THE BREZIS-NIRENBERG RESULT FOR THE FRACTIONAL LAPLACIAN

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Abstract. Aim of this paper is to deal with the non-local fractional counterpart of the Laplace equation involving critical nonlinearities studied in the famous paper of Brezis and Nirenberg [3]. Namely, our model is the following equation

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplace operator, $s \in (0,1)$, Ω is an open bounded set of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, $\lambda > 0$ is a real parameter and $2^* = 2n/(n-2s)$ is a fractional critical Sobolev exponent.

In this paper we first study the problem in a general framework, indeed we consider the equation

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $\mathcal{L}_K$ is a general non-local integrodifferential operator and f is a lower order perturbation of the critical power $|u|^{2^*-2}u$. In this setting we prove an existence result through variational techniques.

Then, as a concrete example, we derive a Brezis-Nirenberg type result for our model equation, that is we show that, if $\lambda_{1,s}$ is the first eigenvalue of the non-local operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum, then for any $\lambda \in (0, \lambda_{1,s})$ there exists a non-trivial solution of the above model equation, provided $n \geqslant 4s$. In this sense the present work may be seen as the extension of the classical Brezis-Nirenberg result to the case of non-local fractional operators.

Contents

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1. Introduction

1.1. **Fractional Laplace equations with critical nonlinearities.** Recently, a great attention has been devoted to fractional and non-local operators of elliptic type, both for their interesting theoretical structure and in view of concrete applications (for several motivations, an elementary introduction to this topic and a – still not exhaustive – list of references see, e.g., [6]). Motivated by the interest shared by the mathematical community in this topic, we consider here the non-local counterpart of the following nonlinear elliptic partial differential critical equation

$$\begin{cases} -\Delta u - \lambda u = |u|^{2_*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \quad 2_* = 2n/(n-2)\,, n > 2\,, \end{cases} \tag{1.1}$$

namely the non-local fractional equation

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.2}$$

where $s \in (0,1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\,, \quad x \in \mathbb{R}^n \tag{1.3}$$

(see for instance [6] and references therein for further details on the fractional Laplacian), $\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary, $\lambda > 0$ and

$$2^* = \frac{2n}{n-2s}\,, \tag{1.4}$$

is the fractional critical Sobolev exponent.[1] The homogeneous Dirichlet datum in (1.2) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, as it happens in (1.1), consistently with the non-local character of the operator $(-\Delta)^s$.

In the case when the nonlinear term in problems like (1.1) has a subcritical growth (i.e., for instance, is of the form $|u|^{q-2}u$, with $q \in (2, 2_*)$) the existence of solutions was investigated by many authors (see [12, 14] and references therein), while the fractional analogue of these results for problem (1.2) can be found in [10, 11].

Aim of this paper is to investigate the critical case. In the classical critical setting in the famous paper [3] Brezis and Nirenberg proved that

a) if $n \geqslant 4$, then for any $\lambda \in \big(0, \lambda_1(-\Delta)\big)$ problem (1.1) has a positive solution;

b) if $n = 3$ then there exists a constant $\lambda_* \in \big(0, \lambda_1(-\Delta)\big)$ such that for any $\lambda \in \big(\lambda_*, \lambda_1(-\Delta)\big)$ problem (1.1) has a positive solution.

In the case when Ω is a ball in *b*) they calculated explicitly λ_* and proved that the following result (stronger than *b*)) holds true:

c) when $n = 3$ and Ω is a ball, problem (1.1) has a positive solution if and only if $\lambda \in \big(\lambda_1(-\Delta)/4, \lambda_1(-\Delta)\big)$.

Here $\lambda_1(-\Delta)$ is the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions. In [4] Capozzi, Fortunato and Palmieri extended the existence result of [3], proving that if $n \geqslant 4$, then problem (1.1) has a solution for any $\lambda > 0$.

[1] We remark that

$$2_* = \frac{2n}{n-2} > \frac{2n}{n-2s} = 2^* > 2\,.$$

In our framework, the exponent 2^* plays the role of a fractional critical Sobolev exponent (see, e.g. [6, Theorem 6.5]).

Our aim is to show that the results of [3] can be extended to problem (1.2). In order to do this, as in the classical case, we will use a variational technique: we can approach the problem at least in two different ways. The first one consists of noting that the solutions of (1.2) correspond to critical points of the 'free' functional defined as follows

$$u \mapsto \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\,.$$

The second variational approach consists of looking for critical points of the functional

$$u \mapsto \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |u(x)|^2\,dx$$

on the sphere $\big\{u : \|u\|_{L^{2^*}(\Omega)} = 1\big\}$. This second approach is related to the existence of the best fractional critical Sobolev constant.

In the model case (1.2) these two approaches are equivalent. On the other hand, for more general nonlinearities with critical growth this is not true, since it is not always possible to reduce a boundary value problem to a constrained minimization problem: in this case, the use of the 'free' functional becomes, therefore, somewhat unavoidable.

In the present paper we first study the following general integrodifferential equation

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.5}$$

where $s \in (0,1)$ is fixed, $\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary, $\lambda > 0$, and $\mathcal{L}_K$ is the non-local operator defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big)K(y)\,dy\,, \quad x \in \mathbb{R}^n\,. \tag{1.6}$$

Here $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.7}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.8}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.9}$$

A model for K is given by $K(x) = |x|^{-(n+2s)}$ and, in this case, $\mathcal{L}_K$ is – up to a normalization constant – the fractional Laplace operator $-(-\Delta)^s$ defined in (1.3).

The nonlinear term in equation (1.5) is a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verifying the following conditions:

$$\sup\Big\{|f(x,t)| : \text{a.e. } x \in \Omega\,,\ |t| \leqslant M\Big\} < +\infty \ \text{ for any } M > 0\,; \tag{1.10}$$

$$\lim_{t \to 0} \frac{f(x,t)}{t} = 0 \text{ uniformly in } x \in \Omega\,; \tag{1.11}$$

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{2^*-1}} = 0 \text{ uniformly in } x \in \Omega\,. \tag{1.12}$$

The term f represents a lower order perturbation of the critical nonlinearity $|u|^{2^*-2}u$. As a model for f we can take the nonlinearity $f(x,t) = a(x)|t|^q$, with $a \in L^\infty(\Omega)$ and $q \in (1, 2^*-1)$.

In particular we are interested in the weak formulation of (1.5) given by the following problem (for this, it is convenient to assume (1.9))

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad \displaystyle = \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx + \int_\Omega f(x,u(x))\varphi(x)dx \quad \forall\,\varphi\in X_0 \\ u\in X_0\,. \end{cases} \tag{1.13}$$

Here the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y)\mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega), dxdy\big)\,,$$

where $\mathcal{C}\Omega := \mathbb{R}^n\setminus\Omega$, while

$$X_0 = \{g\in X : g=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\}\,.$$

We note that

$$C^2_0(\Omega)\subseteq X_0, \tag{1.14}$$

see, e.g., [9, Lemma 11] (for this we need condition (1.7)), and so X and X_0 are non-empty.

1.2. **Main results of the paper.** Assumption (1.11) implies that $f(x,0)=0$, so that the function $u\equiv 0$ is a (trivial) solution of (1.5). The aim of this paper will be, then, to find non-trivial solutions for (1.5). For this, we will adapt the variational techniques of [3] to the non-local setting. More precisely, we will study the critical points of the functional $\mathcal{J}_{K,\lambda} : X_0\to\mathbb{R}$ defined as follows

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ -\frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx - \int_\Omega F(x,u(x))dx\,, \end{aligned} \tag{1.15}$$

where F is given by

$$F(x,t) = \int_0^t f(x,\tau)d\tau\,. \tag{1.16}$$

As in the classical case of the Laplacian, in the non-local setting the issue that needs to be overcome is that the functional $\mathcal{J}_{K,\lambda}$ does not satisfy the Palais–Smale condition. This lack of compactness is due to the fact that the embedding $X_0\hookrightarrow L^{2^*}(\mathbb{R}^n)$ (or, in the case of the fractional Laplacian, $H^s(\mathbb{R}^n)\hookrightarrow L^{2^*}(\mathbb{R}^n)$) is not compact. For this the functional $\mathcal{J}_{K,\lambda}$ does not verify the Palais–Smale condition globally, but – as we will see in the sequel – only in a suitable range related to the best fractional critical Sobolev constant in the embedding $X_0\hookrightarrow L^{2^*}(\mathbb{R}^n)$. For this, it is convenient to define

$$S_K := \inf_{v\in X_0\setminus\{0\}} S_K(v)\,, \tag{1.17}$$

where for any $v\in X_0\setminus\{0\}$

$$S_K(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2K(x-y)\,dx\,dy}{\displaystyle\Big(\int_\Omega |v(x)|^{2^*}dx\Big)^{2/2^*}} = \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2K(x-y)\,dx\,dy}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}dx\Big)^{2/2^*}}\,. \tag{1.18}$$

Along the paper we also need the following constant

$$S_{K,\lambda} := \inf_{v\in X_0\setminus\{0\}} S_{K,\lambda}(v)\,, \tag{1.19}$$

where for any $v \in X_0 \setminus \{0\}$

$$
\begin{aligned}
S_{K,\lambda}(v) := & \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda \int_\Omega |v(x)|^2\,dx}{\displaystyle\Big(\int_\Omega |v(x)|^{2^*}\,dx\Big)^{2/2^*}} \\
= & \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda \int_{\mathbb{R}^n} |v(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,.
\end{aligned} \tag{1.20}
$$

Of course, $S_{K,\lambda}(v) < S_K(v)$ for any $v \in X_0 \setminus \{0\}$ and any $\lambda > 0$, so $S_{K,\lambda} \leqslant S_K$. The topic of research of this paper will actually focus on the case when the *strict* inequality occurs (see (1.27) below).

In the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ the counterpart of S_K and $S_{K,\lambda}$ are the constants S_s and $S_{s,\lambda}$ defined as follows

$$
S_s := \inf_{v \in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v)\,, \tag{1.21}
$$

where for any $v \in H^s(\mathbb{R}^n) \setminus \{0\}$

$$
S_s(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,, \tag{1.22}
$$

and

$$
S_{s,\lambda} := \inf_{v \in H^s(\mathbb{R}^n)\setminus\{0\}} S_{s,\lambda}(v)\,, \tag{1.23}
$$

where for any $v \in H^s(\mathbb{R}^n) \setminus \{0\}$

$$
S_{s,\lambda}(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_{\mathbb{R}^n} |v(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{1.24}
$$

We stress that $S_{K,\lambda}$ and S_K depend on Ω (since so does X_0), while $S_{s,\lambda}$ and S_s do not (because the minimization occurs on the whole of $H^s(\mathbb{R}^n)$). The consistency of all these definitions will be discussed in next results (see Lemmas 9, 10, 13 and 14 in Subsection 2.3). Notice that $S_{s,\lambda}(v) < S_s(v)$ for any $v \in H^s(\mathbb{R}^n) \setminus \{0\}$ and any $\lambda > 0$, so $S_{s,\lambda} \leqslant S_s$: in the forthcoming Theorems 3 and 4 we will consider the case in which the *strict* inequality occurs.

The first result of the present paper is the following one:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of the non-local operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions.*

Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying conditions (1.7)–(1.9) and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying (1.10)–(1.12). Finally, assume that

$$
\begin{gathered}
\text{there exists } u_0 \in X_0 \setminus \{0\} \text{ with } u_0 \geqslant 0 \text{ a.e. in } \mathbb{R}^n\,,\ \text{such that} \\
\sup_{\zeta \geqslant 0} \mathcal{J}_{K,\lambda}(\zeta u_0) < \frac{s}{n}\, S_K^{n/(2s)}\,.
\end{gathered} \tag{1.25}
$$

Then, problem (1.13) admits a solution $u \in X_0$, which is not identically zero.

In the case when there is no perturbation of the critical power nonlinearity, that is $f \equiv 0$ in $\Omega \times \mathbb{R}$, problem (1.13) becomes the critical equation

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad = \displaystyle\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \quad \forall\, \varphi \in X_0 \\ u \in X_0\,, \end{cases} \tag{1.26}$$

for which Theorem 1 yields the following existence result:

Theorem 2. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.7)–(1.9) *and let $\lambda \in (0,\lambda_1)$, where λ_1 denotes the first eigenvalue of the non-local operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. Assume that*

$$S_{K,\lambda} < S_K\,. \tag{1.27}$$

Then, problem (1.26) *admits a solution $u \in X_0$, which is not identically zero.*

In a sense, Theorem 2 clarifies condition (1.25): indeed, as we will see in Section 3.2, condition (1.25), which looks artificial at a first glance, boils down to the more natural condition (1.27) when $f \equiv 0$.

In the non-local framework, the simplest example we can deal with is given by the fractional Laplacian $(-\Delta)^s$, for which we have the following critical equation with a lower order perturbation:

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad = \displaystyle\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx + \int_\Omega f(x,u(x))\,\varphi(x)\,dx \\ \qquad\qquad\qquad \forall\, \varphi \in H^s(\mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \\ u \in H^s(\mathbb{R}^n)\,. \end{cases} \tag{1.28}$$

This problem represents the Euler-Lagrange equation of the functional $\mathcal{J}_{s,\lambda} : H^s(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$\begin{aligned} \mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx \\ - \int_\Omega F(x,u(x))\,dx\,, \end{aligned} \tag{1.29}$$

with F as in (1.16). In this setting our existence result reads as follows:

Theorem 3. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $\lambda \in (0,\lambda_{1,s})$, where $\lambda_{1,s}$ is the first eigenvalue of the non-local operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying* (1.10)–(1.12) *and assume that*

$$\begin{gathered} \text{there exists } u_0 \in H^s(\mathbb{R}^n) \setminus \{0\} \text{ with } u_0 \geqslant 0 \text{ a.e. in } \Omega \\ \text{and } u_0 = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\,, \text{ such that} \\ \sup_{\zeta \geqslant 0} \mathcal{J}_{s,\lambda}(\zeta u_0) < \frac{s}{n}\,S_s^{n/(2s)}\,. \end{gathered} \tag{1.30}$$

Then, problem (1.28) *admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.*

When $f \equiv 0$ in $\Omega \times \mathbb{R}^n$ problem (1.28) becomes our critical model equation (1.2), whose weak formulation is given by

$$
(1.31) \quad \begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad \displaystyle = \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \\ \qquad\qquad\qquad\qquad \forall\, \varphi \in H^s(\mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \\ u \in H^s(\mathbb{R}^n)\,. \end{cases}
$$

In this framework, assumption (1.30) may be checked directly and Theorem 3 gives:

Theorem 4. *Let $s \in (0,1)$, $n \geqslant 4s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary.*

Then, denoting by $\lambda_{1,s}$ the first eigenvalue of the non-local operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum, for any $\lambda \in (0, \lambda_{1,s})$ problem (1.31) admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

For the proof of Theorem 4 we use the fact the infimum in (1.21) is attained by a suitable function $\widetilde{u}$, as it is proved in [5, Theorem 1.1]. Then, starting from $\widetilde{u}$ we define a family u_ε and we estimate $S_{s,\lambda}(u_\varepsilon)$, see Subsection 4.2 for the details. With respect to the classical case of the Laplacian, here this estimate is more delicate, due to the non-local nature of the operator $(-\Delta)^s$. In particular, in the non-local framework, in order to control the Gagliardo seminorm of u_ε, pointwise estimates on u_ε are not enough. Indeed, we also need some integral estimates which allow us to control the interactions between B_δ and $\mathbb{R}^n \setminus B_\delta$ (for more details, see Proposition 21) .

As we said before, in this paper we adapt the variational approach used in [3] to the non-local framework. For this we will work in a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation. Also the functional setting we will use allow us to overcome the problems related to the lack of compactness and to show that the Palais–Smale condition holds true in a suitable range related to the best fractional Sobolev constant in the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ (or, in the case of the fractional Laplacian, $H^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$), see Subsection 4.2.2 for more details. In fact, this range of validity of the Palais–Smale condition may be explicitly related to the strict sign of the inequality obtained from the Sobolev constants, as we discuss in Proposition 24.

Finally, note that when $s = 1$ Theorem 4 reads as the results obtained in [3] for the Laplacian. For all these reasons we think that Theorem 4 may be seen as the natural extension of the results proved in [3] (see also [12, 14]) to the non-local fractional framework. With this respect, we recall the recent paper [13], where a non-local version of the Brezis-Nirenberg result was also considered (in the case $s = 1/2$ and for an operator conceptually different than the one treated here).

The paper is organized as follows. In Section 2 we give some basic estimates on the nonlinearity f and its primitive. Moreover, we introduce the functional setting we will work in and we discuss some embeddings of the spaces X_0 and $H^s(\mathbb{R}^n)$ into the usual Lebesgue spaces. At the end of the section, we deal with an eigenvalue problem driven by the non-local integrodifferential operator $-\mathcal{L}_K$ and we recall some properties of its first eigenvalue. In Section 3 we prove Theorems 1 and 2 by performing a variational method (in fact, we reduce Theorem 2 to Theorem 1 proving the equivalence of conditions (1.25) and (1.27) when $f \equiv 0$). Finally, in Section 4, as an application, we consider the case of an equation driven by the fractional Laplacian operator and we prove Theorems 3 and 4 (for this, condition (1.30) will be reduced to (1.25) and so Theorem 1 will be applied to prove Theorems 3, which, in turn, will imply Theorem 4).

The proofs are, in several points, different than the ones of the classical case of the Laplacian: these additional complications are not only technical, but, sometimes, they somehow reflect the different distribution of the energy density due to the non-local interactions. For instance, the far-away part of the energy in our case, though it is small, gives a contribution to the functional that cannot be neglected, since it is of order comparable with the perturbation.

2. Some preliminary results

In this section we prove some preliminary results which will be useful in the sequel.

2.1. **Estimates on the nonlinearity and its primitive.** Here we gather some elementary results which will be useful in the main estimates of the paper. We use the behavior of f at zero and at infinity in order to deduce some estimates on the nonlinear term and its primitive (with respect to its second variable) F defined in (1.16). This part is quite standard and does not take into account the non-local nature of the problem: the reader familiar with these topics may skip it and go directly to Subsection 2.2.

Lemma 5. *Assume $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions* (1.10)–(1.12). *Then, for any $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$*

$$|f(x,t)| \leqslant 2^*\varepsilon|t|^{2^*-1} + M(\varepsilon) \tag{2.1}$$

and so, as a consequence,

$$|F(x,t)| \leqslant \varepsilon\,|t|^{2^*} + M(\varepsilon)|t|\,, \tag{2.2}$$

where F is defined as in (1.16).

Proof. By assumption (1.12) for any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that for any $t \in \mathbb{R}$ with $|t| > \sigma$ and a.e. $x \in \Omega$ we get

$$|f(x,t)| \leqslant 2^*\varepsilon|t|^{2^*-1}\,. \tag{2.3}$$

Moreover, by (1.10) there exists $M(\varepsilon) = M(\sigma(\varepsilon)) > 0$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$ with $|t| \leqslant \sigma$ we have

$$|f(x,t)| \leqslant M(\varepsilon)\,. \tag{2.4}$$

Combining (2.3) and (2.4) it easily follows (2.1). Using the definition of F (see (1.16)), we also get (2.2). ☐

Lemma 6. *Assume $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions* (1.10)–(1.12). *Then, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$*

$$|f(x,t)| \leqslant 2\varepsilon|t| + 2^*\delta(\varepsilon)|t|^{2^*-1} \tag{2.5}$$

and so, as a consequence,

$$|F(x,t)| \leqslant \varepsilon\,|t|^2 + \delta(\varepsilon)\,|t|^{2^*}\,, \tag{2.6}$$

where F is defined as in (1.16).

Proof. By assumption (1.11) for any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that for any $t \in \mathbb{R}$ with $|t| < \sigma$ and a.e. $x \in \Omega$ we get

$$|f(x,t)| \leqslant 2\varepsilon|t|\,. \tag{2.7}$$

Moreover, by (2.1) with $\varepsilon = 1$ we get that

$$|f(x,t)| \leqslant 2^*|t|^{2^*-1} + M(1) = |t|^{2^*-1}\Big(2^* + \frac{M(1)}{|t|^{2^*-1}}\Big)\,,$$

so that there exists $\delta(\varepsilon)=\delta(\sigma(\varepsilon))>0$ such that a.e. $x\in\Omega$ and for any $t\in\mathbb{R}$ with $|t|\geqslant\sigma$ we have

$$|f(x,t)|\leqslant 2^*\delta(\varepsilon)|t|^{2^*-1}\,. \tag{2.8}$$

Combining (2.7) and (2.8) it easily follows (2.5). Using the definition of F given in (1.16), we also deduce (2.6). □

2.2. **The functional setting.** Here, we give some basic results on the spaces X and X_0. The reader familiar with the fractional Sobolev spaces and their extensions (as in [10, 11]) may go directly to Section 3. In the sequel we set $Q=\mathbb{R}^{2n}\setminus\mathcal{O}$, where

$$\mathcal{O}=(\mathcal{C}\Omega)\times(\mathcal{C}\Omega)\subset\mathbb{R}^{2n}\,, \tag{2.9}$$

and $\mathcal{C}\Omega=\mathbb{R}^n\setminus\Omega$.

The space X is endowed with the norm defined as

$$\|g\|_X=\|g\|_{L^2(\Omega)}+\Big(\int_Q|g(x)-g(y)|^2K(x-y)dx\,dy\Big)^{1/2}\,. \tag{2.10}$$

It is easily seen that $\|\cdot\|_X$ is a norm on X (see, for instance, [10] for a proof).

In the following we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)}=\|g\|_{L^2(\Omega)}+\Big(\int_{\Omega\times\Omega}\frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\,. \tag{2.11}$$

We remark that, even in the model case in which $K(x)=|x|^{-(n+2s)}$, the norms in (2.10) and (2.11) are not the same, because $\Omega\times\Omega$ is strictly contained in Q (this makes the classical fractional Sobolev space approach not sufficient for studying the problem).

For further details on the fractional Sobolev spaces we refer to [6] and to the references therein.

In the next result we give some connections between the space X_0 and the usual fractional Sobolev spaces $H^s(\mathbb{R}^n)$, which will be useful along the paper.

Lemma 7. *The following assertions hold true:*

a) let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions (1.7)–(1.9). Then $X_0\subseteq H^s(\mathbb{R}^n)$ and, moreover

$$\|v\|_{H^s(\Omega)}\leqslant\|v\|_{H^s(\mathbb{R}^n)}\leqslant c(\theta)\|v\|_X\,,$$

where $c(\theta)=\max\{1,\theta^{-1/2}\}$, with θ given in (1.8);

b) let $K(x)=|x|^{-(n+2s)}$. Then

$$X_0=\big\{v\in H^s(\mathbb{R}^n): v=0 \ \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}\,.$$

Proof. Part *a*) was proved in [10, Lemma 5-*b*)]. Let us show part *b*). By assertion *a*) it is easily seen that $X_0\subseteq\big\{v\in H^s(\mathbb{R}^n): v=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}$.

Conversely, let $v\in H^s(\mathbb{R}^n)$ be such that $v=0$ a.e. in $\mathbb{R}^n\setminus\Omega$. Since $v\in H^s(\mathbb{R}^n)$, we have that

$$\int_{\mathbb{R}^n}|v(x)|^2\,dx<+\infty$$

and

$$\int_{\mathbb{R}^{2n}}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy<+\infty\,,$$

from which, using the property that $v=0$ a.e. in $\mathbb{R}^n\setminus\Omega$, we deduce

$$\int_\Omega|v(x)|^2\,dx=\int_{\mathbb{R}^n}|v(x)|^2\,dx<+\infty$$

and

$$\int_Q\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy=\int_{\mathbb{R}^{2n}}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy<+\infty\,.$$

Hence, $v \in X_0$ and this ends the proof of Lemma 7-*b*) . □

In the sequel we consider the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \Big(\int_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2} \tag{2.12}$$

and we take (2.12) as norm on X_0. The following result, proved in [10, Lemmas 6 and 7], holds true:

Lemma 8. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (1.7)–(1.9) *. Then*

a) there exists a positive constant c, depending only on n and s, such that for any $v \in X_0$

$$\|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant c \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,,$$

where 2^* is given in (1.4) *;*

b) there exists a constant $C > 1$, depending only on n, s, θ and Ω, such that for any $v \in X_0$

$$\int_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy \leqslant \|v\|^2_X \leqslant C \int_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy\,,$$

that is formula (2.12) *defines a norm on X_0 equivalent to the usual one given in* (2.10)*;*

c) $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with scalar product

$$\langle u,v\rangle_{X_0} = \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)\,dx\,dy.$$

Note that in (2.12) (and in the related scalar product, see Lemma 8-*c*)) the integral can be extended to all $\mathbb{R}^{2n}$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

2.3. **Some embeddings into the usual Lebesgue spaces.** In this subsection we discuss some results related to the embeddings of the spaces X_0 and $H^s(\mathbb{R}^n)$ into the usual Lebesgue spaces, which allow us to define the best fractional critical Sobolev constants given in formulas (1.17), (1.19), (1.21) and (1.23) .

Lemma 9. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (1.7)–(1.9) *. Then, the following assertions hold true:*

a) if Ω has a Lipschitz boundary, then the embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is compact for any $\nu \in [1, 2^*)$;

b) the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous.

Proof. Part *a*) is proved, for instance, in [10, Lemma 8]. Let us show *b*) . For this, by Lemma 8-*a*) and assumption (1.8) we get that for any $v \in X_0$

$$\begin{aligned}\|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^n)} &\leqslant c \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\leqslant \frac{c}{\theta}\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy \\ &= \frac{c}{\theta}\|v\|^2_{X_0}\,,\end{aligned}$$

for a suitable positive constant c, depending only on n and s. Hence the assertion *b*) is proved. □

Thanks to Lemma 9 we can define the constant S_K given in formula (1.17) and see that $S_K > 0$. Moreover, note that, since in formula (1.18) the integral over Ω can be extended to all $\mathbb{R}^n$ (being $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$), then the function $v \mapsto S_K(v)$ does not depend on the domain Ω, while, in general, S_K does (indeed, X_0 depends on Ω).

The counterpart of Lemma 9 in the usual fractional Sobolev spaces is given by the following result proved, for example, in [6, Theorem 6.5]:

Lemma 10. *The embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [2, 2^*]$.*

Lemma 10 allows us to define the constant S_s given in formula (1.21) and see that $S_s > 0$.

Remark 11. In the particular case when $K(x) = |x|^{-(n+2s)}$, it is easily seen that

$$\inf_{v \in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v) \leqslant \inf_{v \in X_0 \setminus\{0\}} S_K(v),$$

since $X_0 \subseteq H^s(\mathbb{R}^n)$ by Lemma 7-*a*).

As a final observation, we note that, with respect to the embeddings in the Lebesgue spaces, the fractional Sobolev space $H^s(\mathbb{R}^n)$ behaves like the usual Sobolev space $H^1(\mathbb{R}^n)$, while X_0 like $H^1_0(\Omega)$ (this is due to the fact that the functions $v \in X_0$ are such that $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and so X_0 may be seen somehow as a space of functions defined in the bounded set Ω).

Due to the presence of the term λu in the equations we are interested in, it is convenient to define another norm in the spaces X_0 and $H^s(\mathbb{R}^n)$, respectively, as follows:

$$X_0 \ni v \mapsto \|v\|_{X_0,\lambda} := \left(\int_{\mathbb{R}^{2n}} |v(x) - v(y)|^2 K(x-y)\, dx\, dy - \lambda \int_\Omega |v(x)|^2\, dx \right)^{1/2}, \tag{2.13}$$

and

$$H^s(\mathbb{R}^n) \ni v \mapsto \|v\|_{H^s(\mathbb{R}^n),\lambda} := \left(\int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy - \lambda \int_{\mathbb{R}^n} |v(x)|^2\, dx \right)^{1/2}. \tag{2.14}$$

In [11, Lemma 10] we proved the following result:

Lemma 12. *Let $K : \mathbb{R}^n\setminus\{0\} \to (0, +\infty)$ satisfy assumptions* (1.7)–(1.9) *and let $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. Then, formula* (2.13) *defines a norm on X_0 which is equivalent to the usual one given in* (2.12) *in the sense that for any $v \in X_0$*

$$\left(1 - \frac{\lambda}{\lambda_1}\right) \|v\|^2_{X_0} \leqslant \|v\|^2_{X_0,\lambda} \leqslant \|v\|^2_{X_0}. \tag{2.15}$$

As a consequence of Lemmas 9-*b*) and 12 we have the following result:

Lemma 13. *Let $K : \mathbb{R}^n\setminus\{0\} \to (0, +\infty)$ satisfy assumptions* (1.7)–(1.9) *and let $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. Then the embedding $(X_0, \|\cdot\|_{X_0,\lambda}) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous.*

Thanks to Lemma 13 we can define the constant $S_{K,\lambda}$ as in formula (1.19) and get that $S_{K,\lambda} > 0$. Also in this case we note that, since in formula (1.20) the integral over Ω can be extended to all $\mathbb{R}^n$ (being $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$), then the function $v \mapsto S_{K,\lambda}(v)$ does not depend on the domain Ω, while, in general, $S_{K,\lambda}$ does.

The counterpart of Lemmas 12 and 13 in the usual fractional Sobolev spaces is given by the following result:

Lemma 14. *Let $\lambda \in (0, \lambda_{1,s})$, where $\lambda_{1,s}$ is the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary conditions. Then formula* (2.14) *defines a norm on $H^s(\mathbb{R}^n)$ equivalent to the usual one given in* (2.11).

Moreover, the embedding $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n),\lambda}) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous.

Proof. To see that formula (2.14) defines a norm on $H^s(\mathbb{R}^n)$ equivalent to the usual one, it is enough to argue as in the proof of [11, Lemma 10].

The continuity of the embedding $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n),\lambda}) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ comes from the fact that the norms (2.11) and (2.14) are equivalent and by Lemma 10. $\square$

Thanks to Lemma 14 we can define the constant $S_{s,\lambda}$ given in formula (1.23), and conclude that $S_{s,\lambda} > 0$.

Remark 15. In the model case when $K(x) = |x|^{-(n+2s)}$, it is easy to prove that

$$\inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_{s,\lambda}(v) \leqslant \inf_{v\in X_0\setminus\{0\}} S_{K,\lambda}(v)\,,$$

since $X_0 \subseteq H^s(\mathbb{R}^n)$ by Lemma 7-*a*).

Remark 16. In the case when $\lambda = 0$, we get that

$$S_{K,\lambda} = S_K \qquad \text{and} \qquad S_{s,\lambda} = S_s\,,$$

while for $\lambda > 0$ we have that

$$S_{K,\lambda} \leqslant S_K \qquad \text{and} \qquad S_{s,\lambda} \leqslant S_s\,.$$

2.4. **An eigenvalue problem.** Here we consider the following eigenvalue problem driven by the non-local integrodifferential operator $-\mathcal{L}_K$

$$\begin{cases} -\mathcal{L}_K u = \lambda\, u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{2.16}$$

where $s \in (0,1)$, $n > 2s$, Ω is an open bounded set of $\mathbb{R}^n$ and $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function satisfying (1.7)–(1.9).

More precisely, we are interested in the weak formulation of (2.16), which consists in the following eigenvalue problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy = \lambda \int_\Omega u(x)\varphi(x)\,dx \quad \forall\, \varphi \in X_0 \\ u \in X_0\,. \end{cases} \tag{2.17}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\mathcal{L}_K$ provided there exists a non-trivial solution $u \in X_0$ of problem (2.16) – in fact, of its weak formulation (2.17) – and, in this case, any solution will be called an eigenfunction corresponding to the eigenvalue λ.

For the purpose of this paper we need only the properties of the first eigenvalue of $-\mathcal{L}_K$ and of its eigenfunction, that is the following result, whose proof can be found in [11, Proposition 9 and Appendix A], where a complete study of the spectrum of $-\mathcal{L}_K$ is presented:

Proposition 17 (First eigenvalue and first eigenfunction of $-\mathcal{L}_K$)**.** *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded set of $\mathbb{R}^n$ and let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying assumptions* (1.7)–(1.9)*. Then,*

a) problem (2.17) *admits an eigenvalue λ_1 which is positive and that can be characterized as follows*

$$\lambda_1 = \min_{\substack{u\in X_0 \\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy\,, \tag{2.18}$$

or, equivalently,

$$\lambda_1 = \min_{u\in X_0\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}\,; \tag{2.19}$$

b) there exists a non-negative function $e_1 \in X_0$*, which is an eigenfunction corresponding to* λ_1*, attaining the minimum in* (2.18)*, that is* $\|e_1\|_{L^2(\Omega)} = 1$ *and*

$$\lambda_1 = \int_{\mathbb{R}^{2n}} |e_1(x) - e_1(y)|^2 K(x-y)\,dx\,dy\,; \tag{2.20}$$

c) λ_1 *is simple, that is if* $u \in X_0$ *is a solution of the following equation*

$$\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy = \lambda_1 \int_\Omega u(x)\varphi(x)\,dx \quad \forall\, \varphi \in X_0\,, \tag{2.21}$$

then $u = \zeta e_1$*, with* $\zeta \in \mathbb{R}$*.*

3. A general critical integrodifferential equation

In this section we study problem (1.13) which is the Euler-Lagrange equation of the functional $\mathcal{J}_{K,\lambda}$ defined in (1.15). Notice that this functional is well defined thanks to assumptions (1.10) and (1.12), to the fact that Ω is bounded, and to Lemma 9-*b*). Moreover, $\mathcal{J}_{K,\lambda}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned}\langle \mathcal{J}'_{K,\lambda}(u), \varphi\rangle = \int_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ - \int_\Omega |u(x)|^{2^*-1}(x)\varphi(x)\,dx - \int_\Omega f(x,u(x))\varphi(x)\,dx\,.\end{aligned}$$

Thus, critical points of $\mathcal{J}_{K,\lambda}$ are solutions to problem (1.13). In order to find these critical points, we will apply a variant of the Mountain Pass Theorem without the Palais-Smale condition, as given in [1] (see also [3, Theorem 2.2]).

Indeed, here we can not apply the classical Mountain Pass Theorem, since, due to the lack of compactness in the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ (see Lemma 9-*b*)), the functional $\mathcal{J}_{K,\lambda}$ does not verify the Palais-Smale condition globally, but only in an energy range determined by the best fractional critical Sobolev constant S_K given in formula (1.17) (see also Lemma 9-*b*)).

This variant of the Mountain Pass Theorem requires that the functional $\mathcal{J}_{K,\lambda}$ has a suitable *geometric structure*, as stated, e.g., in conditions (2.9) and (2.10) of [3, Theorem 2.2].

3.1. **The critical case with a lower order perturbation.**

This subsection is devoted to problem (1.13) and to the proof of Theorem 1.

First of all, we prove that the functional $\mathcal{J}_{K,\lambda}$ has the geometric features required by [3, Theorem 2.2].

Proposition 18. *Let* $\lambda \in (0, \lambda_1)$ *and let* f *be a Carathéodory function satisfying conditions* (1.10)–(1.12)*. Then, there exist* $\rho > 0$ *and* $\beta > 0$ *such that for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$ *it results that* $\mathcal{J}_{K,\lambda}(u) \geqslant \beta$*.*

Proof. Let u be a function in X_0. By (2.6) we get that for any $\varepsilon > 0$

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx \\
&\qquad - \varepsilon\int_\Omega |u(x)|^2 dx - \delta(\varepsilon)\int_\Omega |u(x)|^{2^*}dx \\
&\geqslant \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \varepsilon\|u\|^2_{L^2(\Omega)} \\
&\qquad - \Big(\frac{1}{2^*}+\delta(\varepsilon)\Big)\|u\|^{2^*}_{L^{2^*}(\Omega)} \\
&\geqslant \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \varepsilon|\Omega|^{(2^*-2)/2^*}\|u\|^2_{L^{2^*}(\Omega)} \\
&\qquad - \Big(\frac{1}{2^*}+\delta(\varepsilon)\Big)\|u\|^{2^*}_{L^{2^*}(\Omega)}\,,
\end{aligned} \tag{3.1}$$

thanks to Lemma 12 (here we need $0 < \lambda < \lambda_1$) and to the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously (being Ω bounded and $2 < 2^*$).

Using (1.8) and Lemma 8-*a*)-*b*), we deduce from (3.1) that for any $\varepsilon > 0$

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&\quad - \varepsilon c|\Omega|^{(2^*-2)/2^*}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \Big(\frac{1}{2^*}+\delta(\varepsilon)\Big)c^{2^*/2}\Big(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{2^*/2} \\
&\geqslant \Big[\frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big) - \frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\Big]\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&\quad - \Big(\frac{1}{2^*}+\delta(\varepsilon)\Big)\Big(\frac{c}{\theta}\Big)^{2^*/2}\Big(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy\Big)^{2^*/2} \\
&= \Big[\frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big) - \frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\Big]\|u\|^2_{X_0} - \Big(\frac{1}{2^*}+\delta(\varepsilon)\Big)\Big(\frac{c}{\theta}\Big)^{2^*/2}\|u\|^{2^*}_{X_0}\,.
\end{aligned} \tag{3.2}$$

Choosing $\varepsilon > 0$ such that $2\varepsilon c|\Omega|^{(2^*-2)/2^*} < \theta(1-\lambda/\lambda_1)$, by (3.2) it easily follows that

$$\mathcal{J}_{K,\lambda}(u) \geqslant \alpha\|u\|^2_{X_0}\Big(1-\kappa\|u\|^{2^*-2}_{X_0}\Big)\,,$$

for suitable positive constants α and κ.

Now, let $u \in X_0$ be such that $\|u\|_{X_0} = \rho > 0$. Since $2^* > 2$, we can choose ρ sufficiently small (i.e. ρ such that $1-\kappa\rho^{2^*-2} > 0$), so that

$$\inf_{\substack{u\in X_0\\ \|u\|_{X_0}=\rho}} \mathcal{J}_{K,\lambda}(u) \geqslant \alpha\rho^2(1-\kappa\rho^{2^*-2}) =: \beta > 0\,.$$

Hence, Proposition 18 is proved. □

Proposition 19. *Let $\lambda \in (0,\lambda_1)$ and let f be a Carathéodory function satisfying conditions* (1.10)–(1.12). *Then, there exists $e \in X_0$ such that $e \geqslant 0$ a.e. in $\mathbb{R}^n$, $\|e\|_{X_0} > \rho$ and $\mathcal{J}_{K,\lambda}(e) < \beta$, where ρ and β are given in Proposition 18.*

In particular, if we assume condition (1.25), *we can construct e as follows*

$$e = \zeta_0 u_0 \tag{3.3}$$

with u_0 as in (1.25) *and $\zeta_0 > 0$ large enough.*

Proof. We fix $u \in X_0$ such that $\|u\|_{X_0} = 1$ and $u \geqslant 0$ a.e. in $\mathbb{R}^n$: we remark that this choice is possible thanks to (1.14) (alternatively, one can replace any $u \in X_0$ with its positive part, which belongs to X_0 too, thanks to [9, Lemma 12]).

Now, let $\zeta > 0$. By Lemma 12 (here we use again the fact that $\lambda < \lambda_1$) and Lemma 5 (see formula (2.2) used here with $\varepsilon = 1/(2 \cdot 2^*)$) we have

$$\begin{aligned}\mathcal{J}_{K,\lambda}(\zeta u) &= \frac{\zeta^2}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\zeta^2 \int_\Omega |u(x)|^2\,dx \\ &\qquad - \frac{\zeta^{2^*}}{2^*}\int_\Omega |u(x)|^{2^*}\,dx - \int_\Omega F(x,\zeta u(x))\,dx \\ &\leqslant \frac{\zeta^2}{2} - \Big(\frac{1}{2^*} - \frac{1}{2\cdot 2^*}\Big)\zeta^{2^*}\int_\Omega |u(x)|^{2^*}\,dx + M\big(1/(2\cdot 2^*)\big)\int_\Omega |\zeta u(x)|\,dx \\ &= \frac{\zeta^2}{2} - \frac{\zeta^{2^*}}{2\cdot 2^*}\int_\Omega |u(x)|^{2^*}\,dx + \zeta M\big(1/(2\cdot 2^*)\big)\int_\Omega |u(x)|\,dx\end{aligned}$$

Since $2^* > 2 > 1$, passing to the limit as $\zeta \to +\infty$, we get that $\mathcal{J}_{K,\lambda}(\zeta u) \to -\infty$, so that the assertion follows taking $e = \zeta u$, with ζ sufficiently large.

In particular, under condition (1.25), we can take $u = u_0/\|u_0\|_{X_0} \in X_0$ (note that $u_0 \not\equiv 0$ by assumption) so that we can choose $e = \zeta_0 u_0$ with ζ_0 large enough. □

3.1.1. *End of the proof of Theorem 1.* Propositions 18 and 19 give that the geometry of the variant of the Mountain Pass Theorem stated in [3, Theorem 2.2] is fulfilled by $\mathcal{J}_{K,\lambda}$. Moreover, since $F(0) = 0$, we easily get that $\mathcal{J}_{K,\lambda}(0) = 0 < \beta$, with β given in Proposition 18. Now, set

$$c = \inf_{P\in\mathcal{P}} \sup_{v\in P([0,1])} \mathcal{J}_{K,\lambda}(v)\,, \tag{3.4}$$

where

$$\mathcal{P} = \Big\{P \in C([0,1];X_0) : P(0) = 0\,,\ P(1) = e\Big\}$$

with $e = \zeta_0 u_0$ given in Proposition 19. In order to prove Theorem 1 we proceed by steps.

Claim 1. *The constant c given in* (3.4) *is such that*

$$\beta \leqslant c < \frac{s}{n} S_K^{n/(2s)}\,,$$

where β is given in Proposition 18 and S_K is defined in formula (1.17) *(see also Lemma 9-b)).*

Proof. We observe that, for any $P \in \mathcal{P}$, the function $t \mapsto \|P(t)\|_{X_0}$ is continuous in $[0,1]$, that $\|P(0)\|_{X_0} = \|0\|_{X_0} = 0 < \rho$, and that $\|P(1)\|_{X_0} = \|e\|_{X_0} > \rho$, with ρ as in Proposition 18. Accordingly, there exists $\bar t \in (0,1)$ such that $\|P(\bar t)\|_{X_0} = \rho$. Thus,

$$\sup_{v\in P([0,1])} \mathcal{J}_{K,\lambda}(v) \geqslant \mathcal{J}_{K,\lambda}(P(\bar t)) \geqslant \inf_{\substack{v\in X_0\\ \|v\|_{X_0}=\rho}} \mathcal{J}_{K,\lambda}(v)\,.$$

As a consequence $c \geqslant \beta$, where β is given in Proposition 18.

Now, we recall that $e = \zeta_0 u_0$, thanks to (3.3), and so the map $[0,1] \ni t \mapsto t\,\zeta_0 u_0$ belongs to $\mathcal{P}$. Hence, by assumption (1.25), we conclude that

$$\inf_{P\in\mathcal{P}} \sup_{v\in P([0,1])} \mathcal{J}_{K,\lambda}(v) \leqslant \sup_{\zeta\geqslant 0} \mathcal{J}_{K,\lambda}(\zeta u_0) < \frac{s}{n} S_K^{n/(2s)}$$

so that

$$c < \frac{s}{n} S_K^{n/(2s)}\,. \tag{3.5}$$

Hence, Claim 1 is proved. □

By [3, Theorem 2.2] there exists a sequence u_j in X_0 such that

$$\mathcal{J}_{K,\lambda}(u_j) \to c \tag{3.6}$$

and

$$\sup\Big\{\big|\langle\, \mathcal{J}'_{K,\lambda}(u_j), \varphi\,\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \tag{3.7}$$

as $j \to +\infty$. Now we prove some properties of the sequence u_j.

Claim 2. *The sequence u_j is bounded in X_0.*

Proof. For any $j \in \mathbb{N}$ by (3.6) and (3.7) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_{K,\lambda}(u_j)| \leqslant \kappa\,, \tag{3.8}$$

and

$$\Big|\langle \mathcal{J}'_{K,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa\,. \tag{3.9}$$

As a consequence of (3.8) and (3.9) we have

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \leqslant \kappa\,(1 + \|u_j\|_{X_0})\,. \tag{3.10}$$

By Hölder inequality, Lemma 8-a) and (1.8)

$$\int_\Omega |u_j(x)|\,dx \leqslant |\Omega|^{(2^*-1)/2^*}\left(\int_\Omega |u_j(x)|^{2^*}\,dx\right)^{1/2^*} \leqslant \widetilde{\kappa}|\Omega|^{(2^*-1)/2^*}\|u_j\|_{X_0}\,, \tag{3.11}$$

for some positive constant $\widetilde{\kappa}$. Accordingly, by Lemma 5 and (3.11) we obtain

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle &= -\left(\frac{1}{2^*} - \frac{1}{2}\right)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} - \int_\Omega F(x, u_j(x))\,dx \\ &\qquad + \frac{1}{2}\int_\Omega f(x, u_j(x))\,u_j(x)\,dx \\ &\geqslant \frac{s}{n}\,\|u_j\|^{2^*}_{L^{2^*}(\Omega)} - \varepsilon\Big(1 + \frac{2^*}{2}\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \\ &\qquad - \frac{3M(\varepsilon)}{2}\int_\Omega |u_j(x)|\,dx \\ &\geqslant \frac{s}{n}\,\|u_j\|^{2^*}_{L^{2^*}(\Omega)} - \varepsilon\Big(1 + \frac{2^*}{2}\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \\ &\qquad - C_\varepsilon\|u_j\|_{X_0}\,,\end{aligned} \tag{3.12}$$

for a suitable $C_\varepsilon > 0$, possibly depending on $|\Omega|$. Choosing $\varepsilon > 0$ small enough (i.e. ε such that $\frac{s}{n} > \varepsilon\Big(1 + \frac{2^*}{2}\Big)$), (3.10) and (3.12) imply that, for any $j \in \mathbb{N}$

$$\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \leqslant \kappa_*\,(1 + \|u_j\|_{X_0}) \tag{3.13}$$

for a suitable positive constant κ_*.

Finally, by Lemma 12 (which holds true, since $0 < \lambda < \lambda_1$) and Lemma 5 (used here with $\varepsilon = 1$) it follows that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j) &\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_j\|^2_{X_0} - \frac{1}{2^*}\|u_j\|^{2^*}_{L^{2^*}(\Omega)} - \int_\Omega F(x, u_j(x))\,dx \\ &\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_j\|^2_{X_0} - \Big(\frac{1}{2^*} + 1\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} - M(1)\int_\Omega |u_j(x)|\,dx\,.\end{aligned} \tag{3.14}$$

Combining (3.8), (3.13) and (3.14) we conclude that

$$\begin{aligned}\kappa \geqslant \mathcal{J}_{K,\lambda}(u_j) &\geqslant \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)\|u_j\|^2_{X_0} - \Big(\frac{1}{2^*}+1\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} - M(1)\int_\Omega |u_j(x)|\,dx \\ &\geqslant \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)\|u_j\|^2_{X_0} - \kappa_*\Big(\frac{1}{2^*}+1\Big)\left(1+\|u_j\|_{X_0}\right) - M(1)\int_\Omega |u_j(x)|\,dx\,,\end{aligned}$$

so that, recalling (3.11), we see that, for any $j\in\mathbb{N}$

$$\|u_j\|^2_{X_0} \leqslant \kappa_{**}\left(1+\|u_j\|_{X_0}\right)$$

for a suitable positive constant κ_{**}. Hence, the proof of Claim 2 is complete. □

Claim 3. *Problem* (1.13) *admits a solution* $u_\infty\in X_0$.

Proof. Since u_j is bounded in X_0 (thanks to Claim 2) and X_0 is a reflexive space (being a Hilbert space, by Lemma 8-*c*)), up to a subsequence, still denoted by u_j, there exists $u_\infty\in X_0$ such that $u_j\to u_\infty$ weakly in X_0, that is

$$\begin{aligned}&\int_{\mathbb{R}^{2n}}\big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \to \\ &\qquad\int_{\mathbb{R}^{2n}}\big(u_\infty(x)-u_\infty(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \quad \text{for any } \varphi\in X_0\end{aligned} \tag{3.15}$$

as $j\to+\infty$. Moreover, since u_j is bounded in X_0, we deduce from (3.13) that

$$u_j \text{ is bounded in } L^{2^*}(\Omega)\,. \tag{3.16}$$

Consequently, by Lemma 9-*b*) and the fact that $L^{2^*}(\mathbb{R}^n)$ is a reflexive space we have that, up to a subsequence

$$u_j\to u_\infty \quad \text{weakly in } L^{2^*}(\mathbb{R}^n) \tag{3.17}$$

as $j\to+\infty$, while by Lemma 9-*a*), up to a subsequence,

$$u_j\to u_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \tag{3.18}$$

$$u_j\to u_\infty \quad \text{a.e. in } \mathbb{R}^n \tag{3.19}$$

as $j\to+\infty$ and there exists $\ell\in L^\nu(\mathbb{R}^n)$ such that

$$|u_j(x)|\leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j\in\mathbb{N} \tag{3.20}$$

for any $\nu\in[1,2^*)$ (see, for instance [2, Theorem IV.9]).

By (3.17) and the fact that $|u_j|^{2^*-2}u_j$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$ we see that

$$|u_j|^{2^*-2}u_j\to|u_\infty|^{2^*-2}u_\infty \quad \text{weakly in } L^{2^*/(2^*-1)}(\Omega) \tag{3.21}$$

as $j\to+\infty$.

Furthermore, by (3.16) and Lemma 5 (see formula (2.1)) and the fact that Ω is bounded we get that

$$f(\cdot,u_j(\cdot)) \text{ is bounded in } L^{2^*/(2^*-1)}(\Omega)\,.$$

More precisely, by (3.19), (3.20), the fact that the map $t\mapsto f(\cdot,t)$ is continuous in $t\in\mathbb{R}$, (2.1) and the Dominated Convergence Theorem applied in $L^{2^*/(2^*-1)}(\Omega)$ (note that $2^*/(2^*-1)<2^*$, being $2<2^*$) we get

$$f(\cdot,u_j(\cdot))\to f(\cdot,u_\infty(\cdot)) \quad \text{in} \quad L^{2^*/(2^*-1)}(\Omega) \tag{3.22}$$

as $j\to+\infty$. Since the dual of $L^{2^*/(2^*-1)}(\Omega)$ is $L^{2^*}(\Omega)$, by using (3.22) it is easily seen that

$$\int_\Omega f(x,u_j(x))\varphi(x)\,dx\to\int_\Omega f(x,u_\infty(x))\varphi(x)\,dx \quad \text{for any} \quad \varphi\in L^{2^*}(\Omega)\,,$$

and so, in particular,

$$\int_\Omega f(x,u_j(x))\varphi(x)\,dx\to\int_\Omega f(x,u_\infty(x))\varphi(x)\,dx \quad \text{for any} \quad \varphi\in X_0 \tag{3.23}$$

as $j\to+\infty$ (here we use Lemma 9-b)).

Since (3.7) holds true, for any $\varphi\in X_0$

$$0\leftarrow\langle\mathcal{J}'_{K,\lambda}(u_j),\varphi\rangle=\int_{\mathbb{R}^{2n}}\big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy-\lambda\int_\Omega u_j(x)\varphi(x)\,dx$$
$$-\int_\Omega|u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx-\int_\Omega f(x,u_j(x))\varphi(x)\,dx\,,$$

so that, passing to the limit in this expression as $j\to+\infty$ and taking into account (3.15), (3.18), (3.21) and (3.23) we get

$$\int_{\mathbb{R}^{2n}}\big(u_\infty(x)-u_\infty(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy-\lambda\int_\Omega u_\infty(x)\varphi(x)\,dx$$
$$-\int_\Omega|u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx-\int_\Omega f(x,u_\infty(x))\varphi(x)\,dx=0$$

for any $\varphi\in X_0$, that is u_∞ is a solution of problem (1.26) and Claim 3 follows. □

Claim 4. *The solution u_∞ is non-trivial, i.e. $u_\infty\not\equiv0$ in Ω.*

Proof. Suppose, by contradiction, that $u_\infty\equiv0$ in Ω (and so $u_\infty\equiv0$ in $\mathbb{R}^n$, being $u_\infty\in X_0$).

Using Lemma 5 and (3.16), we see that for any $\varepsilon>0$ and $j\in\mathbb{N}$

$$\begin{aligned}\left|\int_\Omega f(x,u_j(x))u_j(x)\,dx\right|&\leqslant 2^*\varepsilon\int_\Omega|u_j(x)|^{2^*}dx+M(\varepsilon)\int_\Omega|u_j(x)|\,dx\\&\leqslant 2^*\varepsilon C+M(\varepsilon)\|u_j\|_{L^1(\Omega)}\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}\left|\int_\Omega F(x,u_j(x))\,dx\right|&\leqslant \varepsilon\int_\Omega|u_j(x)|^{2^*}dx+M(\varepsilon)\int_\Omega|u_j(x)|\,dx\\&\leqslant \varepsilon C+M(\varepsilon)\,\|u_j\|_{L^1(\Omega)}\,.\end{aligned}\tag{3.25}$$

Keeping ε fixed and taking the limit in j, we obtain

$$\limsup_{j\to+\infty}\left|\int_\Omega f(x,u_j(x))u_j(x)\,dx\right|\leqslant 2^*\varepsilon C$$
$$\text{and}\qquad\limsup_{j\to+\infty}\left|\int_\Omega F(x,u_j(x))\,dx\right|\leqslant\varepsilon C\,,$$

due to (3.18). Hence, by sending $\varepsilon\to0^+$, we obtain

$$\int_\Omega f(x,u_j(x))u_j(x)\,dx\to0\tag{3.26}$$

and

$$\int_\Omega F(x,u_j(x))\,dx\to0\tag{3.27}$$

as $j\to+\infty$.

Moreover, being u_j bounded in X_0 by Claim 2, from (3.7) it follows that

$$0\leftarrow\langle\mathcal{J}'_{K,\lambda}(u_j),u_j\rangle=\int_{\mathbb{R}^{2n}}|u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy-\lambda\int_\Omega|u_j(x)|^2\,dx$$
$$-\int_\Omega|u_j(x)|^{2^*}dx-\int_\Omega f(x,u_j(x))u_j(x)\,dx\,,$$

which, thanks to (3.18) and (3.26), gives

$$\int_{\mathbb{R}^{2n}}|u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy-\int_\Omega|u_j(x)|^{2^*}dx\to0\tag{3.28}$$

as $j\to+\infty$.

Now, by Claim 2 the sequence $\|u_j\|_{X_0}$ is bounded in $\mathbb{R}$. Hence, up to a subsequence, if necessary, we can assume that

$$\|u_j\|^2_{X_0} = \int_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)\,dx\,dy \to L \tag{3.29}$$

and so, as a consequence of (3.28)

$$\int_\Omega |u_j(x)|^{2^*}\,dx \to L \tag{3.30}$$

as $j \to +\infty$. Of course $L \in [0, +\infty)$. Furthermore, by (3.6) we have that, as $j \to +\infty$,

$$\frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u_j(x)|^2\,dx \\ - \frac{1}{2^*}\int_\Omega |u_j(x)|^{2^*}\,dx - \int_\Omega F(x, u_j(x))\,dx \to c\,,$$

so that, using (3.18) with $u_\infty \equiv 0$ in $\mathbb{R}^n$, (3.27), (3.29) and (3.30), it follows

$$c = \Big(\frac{1}{2} - \frac{1}{2^*}\Big)L = \frac{s}{n}\,L\,. \tag{3.31}$$

Since $c \geqslant \beta > 0$ by Claim 1, it is easily seen that $L > 0$.

Moreover, by Lemma 9-b) and (1.17)

$$\int_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)\,dx\,dy \geqslant S_K \|u_j\|^2_{L^{2^*}(\Omega)}\,,$$

so that, passing to the limit as $j \to +\infty$, and taking into account (3.29) and (3.30), we get

$$L \geqslant S_K L^{2/2^*}\,,$$

which, combined with (3.31) gives

$$c \geqslant \frac{s}{n}\,S_K^{2^*/(2^*-2)} = \frac{s}{n}\,S_K^{n/(2s)}\,.$$

This contradicts Claim 1. Hence, $u_\infty \not\equiv 0$ in Ω and this ends the proof of Claim 4.

The proof of Theorem 1 is complete. □

3.2. **The critical case $\mathcal{L}_K u + \lambda u + |u|^{2^*-2}u = 0$.** In this subsection we consider problem (1.13) with $f \equiv 0$, that is we study (1.26) and we prove Theorem 2.

In order to do this it is enough to show that, when $f \equiv 0$, conditions (1.25) and (1.27) are equivalent. For this we need the following proposition:

Proposition 20. *For any $u_0 \in X_0 \setminus \{0\}$,*

$$\sup_{\zeta \geqslant 0}\Big(\frac{\zeta^2}{2}\int_{\mathbb{R}^{2n}} |u_0(x) - u_0(y)|^2 K(x-y)\,dx\,dy - \frac{\zeta^2\lambda}{2}\int_\Omega |u_0(x)|^2\,dx \\ -\frac{\zeta^{2^*}}{2^*}\int_\Omega |u_0(x)|^{2^*}\,dx\Big) = \frac{s}{n}\,S_{K,\lambda}^{n/(2s)}(u_0)\,, \tag{3.32}$$

where the function $X_0 \setminus \{0\} \ni v \mapsto S_{K,\lambda}(v)$ is defined in (1.20).

Proof. Let $g : [0, +\infty) \to \mathbb{R}$ be the following function

$$g(\zeta) = \frac{\zeta^2}{2}\int_{\mathbb{R}^{2n}} |u_0(x) - u_0(y)|^2 K(x-y)\,dx\,dy - \frac{\zeta^2\lambda}{2}\int_\Omega |u_0(x)|^2\,dx - \frac{\zeta^{2^*}}{2^*}\int_\Omega |u_0(x)|^{2^*}\,dx\,.$$

Note that g is differentiable in $(0, +\infty)$ and

$$g'(\zeta) = \zeta\int_{\mathbb{R}^{2n}} |u_0(x) - u_0(y)|^2 K(x-y)\,dx\,dy - \zeta\lambda\int_\Omega |u_0(x)|^2\,dx - \zeta^{2^*-1}\int_\Omega |u_0(x)|^{2^*}\,dx\,,$$

so that $g'(\zeta) \geqslant 0$ if and only if

$$\zeta \leqslant \bar{\zeta} = \left(\frac{\displaystyle\int_{\mathbb{R}^{2n}} |u_0(x) - u_0(y)|^2 K(x-y)\, dx\, dy - \lambda \int_\Omega |u_0(x)|^2\, dx}{\displaystyle\int_\Omega |u_0(x)|^{2^*}\, dx} \right)^{1/(2^*-2)}.$$

Therefore $\bar{\zeta}$ is a maximum for g and

$$\sup_{\zeta \geqslant 0} g(\zeta) = \max_{\zeta \geqslant 0} g(\zeta) = g(\bar{\zeta}) = \frac{s}{n}\, S_{K,\lambda}^{n/(2s)}(u_0)\,.$$

This concludes the proof of Proposition 20 . □

3.2.1. *End of the proof of Theorem 2.* By Proposition 20 we deduce that, when $f \equiv 0$ in $\Omega \times \mathbb{R}$, condition (1.25) reads as follows

$$\begin{aligned} &\text{there exists } u_0 \in X_0 \setminus \{0\} \text{ with } u_0 \geqslant 0 \text{ a.e. in } \mathbb{R}^n\,, \\ &\qquad\qquad \text{such that } S_{K,\lambda}(u_0) < S_K\,. \end{aligned} \tag{3.33}$$

It is easily seen that conditions (3.33) and (1.27) are equivalent. Indeed, suppose that (3.33) holds true. Then,

$$S_{K,\lambda} := \inf_{v \in X_0 \setminus \{0\}} S_{K,\lambda}(v) \leqslant S_{K,\lambda}(u_0) < S_K\,. \tag{3.34}$$

While, if (1.27) is valid, then there exists $\widetilde{v} \in X_0 \setminus \{0\}$ such that

$$S_{K,\lambda}(\widetilde{v}) < S_K\,. \tag{3.35}$$

Thus, condition (3.33) follows by taking $u_0 = |\widetilde{v}| \in X_0$: we remark that this choice is possible since for any $v \in X_0$ its positive part $v^+ = \max\{v, 0\}$ and its negative part $v^- = \max\{-v, 0\}$ belong to X_0 too, by [9, Lemma 12] (hence, $|v| = v^+ + v^- \in X_0$, being X_0 a linear space). Indeed, by triangle inequality, a.e. $x, y \in \mathbb{R}^n$

$$\Big| |\widetilde{v}(x)| - |\widetilde{v}(y)| \Big| \leqslant |\widetilde{v}(x) - \widetilde{v}(y)|.$$

This and (3.35) give

$$S_{K,\lambda}(u_0) = S_{K,\lambda}(|\widetilde{v}|) \leqslant S_{K,\lambda}(\widetilde{v}) < S_K\,.$$

Hence, conditions (3.33) and (1.27) are equivalent and so Theorem 2 comes from Theorem 1 .

4. A critical fractional Laplace equation

Aim of this section is to consider the particular fractional kernel

$$K(x) = |x|^{-(n+2s)},$$

which gives rise to the fractional Laplace operator $-(-\Delta)^s$ defined is (1.3) . Hence, here we study

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2} u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

which reduces, when $f \equiv 0$ in $\Omega \times \mathbb{R}$, to problem (1.2) . More precisely, we are interested in the weak formulation of these problems, which is given by (1.28) and (1.31), respectively.

4.1. **The critical case for $(-\Delta)^s$ with a lower order perturbation: proof of Theorem 3.** In order to prove Theorem 3 it is enough to note that, in this setting, condition (1.30) implies (1.25), thanks to Lemma 7-*b*) and Remark 11 .

Hence, Theorem 3 follows from Theorem 1 and from the fact that $X_0 \subseteq H^s(\mathbb{R}^n)$ by Lemma 7-*a*) (alternatively, we can use Lemma 7-*b*), since the solution u provided by Theorem 1 is such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$) .

4.2. **The critical case** $(-\Delta)^s u - \lambda u = |u|^{2^*-2}u$**: proof of Theorem 4.** The idea consists of applying Theorem 3 with $f \equiv 0$ in $\Omega \times \mathbb{R}$.

First of all, note that by Proposition 20 with $K(x) = |x|^{-(n+2s)}$ – see also Lemma 7-*b*) – we have that if $u_0 \in H^s(\mathbb{R}^n) \setminus \{0\}$, with $u_0 = 0$ a.e. outside Ω, then

$$\begin{aligned} \sup_{\zeta \geqslant 0} \Bigg(\frac{\zeta^2}{2} \int_{\mathbb{R}^{2n}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{+2s}} \, dx\, dy - \frac{\zeta^2 \lambda}{2} \int_\Omega |u_0(x)|^2 \, dx \\ - \frac{\zeta^{2^*}}{2^*} \int_\Omega |u_0(x)|^{2^*} \, dx \Bigg) = \frac{s}{n} S_{s,\lambda}^{n/(2s)}(u_0)\,, \end{aligned} \tag{4.1}$$

where the function $H^s(\mathbb{R}^n) \setminus \{0\} \ni v \mapsto S_{s,\lambda}(v)$ is defined in (1.24).

Therefore, when $f \equiv 0$ in $\Omega \times \mathbb{R}$, condition (1.30) reads as follows

$$\begin{gathered} \text{there exists } u_0 \in H^s(\mathbb{R}^n) \setminus \{0\} \text{ with } u_0 \geqslant 0 \text{ a.e. in } \mathbb{R}^n \text{ and } u_0 = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\,, \\ \text{such that } S_{s,\lambda}(u_0) < S_s\,. \end{gathered} \tag{4.2}$$

Hence, in order to prove Theorem 4 it is enough to show that condition (4.2) is satisfied for some appropriate function u_0. For this, we proceed again by steps.

Claim 5. *The infimum in formula* (1.21) *is attained, that is*

$$S_s = S_s(\widetilde{u})\,,$$

where

$$\widetilde{u}(x) = \kappa \Big(\mu^2 + |x - x_0|^2\Big)^{-(n-2s)/2}, \quad x \in \mathbb{R}^n \tag{4.3}$$

with $\kappa \in \mathbb{R} \setminus \{0\}$, $\mu > 0$ *and* $x_0 \in \mathbb{R}^n$ *fixed constants.*

Equivalently, the function $\bar{u}$ *defined as*

$$\bar{u}(x) = \frac{\widetilde{u}(x)}{\|\widetilde{u}\|_{L^{2^*}(\mathbb{R}^n)}} \tag{4.4}$$

is such that

$$S_s = \inf_{\substack{v \in H^s(\mathbb{R}^n) \\ \|v\|_{L^{2^*}(\mathbb{R}^n)} = 1}} \int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx\, dy = \int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+2s}} \, dx\, dy\,. \tag{4.5}$$

Proof. This assertion comes from [5, Theorem 1.1]. □

As in the classical case of the Laplacian, starting from $\bar{u}$ we can construct an explicit solution of the limiting problem

$$(-\Delta)^s u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n \tag{4.6}$$

as we show in the following result:

Claim 6. *The function*

$$u^*(x) = \bar{u}\left(\frac{x}{S_s^{1/(2s)}}\right), \qquad x \in \mathbb{R}^n \tag{4.7}$$

is a solution of problem (4.6) *satisfying the property*

$$\|u^*\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = S_s^{n/(2s)}\,. \tag{4.8}$$

Proof. By Claim 5 there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^{2n}} \frac{(\bar{u}(x) - \bar{u}(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx\, dy = \gamma \int_{\mathbb{R}^n} |\bar{u}(x)|^{2^*-2} \bar{u}(x) \varphi(x) dx \tag{4.9}$$

for any $\varphi \in H^s(\mathbb{R}^n)$.

Taking $\varphi = \bar u$ as a test function in (4.9) and using again Claim 5 we get

$$S_s = \int_{\mathbb{R}^{2n}} \frac{|\bar u(x) - \bar u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \gamma\,, \tag{4.10}$$

since $\|\bar u\|_{L^{2^*}(\mathbb{R}^n)} = 1$ by construction (see (4.4)).

Let u^* be the function defined as in (4.7). Note that $u^* \in H^s(\mathbb{R}^n) \setminus \{0\}$. Moreover, for any $\varphi \in H^s(\mathbb{R}^n)$, by (4.9) and (4.10) we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \frac{(u^*(x) - u^*(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&\qquad = S_s^{(n-2s)/(2s)} \int_{\mathbb{R}^{2n}} \frac{(\bar u(x) - \bar u(y))\big(\varphi(S_s^{1/(2s)}x) - \varphi(S_s^{1/(2s)}y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\
&\qquad = S_s^{n/(2s)} \int_{\mathbb{R}^n} |\bar u(x)|^{2^*-2} \bar u(x) \varphi(S_s^{1/(2s)}x) dx \\
&\qquad = \int_{\mathbb{R}^n} |u^*(x)|^{2^*-2} u^*(x) \varphi(x) dx\,,
\end{aligned} \tag{4.11}$$

that is u^* is a solution of problem (4.6).

Finally, taking $\varphi = u^*$ as a test function in (4.11) and using (4.4) and (4.7) we get

$$\int_{\mathbb{R}^{2n}} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} |u^*(x)|^{2^*}\, dx = S_s^{n/(2s)} \int_{\mathbb{R}^n} |\bar u(x)|^{2^*}\, dx = S_s^{n/(2s)}\,,$$

so that the proof of Claim 6 is complete. □

Now, we consider the following family of functions U_ε defined as

$$U_\varepsilon(x) = \varepsilon^{-(n-2s)/2}\, u^*(x/\varepsilon)\,, \qquad x \in \mathbb{R}^n \tag{4.12}$$

for any $\varepsilon > 0$. The scaling invariance of the problem is pointed out in the following observation:

Claim 7. *The function U_ε is a solution of problem* (4.6) *and verifies the following equalities*

$$\int_{\mathbb{R}^{2n}} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} |U_\varepsilon(x)|^{2^*}\, dx = S_s^{n/(2s)} \tag{4.13}$$

for any $\varepsilon > 0$.

Proof. By (4.12) and Claim 6 it easily follows that $U_\varepsilon \in H^s(\mathbb{R}^n)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \frac{(U_\varepsilon(x) - U_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&\qquad = \varepsilon^{(n-2s)/2} \int_{\mathbb{R}^{2n}} \frac{(u^*(x) - u^*(y))\big(\varphi(\varepsilon x) - \varphi(\varepsilon y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\
&\qquad = \varepsilon^{(n-2s)/2} \int_{\mathbb{R}^n} |u^*(x)|^{2^*-2} u^*(x) \varphi(\varepsilon x) dx \\
&\qquad = \int_{\mathbb{R}^n} |U_\varepsilon(x)|^{2^*-2} U_\varepsilon(x) \varphi(x) dx\,,
\end{aligned} \tag{4.14}$$

for any $\varphi \in H^s(\mathbb{R}^n)$ and $\varepsilon > 0$. Hence, U_ε is a solution of problem (4.6) for any $\varepsilon > 0$.

Taking $\varphi = U_\varepsilon$ in (4.14) (this choice is admissible, since $U_\varepsilon \in H^s(\mathbb{R}^n)$), we get

$$\int_{\mathbb{R}^{2n}} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} |U_\varepsilon(x)|^{2^*}\, dx$$

for any $\varepsilon > 0$. The scale invariance of the norm in $L^{2^*}(\mathbb{R}^n)$ under the scaling $u^* \mapsto U_\varepsilon$ and (4.8) give

$$\int_{\mathbb{R}^n} |U_\varepsilon(x)|^{2^*}\, dx = \int_{\mathbb{R}^n} |u^*(x)|^{2^*}\, dx = S_s^{n/(2s)}$$

for any $\varepsilon > 0$, and so Claim 7 is proved. □

In order to check (4.2), it is now necessary to appropriately put U_ε to zero outside Ω, according to the following procedure. Let us fix $\delta > 0$ such that

$$B_{4\delta} \subset \Omega \tag{4.15}$$

and let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^n$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathcal{C}B_{2\delta}$, where $B_\delta = B(0,\delta)$ and $\mathcal{C}B_\delta = \mathbb{R}^n \setminus B_\delta$. For every $\varepsilon > 0$ we denote by u_ε the following function

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x), \quad x \in \mathbb{R}^n, \tag{4.16}$$

where U_ε is given in (4.12).

In order to prove Theorem 4 we need to check condition (4.2) for u_ε (at least for some suitably small ε), and for this we will estimate $S_{s,\lambda}(u_\varepsilon)$. In the setting of the fractional Laplacian this estimate is more delicate than in the case of the Laplacian, due to the non-local nature of the operator $(-\Delta)^s$.

For this we need some preliminary claims. In what follows we suppose that, up to a translation, $x_0 = 0$ in (4.3).

Claim 8. *Let $\varrho > 0$ and μ be as in (4.3). If $x \in \mathcal{C}B_\varrho$, then*

$$\big|u_\varepsilon(x)\big| \leqslant \big|U_\varepsilon(x)\big| \leqslant C\varepsilon^{(n-2s)/2} \tag{4.17}$$

for any $\varepsilon > 0$ and for some positive constant C, possibly depending on μ, ϱ, s and n.

Proof. Using (4.12), (4.7), (4.4) and (4.3) (with $x_0 = 0$), we have that

$$U_\varepsilon(x) = \widetilde{\kappa}\varepsilon^{-(n-2s)/2} \left(\mu^2 + \left|\frac{x}{\varepsilon S_s^{1/(2s)}}\right|^2\right)^{-(n-2s)/2}, \tag{4.18}$$

for a suitable $\widetilde{\kappa} \in \mathbb{R} \setminus \{0\}$. If $x \in \mathcal{C}B_\varrho$, we deduce that

$$\big|U_\varepsilon(x)\big| \leqslant C\varepsilon^{-(n-2s)/2} \left(\mu^2 + \left(\frac{\varrho}{\varepsilon S_s^{1/(2s)}}\right)^2\right)^{-(n-2s)/2} \leqslant C\varepsilon^{(n-2s)/2},$$

and so Claim 8 follows, being $\eta \leqslant 1$ in $\mathbb{R}^n$. □

Claim 9. *Let $\varrho > 0$ and μ be as in (4.3). If $x \in \mathcal{C}B_\varrho$, then*

$$\big|\nabla u_\varepsilon(x)\big| \leqslant C\varepsilon^{(n-2s)/2} \tag{4.19}$$

for any $\varepsilon > 0$ and for some positive constant C, possibly depending on μ, ϱ, s and n.

Proof. First of all we observe that, for any $|\xi| \geqslant \varrho$, we have that

$$\begin{aligned}
&\left(1 + \left|\frac{\xi}{\varepsilon}\right|^2\right)^{-(n-2s)/2} + \frac{|\xi|}{\varepsilon^2}\left(1 + \left|\frac{\xi}{\varepsilon}\right|^2\right)^{-1-(n-2s)/2} \\
&\leqslant \left(1 + \left|\frac{\xi}{\varepsilon}\right|^2\right)^{-(n-2s)/2} + \frac{1}{\varrho}\left|\frac{\xi}{\varepsilon}\right|^2\left(1 + \left|\frac{\xi}{\varepsilon}\right|^2\right)^{-1-(n-2s)/2} \\
&\leqslant \left(1 + \frac{1}{\varrho}\right)\left(1 + \left|\frac{\xi}{\varepsilon}\right|^2\right)^{-(n-2s)/2} \\
&\leqslant \left(1 + \frac{1}{\varrho}\right)\left(1 + \left|\frac{\varrho}{\varepsilon}\right|^2\right)^{-(n-2s)/2} \\
&\leqslant \left(1 + \frac{1}{\varrho}\right)\left(\frac{\varepsilon}{\varrho}\right)^{n-2s}.
\end{aligned} \tag{4.20}$$

As a consequence of (4.16) and (4.20) we have that, for any $x \in \mathcal{C}B_\varrho$,

$$\begin{aligned}\big|\nabla u_\varepsilon(x)\big| &\leqslant C\varepsilon^{-(n-2s)/2}\left[\left(\mu^2+\left|\frac{x}{\varepsilon S_s^{1/(2s)}}\right|^2\right)^{-(n-2s)/2}\right.\\ &\qquad\left.+\frac{1}{\varepsilon S_s^{1/(2s)}}\left|\frac{x}{\varepsilon S_s^{1/(2s)}}\right|\left(\mu^2+\left|\frac{x}{\varepsilon S_s^{1/(2s)}}\right|^2\right)^{-1-(n-2s)/2}\right]\\ &\leqslant C\varepsilon^{-(n-2s)/2}\cdot\varepsilon^{n-2s}=C\varepsilon^{(n-2s)/2},\end{aligned}$$

which proves Claim 9. □

Claim 10. *Let δ be as in* (4.15) *and μ be as in* (4.3). *The following assertions hold true*

a) for any $x \in \mathbb{R}^n$ and $y \in \mathcal{C}B_\delta$, with $|x-y| \leqslant \delta/2$,

$$\big|u_\varepsilon(x)-u_\varepsilon(y)\big| \leqslant C\varepsilon^{(n-2s)/2}|x-y|\,; \tag{4.21}$$

b) for any $x, y \in \mathcal{C}B_\delta$

$$\big|u_\varepsilon(x)-u_\varepsilon(y)\big| \leqslant C\varepsilon^{(n-2s)/2}\min\{1,|x-y|\} \tag{4.22}$$

for any $\varepsilon > 0$ and for some positive constant C, possibly depending on μ, δ, s and n.

Proof. Let us start by proving assertion $a)$. For this let $x \in \mathbb{R}^n$ and $y \in \mathcal{C}B_\delta$ with $|x-y| \leqslant \delta/2$ and let ξ be any point on the segment joining x and y. Then

$$\xi = tx + (1-t)y, \qquad \text{for some } t \in [0,1]\,,$$

so that

$$|\xi| = |y + t(x-y)| \geqslant |y| - t|x-y| \geqslant \delta - t(\delta/2) \geqslant \delta/2\,.$$

This and Claim 9 (here used with $\varrho = \delta/2$) imply that $|\nabla u_\varepsilon(\xi)| \leqslant C\varepsilon^{(n-2s)/2}$, and so, by a first order Taylor expansion,

$$\big|u_\varepsilon(x)-u_\varepsilon(y)\big| \leqslant C\varepsilon^{(n-2s)/2}|x-y|\,,$$

which proves (4.21).

Now, we show $b)$. For this let $x, y \in \mathcal{C}B_\delta$. If $|x-y| \leqslant \delta/2$ then $b)$ follows from $a)$, so we may suppose $|x-y| > \delta/2$: then

$$\big|u_\varepsilon(x)-u_\varepsilon(y)\big| \leqslant \big|u_\varepsilon(x)\big| + \big|u_\varepsilon(y)\big| \leqslant C\varepsilon^{(n-2s)/2}\,,$$

thanks to Claim 8 (here used with $\varrho = \delta$), and this completes the proof of (4.22). □

Now we can estimate the Gagliardo seminorm of u_ε according to the following result:

Proposition 21. *Let $s \in (0,1)$ and $n > 2s$. Then, the following estimate holds true*

$$\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s})$$

as $\varepsilon \to 0$.

Proof. The proof makes use of the previous estimates and it is a bit complicated – definitely more difficult than the one for similar results in the case of the Laplacian. The additional complications arise not only from technical obstacles (differentiating is of course simpler than integrating), but also from a different distribution of the energy density: in particular, in the fractional case, the interaction between the interior and the exterior of the cut-off in (4.16) is small but not anymore negligible (indeed it will contribute to the total energy according to the forthcoming estimate (4.27), (4.28) and (4.31)). To keep track of these non-local interactions, we introduce the notation

$$\mathbb{D} := \big\{(x,y) \in \mathbb{R}^{2n} \,:\, x \in B_\delta,\ y \in \mathcal{C}B_\delta \text{ and } |x-y| > \delta/2\big\}$$

and

$$\mathbb{E} := \left\{(x,y) \in \mathbb{R}^{2n} \,:\, x \in B_\delta,\ y \in \mathcal{C}B_\delta \text{ and } |x-y| \leqslant \delta/2\right\},$$

where δ is as in (4.15).

By (4.16) we have that

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = & \int_{B_\delta\times B_\delta} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + 2\int_{\mathbb{D}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + 2\int_{\mathbb{E}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + \int_{(\mathcal{C}B_\delta)\times(\mathcal{C}B_\delta)} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,.\end{aligned} \tag{4.23}$$

By Claim 10 (here, in particular, we use (4.22)) we have

$$\int_{(\mathcal{C}B_\delta)\times(\mathcal{C}B_\delta)} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant C\varepsilon^{n-2s}\int_{\mathbb{R}^{2n}} \frac{\min\{1,|x-y|^2\}}{|x-y|^{n+2s}}\,dx\,dy = \mathcal{O}(\varepsilon^{n-2s})\,, \tag{4.24}$$

while, by (4.21)

$$\begin{aligned}\int_{\mathbb{E}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy & \leqslant C\varepsilon^{n-2s}\int_{\substack{x\in B_\delta,\, y\in\mathcal{C}B_\delta \\ |x-y|\leqslant\delta/2}} \frac{|x-y|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & \leqslant C\varepsilon^{n-2s}\int_{|x|\leqslant\delta} dx \int_{|\xi|\leqslant\delta/2} \frac{1}{|\xi|^{n+2s-2}}\,d\xi \\ & = \mathcal{O}(\varepsilon^{n-2s})\,,\end{aligned} \tag{4.25}$$

as $\varepsilon \to 0$. In both these estimates we use that $s \in (0,1)$.

Now, in (4.23) it remains to estimate the integral on $\mathbb{D}$, that is

$$\int_{\mathbb{D}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,. \tag{4.26}$$

For this, recalling that $u_\varepsilon(x) = U_\varepsilon(x)$ for any $x \in B_\delta$ thanks to (4.16), we note that for any $(x,y) \in \mathbb{D}$

$$\begin{aligned}|u_\varepsilon(x)-u_\varepsilon(y)|^2 & = |U_\varepsilon(x)-u_\varepsilon(y)|^2 \\ & = \big|\big(U_\varepsilon(x)-U_\varepsilon(y)\big)+\big(U_\varepsilon(y)-u_\varepsilon(y)\big)\big|^2 \\ & \leqslant |U_\varepsilon(x)-U_\varepsilon(y)|^2 + |U_\varepsilon(y)-u_\varepsilon(y)|^2 + 2|U_\varepsilon(x)-U_\varepsilon(y)|\,|U_\varepsilon(y)-u_\varepsilon(y)|\,,\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbb{D}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant & \int_{\mathbb{D}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + \int_{\mathbb{D}} \frac{|U_\varepsilon(y)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + 2\int_{\mathbb{D}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|\,|U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy\,.\end{aligned} \tag{4.27}$$

Hence, in order to estimate (4.26), we bound the last two terms in the right-hand side of (4.27).

By exploiting Claim 8 (here used with $\varrho=\delta$) , we obtain

$$
\begin{aligned}
\int_{\mathbb{D}} \frac{|U_\varepsilon(y)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\leqslant \int_{\mathbb{D}} \frac{\big(|U_\varepsilon(y)|+|u_\varepsilon(y)|\big)^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\leqslant 4\int_{\mathbb{D}} \frac{|U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\leqslant C\varepsilon^{n-2s}\int_{\substack{x\in B_\delta,\, y\in \mathcal{C}B_\delta\\ |x-y|>\delta/2}} \frac{1}{|x-y|^{n+2s}}\,dx\,dy \\
&\leqslant C\varepsilon^{n-2s}\int_{|\zeta|\leqslant\delta} d\zeta\int_{|\xi|>\delta/2}\frac{1}{|\xi|^{n+2s}}\,d\xi \\
&= \mathcal{O}(\varepsilon^{n-2s})\,,
\end{aligned}
\tag{4.28}
$$

as $\varepsilon\to 0$.

In order to estimate the last term in the right-hand side of (4.27), first of all we note that, once more by (4.18) (which is valid for any $x\in\mathbb{R}^n$) and Claim 8 ,

$$
|U_\varepsilon(x)|\,|U_\varepsilon(y)| \leqslant C\left(\mu^2+\left|\frac{x}{\varepsilon S_s^{1/(2s)}}\right|^2\right)^{-(n-2s)/2}
$$

for any $(x,y)\in\mathbb{D}$.

Therefore, by using the short-hand notation

$$
\delta_\varepsilon := \delta/(\varepsilon S_s^{1/(2s)})
$$

and the change of variable $\zeta:=x/(\varepsilon S_s^{1/(2s)})$ and $\xi:=x-y$ (and up to renaming C), we have that

$$
\begin{aligned}
\int_{\mathbb{D}} \frac{|U_\varepsilon(x)|\,|U_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy &\leqslant C\int_{\mathbb{D}}\left(\mu^2+\left|\frac{x}{\varepsilon S_s^{1/(2s)}}\right|^2\right)^{-(n-2s)/2} |x-y|^{-(n+2s)}\,dx\,dy \\
&\leqslant C\varepsilon^n \int_{\substack{\zeta\in B_{\delta_\varepsilon}\\ |\xi|>\delta/2}} \left(\mu^2+|\zeta|^2\right)^{-(n-2s)/2} |\xi|^{-(n+2s)}\,d\zeta\,d\xi \\
&\leqslant C\varepsilon^n \int_{\zeta\in B_{\delta_\varepsilon}} \left(\mu^2+|\zeta|^2\right)^{-(n-2s)/2}\,d\zeta \\
&\leqslant C\varepsilon^n \left[1+\int_{\zeta\in B_{\delta_\varepsilon}\setminus B_1} \left(\mu^2+|\zeta|^2\right)^{-(n-2s)/2}\,d\zeta\right] \\
&\leqslant C\varepsilon^n \left[1+\int_{\zeta\in B_{\delta_\varepsilon}\setminus B_1} |\zeta|^{-(n-2s)}\,d\zeta\right] \\
&= C\varepsilon^n \left[1+\int_1^{\delta/(\varepsilon S_s^{1/(2s)})} \rho^{-(n-2s)+(n-1)}\,d\rho\right] \\
&= C\varepsilon^n\big(1+\varepsilon^{-2s}\big) \\
&= \mathcal{O}(\varepsilon^{n-2s})\,,
\end{aligned}
$$

as $\varepsilon\to 0$. Here we use again that $s\in(0,1)$.

As a consequence, recalling (4.16) and Claim 8, we get

$$\begin{aligned}\int_{\mathbb{D}} \frac{|U_\varepsilon(x)|\,|U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy &\leqslant \int_{\mathbb{D}} \frac{|U_\varepsilon(x)|\,\big(|U_\varepsilon(y)|+|u_\varepsilon(y)|\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &\leqslant 2\int_{\mathbb{D}} \frac{|U_\varepsilon(x)|\,|U_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy \\ &= \mathcal{O}(\varepsilon^{n-2s}),\end{aligned} \tag{4.29}$$

as $\varepsilon \to 0$.

On the other hand, again by (4.16) and Claim 8,

$$\begin{aligned}\int_{\mathbb{D}} \frac{|U_\varepsilon(y)|\,|U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy &\leqslant 2\int_{\mathbb{D}} \frac{|U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\leqslant C\varepsilon^{n-2s}\int_{\substack{x\in B_\delta,\, y\in \mathcal{C}B_\delta\\ |x-y|>\delta/2}} \frac{1}{|x-y|^{n+2s}}\,dx\,dy \\ &= \mathcal{O}(\varepsilon^{n-2s}),\end{aligned} \tag{4.30}$$

as $\varepsilon \to 0$ (here we argue as in (4.28)). Putting together (4.29) with (4.30), we infer that

$$\begin{aligned}\int_{\mathbb{D}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|\,|U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy &\leqslant \int_{\mathbb{D}} \frac{|U_\varepsilon(x)|\,|U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + \int_{\mathbb{D}} \frac{|U_\varepsilon(y)|\,|U_\varepsilon(y)-u_\varepsilon(y)|}{|x-y|^{n+2s}}\,dx\,dy \\ &= \mathcal{O}(\varepsilon^{n-2s}),\end{aligned} \tag{4.31}$$

as $\varepsilon \to 0$.

Finally, by (4.23)–(4.25), (4.27), (4.28) and (4.31) we get[2]

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &= \int_{B_\delta\times B_\delta} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + 2\int_{\mathbb{D}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathcal{O}(\varepsilon^{n-2s}) \\ &\leqslant \int_{\mathbb{R}^{2n}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathcal{O}(\varepsilon^{n-2s}),\end{aligned}$$

as $\varepsilon \to 0$. Then, the desired result now follows from Claim 7. □

Now we consider the L^2 and the L^{2^*}– norms of the function u_ε: in this case our estimates can be proved exactly as in the case of the Laplacian, but we prefer to repeat them, for reader's convenience.

Proposition 22. *Let $s \in (0,1)$ and $n > 2s$. Then, the following estimates hold true*

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^2\,dx \geqslant \begin{cases} C_s\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) & \text{if} \quad n > 4s \\ C_s\varepsilon^{2s}\big|\log\varepsilon\big| + \mathcal{O}(\varepsilon^{2s}) & \text{if} \quad n = 4s \\ C_s\varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s}) & \text{if} \quad n < 4s, \end{cases} \tag{4.32}$$

and

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*}\,dx = S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n}), \tag{4.33}$$

as $\varepsilon \to 0$, for some positive constant C_s depending on s.

[2]It is interesting to observe that the energy interaction outside $B_\delta \times B_\delta$ is not negligible: indeed, while the contributions in (4.24), (4.25), (4.28) and (4.31) are all $\mathcal{O}(\varepsilon^{n-2s})$, the integral in $\mathbb{D}$ provides a relevant part that needs to be taken into account in the full energy.

Proof. By definition of u_ε and (4.12) we get

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\varepsilon(x)|^2\,dx &= \int_{B_\delta} |U_\varepsilon(x)|^2\,dx + \int_{B_{2\delta}\setminus B_\delta} |\eta(x)U_\varepsilon(x)|^2\,dx \\ &\geqslant \varepsilon^{-(n-2s)}\int_{B_\delta} |u^*(x/\varepsilon)|^2\,dx \\ &\geqslant \varepsilon^{2s}\int_R^{\delta/\varepsilon} |u^*(r)|^2\,r^{n-1}\,dr\end{aligned}$$

for any $0<R<\delta/\varepsilon$.

Now, by (4.7) and Claim 5 we get

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\varepsilon(x)|^2\,dx &\geqslant \varepsilon^{2s}\int_R^{\delta/\varepsilon} |u^*(r)|^2\,r^{n-1}\,dr \\ &\cong \varepsilon^{2s}\int_R^{\delta/\varepsilon} r^{-n+4s-1}dr \\ &= \begin{cases} -C_{1,s}\varepsilon^{n-2s}+C_{2,s}\varepsilon^{2s} & \text{if} \quad n>4s \\ C_{1,s}\varepsilon^{2s}\big|\log\varepsilon\big|+C_{2,s}\varepsilon^{2s} & \text{if} \quad n=4s \\ C_{1,s}\varepsilon^{n-2s}-C_{2,s}\varepsilon^{2s} & \text{if} \quad n<4s\,, \end{cases} \\ &= \begin{cases} C_s\varepsilon^{2s}+\mathcal{O}(\varepsilon^{n-2s}) & \text{if} \quad n>4s \\ C_s\varepsilon^{2s}\big|\log\varepsilon\big|+\mathcal{O}(\varepsilon^{2s}) & \text{if} \quad n=4s \\ C_s\varepsilon^{n-2s}+\mathcal{O}(\varepsilon^{2s}) & \text{if} \quad n<4s\,, \end{cases}\end{aligned}$$

as $\varepsilon\to 0$, for some positive constant $C_{1,s}$, $C_{2,s}$ and C_s, depending on s, and for ε sufficiently small. This proves (4.32) and we now prove (4.33). Arguing in the same way and using Claim 7, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*}\,dx &= \int_{B_\delta} |U_\varepsilon(x)|^{2^*}\,dx + \int_{B_{2\delta}\setminus B_\delta} |\eta(x)U_\varepsilon(x)|^{2^*}\,dx \\ &\leqslant \int_{\mathbb{R}^n} |U_\varepsilon(x)|^{2^*}\,dx + \int_{B_{2\delta}\setminus B_\delta} |U_\varepsilon(x)|^{2^*}\,dx \\ &= S_s^{n/(2s)} + \varepsilon^{-n}\int_{B_{2\delta}\setminus B_\delta} |u^*(x/\varepsilon)|^{2^*}\,dx \\ &\leqslant S_s^{n/(2s)} + C\varepsilon^{n}\int_{B_{2\delta}\setminus B_\delta} |x|^{-2n}\,dx \\ &= S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\end{aligned} \tag{4.34}$$

as $\varepsilon\to 0$. This ends the proof of Proposition 22. □

With Propositions 21 and 22 we can conclude the proof of Theorem 4.

4.2.1. *End of the proof of Theorem 4.* We prove that u_ε satisfies condition (4.2) if ε is suitably small (recall also that u_ϵ vanishes outside Ω because of (4.15)). For this, let us consider two different cases.

Case 1: $n > 4s$**.** By the definition of the function $S_{s,\lambda}(\cdot)$ given in formula (1.24) and by Propositions 21 and 22 we get

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda C_s \varepsilon^{2s}}{\Big(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\Big)^{2/2^*}} \\ &\leqslant S_s + \mathcal{O}(\varepsilon^{n-2s}) - \lambda \widetilde{C}_s \varepsilon^{2s} \\ &= S_s + \varepsilon^{2s}\Big(\mathcal{O}(\varepsilon^{n-4s}) - \lambda \widetilde{C}_s\Big) < S_s\,, \end{aligned}$$

for any $\lambda > 0$, if $\varepsilon > 0$ is sufficiently small, for a suitable positive constant $\widetilde{C}_s$. Hence, condition (4.2) is satisfied (with $u_0 = u_\varepsilon$) for any $\lambda > 0$ when $n > 4s$. Thus, by Theorem 3, for any $\lambda \in (0, \lambda_{1,s})$ problem (1.31) admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

Case 2: $n = 4s$**.** Arguing as in Case 1 and taking into account that $n = 4s$ for some positive constant $\widetilde{C}_s$ we get

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda C_s \varepsilon^{2s}\big|\log\varepsilon\big| + \mathcal{O}(\varepsilon^{2s})}{\Big(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\Big)^{2/2^*}} \\ &\leqslant S_s + \mathcal{O}(\varepsilon^{2s}) - \lambda \widetilde{C}_s \varepsilon^{2s}\big|\log\varepsilon\big| \\ &= S_s + \varepsilon^{2s}\Big(\mathcal{O}(1) - \lambda \widetilde{C}_s\big|\log\varepsilon\big|\Big) < S_s\,, \end{aligned}$$

for any $\lambda > 0$, if $\varepsilon > 0$ is sufficiently small. Hence, condition (4.2) is satisfied with $u_0 = u_\varepsilon$ for any $\lambda > 0$ when $n = 4s$. Therefore, thanks to Theorem 3, for any $\lambda \in (0, \lambda_{1,s})$ problem (1.31) admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

This completes the proof of Theorem 4.

Remark 23. We think that it is an interesting open problem to check whether the same result of Theorem 4 holds true when $2s < n < 4s$. In this last case, arguing as in Cases 1 and 2, we have

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda C_s \varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s})}{\Big(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\Big)^{2/2^*}} \\ &\leqslant S_s + \varepsilon^{n-2s}\Big(\mathcal{O}(1) - \lambda \widetilde{C}_s\Big) + \mathcal{O}(\varepsilon^{2s}) < S_s\,, \end{aligned} \tag{4.35}$$

if λ is large enough, say $\lambda > \lambda_s > 0$, when $\varepsilon > 0$ is sufficiently small. Hence, as a consequence of Theorem 3, when $2s < n < 4s$, if

$$\lambda_s < \lambda_{1,s} \tag{4.36}$$

then problem (1.31) admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, for any $\lambda \in (\lambda_s, \lambda_{1,s})$. The difficulty here is to fix the competition in the above choices of λ_s, i.e. to show that it is possible to choose λ_s large enough, according to (4.35), but not too large, according to (4.36).

4.2.2. *Some remarks on condition* $S_{s,\lambda} < S_s$. As in the classical case (see the discussion in [3, pages 458–459]), also in the non-local setting the Palais–Smale condition may globally fail for the functional $\mathcal{J}_{s,\lambda}$. Therefore, it is necessary to restrict our attention only to a suitable range related to the best fractional critical Sobolev constant S_s of the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$. Indeed, this functional satisfies the Palais–Smale condition at any level c, with c satisfying

$$c < \frac{s}{n}\, S_s^{n/(2s)}\,. \tag{4.37}$$

As discussed in the introduction, another natural condition to look at is given by

$$S_{s,\lambda} < S_s\,. \tag{4.38}$$

It turns out that conditions (4.37) and (4.38) are related as follows:

Proposition 24. *Let $0 < \lambda < \lambda_{1,s}$, where $\lambda_{1,s}$ is the first eigenvalue of the operator $(-\Delta)^s$with homogeneous Dirichlet boundary conditions and let*

$$c = \inf_{P\in\mathcal{P}} \sup_{v\in P([0,1])} \mathcal{J}^0_{s,\lambda}(v)\,, \tag{4.39}$$

where

$$\mathcal{J}^0_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\,,$$

and

$$\mathcal{P} = \Big\{P\in C([0,1];X_0) : P(0)=0\,,\ P(1)=e\Big\}$$

with $e\in H^s(\mathbb{R}^n)$ such that $\mathcal{J}^0_{s,\lambda}(e)<0$.

Then, conditions (4.37) *and* (4.38) *are equivalent.*

Proof. It is enough to show that

$$c = \frac{s}{n}\,S_{s,\lambda}^{n/(2s)}\,. \tag{4.40}$$

For this, let $u\in H^s(\mathbb{R}^n)\setminus\{0\}$ and let $P(\zeta)=\zeta u$ with $\zeta\in[0,1]$. By (4.1) and (4.39) it is easily seen that

$$c \leqslant \sup_{\zeta\geqslant 0} \mathcal{J}_{s,\lambda}(\zeta u) = \frac{s}{n}\,S_{s,\lambda}^{n/(2s)}(u)\,,$$

so that, taking the infimum and using the definition (1.23), we obtain

$$c \leqslant \frac{s}{n}\,S_{s,\lambda}^{n/(2s)}\,. \tag{4.41}$$

Now, let us prove the other inequality. In order to do this let us consider the continuous function

$$[0,1]\ni\zeta\mapsto \langle(\mathcal{J}^0_{s,\lambda})'(P(\zeta)),P(\zeta)\rangle\,,$$

for any fixed $P\in\mathcal{P}$. Let

$$\widetilde{\zeta} := \sup\{\zeta\in[0,1]\ :\ P(\zeta)=0\}\,. \tag{4.42}$$

Since $P(0)=0$, $P(1)=e\neq 0$ and P is continuous, it follows that

$$\widetilde{\zeta}\in[0,1)\,. \tag{4.43}$$

Since $2<2^*$ and $\mathcal{J}^0_{s,\lambda}(e)<0$ by assumption, we have that

$$\begin{aligned}\langle(\mathcal{J}^0_{s,\lambda})'(e),e\rangle &= \int_{\mathbb{R}^{2n}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |e(x)|^2\,dx - \int_\Omega |e(x)|^{2^*}\,dx\\ &= 2\mathcal{J}^0_{s,\lambda}(e) + \Big(\frac{2}{2^*}-1\Big)\int_\Omega |e(x)|^{2^*}\,dx\\ &< 2\mathcal{J}^0_{s,\lambda}(e) < 0\,.\end{aligned} \tag{4.44}$$

On the other hand, since $0<\lambda<\lambda_{1,s}$, we get

$$\begin{aligned}\langle(\mathcal{J}^0_{s,\lambda})'(P(\zeta)),P(\zeta)\rangle &= \int_{\mathbb{R}^{2n}} \frac{|P(\zeta)(x)-P(\zeta)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |P(\zeta)(x)|^2\,dx\\ &\qquad - \int_\Omega |P(\zeta)(x)|^{2^*}\,dx\\ &\geqslant \Big(1-\frac{\lambda}{\lambda_{1,s}}\Big)\int_{\mathbb{R}^{2n}} \frac{|P(\zeta)(x)-P(\zeta)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \|P(\zeta)\|^{2^*}_{L^{2^*}(\Omega)}\,.\end{aligned}$$

So, by Lemma 8-a) and the fact that $2 < 2^*$,

$$\langle (\mathcal{J}^0_{s,\lambda})'(P(\zeta)), P(\zeta)\rangle \geqslant \frac{1}{c}\Big(1-\frac{\lambda}{\lambda_{1,s}}\Big)\|P(\zeta)\|^2_{L^{2^*}(\Omega)} - \|P(\zeta)\|^{2^*}_{L^{2^*}(\Omega)} > 0\,,$$

provided that

$$\|P(\zeta)\|_{L^{2^*}(\Omega)} \in \left(0, \left[\frac{1}{c}\Big(1-\frac{\lambda}{\lambda_{1,s}}\Big)\right]^{1/(2^*-2)}\right).$$

In particular, since P is continuous, recalling (4.42) and (4.43), we obtain that there exists $\widehat{\zeta} \in (\widetilde{\zeta}, 1)$ such that

$$\langle (\mathcal{J}^0_{s,\lambda})'(P(\zeta)), P(\zeta)\rangle > 0 \text{ for any } \zeta \in (\widetilde{\zeta}, \widehat{\zeta})\,. \tag{4.45}$$

As a consequence of (4.44) and (4.45) we get that there exists $\bar{\zeta} \in (\widehat{\zeta}, 1)$ such that $\bar{u} := P(\bar{\zeta})$ satisfies

$$\langle (\mathcal{J}^0_{s,\lambda})'(\bar{u}), \bar{u}\rangle = \int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |\bar{u}(x)|^2\,dx - \int_\Omega |\bar{u}(x)|^{2^*}\,dx = 0\,. \tag{4.46}$$

We stress that

$$\bar{u} \neq 0\,, \tag{4.47}$$

because $\bar{\zeta} > \widehat{\zeta} > \widetilde{\zeta}$ by construction, and so $P(\bar{\zeta}) \neq 0$ by (4.42).

Also, using the definition of $\mathcal{J}^0_{s,\lambda}$ and (4.46) we easily get

$$\mathcal{J}^0_{s,\lambda}(\bar{u}) = \Big(\frac{1}{2}-\frac{1}{2^*}\Big)\|\bar{u}\|^{2^*}_{L^{2^*}} = \frac{s}{n}\,\|\bar{u}\|^{2^*}_{L^{2^*}}\,. \tag{4.48}$$

Hence, by (1.23), it holds true that

$$\begin{aligned}
S_{s,\lambda} &\leqslant S_{s,\lambda}(\bar{u}) \\
&= \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_{\mathbb{R}^n}|\bar{u}(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n}|\bar{u}(x)|^{2^*}\,dx\Big)^{2/2^*}} \\
&\leqslant \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_{\Omega}|\bar{u}(x)|^2\,dx}{\displaystyle\Big(\int_{\Omega}|\bar{u}(x)|^{2^*}\,dx\Big)^{2/2^*}} \\
&= \Big(\int_\Omega |\bar{u}(x)|^{2^*}\,dx\Big)^{(2^*-2)/2^*} \\
&= \Big(\frac{n}{s}\,\mathcal{J}^0_{s,\lambda}(\bar{u})\Big)^{2s/n} \\
&\leqslant \Big(\frac{n}{s}\sup_{v\in P([0,1])}\mathcal{J}^0_{s,\lambda}(v)\Big)^{2s/n}\,,
\end{aligned}$$

thanks to (4.46)–(4.48). Taking the infimum in $P \in \mathcal{P}$, we conclude that

$$S_{s,\lambda} \leqslant \Big(\frac{n}{s}\,c\Big)^{2s/n}\,,$$

that is

$$c \geqslant \frac{s}{n}\,S^{n/(2s)}_{s,\lambda}\,. \tag{4.49}$$

By (4.41) and (4.49) we deduce (4.40), and this ends the proof of Proposition 24. □

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