EIGENVALUE ESTIMATES FOR THE PERTURBED ANDERSSON MODEL

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ABSTRACT. Consider the operator

$$H = -\Delta + p_\omega - V,$$

where p_ω is the potential of the Andersson type and $V \geq 0$ is a function that decays slowly at the infinity. We study the rate of accumulation of eigenvalues of H to the bottom of the essential spectrum.

1. STATEMENT OF THE MAIN RESULT

The question we study is rooted in two different areas of mathematics: the spectral theory of differential operators and the theory of percolations. We begin with a discussion of the topics that are more important for the Cwikel-Lieb-Rozenblum and Lieb-Thirring estimates.

Let E_j be the negative eigenvalues of the operator

$$-\Delta - V(x), \qquad V \geq 0.$$

Then

$$\sum_j |E_j|^\gamma \leq C \int_{\mathbb{R}^d} V^{d/2+\gamma} dx, \qquad d \geq 3, \quad \gamma \geq 0.$$

The constant C in the latter inequality depends on γ and d. If $\gamma = 0$, this inequality is called Cwikel-Lieb-Rozenblum estimate (see [1], [5], [6] and [11]). If $\gamma > 0$, then it is called the Lieb-Thirring bound [8].

These inequalities are generalized to the case when the operator $-\Delta$ is replaced by $-\Delta + p(x)$, where p is any positive bonded function. However, it is expected that V might decay at the infinity slower than in the case $p = 0$. In the present paper, we study spectral properties of the Schrödinger operator

$$H = -\Delta + p_\omega - V,$$

where p_ω is a positive random potential of the form

$$p_\omega = \sum_{n \in \mathbb{Z}^d} \omega_n \chi(x - n), \qquad \omega_n \geq 0,$$

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$V \geq 0$ is a decaying real valued function and ω_n are independent identically distributed random variables. Here χ is the characteristic function of the unit cube $[0,1)^d$.

As it is known, the negative spectrum of H is discrete, i.e. consists of isolated eigenvalues $\lambda_j < 0$, possibly accumulating to zero. In [10], Molchanov and Vainberg studied the question how fast should $V \geq 0$ decay at the infinity in order that H had at most finite number of negative eigenvalues. It turns out that it is true for potentials decaying faster than $c \ln^{-2/d} |x|$. More precisely, operators with potentials of the form $V = c \ln^{-2/d} |x|$ (here $|x| > 1$) have finite number of negative eigenvalues if $c > 0$ is sufficiently small, and the number of eigenvalues is infinite if c is sufficiently large. We shall try to answer the question how fast do these eigenvalues accumulate to zero in the case when V decays at the infinity in a very slow manner. Our assumptions on ω_n will be the following

$$\begin{aligned} \text{Prob}\{\omega_n = 0\} = q < 1, \\ \text{Prob}\{\omega_n = 1\} = 1 - q, \end{aligned} \tag{1.1}$$

which implies that $p_\omega(x)$ takes only two values: zero and one. For simplicity, we assume that

$$q < (3^d - 2)^{-1} \tag{1.2}$$

This assumption is needed because of the following reason. Consider the largest connected domain where $p_\omega(x) = 0$ containing the cube $[0,1)^d$. Notice that this domain might be empty and the probability of the event that this domain is not empty equals q. We might ask a more subtle question: what is the probability that this set contains n cubes? It turns out that it can be estimated by $C\Big(q(3^d-2)\Big)^n$, so under assumption (1.2) this probability tends to zero as $n \to \infty$.

One should mention that we understand the notion of connectedness in a somewhat different way, compared to how it is usually done. We consider two different cubes of the form $[0,1)^d + n$ connected to each other if their closures have at least one common point. It is clear that in this case the distance between their centers is less or equal to $\sqrt{d}$, where d is the dimension of the space $\mathbb{R}^d$.

The following lemma (cf [10]) plays a key role in the proof of Theorem 1.1

Lemma 1.1. *Let Ω be an open domain with a smooth boundary having a bounded curvature smaller than $k_0 > 0$. Let W be a positive function such that $W(x) = 1$ if $\operatorname{dist}(x, \partial\Omega) \leq 1/2$. Then the lowest eigenvalue μ_0 of the operator*

$$-\Delta + W$$

with the Neumann boundary condition on $\partial\Omega$ satisfies the inequality

$$\mu_0 \geq \frac{\nu}{|\Omega|^{2/d}} \tag{1.3}$$

where the constant $\nu > 0$ is independent of the domain Ω.

Let us formulate our main result which deals with the case when

$$V(x) = \frac{v_0}{\ln^s |x|}, \qquad |x| > 1,\ 0 < s < 2/d. \tag{1.4}$$

One of the consequences of the theorem formulated below is that all eigenvalue sums of the form

$$\sum_j |\lambda_j|^\gamma = \infty, \qquad \gamma > 0,$$

are divergent if $s < 2/d$, while it is known that H has only finite number of eigenvalues if $s > 2/d$. This is an unusual transition from "finite" to "infinite", for it would be natural to expect that an intermediate case $\sum_j |\lambda_j|^\gamma < \infty$ occurs at least for some $s \geq 2/d$. Perhaps, we should seek for the key to this puzzle by studying carefully the case $s = 2/d$ where a similar transition takes place when and we change the coefficient v_0.

Theorem 1.1. *Let V be given by* (1.4) *and let q satisfy the condition* (1.2). *Then the negative eigenvalues λ_j of the operator H satisfy*

$$\sum_j \exp\Big(-\frac{v_0^{1/s} d}{|\lambda_j|^{1/s}} + \frac{\alpha}{|\lambda_j|^{d/2}}\Big) < \infty \quad \text{for} \qquad \alpha < -2^{-1}\nu^{d/2}\ln(q(3^d-2)), \tag{1.5}$$

and

$$\sum_j e^{-\tau|\lambda_j|^{-1/s}} = \infty \quad \text{for} \quad \tau < v_0^{1/s} d. \tag{1.6}$$

Proof. First, let us prove (1.5). Decompose the whole space $\mathbb{R}^d$ into the disjoint union of the layers

$$\Omega_n = \{x \in \mathbb{R}^d :\ e^n < |x| \leq e^{n+1}\}$$

Note that

$$V(x) \leq v_0 n^{-s} \qquad x \in \Omega_n.$$

According to Lemma 1.1 we should pay attention to the domains $\Omega \subset \Omega_n$ where $p_\omega(x) = 0$ and

$$\frac{\nu}{|\Omega|^{2/d}} \leq v_0 n^{-s}.$$

Consequently, such domains should contain not less than

$$\mathfrak{N}(n) = \Big(\nu/v_0\Big)^{d/2} n^{sd/2}$$

cubes of the form $[0,1)^d + j$ where $j \in \mathbb{Z}^d$.

Let $x \in \Omega_n$ be a fixed point. Then the probability of the event that x belongs to a connected domain $\Omega(x)$, having the volume not smaller than $\mathfrak{N}(n)$ and consisting only of points where $p_\omega = 0$, is not bigger than

$$C\Big(q(3^d-2)\Big)^{\mathfrak{N}(n)}$$

(here the constant C already depends on q but is independent of n).

It is clear that $\Omega(x)$ is empty with probability larger than $1 - C\Big(q(3^d - 2)\Big)^{\mathfrak{N}(n)}$. Therefore different domains $\Omega(x)$ are separated form each other by very large regions. It turns out that one can estimate the distance between two different non-empty domains $\Omega(x)$ and $\Omega(x')$ using the Borel-Cantelli lemma.

Indeed, let us decompose Ω_n into the union of disjoint cubes Q_j whose volume equals

$$|Q_j| = \Big(q(3^d - 2)\Big)^{-2^{-1}\mathfrak{N}(n)}.$$

Note that the number of such cubes in Ω_n does not exceed

$$\frac{|\Omega_n|}{|Q_j|} \le \frac{e^{dn}}{\Big(q(3^d - 2)\Big)^{-2^{-1}\mathfrak{N}(n)}} \tag{1.7}$$

Then the probability that a cube of this size contains $\mathfrak{m}$ non-empty domains $\Omega(x)$ is not larger than

$$C\Big(q(3^d - 2)\Big)^{2^{-1}\mathfrak{N}(n)\mathfrak{m}}.$$

Now choose

$$\mathfrak{m} = \mathfrak{m}(n) := \frac{2(1+\epsilon)dn}{\mathfrak{N}(n)\ln(1/q(3^d-2))}, \qquad \epsilon > 0.$$

Then the probability that there exists at least one cube $Q_j \subset \Omega_n$ containing not less than $\mathfrak{m}(n)$ non-empty "clearing" domains $\Omega(x)$ is not bigger than $Ce^{-n\epsilon d}$.

Since $\sum_n e^{-n\epsilon d} < \infty$, we obtain that the sum of these probabilities over set all possible indexes $n > 1$ is convergent. According to Borel-Cantelli lemma, this implies that there exist such an index n_0 that all cubes $Q_j \subset \Omega_n$ contain less than $\mathfrak{m}(n)$ non-empty clearing domains $\Omega(x)$ as soon as $n > n_0$.

Now we need to know how large can $\Omega(x)$ be. It was proven by Molchanov and Vainberg in [10], that

$$|\Omega(x)| < c\ln|x|, \qquad \text{for} \quad |x| > R(c,\omega)$$

for any $c > c_0 = -d/\ln((3^d-2)q)$. In particular, if $\Omega(x) \subset \Omega_n$, then

$$|\Omega(x)| < c(n+1), \qquad \text{for} \quad |x| > R(c,\omega). \tag{1.8}$$

We are going to apply the standard trick imposing the Neumann conditions on the boundary of each cube of the form $[0,1)^d + j$, $j \in \mathbb{Z}^d$. The eigenvalues of the resulting orthogonal sum of operators will be lower than the eigenvalues of the original operator. On the other hand, without loss of generality we can assume that the operator $-\Delta - V$ on the cube $\big([0,1)^d + j\big) \in \Omega_n$ has only one negative eigenvalue. This eigenvalue equals

$$\lambda = -v_0 n^{-s}. \tag{1.9}$$

It is actually true, if n is sufficiently large.

Consequently, according to (1.7), (1.8) and (1.9), we obtain

$$\sum_j e^{-\tau/|\lambda_j|^\gamma} e^{\alpha/|\lambda_j|^\beta} \leq \sum_n \frac{Ce^{dn}(c_0+\epsilon)(n+1)\mathfrak{m}(n)}{\Big(q(3^d-2)\Big)^{-2^{-1}\mathfrak{N}(n)}} e^{-\tau/|v_0 n^{-s}|^\gamma} e^{\alpha/|v_0 n^{-s}|^\beta}$$

Choosing $s\gamma = 1$ and $\tau/v_0^\gamma = d$, we obtain that

$$\sum_j e^{-\tau/|\lambda_j|^\gamma} e^{\alpha/|\lambda_j|^\beta} \leq \sum_n \frac{C(c_0+\epsilon)(n+1)\mathfrak{m}(n)}{\Big(q(3^d-2)\Big)^{-2^{-1}\mathfrak{N}(n)}} e^{\alpha/|v_0 n^{-s}|^\beta}$$

This series converges if $\beta = d/2$ and $\alpha < -2^{-1}\nu^{d/2}\ln(q(3^d-2))$. This proves (1.5).

Let us now prove (1.6). Again, we decompose the whole space $\mathbb{R}^d$ into the disjoint union of the layers

$$\Omega_n = \{x \in \mathbb{R}^d : \ e^n < |x| \leq e^{n+1}\}$$

and note that on each such layer $V(x)$ admits the following estimate

$$V(x) \geq v_0(n+1)^{-s}, \qquad \forall x \in \Omega_n.$$

Since we are going to establish an estimate of a different type, instead of Lemma 1.1 which says that the lowest eigenvalue can not be too small, we will use a different fact claiming that it can not be too large. This fact is needed only for the case when the domain is a cube, which makes things even simpler, because the eigenvalues of the Laplace operator on a cube are computed explicitly. Indeed, let μ_1 be the lowest eigenvalue of the Laplace operator with Dirichlet boundary conditions on the boundary of the cube $Q = [0, L)^d$ (Obviously, $|Q| = L^d$ in this case). Then

$$\mu_1 = \nu_1 L^{-2} = \nu_1 |Q|^{-2/d},$$

where $\nu_1 = \pi^2 d$.

Our method is rather rough because we will ignore the shape of the clearings domains, i.e. domains consisting of x such that $p_\omega(x) = 0$, and we will pay attention only to the cubes $\Omega \subset \Omega_n$ where $p_\omega(x) = 0$ and

$$\frac{\nu_1}{|\Omega|^{2/d}} \leq v_0(n+1)^{-s}. \tag{1.10}$$

Only on such cubes, the operator $-\Delta - V$ might have negative eigenvalues. Relation (1.10) is a restriction on the size of Ω that allows one to estimate the number of elementary cubes the form $[0,1) + j$, $j \in \mathbb{Z}^d$ that are contained in Ω. Namely, (1.10) implies that

$$|\Omega| \geq \Big(\nu_1/v_0\Big)^{d/2}(n+1)^{sd/2}.$$

However, in order to estimate the size of the lowest eigenvalue of the operator $-\Delta - V$ on Ω, we need to consider slightly bigger domains. In order to do that, we introduce

$$\mathfrak{N}_1(n) := \Big((1-\varepsilon)^{-1}\nu_1/v_0\Big)^{d/2}(n+1)^{sd/2}, \qquad \varepsilon \in (0,1).$$

Note that if Ω contains $\mathfrak{N}_1(n)$ cubes of the form $[0,1)+j$, $j\in\mathbb{Z}^d$, then not only (1.10) holds, but the lowest eigenvalue λ of $\Delta - V$ on the cube $\Omega\subset\Omega_n$ satisfies the estimate

$$\lambda \le -\varepsilon v_0(n+1)^{-s}. \tag{1.11}$$

Now we will find cubes Q_j, that are likely to contain at least one set Ω of the size $|\Omega| = \mathfrak{N}_1(n)$ having the property that $p_\omega(x)=0$ for all $x\in\Omega$. These cubes will be sufficiently large and they will have the same volume:

$$|Q_j| = q^{-(1+\delta)\mathfrak{N}_1(n)}, \qquad \delta>0.$$

Moreover, Q_j will not intersect each other, which means that we can decompose Ω_n into the union of disjoint cubes Q_j Note that the number of sets Q_j in Ω_n does not exceed the quantity

$$\frac{|\Omega_n|}{|Q_j|} \le \frac{C_0 e^{dn}}{q^{-(1+\delta)\mathfrak{N}_1(n)}}. \tag{1.12}$$

The probability that a fixed cube Q_j does not contain a smaller cube Ω of the size $|\Omega| = \mathfrak{N}_1(n)$ consisting of points x at which $p_\omega(x)=0$ is not larger than

$$\Big(1-q^{\mathfrak{N}_1(n)}\Big)^{|Q_j|/\mathfrak{N}_1(n)} \le \exp\Big(-\frac{q^{-\delta\mathfrak{N}_1(n)}}{\mathfrak{N}_1(n)}\Big).$$

Due to (1.22), the probability that there exists at least one Q_j in Ω_n having the property described above is not larger than

$$C\exp\Big(-\frac{q^{-\delta\mathfrak{N}_1(n)}}{\mathfrak{N}_1(n)}\Big)e^{dn}.$$

Since

$$\sum_n \exp\Big(-\frac{q^{-\delta\mathfrak{N}_1(n)}}{\mathfrak{N}_1(n)}\Big)e^{dn} < \infty$$

we conclude that no such $Q_j\subset\Omega_n$ exist if n is sufficiently large. Put it differently, all cubes Q_j contain smaller cubes Ω of the size $|\Omega| = \mathfrak{N}_1(n)$ on which $p_\omega = 0$.

If we impose the Dirichlet conditions on boundary of each set Ω, then the spectrum of the resulting operator is lifted up. Therefore, it is sufficient to prove that the corresponding eigenvalue sum diverges for the operator with such boundary conditions. On the other hand we know that the lowest eigenvalue of $\Delta - V$ on $\Omega\subset\Omega_n$ satisfies (1.21). Which implies that

$$e^{-\tau|\lambda|^{-1/s}} \ge e^{-\tau(\varepsilon v_0)^{-1/s}(n+1)}$$

Since ε can be chosen as close to 1 as we wish, we can assume that $\tau(\varepsilon v_0)^{-1/s} < (1-\delta_0)d$ where δ_0 is a sufficiently small number. Consequently,

$$e^{-\tau|\lambda|^{-1/s}} \geq e^{-d(n+1)(1-\delta_0)}$$

Therefore the eigenvalue sum in (1.6) can be estimated by

$$\sum_j e^{-\tau|\lambda_j|^{-1/s}} \geq \sum_n e^{-d(n+1)(1-\delta_0)} \frac{|\Omega_n|}{|Q_j|} \geq \sum_n e^{2^{-1}d\delta_0(n+1)} = \infty.$$

The proof is completed. □

Let us formulate our next result which deals with the case when

$$V(x) = \frac{v_0}{\ln^{2/d}|x|}, \qquad |x| > 1. \tag{1.13}$$

Molchanov and Vainberg [10] proved that H has finitely many negative eigenvalues if

$$\frac{dv_0^{d/2}}{\nu^{d/2}\ln(1/q(3^d-2))} < 1.$$

The results of the same paper say that the number of negative eigenvalues of H is infinite, if $v_0 > 0$ is sufficiently large. Therefore we are going to consider only the case when

$$\frac{dv_0^{d/2}}{\nu^{d/2}\ln(1/q(3^d-2))} > 1. \tag{1.14}$$

Theorem 1.2. *Let V be given by* (1.13) *and let q satisfy the condition* (1.2). *Assume that ν in Lemma 1.1 fulfills the condition* (1.14) *Then the negative eigenvalues λ_j of the operator H satisfy*

$$\sum_j \exp\Big(\frac{\alpha - v_0^{d/2}d}{|\lambda_j|^{d/2}}\Big) < \infty \quad \text{for} \qquad \alpha < -2^{-1}\nu^{d/2}\ln(q(3^d-2)). \tag{1.15}$$

Moreover, if

$$v_0 > 2\pi^2 d^{1-2/d}\ln^{2/d}(\frac{1}{q}),$$

then

$$\sum_j e^{-\tau|\lambda_j|^{-d/2}} = \infty \quad \text{for} \quad \tau < 2^{-d/2}v_0^{d/2}\Big(d - \Big(\frac{2\pi^2 d}{v_0}\Big)^{d/2}\ln\frac{1}{q}\Big). \tag{1.16}$$

Proof. First, let us prove (1.15). Again, as before, we decompose the whole space $\mathbb{R}^d$ into the disjoint union of the layers

$$\Omega_n = \{x \in \mathbb{R}^d : \ e^n < |x| \leq e^{n+1}\}$$

Note that

$$V(x) \leq v_0 n^{-2/d} \qquad x \in \Omega_n.$$

According to Lemma 1.1 we should pay attention to the domains $\Omega \subset \Omega_n$ where $p_\omega(x) = 0$ and

$$\frac{\nu}{|\Omega|^{2/d}} \leq \frac{v_0}{n^{2/d}}.$$

Consequently, such domains should contain not less than

$$\mathfrak{N}(n) = \Big(\nu/v_0\Big)^{d/2} n$$

cubes of the form $[0,1)^d + j$ where $j \in \mathbb{Z}^d$.

Let $x \in \Omega_n$ be a fixed point. Then the probability of the event that x belongs to a connected domain $\Omega(x)$, having the volume not smaller that $\mathfrak{N}(n)$ and consisting only of points where $p_\omega = 0$, is not bigger than

$$C\Big(q(3^d - 2)\Big)^{\mathfrak{N}(n)}$$

(here the constant C already depends on q but is independent of n).

It is clear that $\Omega(x)$ is empty with probability larger than $1 - C\big(q(3^d - 2)\big)^{\mathfrak{N}(n)}$. Therefore different domains $\Omega(x)$ are separated form each other by very large regions. It turns out that one can estimate the distance between two different non-empty domains $\Omega(x)$ and $\Omega(x')$ using the Borel-Cantelli lemma.

Indeed, let us decompose Ω_n into the union of disjoint cubes Q_j whose volume equals

$$|Q_j| = \Big(q(3^d - 2)\Big)^{-2^{-1}\mathfrak{N}(n)}.$$

Note that $|Q_j| < e^{dn/2}$ and the number of such cubes in Ω_n does not exceed the ratio

$$\frac{|\Omega_n|}{|Q_j|} \leq \frac{e^{dn}}{\Big(q(3^d - 2)\Big)^{-2^{-1}\mathfrak{N}(n)}} \tag{1.17}$$

Then the probability that a cube of this size contains $\mathfrak{m}$ non-empty domains $\Omega(x)$ is not larger than

$$C\Big(q(3^d - 2)\Big)^{2^{-1}\mathfrak{N}(n)\mathfrak{m}}.$$

Now choose

$$\mathfrak{m} := \frac{2(1+\epsilon)dn}{\mathfrak{N}(n)\ln(1/q(3^d - 2))} = \frac{2(1+\epsilon)dv_0^{d/2}}{\nu^{d/2}\ln(1/q(3^d - 2))}, \qquad \epsilon > 0.$$

Under conditions of Theorem 1.2, we have $\mathfrak{m} > 2$. Then the probability that there exists at least one cube $Q_j \subset \Omega_n$ containing not less than $\mathfrak{m}$ non-empty "clearing" domains $\Omega(x)$ is not bigger than $Ce^{-n\epsilon d}$.

Since $\sum_n e^{-n\epsilon d} < \infty$, we obtain that the sum of these probabilities over the set of all possible indexes $n > 1$ is convergent. According to Borel-Cantelli lemma, this implies that there exist such an index n_0 that all cubes $Q_j \subset \Omega_n$ contain less than $\mathfrak{m}$ non-empty clearing domains $\Omega(x)$ as soon as $n > n_0$.

Now we need to know how large can $\Omega(x)$ be. It was proven by Molchanov and Vainberg in [10], that

$$|\Omega(x)| < c \ln |x|, \qquad \text{for} \quad |x| > R(c, \omega)$$

for any $c > c_0 = -d/\ln((3^d - 2)q)$. In particular, if $\Omega(x) \subset \Omega_n$, then

$$|\Omega(x)| < c(n+1), \qquad \text{for} \quad |x| > R(c, \omega). \tag{1.18}$$

We are going to apply the standard trick imposing the Neumann conditions on the boundary of each cube of the form $[0,1)^d + j$, $j \in \mathbb{Z}^d$. The eigenvalues of the resulting orthogonal sum of operators will be lower than the eigenvalues of the original operator. On the other hand, without loss of generality we can assume that the operator $-\Delta - V$ on the cube $\big([0,1)^d + j\big) \in \Omega_n$ has only one negative eigenvalue. This eigenvalue equals

$$\lambda = -v_0 n^{-2/d}. \tag{1.19}$$

It is actually true, if n is sufficiently large.

Consequently, according to (1.17), (1.18) and (1.19), we obtain

$$\sum_j e^{-\tau/|\lambda_j|^\gamma} e^{\alpha/|\lambda_j|^\beta} \le \sum_n \frac{Ce^{dn}(c_0 + \epsilon)(n+1)\mathfrak{m}(n)}{\Big(q(3^d - 2)\Big)^{-2^{-1}\mathfrak{N}(n)}} e^{-\tau/|v_0 n^{-2/d}|^\gamma} e^{\alpha/|v_0 n^{-2/d}|^\beta}$$

Choosing $\gamma = d/2$ and $\tau = dv_0^{d/2}$, we obtain that

$$\sum_j e^{-\tau/|\lambda_j|^\gamma} e^{\alpha/|\lambda_j|^\beta} \le \sum_n \frac{C(c_0 + \epsilon)(n+1)\mathfrak{m}(n)}{\Big(q(3^d - 2)\Big)^{-2^{-1}\mathfrak{N}(n)}} e^{\alpha/|v_0 n^{-s}|^\beta}$$

This series converges if $\beta = d/2$ and $\alpha < -2^{-1}\nu^{d/2}\ln(q(3^d - 2))$. This proves (1.15).

Let us now prove (1.16). Again, we decompose the whole space $\mathbb{R}^d$ into the disjoint union of the layers

$$\Omega_n = \{x \in \mathbb{R}^d : \; e^n < |x| \le e^{n+1}\}$$

and note that on each such layer $V(x)$ admits the following estimate

$$V(x) \ge v_0(n+1)^{-2/d}, \qquad \forall x \in \Omega_n.$$

Since we are going to establish an estimate of a different type, instead of Lemma 1.1 which says that the lowest eigenvalue can not be too small, we will use a different fact claiming that it can not be too large. This fact is needed only for the case when the domain is a cube, which makes things even simpler, because the eigenvalues of the Laplace operator on a cube are computed explicitly. Indeed, let μ_1 be the lowest eigenvalue of the Laplace operator with Dirichlet boundary conditions on the boundary of the cube $Q = [0, L)^d$ (Obviously, $|Q| = L^d$ in this case). Then

$$\mu_1 = \nu_1 L^{-2} = \nu_1 |Q|^{-2/d},$$

where $\nu_1 = \pi^2 d$.

Our method is rather rough because we will ignore the shape of the clearings domains, i.e. domains consisting of x such that $p_\omega(x) = 0$, and we will pay attention only to the cubes $\Omega \subset \Omega_n$ where $p_\omega(x) = 0$ and

$$\frac{\nu_1}{|\Omega|^{2/d}} \leq \frac{v_0}{(n+1)^{2/d}}. \tag{1.20}$$

Only on such cubes, the operator $-\Delta - V$ might have negative eigenvalues. Relation (1.10) is a restriction on the size of Ω that allows one to estimate the number of elementary cubes the form $[0, 1) + j$, $j \in \mathbb{Z}^d$ that are contained in Ω. Namely, (1.10) implies that

$$|\Omega| \geq \Big(\nu_1/v_0\Big)^{d/2}(n+1).$$

However, in order to have a nice estimate for the size of the lowest eigenvalue of the operator $-\Delta - V$ on Ω, we need to consider slightly bigger domains. In order to do that, we introduce

$$\mathfrak{N}_1(n) := \Big(2\nu_1/v_0\Big)^{d/2}(n+1).$$

Note that if Ω contains $\mathfrak{N}_1(n)$ cubes of the form $[0, 1) + j$, $j \in \mathbb{Z}^d$, then not only (1.20) holds, but the lowest eigenvalue λ of $\Delta - V$ on the cube $\Omega \subset \Omega_n$ satisfies the estimate

$$\lambda \leq -2^{-1} v_0 (n+1)^{-2/d}. \tag{1.21}$$

Now we will find cubes Q_j, that are likely to contain at least one set Ω of the size $|\Omega| = \mathfrak{N}_1(n)$ having the property that $p_\omega(x) = 0$ for all $x \in \Omega$. These cubes will be sufficiently large and they will have the same volume:

$$|Q_j| = q^{-(1+\delta)\mathfrak{N}_1(n)}, \qquad \delta > 0.$$

Moreover, Q_j will not intersect each other, which means that we can decompose Ω_n into the union of disjoint cubes Q_j Note that the number of sets Q_j in Ω_n does not exceed the quantity

$$\frac{|\Omega_n|}{|Q_j|} \leq \frac{C_0 e^{dn}}{q^{-(1+\delta)\mathfrak{N}_1(n)}}. \tag{1.22}$$

Observe that this quantity grows with the growth of n. The probability that a fixed cube Q_j does not contain a smaller cube Ω of the size $|\Omega| = \mathfrak{N}_1(n)$ consisting of points x at which $p_\omega(x) = 0$ is not larger than

$$\Big(1 - q^{\mathfrak{N}_1(n)}\Big)^{|Q_j|} \leq \exp\Big(-q^{-\delta\mathfrak{N}_1(n)}\Big).$$

Due to (1.22), the probability that there exists at least one Q_j in Ω_n having the property described above is not larger than

$$C \exp\Big(-q^{-\delta \mathfrak{N}_1(n)}\Big) e^{dn}.$$

Since

$$\sum_n \exp\Big(-q^{-\delta \mathfrak{N}_1(n)}\Big) e^{dn} < \infty$$

we conclude that no such $Q_j \subset \Omega_n$ exist if n is sufficiently large. Put it differently, all cubes Q_j contain smaller cubes Ω of the size $|\Omega| = \mathfrak{N}_1(n)$ on which $p_\omega = 0$.

If we impose the Dirichlet conditions on boundary of each set Ω, then the spectrum of the resulting operator is lifted up. Therefore, it is sufficient to prove that the corresponding eigenvalue sum diverges for the operator with such boundary conditions. On the other hand we know that the lowest eigenvalue of $\Delta - V$ on $\Omega \subset \Omega_n$ satisfies (1.21). Which implies that

$$e^{-\tau|\lambda|^{-d/2}} \geq e^{-\tau(2^{-1}v_0)^{-d/2}(n+1)}$$

Due to the conditions of the theorem, we can assume that

$$\tau(2^{-1}v_0)^{-d/2} < d - (1+\delta)\ln(\frac{1}{q})\Big(\frac{2\nu_1}{v_0}\Big)^{d/2} - \varepsilon$$

where $\varepsilon > 0$ is a sufficiently small but positive number. Consequently,

$$e^{-\tau|\lambda|^{-d/2}} \frac{|\Omega_n|}{|Q_j|} \geq e^{\varepsilon n}$$

Therefore the eigenvalue sum in (1.16) can be estimated by

$$\sum_j e^{-\tau|\lambda_j|^{-d/2}} \geq \sum_n e^{\varepsilon n} = \infty.$$

The proof is completed. □

Theorem 1.2 implies that

$$\sum_j |\lambda_j|^\gamma = \infty$$

if $v_0 > 0$ is sufficiently large, while the number of negative eigenvalues of H

$$N(V) := \#\{\, j : \ \ \lambda_j < 0 \,\} < \infty$$

is finite if v_0 is sufficiently small. This transition from "finite" to "infinite" is somewhat unusual. It does not let us to agree that this result is complete. That is why we formulate the following question

Question: Are there parameters $v_0 > 0$ and $\gamma > 0$ such that the series

$$\sum_j |\lambda_j|^\gamma < \infty$$

converges, while

$$N(V) = \infty \ ?$$

We remind the reader that, in the multi-dimensional situation, the sharp borderline between the cases $N(V) < \infty$ and $N(V) = \infty$ is not yet established in terms of the values of v_0. Therefore, it is better to start solving this problem in the case $d = 1$ (see, for instance, the paper [4] which deals with the quantity $N(V)$).

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