INFINITELY MANY SOLUTIONS FOR FRACTIONAL LAPLACE EQUATIONS WITH SUBCRITICAL NONLINEARITY

RAFFAELLA SERVADEI

Dedicated to Patrizia Pucci
on the occasion of her sixtieth birthday,
with all my affection, esteem and gratitude.

Abstract. In this paper we discuss the existence of infinitely many solutions for a nonlocal, nonlinear equation with homogeneous Dirichlet boundary data. Our model problem is the following one

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{q-2}u + h & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $s \in (0,1)$ is a fixed parameter, $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n\,,$$

while λ is a real parameter, the exponent $q \in (2, 2^*)$, with $2^* = 2n/(n-2s)$, $n > 2s$, the function h belongs to the space $L^2(\Omega)$ and, finally, the set Ω is an open, bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Here the solution is sought to satisfy $u = 0$ on $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the fractional Laplace operator.

Adapting the classical variational techniques used in order to study the standard Laplace equation with subcritical growth nonlinearities to the nonlocal framework, along the present paper we prove that this problem admits infinitely many weak solutions u_k, with the property that their Sobolev norm goes to infinity as $k \to +\infty$, provided the exponent $q < 2^* - 2s/(n-2s)$. In this sense, the results presented here may be seen as the extension of some classical nonlinear analysis theorems to the case of fractional operators.

Contents

Key words and phrases. Fractional Laplacian, integrodifferential operators, variational and topological techniques, Palais–Smale condition, subcritical nonlinearities.

2010 AMS Subject Classification: Primary: 49J35, 35A15, 35S15; Secondary: 47G20, 45G05.

The author was supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena* and by the ERC grant ϵ (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*).

1. Introduction

The starting point of the present paper is represented by the following standard Laplace equation

$$\begin{cases} -\Delta u - \lambda u = |u|^{q-2}u + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where λ is a real parameter, the exponent $q \in (2, 2_*)$, 2_* is the classical critical Sobolev exponent given by $2_* = 2n/(n-2)$, $n > 2$, $h \in L^2(\Omega)$ and Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary.

This problem was widely studied and in the literature there are many existence, non-existence and multiplicity results about it (see, for instance, [6, 17, 18] and the references therein).

Since, recently, a great attention has been focused on the study of fractional Laplacian equations, the aim of the present paper is to consider the nonlocal counterpart of equation (1.1), namely the following problem

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{q-2}u + h & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.2}$$

Here, the parameter $s \in (0, 1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n \tag{1.3}$$

(see [5] and references therein for further details on the fractional Laplacian), while $q \in (2, 2^*)$, $2^* = 2n/(n-2s)$ is the fractional critical Sobolev exponent, $n > 2s$, $h \in L^2(\Omega)$ and Ω is an open, bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Here, the standard Dirichlet condition $u = 0$ in $\partial\Omega$ is replaced with the condition that the function u vanishes outside Ω, consistently with the non-local character of the operator $(-\Delta)^s$.

In [10, 11] the case $h \equiv 0$ in (1.2) was considered and the existence of a nontrivial weak solution was obtained using classical minimax theorems, namely the Mountain Pass and the Linking Theorems (see [1, 7]), according to different values of the parameter λ.

A natural question is whether or not classical existence results for equation (1.1) still hold in the nonlocal framework of (1.2). The aim of this paper is to answer this question, with respect to the existence of infinitely many solutions for the subcritical equation (1.2).

In [2, 3, 16] (see also [6, 17]) problem (1.1) was studied using different methods and the existence of infinitely many weak solutions for (1.1) (with the property that the L^2–norm of their gradient goes to infinity) was proved. Along this paper we will show that this result still holds true in the nonlocal setting.

Precisely, our existence result with respect to problem (1.2) can be stated as follows:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Assume also that $h \in L^2(\Omega)$ and $q \in (2, 2^* - 2s/(n-2s))$, with $2^* = 2n/(n-2s)$.*

Then, for any $\lambda \in \mathbb{R}$ problem (1.2) admits infinitely many weak solutions $u_k \in H^s(\mathbb{R}^n)$ such that $u_k = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \to +\infty$$

as $k \to +\infty$.

For a weak solution of problem (1.2) we mean a function u such that

$$(1.4)\qquad \begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\displaystyle = \int_\Omega |u(x)|^{q-2}u(x)\varphi(x)\,dx + \int_\Omega h(x)\varphi(x)\,dx \\ \qquad\qquad \forall\ \varphi \in H^s(\mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega \\ u \in H^s(\mathbb{R}^n) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Equation (1.2) represents only a model for the general problem studied along this work. Indeed, in the present paper we consider the following equation

$$(1.5)\qquad \begin{cases} \mathcal{L}_K u + \lambda u + f(x,u) + h(x) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega\,, \end{cases}$$

where the set $\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary and $s \in (0,1)$ is fixed, while $\mathcal{L}_K$ is the non-local operator defined as follows:

$$(1.6)\qquad \mathcal{L}_K u(x) = \int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big)K(y)\,dy\,, \quad x\in\mathbb{R}^n\,.$$

Here the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function such that

$$(1.7)\qquad mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,;$$

(1.8)

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\} ;$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\} . \tag{1.9}$$

A typical model for K is given by $K(x) = |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$ defined in (1.3).

Moreover, the nonlinear term in equation (1.5) is a function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verifying the following conditions:

$$f \text{ is a continuous function in } \overline{\Omega} \times \mathbb{R} ; \tag{1.10}$$

$$f(x, \cdot) \text{ is a } C^1(\mathbb{R})\text{–continuous function for any } x \in \overline{\Omega} ; \tag{1.11}$$

$$\begin{aligned} &\text{there exist } a_1, a_2 > 0 \text{ and } q \in (2, 2^*), 2^* = 2n/(n-2s) , \text{ such that} \\ &\qquad |f(x,t)| \leqslant a_1 + a_2 |t|^{q-1} \text{ for any } x \in \overline{\Omega},\ t \in \mathbb{R} ; \end{aligned} \tag{1.12}$$

$$\lim_{t \to 0} \frac{f(x,t)}{t} = 0 \text{ uniformly in } x \in \Omega ; \tag{1.13}$$

(1.14)

$$\text{there exist } \beta \in (0,1) \text{ and } r > 0 \text{ such that for any } x \in \overline{\Omega},\ t \in \mathbb{R},\ |t| \geqslant r$$
$$0 < \frac{f(x,t)}{t} \leqslant \beta f_t(x,t) ,$$

where f_t denotes the derivative of f with respect to the second variable, i.e.

$$f_t(x,t) = \frac{\partial f}{\partial t}(x,t) .$$

Note that these assumptions are the standard ones, when dealing with partial differential equations driven by the Laplace operator (or, more generally, by uniformly elliptic operators) with homogeneous Dirichlet boundary conditions. For this we refer to [2, Theorem 6.1], [6, p.233] and [17, Chapter II, Theorem 7.2] (see also [3, 16]) .

As a model for f we can take the odd nonlinearity $f(x,t) = a(x)|t|^{q-2}t$, with $a \in C(\overline{\Omega})$, $a > 0$ in $\overline{\Omega}$, and $q \in (2, 2^*)$. Finally, the perturbation $h : \Omega \to \mathbb{R}$ is a function such that

$$h \in L^2(\Omega) . \tag{1.15}$$

In order to give the weak formulation of problem (1.5), we need to work in a special functional space. Indeed, one of the difficulty in treating problem (1.5) is related to the encoding the Dirichlet boundary condition in the variational formulation. With this respect the standard fractional Sobolev spaces are not enough in order to study the problem. We overcome this difficulty working in a new functional space, whose definition will be recalled here below.

In the sequel the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

the map $(x,y)\mapsto (g(x)-g(y))\sqrt{K(x-y)}$ is in $L^2\big((\mathbb{R}^n\times\mathbb{R}^n)\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega), dxdy\big)$,

where $\mathcal{C}\Omega := \mathbb{R}^n\setminus\Omega$.

Moreover,

$$X_0 = \{g\in X : g=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\}\,.$$

Both X and X_0 are non-empty, since they contain the space $C^2_0(\Omega)$ (see, e.g., [9, Lemma 11], for this we need condition (1.7)). The spaces X and X_0 were introduced in [9] (see also [8, 10, 11, 12] for the definitions and the properties of these spaces).

The weak formulation of (1.5) is given by the following problem:

$$\begin{cases} u\in X_0, \\ \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy \\ \displaystyle -\lambda\int_\Omega u(x)\varphi(x)\,dx = \int_\Omega f(x,u(x))\varphi(x)dx + \int_\Omega h(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\,\varphi\in X_0\,. \end{cases} \tag{1.16}$$

In order to write problem (1.16) we need to assume (1.9).

The main result of the present paper is given by the following one:

Theorem 2. *Let $s\in(0,1)$, $n>2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ be a function satisfying conditions* (1.7)–(1.9)*, let $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ verify* (1.10)–(1.14) *with $q\in(2,2^*-2s/(n-2s))$, $2^*=2n/(n-2s)$, and let $h:\Omega\to\mathbb{R}$ satisfy* (1.15).

Then, for any $\lambda\in\mathbb{R}$ problem (1.5) *admits infinitely many weak solutions $u_k\in X_0$, such that*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_k(x)-u_k(y)|^2K(x-y)\,dx\,dy\to+\infty$$

as $k\to+\infty$.

Of course, when K and f are exactly as in the model, that is $K(x)=|x|^{-(n+2s)}$ and $f(x,t)=|t|^{q-2}t$, $t\in(2,2^*)$, then Theorem 2 reduces to Theorem 1.

We would like to note that Theorem 1 establishes the existence of infinitely many weak solutions for problem (1.5), provided $q<2^*-2s/(n-2s)$. When $s=1$, equation (1.2) reduces to the standard semilinear Laplace partial differential equation (1.1). In this framework the bound from above on the exponent q becomes $q<2(n-1)/(n-2)$, which is the usual one, when dealing with the Laplace equation (see [6, Chapter 5, Theorem 4.6 and Remark 4.7]). For all these reasons Theorems 1 and 2 may be seen as the fractional version of the classical existence results given in [6, Chapter 5, Theorem 4.6] in the model case $f(x,t) = |t|^{q-2}t$ and in [2, Theorem 6.1] for more general nonlinearities (see also [17, Chapter II, Theorem 7.2] and [3, 16]). Also, when $h \equiv 0$ (i.e. when there is no perturbation) Theorem 1 represents the nonlocal counterpart of the classical results obtained, for instance, in [1] (see also [17] and references therein).

The classical results stated in [6, Chapter 5, Theorem 4.6] and in [2, Theorem 6.1] are proved by means of variational and topological methods. In the present paper we will adapt these techniques to the nonlocal framework in order to get Theorem 2. The main difficulty, as in the standard case of the Laplacian, will be to prove the Palais–Smale condition for the energy functional associated with the problem.

The paper is organized as follows. In Section 2 we will give some notations, while Section 3 we will be devoted to the strategy used along the paper in order to get the main results. In Section 4 we will show that the energy functional satisfies the Palais-Smale condition and in Section 5 we will prove the main results of the paper.

2. Some notations

This section is devoted to the notations used along the present paper. In the sequel we set $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, with $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$.

We recall that the spaces X and X_0 are endowed, respectively, with the norms defined by

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x) - g(y)|^2 K(x-y) dx\, dy \Big)^{1/2}, \tag{2.1}$$

and

$$\|g\|_{X_0} = \Big(\int_Q |g(x) - g(y)|^2 K(x-y)\, dx\, dy \Big)^{1/2}. \tag{2.2}$$

With the norm given in formula (2.2), X_0 is a Hilbert space with scalar product defined as

$$\langle u, v \rangle_{X_0} = \int_Q \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, K(x-y)\, dx\, dy\,. \tag{2.3}$$

For this see [10, Lemma 7].

We also denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \Big)^{1/2}. \tag{2.4}$$

We remark that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.4) are not the same, because $\Omega \times \Omega$ is strictly contained in Q. This makes the classical fractional Sobolev space approach not sufficient for studying the problem.

For further details on the spaces X and X_0 and for their properties we refer to [8, 9, 10, 11, 12], while for the fractional Sobolev spaces we refer to [5] and to the references therein.

Finally, along the paper we denote by

$$\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$$

the divergent sequence of eigenvalues of $-\mathcal{L}_K$ and by e_k the k-th eigenfunction corresponding to the eigenvalue λ_k , namely

$$\begin{cases} -\mathcal{L}_K e_k = \lambda_k e_k & \text{in } \Omega \\ e_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega \,. \end{cases}$$

For the properties of the eigenvalues and the eigenfunctions of the non-local integrodifferential operator $-\mathcal{L}_K$ (and, in particular, of the fractional Laplacian $(-\Delta)^s$) we refer to [11, Proposition 9 and Appendix A] (see also [8, 13, 14]).

3. Plan for proving the main results

In this section, for reader's convenience, we illustrate our strategy in order to prove Theorem 2 .

First of all, we observe that problem (1.16) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_{K,\lambda} : X_0 \to \mathbb{R}$ defined as follows

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ - \int_\Omega F(x,u(x))\,dx - \int_\Omega h(x)u(x)\,dx \,. \end{aligned}$$

Here the function F is the primitive of f with respect to its second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)d\tau \,. \tag{3.1}$$

Notice that the functional $\mathcal{J}_{K,\lambda}$ is well defined thanks to assumptions (1.10) and (1.12), to [12, Lemma 9-a)] and also thanks to condition (1.15) . Moreover, $\mathcal{J}_{K,\lambda}$ is Fréchet differentiable at $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned} \langle \mathcal{J}'_{K,\lambda}(u), \varphi\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big) K(x-y)\,dx\,dy \\ - \lambda\int_\Omega u(x)\varphi(x)\,dx - \int_\Omega f(x,u(x))\varphi(x)\,dx - \int_\Omega h(x)\varphi(x)\,dx \,. \end{aligned}$$

Hence, in order to prove Theorem 2, we will look for critical points of the functional $\mathcal{J}_{K,\lambda}$. To this purpose, let us introduce the functional $\mathcal{I}_{K,\lambda} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{I}_{K,\lambda}(u) = \max_{\tau>0} \mathcal{J}_{K,\lambda}(\tau u) \,.$$

As we will show below, the critical points of functionals $\mathcal{J}_{K,\lambda}$ and $\mathcal{I}_{K,\lambda}$ are strictly related the ones to the others, indeed the following result holds true:

Proposition 3. *Let $\lambda \in \mathbb{R}$, $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.7)–(1.9)*, let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verify* (1.10)–(1.14) *and let $h : \Omega \to \mathbb{R}$ satisfy* (1.15) *.*

Then, there exists a positive constant α such that

a) the functional $\mathcal{I}_{K,\lambda}$ is of class C^1 in $\{\mathcal{I}_{K,\lambda} > \alpha\}$;

b) for any $v \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ there exists a unique $\sigma(v) > 0$ such that

$$\mathcal{I}_{K,\lambda}(v) = \mathcal{J}_{K,\lambda}(\sigma(v)v)\,.$$

As a consequence, for any $v \in \{\mathcal{I}_{K,\lambda} > \alpha\}$

$$\mathcal{I}'_{K,\lambda}(v) = \sigma(v)\mathcal{J}'_{K,\lambda}(\sigma(v)v)\,.$$

Proof. Let $v \in X_0$. First of all, note that, by definition of $\mathcal{I}_{K,\lambda}$ we have

$$\mathcal{I}_{K,\lambda}(tv) = \mathcal{I}_{K,\lambda}(v) \text{ for any } t > 0\,.$$

Hence, without loss of generality, we can assume that $\|v\|_{L^q(\Omega)} = 1$.

From now on, thanks in particular to assumptions (1.11) and (1.14), we can argue exactly as in [2, Propositions 6.2 and 6.3], where the classical case of the Laplacian (in general of second order elliptic operators) was considered: we have just to replace the L^2–norm of the gradient with the X_0–norm. □

As a consequence of Proposition 3 we have the next result, which relates the critical points of $\mathcal{I}_{K,\lambda}$ to the ones of $\mathcal{J}_{K,\lambda}$. In the following M will denote the set

$$M := \big\{u \in X_0 : \|u\|_{L^q(\Omega)} = 1\big\}\,,$$

where $q \in (2, 2^*)$. Note that M is well defined, since the space X_0 is compactly embedded into $L^q(\Omega)$, by [12, Lemma 9].

Corollary 4. *Let all the assumptions of Proposition 3 be satisfied and let α and $\sigma(v)$ be as in Proposition 3. Then, the following assertions hold true:*

a) if $v \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ is a critical point of $\mathcal{I}_{K,\lambda}$, then $\sigma(v)v$ is a critical point of $\mathcal{J}_{K,\lambda}$;

b) if $v \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ is a critical point of $\mathcal{I}_{K,\lambda}$ on M, then v is a critical point of $\mathcal{I}_{K,\lambda}$ on X_0 .

Proof. Assertion *a)* is a consequence of the fact that $\sigma(v) > 0$ and of Proposition 3-*b)*. For part *b)* note that if $v \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ is a critical point of $\mathcal{I}_{K,\lambda}$ on M, then there exists $\varrho \in \mathbb{R}$ such that

$$\langle \mathcal{I}'_{K,\lambda}(v), \varphi\rangle = \varrho \int_\Omega |v(x)|^{q-2}v(x)\varphi(x)\,dx \tag{3.2}$$

for any $\varphi \in X_0$. In particular, taking $\varphi = v$, since $v \in M$ we get

$$\varrho = \langle \mathcal{I}'_{K,\lambda}(v), v \rangle .$$

Hence, by Proposition 3-*b*) and by part *a*) we have

$$\varrho = \langle \mathcal{I}'_{K,\lambda}(v), v \rangle = \sigma(v)\langle \mathcal{J}'_{K,\lambda}(\sigma(v)v), v \rangle = 0 .$$

As a consequence of this and of (3.2) we deduce that $\langle \mathcal{I}'_{K,\lambda}(v), \varphi \rangle = 0$ for any $\varphi \in X_0$, which proves assertion *b*) . □

Remark 5. We would like to note that in the model case $f(x,t) = |t|^{q-2}t$, $q \in (2, 2^*)$, in the assertions of Proposition 3 and of Corollary 4 the open set $\{\mathcal{I}_{K,\lambda} > \alpha\}$ can be replaced with $\{\mathcal{I}_{K,\lambda} > 0\}$.

Thanks to Corollary 4, in order to find critical points of $\mathcal{J}_{K,\lambda}$ it is enough to look for critical points $u \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ of the functional $\mathcal{I}_{K,\lambda}$ on M . Hence, in order to prove Theorem 2, this will be our strategy, namely we will look for critical points $u \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ of $\mathcal{I}_{K,\lambda}$ on M .

To this purpose we will apply [6, Chapter 5, Theorem 2.2] to the functional $\mathcal{I}_{K,\lambda}$. For this we have to prove that

i) the functional $\mathcal{I}_{K,\lambda}$ satisfies the *Palais–Smale compactness condition* on M ;

ii) for $k \in \mathbb{N}$ large enough it hold true that $a_k \in \mathbb{R}$ and $a_k < b_k$, where a_k and b_k are defined as follows

$$a_k := \inf_{A \in \mathcal{A}_k} \max_{v \in A} \mathcal{I}_{K,\lambda}(v) , \tag{3.3}$$

$$b_k := \inf_{B \in \mathcal{B}_k} \max_{v \in B} \mathcal{I}_{K,\lambda}(v) \tag{3.4}$$

with $\mathcal{A}_k$ and $\mathcal{B}_k$ given by

$$\mathcal{A}_k := \Big\{ g(S^{k-1}) \text{ such that } g : S^{k-1} \to M \text{ is continuous and odd} \Big\} ,$$

$$\mathcal{B}_k := \Big\{ g(S^k_+) \text{ such that } g : S^k_+ \to M \text{ is continuous and } g_{|S^k_+} \text{ is odd} \Big\} .$$

Here S^k is the unit ball in $\mathbb{R}^{k+1}$, while S^k_+ is the northern hemisphere of S^k that is the set

$$S^k_+ = \Big\{ (u_1, \dots, u_{k+1}) \in S^k \text{ such that } u_{k+1} \geqslant 0 \Big\} .$$

In order to prove the properties stated above with respect to the functional $\mathcal{I}_{K,\lambda}$ we need some preliminary results.

3.1. **Some preliminary lemmas.** In this subsection we prove some auxiliary results on $\mathcal{I}_{K,\lambda}$ which will be useful in the sequel. Here assumptions (1.11) and (1.14) will be crucial.

First we need some notations. In what follows $\mathcal{I}^0_{K,\lambda}$ will be the even functional $\mathcal{I}_{K,\lambda}$ obtained in the particular case when $h \equiv 0$, that is

$$\mathcal{I}^0_{K,\lambda}(u) = \max_{\tau > 0} \mathcal{J}^0_{K,\lambda}(\tau u) ,$$

where

$$\mathcal{J}^0_{K,\lambda}(u)=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dx\,dy-\frac{\lambda}{2}\int_\Omega|u(x)|^2\,dx \\ -\int_\Omega F(x,u(x))\,dx\,.$$

Moreover, in this case we also will denote by $\sigma_0(v)$ the element given in Proposition 3-b), i.e. $\sigma_0(v)$ will be such that

$$\mathcal{I}^0_{K,\lambda}(v)=\mathcal{J}^0_{K,\lambda}(\sigma_0(v)v)\,.$$

According to these notations we have the following result:

Lemma 6. *Let $\lambda\in\mathbb{R}$, $s\in(0,1)$, $n>2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ be a function satisfying conditions* (1.7)*–*(1.9)*, let $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ verify* (1.10)*–*(1.14) *and let $h:\Omega\to\mathbb{R}$ satisfy* (1.15)*. Finally, let α be as in Proposition 3.*

Then, there exist two positive constants $\bar{R}$ and $\bar{c}$, depending only on α, λ, q, $|\Omega|$ and $\|h\|_{L^2(\Omega)}$, such that for any $v\in M$ with $\|v\|_{X_0}\geqslant\bar{R}$, the following assertions hold true:

a) $\mathcal{I}_{K,\lambda}(v)>\alpha$*;*
b) $|\mathcal{I}_{K,\lambda}(v)-\mathcal{I}^0_{K,\lambda}(v)|\leqslant\bar{c}\left(1+\mathcal{I}_{K,\lambda}(v)^{1/q}\right)$*;*
c) $|\mathcal{I}_{K,\lambda}(v)-\mathcal{I}^0_{K,\lambda}(v)|\leqslant\bar{c}\Big(1+\mathcal{I}^0_{K,\lambda}(v)^{1/q}\Big)$.

Proof. Let us start by proving assertion a). Thanks to the definition of $\mathcal{I}_{K,\lambda}$, in order to get our goal it is enough to show that

$$\mathcal{J}_{K,\lambda}(v)>\alpha\,, \tag{3.5}$$

when $v\in M$ is such that $\|v\|_{X_0}$ is large enough.

To this purpose, note that (1.12), (1.15) and the Hölder inequality yield

$$\begin{aligned}\mathcal{J}_{K,\lambda}(v)&=\frac{1}{2}\|v\|^2_{X_0}-\frac{\lambda}{2}\|v\|^2_{L^2(\Omega)}-\int_\Omega F(x,u_j(x))\,dx-\int_\Omega h(x)v(x)\,dx\\&\geqslant\frac{1}{2}\|v\|^2_{X_0}-\frac{\lambda}{2}\|v\|^2_{L^2(\Omega)}-a_1\|v\|_{L^1(\Omega)}-\frac{a_2}{q}\|v\|^q_{L^q(\Omega)}-\|h\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\&\geqslant\frac{1}{2}\|v\|^2_{X_0}-\kappa-a_1|\Omega|^{(q-1)/q}\|v\|_{L^q(\Omega)}-\frac{a_2}{q}\|v\|^q_{L^q(\Omega)}\\&=\frac{1}{2}\|v\|^2_{X_0}-\kappa-a_1|\Omega|^{(q-1)/q}-\frac{a_2}{q}\,,\end{aligned}$$

when $v\in M$. Here κ is a positive constant given by

$$\kappa=\max\left\{\left(\lambda\,|\Omega|^{(q-2)/q}\right)/2+|\Omega|^{(q-2)/(2q)}\|h\|_{L^2(\Omega)}\,,|\Omega|^{(q-2)/(2q)}\|h\|_{L^2(\Omega)}\right\}.$$

Hence, if $\|v\|_{X_0}$ is sufficiently large, we get (3.5) and this ends the proof of a).

Taking into account that part a) holds true, assertions b) and c) can be proved as in [2, Lemma 6.8]. □

Also, in the sequel we denote by a_k^0 and b_k^0 the quantity

$$a_k^0 := \inf_{A\in\mathcal{A}_k} \max_{v\in A} \mathcal{I}^0_{K,\lambda}(v)\,, \tag{3.6}$$

$$b_k^0 := \inf_{B\in\mathcal{B}_k} \max_{v\in B} \mathcal{I}^0_{K,\lambda}(v) \tag{3.7}$$

with $\mathcal{A}_k$ and $\mathcal{B}_k$ given as above, for any $k \in \mathbb{N}$.

About the sequence a_k defined in (3.3) we have the following result:

Lemma 7. *Let $\lambda \in \mathbb{R}$, $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.7)–(1.9)*, let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verify* (1.10)–(1.14) *and let $h : \Omega \to \mathbb{R}$ satisfy* (1.15) *.*

Then, there exist $\tilde{c} > 0$ and $\tilde{\kappa} \in \mathbb{N}$ such that for any $k \geqslant \tilde{\kappa}$

$$a_k \geqslant \tilde{c}\, k^{2sq/(n(q-2))}\,.$$

In particular, $a_k \to +\infty$ as $k \to +\infty$.

Proof. Here we can argue exactly as in [2, Proposition 6.10]. To this purpose we can use also the properties of the eigenvalues of $(-\Delta)^s$ (see [15]). $\square$

In the sequel we also need the following result, which is a sort of Gronwall Lemma in a discrete framework: its proof can be found in [6, Chapter 5, Lemma 4.5].

Lemma 8. *Let $p > 1$ and α_k be a sequence such that $\alpha_k > 0$ for any $k \in \mathbb{N}$. Assume that there exist a positive constant c_o and $\kappa_o \in \mathbb{N}$ such that*

$$\alpha_{k+1} \leqslant \alpha_k + c_o(1 + \alpha_k^{1/(p+1)}) \ \textit{for any}\ k \geqslant \kappa_o\,.$$

Then, there exist a positive constant c^* and $\kappa^* \in \mathbb{N}$ such that

$$\alpha_k \leqslant c^* k^{(p+1)/p} \quad \textit{for any}\ k \geqslant \kappa^*\,. \tag{3.8}$$

Now, we are ready to prove the properties stated above in $i)$ and $ii)$: this will be done in the forthcoming sections.

4. Some compactness conditions

In this section we prove that the functional $\mathcal{I}_{K,\lambda}$ satisfies the property $i)$, that is the Palais–Smale compactness condition on M . To this purpose, first of all we need to show that the energy functional $\mathcal{J}_{K,\lambda}$ associated with problem (1.6) satisfies the Palais–Smale condition in X_0 at any level $c \in \mathbb{R}$. This will be accomplished in the forthcoming Subsection 4.1 .

4.1. **The Palais–Smale condition for $\mathcal{J}_{K,\lambda}$.** This subsection is devoted to the proof of the Palais–Smale condition for the functional $\mathcal{J}_{K,\lambda}$. The proof can be performed as in [11, Propositions 13, 14 and 20]. Due to the presence of the extra term $\int_\Omega h(x)u(x)\,dx$ in the functional $\mathcal{J}_{K,\lambda}$, we prefer to repeat the calculation, for reader's convenience and also for the sake of clarity.

In order to perform our proof we need some estimates on the nonlinearity f and its primitive F, given in the following result whose proof can be found in [10, Lemma 3]:

Lemma 9. *Assume $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a function satisfying conditions* (1.10)*,* (1.12) *and* (1.13)*. Then, for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)$ such that for any $x\in\overline{\Omega}$ and $t\in\mathbb{R}$*

$$|f(x,t)|\leqslant 2\varepsilon|t|+q\delta(\varepsilon)|t|^{q-1}$$

and so, as a consequence,

$$|F(x,t)|\leqslant \varepsilon\,|t|^2+\delta(\varepsilon)\,|t|^q\,,$$

where F is defined as in (3.1)*.*

With the next result we show that the function F (i.e. the primitive of f, see (3.1)) satisfies a superquadratic growth condition and, moreover, F and f verify a sort of Ambrosetti–Rabinowitz condition (see [1, 7]).

Lemma 10. *Assume $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a function satisfying conditions* (1.10)*,* (1.11) *and* (1.14)*. Let F be as in* (3.1) *and $\mu=(\beta+1)/\beta>2$. Then,*

a) there exist two positive constant a_3 and a_4 such that for any $x\in\overline{\Omega}$ and $t\in\mathbb{R}$
$$|F(x,t)|\geqslant a_3|t|^\mu-a_4\,;$$
b) there exists a positive constant a_5 such that for any $x\in\overline{\Omega}$ and $t\in\mathbb{R}$, $|t|\geqslant r$
$$0<\mu F(x,t)\leqslant tf(x,t)+a_5\,.$$

Proof. First of all, let us prove assertion $a)$. Let $r>0$ be as in (1.14): then, for any $x\in\overline{\Omega}$ and $t\in\mathbb{R}$ with $|t|\geqslant r>0$

$$\frac{f_t(x,t)}{f(x,t)}\geqslant\frac{1}{\beta t}\,.$$

Here we use also the fact that $tf(x,t)>0$ for $|t|\geqslant r$.

Now, suppose that $t>r$. Integrating both terms in $[r,t]$ we obtain

$$|f(x,t)|=f(x,t)\geqslant\frac{f(x,r)}{r^{1/\beta}}\,t^\beta\,.$$

With the same arguments and using again the fact that $tf(x,t)>0$ when $|t|\geqslant r$ it is easy to prove that if $x\in\overline{\Omega}$ and $t<-r$ then it holds

$$|f(x,t)|\geqslant\frac{|f(x,-r)|}{r^{1/\beta}}\,|t|^{1/\beta}\,,$$

so that for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r$ we get

$$|f(x,t)| \geqslant \tilde{m}(x)\,|t|^{1/\beta}\,, \tag{4.1}$$

where $\tilde{m}(x) = r^{-1/\beta} \min\{f(x,r),\, |f(x,-r)|\}$.

Note that $\tilde{m}$ is a continuous functions, since $x \mapsto f(x,\cdot)$ does. Moreover, $\tilde{m}$ is positive, being $f(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \geqslant r$ (see (1.14)). Hence, for any $x \in \overline{\Omega}$

$$\tilde{m}(x) \geqslant \min_{x \in \overline{\Omega}} \tilde{m}(x) =: \tilde{m} > 0\,.$$

This and (4.1) give

$$|f(x,t)| \geqslant \tilde{m}\,|t|^{1/\beta} \tag{4.2}$$

for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \geqslant r$.

Now, let $t > r$. Integrating both terms in (4.2) in $[r,t]$ we get

$$F(x,t) \geqslant a_3\,|t|^{\mu} + F(x,r) - a_3 r^{\mu}\,, \tag{4.3}$$

where a_3 is a positive constant and $\mu = (\beta+1)/\beta$. Note that $\mu > 2$, since $\beta \in (0,1)$ by assumption.

Arguing in the same way it is easily seen that if $t < -r$ then

$$F(x,t) \geqslant a_3\,|t|^{\mu} + F(x,-r) - a_3 r^{\mu}\,,$$

so that, by this and (4.3) we obtain that for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r$

$$F(x,t) \geqslant a_3\,|t|^{\mu} + \hat{m} - a_3 r^{\mu} \geqslant a_3\,|t|^{\mu} - a_3 r^{\mu}\,, \tag{4.4}$$

where $\hat{m}(x) = \min\{F(x,r),\, F(x,-r)\} > 0$ (note that $F(x,r) > 0$ since $tf(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, with $|t| \geqslant r$).

Also, the function F is continuous for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \leqslant r$, so that, by the Weierstrass Theorem, it is bounded, say

$$|F(x,t)| \leqslant \widetilde{M} \quad \text{in } \overline{\Omega} \times \{|t| \leqslant r\}\,, \tag{4.5}$$

where $\widetilde{M}$ is a positive constant.

Taking $a_4 := \widetilde{M} + a_3 r^{\mu} > 0$ from (4.4) and (4.5) it follows that

$$F(x,t) \geqslant a_3\,|t|^{\mu} - a_4 \tag{4.6}$$

for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. This concludes the proof of part *a*).

Now, let us show assertion *b*). For this note that, since $tf(x,t) > 0$ when $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r$, then $F(x,t) > 0$ in the same set.

Let $r > 0$ be as in (1.14): then, for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r > 0$

$$f(x,t) \leqslant \beta t f_t(x,t)\,.$$

Suppose $t > r$. Integrating both terms in $[r,t]$ we obtain

$$0 < \mu F(x,t) \leqslant t f(x,t) + \mu F(x,r) - r f(x,r)\,,$$

so that, taking into account that $rf(x,r) > 0$, we have

$$0 < \mu F(x,t) \leqslant t f(x,t) + \mu F(x,r) \tag{4.7}$$

for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, $|t| \geqslant r$.

With the same arguments it is easy to prove that if $t < -r$ then it holds

$$0 < \mu F(x,t) \leqslant tf(x,t) + \mu F(x,-r) + rf(x,-r)\,,$$

and so

$$0 < \mu F(x,t) \leqslant tf(x,t) + \mu F(x,-r)\,,$$

since $rf(x,-r) < 0$.

Hence, from this and (4.7) for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r$ we get

$$0 < \mu F(x,t) \leqslant tf(x,t) + M(x)\,, \tag{4.8}$$

where $M(x) = \mu \max\{F(x,r), F(x,-r)\} > 0$, being $F(x,t) > 0$.

Taking $a_5 := \max_{x\in\overline{\Omega}} M(x)$, we get assertion *b*) . Note that a_5 exists by assumption (1.10) and the Weierstrass Theorem. □

Assertion *b*) in Lemma 10 is classical and it is a sort of Ambrosetti–Rabinowitz condition, which, usually, it is assumed in order to prove that the energy functional satisfies the Palais–Smale compactness property. Of course, our condition (1.14) is stronger than assertion *b*), but, in a sense, they have the same nature.

Now we are ready to prove the main result of the present subsection.

Proposition 11. *Let $\lambda \in \mathbb{R}$, $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.7)*–*(1.9)*, let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verify* (1.10)*–*(1.14) *and let $h : \Omega \to \mathbb{R}$ satisfy* (1.15) *. Let $c \in \mathbb{R}$ and let u_j be a sequence in X_0 such that*

$$\mathcal{J}_{K,\lambda}(u_j) \to c \tag{4.9}$$

and

$$\sup\Big\{\big|\langle\, \mathcal{J}'_{K,\lambda}(u_j), \varphi\,\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \tag{4.10}$$

as $j \to +\infty$.

Then, the sequence u_j is bounded in X_0 .

Proof. For any $j \in \mathbb{N}$ by (4.9) and (4.10) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_{K,\lambda}(u_j)| \leqslant \kappa\,, \tag{4.11}$$

and

$$\Big|\langle \mathcal{J}'_{K,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa\,. \tag{4.12}$$

As a consequence of (4.11) and (4.12) we also have

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\mu}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \leqslant \kappa\,(1 + \|u_j\|_{X_0})\ . \tag{4.13}$$

Moreover, by Lemma 9 applied with $\varepsilon = 1$ we have that

$$\begin{aligned}\Big|\int_{\Omega\cap\{|u_j|\leqslant r\}}&\Big(F(x,u_j(x)) - \frac{1}{\mu}\,f(x,u_j(x))\,u_j(x)\Big)\,dx\Big| \\ &\leqslant \Big(r^2+\delta(1)r^q+\frac{2}{\mu}\,r+\frac{q}{\mu}\,\delta(1)r^{q-1}\Big)\,|\Omega| =: \tilde{\kappa}\,.\end{aligned} \tag{4.14}$$

Now, we have to distinguish two different cases, according to the fact that λ is less or greater than the first eigenvalue λ_1 of $-\mathcal{L}_K$ (see Section 2 for the notation), namely $\lambda < \lambda_1$ or $\lambda \geqslant \lambda_1$.

Let us start with the case when $\lambda < \lambda_1$. Then, by the variational characterization of λ_1 (see [11, Proposition 9-*a*)]), Lemma 10-*b*), (1.15), (4.14) and Hölder inequality we get

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\mu}\langle \mathcal{J}'_{K,\lambda}(u_j),u_j\rangle &= \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\Big(\|u_j\|^2_{X_0}-\lambda\|u\|^2_{L^2(\Omega)}\Big) \\ &\quad - \frac{1}{\mu}\int_\Omega \Big(\mu F(x,u_j(x)) - f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\quad - \Big(1-\frac{1}{\mu}\Big)\int_\Omega h(x)u_j(x)\,dx \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\,m_1^\lambda\|u_j\|^2_{X_0} \\ &\quad - \int_{\Omega\cap\{|u_j|\leqslant r\}}\Big(F(x,u_j(x)) - \frac{1}{\mu}\,f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\quad - a_5 - \Big(1-\frac{1}{\mu}\Big)\|h\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\,m_1^\lambda\|u_j\|^2_{X_0} - \bar{\kappa} - \hat{\kappa}\|h\|_{L^2(\Omega)}\|u_j\|_{X_0}\end{aligned} \tag{4.15}$$

for some $\bar{\kappa}$ and $\hat{\kappa}$ positive. Here $m_1^\lambda := \min\big\{1, 1-\lambda/\lambda_1\big\}$.

By (4.13) and (4.15), for any $j \in \mathbb{N}$

$$\|u_j\|^2_{X_0} \leqslant \kappa_*\,(1+\|u_j\|_{X_0})$$

for a suitable positive constant κ_*. Hence, the assertion of Proposition 11 is proved, when $\lambda < \lambda_1$.

Now, let us consider the case when $\lambda \geqslant \lambda_1$. In this context the arguments are the same as when $\lambda < \lambda_1$. Nevertheless, in this situation we can not use the variational characterization of the first eigenvalues of $-\mathcal{L}_K$. For this reason some technical difficulties arise and, in order to overcome these problems, we need to introduce an additional parameter γ.

Note that, as we said in Section 2, the sequence λ_k of the eigenvalues of $-\mathcal{L}_K$ diverges as $k \to +\infty$. Hence, since $\lambda \geqslant \lambda_1$, we can suppose that

$$\lambda \in [\lambda_k, \lambda_{k+1}) \ \text{ for some } k \in \mathbb{N}\,.$$

Now, let us fix $\gamma\in(2,\mu)$, where $\mu>2$ is given in Lemma 10. As in (4.14), by Lemma 9 applied with $\varepsilon=1$ we have that

$$\begin{aligned}\Big|\int_{\Omega\cap\{|u_j|\leqslant r\}}\Big(F(x,u_j(x))-\frac{1}{\gamma}f(x,u_j(x))\,u_j(x)\Big)\,dx\Big|&\\ \leqslant\Big(r^2+\delta(1)r^q+\frac{2}{\gamma}\,r+\frac{q}{\gamma}\,\delta(1)r^{q-1}\Big)\,|\Omega|=:\tilde{\kappa}\,,&\end{aligned}\tag{4.16}$$

so that, using also Lemma 10-b), Hölder inequality, (1.15) and Lemma 9,

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j)-\frac{1}{\gamma}\langle\mathcal{J}'_{K,\lambda}(u_j),u_j\rangle=&\Big(\frac12-\frac1\gamma\Big)\Big(\|u_j\|^2_{X_0}-\lambda\|u_j\|^2_{L^2(\Omega)}\Big)\\ &-\int_\Omega\Big(F(x,u_j(x))-\frac1\gamma f(x,u_j(x))\,u_j(x)\Big)\,dx\\ &-\Big(1-\frac1\gamma\Big)\int_\Omega h(x)u_j(x)\,dx\\ \geqslant&\Big(\frac12-\frac1\gamma\Big)\Big(\|u_j\|^2_{X_0}-\lambda\|u_j\|^2_{L^2(\Omega)}\Big)\\ &+\Big(\frac\mu\gamma-1\Big)\int_{\Omega\cap\{|u_j|\geqslant r\}}F(x,u_j(x))\,dx-a_5\\ &-\int_{\Omega\cap\{|u_j|\leqslant r\}}\Big(F(x,u_j(x))-\frac1\gamma f(x,u_j(x))\,u_j(x)\Big)\,dx\\ &-\kappa^*\|h\|_{L^2(\Omega)}\|u_j\|_{X_0}\\ \geqslant&\Big(\frac12-\frac1\gamma\Big)\Big(\|u_j\|^2_{X_0}-\lambda\|u_j\|^2_{L^2(\Omega)}\Big)\\ &+\Big(\frac\mu\gamma-1\Big)\int_{\Omega\cap\{|u_j|\geqslant r\}}F(x,u_j(x))\,dx-\bar\kappa\\ &-\kappa^*\|h\|_{L^2(\Omega)}\|u_j\|_{X_0}\\ \geqslant&\Big(\frac12-\frac1\gamma\Big)\Big(\|u_j\|^2_{X_0}-\lambda\|u_j\|^2_{L^2(\Omega)}\Big)\\ &+a_3\Big(\frac\mu\gamma-1\Big)\|u_j\|^\mu_{L^\mu(\Omega)}-a_4\Big(1-\frac\mu\gamma\Big)\,|\Omega|-\bar\kappa\\ &-\kappa^*\|h\|_{L^2(\Omega)}\|u_j\|_{X_0}\,.\end{aligned}\tag{4.17}$$

Here $\bar\kappa$ and κ^* are positive constants. Moreover, for any $\varepsilon>0$ the Young inequality (applied here with $\mu/2$ and its conjugate $\mu/(\mu-2)$, $\mu>2$ by assumption, see Lemma 10) gives

$$\|u_j\|^2_{L^2(\Omega)}\leqslant\frac{2\varepsilon}{\mu}\,\|u_j\|^\mu_{L^\mu(\Omega)}+\frac{\mu-2}{\mu}\,\varepsilon^{-2/(\mu-2)}\,|\Omega|\,.\tag{4.18}$$

Hence, by (4.17) and (4.18) we deduce that

$$
\begin{aligned}
\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\gamma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \geqslant & \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_j\|^2_{X_0} \\
& - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2\varepsilon}{\mu}\|u_j\|^{\mu}_{L^\mu(\Omega)} \\
& - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{\mu-2}{\mu}\,\varepsilon^{-2/(\mu-2)}\,|\Omega| \\
& + a_3\left(\frac{\mu}{\gamma}-1\right)\|u_j\|^{\mu}_{L^\mu(\Omega)} \\
& - a_4\left(1-\frac{\mu}{\gamma}\right)|\Omega| - \bar{\kappa} \\
& - \kappa^*\|h\|_{L^2(\Omega)}\|u_j\|_{X_0} \\
= & \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_j\|^2_{X_0} \\
& + \left[a_3\left(\frac{\mu}{\gamma}-1\right) - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2\varepsilon}{\mu}\right]\|u_j\|^{\mu}_{L^\mu(\Omega)} \\
& - C_\varepsilon - \kappa^*\|h\|_{L^2(\Omega)}\|u_j\|_{X_0}\,,
\end{aligned}
\tag{4.19}
$$

where C_ε is a constant such that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$, being $\mu > \gamma > 2$.

Now, choosing ε so small that

$$
a_3\left(\frac{\mu}{\gamma}-1\right) - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2\varepsilon}{\mu} > 0\,,
$$

by (4.19) we get

$$
\begin{aligned}
\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\gamma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \geqslant & \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_j\|^2_{X_0} - C_\varepsilon \\
& - \kappa^*\|h\|_{L^2(\Omega)}\|u_j\|_{X_0}\,.
\end{aligned}
\tag{4.20}
$$

By (4.13) and (4.20) for any $j \in \mathbb{N}$

$$
\|u_j\|^2_{X_0} \leqslant \kappa_*\,(1+\|u_j\|_{X_0})
$$

for a suitable positive constant κ_*. Hence, the assertion of Proposition 11 is proved also in the case when $\lambda \geqslant \lambda_1$. This ends the proof of Proposition 11. □

Now, we are ready to prove the validity of the Palais–Smale condition on X_0 for the functional $\mathcal{J}_{K,\lambda}$ for any value of the parameter $\lambda \in \mathbb{R}$.

Proposition 12. *Let $\lambda \in \mathbb{R}$, $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.7)*–*(1.9)*, let $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (1.10)*–*(1.14) *and let $h : \Omega \to \mathbb{R}$ verify* (1.15)*. Let u_j be a sequence in X_0 such that u_j is bounded in X_0 and* (4.10) *holds true. Then,* *there exists* $u_\infty \in X_0$ *such that, up to a subsequence,* $\|u_j - u_\infty\|_{X_0} \to 0$ *as* $j \to +\infty$ *.*

Proof. Since u_j is bounded in X_0 and X_0 is a reflexive space (being a Hilbert space, by [10, Lemma 7]), up to a subsequence, still denoted by u_j, there exists $u_\infty \in X_0$ such that $u_j \rightharpoonup u_\infty$ weakly in X_0, that is

$$\begin{aligned}(4.21)\qquad &\int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x) - u_j(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \to \\ &\qquad \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_\infty(x) - u_\infty(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy\end{aligned}$$

for any $\varphi \in X_0$ as $j \to +\infty$. Moreover, by [10, Lemma 8], up to a subsequence,

$$(4.22)\qquad \begin{aligned} u_j &\to u_\infty \quad \text{in } L^2(\mathbb{R}^n)\\ u_j &\to u_\infty \quad \text{in } L^q(\mathbb{R}^n)\\ u_j &\to u_\infty \quad \text{a.e. in } \mathbb{R}^n\end{aligned}$$

as $j \to +\infty$ and there exists $\ell \in L^q(\mathbb{R}^n)$ such that

$$(4.23)\qquad |u_j(x)| \leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}$$

(see, for instance [4, Theorem IV.9]).

By (1.12), (4.21)–(4.23), the fact that the map $t \mapsto f(\cdot, t)$ is continuous in $t \in \mathbb{R}$ (see assumption (1.10)) and the Dominated Convergence Theorem we get

$$(4.24)\qquad \int_\Omega f(x, u_j(x))u_j(x)\,dx \to \int_\Omega f(x, u_\infty(x))u_\infty(x)\,dx$$

and

$$(4.25)\qquad \int_\Omega f(x, u_j(x))u_\infty(x)\,dx \to \int_\Omega f(x, u_\infty(x))u_\infty(x)\,dx$$

as $j \to +\infty$. Moreover, by (4.10) and the boundedness of the sequence u_j in X_0 we have that $\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \to 0$, that is

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y)\,dx\,dy - \lambda \int_\Omega |u_j(x)|^2\,dx \\ &\qquad - \int_\Omega f(x, u_j(x))u_j(x)\,dx - \int_\Omega h(x)u_j(x)\,dx \to 0\end{aligned}$$

as $j \to +\infty$. Consequently, recalling also (4.22) and (4.24), we deduce that

$$\begin{aligned}(4.26)\qquad &\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y)\,dx\,dy \to \lambda \int_\Omega |u_\infty(x)|^2\,dx \\ &\qquad + \int_\Omega f(x, u_\infty(x))u_\infty(x)\,dx + \int_\Omega h(x)u_\infty(x)\,dx\end{aligned}$$

as $j \to +\infty$.

Furthermore, using again (4.10), we have $\langle \mathcal{J}'_{K,\lambda}(u_j), u_\infty\rangle \to 0$, that is

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy \\
&\qquad -\lambda\int_\Omega u_j(x)u_\infty(x)\,dx - \int_\Omega f(x,u_j(x))u_\infty(x)\,dx \\
&\qquad -\int_\Omega h(x)u_\infty(x)\,dx \to 0
\end{aligned}\tag{4.27}$$

as $j\to+\infty$. By (4.21), (4.22), (4.25) and (4.27) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy &= \lambda\int_\Omega |u_\infty(x)|^2\,dx \\
&\quad+\int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx + \int_\Omega h(x)u_\infty(x)\,dx\,.
\end{aligned}\tag{4.28}$$

Thus, (4.26) and (4.28) give that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy \to \int_{\mathbb{R}^n\times\mathbb{R}^n} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy\,,$$

namely

$$\|u_j\|_{X_0}\to\|u_\infty\|_{X_0}$$

as $j\to+\infty$. Since $u_j\rightharpoonup u_\infty$ weakly in the Hilbert space X_0 and $\|u_j\|_{X_0}\to\|u_\infty\|_{X_0}$, we conclude that $u_j\to u_\infty$ strongly in X_0 as $j\to+\infty$. □

As a consequence of Proposition 12 we can prove that the functional $\mathcal{I}_{K,\lambda}$ satisfies the Palais–Smale condition on M for any value of the parameter $\lambda\in\mathbb{R}$. This will be done in the next subsection.

4.2. The Palais–Smale condition on M for $\mathcal{I}_{K,\lambda}$.

Along this subsection we show that the functional $\mathcal{I}_{K,\lambda}$ satisfies the Palais–Smale condition on M, namely the following result holds true:

Proposition 13. *Let $\lambda\in\mathbb{R}$, $s\in(0,1)$, $n>2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ be a function satisfying conditions* (1.7)*–*(1.9)*, let $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ verify* (1.10)*–*(1.14) *and let $h:\Omega\to\mathbb{R}$ satisfy* (1.15)*. Finally, let $\bar R>0$ be the constant given in Lemma 6. Let $c\in\mathbb{R}$ and let (u_j,ρ_j) be a sequence in $M\times\mathbb{R}$ such that $\|u_j\|_{X_0}\geqslant\bar R$ for $j\in\mathbb{N}$ sufficiently large,*

$$\mathcal{I}_{K,\lambda}(u_j)\to c\tag{4.29}$$

and

$$\sup\Big\{\big|\langle \mathcal{I}'_{K,\lambda}(u_j)+\rho_j|u_j|^{q-2}u_j,\varphi\rangle\big| \,:\, \varphi\in X_0\,, \|\varphi\|_{X_0}=1\Big\}\to 0\tag{4.30}$$

as $j\to+\infty$.

Then, there exists $u_\infty\in X_0$ such that, up to a subsequence,

$$\|u_j-u_\infty\|_{X_0}\to 0$$

and

$$\rho_j \to 0$$

as $j \to +\infty$ *.*

Proof. First of all, we want to show that the sequence u_j is bounded in X_0. By (4.29) there exists $\kappa > 0$ such that for any $j \in \mathbb{N}$

$$|\mathcal{I}_{K,\lambda}(u_j)| \leqslant \kappa \,. \tag{4.31}$$

Also, by definition of $\mathcal{I}_{K,\lambda}$, it is easily seen that

$$\mathcal{J}_{K,\lambda}(u_j) \leqslant \mathcal{I}_{K,\lambda}(u_j)\,,$$

so that, by this and (4.31)

$$\mathcal{J}_{K,\lambda}(u_j) \leqslant \kappa$$

for any $j \in \mathbb{N}$.

Hence,

$$\begin{aligned} \|u_j\|^2_{X_0} &= 2\mathcal{J}_{K,\lambda}(u_j) + \lambda\,\|u_j\|^2_{L^2(\Omega)} + 2\int_\Omega F(x,u_j(x))\,dx + 2\int_\Omega h(x)u_j(x)\,dx \\ &\leqslant 2\kappa + \lambda\,\|u_j\|^2_{L^2(\Omega)} + 2a_1\,\|u_j\|_{L^1(\Omega)} + \frac{2a_2}{q}\,\|u_j\|^q_{L^q(\Omega)} + 2\|h\|_{L^2(\Omega)}\,\|u_j\|_{L^2(\Omega)} \\ &\leqslant 2\kappa + \kappa^* + 2a_1\,|\Omega|^{(q-1)/q}\,\|u_j\|_{L^q(\Omega)} + \frac{2a_2}{q}\,\|u_j\|^q_{L^q(\Omega)} \\ &= 2\kappa + \kappa^* + 2a_1\,|\Omega|^{(q-1)/q} + \frac{2a_2}{q}\,, \end{aligned}$$

thanks to condition (1.12), Hölder inequality (here used with $q/2 > 1$ and its conjugate and also with q and its conjugate) and since $u_j \in M$. Here $\kappa^* = \max\big\{\big(\lambda\,|\Omega|^{(q-2)/q}\big) + 2|\Omega|^{(q-2)/2q}\,\|h\|_{L^2(\Omega)}\,, 2|\Omega|^{(q-2)/2q}\,\|h\|_{L^2(\Omega)}\big\} > 0$. Hence,

$$\text{the sequence } u_j \text{ is bounded in } X_0\,. \tag{4.32}$$

We would like to note that, since $\|u_j\|_{X_0} \geqslant \bar{R}$ for j sufficiently large, say $j \geqslant j^*$, with $j^* \in \mathbb{N}$, then $u_j \not\equiv 0$ for any $j \geqslant j^*$. Moreover, by Lemma 6

$$\mathcal{I}_{K,\lambda}(u_j) > \alpha \text{ for any } j \geqslant j^*\,. \tag{4.33}$$

Hence, the functional $\mathcal{I}_{K,\lambda}$ is differentiable in u_j for $j \geqslant j^*$, by Proposition 3.

Now, we claim that

$$\langle \mathcal{I}'_{K,\lambda}(u_j), u_j\rangle = 0 \quad \text{for } j \geqslant j^*\,. \tag{4.34}$$

To this purpose, fix $j \geqslant j^*$ and let $g(t) := \mathcal{I}_{K,\lambda}(tu_j)$ for any $t > 0$. Then, by definition of $\mathcal{I}_{K,\lambda}$, the function g turns out to be constant, and so

$$0 = g'(t) = \langle \mathcal{I}'_{K,\lambda}(tu_j), u_j\rangle$$

for any $t > 0$. Taking $t = 1$ we get (4.34).

By (4.30)–(4.34) and the fact that $u_j \in M$ it is easily seen that

$$\begin{aligned}|\rho_j| &= \rho_j \|u_j\|^q_{L^q(\Omega)} \\ &= |\langle \mathcal{I}'_{K,\lambda}(u_j) + \rho_j |u_j|^{q-2} u_j, u_j \rangle| \\ &\leqslant \sup \Big\{ \big|\langle \mathcal{I}'_{K,\lambda}(u_j) + \rho_j |u_j|^{q-2} u_j, \varphi \rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1 \Big\} \|u_j\|_{X_0} \to 0\end{aligned}$$

as $j \to +\infty$. Thus, one of the assertion of Proposition 13 is proved.

Now, it remains to show that, up to a subsequence, u_j converges strongly to some u_∞ in X_0. To this purpose, for any $j \geqslant j^*$ let us denote by v_j the element

$$v_j := \sigma(u_j) u_j\,, \tag{4.35}$$

where $\sigma(u_j) > 0$ is the unique constant (see Proposition 3-*b*)) such that

$$\mathcal{I}_{K,\lambda}(u_j) = \mathcal{J}_{K,\lambda}(v_j)\,. \tag{4.36}$$

Note that we can apply Proposition 3, since (4.33) holds true.

By (4.29) and (4.36) we get

$$\mathcal{J}_{K,\lambda}(v_j) \to c \quad \text{as} \quad j \to +\infty\,. \tag{4.37}$$

Moreover, again by Proposition 3-*b*), (4.30), (4.32) and the fact that $\rho_j \to 0$, we have that for any $\varphi \in X_0$

$$\sigma(u_j) \langle \mathcal{J}'_{K,\lambda}(v_j), \varphi \rangle = \langle \mathcal{I}'_{K,\lambda}(u_j), \varphi \rangle \to 0 \tag{4.38}$$

as $j \to +\infty$.

We claim that

$$\text{there exists } \delta > 0 \text{ such that } \sigma(u_j) \geqslant \delta \text{ for } j \text{ large enough.} \tag{4.39}$$

For this, we argue by contradiction and we suppose that

$$\sigma(u_j) \to 0 \tag{4.40}$$

as $j \to +\infty$. Then, by this, (4.32) and (4.35) we deduce that $v_j \rightharpoonup 0$ weakly in X_0, so that, by [10, Lemma 8], up to a subsequence,

$$\begin{aligned} &v_j \to 0 \quad \text{in } L^\nu(\mathbb{R}^n) \\ &v_j \to 0 \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.41}$$

as $j \to +\infty$ and by [4, Theorem IV.9] there exists $\ell_\nu \in L^\nu(\mathbb{R}^n)$ such that

$$|u_j(x)| \leqslant \ell_\nu(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N} \tag{4.42}$$

for any $\nu \in [1, 2^*)$.

By Lemma 9, (4.41) and the Dominated Convergence Theorem we have that

$$\int_\Omega F(x, v_j(x))\, dx \to \int_\Omega F(x, 0)\, dx = 0 \tag{4.43}$$

as $j \to +\infty$, since $F(\cdot, 0) = 0$. Thus, by (4.37), (4.41) and (4.43) we get

$$\begin{aligned} c &= \lim_{j\to+\infty} \mathcal{J}_{K,\lambda}(v_j) \\ &= \lim_{j\to+\infty} \Big(\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} |v_j(x)-v_j(y)|^2 K(x-y)\,dx\,dy \\ &\qquad -\frac{\lambda}{2}\int_\Omega |v_j(x)|^2\,dx - \int_\Omega F(x, v_j(x))\,dx - \int_\Omega h(x)v_j(x)\,dx \Big) \\ &= \frac{1}{2}\lim_{j\to+\infty} \|v_j\|^2_{X_0}, \end{aligned}$$

namely

$$\|v_j\|^2_{X_0} \to 2c \geqslant 2\alpha > 0 \tag{4.44}$$

as $j \to +\infty$. For the last inequality we use the fact that $c = \lim_{j\to+\infty} \mathcal{I}_{K,\lambda}(u_j) \geqslant \alpha$, thanks to (4.29) and (4.33).

By (4.35) and the fact that $\sigma(u_j) > 0$ by construction, we deduce that

$$\|u_j\|_{X_0} = \big(\sigma(u_j)\big)^{-1}\|v_j\|_{X_0},$$

so that, by (4.40) and (4.44) we conclude that

$$\|u_j\|_{X_0} \to +\infty,$$

as $j \to +\infty$. This contradicts (4.32) and so assertion (4.39) is proved.

As a consequence of (4.38) and (4.39) we deduce that for any $\varphi \in X_0$

$$\langle \mathcal{J}'_{K,\lambda}(v_j), \varphi\rangle \to 0 \tag{4.45}$$

as $j \to +\infty$.

Since (4.37) and (4.45) hold true, by Proposition 11 we get that the sequence v_j is bounded in X_0. Hence, the sequence v_j is bounded in $L^q(\Omega)$ by the embedding properties of X_0 into the classical Lebesgue spaces (see [10, Lemma 8]). As a consequence of this, of (4.35) and of the fact that $u_j \in M$, we get

$$\sigma(u_j) = \sigma(u_j)\|u_j\|_{L^q(\Omega)} = \|v_j\|_{L^q(\Omega)} < \kappa$$

for any $j \geqslant j^*$, for a suitable positive constant κ, that is the sequence

$$\sigma(u_j) \ \text{ is bounded in } \ [\delta, +\infty). \tag{4.46}$$

Now, we can apply Proposition 12 to the sequence v_j (remember that that the sequence v_j is bounded in X_0 and again (4.45) holds true). Then, this and (4.46) yield that there exist $v_\infty \in X_0$ and $\sigma^* \in \mathbb{R}$ such that, up to subsequences (still denoted by v_j and $\sigma(u_j)$),

$$v_j \to v_\infty \ \text{ strongly in } \ X_0 \tag{4.47}$$

and

$$\sigma(u_j) \to \sigma^* \ \text{ in } \ [\delta, +\infty) \tag{4.48}$$

as $j \to +\infty$.

Note that, by (4.39)

$$\sigma^* \geqslant \delta > 0\,. \tag{4.49}$$

Finally, as a consequence of (4.35) (remember that $\sigma(u_j) > 0$ for any $j \geqslant j^*$ by Proposition 3-*b*)), (4.47)–(4.49), we have

$$u_j = \big(\sigma(u_j)\big)^{-1} v_j \to \big(\sigma^*\big)^{-1} v_\infty =: u_\infty \quad \text{strongly in } X_0$$

as $j \to +\infty$. This concludes the proof of Proposition 13. □

Remark 14. Note that the function u_∞ given in Proposition 13 is not the trivial one, i.e. $u_\infty \not\equiv 0$ in X_0. Indeed, by (4.47) and [10, Lemma 8], up to a subsequence,

$$\begin{aligned} &v_j \to v_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \\ &v_j \to v_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.50}$$

as $j \to +\infty$ and, by [4, Theorem IV.9], there exists $\ell_\nu \in L^\nu(\mathbb{R}^n)$ such that

$$|v_j(x)| \leqslant \ell_\nu(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}$$

for any $\nu \in [1, 2^*)$. Then, by the Dominated Convergence Theorem applied in $L^\nu(\Omega)$ (here with $\nu = 1$ and $\nu = q$) and thanks to assumption (1.12), we get that

$$\int_\Omega F(x, v_j(x))\,dx \to \int_\Omega F(x, v_\infty(x))\,dx$$

as $j \to +\infty$, so that

$$\mathcal{J}_{K,\lambda}(v_j) \to \mathcal{J}_{K,\lambda}(v_\infty)\,,$$

as $j \to +\infty$, by (4.47), (4.50) and (1.15). That is, by (4.37)

$$\mathcal{J}_{K,\lambda}(v_\infty) = c\,.$$

From this it is easy to see that $v_\infty \not\equiv 0$, since $\mathcal{J}_{K,\lambda}(0) = 0 < \alpha \leqslant c = \mathcal{J}_{K,\lambda}(v_\infty)$, thanks to the fact that $F(\cdot, 0) = 0$.

Finally, (4.49) and $v_\infty \not\equiv 0$ give that $u_\infty := \big(\sigma^*\big)^{-1} v_\infty$ is not the trivial function in X_0.

5. Existence of infinitely many solutions

This section is devoted to the proof of Theorems 1 and 2. The arguments we will use here are variational ones. In order to perform our proof, we will use the preliminary results stated in Subsection 3.1 and we will apply [6, Chapter 5, Theorem 2.2]. For this, as we already explained, it is enough to verify that conditions *i*) and *ii*), stated in Section 3, are satisfied.

In the proof of Theorem 2 the assumptions on q will be crucial for our argument. In particular, we will use the fact that

$$q < 2^* - \frac{2s}{n-2s}\,. \tag{5.1}$$

We would like to note that, until now, we never used this condition and all the results proved in the previous sections are valid for any $q \in (2, 2^*)$. Assumption (5.1) is fundamental for the proof of the validity of $ii)$.

If we consider the classical setting of equation (1.1), the existence of infinitely many solutions for this problem was proved under some restriction on q, namely for any $q > 2$ such that

$$q < \frac{2(n-1)}{n-2}.$$

For more details on this we refer to [6, Chapter 5, Remark 4.7]. Note that this bound from above on q corresponds to (5.1) when $s = 1$ (which gives the classical Laplacian case).

5.1. **Proof of Theorem 2.** As we said in Section 3, since problem (1.5) has a variational nature, the proof of Theorem 2 reduces to find critical points of the functional $\mathcal{J}_{K,\lambda}$. To this purpose, by Corollary 4, it is enough to look for critical points $u \in \{\mathcal{I}_{K,\lambda} > \alpha\}$ of the functional $\mathcal{I}_{K,\lambda}$ on M (here $\alpha > 0$ is the constant given in Proposition 3). For this we will apply [6, Chapter 5, Theorem 2.2].

First of all, note that the functional $\mathcal{I}_{K,\lambda} \in C^1(\{\mathcal{I}_{K,\lambda} > \alpha\})$ by Proposition 3-a) and it verifies the Palais–Smale condition on M, thanks to Proposition 13. Hence, by [6, Chapter 5, Theorem 2.2], we get that for any $k \in \mathbb{N}$

$$a_k \leqslant b_k\,,$$

where a_k and b_k are as in (3.3) and (3.4), respectively.

Now, in order to get our goal, we need to prove that $a_k \in \mathbb{R}$ and that $a_k < b_k$ for $k \in \mathbb{N}$ sufficiently large.

To this purpose, first of all note that, by Lemma 7 and the choice of $q > 2$, we have that $a_k \to +\infty$ as $k \to +\infty$, so that $a_k > \alpha$ for $k \geqslant \tilde{\kappa}$, with $\tilde{\kappa} \in \mathbb{N}$. This means that $a_k \in \mathbb{R}$ for any $k \geqslant \tilde{\kappa}$.

Now, it remains to show that $a_k < b_k$ for any $k \in \mathbb{N}$ large enough. To this purpose, first of all note that for any $k \in \mathbb{N}$ and any $A \in \mathcal{A}_k$ there exists $u_A \in A \cap \mathbb{P}_k$ (for this see [6, Chapter 2, Theorem 4.4 and the proof of Chapter 5, Lemma 3.3]), where

$$\mathbb{P}_k := \big\{u \in X_0 \text{ s.t. } \langle u, e_j\rangle_{X_0} = 0 \quad \forall j = 1, \dots, k-1\big\}.$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in X_0 (see formula (2.3)), while e_j are the eigenfunctions of $-\mathcal{L}_K$ (see Section 2).

Let us fix $k \in \mathbb{N}$. By the variational characterization of λ_k (see [11, Proposition 9]), we have that

$$\|u_A\|_{X_0} \geqslant \sqrt{\lambda_k}\,\|u_A\|_{L^2(\Omega)}\,. \tag{5.2}$$

Also, by Hölder inequality applied in $L^2(\Omega)$ (for this we need that $2 < q < 2^* - 2s/(n-2s)$, which is our assumption) and the embedding properties of the space X_0 (see [10, Lemma 8]) we get

$$\begin{aligned}\|u_A\|^q_{L^q(\Omega)} &= \int_\Omega |u_A(x)|^{q-1}|u_A(x)|\,dx \\ &\leqslant \|u_A\|_{L^2(\Omega)}\,\|u_A\|^{q-1}_{L^{2(q-1)}(\Omega)} \\ &\leqslant C\|u_A\|_{L^2(\Omega)}\,\|u_A\|^{q-1}_{X_0}\end{aligned}$$

for a suitable $C>0$. As a consequence of this and taking into account that $u_A \in A \subseteq M$ we get

$$\|u_A\|_{L^2(\Omega)} \geqslant C^{-1}\|u_A\|^q_{L^q(\Omega)}\,\|u_A\|^{1-q}_{X_0} = C^{-1}\|u_A\|^{1-q}_{X_0}\,,$$

so that this and (5.2) yield

$$\|u_A\|_{X_0} \geqslant C^{-1}\sqrt{\lambda_k}\,\|u_A\|^{1-q}_{X_0}\,,$$

namely

$$\|u_A\|_{X_0} \geqslant C^{-1/q}\lambda_k^{1/(2q)}\,.$$

Since $\lambda_k \to +\infty$ as $k \to +\infty$ (see [11, Proposition 9]), for k large enough we get that

$$\|u_A\|_{X_0} \geqslant \bar R\,,$$

where $\bar R$ is the positive constant given in Lemma 6. This proves that for $k \in \mathbb{N}$ sufficiently large, say $k \geqslant \breve\kappa$, with $\breve\kappa \in \mathbb{N}$, and for any $A \in \mathcal{A}_k$

$$\text{the set } A \cap \big\{\|v\|_{X_0} \geqslant \bar R\big\} \text{ is not empty.}$$

With this assertion and using the fact that $\mathcal{I}_{K,\lambda}(v) = \mathcal{I}_{K,\lambda}(tv)$ for any $t>0$ it is easy to see that for $k \geqslant \breve\kappa$

$$a_k = \inf_{A\in\mathcal{A}_k} \max_{\substack{v\in A\\ \|v\|_{X_0}\geqslant \bar R}} \mathcal{I}_{K,\lambda}(v) \tag{5.3}$$

and

$$b_k = \inf_{B\in\mathcal{B}_k} \max_{\substack{v\in B\\ \|v\|_{X_0}\geqslant \bar R}} \mathcal{I}_{K,\lambda}(v)\,. \tag{5.4}$$

Also the same holds true for a_k^0 and b_k^0 (see formulas (3.6) and (3.7)).

Finally, we can show that

$$a_k < b_k \text{ for any } k \in \mathbb{N} \text{ sufficiently large.} \tag{5.5}$$

For this we argue by contradiction and we suppose that there exists $\bar\kappa \in \mathbb{N}$ such that $a_k = b_k$ for any $k \geqslant \bar\kappa$ (note that we know that $a_k \leqslant b_k$, for any $k \in \mathbb{N}$). Let $\hat\kappa = \max\{\bar\kappa, \tilde\kappa, \breve\kappa\}$.

By (5.3) and (5.4) and the analogous formulas for a_k^0 and b_k^0 and Lemma 6 we get that for any $k \in \mathbb{N}$ with $k \geqslant \breve\kappa$

$$a^0_{k+1} = b^0_k \leqslant b_k + \bar c\big(1 + b_k^{1/q}\big)$$

and

$$a_{k+1} \leqslant a^0_{k+1} + \bar c\big(1 + (a^0_{k+1})^{1/q}\big)\,,$$

so that

$$a_{k+1} \leqslant b_k + \bar{c}\big(1 + b_k^{1/q}\big) + \bar{c}\big[1 + \big(b_k + \bar{c}\big(1 + b_k^{1/q}\big)\big)^{1/q}\big]\,.$$

As a consequence of this and using the assumption that $a_k = b_k$ for any $k \geqslant \hat{\kappa}$, we deduce that

$$a_{k+1} \leqslant a_k + \bar{c}\big(1 + a_k^{1/q}\big) + \bar{c}\big[1 + \big(a_k + \bar{c}\big(1 + a_k^{1/q}\big)\big)^{1/q}\big]\,,$$

which gives

$$a_{k+1} \leqslant a_k + \hat{c}\big(1 + a_k^{1/q}\big) \tag{5.6}$$

for a suitable positive constant $\hat{c}$ and for any $k \geqslant \hat{\kappa}$. Here we use also the fact that a_k is large enough (since $a_k \to +\infty$ as $k \to +\infty$).

By (5.6) and Lemma 8 (applied here with $p = q - 1 > 1$) there exist $c^* > 0$ and $\kappa^* \in \mathbb{N}$ such that

$$a_k \leqslant c^* k^{q/(q-1)} \quad \text{for } \ k \geqslant \kappa^*\,. \tag{5.7}$$

Moreover, combining (5.7) and Lemma 7 we deduce that, for k sufficiently large

$$\tilde{c}\, k^{2sq/n(q-2)} \leqslant a_k \leqslant c^* k^{q/(q-1)}\,,$$

which implies that

$$\frac{2sq}{n(q-2)} \leqslant \frac{q}{q-1}\,,$$

that is

$$q \geqslant 1 + n/(n-2s) = 2^* - 2s/(n-2s)\,.$$

Clearly this contradicts the assumption on q (see (5.1)). Thus, (5.5) is proved.

Hence, by [6, Chapter 5, Theorem 2.2], the functional $\mathcal{I}_{K,\lambda}$ admits a sequence of critical points v_k on M with critical value $c_k = \mathcal{I}_{K,\lambda}(v_k) \geqslant b_k$. Since $\mathcal{I}_{K,\lambda}(v_k) \geqslant b_k > a_k > \alpha$ for k large enough (as $a_k \to +\infty$ as $k \to +\infty$), then by Corollary 4-a) the sequence u_k defined as

$$u_k := \sigma(v_k)v_k\,,$$

where $\sigma(v_k)$ is given as in Proposition 3-b), is a sequence of critical points of $\mathcal{J}_{K,\lambda}$ on X_0. In addition, again by Proposition 3-b),

$$\mathcal{J}_{K,\lambda}(u_k) = \mathcal{J}_{K,\lambda}(\sigma(v_k)v_k) = \mathcal{I}_{K,\lambda}(v_k) \geqslant b_k > a_k \to +\infty \tag{5.8}$$

as $k \to +\infty$.

Finally, we have to show that $\|u_k\|_{X_0} \to +\infty$ as $k \to +\infty$. For this, we can argue again by contradiction. Indeed, if the sequence u_k were bounded in X_0, then, by the embeddings properties of X_0 into the usual Lebesgue spaces (see [10, Lemma 8]) and assumption (1.12), we would get that the sequence $\mathcal{J}_{K,\lambda}(u_k)$ is bounded in $\mathbb{R}$, but this contradicts (5.8). Thus, $\|u_k\|_{X_0} \to +\infty$ as $k \to +\infty$ and this concludes the proof of Theorem 2.

5.2. **Proof of Theorem 1.** This is a consequence of Theorem 2, since, in the model case $K(x) = |x|^{-(n+2s)}$, we have that $X_0 = \{v \in H^s(\mathbb{R}^n) : v = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega\}$, see [12, Lemma 7] .

Acknowledgements. The author would like to thank Otared Kavian for his time and his useful and interesting advices during the preparation of this paper.

References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14, 349–381 (1973).

[2] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc., 267, no. 1, 1?-32 (1981).

[3] A. Bahri and P.L. Lions, *Morse index of some min-max critical points. I. Application to multiplicity results*, Comm. Pure Appl. Math., 41, no. 8, 1027?-1037 (1988).

[4] H. Brézis, Analyse fonctionelle. Théorie et applications, *Masson*, Paris (1983).

[5] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136, no. 5, 521–573 (2012).

[6] O. Kavian, Introduction à la théorie des points critiques et applications aux problèmes elliptiques, *Mathématiques & Applications*, Springer-Verlag, Paris (1993).

[7] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.*, 65, *American Mathematical Society*, Providence, RI (1986).

[8] R. Servadei, *The Yamabe equation in a non-local setting*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-40` .

[9] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, to appear in Rev. Mat. Iberoam., available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-103` .

[10] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl., 389, 887–898 (2012).

[11] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, to appear in Discrete Contin. Dyn. Syst., available at `http://www.math.utexas.edu/mp_arc-bin/mpa?yn=11-131`.

[12] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, to appear in Trans. AMS., available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-196`.

[13] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-41`.

[14] R. Servadei and E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-82`.

[15] R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators*, in preparation.

[16] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math., 32, no. 3-4, 335?-364 (1980).

[17] M. Struwe, Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, *Springer Verlag*, Berlin–Heidelberg (1990).

[18] M. Willem, Minimax theorems, *Progress in Nonlinear Differential Equations and their Applications*, 24, Birkhäuser, Boston (1996).

Dipartimento di Matematica, Università della Calabria, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende (Cosenza), Italy
E-mail address: servadei@mat.unical.it