists $T \in (0, +\infty)$ such that the equation* (3.1) *has a unique solution $\psi \in C([0,T), D(H)) \cap C^1([0,T), L^2(\mathbb{R}))$. Moreover, equation* (3.1) *has a maximal solution $\psi(t)$ defined on the interval $[0, T^\star)$. For such a solution, the following alternative holds: either $T^\star = +\infty$ or*

$$\lim_{t \to T^\star} \|\psi(t)\|_H \;=\; +\infty.$$

Moreover, if $0 < \mu < 2$ and $\psi_0 \in D(H_\gamma)$ the equation (3.1) *has a unique global solution $\psi \in C(\mathbb{R}, D(H_\gamma)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}))$.*
Finally, mass and energy are conserved quantities for both weak and strong solutions.

4 Existence of a ground state

In this section we show the existence of a solution to problem 2.3 for any $\omega > \omega_0$. More precisely, we prove the following theorem:

Theorem 4.1. *Let $\omega > \frac{4}{\gamma^2}$. There exists $\psi \in Q\backslash\{0\}$, with $I_\omega(\psi) = 0$, that solves problem 2.4, namely*

$$S_\omega(\psi) \;=\; d(\omega).$$

In order to prove theorem 4.1 we need four preliminary lemmas. In the first lemma we show that problems 2.3 and 2.4 share the same solutions.

Lemma 4.2. *For the functionals S_ω and $\widetilde{S}$, defined respectively in* (2.9) *and* (2.12)*, the following equalities hold:*

$$d(\omega) \;:=\; \inf\{S_\omega(\psi),\, \psi \in Q\backslash\{0\},\, I_\omega(\psi) = 0\} \;=\; \inf\{\widetilde{S}(\psi),\, \psi \in Q\backslash\{0\},\, I_\omega(\psi) \le 0\}. \qquad (4.1)$$

Furthermore, a function $\phi \in Q\backslash\{0\}$ satisfies $\widetilde{S}(\phi) = d(\omega)$ and $I_\omega(\phi) \le 0$ if and only if $S_\omega(\phi) = d(\omega)$ and $I_\omega(\phi) = 0$.

Proof. Let ϕ be a nonzero element of Q such that $I_\omega(\phi) = 0$. Then, by (2.11), $S_\omega(\phi) = \widetilde{S}(\phi)$, therefore

$$\inf\{S_\omega(\psi),\, \psi \in Q\backslash\{0\},\, I_\omega(\psi) = 0\} \;\ge\; \inf\{\widetilde{S}(\psi),\, \psi \in Q\backslash\{0\},\, I_\omega(\psi) \le 0\}. \qquad (4.2)$$

On the other hand, let ϕ be an element of Q such that $I_\omega(\phi) < 0$. Defined

$$\alpha(\phi) := \frac{[F_\gamma(\phi) + \omega\|\phi\|^2]^{\frac{1}{2\mu}}}{\lambda^{\frac{1}{2\mu}} \|\phi\|_{2\mu+2}^{1+\frac{1}{\mu}}}, \qquad (4.3)$$

one can directly verify that $\alpha(\phi) < 1$, $I_\omega(\alpha(\phi)\phi) = 0$, and then, by (2.11),

$$S_\omega(\alpha(\phi)\phi) \;=\; \widetilde{S}(\alpha(\phi)\phi) \;=\; \alpha(\phi)^{2\mu+2}\widetilde{S}(\phi) \;<\; \widetilde{S}(\phi),$$

so

$$\inf\{S_\omega(\psi),\, \psi \in Q\backslash\{0\},\, I_\omega(\psi) = 0\} \;\le\; \inf\{\widetilde{S}(\psi)),\, \psi \in Q\backslash\{0\},\, I_\omega(\psi) \le 0\}. \qquad (4.4)$$

From (4.2) and (4.4), identity (4.1) is proven.

From (4.1), it is obvious that, if ϕ minimizes S_ω on the set $I_\omega = 0$, then it minimizes $\widetilde{S}$ on the set $I_\omega \leq 0$ too.
Suppose now that $\widetilde{S}(\phi) = d(\omega)$ (then ϕ minimizes $\widetilde{S}$ on the set $I_\omega \leq 0$) and $I_\omega(\phi) < 0$. Defining $\alpha(\phi)$ like in (4.3) one finds $\alpha(\phi) < 1$, $I_\omega(\alpha(\phi)\phi) = 0$, and $S_\omega(\alpha(\phi)\phi) = \alpha(\phi)^{2\mu+2}\widetilde{S}(\phi) < d(\omega)$ again, that contradicts the definition of $d(\omega)$. So the lemma is proven. □

Lemma 4.3. *The quantity $d(\omega)$ defined in* (2.13) *is strictly positive.*

Proof. First notice that, for any $a > 0$,

$$\begin{aligned} |\psi(0+) - \psi(0-)|^2 &\leq 2|\psi(0+)|^2 + 2|\psi(0-)|^2 \;\leq\; 2\|\psi_+\|\|\psi'_+\| + 2\|\psi_-\|\|\psi'_-\| \\ &\leq a(\|\psi_+\|^2 + \|\psi_-\|^2) + a^{-1}(\|\psi'_+\|^2 + \|\psi'_-\|^2) \;=\; 2a\|\psi\|^2 + 2a^{-1}\|\psi'\|^2 \end{aligned} \tag{4.5}$$

where we used decomposition (2.3), estimate (3.6) with $p = \infty$, and Cauchy-Schwarz inequality. Therefore,

$$F_\gamma(\psi) + \omega\|\psi\|^2 \;\geq\; \left(1 - \frac{2}{\gamma a}\right)\|\psi'\|^2 + \left(\omega - \frac{2a}{\gamma}\right)\|\psi\|^2. \tag{4.6}$$

Since $\omega > \frac{4}{\gamma^2}$, we can fix the parameter a in such a way that

$$\frac{2}{\gamma} \;<\; a \;<\; \frac{\gamma\omega}{2},$$

so it is proven that

$$F_\gamma(\psi) + \omega\|\psi\|^2 \;\geq\; C\|\psi\|_Q^2. \tag{4.7}$$

Then, by the estimate (3.7)

$$I_\omega(\psi) \;\geq\; C\|\psi\|_Q^2 - \lambda\|\psi\|_{2\mu+2}^{2\mu+2} \;\geq\; C_1\|\psi\|_{2\mu+2}^2 - \lambda\|\psi\|_{2\mu+2}^{2\mu+2}.$$

It appears that, if $I_\omega(\psi) \leq 0$, then either $\psi = 0$ or

$$\|\psi\|_{2\mu+2} \;\geq\; \left(\frac{C_1}{\lambda}\right)^{\frac{1}{2\mu}} \;>\; 0.$$

Therefore, since in problem 2.4 we are looking for a non vanishing minimizer, and owing to the fact that on the Nehari manifold $S_\omega = \widetilde{S}$, it must be $d(\omega) > 0$.

□

In the third lemma we consider a pair of functionals S_ω^0, I_ω^0, that correspond to the functionals S_ω, I_ω in the absence of the point interaction:

$$\begin{aligned} S_\omega^0(\psi) &= \frac{1}{2}\|\psi'\|^2 + \frac{\omega}{2}\|\psi\|^2 - \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2} \\ I_\omega^0(\psi) &= \|\psi'\|^2 + \omega\|\psi\|^2 - \lambda\|\psi\|_{2\mu+2}^{2\mu+2}. \end{aligned}$$

Lemma 4.4. *The set of the minimizers of the functional S_ω^0 among the functions in $Q\backslash\{0\}$, satisfying the constraint $I_\omega^0 = 0$, is given by*

$$\{e^{i\theta}\chi_+\phi_\omega^0,\; e^{i\theta}\chi_-\phi_\omega^0,\; \theta \in [0, 2\pi)\},$$

where the function ϕ_ω^0 has been defined in (2.1).

Proof. First notice that, reasoning like in the proof of lemma 4.2, one can prove that the search for the minimizers of S^0_ω among the nonzero functions in Q that satisfy $I^0_\omega = 0$, is equivalent to the search for the minimizers of $\widetilde{S}$ among the nonzero functions in Q that satisfy $I^0_\omega \le 0$.

Let us define the real function of a real variable

$$d^0(\omega) \;:=\; \inf\{\widetilde{S}(\psi),\ \psi \in Q\backslash\{0\},\ I^0_\omega(\psi) \le 0\}.$$

Proceeding like in lemma 4.3 one can show that $d^0(\omega) > 0$.

Besides, we recall that ϕ^0_ω minimizes the functional $\widetilde{S}$ among all functions in $H^1(\mathbb{R})\backslash\{0\}$ such that $I^0_\omega = 0$. Now, making resort to the representation (2.3), let us consider a generic function of $Q\backslash\{0\}$ supported on $\mathbb{R}^+$, call it $\chi_+\psi_+$, with $\psi_+ \in H^1(\mathbb{R})$ and even, and suppose that $I^0_\omega(\chi_+\psi_+) \le 0$. One immediately has

$$\widetilde{S}(\chi_+\psi_+) \;=\; \frac{1}{2}\widetilde{S}(\psi_+) \;\ge\; \frac{1}{2}\widetilde{S}(\phi^0_\omega) \;=\; \widetilde{S}(\chi_+\phi^0_\omega),$$

so $\chi_+\phi^0_\omega$ is a minimizer of $\widetilde{S}$ among the functions of $Q\backslash\{0\}$, supported on $\mathbb{R}^+$ and satisfying $I^0_\omega \le 0$. Notice that the equal sign holds if and only if $\psi_+ = \phi^0_\omega$. Otherwise, ψ_+ would not belong to the family (2.1), nevertheless, as $S^0_\omega(\psi_+) = S^0_\omega(\phi^0_\omega)$, it would be a minimizer of S^0_ω among the nonzero functions in $H^1(\mathbb{R})$ that satisfy $I^0_\omega = 0$, which is impossible.
Thus, for any function $\psi \in Q\backslash\{0\}$, with $\psi \;=\; \chi_+\psi_+ + \chi_-\psi_-$, and $I^0_\omega(\psi) \le 0$, the following alternative holds: either $I^0_\omega(\chi_+\psi_+) \le 0$ and so

$$\widetilde{S}(\psi) \;\ge\; \widetilde{S}(\chi_+\psi_+) \;\ge\; \widetilde{S}(\chi_+\phi^0_\omega), \tag{4.8}$$

or $I^0_\omega(\chi_-\psi_-) \le 0$ and so

$$\widetilde{S}(\psi) \;\ge\; \widetilde{S}(\chi_-\psi_-) \;\ge\; \widetilde{S}(\chi_-\phi^0_\omega), \tag{4.9}$$

and the equality in the last step of (4.8) and (4.9) holds if and only if $|\psi_+| = \phi^0_\omega$, or $|\psi_-| = \phi^0_\omega$, respectively.

Taking into account the $U(1)-$symmetry of the problem, the proof is complete. □

Lemma 4.5. *For the infimum of problem 2.4 the following inequality holds*

$$d(\omega) \;<\; \widetilde{S}(e^{i\theta}\chi_+\phi^0_\omega) \;=\; \frac{1}{2}\left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}} \omega^{\frac{1}{2}+\frac{1}{\mu}} \int_0^1 (1-u^2)^{\frac{1}{\mu}}\, du. \tag{4.10}$$

Proof. We notice that the last identity in (4.10) can be obtained by direct computation. Furthermore,

$$I_\omega(\chi_+\phi^\omega_0) \;=\; I^0_\omega(\chi_+\phi^\omega_0) - \frac{1}{\gamma}\left(\frac{(\mu+1)\omega}{\lambda}\right)^{\frac{1}{\mu}} \;=\; -\frac{1}{\gamma}\left(\frac{(\mu+1)\omega}{\lambda}\right)^{\frac{1}{\mu}} \;<\; 0.$$

Following the proof of lemma 4.2 we define

$$\alpha := \frac{[F_\gamma(\chi_+\phi^0_\omega) + \omega\|\chi_+\phi^0_\omega\|^2]^{\frac{1}{2\mu}}}{\lambda^{\frac{1}{2\mu}}\|\chi_+\phi^0_\omega\|^{1+\frac{1}{\mu}}_{2\mu+2}} \;=\; \left(1 - \frac{1}{\gamma\lambda}\frac{2|\phi^0_\omega(0+)|^2}{\|\phi^0_\omega\|^{2\mu+2}_{2\mu+2}}\right)^{\frac{1}{2\mu}} \;=\; \left(1 - \frac{\mu}{\gamma(\mu+1)\omega^{\frac{1}{2}}\int_0^1(1-u^2)^{\frac{1}{\mu}}\,du}\right)^{\frac{1}{2\mu}}$$
$$<\; 1.$$

Therefore, $I^0_\omega(\alpha\chi_+\phi^\omega_0) = 0$ and

$$\widetilde{S}(\alpha\chi_+\phi^\omega_0) \;=\; \frac{\alpha^{2\mu+2}}{2}\left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}} \omega^{\frac{1}{2}+\frac{1}{\mu}} \int_0^{+\infty} (1-u^2)^{\frac{1}{\mu}}\, du,$$

and, since $\alpha < 1$, we get

$$d(\omega) \leq \frac{\alpha^{2\mu+2}}{2}\left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}}\omega^{\frac{1}{2}+\frac{1}{\mu}}\int_0^{+\infty}(1-u^2)^{\frac{1}{\mu}}\,du \;<\; \frac{1}{2}\left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}}\omega^{\frac{1}{2}+\frac{1}{\mu}}\int_0^{+\infty}(1-u^2)^{\frac{1}{\mu}}\,du,$$

and the proof is complete. □

Now we can prove theorem 4.1.

Proof. Let $\{\psi_n\}$ be a minimizing sequence for the functional $\widetilde{S}$ on the set $I_\omega \leq 0$. We show that there exists a subsequence of $\{\psi_n\}$ that converges weakly in Q. First, notice that $\|\psi_n\|_Q$ is bounded. Indeed, the sequence $\|\psi_n\|_{2\mu+2}$ is bounded as it converges. Furthermore, by the lower boundedness of the form F_γ, and recalling that $I_\omega(\psi_n) \leq 0$, we have

$$0 \;\leq\; \left(\omega - \frac{4}{\gamma^2}\right)\|\psi_n\|^2 \;\leq\; F_\gamma(\psi_n) + \omega\|\psi_n\|^2 \;\leq\; \lambda\,\|\psi_n\|_{2\mu+2}^{2\mu+2} \;\leq\; C,$$

so $\|\psi_n\| \leq C$.

Then, using $I_\omega(\psi_n) \leq 0$ again, by the decomposition (2.3) and estimate (3.6),

$$\begin{aligned}\|\psi_n'\|^2 \;&\leq\; \lambda\,\|\psi_n\|_{2\mu+2}^{2\mu+2} - \omega\|\psi_n\|^2 + \frac{1}{\gamma}|\psi_n(0+) - \psi_n(0-)|^2 \;\leq\; C + \frac{2}{\gamma}(|\psi_{n,+}(0)|^2 + |\psi_{n,-}(0)|^2)\\ &\leq\; C + C\|\psi_{n,+}\|\|\psi_{n,+}'\| + C\|\psi_{n,-}\|\|\psi_{n,-}'\| \;\leq\; C + C\left(\frac{1}{\varepsilon}\|\psi_n\|^2 + \varepsilon\|\psi_n'\|^2\right).\end{aligned} \tag{4.11}$$

Choosing ε sufficiently small, we obtain that $\|\psi_n'\|^2$ is bounded, so the sequence $\{\psi_n\}$ is bounded in Q, and then, by Banach-Alaoglu theorem, there exists a converging subsequence, that we call $\{\psi_n\}$ again, in the weak topology of Q. We call ψ_∞ the weak limit of the sequence $\{\psi_n\}$.

We prove that $\psi_\infty \neq 0$. To this aim, we show, first, that the sequences $\{\psi_n(0\pm)\}$ converge to $\psi_\infty(0\pm)$, and, second, that $\lim_{n\to\infty} I_\omega(\psi_n) = 0$.

Let us define the functions $\varphi_\pm(x) := \chi_\pm(x)e^{\mp x}$. Then, integrating by parts,

$$(\varphi_+, \psi_n)_Q \;=\; \int_0^{+\infty} e^{-x}\psi_n(x)\,dx - \int_0^{+\infty} e^{-x}\psi_n'(x)\,dx \;=\; \psi_n(0+).$$

Analogously, $(\varphi_-, \psi_n)_Q \;=\; \psi_n(0-)$. Therefore, by weak convergence,

$$\psi_n(0\pm) \;=\; (\varphi_\pm, \psi_n)_Q \;\to\; (\varphi_\pm, \psi_\infty)_Q \;=\; \psi_\infty(0\pm), \tag{4.12}$$

and the first preliminary claim is proven. We prove the second claim by contradiction, i.e., supposing that $I_\omega(\psi_n) \to 0$ is false. Then, there must be a subsequence of $\{\psi_n\}$, denoted by $\{\psi_n\}$ too, such that

$$\lim_{n\to\infty} I_\omega(\psi_n) \;=\; -\beta \;<\; 0.$$

We define the sequence $\zeta_n := \nu_n\psi_n$, with

$$\nu_n := \frac{[F_\gamma(\psi_n) + \omega\|\psi_n\|^2]^{\frac{1}{2\mu}}}{\lambda^{\frac{1}{2\mu}}\|\psi_n\|_{2\mu+2}^{1+\frac{1}{\mu}}} \;<\; 1.$$

Since

$$\lim_{n\to\infty} \nu_n \;=\; \lim_{n\to\infty} \left[1 + \frac{I_\omega(\psi_n)}{\lambda \|\psi_n\|_{2\mu+2}^{2\mu+2}}\right]^{\frac{1}{2\mu}} \;=\; \left[1 - \frac{\beta\mu}{2(\mu+1)d(\omega)}\right]^{\frac{1}{2\mu}} \;<\; 1,$$

we obtain

$$\lim_{n\to\infty} \widetilde{S}(\zeta_n) \;=\; \nu^{2\mu+2} \widetilde{S}(\psi_n) \;<\; \widetilde{S}(\psi_n)$$

and, since $I_\omega(\zeta_n) = 0$, it follows that the assumption that $\{\psi_n\}$ is a minimizing sequence is false. Therefore, it must be

$$\lim_{n\to\infty} I_\omega(\psi_n) = 0. \tag{4.13}$$

To prove that $\psi_\infty \neq 0$ we proceed by contradiction again. Assume that $\psi_\infty = 0$. Define the sequence $\eta_n := \rho_n \psi_n$ with

$$\rho_n := \frac{\left[\|\psi_n'\|^2 + \omega\|\psi_n\|^2\right]^{\frac{1}{2\mu}}}{\lambda^{\frac{1}{2\mu}} \|\psi_n\|_{2\mu+2}^{1+\frac{1}{\mu}}} \tag{4.14}$$

Using (4.12) and (4.13) we obtain

$$\lim_{n\to\infty} \rho_n \;=\; \lim_{n\to\infty} \left[1 + \frac{I_\omega(\psi_n) + \gamma^{-1}|\psi_n(0+) - \psi_n(0-)|^2}{\lambda \|\psi_n\|_{2\mu+2}^{2\mu+2}}\right]^{\frac{1}{2\mu}} \;=\; 1,$$

and therefore

$$\lim_{n\to\infty} \widetilde{S}(\eta_n) \;=\; \lim_{n\to\infty} \rho_n^{2\mu+2} \widetilde{S}(\psi_n) \;=\; d(\omega).$$

Moreover, owing to definition (4.14),

$$I_\omega^0(\eta_n) \;=\; I_\omega^0(\rho_n\psi_n) \;=\; \rho_n^2 \left(\|\psi_n'\|^2 + \omega\|\psi_n\|^2 - \lambda\rho_n^{2\mu}\|\psi_n\|_{2\mu+2}^{2\mu+2}\right) \;=\; 0,$$

so, due to Lemma 4.4,

$$d(\omega) \;\geq\; S_\omega^0(\chi_+\phi_0). \tag{4.15}$$

On the other hand, by Lemma 4.5 we conclude

$$d(\omega) \;\;<\;\; \widetilde{S}(\chi_+\phi_0) \;\leq\; \widetilde{S}(\eta_n).$$

that contradicts (4.15). So the hypothesis $\psi_\infty = 0$ cannot hold.

Now we prove that $I_\omega(\psi_\infty) \leq 0$. To this purpose, we follow the last lines in the proof of proposition 2 in [17]. First, we recall an inequality due to Brezis and Lieb ([7]): if u_n converges to u_∞ weakly in L^p, then

$$\|u_n\|_p^p - \|u_n - u_\infty\|_p^p - \|u_\infty\|_p^p \;\;\longrightarrow\;\; 0, \quad \forall\, 1 < p < \infty. \tag{4.16}$$

First, we notice that if $u_n = \psi_n$ and $p = 2\mu + 2$, then (4.16) yields

$$\widetilde{S}(\psi_n) - \widetilde{S}(\psi_n - \psi_\infty) - \widetilde{S}(\psi_\infty) \;\longrightarrow\; 0. \tag{4.17}$$

Further applying (4.16) to the sequence $\{\psi_n\}$ and to the sequence $\{\psi_n'\}$ with $p = 2$, and using (4.12), yields

$$I_\omega(\psi_n) - I_\omega(\psi_n - \psi_\infty) - I_\omega(\psi_\infty) \;\;\longrightarrow\;\; 0. \tag{4.18}$$

Suppose $I_\omega(\psi_\infty) > 0$. Then, by (4.18) and (4.13),

$$\lim_{n\to\infty} I_\omega(\psi_n - \psi_\infty) \;=\; \lim_{n\to\infty} I_\omega(\psi_n) - I_\omega(\psi_\infty) \;=\; -I_\omega(\psi_\infty) \;<\; 0.$$

Choose $\bar{n}$ such that $I_\omega(\psi_n - \psi_\infty) < 0$ for any $n > \bar{n}$. Then, by definition of $d(\omega)$ we have

$$d(\omega) \;\le\; \widetilde{S}(\psi_n - \psi_\infty), \quad \forall n > \bar{n}, \tag{4.19}$$

but, on the other hand, $\psi_\infty \neq 0$ implies $\widetilde{S}(\psi_\infty) > 0$, and, together with (4.17), this yields

$$\lim_{n\to\infty} \widetilde{S}(\psi_n - \psi_\infty) \;=\; \lim_{n\to\infty} \widetilde{S}(\psi_n) - \widetilde{S}(\psi_\infty) \;=\; d(\omega) - S(\psi_\infty) \;<\; d(\omega),$$

that contradicts (4.19), and so it must be $I_\omega(\psi_\infty) \;\le\; 0$. As a consequence, by definition of $d(\omega)$,

$$\widetilde{S}(\psi_\infty) \;\ge\; d(\omega).$$

Now, since ψ_∞ is the weak limit of $\{\psi_n\}$ in $L^{2\mu+2}$, we must have

$$\widetilde{S}(\psi_\infty) \;=\; \frac{\lambda\mu}{2(\mu+1)}\|\psi_\infty\|_{2\mu+2}^{2\mu+2} \;\le\; \lim_{n\to\infty} \frac{\lambda\mu}{2(\mu+1)}\|\psi_n\|_{2\mu+2}^{2\mu+2} \;=\; d(\omega)$$

which implies

$$\widetilde{S}(\psi_\infty) = d(\omega), \tag{4.20}$$

and so ψ_∞ is a solution to the minimization problem 2.4, and therefore, to the minimization problem 2.3 too. The proof is complete.

□

Corollary 4.6 (Strong convergence)**.** *If a minimizing sequence $\{\psi_n\}$ for the problem 2.4 converges weakly in Q, then it converges strongly in Q.*

Proof. Formulas (4.16) and (4.20) prove that $\{\psi_n\}$ converges strongly in $L^{2\mu+2}$. As a consequence,

$$F_\gamma(\psi_n) + \omega\|\psi_n\|^2 \;=\; 2\frac{\mu}{\mu+1}\widetilde{S}(\psi_n) + I_\omega(\psi_n) \;\longrightarrow\; \frac{\mu}{\mu+1}\widetilde{S}(\psi_\infty) \;\longrightarrow\; F_\gamma(\psi_\infty) + \omega\|\psi_\infty\|^2,$$

and by (4.7) this is equivalent to the strong convergence in Q. □

We end this section by adding some remarks on the variational problem 2.2, i.e. to minimize the energy at fixed norm.
To be precise, let us define the manifold

$$\Gamma_m = \left\{\psi \in Q : \|\psi\|^2 = m\right\} \tag{4.21}$$

and

$$-\mathcal{E}_m = \inf\left\{E(\psi) \mid \;\; \psi \in \Gamma_m\right\} \;. \tag{4.22}$$

In the following results it is shown that in the supercritical regime the constrained energy is unbounded from below and in the subcritical regime its infimum is finite and negative. Moreover the energy is controlled from below by $\|\cdot\|_Q$ norm.

Lemma 4.7. (Behaviour of the constrained energy).
1) Let $\mu > 2$*; then* $\mathcal{E}_m = +\infty$ *and the energy* E *is unbounded from below in* Γ_m*;*
2) Let $\mu < 2$*; then* $0 < \mathcal{E}_m < +\infty$*; moreover there exist positive and finite constants* C_1*,* C_2 *(depending on* μ, γ, m*) such that*

$$E(\psi) > C_1\|\psi\|_Q^2 - C_2 \qquad \forall\ \psi \in \Gamma_m; \tag{4.23}$$

3) Let $\mu = 2$*; then, there exists* $m^* > 0$ *such that for* $m < m^*$ *inequality* (4.23) *holds true.*

Proof. To show 1) let us consider the trial function

$$\Phi(\sigma, x) = \frac{\sqrt{m}}{(2\pi\sigma^2)^{\frac{1}{4}}}\ e^{-\frac{|x|^2}{4\sigma^2}}\ .$$

A direct calculation shows that $\Phi(\sigma, x) \in \Gamma_m$, and

$$E(\Phi(\sigma, x)) = \frac{1}{2\sigma^2}\|\Phi'(1, \cdot)\|^2 - \frac{\sigma^{-\mu}}{2\mu + 2}\|\Phi(1, \cdot)\|_{2\mu+2}^{2\mu+2}\ .$$

This proves that for $\mu > 2$ the energy is unbounded from below. Moreover, for $\mu < 2$ and σ big enough, $E(\Phi(\sigma, \cdot)) < 0$.
Now, let us show the bound (4.23). The nonlinear term and the point interaction term in the energy are dominated by the kinetic energy. Let us consider first the nonlinear term.
Gagliardo-Nirenberg estimate (3.6) jointly with the condition $\psi \in \Gamma_m$ give

$$\|\psi\|_{2\mu+2}^{2\mu+2} \leq C\|\psi'\|^\mu\|\psi\|^{\mu+2} = C(\|\psi'\|^2)^{\frac{\mu}{2}}(\|\psi\|^2)^{1-\frac{\mu}{2}+\mu} = Cm^\mu\ (\|\psi'\|^2)^{\frac{\mu}{2}}\ (\|\psi\|^2)^{1-\frac{\mu}{2}} \equiv *\ .$$

With the use of the classical elementary inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}\ , \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

one obtains, for any $\varepsilon > 0$,

$$\begin{aligned} * &= Cm^\mu\ \left(\|\psi'\|^2\ \varepsilon\right)^{\frac{\mu}{2}}\ \left(\|\psi\|^2\varepsilon^{-\frac{\mu}{2-\mu}}\right)^{1-\frac{\mu}{2}} \leq Cm^\mu\ \left[\frac{\|\psi'\|^2\varepsilon}{\frac{2}{\mu}} + \frac{\|\psi\|^2\ \varepsilon^{-\frac{\mu}{2-\mu}}}{\frac{2}{2-\mu}}\right] \\ &= Cm^\mu\frac{\mu}{2}\varepsilon\|\psi'\|^2 + Cm^{\mu+1}\frac{2-\mu}{2}\varepsilon^{-\frac{\mu}{2-\mu}}\ . \end{aligned} \tag{4.24}$$

from which it follows

$$\|\psi\|_{2\mu+2}^{2\mu+2} \leq \tilde{C}_1\varepsilon\|\psi'\|_Q^2 + \tilde{C}_2 \tag{4.25}$$

In an analogous way one can treat the point interaction part of the energy. Again taking in account that $\psi \in \Gamma_m$ and by use of Sobolev embedding in one dimension and elementary inequalities, one has

$$\frac{1}{\gamma}|\psi(0^+) - \psi(0^-)|^2 \leq \frac{1}{\gamma}\left[\varepsilon\|\psi'\|^2 + \frac{\|\psi\|^2}{\varepsilon}\right] \leq \frac{1}{\gamma}\varepsilon\|\psi\|_Q^2 + \delta \tag{4.26}$$

Collecting the estimates for the nonlinear part and for the point interaction part of the energy and choosing ε small enough one gets points 2).

To prove 3), it suffices to notice that from (3.6) one immediately has

$$E(\psi) \geq \frac{1}{2}\|\psi'\|^2 - \frac{C\lambda m}{6}\|\psi'\|^2 - \frac{\sqrt{m}}{\gamma}\|\psi'\|,$$

for any $\psi \in \Gamma_m$, and the proof is complete. ∎

Note that the constrained action attains its minimum for every positive value of μ, at variance with the constrained energy, which is unbounded from below for $\mu > 2$. Even for $\mu < 2$ it is not guaranteed that the energy constrained on Γ_m has a minimum, i.e. that there exist a solution to the variational problem

$$-\mathcal{E}_m = \min \{E(\psi) \mid \psi \in \Gamma_m\} \ .$$

An analysis of this problem for NLS with point interactions will be given in [2]. However, let us note that if a minimum exists at ψ_m, and the constraint Γ_m is regular at ψ_m, there exists a Lagrange multiplier Λ_m such that $E'(\psi_m) + \Lambda_m M'(\psi_m) = 0$. This means that ψ_m is a stationary point for S_ω with $\omega = \Lambda_m$ and so $\psi_m \in I_\omega$.

5 Identification of the ground state: bifurcation

Proposition 5.1. *Any solution ψ to Problem 2.3 has the form*

$$\psi_\omega^{x_1,x_2,\theta}(x) = \begin{cases} -e^{i\theta}\phi_\omega^{x_1}(x), & x < 0 \\ e^{i\theta}\phi_\omega^{x_2}(x), & x > 0 \end{cases} , \tag{5.1}$$

where the functions $\phi_\omega^{x_i}$ have been defined in (2.1)*, θ can be arbitarily chosen, and the couple (x_1, x_2) is determined in the following way: denoted $t_i = \tanh(\mu\sqrt{\omega}|x_i|)$, then (t_1, t_2) solves the system*

$$\begin{cases} t_1^{2\mu} - t_1^{2\mu+2} &= t_2^{2\mu} - t_2^{2\mu+2} \\ t_1^{-1} + t_2^{-1} &= \gamma\sqrt{\omega} \end{cases} . \tag{5.2}$$

Proof. Consider the functional $J_{\omega,\nu} = S_\omega + \nu I_\omega$, with ν a Lagrange multiplier. Any solution of problem 2.3 must then be a stationary point for $J_{\omega,\nu}$. Let ψ be one of such solutions. Then,

$$S'_\omega(\psi)\psi \ = \ I_\omega(\psi) \ = \ 0.$$

Furthermore,

$$I'_\omega(\psi)\psi \ = \ -2\mu\lambda\|\psi\|_{2\mu+2}^{2\mu+2},$$

and therefore $J'_{\omega,\nu}(\psi)\psi = -2\nu\mu\lambda\|\psi\|_{2\mu+2}^{2\mu+2}$. Thus, for nontrivial solutions it must be $\nu = 0$. We conclude that any non vanishing minimizer ψ of the functional $J_{\mu,\nu}$ must fulfil $S'(\psi) = 0$, i.e.

$$S'_\omega(\psi)\eta \ = \ 0, \quad \forall \eta \in Q. \tag{5.3}$$

Applying (5.3) first to η, then to $\xi = -i\eta$, and summing the two expressions, we find

$$B_\gamma(\psi, \eta) - \lambda(|\psi|^{2\mu}\psi, \eta) + \omega(\psi, \eta) \ = \ 0, \tag{5.4}$$

where we used the shorthand notation

$$B_\gamma(\psi, \eta) \ := \ (\psi', \eta') - \frac{1}{\gamma}(\overline{\psi(0+)} - \overline{\psi(0-)})(\eta(0+) - \eta(0-)).$$

So, from (5.4) the following estimate holds.

$$|B_\gamma(\psi, \eta)| \ \leq \ \lambda\|\psi\|_\infty^{2\mu}\|\psi\|\|\eta\| \ \leq \ C_\psi\|\eta\|, \qquad \forall \eta \in Q. \tag{5.5}$$

Notice that, letting η vary among the functions vanishing in a neighbourhood of zero, we conclude from (5.5) that $\psi \in H^2(\mathbb{R}\backslash\{0\})$. Thus, for a generic $\eta \in Q$ a straightforward computation gives

$$B_\gamma(\psi,\eta) \;=\; (-\psi'',\eta) - (\eta(0+)-\eta(0-))\left(\frac{\overline{\psi(0+)}-\overline{\psi(0-)}}{\gamma} + \overline{\psi'(0-)}\right), \tag{5.6}$$

where we used the notation $\psi'' := \chi_+\psi''_+ + \chi_-\psi''_-$. So, from (5.5) and (5.6), we conclude that ψ belongs to the domain $D(H_\gamma)$ (see definition (2.2)). As a consequence, the function $H_\gamma\psi - \lambda|\psi|^{2\mu}\psi + \omega\psi$ belongs to $L^2(\mathbb{R})$.
Furthermore, from (5.4)

$$(H_\gamma\psi - \lambda|\psi|^{2\mu}\psi + \omega\psi, \eta) \;=\; 0, \qquad \forall\eta \in Q,$$

and, since Q is dense in $L^2(\mathbb{R})$,

$$H_\gamma\psi - \lambda|\psi|^{2\mu}\psi + \omega\psi \;=\; 0 \quad \text{in } L^2(\mathbb{R}). \tag{5.7}$$

Since ψ lies in the domain of H_γ, equation (5.7) can be rewritten as

$$\left\{\begin{array}{ccc} -\psi'' - \lambda|\psi|^{2\mu}\psi + \omega\psi & = & 0, \quad x \neq 0, \quad \psi \in H^2(\mathbb{R}\backslash\{0\}) \\ \psi'(0+) & = & \psi'(0-) \\ \psi(0+) - \psi(0-) & = & -\gamma\psi'(0+) \end{array}\right. . \tag{5.8}$$

Consider first the case of a real ψ. Then, the first equation can be interpreted as the law of motion of a point particle with unitary mass, moving on the line under the action of the double-well potential $V(x) = \frac{\lambda}{2\mu+2}x^{2\mu+2} - \frac{\omega}{2}x^2$. By standard methods of classical mechanics (see e.g. [19]) one immediately sees that the only solutions that vanish at infinity correspond to the zero-energy orbits, whose shape is given by (2.1), where x_0 is a free parameter that, in the mechanical problem, embodies the invariance under time translation.
Consider now the possibility of complex solutions. Writing $\psi(x) = e^{i\theta(x)}\rho(x)$, the first equation in (5.8) yields

$$-\rho'' - 2i\theta'\rho' - \lambda\rho^{2\mu+1} + (\omega + \theta'')\rho \;=\; 0,$$

thus, in order to make the imaginary part vanish, either ρ' or θ' must be identically equal to zero. If $\rho' = 0$, then ψ either vanishes or is not an element of $L^2(\mathbb{R})$. So it must be $\theta' = 0$, and since $\mathbb{R}\backslash\{0\}$ is not connected, one can choose a value for the phase in the positive halfline and another value in the negative halfline. One then obtains that all possible solutions to (5.8) must be given by

$$\psi_\omega^{x_1,x_2,\theta_1,\theta_2}(x) = \left\{\begin{array}{ll} e^{i\theta_1}\phi_\omega^{x_1}(x), & x<0 \\ e^{i\theta_2}\phi_\omega^{x_2}(x), & x>0 \end{array}\right. , \tag{5.9}$$

where x_1 and x_2 are to be chosen in order to satisfy the matching conditions at zero.
We remark that, among the functions in (5.9), once fixed x_1 and x_2 the minimum of S_ω is accomplished if the condition $e^{i\theta_1} = -e^{i\theta_2}$ is fulfilled. Indeed, it is clear that such a condition minimizes the quantity $-2\gamma^{-1}|\psi(0+) - \psi(0-)|^2$, while the other terms in the definition (2.9) of the functional S_ω are the same. This explains the phase factor in (5.1).

Owing to the phase invariance of the problem, without losing generality we can choose $\theta_1 = \pi$, $\theta_2 = 0$, so the matching conditions in (2.2) yield the following system for the unknowns x_1, x_2, and

ω:

$$\begin{cases} \dfrac{\tanh(\mu\sqrt{\omega}x_1)}{\cosh^{\frac{1}{\mu}}(\mu\sqrt{\omega}x_1)} + \dfrac{\tanh(\mu\sqrt{\omega}x_2)}{\cosh^{\frac{1}{\mu}}(\mu\sqrt{\omega}x_2)} & = & 0 \\ \\ \dfrac{1}{\cosh^{\frac{1}{\mu}}(\mu\sqrt{\omega}x_2)} + \dfrac{1}{\cosh^{\frac{1}{\mu}}(\mu\sqrt{\omega}x_1)} & = & \gamma\sqrt{\omega}\dfrac{\tanh(\mu\sqrt{\omega}x_1)}{\cosh^{\frac{1}{\mu}}(\mu\sqrt{\omega}x_1)}. \end{cases} \tag{5.10}$$

By the first equation of system (5.10), x_1 and x_2 must have opposite sign. Furthermore, the second equation gives $x_1 > 0$. So it is proven that $x_2 < 0 < x_1$.

Denoting $t_i = \tanh(\mu\sqrt{\omega}|x_i|)$, and exploiting elementary relations between hyperbolic functions, system (5.10) gives (5.2) and the proof is complete.

□

Before explicitly showing the solutions to the problem (5.1), (5.2), we prove a preliminary lemma.

Lemma 5.2. *For any $\mu > 0$, $a > 2\sqrt{\frac{\mu+1}{\mu}}$, there exists a unique $\bar{x} \in \left(\frac{2}{a}, 1\right]$ such that*

$$\frac{(a^2-1)\bar{x}^2 - 2a\bar{x} + 1}{(a\bar{x}-1)^{2\mu+2}} + \bar{x}^2 - 1 \;=\; 0$$

Proof. Let us denote $w(x) = \frac{(a^2-1)x^2-2ax+1}{(ax-1)^{2\mu+2}} + x^2 - 1$. First, notice that $w\left(\frac{2}{a}\right) = 0$. Furthermore,

$$w'\left(\frac{2}{a}\right) \;=\; \frac{2}{a}(4(\mu+1) - \mu a^2)$$

as $a > 2\sqrt{\frac{\mu}{\mu+1}}$. Therefore $w(x) < 0$ in some right neighbourhood of $\frac{2}{a}$. On the other hand, $w(1) > 0$, so the set Ξ whose element are the zeroes of w in $\left(\frac{2}{a}, 1\right]$, is not empty. Let us denote $x_1 := \min \Xi$. Then, since w is regular in $\left(\frac{2}{a}, +\infty\right)$, it must be either $w(x) < 0$ or $w(x) > 0$ in some right neighbourhood of x_1. In the first case, x_1 is a local maximum for w. Besides, since $w(1) > 0$, there exists $x_2 > x_1$ such that $w(x_2) = 0$. As a consequence, there are two local minima $y_1 \in \left(\frac{2}{a}, x_1\right)$, $y_2 \in (x_1, x_2)$. Owing to the mean value lemma, there exist three points z_1, z_2 and z_3, lying respectively in a neighbourhood of y_1, x_1 and x_2, such that $w''(z_1) > 0$, $w''(z_2) < 0$ and $w''(z_3) > 0$. Owing to the mean value lemma again, there exist $s_1 \in (z_1, z_2)$ and $s_2 \in (z_2, z_3)$ such that $w'''(s_1) < 0$ and $w'''(s_2) > 0$. From the explicit expression

$$\begin{aligned} w'''(x) \;&= [-4\mu a^3(a^2-1)(2\mu+1)(\mu+1)x^2 + 4a^2(\mu+1)(2\mu+1)(2\mu a^2+3)x \\ &\quad + 4a(5\mu^2 a^2 - 2\mu^3 a^2 + \mu + 3\mu a^2 - 1)](ax-1)^{-2\mu-5} \end{aligned} \tag{5.11}$$

it is clear that $w'''(x) < 0$ for large x.It follows that w''' undergoes at least two changes of sign in the interval $\left(\frac{2}{a}, +\infty\right)$, but the expression (5.11) shows that in the interval $\left(\frac{1}{a}, +\infty\right)$ there is a single change of sign only. As a consequence, our starting assumption is false and it must be $w(x) > 0$ in some neighbourhood of x_1.

Let us suppose that there is a point $x_2 > x_1$ such that $w(x_2) = 0$. Following the same reasoning as before, we conclude that w''' must change sign at least twice in $\left(\frac{1}{a}, +\infty\right)$, that contradicts (5.11). As a consequence, there is only one zero (i.e. x_1) of w in $\left(\frac{2}{a}, +\infty\right)$, so the lemma is proven. □

Theorem 5.3. *If $\omega_0 < \omega \leq \omega^*$, then the solutions to Problem 2.3 are given by $\psi_\omega^{y,-y,\theta}$ (see definition (5.1)), with $\theta \in \mathbb{R}$ and*

$$y = \frac{1}{2\mu\sqrt{\omega}} \log \frac{\gamma\sqrt{\omega}+2}{\gamma\sqrt{\omega}-2}. \tag{5.12}$$

If $\omega > \omega^*$, then the solutions to Problem 2.3 are given by $\psi_\omega^{y_1,-y_2,\theta}$ and $\psi_\omega^{y_2,-y_1,\theta}$, with $\theta \in \mathbb{R}$ and

$$y_1 = \frac{1}{2\mu\sqrt{\omega}} \log\left|\frac{1+t_1}{1-t_1}\right|, \qquad y_2 = \frac{1}{2\mu\sqrt{\omega}} \log\left|\frac{1+t_2}{1-t_2}\right|, \tag{5.13}$$

where the couple (t_1, t_2), with $t_1 < t_2$, solves the system (5.2).

Proof. The function

$$f(t) := t^{2\mu} - t^{2\mu+2} \tag{5.14}$$

vanishes at the points 0 and 1, and is strictly positive in the interval $(0,1)$. Furthermore, in the interval $(0,1)$ its only stationary point is $\bar{t} := \sqrt{\frac{\mu}{\mu+1}}$, where the function f has a local maximum and takes the value $m := \frac{\mu^\mu}{(\mu+1)^{\mu+1}}$.
As a consequence, given $a > 0$, the system

$$a = f(t_1) = f(t_2) \tag{5.15}$$

in the unknowns t_1 and t_2, has no solutions for $a > m$, the unique solution $t_1 = t_2 = \bar{t}$ for $a = m$, and, imposing $t_1 < t_2$, three solutions for $0 \le a < m$: indeed, there exists a unique couple t_1, t_2, with $t_1 \in [0, \bar{t})$, $t_2 \in (\bar{t}, 1]$, such that $f(t_1) = f(t_2) = a$. Therefore, the three couples (t_1, t_1) (t_2, t_2), (t_1, t_2), solve (5.15).

So the set of the solutions to the first equation in (5.2) with $t_1 \le t_2$ consists of the union of

$$T_1 := \{0 \le t_1 = t_2 \le 1\}$$

and

$$T_2 := \{(t_1, t_2),\ 0 \le t_1 < \bar{t} < t_2 < 1,\ f(t_1) = f(t_2)\}. \tag{5.16}$$

Due to the regularity of f, T_2 is a regular curve (see Figure 1).

We consider the second equation in (5.2). Varying the parameter ω, it describes a family of hyperboles in the plane (t_1, t_2), whose intersections with T_1 and T_2 provide the required solutions to the system (5.2).

First, observe that

$$\min_{t_1, t_2 \in T_1} (t_1^{-1} + t_2^{-1}) = 2, \tag{5.17}$$

and such a minimum is attained at $t_1 = t_2 = 1$.
Second, we claim that

$$\inf_{t_1, t_2 \in T_2} (t_1^{-1} + t_2^{-1}) = 2\sqrt{\frac{\mu+1}{\mu}}, \tag{5.18}$$

and such a value is attained at $t_1 = t_2 = \bar{t}$. To show this, we use the Lagrange multiplier method, and find that any stationary point of the function $t_1^{-1} + t_2^{-1}$ constrained on $f(t_1) = f(t_2)$ must satisfy the condition

$$t_1^2 f'(t_1) = -t_2^2 f'(t_2), \quad t_1 \text{ and } t_2 \neq \bar{t}.$$

Let us define $g(t) := t^2 f'(t)$. Notice that $g > 0$ in $(0, \bar{t})$, and $g < 0$ in $(\bar{t}, 1]$. Therefore, the condition $g(t_1) = -g(t_2)$ with $0 < t_1 < \bar{t} < t_2 < 1$ is equivalent to $g^2(t_1) = g^2(t_2)$, $t_1 < t_2$.

Observe that

$$g^2(t) = \int_{\bar{t}}^t \frac{d}{ds} g^2(s)\, ds = \int_{\bar{t}}^t (4s^3 (f'(s))^2 + 2s^4 f'(s) f''(s))\, ds. \tag{5.19}$$

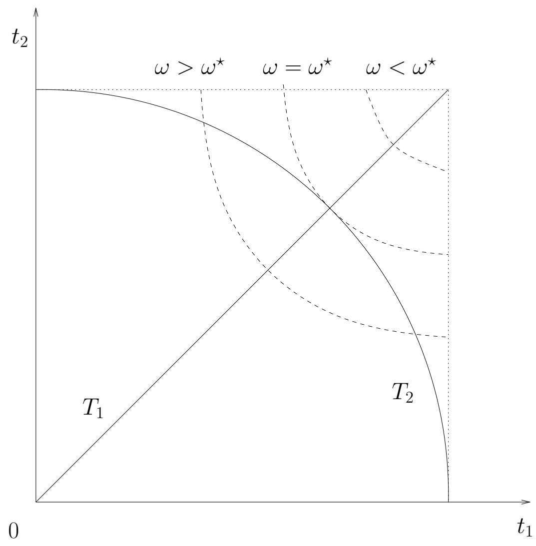

Figure 1: **The system** (5.2). Full lines represent the solutions to the first equation, while dashed lines represent the family of hyperboles given by the second equation. The concavity of the curve represented T_2 refers to a sufficiently large value of μ (for instance, $\mu > 1/2$). For μ near zero, T_2 can exhibit some changes in convexity, but the result on bifurcation and on the number of solutions still holds.

In the interval $[0, \bar{t}]$ the function f is monotone, so it is possible to perform the change of variable $t \to y = f(t)$. Then

$$g^2(t_1) \;=\; \int_m^{f(t_1)} p((\tau(y))\, dy \tag{5.20}$$

where we introduced the function

$$p(t) \;:=\; 4t^3 f'(t) + 2t^4 f''(t) \;=\; (8\mu^2 + 4\mu)t^{2\mu+2} - (8\mu^2 + 20\mu + 12)t^{2\mu+4} \tag{5.21}$$

and the function τ, that is the inverse of f in the interval $[0, \bar{t}]$.

Analogously, exploiting the monotonicity of f in the interval $[\bar{t}, 1]$, one can change the variable in the integral in (5.19) and obtain

$$g^2(t_2) \;=\; \int_m^{f(t_2)} p(\sigma(y))\, dy \tag{5.22}$$

where the function σ is the inverse of f in the interval $[\bar{t}, 1]$.

The function p is non negative in $\left[0, \sqrt{\frac{2\mu^2+\mu}{2\mu^2+5\mu+3}}\right]$, and vanishes at the endpoints of the same interval. Its only stationary point in the interval $(0, 1)$ is located at $\widetilde{t} := \sqrt{\frac{2\mu^2+\mu}{2\mu^2+7\mu+6}}$ and is a local maximum, where

$$p\left(\widetilde{t}\right) \;=\; \frac{4}{(2\mu+3)^{\mu+1}} \left(\frac{2\mu^2+\mu}{\mu+2}\right)^{\mu+2}. \tag{5.23}$$

In the interval $(\widetilde{t}, 1]$ the function p is negative and monotonically decreasing. As a consequence, from (5.22) one gets

$$g^2(t_2) \;=\; \int_{f(t_2)}^m |p(\sigma(y))|\, dy. \tag{5.24}$$

Besides, since

$$p(\bar{t}) \;=\; -8\frac{\mu^{\mu+2}}{(\mu+1)^{\mu+1}}. \tag{5.25}$$

one immediately has $\bar{t} > \sqrt{\frac{2\mu^2+\mu}{2\mu^2+5\mu+3}} > \widetilde{t}$.

Comparing equations (5.23) and (5.25), $p\left(\widetilde{t}\right) < |p(\bar{t})|$ if and only if

$$(\mu+1)^{\mu+1}\left(\mu+\frac{1}{2}\right)^{\mu+2} \;<\; \left(\mu+\frac{3}{2}\right)^{\mu+1}(\mu+2)^{\mu+2}$$

which holds for any value of μ. Therefore, $\max_{t\in[0,\bar{t}]}|p(t)| = |p(\bar{t})|$.
As a consequence, if $s_1 < \bar{t} < s_2$, then

$$|p(s_1)| \;<\; |p(\bar{t})| \;<\; |p(s_2)|,$$

and recalling that t_1 and t_2 are defined to fulfil $f(t_1) = f(t_2)$,

$$g^2(t_1) \;=\; 2\int_m^{f(t_1)} p(\tau(y))\,dy \;<\; 2\int_{f(t_2)}^m |p(\sigma(y))|\,dy \;=\; g^2(t_2). \tag{5.26}$$

where in the last identity we used (5.24).
It follows that the only point where $g^2(t_1) = g^2(t_2)$ is given by $t_1 = t_2 = \bar{t}$, so there are no stationary points of the function $t_1^{-1} + t_2^{-1}$ on the set T_2. By comparison with the endpoints $(0,1)$ and $(1,0)$ one immediately has that $t_1 = t_2 = \bar{t}$ corresponds to a minimum, so by direct computation our claim (5.18) is proved.

As a consequence, from (5.17) and (5.18) we have that:

- if $\omega \leq \omega_0$, then the system (5.2) has no solutions;
- if $\omega_0 < \omega \leq \omega^*$, then the only solution to (5.2) lies in T_1 and reads $t_1 = t_2 = \frac{\gamma}{2\sqrt{\omega}}$;
- if $\omega > \omega^*$, then the system (5.2) exhibits three solutions: the first one lies in T_1 and is given by $t_1 = t_2 = \frac{\gamma}{2\sqrt{\omega}}$.

  Furthermore, in the set T_2 consider the region $t_1 < \frac{\gamma}{2\sqrt{\omega}} < t_2$; expressing t_1 from the second equation of (5.2) and plugging it into the first one, one obtains the equation $w(t_2) = 0$, with w defined as in the proof of lemma 5.2 and $a = \gamma\sqrt{\omega}$. By virtue of the same lemma, such equation has a unique solution t_2^* in the considered interval. The unique value of t_1 such that (t_1, t_2^*) is a solution to (5.2) is given by $(t_1^*)^{-1} = \gamma\sqrt{\omega} - (t_2^*)^{-1}$.

  Due to the symmetry of (5.2) under exchange of t_1 and t_2, the third and last solution is given by (t_2^*, t_1^*).

Any solution to (5.2) singles out a stationary point of the functional S_ω on the Nehari manifold, that is unique up to multiplication by a phase. Obviously, the value of S_ω on such functions is independent of the phase.

Defining y like in (5.12), y_1 and y_2 like in (5.13), and owing to (5.1), we conclude that:

- if $\omega_0 < \omega \leq \omega^*$, then the only stationary point (up to a phase) for the functional S_ω is given by $\psi_\omega^{y,-y,0}$. Due to its uniqueness, it must be the minimizer for S_ω whose existence is established by theorem 4.1. The explicit expression for y given in (5.12) is found imposing $t_1 = t_2$ in the second equation of (5.2).

- For $\omega > \omega^*$ two further solutions appear. Keeping into account the signs of x_1 and x_2 established in proposition (5.1), the two related families of solutions can be denoted by $\psi_\omega^{y_1,-y_2,\theta}$, $\psi_\omega^{y_2,-y_1,\theta}$, with y_1 and y_2 positive numbers. Obviously, the functional S_ω takes the same value on them. In order to establish which stationary point is the minimizer we must compare $S_\omega(\psi_\omega^{y_1,-y_2,\theta})$ with $S_\omega(\psi_\omega^{y,-y,\theta})$.

Let us proceed with such a comparison. From (2.12) we know that the functional S_ω reduces to $\widetilde{S}$ when evaluated on stationary states. We have

$$S_\omega(\psi_\omega^{y,-y,\theta}) \;=\; \frac{\omega^{\frac{1}{2}+\frac{1}{\mu}}(\mu+1)^{\frac{1}{\mu}}}{2\lambda^{\frac{1}{\mu}}}\left[\int_{-1}^{1}(1-t^2)^{\frac{1}{\mu}}\,dt \;-\; \int_{-\frac{2}{\gamma\sqrt{\omega}}}^{\frac{2}{\gamma\sqrt{\omega}}}(1-t^2)^{\frac{1}{\mu}}\,dt\right],$$

and

$$S_\omega(\psi_\omega^{y_1,-y_2,\theta}) \;=\; \frac{\omega^{\frac{1}{2}+\frac{1}{\mu}}(\mu+1)^{\frac{1}{\mu}}}{2\lambda^{\frac{1}{\mu}}}\left[\int_{-1}^{1}(1-t^2)^{\frac{1}{\mu}}\,dt \;-\; \int_{-t_1}^{t_2}(1-t^2)^{\frac{1}{\mu}}\,dt\right].$$

Introducing the function $\varphi(t) := -\frac{t}{\gamma\sqrt{\omega}t-1}$, we obtain

$$S_\omega(\psi_\omega^{y,-y,\theta}) \;=\; \frac{\omega^{\frac{1}{2}+\frac{1}{\mu}}(\mu+1)^{\frac{1}{\mu}}}{2\lambda^{\frac{1}{\mu}}}\left[\int_{-1}^{1}(1-t^2)^{\frac{1}{\mu}}\,dt \;-\; \int_{\varphi\left(\frac{2}{\gamma\sqrt{\omega}}\right)}^{\frac{2}{\gamma\sqrt{\omega}}}(1-t^2)^{\frac{1}{\mu}}\,dt\right]$$

$$S_\omega(\psi_\omega^{y_1,-y_2,\theta}) \;=\; \frac{\omega^{\frac{1}{2}+\frac{1}{\mu}}(\mu+1)^{\frac{1}{\mu}}}{2\lambda^{\frac{1}{\mu}}}\left[\int_{-1}^{1}(1-t^2)^{\frac{1}{\mu}}\,dt \;-\; \int_{\varphi(t_2)}^{t_2}(1-t^2)^{\frac{1}{\mu}}\,dt\right].$$

We define the function

$$q(t) \;:=\; \int_{\varphi(t)}^{t}(1-\nu^2)^{\frac{1}{\mu}}\,d\nu, \tag{5.27}$$

thus

$$S_\omega(\psi_\omega^{y,-y,\theta}) \;=\; \frac{\omega^{\frac{1}{2}+\frac{1}{\mu}}(\mu+1)^{\frac{1}{\mu}}}{2\lambda^{\frac{1}{\mu}}}\left[\int_{-1}^{1}(1-t^2)^{\frac{1}{\mu}}\,dt \;-\; q\left(\frac{2}{\gamma\sqrt{\omega}}\right)\right]$$

$$S_\omega(\psi_\omega^{y_1,-y_2,\theta}) \;=\; \frac{\omega^{\frac{1}{2}+\frac{1}{\mu}}(\mu+1)^{\frac{1}{\mu}}}{2\lambda^{\frac{1}{\mu}}}\left[\int_{-1}^{1}(1-t^2)^{\frac{1}{\mu}}\,dt \;-\; q(t_2)\right].$$

Since

$$q'(t) \;=\; (1-t^2)^{\frac{1}{\mu}} - (1-\varphi^2(t))^{\frac{1}{\mu}}\frac{\varphi^2(t)}{t^2},$$

the stationary points of q must solve the equation

$$t^2(1-t^2)^{\frac{1}{\mu}} - (1-\varphi^2(t))^{\frac{1}{\mu}}\varphi^2(t) \;=\; 0, \tag{5.28}$$

which is equivalent to the first equation of (5.2) in the unknowns $(t,-\varphi(t))$. Furthermore, from (5.27),

$$t^{-1} - \varphi^{-1}(t) \;=\; \gamma\sqrt{\omega}. \tag{5.29}$$

Thus, from (5.28) and (5.29) we conclude that the couple $(t,-\varphi(t))$ solves the system (5.2). This implies that the functional $\widetilde{S}$, restricted to functions of the kind (5.1), has stationary points only in

correspondence with the solutions of (5.2). In other words, for $\omega > \omega^*$ there are three stationary points for q, and they coincide with y, y_1, y_2.
A straightforward computation provides

$$q''\left(\frac{2}{\gamma\sqrt{\omega}}\right) \;=\; \frac{2}{\gamma\sqrt{\omega}}\left(1-\frac{4}{\gamma^2\omega}\right)\left(\gamma^2\omega - 4\,\frac{\mu+1}{\mu}\right)$$

which is positive if $\omega > \omega^*$. Then, $\frac{2}{\gamma\sqrt{\omega}}$ is a minimum for q, and since q is regular and there are no other stationary points, it must be

$$S_\omega(\psi_\omega^{y_1,-y_2,\theta}) \;<\; S_\omega(\psi_\omega^{y,-y,\theta}).$$

Thus we conclude that $\psi_\omega^{y_1,-y_2,\theta}$ and $\psi_\omega^{y_2,-y_1,\theta}$ are the minimizers, and the proof is complete. □

We end this section showing that the branch of nonlinear standing waves bifurcates from the trivial (vanishing) stationary state for $\omega > \omega_0 = \frac{4}{\gamma^2}$.

Proposition 5.4. *Let us consider the branch of symmetric standing waves*

$$(\omega_0, \omega^*) \ni \omega \mapsto \psi_\omega^{y,-y,\theta} \in L^2(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) \;.$$

The following relations hold true

$$\lim_{\omega\to\omega_0} \;\|\psi_\omega^{y,-y,\theta}\| \;=\; \lim_{\omega\to\omega_0} \;\|\psi_\omega^{y,-y,\theta}\|_Q = \lim_{\omega\to\omega_0} \;\|\psi_\omega^{y,-y,\theta}\|_{H^2(\mathbb{R}\backslash\{0\})} \;=\; 0$$

Proof. The result immediately follows by observing that the solution $\left(\frac{2}{\gamma\sqrt{\omega}}, \frac{2}{\gamma\sqrt{\omega}}\right)$ to (5.2) tends to $(1,1)$ as $\omega \to \omega_0 + 0$. So, in the same limit y tends to $+\infty$, and the result immediately follows from the explicit expression ((5.9), (5.12)) for $\psi_\omega^{y,-y,0}$. □

6 Stability and instability of the ground states

6.1 The linearized evolution around a stationary solution

We study the second variation of the functional S_ω. Indeed, for any $w \in Q$

$$\frac{d}{ds}S_\omega(\psi + sw) \;=\; \mathrm{Re}\;(H_\gamma\psi + \omega\psi - \lambda|\psi|^{2\mu}\psi, w),$$

which is often referred to as the fact that $S'_\omega[\psi] \;=\; H_\gamma\psi + \omega\psi - \lambda|\psi|^{2\mu}\psi$ in a weak sense. Therefore, linearizing the equation (1.1) around the *real* function ψ can be made by computing the second variation of the functional S_ω. Given $w \in Q$, with $u = \mathrm{Re}\,w$ and $v = \mathrm{Im}\,w$ one has

$$\begin{aligned}
\frac{d^2}{ds^2}S_\omega(\psi + sw) \;&=\; \|w'\|^2 + \omega\|w\|^2 - \frac{1}{\gamma}|w(0+) - w(0-)|^2 - \lambda\int_{\mathbb{R}} |w(x)|^2|\psi(x)|^{2\mu}dx \\
&\quad - 2\mu\lambda\int_{\mathbb{R}} u^2(x)|\psi(x)|^{2\mu}dx \\
&= \|u'\|^2 + \omega\|u\|^2 - \frac{1}{\gamma}|u(0+) - u(0-)|^2 - \lambda(2\mu+1)\int_{\mathbb{R}} u(x)^2|\psi(x)|^{2\mu}dx \\
&\quad + \|v'\|^2 + \omega\|v\|^2 - \frac{1}{\gamma}|v(0+) - v(0-)|^2 - \lambda\int_{\mathbb{R}} v(x)^2|\psi(x)|^{2\mu}dx.
\end{aligned} \tag{6.1}$$

Defining the operators

$$\begin{aligned} L_1^{\gamma,\omega} u &= H_\gamma u + \omega u - (2\mu+1)\lambda|\psi|^{2\mu} u \\ L_2^{\gamma,\omega} v &= H_\gamma v + \omega v - \lambda|\psi|^{2\mu} v \end{aligned} \tag{6.2}$$

on the domain $D(L_1^{\gamma,\omega}) = D(L_2^{\gamma,\omega}) = D(H_\gamma)$ (see (2.2)), we get

$$\frac{d^2}{ds^2} S_\omega(\psi + sw) = (u, L_1^{\gamma,\omega} u) + (v, L_2^{\gamma,\omega} v). \tag{6.3}$$

Now we derive the general spectral properties of the operators $L_1^{\gamma,\omega}$ and $L_2^{\gamma,\omega}$, needed to prove stability or instability of the stationary states.

It is easy to show that $L_1^{\gamma,\omega}$ and $L_2^{\gamma,\omega}$ are self-adjoint operators in $L^2(\mathbb{R})$. In fact they are abstract Schrödinger operators of the form $(H_\gamma + \omega) + V_i(x)$, where the perturbation $V_i(x) = \mathrm{c}_i|\psi|^{2\mu}(x)$ is given by a bounded and rapidly decaying function, and $\mathrm{c}_1 = 2\mu + 1$, $\mathrm{c}_2 = 1$. Let us consider the couple of operators $L_2^{\gamma,\omega}$ and $-\frac{d^2}{dx^2} + \omega - \lambda|\psi|^{2\mu}$. Both are self-adjoint extensions of the same closed symmetric operator with defect indices $(2,2)$; so their resolvents differ for a finite rank operator. As a consequence, thanks to the Weyl's theorem (see [31], Theorem XIII.4), the essential spectra of $L_2^{\gamma,\omega}$ and $-\frac{d^2}{dx^2} + \omega - \lambda|\psi|^{2\mu}$ coincide. Moreover, $\sigma_e(-\frac{d^2}{dx^2} + \omega - \lambda|\psi|^{2\mu}) = \sigma_e(-\frac{d^2}{dx^2} + \omega) = [\omega + \infty)$, because the potential V_2 is $(-\frac{d^2}{dx^2} + \omega)$-compact. The same reasoning holds for the operator $L_1^{\gamma,\omega}$, so we can conclude

$$\sigma_e(L_1^{\gamma,\omega}) = \sigma_e(L_2^{\gamma,\omega}) = [\omega, +\infty).$$

Moreover, the fact that $L_1^{\gamma,\omega}$ and $L_2^{\gamma,\omega}$ are symmetric and relatively compact perturbations of the self-adjoint nonnegative operator $H_\gamma + \omega$ allows to conclude (see for example [24], Theorem 6.32) that the possible discrete spectrum is finite or accumulates at the border of the essential spectrum, which in our case is positive. So the negative spectrum is finite.

We will often use the previous remarks without repeating the argument.

We need more detailed spectral information on the operators $L_1^{\gamma,\omega}$ and $L_2^{\gamma,\omega}$, in particular concerning the number of negative eigenvalues. A standard technique to deal with this sort of problems in the case of operators with domain elements which are regular enough (typically Schrödinger operator with a smooth enough potential) relies on the Sturm oscillation theorem which relates the number of nodes of an eigenfunction to the ordering of the corresponding eigenvalue. So, if ψ is positive, then it coincides with the first eigenfunction, which is simple and corresponds to the ground state. This reasoning is not applicable in our case, due to the singular character of H_γ, with possibly discontinuous domain elements. By the way, the problem is not completely settled neither for the case of the milder δ interaction, so we give an independent proof of the relevant spectral properties for this case too.

The results are based on a generalization of a ground state transformation for the singular operator H_γ.

Proposition 6.1. *Let $e^{i\omega t}\psi(x)$ be a stationary solution to problem* (3.1) *with* $\psi(0+)\psi(0-) < 0$. *Then, for the operator $L_2^{\gamma,\omega}$ defined in* (6.2), *the following statements hold:*
a) Ker $L_2^{\gamma,\omega} =$ Span $\{\psi\}$
b) $L_2^{\gamma,\omega} \geq 0$

Proof. Along this proof we denote the operator $L_2^{\gamma,\omega}$ by L_2. Proceeding like in the proof of Proposition 5.1 up to formula (5.7), one obtains that the function ψ is regular in $\mathbb{R}\backslash\{0\}$ and fulfils the boundary

conditions defined by the δ'-interaction. So, by the equation for the stationary states (5.7) again, one immediately verifies that $L_2\psi = 0$ and point $a)$ is proven.
To prove $b)$, notice that for any $\phi \in D(L_2)$ the following identity holds at any point $x \neq 0$:

$$-\phi'' + \omega\phi - |\psi|^{2\mu}\phi = -\frac{1}{\psi}\frac{d}{dx}\left(\psi^2 \frac{d}{dx}\left(\frac{\phi}{\psi}\right)\right) \ ;$$

integrating by parts,

$$(\phi, L_2\phi) \ = \ \int_{-\infty}^{0} \psi^2 \left|\frac{d}{dx}\left(\frac{\phi}{\psi}\right)\right|^2 dx + \int_{0}^{\infty} \psi^2 \left|\frac{d}{dx}\left(\frac{\phi}{\psi}\right)\right|^2 dx + \lim_{\varepsilon\to 0}\left[\phi'\overline{\phi} - \frac{\psi'}{\psi}|\phi|^2\right]_{-\varepsilon}^{+\varepsilon}. \tag{6.4}$$

The integral terms in (6.4) are non negative and equal zero if and only if $\phi = \psi$. Let us focus on the contribution of the boundary, that consists of two terms. Using boundary conditions, the first term gives

$$\lim_{\varepsilon\to 0}\left[\phi'\bar{\phi}\right]_{-\varepsilon}^{\varepsilon} \ = \ -\gamma|\phi'(0+)|^2 \tag{6.5}$$

For the second term we immediately get

$$-\lim_{\varepsilon\to 0}\left[\frac{\psi'}{\psi}|\phi|^2\right]_{-\varepsilon}^{\varepsilon} \ = \ \frac{\psi'(0-)\psi(0+)|\phi(0-)|^2 - \psi'(0+)\psi(0-)|\phi(0+)|^2}{\psi(0+)\psi(0-)} \tag{6.6}$$

Summing (6.5) and (6.6), and using the matching condition for both ψ and ϕ we finally get

$$\begin{aligned} -\lim_{\varepsilon\to 0}\left[\frac{\psi'}{\psi}|\phi|^2\right]_{-\varepsilon}^{\varepsilon} \ &= \ -\frac{\psi^2(0+)|\phi(0-)|^2 + \psi^2(0-)|\phi(0+)|^2 - 2\psi(0+)\psi(0-)\mathrm{Re}(\phi(0+)\overline{\phi(0-)})}{\gamma\psi(0+)\psi(0-)} \\ &= \ -\frac{|\psi(0+)\phi(0-) - \psi(0-)\phi(0+)|^2}{\gamma\psi(0+)\psi(0-)} \end{aligned}$$

Due to the hypothesis $\psi(0+)\psi(0-) < 0$, verified by all ground states, we conclude that the boundary term in (6.4) is non negative, and this completes the proof. $\square$

Remark 6.2. An analogous proposition holds in the case, treated in [17], [16], [28] of a δ interaction with strength α (no matter if attractive or repulsive). One has that (the meaning of the symbols is the obvious one)
a) Ker $L_2^{\alpha,\omega} = \mathrm{Span}\,\{\psi\}$
b) $L_2^{\alpha,\omega} \geq 0$
In this case, the boundary term in (6.4) vanishes.

Now we prove the spectral features of interest for the operator $L_1^{\gamma,\omega}$ defined in (6.2).
Consider first the case of the symmetric stationary state $\psi = \psi_\omega^{y,-y,\theta}$. We define

$$L_{1,\mathrm{sym}}^{\gamma,\omega} \ = \ H_\gamma + \omega - \frac{\omega(\mu+1)(2\mu+1)}{\cosh^2(\mu\sqrt{\omega}(|x|+y))} \tag{6.7}$$

and

$$L_{2,\mathrm{sym}}^{\gamma,\omega} \ = \ H_\gamma + \omega - \frac{\omega(\mu+1)}{\cosh^2(\mu\sqrt{\omega}(|x|+y))}, \tag{6.8}$$

where $\tanh(\mu\sqrt{\omega}y) = \frac{2}{\gamma\sqrt{\omega}}$.

Proposition 6.3. *Fixed $\mu > 0$, the operator $L^{\gamma,\omega}_{1,\mathrm{sym}}$ has:*
a) A trivial kernel and one simple negative eigenvalue, if $\omega < \omega^*$;
b) A one-dimensional kernel, spanned by the function

$$\xi_{-1}(x) \;=\; \frac{\sinh(\mu\sqrt{\omega}(|x|+y))}{\cosh^{1+\frac{1}{\mu}}(\mu\sqrt{\omega}(|x|+y))},$$

where y has been defined in (5.12)*, and one simple negative eigenvalue, if $\omega = \omega^*$;*
c) A trivial kernel and two simple negative eigenvalues or a double negative eigenvalue, if $\omega > \omega^*$.

Proof. For shorthand, in this proof we will denote the operators $L^{\gamma,\omega}_{1,\mathrm{sym}}$ and $L^{\gamma,\omega}_{2,\mathrm{sym}}$ by L_1 and L_2, respectively. Furthermore, the function $\psi^{y,-y,0}_{\omega}$ will be denoted by ψ.

Consider first the case $\omega \le \omega^*$. By stationarity of ψ, the following identity must hold up to higher order terms in w:

$$S_\omega(\psi + w) \;=\; S_\omega(\psi) + \frac{1}{2}(u, L_1 u) + \frac{1}{2}(v, L_2 v), \tag{6.9}$$

for any $w = u + iv$ in Q, with u and v real.
By Proposition 6.1, the operator L_2 is non-negative, and by Weyl's theorem on the stability of the essential spectrum (see for example [23] or Theorem XIII.4 in [31]), one has $\sigma_{ess}(L_1) = \sigma_{ess}(H_\gamma + \omega) = [\omega, +\infty)$. Furthermore, Theorem 5.3 guarantees that ψ minimizes the functional S_ω on the Nehari manifold. Thus, since the Nehari manifold has codimension one, the operator L_1 has at most one negative eigenvalue.
On the other hand,

$$(\psi, L_1\psi) \;=\; -2\mu\lambda\|\psi\|^{2\mu+2}_{2\mu+2} \;<\; 0, \tag{6.10}$$

so we can conclude that for $\omega \le \omega^*$ the operator L_1 has exactly one negative eigenvalue.

Concerning the kernel of L_1, we recall that the only square-integrable solution to the linear differential equation

$$-\xi'' + \omega\xi - \frac{\omega(\mu+1)(2\mu+1)}{\cosh^2(\mu\sqrt{\omega}\,\cdot)}\xi \;=\; 0 \tag{6.11}$$

is given, up to a factor, by

$$\xi(x) \;=\; \frac{\sinh(\mu\sqrt{\omega}x)}{\cosh^{1+\frac{1}{\mu}}(\mu\sqrt{\omega}x)}. \tag{6.12}$$

Furthermore, there cannot be a solution $\zeta \notin \mathrm{Span}(\xi)$ to equation (6.11) such that $\int_a^\infty |\zeta(x)|^2 dx < \infty$ for some finite a, otherwise, by invariance under reflection of (6.11), the function $\zeta(-x)$ would be a solution to (6.12) too, satisfying $\int_{-\infty}^{-a} |\zeta(x)|^2 dx < \infty$, so we would obtain three linearly independent solutions to (6.11). As a consequence, the possible solutions to the equation

$$L_1\xi + \omega\xi - \frac{\omega(\mu+1)(2\mu+1)}{\cosh^2(\mu\sqrt{\omega}\;(|\cdot|+y))}\xi \;=\; 0 \tag{6.13}$$

are given by $\xi_a(x) \;=\; \chi_+(x)\xi(x+y) + a\chi_-(x)\xi(x-y)$, with $a \in \mathbb{C}$, provided that they fulfil the matching condition of the δ' interaction. Such conditions prescribe the identity of the left and the right derivative at zero, namely $\xi'(y) = a\xi'(-y)$. Since ξ' is even, this implies either $a = 1$ or $\xi'(y) = 0$. In the first case, imposing the boundary condition $\xi_1(0+) - \xi_1(0-) = -\gamma\xi_1'(0+)$ leads to the equation $2\sinh(\mu\sqrt{\omega}y)\cosh(\mu\sqrt{\omega}y) = \gamma\sqrt{\omega}[\sinh^2(\mu\sqrt{\omega}y) - \mu]$, that cannot be solved in y for any $\mu > 0$. In the second case, one has $\xi_a'(0)=0$, which is fulfilled, as $\xi_a'(0+)=\sqrt{\omega}\cosh^{-2-\frac{1}{\mu}}(\mu\sqrt{\omega}y)[\mu-\sinh^2(\mu\sqrt{\omega}y)]$, if and only if $\sinh^2(\mu\sqrt{\omega}y)=\mu$. This is equivalent to $\tanh(\mu\sqrt{\omega}y)=\sqrt{\frac{\mu}{\mu+1}}$, that, owing to definition of y in (6.7), is verified only for $\omega=\omega^*$. Furthermore, since zero is a stationary point, ξ_a must be continuous at zero, so $a=-1$.

Thus we proved points a) and b), and the case $\omega\leq\omega^*$ is exhausted.

In order to prove point c), let us write the spectrum of L_1 as

$$\sigma(L_1) \ := \ \{\nu_1,\dots,\nu_n\}\cup\{\tau_1,\dots,\tau_m\}\cup[\omega,+\infty),$$

where $\nu_i<\nu_j<0$ and $\tau_l>\tau_m\geq 0$ for any $i<j$ and any $l>m$. The boundedness and fast decay in x of the last term in the l.h.s. of (6.13) ensures that both m and n are finite. Moreover the essential spectrum coincides with the one of L_1, thanks to the Weyl's theorem again.

By (6.10), we know that $n>0$. Denoted by P_α the orthogonal projection in L^2 on the eigenspace associated to the eigenvalue α, we define the following operators:
- P_- is the projection on the space $\bigoplus_j P_{\nu_j}$;
- P_+ is the projection on the space $\bigoplus_j P_{\tau_j}$;
- P_c is the projection on the space associated to the essential spectrum $[\omega,+\infty)$ of L_1.

Let us suppose that $n=1$. Then, denoted by F_1 the quadratic form associated to the operator L_1, there exists at least a non vanishing combination η of ψ and ξ_{-1} that satisfies $F_1(\eta)\geq 0$. Indeed, denoted by ψ_1 the only (up to a phase) normalized eigenfunction associated to the eigenvalue $-\nu_1$, we define the function

$$\eta \ := \ -\frac{(\psi_1,\xi_{-1})}{(\psi_1,\psi)}\psi+\xi_{-1}.$$

Notice first that, since by (6.10) $F_1(\psi)<0$, ψ and ψ_1 cannot be orthogonal, therefore η is well defined. Furthermore, ψ and ξ_{-1} are linearly independent as they have different parity, so it must be $\eta\neq 0$. Since $(\psi_1,\eta)=0$, η has no components in the negative part of the spectrum of L_1, so $F_1(\eta)\geq 0$.

But this is not the case. Indeed, for a generic combination $\phi=\alpha\psi+\beta\xi_-$, we have

$$F_1(\phi) \ = \ |\alpha|^2F_1(\psi)+|\beta|^2F_1(\xi_{-1})+2\mathrm{Re}\bar{\alpha}\beta\langle L_1\psi,\xi_{-1}\rangle \ = \ |\alpha|^2F_1(\psi)+|\beta|^2F_1^{\gamma,\omega}(\xi_{-1}), \qquad (6.14)$$

as the mixed term vanishes, being the scalar product of an even and an odd function.

Now we compute $F_1(\xi_{-1})$. We notice that, due to the continuity of ξ_{-1}, the term related to the point interaction vanishes, so, after integrating by parts, we get

$$\begin{aligned} F_1(\xi_{-1}) \ &= \ \lim_{\varepsilon\to 0}\Bigg[\int_{-\infty}^{-\varepsilon}\xi_{-1}(x)\left(-\xi_{-1}''(x)+\omega\xi_{-1}-\frac{\omega(2\mu+1)(\mu+1)}{\cosh^2(\mu\sqrt{\omega}(x-y))}\xi_{-1}(x)\right)dx \\ &\quad +\int_{\varepsilon}^{\infty}\xi_{-1}(x)\left(-\xi_{-1}''(x)+\omega\xi_{-1}-\frac{\omega(2\mu+1)(\mu+1)}{\cosh^2(\mu\sqrt{\omega}(x+y))}\xi_-(x)\right)dx \\ &\quad +\xi_{-1}(0)(\xi_{-1}'(0-)-\xi_{-1}'(0+))\Bigg] \\ &= \xi_{-1}(0)(\xi_-'(0-)-\xi_{-1}'(0+)), \end{aligned} \qquad (6.15)$$

where we used the fact that, by definition of ξ_{-1},

$$-\xi_{-1}''(x)+\omega\xi_{-1}(x)-\frac{\omega(2\mu+1)(\mu+1)}{\cosh^2(\mu\sqrt{\omega}(|x|+y))}\xi_{-1}(x) \ = \ 0, \qquad \forall x\neq 0.$$

Then, one can directly compute

$$F_1(\xi_{-1}) \;=\; -\frac{4}{\gamma}\left(1-\frac{4}{\gamma^2\omega}\right)^{\frac{1}{\mu}}\left[\mu-(\mu+1)\frac{4}{\gamma^2\omega}\right]$$

which is negative if and only if $\omega > \omega^*$. Thus, as a consequence of (6.14), beyond the bifurcation frequency ω^*, for any linear combination ϕ of ξ_{-1} and ψ we have $F_1(\phi) < 0$, that contradicts the hypothesis of having only one simple negative eigenvalue in the spectrum of L_1.

In order to prove that actually either $n = 2$ or $n = 1$ and ν_1 is a double eigenvalue, we prove that ψ minimizes S_ω on the Nehari manifold with the additional constraint $\varphi(0+) = -\varphi(0-)$. To this aim, we first observe that, if $\varphi \in Q$ fulfils $\varphi(0+) = -\varphi(0-)$, then

$$\begin{aligned} S_\omega(\varphi) \;&=\; \frac{1}{2}\|\varphi'\|^2 + \frac{\omega}{2}\|\varphi\|^2 - \frac{\lambda}{2\mu+2}\|\varphi\|_{2\mu+2}^{2\mu+2} - \frac{2}{\gamma}|\varphi(0+)|^2 \\ I_\omega(\varphi) \;&=\; \|\varphi'\|^2 + \omega\|\varphi\|^2 - \lambda\|\varphi\|_{2\mu+2}^{2\mu+2} - \frac{4}{\gamma}|\varphi(0+)|^2 \;=\; 0 \end{aligned} \tag{6.16}$$

Consider the unitary transformation $U_\sharp$ of the space Q, defined by

$$\varphi_\sharp(x) := (U_\sharp\varphi)(x) \;:=\; \epsilon(x)\varphi(x)$$

and notice that, if $\varphi(0+) = -\varphi(0-)$, then $\varphi_\sharp$ belongs to $H^1(\mathbb{R})$, so the minimization problem is mapped into the problem of minimizing the functional

$$S_{\omega,\sharp}(\varphi_\sharp) \;=\; \frac{1}{2}\|\varphi_\sharp'\|^2 + \frac{\omega}{2}\|\varphi_\sharp\|^2 - \frac{\lambda}{2\mu+2}\|\varphi_\sharp\|_{2\mu+2}^{2\mu+2} - \frac{2}{\gamma}|\varphi_\sharp(0)|^2 \tag{6.17}$$

among the functions in $H^1(\mathbb{R})$ that satisfy the constraint

$$I_{\omega,\sharp}(\varphi_\sharp) \;=\; \|\varphi_\sharp'\|^2 + \omega\|\varphi_\sharp\|^2 - \lambda\|\varphi_\sharp\|_{2\mu+2}^{2\mu+2} - \frac{4}{\gamma}|\varphi_\sharp(0)|^2 \;=\; 0.$$

Problem (6.17) corresponds to the issue of finding the ground state for a nonlinear Schrödinger equation in the presence of a δ-type defect of strength $-4/\gamma$. By [17] and [16], we know that the solution reads

$$\phi_{\omega,\sharp} \;:=\; \left[\frac{(\mu+1)\omega}{\lambda}\right]^{\frac{1}{2\mu}}\cosh^{-\frac{1}{\mu}}(|x|+y)\,,$$

so we obtain, still up to a phase,

$$\phi_{\omega,\sharp} \;=\; U_\sharp\psi$$

and, as a consequence, ψ minimizes the action on the Nehari manifold with the additional condition $\psi(0+) = -\psi(0-)$.

It remains to prove that such a constraint has codimension two. We denote the constraint by

$$\mathcal{M} \;:=\; \{\varphi \in Q,\; I_\omega(\varphi) = 0,\;\; \varphi(0+) = -\varphi(0-)\}. \tag{6.18}$$

By the operator $U_\sharp$ such a constraint is mapped to

$$\mathcal{M}_\sharp \;:=\; \{\varphi \in H^1(\mathbb{R}),\; I_{\omega,\sharp}(\varphi) = 0\}. \tag{6.19}$$

It is well-known ([17], [16]) that $\mathcal{M}_\sharp$ has codimension one as a subspace of $H^1(\mathbb{R})$. On the other hand, since any function ζ in Q can be decomposed as

$$\zeta \;=\; \frac{1}{2}(\zeta(0+)-\zeta(0-))\epsilon(\cdot)e^{-|\cdot|}+\widetilde{\zeta},$$

with $\widetilde{\zeta}\in H^1(\mathbb{R})$, we have

$$Q \;=\; H^1(\mathbb{R})\oplus \mathrm{Span}\{\epsilon(\cdot)e^{-|\cdot|}\},$$

so $H^1(\mathbb{R})$ has codimension one as a subspace of Q. Therefore, $\mathcal{M}_\sharp$ has codimension two as a subspace of Q. Thus, by unitarity of $U_\sharp$, $\mathcal{M}$ has codimension two as a subspace of Q too.

We then proved that the negative space of the operator L_1 has dimension at most two and that for $\omega>\omega^*$ it equals exactly two. The proof is concluded. $\square$

Now we consider the case of an asymmetric ground state $\psi_\omega^{y_1,-y_2,\theta}$. We define

$$L_{1,\mathrm{asym}}^{\gamma,\omega} \;=\; H_\gamma+\omega-\frac{\omega(\mu+1)(2\mu+1)}{\cosh^2\left(\mu\sqrt{\omega}\,(x+\chi_+(x)y_2-\chi_-(x)y_1)\right)}$$

and

$$L_{2,\mathrm{asym}}^{\gamma,\omega} \;=\; H_\gamma+\omega-\frac{\omega(\mu+1)}{\cosh^2\left(\mu\sqrt{\omega}\,(x+\chi_+(x)y_2-\chi_-(x)y_1)\right)}$$

where $\tanh(\mu\sqrt{\omega}y_j)=t_j$, and t_1,t_2 are the unique positive solutions to (5.2) with $t_1<t_2$.

Proposition 6.4. *Fixed $\mu>0$, for any $\omega>\omega^*$ the operator $L_{1,\mathrm{asym}}^{\gamma,\omega}$ has trivial kernel and one simple negative eigenvalue.*

Proof. For shorthand, in this proof we will denote the operator $L_{1,\mathrm{asym}}^{\gamma,\omega}$ by L_1. Furthermore, the function $\psi_\omega^{y_1,-y_2,0}$ will be denoted by ψ.

By Theorem 5.3, if $\omega>\omega^*$ then ψ is a local minimizer for the functional S_ω on the Nehari manifold. As a consequence, one can prove that L_1 has one simple negative eigenvalue by following the proof of Proposition 6.3 up to (6.10).

Concerning the kernel of L_1, one can follow the reasoning carried out in the proof of 6.3 through (6.11), (6.12) and conclude that the only solutions to the equation $L_1\xi=0$ can be given by $\xi_a(x)=\chi_+\xi(x+y_2)+a\chi_-\xi(x-y_1)$, where a is a complex number, provided that ξ_a fulfils the matching conditions at zero, that translate into the system

$$\begin{cases} \frac{\mu-\sinh^2(\mu\sqrt{\omega}y_2)}{\cosh^{2+\frac{1}{\mu}}(\mu\sqrt{\omega}y_2)} &=\; a\frac{\mu-\sinh^2(\mu\sqrt{\omega}y_1)}{\cosh^{2+\frac{1}{\mu}}(\mu\sqrt{\omega}y_1)} \\ \frac{\sinh(\mu\sqrt{\omega}y_2)}{\cosh^{1+\frac{1}{\mu}}(\mu\sqrt{\omega}y_2)}+a\frac{\sinh(\mu\sqrt{\omega}y_1)}{\cosh^{1+\frac{1}{\mu}}(\mu\sqrt{\omega}y_1)} &=\; -\gamma\sqrt{\omega}\frac{\sinh(\mu\sqrt{\omega}y_2)}{\cosh^{2+\frac{1}{\mu}}(\mu\sqrt{\omega}y_2)} \end{cases} \tag{6.20}$$

Expliciting a from the first equation, plugging it into the second, and denoting as customary $t_i=\tanh(\mu\sqrt{\omega}y_i)$, we get the equation

$$\frac{t_1}{\mu-(\mu+1)t_1^2}+\frac{t_2}{\mu-(\mu+1)t_2^2} \;=\; -\gamma\sqrt{\omega},$$

that, using the second equation in (5.2), gives

$$\frac{1-t_1^2}{t_1(\mu-(\mu+1)t_1^2)}+\frac{1-t_2^2}{t_2(\mu-(\mu+1)t_2^2)} \;=\; 0.$$

Finally, by the first equation in (5.2) one gets

$$\frac{1}{t_1^{2\mu+1}(\mu-(\mu+1)t_1^2)}+\frac{1}{t_2^{2\mu+1}(\mu-(\mu+1)t_2^2)} \;=\; 0.$$

Such a problem translates to the problem of finding t_1 and t_2 such that $0 \le t_1 < \bar{t} < t_2 \le 1$, and $g^2(t_1) = g^2(t_2)$, where $g(t) := t^2 f'(t)$ and $f(t) = t^{2\mu} - t^{2\mu+2}$.
The problem was treated in the last form in the proof of Theorem 5.3, from formula (5.19) up to formula (5.26). The conclusion is that it has no solutions, so none among the functions ξ_a lies in the kernel of L_1, which is therefore trivial. This concludes the proof. □

6.2 The sign of $d''(\omega)$

Proposition 6.5. *Given $\mu > 0$, the sign of the second derivative of the function d, defined in (2.13), is determined as follows:*

1. *if $0 < \mu \le 2$, then $d''(\omega) > 0$ for any $\omega \in (\omega_0, \omega^*) \cup (\omega^*, +\infty)$. Furthermore, $0 < d''(\omega^*+0) < d''(\omega^*-0)$;*
2. *if $\mu > 2$, then $d''(\omega) > 0$ for $\omega \in (\omega_0, \omega^*)$. Besides, $d''(\omega^*-0) > 0$;*
3. *there exists $\mu^\star \in (2, 2.5)$ such that*
   - *(a) If $\mu < \mu^\star$, then $d''(\omega^*+0) > 0$, so there exists $\omega_1(\mu) \ge \omega^*$ such that $d''(\omega) > 0$ for any $\omega \in (\omega^*, \omega_1(\mu))$;*
   - *(b) if $\mu = \mu^\star$, then $d''(\omega^*+0) = 0$;*
   - *(c) if $\mu > \mu^\star$, then $d''(\omega^*+0) < 0$;*
4. *if $\mu > 2$, then there exists $\omega_2(\mu) \ge \omega^*$ such that, if $\omega > \omega_2(\mu)$, then $d''(\omega) < 0$.*

Proof. First, we notice that, given $\mu > 0$, $\omega > \omega_0$, and denoted by ψ_ω a solution to the problem 2.3 corresponding to the chosen value of μ and ω, one has

$$d'(\omega) \;=\; \frac{1}{2}\|\psi_\omega\|^2.$$

Indeed, from definition (2.13), using the stationarity of ψ_ω we get

$$d'(\omega) \;=\; \frac{d}{d\omega}S_\omega(\psi_\omega) \;=\; \frac{d}{d\omega}E(\psi_\omega)+\frac{1}{2}M(\psi_\omega)+\frac{\omega}{2}\frac{d}{d\omega}M(\psi_\omega) \;=\; S'_\omega[\psi_\omega]\frac{d}{d\omega}\psi_\omega+\frac{1}{2}M(\psi_\omega) \;=\; \frac{1}{2}M(\psi_\omega).$$

Expliciting $M(\psi_\omega)$ one obtains

$$d'(\omega) \;=\; \left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}}\frac{\omega^{\frac{1}{\mu}-\frac{1}{2}}}{2\mu}\left[\int_{\zeta_1(\omega)}^{1}(1-t^2)^{\frac{1}{\mu}-1}\,dt+\int_{\zeta_2(\omega)}^{1}(1-t^2)^{\frac{1}{\mu}-1}\,dt\right], \tag{6.21}$$

provided that the change of variable $t = \tanh(\mu\sqrt{\omega}(x-x_1))$ ($t = \tanh(\mu\sqrt{\omega}(x-x_2))$) has been performed in the first (second) integral. The lower bounds in the intervals of integration are defined by

$$\zeta_i(\omega) \;=\; \begin{cases} \frac{2}{\gamma\sqrt{\omega}}, & \omega \in (\omega_0, \omega^*] \\ t_i, & \omega \in (\omega^*, +\infty) \end{cases}, \quad i = 1,2 \tag{6.22}$$

where the couple t_1, t_2 is the unique solution to the system (5.2) such that $t_1 < \bar{t} < t_2$, where $\bar{t} = \sqrt{\frac{\mu}{\mu+1}}$.

Differentiating (6.21) yields

$$\begin{aligned} d''(\omega) &= \left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}} \frac{\omega^{\frac{1}{\mu}-\frac{3}{2}}}{2\mu} \left\{ \left(\frac{1}{\mu}-\frac{1}{2}\right) \left[\int_{\zeta_1(\omega)}^1 (1-t^2)^{\frac{1}{\mu}-1}\, dt + \int_{\zeta_2(\omega)}^1 (1-t^2)^{\frac{1}{\mu}-1}\, dt\right] \right. \\ & \left. -\omega \left[\zeta_1'(\omega)(1-\zeta_1^2(\omega))^{\frac{1}{\mu}-1} + \zeta_2'(\omega)(1-\zeta_2^2(\omega))^{\frac{1}{\mu}-1}\right] \right\}. \end{aligned} \tag{6.23}$$

Let us denote

$$\begin{aligned} (I) &:= \left(\frac{1}{\mu}-\frac{1}{2}\right) \left[\int_{\zeta_1(\omega)}^1 (1-t^2)^{\frac{1}{\mu}-1}\, dt + \int_{\zeta_2(\omega)}^1 (1-t^2)^{\frac{1}{\mu}-1}\, dt\right] \\ (II) &:= -\,\omega \left[\zeta_1'(\omega)(1-\zeta_1^2(\omega))^{\frac{1}{\mu}-1} + \zeta_2'(\omega)(1-\zeta_2^2(\omega))^{\frac{1}{\mu}-1}\right]. \end{aligned} \tag{6.24}$$

1. $0 < \mu \le 2$.

The quantity (I) is positive for any $\omega > \omega_0$. Moreover, by (6.22), for $\omega \in (\omega_0, \omega^*)$ the quantity (II) can be explicitly evaluated as

$$(II) \;=\; \frac{2}{\gamma\sqrt{\omega}}\left(1-\frac{4}{\gamma^2\omega}\right)^{\frac{1}{\mu}-1} \tag{6.25}$$

which is positive too. Therefore, by (6.23) and (6.24), we conclude that $d''(\omega) > 0$, if $\omega_0 < \omega < \omega^*$. To determine the sign of (II) for $\omega > \omega^*$, it is convenient to distinguish between the cases $\mu \le 2$ and $\mu > 2$. Let us start with $\mu \le 2$. Rewriting the first equation in (5.2) as

$$t_1^2(1-t_1^2)^{\frac{1}{\mu}} \;=\; t_2^2(1-t_2^2)^{\frac{1}{\mu}}$$

and differentiating with respect to ω, we obtain

$$t_1' \;=\; \frac{\mu t_2 - (\mu+1)t_2^3}{\mu t_1 - (\mu+1)t_1^3} \left(\frac{1-t_2^2}{1-t_1^2}\right)^{\frac{1}{\mu}-1} t_2',$$

where we neglected in the notation the dependence on ω. Therefore

$$(II) \;=\; -\omega t_2'(1-t_2^2)^{\frac{1}{\mu}-1}\left(1+\frac{\mu t_2-(\mu+1)t_2^3}{\mu t_1-(\mu+1)t_1^3}\right).$$

We prove that such a quantity is positive for $\mu < 2$. As $t_1 < \bar{t} < t_2$, this reduces to prove

$$(\mu+1)t_2^3 - \mu t_2 \;>\; \mu t_1 - (\mu+1)t_1^3. \tag{6.26}$$

To this aim, we define the function

$$\Gamma(t) \;:=\; [(\mu+1)t^3-\mu t]^2 \;=\; \frac{1}{4}t^{4-4\mu}[f'(t)]^2,$$

where f is defined as in (5.14), namely by $f(t) := t^{2\mu} - t^{2\mu+2}$. By the fundamental theorem of the integral calculus, for any $t \in [0, 1]$

$$\Gamma(t) \;=\; \int_{\bar{t}}^t \left[(1-\mu)s^{3-4\mu}[f'(s)]^2 + \frac{1}{2}s^{4-4\mu}f'(s)f''(s)\right] ds, \tag{6.27}$$

where $\bar{t} = \sqrt{\frac{\mu}{\mu+1}}$. Now we proceed like in the proof of theorem 5.3, formulas (5.20)-(5.25). First, we define the function s_1 as the inverse of f in the interval $[0, \bar{t}]$, as well as the function s_2 that is the inverse of f in $[\bar{t}, 1]$. Then, performing the change of variable $u = f(s)$, (6.27) gives

$$\Gamma(t) \;=\; \int_m^{f(t)} \Sigma(s_i(u))\, du \tag{6.28}$$

where $i = 1$ if $t \in [0, \bar{t}]$, $i = 2$ if $t \in (\bar{t}, 1]$, $m := f(\bar{t}) = \frac{\mu^\mu}{(\mu+1)^{\mu+1}}$, and

$$\Sigma(s) \;:=\; (1-\mu)s^{3-4\mu} f'(s) + \frac{1}{2} s^{4-4\mu} f''(s) \;=\; \mu s^{2-2\mu} - 3(\mu+1)s^{4-2\mu}.$$

Consider the case $t_1 \in [\bar{t}/\sqrt{3}, \bar{t})$. The study of the sign of Σ and Σ' shows that Σ is negative and strictly decreasing in $(\bar{t}/\sqrt{3}, 1)$, for any $u \in [f(t_1), m)$ one has

$$0 \;>\; \Sigma(s_1(u)) \;>\; \Sigma(\bar{t}) \;=\; -2\frac{\mu^{2-\mu}}{(1+\mu)^{1-\mu}} \;>\; \Sigma(s_2(u)),$$

and therefore, denoting $a = f(t_1) = f(t_2)$,

$$\Gamma(t_1) \;=\; -\int_a^m \Sigma(s_1(u))\, du \;<\; -\int_a^m \Sigma(s_2(u))\, du \;=\; \Gamma(t_2). \tag{6.29}$$

Second, consider the case $t_1 < \bar{t}/\sqrt{3}$. Write $\Gamma(t_1)$ as

$$\Gamma(t_1) \;=\; \int_m^{f(\bar{t}/\sqrt{3}))} \Sigma(s_1(u))\, du + \int_{f(\bar{t}/\sqrt{3})}^{f(t_1)} \Sigma(s_1(u))\, du.$$

Notice that the first integral in the r.h.s. is positive, while the second is negative, owing to the facts that Σ is positive in $(0, \bar{t}/\sqrt{3})$ and that $f(t_1) < f(\bar{t}/\sqrt{3})$. Then, denoting by $\bar{t}_2$ the only point in $(t_1, 1]$ such that $f(\bar{t}/\sqrt{3}) = f(\bar{t}_2)$, by (6.29),

$$\Gamma(t_1) \;< \Gamma(\bar{t}/\sqrt{3}) \;\le\; \Gamma(\bar{t}_2). \tag{6.30}$$

Furthermore, since $t_1 < \bar{t}/\sqrt{3}$, it must be $\bar{t}_2 < t_2$, and since Σ is negative in the interval $(\bar{t}_2, t_2)$, we obtain $\Gamma(t_2) = -\int_a^{f(\bar{t}_2)} \Sigma(s_2(u))du + \Gamma(\bar{t}_2) > \Gamma(\bar{t}_2)$ that, together with (6.30) and (6.29) yields $\Gamma(t_1) < \Gamma(t_2)$ for any $t_1 \in [0, \bar{t})$, which is equivalent to (6.26). So we proved $(II) > 0$ if $\mu \le 2$, that, together with the positivity of (I), proves that $d''(\omega) > 0$ for any $\omega > \omega_0$, $\omega \neq \omega^*$.

In order to complete the proof of point *1.*, we need to evaluate $d''(\omega^* \pm 0)$, and to compare them. By (6.23) and (6.24), this amounts to compare the value of the two limits $\lim_{\omega\to\omega^*\pm 0}[(I)+(II)]$. From (6.24) it is clear that (I) is continuous at ω^*, where it takes the value $\left(\frac{2}{\mu} - 1\right) \int_{\sqrt{\frac{\mu}{\mu+1}}}^1 (1-t^2)^{\frac{1}{\mu}-1}\, dt$, so we reduce to study the limits of the term (II) only. The left limit is immediately given by (6.24) and (6.25), and reads

$$\lim_{\omega\to\omega^*-0} (II) \;=\; \frac{\sqrt{\mu}}{(\mu+1)^{\frac{1}{\mu}-\frac{1}{2}}} \;>\; 0. \tag{6.31}$$

Computing the right limit is more complicated. Differentiating both equations in (5.2) one can express the couple (t_1', t_2') as a function of the couple (t_1, t_2), namely

$$t_1' \;=\; -\frac{\gamma f'(t_2) t_1^2 t_2^2}{2\sqrt{\omega}\left(t_1^2 f'(t_1) + t_2^2 f'(t_2)\right)}, \qquad t_2' \;=\; -\frac{\gamma f'(t_1) t_1^2 t_2^2}{2\sqrt{\omega}\left(t_1^2 f'(t_1) + t_2^2 f'(t_2)\right)}, \tag{6.32}$$

where we neglected in notation the dependence of t_1 and t_2 on the variable ω and denoted $f(t) = t^{2\mu} - t^{2\mu+2}$. From (6.32), (6.22), (6.24), and since, by (5.2), $f(t_1) = f(t_2)$, one gets

$$(II) \;=\; \frac{\gamma\sqrt{\omega}}{2} t_1^{2\mu} t_2^{2\mu} f^{\frac{1}{\mu}-1}(t_1) \frac{t_1^{2-2\mu} f'(t_1) + t_2^{2-2\mu} f'(t_2)}{t_1^2 f'(t_1) + t_2^2 f'(t_2)}. \tag{6.33}$$

Here, the only non trivial factor is the last one, in which both numerator and denominator vanish as ω goes to ω^* from the right. In order to compute such a limit we pass from the variable ω to the variable t_1. In other words, we consider t_2 as a function of t_1.
We define the functions

$$\begin{aligned} N(t_1) &:= t_1^{2-2\mu} f'(t_1) + t_2^{2-2\mu}(t_1) f'(t_2(t_1)) \\ D(t_1) &:= t_1^2 f'(t_1) + t_2^2(t_1) f'(t_2(t_1)) \end{aligned} \tag{6.34}$$

and provide a Taylor expansion near $t_1 = \bar{t} = \frac{2}{\gamma\sqrt{\omega}}$ for both of them. One immediately gets

$$\begin{aligned} N'(t_1) &:= (2-2\mu)\left[t_1^{1-2\mu} f'(t_1) + t_2^{1-2\mu} f'(t_2)\dot{t}_2\right] + t_1^{2-2\mu} f''(t_1) + t_2^{2-2\mu} f''(t_2)\dot{t}_2 \\ D'(t_1) &:= 2t_1 f'(t_1) + t_1^2 f''(t_1) + 2t_2 f'(t_2)\dot{t}_2 + t_2^2 f''(t_2)\dot{t}_2 \end{aligned}$$

where we used the notation $\dot{t}_2 := \frac{dt_2}{dt_1}(t_1)$. To evaluate $N'(\bar{t})$ and $D'(\bar{t})$ we must then compute $\frac{dt_2}{dt_1}(\bar{t})$. From $f(t_1) = f(t_2)$ it follows

$$\dot{t}_2 = \frac{f'(t_1)}{f'(t_2)}. \tag{6.35}$$

By de l'Hôpital's Theorem,

$$\lim_{t_1 \to \bar{t}-0} \dot{t}_2 \;=\; \lim_{t_1 \to \bar{t}-0} \frac{f''(t_1)}{f''(t_2)\dot{t}_2},$$

from which one immediately has $\left(\lim_{t_1 \to \bar{t}} \dot{t}_2\right)^2 = 1$. Now, since $f'(t_1) > 0$ and $f'(t_2) < 0$, it must be

$$\lim_{t_1 \to \bar{t}-0} \dot{t}_2 \;=\; -1.$$

As a consequence, $N'(\bar{t}) = D'(\bar{t}) = 0$. Further differentiating N and D, and recalling that $f'(\bar{t}) = 0$, one obtains

$$\begin{aligned} N''(\bar{t}) &:= 8(1-\mu)\bar{t}^{1-2\mu} f''(\bar{t}) + 2\bar{t}^{2-2\mu} f'''(\bar{t}) + t^{2-2\mu} f''(\bar{t}) \frac{d^2 t_2}{dt_1^2}(\bar{t}) \\ D''(\bar{t}) &:= 8\bar{t} f''(\bar{t}) + 2\bar{t}^2 f'''(\bar{t}) + t^2 f''(\bar{t}) \frac{d^2 t_2}{dt_1^2}(\bar{t}). \end{aligned}$$

By (6.35)

$$\frac{d^2 t_2}{dt_1^2}(t_1) \;=\; \frac{f''(t_1)(f')^2(t_2(t_1)) - (f')^2(t_1) f''(t_2(t_1))}{(f')^3(t_2(t_1))},$$

so, again using de l'Hôpital's theorem,

$$\lim_{t_1 \to \bar{t}-0} \;=\; -\frac{2f'''(\bar{t})}{3f''(\bar{t})}.$$

It then follows

$$N''(\bar{t}) \;:=\; 8(1-\mu)\bar{t}^{1-2\mu}f''(\bar{t}) + \frac{4}{3}\bar{t}^{2-2\mu}f'''(\bar{t})$$
$$D''(\bar{t}) \;:=\; 8\bar{t}f''(\bar{t}) + \frac{4}{3}\bar{t}^2 f'''(\bar{t}).$$

So, going back to (6.33), we get

$$\lim_{\omega\to\omega^*+0}(II) \;=\; \frac{\gamma\sqrt{\omega^*}}{2}\bar{t}^{4\mu} f^{\frac{1}{\mu}-1}(\bar{t})\frac{6(1-\mu)\bar{t}^{1-2\mu}f''(\bar{t}) + \bar{t}^{2-2\mu}f'''(\bar{t})}{6\bar{t}f''(\bar{t}) + \bar{t}^2 f'''(\bar{t})},$$

which, recalling that $\bar{t} = \sqrt{\frac{\mu}{\mu+1}}$ and the definition of the function f, gives

$$\lim_{\omega\to\omega^*+0}(II) \;=\; \frac{\sqrt{\mu}}{(\mu+1)^{\frac{1}{\mu}-\frac{1}{2}}}\frac{5-2\mu}{4\mu+5} \;>\; 0. \tag{6.36}$$

Comparing (6.31) and (6.36), and observing that the existence of such limits implies the existence of the left and right derivative and gives their values, one completes the proof of point *1.*

2. $\mu > 2$, $\omega_0 < \omega \le \omega^*$.

In this case, term (I) is negative, so one must compare its size to the size of (II).

From (6.22) and (6.23) we know that

$$d''(\omega) \;=\; \left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}} \frac{\omega^{\frac{1}{\mu}-\frac{3}{2}}}{\mu} r(\omega), \tag{6.37}$$

where

$$r(\omega) \;=\; \frac{2-\mu}{2\mu}\int_{\frac{2}{\gamma\sqrt{\omega}}}^{1}(1-t^2)^{\frac{1}{\mu}-1}dt + \frac{1}{\gamma\sqrt{\omega}}\left(1-\frac{4}{\gamma^2\omega}\right)^{\frac{1}{\mu}-1}. \tag{6.38}$$

Then,

$$r'(\omega) \;=\; -\frac{1}{\gamma\omega^{\frac{3}{2}}}\left(1-\frac{1}{\mu}\right)\left(1-\frac{4}{\gamma^2\omega}\right)^{\frac{1}{\mu}-2} \;<\; 0.$$

We estimate the first term in the r.h.s. of (6.38), for $\omega = \omega^*$, as

$$0 \;>\; \frac{2-\mu}{2\mu}\int_{\sqrt{\frac{\mu}{\mu+1}}}^{1}(1-t^2)^{\frac{1}{\mu}-1}dt \;>\; \frac{2-\mu}{2\mu}\frac{\int_{\sqrt{\frac{\mu}{\mu+1}}}^{1}(1-t)^{\frac{1}{\mu}-1}dt}{\left(1+\sqrt{\frac{\mu}{\mu+1}}\right)^{1-\frac{1}{\mu}}} \;=\; \frac{2-\mu}{2\left(1+\sqrt{\frac{\mu}{\mu+1}}\right)(\mu+1)^{\frac{1}{\mu}}}.$$

So,

$$r(\omega^*) \;>\; \frac{1}{2(\mu+1)^{\frac{1}{\mu}}}\left[\frac{2-\mu}{1+\sqrt{\frac{\mu}{\mu+1}}} + \sqrt{\mu(\mu+1)}\right] \;>\; \frac{2-\mu+\sqrt{\mu(\mu+1)}}{2(\mu+1)^{\frac{1}{\mu}}} \;>\; 0.$$

Thus, since r is monotonically decreasing, we have $r > 0$ for any $r \in (\omega_0, \omega^*]$.

Finally, since $\lim_{\omega\to\omega^*-0} d''(\omega)$ exists and can be recovered by putting $\omega = \omega^*$ in (6.37), one obtains that $d''(\omega^*-0)$ equals such limit and so point 2. is proven.

3. $\mu > 2$.

In order to prove points *3 (a,b,c)* we need to evaluate $d''(\omega^* + 0)$. We compute $\lim_{\omega\to\omega^*+0} d''(\omega)$. Then,

$$\lim_{\omega\to\omega^*+0} d''(\omega) = \lim_{\omega\to\omega^*+0}(I) + \lim_{\omega\to\omega^*+0}(II),$$

with (I) and (II) defined in (6.24), and, by a direct computation,

$$\lim_{\omega\to\omega^*+0}(I) \;=\; \left(\frac{2}{\mu}-1\right)\int_{\sqrt{\frac{\mu}{\mu+1}}}^{1}(1-t^2)^{\frac{1}{\mu}-1}dt \;<\; 0.$$

Exactly as in the proof of point *1.*, one finds that the right limit of the term (II) is given by formula (6.36).

Now, according to (6.23) and (6.24), the sign of $d''(\omega^* + 0)$, is given by the sign of the function

$$w(\mu) \;:=\; \lim_{\omega\to\omega^*+0}[(I)+(II)] \;=\; \left(\frac{2}{\mu}-1\right)\int_{\sqrt{\frac{\mu}{\mu+1}}}^{1}(1-t^2)^{\frac{1}{\mu}-1}\,dt + \frac{\sqrt{\mu}}{(\mu+1)^{\frac{1}{\mu}-\frac{1}{2}}}\frac{5-2\mu}{4\mu+5}.$$

Such a sign is obviously negative for $\mu \geq \frac{5}{2}$, so we restrict to $2 < \mu < \frac{5}{2}$. Then,

$$\begin{aligned} w'(\mu) \;=\; & -\left(\frac{2}{\mu^2}+\frac{\mu-2}{2\mu^{\frac{3}{2}}(\mu+1)^{\frac{3}{2}}}\right)\int_{\sqrt{\frac{\mu}{\mu+1}}}^{1}(1-t^2)^{\frac{1}{\mu}-1}\,dt - \frac{\mu-2}{\mu^3}\int_{\sqrt{\frac{\mu}{\mu+1}}}^{1}(1-t^2)^{\frac{1}{\mu}-1}\log\left(\frac{1}{1-t^2}\right)dt \\ & + \frac{2\mu+1}{2\sqrt{\mu}(\mu+1)^{1+\frac{1}{\mu}}}\frac{5-2\mu}{4\mu+5} - \frac{(\mu+1)^{\frac{1}{2}-\frac{1}{\mu}}}{\mu^{\frac{3}{2}}}\log(\mu+1)\frac{5-2\mu}{4\mu+5} - \frac{1}{\sqrt{\mu}(\mu+1)^{\frac{1}{2}-\frac{1}{\mu}}}\frac{5-2\mu}{4\mu+5} \\ & - 30\frac{\sqrt{\mu}(\mu+1)^{\frac{1}{2}-\frac{1}{\mu}}}{(4\mu+5)^2}. \end{aligned} \tag{6.39}$$

For $2 < \mu < \frac{5}{2}$ the only non negative term in (6.39) is the third one. By elementary computation we find that the sign of the sum of such term with the last one is negative if and only if

$$-16\mu^3 + 12\mu^2 + 25 \;<\; 60\mu^2.$$

Since for $\mu = 2$ the inequality is verified, it must be verified for any $\mu > 2$, owing to the fact, easy to check, that the l.h.s is a monotonically decreasing function, while the r.h.s is monotonically increasing. As a consequence, w is monotonically decreasing for $\mu \in [2, \frac{5}{2}]$, so there exists a unique value of μ that makes $d''(\omega^* + 0)$ vanish. Denoting it by $\mu^\star$, we complete the proof of points *3 (b,c)*. Point *3 (a)* follows by continuity from the fact that, if $\mu < \mu^\star$, then $d''(\omega^* + 0) > 0$.

4. $\mu > 2, \omega \to \infty$.

Point 4. follows from the asymptotics of t_1 and t_2 as ω goes to ∞, in the region $t_2 > t_1 > 0$:

$$t_1 \;=\; \frac{1}{\gamma\sqrt{\omega}} + \mathrm{o}(\omega^{-\frac{1}{2}}) \qquad t_2 \;=\; 1 - \frac{1}{2\gamma^{2\mu}\omega^{\mu}} + \mathrm{o}(\omega^{-\mu}) \tag{6.40}$$

$$t_1' \;=\; -\frac{1}{2\gamma\omega^{\frac{3}{2}}} + \mathrm{o}(\omega^{-\frac{3}{2}}) \qquad t_2' \;=\; \frac{\mu}{2\gamma^{2\mu}\omega^{\mu+1}} + \mathrm{o}(\omega^{-\mu-1}). \tag{6.41}$$

Let us prove such asymptotics. The condition $t_2 > t_1$ selects, among the solutions to the system (5.2), those belonging to the set T_2 defined in (5.16). It is immediately seen that, in such region, $t_1 \to 0$ and $t_2 \to 1$ as $\omega \to \infty$. As a consequence, from the second equation in (5.2) one gets

$$\lim_{\omega\to\infty} \gamma\sqrt{\omega}t_1 \;=\; \lim_{\omega\to\infty} \frac{\gamma\sqrt{\omega}t_2}{\gamma\sqrt{\omega}t_2 - 1} \;=\; 1,$$

so the first formula in (6.40) is proven.
From the first equation in (5.2) we have

$$\lim_{\omega\to\infty} t_1^{-2\mu}(1 - t_2^2) \;=\; \lim_{\omega\to\infty} t_2^{-2\mu}(1 - t_1^2) \;=\; 1,$$

and by the first of (6.40)

$$1 - t_2^2 \;=\; \frac{1}{\gamma^{2\mu}\omega^{\mu}} + \mathrm{o}(\omega^{-\mu})$$

and the second identity in (6.40) immediately follows.
In order to prove (6.41) we differentiate both equations in (5.2), obtaining

$$\begin{cases} t_1' t_1^{2\mu-1}(\mu - (\mu+1)t_1^2) \;=\; t_2' t_2^{2\mu-1}(\mu - (\mu+1)t_2^2) \\ t_1' t_1^{-2} + t_2' t_2^{-2} \;=\; -\frac{\gamma}{2\sqrt{\omega}} \end{cases} \tag{6.42}$$

Expressing t_2' from the second equation and plugging it into the first one we get

$$\omega^{\frac{3}{2}} t_1' \;=\; -\frac{\gamma\omega t_1^2 t_2^{2\mu+1}(\mu - (\mu+1)t_2^2)}{2t_2^{2\mu+1}(\mu - (\mu+1)t_2^2) + 2t_1^{2\mu+1}(\mu - (\mu+1)t_1^2)},$$

that converges to $-\frac{1}{2\gamma}$ as ω goes to infinity, and so the first formula in (6.41) is proved. To prove the second one, it is sufficient to use the first equation in (6.42) and by (6.40), (4.9), the first in (6.41) one finally obtains

$$\lim_{\omega\to\infty} \omega^{\mu+1} t_2' \;=\; \frac{\mu}{2\gamma^{2\mu}}$$

and the proof is complete. □

6.3 Stability and instability of the ground states. Pitchfork bifurcation

We recall the definition of orbital neighbourhood and orbital stability.

Definition 6.6. *The set*

$$U_\eta(\phi) := \{\psi \in Q, \text{ s.t.} \inf_{\theta\in[0,2\pi)} \|\psi - e^{i\theta}\phi\|_Q \le \eta\}$$

is called the orbital spherical neighbourhood with radius η of the function ϕ.

Definition 6.7. *We call* orbitally stable *(in the future) any stationary state ϕ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\inf_{\theta\in[0,2\pi)} \|\psi - e^{i\theta}\phi\|_Q \le \delta \Longrightarrow \sup_{t\ge 0} \inf_{\theta\in[0,2\pi)} \|\psi(t) - e^{i\theta}\phi\|_Q \le \varepsilon,$$

where $\psi(t)$ is the solution to the problem (3.1) *with ψ as initial datum.*

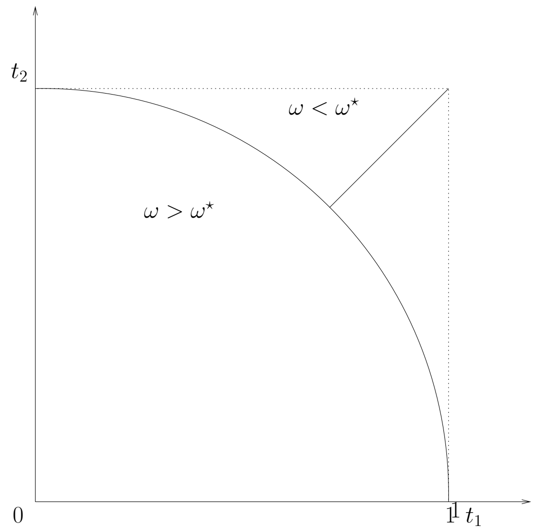

Figure 2: **Hypocritical and critical case (i.e. $\mu \leq 2$).** All ground states are stable.

Definition 6.8. *We call* orbitally unstable *any stationary state that is not orbitally stable.*

Proposition 6.9 (Stability and instability of ground states)**.** *Consider the ground states of the dynamics described by* (1.1)*, defined as the solutions to problem 2.4 and explicitly computed in Theorem 5.3. Then,*

1. *If $0 < \mu \leq 2$, then for any $\omega \in (\omega_0, +\infty)$, $\omega \neq \omega^*$, all ground states are stable.*
2. *If $\mu > 2$, then*
   (a) there exists $\omega_1 > \omega^*$ such that all the ground states with frequency $\omega < \omega_1$ are orbitally stable.
   (b) There exists $\omega_2 \geq \omega_1$ such that if $\omega > \omega_2$ then all ground states with frequency $\omega > \omega_2$ are orbitally unstable.

Proof. Points 1 and 2 (a) follow from Theorem 2 in [21]. Indeed, notice that Assumption 1 in such theorem is proven by Propositions 3.3 and 3.4, while Assumption 2 follows from Proposition 5.1 and Theorem 5.3. Furthermore, owing to propositions 6.1, 6.3 a), b), and 6.4, Assumption 3 is verified for all ground states. Therefore, in order to establish orbital stability, it is sufficient to remark that, for the considered cases, Proposition 6.5 establishes $d''(\omega) > 0$.

Case 2 b) follows from Theorem 4.7 in [21], as we know from Proposition 6.5 that $d''(\omega) < 0$. Thus, the theorem is proven. □

Remark 6.10. By Theorem 3 in [21] we have that, under the hypotheses of Theorem 6.9, the ground states solve Problem 2.2, i.e., they minimize *at least locally* the energy 2.8 among the functions with the same L^2-norm.

Theorem 6.11 (Pitchfork bifurcation)**.** *Given $\mu > 0$, if $\omega > \omega^*$ then the stationary solutions $\psi_\omega^{y,-y,\theta}$ defined in Theorem 5.3 are orbitally unstable.*

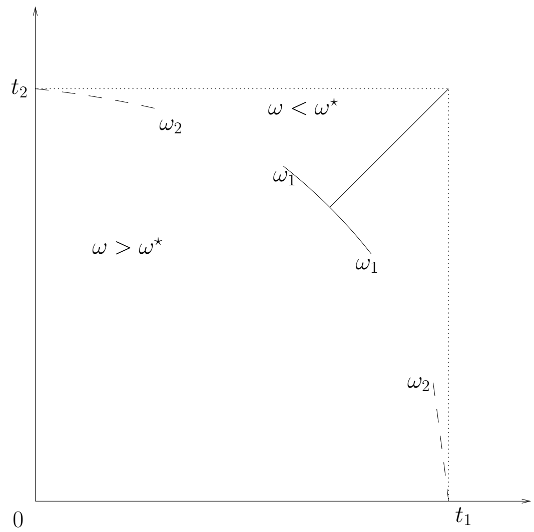

Figure 3: **First hypercritical case (i.e. $2 < \mu < \mu^\star < 2.5$).** Symmetric ground states are stable. Immediately after bifurcation, the two newborn asymmetric states are still stable. At large frequencies, they become unstable.

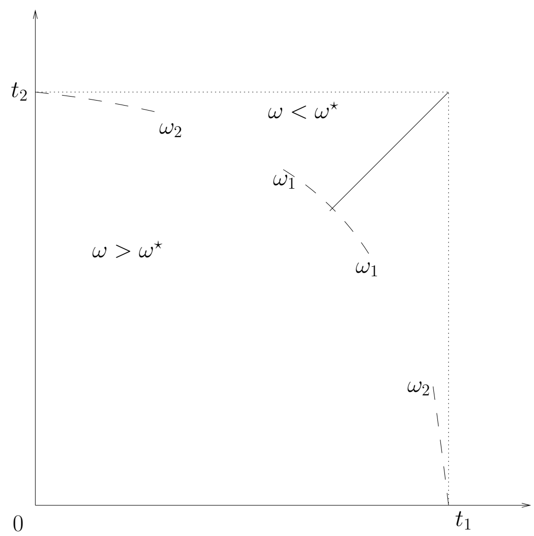

Figure 4: **Second hypercritical case (i.e. $\mu > \mu^\star$).** Symmetric ground states are stable. Change of stability occurs immediately after bifurcation.

Proof. Closely mimicking the computation that led to formula (6.23), one gets

$$\frac{d^2 S_\omega(\psi_\omega^{y,-y,\theta})}{d\omega^2}(\omega) = \left(\frac{\mu+1}{\lambda}\right)^{\frac{1}{\mu}} \frac{\omega^{\frac{1}{\mu}-\frac{3}{2}}}{\mu} \left\{ \left(\frac{1}{\mu}-\frac{1}{2}\right) \int_{\frac{2}{\gamma\sqrt{\omega}}}^{1} (1-t^2)^{\frac{1}{\mu}-1}\, dt + \frac{2}{\gamma\sqrt{\omega}} \left(1-\frac{4}{\gamma^2\omega}\right)^{\frac{1}{\mu}-1} \right\}.$$

As $\omega > \omega^*$, we know by proposition 6.3 we know that the number of negative eigenvalues of the operator $L^{\gamma,\omega}$ equals two. If $\frac{d^2 S_\omega(\psi_\omega^{y,-y,\theta})}{d\omega^2}(\omega)$ is positive (for instance if $\mu \le 2$), then the result follows from Theorem 6.2 in [22].

On the other hand, if $\frac{d^2 S_\omega(\psi_\omega^{y,-y,\theta})}{d\omega^2}(\omega)$ is negative, then we restrict the problem to the space Q_a of antisymmetric functions lying in Q. First, from the explicit knowledge of the propagator of the Schrödinger equation with a δ' interaction, represented by the integral kernel (see [4])

$$e^{-iH_\gamma t}(x,y) = \frac{e^{i\frac{(x-y)^2}{4t}}}{\sqrt{4\pi i t}} + \epsilon(xy)\frac{e^{i\frac{(|x|+|y|)^2}{4t}}}{\sqrt{4\pi i t}} + \frac{\epsilon(xy)}{2\gamma}\int_0^{+\infty} e^{\frac{-2u}{\gamma}} \frac{e^{i\frac{(|x|+|y|-u)^2}{4t}}}{\sqrt{4\pi i t}} + \frac{2\epsilon(xy)}{\gamma} e^{i\frac{4t}{\gamma^2}} e^{-\frac{2}{\gamma}|x|+|y|}$$

it appears that, denoting $\widetilde{g}(x) = g(-x)$, one has

$$\widetilde{e^{-iH_\gamma t}\psi_0} = e^{-iH_\gamma t}\widetilde{\psi}_0$$

Let us consider the problem (1.1), and initial data ψ_0^a such that $\widetilde{\psi}_0^a = -\psi_0^a$. Then, applying Duhamel's formula to (1.1), one finds

$$\psi_t = e^{-iH_\gamma t}\psi_0 + i\lambda \int_0^t e^{-iH_\gamma(t-s)}|\psi_s|^{2\mu}\psi_s,$$

so that

$$\widetilde{\psi}_t = e^{-iH_\gamma t}\widetilde{\psi}_0 + i\lambda \int_0^t e^{-iH_\gamma(t-s)}|\widetilde{\psi}_s|^{2\mu}\widetilde{\psi}_s = -e^{-iH_\gamma t}\psi_0 + i\lambda \int_0^t e^{-iH_\gamma(t-s)}|\widetilde{\psi}_s|^{2\mu}\widetilde{\psi}_s$$

It follows that $\widetilde{\psi}_t$ solves (1.1) with $-\psi_0$ as initial data. Since (1.1) is invariant under multiplication by a phase factor, and since the solution is unique, it must be $\widetilde{\psi}_t = -\psi_t$, and so we have that (1.1) preserves the antisymmetry, and therefore the evolution problem (1.1) is well-defined in Q_a.

Now, as already remarked in the proof of Proposition 6.3, the functions $\psi_\omega^{y,-y,\theta}$ are the minimizers of S_ω among the antisymmetric functions belonging to the Nehari manifold. The time-evolution operator, linearized around them, admits a simple negative eigenvalue, so, by Theorem 4.1 in [21], the stationary state $\psi_\omega^{y,-y,\theta}$ is orbitally unstable in Q_a, so, *a fortiori*, it is orbitally unstable in Q. To complete the proof, we recall that the instability of $\psi_\omega^{y,-y,\theta}$ in the case $\frac{d^2 S_\omega(\psi_\omega^{y,-y,\theta})}{d\omega^2}(\omega) = 0$ can be established by the argument in [12]. □

We are then in the presence of a pitchfork bifurcation, that can be depicted as in Figure 5.

We leave open the issue of optimizing ω_1 and ω_2, and possibly getting $\omega_1 = \omega_2$. In other words, we do not know whether, for a frequency beyond the bifurcation, each ground state undergoes only a change of variable or more.

Finally, we don't treat here the problem of determining the stability features of the ground states at the bifurcation frequency. This will be the subject of a forthcoming note.

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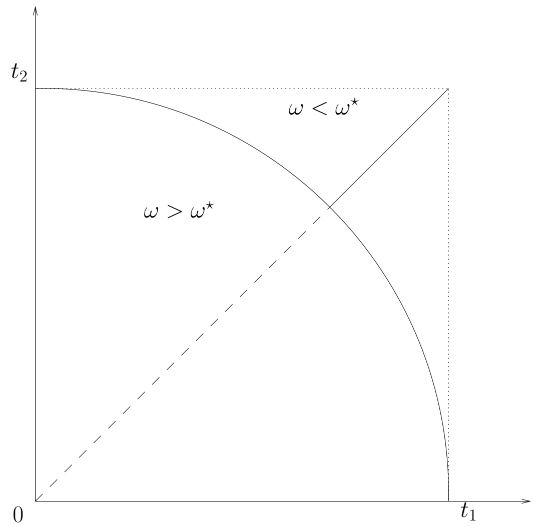

Figure 5: **Pitchfork bifurcation**. The branch of the (anti-)symmetric ground states can be continued beyond bifurcation, for any value of ω, but the corresponding stationary states are orbitally unstable. By the same modification, i.e., adding the unstable straight branch from the origin up to (ω^*, ω^*) to Figures 3 and 4, one obtains the pictures that correspond to the cases $2 < \mu < \mu^\star$ and $\mu > \mu^\star$.

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References

[1] Adami R., Noja D.: *Existence of dynamics for a 1-d NLS equation in dimension one*, J. Phys. A, **42**, 49, 495302, 19pp (2009).

[2] Adami R., Noja D., Visciglia N.: *Variational structure of standing waves for NLS with point interactions: a general result with applications*, in preparation.

[3] Akhmediev N. N.: *Novel class of nonlinear surface waves: asymmetric modes in a symmetric layered structure*, Sov. Phys. JETP **56** 299-303 (1982).

[4] Albeverio S., Brzeźniak Z., Dabrowski L.: *Fundamental solutions of the Heat and Schrödinger Equations with point interaction*, J. Func. An., **130**, 220–254 (1995).

[5] Albeverio S., Gesztesy F., Hoegh-Krohn R., Holden H.: *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York (1988).

[6] Avron J. E., Exner P., Last Y.: *Periodic Schrödinger operators with large gaps and Wannier-Stark ladders*, Phys. Rev. Lett. 72, 896899 (1994).

[7] Brezis H., Lieb E. H.: *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., **88**, 486–490 (1983).

[8] Cazenave T., Lions P.-L.: *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., **85** 549–561 (1982).

[9] Cao Xiang D., Malomed A. B.: *Soliton defect collisions in the nonlinear Schrödinger equation*, Phys. Lett. A, **206**, 177–182 (1995).

[10] Cazenave, T.: *Semilinear Schrödinger Equations*, vol. **10** Courant Lecture Notes in Mathematics *AMS*, Providence (2003).

[11] Cheon T., Shigehara T.: *Realizing discontinuous wave functions with renormalized short-range potentials*, Phys. Lett. A, **243**, 111–116 (1998).

[12] Comech A., Pelinovsky D.: *Purely nonlinear instability of standing waves with minimal energy*, Comm. Pure App. Math., **56**, 1565–1607 (2003).

[13] Exner P., Grosse P.: *Some properties of the one-dimensional generalized point interactions (a torso)*, mp-arc 99-390, math-ph/9910029 (1999).

[14] Exner P., Neidhart H., Zagrebnov V. A.: *Potential approximations to δ': an inverse Klauder phenomenon with norm-resolvent convergence*, Comm. Math. Phys., **224**, 593–612 (2001).

[15] Fibich G., Wang X. P.: *Stability for solitary waves for nonlinear Schrödinger equations with inhomogenous nonlinearities*, Physica D **175**, 96–108 (2003).

[16] Fukuizumi R., Jeanjean L.: *Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential*, Dis. Cont. Dyn. Syst. (A), **21**, 129–144 (2008).

[17] Fukuizumi R., Ohta M, Ozawa T.: *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré - AN, **25**, 837–845 (2008).

[18] Fukuizumi R., Sacchetti A.: *Bifurcation and stability for nonlinear Schrödinger equation with double well potential in the semiclassical limit*, arxiv:1104.1511v1, to appear on J. Stat. Phys. (2011).

[19] Goldstein H., Pole C.P., Safko J.L.: *Classical Mechanics*, Addison Wesley, 2002.

[20] Goodman R. H., Holmes P. J., Weinstein M.I.: *Strong NLS soliton-defect interactions*, Physica D, **192**, 215–248 (2004).

[21] Grillakis M., Shatah J., Strauss W.: *Stability theory of solitary waves in the presence of symmetry - I*, J. Func. An., **74**, 160–197 (1987).

[22] Grillakis M., Shatah J., Strauss W.: *Stability theory of solitary waves in the presence of symmetry - II*, J. Func. An., **94**, 308–348 (1990).

[23] Hislop P.D., Sigal I.M.: *Introduction to spectral theory: With applications to Schrödinger operators*, Springer, New York (1996).

[24] Haroske D.D., Triebel H.: *Distributions, Sobolev Spaces, Elliptic Equations*, European Mathematical Society (2008).

[25] Holmer J., Marzuola J., Zworski M.: Fast soliton scattering by delta impurities, *Comm. Math. Phys*, **274**, 187–216 (2007).

[26] Jackson R. K., Weinstein M.: *Geometric analysis of bifurcation and symmetry breaking in a Gross-Pitaevskii equation*, J. Stat. Phys., **116**, 881–905 (2004).

[27] Kirr E., Kevrekidis P. G., Pelinovsky D. E.: *Symmetry-breaking bifurcation in the nonlinear Schrödinger equation with symmetric potentials*, arxiv:1012.3921, to appear on Comm. Math. Phys. (2010).

[28] Le Coz S., Fukuizumi R., Fibich G., Ksherim B., Sivan Y.: *Instability of bound states of a nonlinear Schrdinger equation with a Dirac potential*, Phys. D, **237** n. 8, 1103–1128 (2008).

[29] Marangell R., Jones C. K. R. T., Susanto H.: *Localized standing waves in inhomogeneous Schrodinger equations*, Nonlinearity, **23**, 9 2059–2080 (2010).

[30] Pelinovsky D. E., Phan T.: *Normal form for the symmetry-breaking bifurcation in the nonlinear Schrödinger equation*, preprint arXiv:1101.5402 (2011).

[31] Reed M., Simon B.: *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press Inc., San Diego (CA) (1978).

[32] Weinstein M.: *Modulational stability of ground states of nonlinear Schroedinger equations*, SIAM J. Math. Anal., **16**, 472–491 (1985).

[33] Weinstein M.: *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., **39** 51–68 (1986).

[34] Witthaut D., Mossmann S., Korsch H. J.: *Bound and resonance states of the nonlinear Schrödinger equation in simple model systems*, J. Phys. A, **38**, 1777-1702 (2005).