HÉNON MAP: SIMPLE SINKS GAINING COEXISTENCE AS $b \to 1$

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Abstract. The quadratic map of the interval displays one attractor for each parameter value. Conservative maps of the plane display infinite coexistence of stability islands around periodic orbits. Between these two extremes, dissipative systems of the plane are known to have infinite coexistence of sinks as a generic property, yet very hard to detect. We investigate how more and more coexistence is gained as the area-contraction rate $b \to 1$. In this paper we show a sequence of simple sinks gaining coexistence, and investigate the convergence properties of its bifurcation values. The sinks are simple, or primary, due to their geometrical structure.

1. Introduction and Notation

In recent years the Hénon model has got again a noticeable attention [Gonchenho & Shilnikov(2003), De Carvalho *et al.*(2005), Gonchenko Marina(2005), Gonchenko *et al*(2005), Gonchenko V.S *et al*(2010), Lyubich & Martens(2011)] stemming from the highly degenerate bifurcation phenomena occurring in the area-preserving case, and from its rescaling structure, in general complicated. Moreover Hénon map is both a map that displays homoclinic and heteroclinic behavior as one of the essential ingredients leading to complicated topology, and also naturally arises as first-return map when describing homoclinic behavior in other systems. Coexistence of infinitely many periodic sinks in a compact set, albeit counterintuitive, is a generic property, consequent to the onset of homoclinic behavior [Newhouse (1974), Newhouse(1979), Gavrilov&Shilnikov(1972), Gavrilov&Shilnikov (1973)] , and very unlikely to be detected [Tedeschini-Lalli & Yorke(1986), Gorodetski & Kaloshin(2008)]. Homoclinic behavior, on the other hand, is ubiquitous in nonlinear systems. We study a sequence of periodic sinks for the Hénon map (Fig.1) gaining coexistence within the family as the map becomes less and less dissipative.

The sequence of sinks is associated with a global bifurcation which is necessarily degenerate, as in the nondegenerate case no coexistence within the same primary homoclinic family is possible. Our sequence accumulates, for $b = 1$, on a heteroclinic loop, a tangency of infinite order. Literature about coexistence of sinks has been reviewed in [Goswami(2005)],

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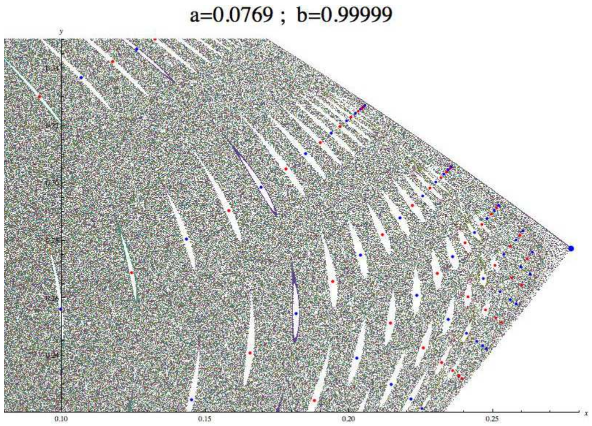

Figure 1: $b = 1 - 10^{-5}$; coexistent sinks and their visible basins in plane (x, y).

with particular attention to possibilities of external control, and to a hierarchical self-similar structure of secondary bifurcations, leading to the description of the coexistence predicted in [Newhouse (1974)]. We study, instead, coexistence within the same primary family.

Previous research on portraits of coexistence of a large number of different attractors include [Feudel*et al*(1996), Feudel*et al*(1998)], respectively for a single and a double rotor system with small dissipation. We choose Hénon map, that links parametrically the well-studied map of the interval and an area-preserving map of the plane; the parameter explicitly bridging between the two is the area-contracting factor b. The area-preserving case of Hénon maps i.e. $b = \pm 1$, is not as well understood as the conservative case of rotor and double rotor, and has attracted much interest in recent years [Gonchenko Marina(2005), Gonchenko *et al*(2005), Gonchenho & Shilnikov(2003)]. Among what has been known for long, is that all Hénon maps create a topological horseshoe as the parameters are varied: in fact, it is known [Devaney& Nitecki(1979)] that there exist in the (a, b) parameter plane a lower (initial) and an upper (final) curve $a_{in}(b)$ and $a_{fin}(b)$ such that all curves (i.e. one-parameter families of maps) joining them, will display bifurcations creating horseshoes, passing from a dynamics with no recurrent set, to a dynamics isomorphic to the full 2-shift, in particular, with 2^p periodic points of period p for each integer p. Such maps necessarily display only saddle-node and period-doubling codimension-1 bifurcations whenever $|b| \neq 1$; this is not the case at $b = 1$, where more types of bifurcations are possible. All one-parameter families of maps under these conditions have been proved to display infinitely many cascades of period-doubling bifurcations [Yorke & Alligood(1985)]. We exploit this aspect and implement a continuation method drawing an arch of periodic solutions that allows to dribble bifurcation degeneracies at $b = \pm 1$. In this paper we study a family of sinks in the (a, b) parameter plane, together with their bifurcation values for $b = 1 - \varepsilon$, and then let $\varepsilon \to 0$, numerically. The family of sinks is a sequence and for

b near 1, it converges as the periods diverge, implying a heteroclinic loop. This result is quite compatible with the first heteroclinic connections of fixed points numerically shown in [Gonchenko *et al*(2005)].

The map of Hénon we study is in the following formulation:

$$(x, y) \to (X(x, y, a, b), Y(x, y, a, b)) \tag{1}$$

where

$$\begin{aligned} X(x, y, a, b) &= q_a(x) - by \\ Y(x, y, a, b) &= x \end{aligned} \tag{2}$$

and $q_a(x) = a - x^2$ is one of the formulations of the well-studied logistic map. The determinant of the Jacobian of the map is $-b$, so $|b|$ is the area-contraction factor. In this paper we address the orientation-reversing case $b > 0$, because it contains the original parameter values of the Hénon attractor $a = 1.4, b = 0.3$ and because it has been less studied in the conservative case; our numerical method applies in any case.

Continuability of curves of bifurcation from $b = 0$ throughout the $(a, b)-$plane up to the borders $b = \pm 1$ has long ago been proved in [Holmes & Whitley(1984)]. On the contrary, there is no general result for curves of bifurcation possibily initializing at $b = 1$

The paper is organized as follows: in section 2 we introduce the sequence of periodic sinks with diverging period p. We illustrate their coexistence properties as the map approaches the conservative limit. Then we go on to study the converging properties, as the period p diverges, of their bifurcation curves in plane (a, b) for a relatively wide choice of b values as $b \to 1$. We also discuss which features are retained in the limit case $b = 1$. Finally, in section 3 we explain the numerical method that allowed us to focus on the bifurcation values of this sequence, which is a method initializing at $b = 1$, the area-preserving map.

2. A Sequence of Sinks Gaining Coexistence as $b \to 1$

2.1. **The sequence, its coexistence property.** The periodic orbits we follow, all of even period, share the geometric property of circuiting exactly once around the period-2 point at $b = 1$, in the area-preserving case. (The method to follow them into the area-contracting zone, to be able to study their bifurcation curves in $(a, b)-$plane, is carefully described in the next section). The intervals of stability of some of these sinks overlap in the $(a, b)-$plane, pointing to the coexistence of simple sinks of the same family Fig.2. We stress that such coexistence within the same homoclinic family is impossible for dissipative systems near a non-degenerate homoclinic tangency. So we investigated more into this.

Starting from the values in (x, y, a, b, p) of sinks for some different period p, we searched and found periodic solutions for growing p, and thus the entire sequence of periodic orbits, for fixed a and b, up to period 190; we then continued the entire sequence of sinks and their bifurcation values in the $(a, b)-$plane. Fixing b and plotting the $a-$bifurcation values against their period Fig.3, we ascertained which and how many of them coexist. In this type of image, vertical lines are actually projections on (a, p) of arches of sinks appearing

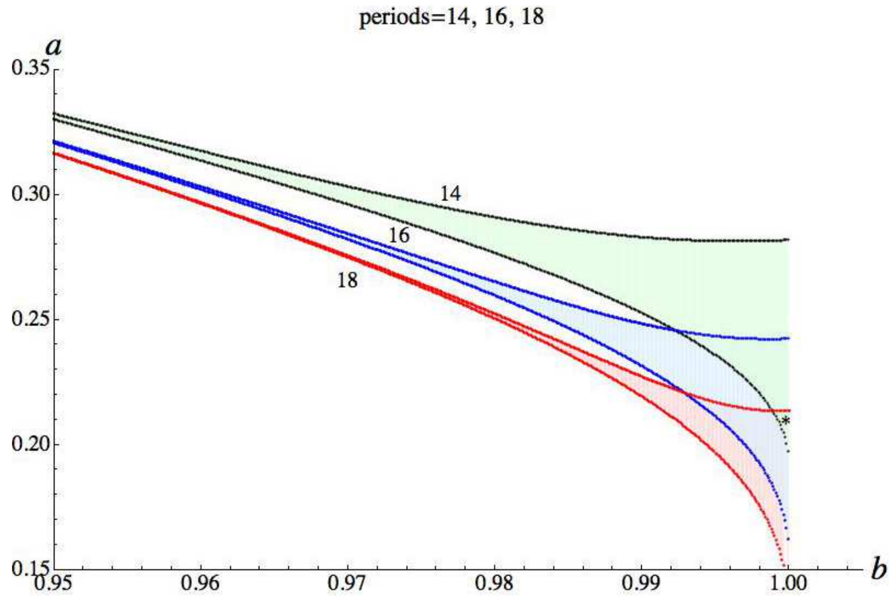

Figure 2: $(a, b)-$plane. bifurcation curves of orbits of period $12, 14, 16$. Their stability ranges overlap: in $*$ these three periodic sinks coexist.

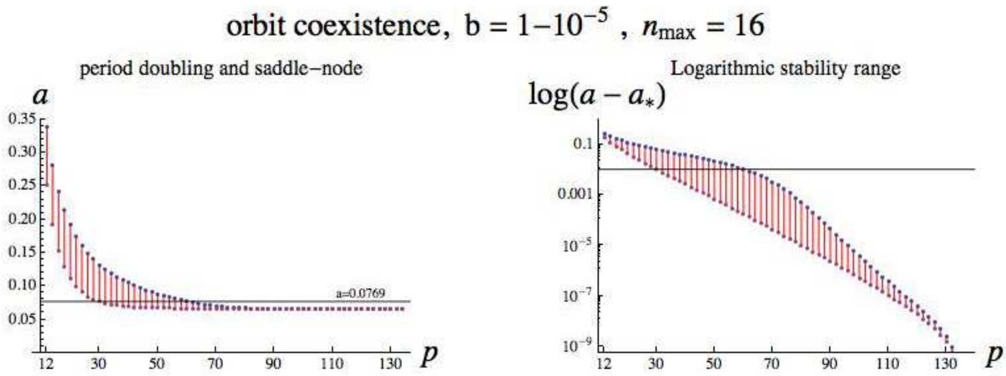

Figure 3: $\varepsilon = 10^{-5}$: Sequence of bifurcation $a-$values of sinks, period p.

via saddle-node and loosing stability via period-doubling: we have checked each of them, a table is included at the end of the paper.

First have a look into the situation at $b = 1 - 10^{-5}$. Here we find a coexistence of 16 simple sinks, Fig.3. The maximum coexistence, at b (or k) fixed, is quantified by tracing a value of a, i.e. a horizontal line, that intersects the filled-in area between curves; this

k	$b = 1 - 10^{-k}$	$N_{max}(b)$	$a_{max}(b)$	$\underline{p}$	$\bar{p}$
3	.999	4	0.15160	20	26
4	.9999	9	0.09875	28	44
5	.99999	16	0.07690	30	60
6	.999999	27	0.05400	42	94
7	.9999999	39	0.04500	46	122

Table 1: Maximum coexistence $N_{max}(b)$ within the sequence, as $b \to 1$

graphic count is obviously much more reliably done in logarithmic scale. The count is then tuned up by a numerical check on the stability of the coexisting periods. At a given b, we label $a_{max}(b)$ a value of a at which maximum coexistence is attained.

More about the convergence properties of the other quantities involved, in the following.

How much and in what way coexistence increases as $b \to 1$, i.e as the map takes on more and more geometrical properties of conservative maps?

Set $b = 1 - \varepsilon$, and $\varepsilon = 10^{-k}$. In Fig.4 the same study of coexistence just illustrated is performed for $k = 3, 4, 5, 6, 7$. The resulting number of coexistent sinks is reported in Table 1; all sinks of even period between $\underline{p}$ and $\bar{p}$ are stable for values of a near a_{max}. The coexistence number N_{max} depends on the area-contraction rate b, and it appears to diverge as $b \to 1$.

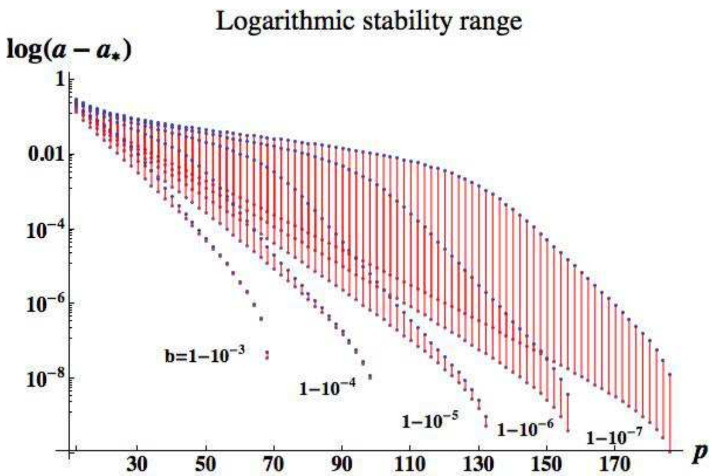

Figure 4: Different fixed b. Sequence of bifurcation $a-$values of sinks, vs. period p. Logarithmic scale enhances gaining of coexistence, quantified in Table 1.

2.2. **The sequence of sinks, and its convergence properties in** (a, b)**.** At fixed b, very close to $b = 1$, the sequence of bifurcation curves accumulates on a limiting curve $a_o(b)$. The accumulation rate changes with b. Observations: the curve $a_o(b)$ is compatible with the curve of heteroclinic connection studied in [Gonchenko *et al*(2005)]; in particular, we find the same cusp singularity at $b = 1$.

All stability windows shrink as period grows:

$$\lim_{p \to \infty} (a_{pd}(b, p) - a_{sn}(b, p)) = 0$$

On each branch in Fig.5, (i.e. at each b) the logarithmic scale clearly highlights two different rates of decay. Maximum coexistence is attained in the first regime, where stability windows decay rather slowly with respect to growing period.

We observe that around the same values in (a, b, p) in which the rate of convergence changes, it becomes hard to guess positions of periodic solutions for greater periods p, pointing to a nontrivial renormalisation scheme in the dynamical plane.

Geometrically, in dynamical plane, periodic points accumulate as $p \to \infty$ on a heteroclinic connection between the two fixed points . On each branch in Fig. 5, in the second regime, i.e. for higher p's, positions are naturally more subject to the accumulation rate on the hyperbolic fixed point. So it is convenient to switch from initializing a guess in the isolated point around the tip of the manifold (first regime), to initialize the guess in the hyperbolic neighborhood of the fixed flip-point. More on this renormalization scheme is under study.

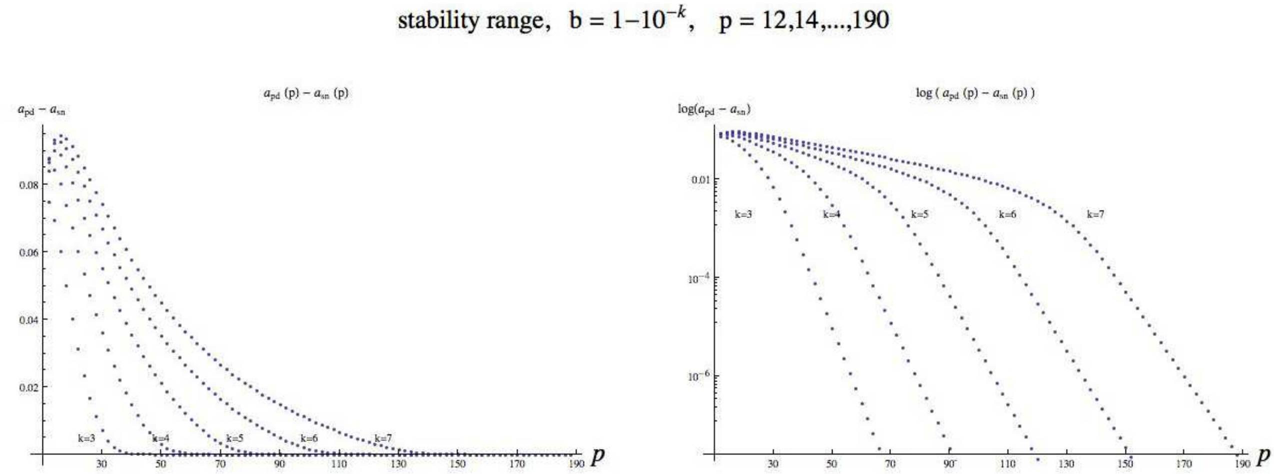

Figure 5: $\varepsilon = 10^{-k}$, $k = 3, 4, 5, 6, 7$. Convergence of stability ranges of $p-$sink

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The first sinks of our sequence (periods 12 through 26) also have the interesting property that their bifurcation curves are continuable in a straightforward fashion down to $b = 0$, meeting no singularity point; that is, they meet no codimension-2 bifurcation value in the $(a, b)-$plane. This is rather unusual for bifurcation curves in the (a, b) plane, so more on this is under study.

2.3. **What of all this is retained at $b = 1$.** In the area-preserving, orientation-reversing case i.e. $b = 1$, more types of bifurcations are possible, many of them degenerate, [Gonchenko Marina(2005)], and the wider scope of our study is to compose these bifurcations into a coherent picture with the neighboring dissipative case. Now, we got to the step of having to handle properties of a sequence of periodic orbits concerning a double limit, in the period and in the area contraction factor, i.e. in $p \to \infty$ and $b \to 1$. There is a degeneracy both in $b = 1$, where our bifurcation curves form a cusp, and also in the period p, as we just saw in the last subsection that convergence in p does not appear to be uniform w.r.t. parameter b, so the limits of quantities linked to this sequence cannot in general be inverted. On the other hand the number of coexistent sinks, $N_{max}(b)$, does appear to diverge as $b \to 1$. So we decided to follow arches of periodic solutions into $b = 1$, with fixed a. What we can say now is that all periodic orbits which are stable and attracting at $b = 1 - 10^{-7}$, $a = a_{max}(b)$ maintain stability as elliptic points into $b = 1$, with the same value of a; so,

$$N_{max}(1) \geq N_{max}(1 - 10^{-7})$$

$$a_{max}(1) \leq a_{max}(1 - 10^{-7})$$

The question is open within this "simple" or "primary" sequence, then, as to whether $a_{max}(1)$ exists and is different from 0.

At $b = 1$ the sequence of (degenerate) saddle-node bifurcations converges to the creation of a heteroclinic loop between the two fixed points. While the saddle-node bifurcation curves of our sequence all tend to take on negative vertical tangent as $b \to 1$, the period-doubling curves of the same periodic solutions intersect $b = 1$ transversely, in fact horizontally for p large enough .

3. A numerical method to dribble degeneracies from $b = 1$

Numerical continuation methods for bifurcation curves are generally based on the Implicit Function Theorem. Bifurcations at $b = 1$ are highly degenerate and one cannot initialize an implicit function theorem in the stable elliptic points, much less so on their bifurcation values. There are topological reasons, though, to expect many of these elliptic points can be continued into the dissipative case. So we devised a combination of numerical methods to "dribble" the degenerate points corresponding to saddle-node bifurcation values at $b = 1$, in the (x, y, a, b) space.

The method is in three steps Fig.6:

-fix a; look at the conservative dynamical picture ($b = 1$) in plane (x, y); initialize a continuation method on a saddle of period p, and continuate the saddle into $b < 1$. Being a saddle is an open condition on the Jacobian, so saddles are always continuable; stop when you deem so, i.e. for a value b you like, or where you can. We reached the first switch.

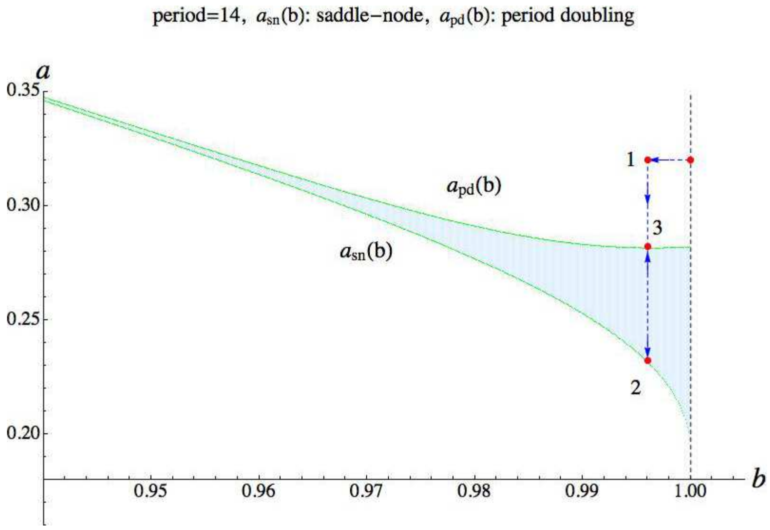

Figure 6: Continuation around conservative: the three switches

-now fix b; follow the saddle downward in a until you necessarily reach [Yorke & Alligood(1985)] its saddle-node bifurcation value $a_{sn}(b,p)$. We are at the second switch.

-still for fixed b, follow the node upward in a until you necessarily reach [Yorke & Alligood(1985)] its period-doubling bifurcation value $a_{pd}(b,p)$. This is the third switch.

There are three switches, and each can be a many-way switch: we are in a 5-dimensional space (x, y, a, b, p). From switch 2 and switch 3, one can also initialize a continuation method for bifurcation curves in the (a, b) plane, and get the bifurcation curve, as long as the method utilised, or the curve itself, allows; this is what we have done. Or else one can search for solutions in p: The important part of the method is we have dribbled the points at $b = 1$ which hold degenerate conditions for continuation methods, and can now study how the different features behave as $b \to 1$, rather than perturb from $b = 1$.

We first applied this method for a relatively small bunch of periodic orbits of consecutive periods, including continuing the bifurcation curves as shown in Fig.2 for periods $p = 14, 16, 18$. From the clearly visible overlap of the stability ranges in (a, b), we then started the study of coexistence. At each fixed (a, b) value one can implement an accurate search for a sequence of periodic orbits following the same geometrical pattern, with diverging period p up to $p = 190$. For each period, then, we study (numerically) the saddle-node and period-doubling bifurcation value. Their convergence properties are illustrated in the previous section. In particular they do not allow a general method to work in all cases as p diverges: more delicate geometrical choices were needed for different ε, as convergence properties change.

Numerical reliability: all conditions on different periodic orbits and bifurcation values were found and checked within a precision of 480 digits. We performed our study on

Mathematica8. We underline that while one usually looks for features which are somewhat stable with respect to numerical precision, here we need to deal with periodic points of rather high period, and with very small dissipation, both situations which suggest a good numerical reliability for reading meaningful results.

4. Conclusions

We studied the existence and convergence properties of a sequence of sinks of the orientation-reversing Hénon map, a sequence gaining more and more coexistence as the system becomes area-preserving. In Table 1 this gain is described quantitatively, as the area-contraction rate b of the map tends to 1. The study of the stability windows of our sinks in parameter plane (a, b) allowed to establish this coexistence quantitatively, and provided a new problem due to a pattern, repeated over different $b-$values, by which convergence rates of the bifurcation values decidedly have a critical turn from some period on. The sequence in question consists of "simple sinks", i.e. the simplest geometrical fashion in which a sequence of periodic orbits can accumulate on a heteroclinic loop: they all share a circuitation of rotation number $\frac{1}{p}$, they all appear via a saddle-node bifurcation, and all loose stability via a period-doubling bifurcation; we expect more complicated coexistence structures to emanate from the successive bifurcations of each of these sinks. There is a double limit handling, then: for b fixed and as $p \to \infty$, the sequence of points accumulates on a heteroclinic connection between the two fixed points of the map. At $b = 1$ this connection is the first heteroclinic connection. On the other hand, as $b \to 1$ the successive saddle-node bifurcation curves form a cusp, while the successive period-doubling bifurcation curves tend to intersect axis $b = 1$ transversely, resulting in an enlargement of the stability window. As to the family of curves in the $(a, b)-$plane, for $b = 1 - \varepsilon$ the bifurcation curves accumulate on a heteroclinic continuation curve in (a, b)-plane, as $p \to \infty$, and this is compatible with the continuation curve in (a, b) plane of the first such heteroclinic connection, demonstrated and pictured by Yuri Kuznetov, for example in [Gonchenko *et al*(2005)].

5. Appendices

5.1. **Known convergence rates for homoclinic families.** For the sake of comparison about rates of convergence, in Fig.6 we illustrate some already established results on other, known homoclinic families, with the same type of graph of convergences shown in Fig.5 above. What is known: if the tangency at which the sequence accumulates is non-degenerate, (both in the parameter and in dynamical plane), and the map is area contracting, a rescaling result was proved for the sequence of homoclinic bifurcations of simple sinks, in [Tedeschini-Lalli & Yorke(1986)]. In this case, the sequence of bifurcation values of the simple homoclinic sinks accumulates at a constant rate, just as the last sequence of sinks arising in the map of the interval (or $b = 0$) and accumulating on the last crisis value $a = 2$. The exact rate depends on the geometry (curvature) of the nondegenerate tangency, and on the linear part of the saddle point creating the tangency. The sequence illustrated in Fig.6a is obtained initializing an implicit function problem for all saddle-node bifurcations of period p in the quadratic map of the interval, from $a = 2$. Beyond this value of a, the map becomes expansive and contains no attractor, or in other words it is the last crisis value, i.e. where the unstable fixed point collides with the attractor, destroying its attractiveness.

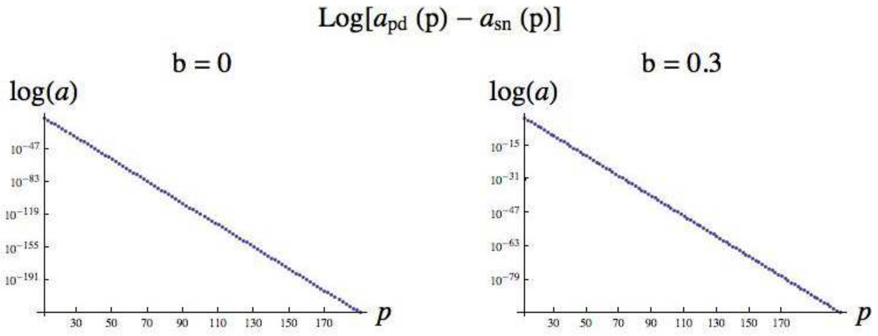

Figure 7: Convergence rate of stability ranges of p-sink. Left: $b = 0$, last crisis value of $a = 2$. Right: $b = 0.3$, first homoclinic tangency value of a fixed point

The sequence illustrated in fig.6b refers to the historical value of $b = 0.3$, and the first homoclinic tangency created by the flip fixed point. The sequence was studied in [Tedeschini-Lalli & Yorke(1986)] up to period $p = 20$, and we just elongated the sequence for higher periods p, as now this can be done easily and reliably.

5.2. **The unstable manifold of the flip fixed point.** What about the basins of attraction of all these sinks? Fig.1 at the beginning of the paper is actually obtained by drawing only the unstable manifold of the upper fixed point. As dissipation decreases and the map tends to the area-preserving case, this unstable manifold behaves as in Fig.8. This deserves a separate study.

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References

[Calleja & Celletti(2010)] Calleja R., Celletti A. *Breakdown of Invariant Attractors for the Dissipative Standard Map* CHAOS, **20** (2010)

[Celletti(2009)] Celletti A. *Periodic and Quasi-periodic Attractors of Weakly-dissipative Nearly-integrable Systems*Reg. Chaotic Dyn., **14**:49-63 (2009)

[Celletti*et al.*(2010)] Celletti A., Froeschle' C., Lega E.:*Dissipative and Weakly-dissipative Regimes in Nearly-integrable Mappings* DCDS-A, **16**: 757-781 (2006).

[De Carvalho *et al.*(2005)] De Carvalho A., Lyubich M., Martens M. *Renormalization in the Hénon Family, I: Universality But Non-Rigidity* Journal of Statistical Physics, **121** (2005)

[Devaney& Nitecki(1979)] Devaney R., Nitecki Z. *Shift Automorphism in the Hénon Map* Comm.Math.Pyhs., **67**:137-146 (1979).

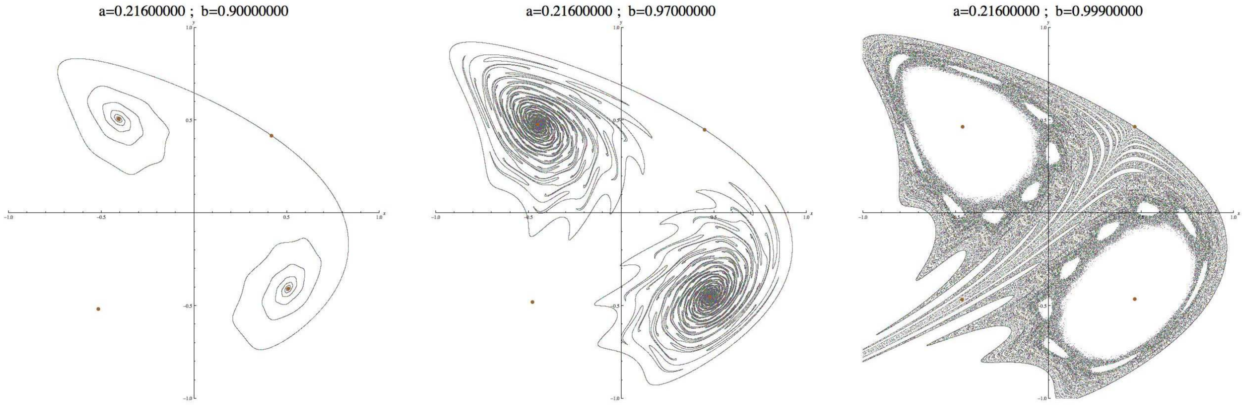

Figure 8: Qualitative change in the unstable manifold of the flip fixed point: everything happens for $0.9 < b < 1$. In this picture a is fixed.

[Feudel*et al*(1996)] Feudel U., Poon L. , Grebogi C., Yorke J. A.*Dynamical Properties of a Simple Mechanical System with a Large Number of Coexisting Periodic Attractors* Chaos, Solitons and Fractals **9**: 171-180(1998).

[Feudel*et al*(1998)] Feudel U., Grebogi C. , Hunt B. R., Yorke J. A. *A Map with More than 100 Coexisting Low-period Periodic Attractors* Phys. Rev. E **54**: 71-81 (1996).

[Gavrilov&Shilnikov(1972)] Gavrilov, N. and Shilnikov, L. *On 3-Dimensional Dynamical Systems Close to Systems With a Structurally Unstable Homoclinic Curve* I. Math. USSR Sb. **88**: 467-485 (1972)

[Gavrilov&Shilnikov (1973)] Gavrilov, N. and Shilnikov, L.P. *On 3-Dimensional Dynamical Systems Close to Systems With a Structurally Unstable Homoclinic Curve, II.* Math. USSR Sb. **90**: 139-156 (1973)

[Gelfreich&Simó(2008)] Gelfreich V., Simó C. *High-precision Computations of Divergent Asymptotic Series and Homoclinic Phenomena* Discrete Contin. Dyn. Syst. Ser. B **10** : 511-536 (2008)

[Gonchenko Marina(2005)] Gonchenko, M.S.*On the Structure of 1:4 Resonances in Hénon Maps* Int.J. Bifurcation and Chaos, **15**:3653-3660, (2005)

[Gonchenko V.S *et al*(2010)] Gonchenko V.S., Gonchenko S.V., Shilnikov L.P. *On a Homoclinic Origin of Hénon-like Maps* Regular and Chaotic Dynamics **15**: 462-481 (2010), (and bibliography therein)

[Gonchenko *et al*(2005)] Gonchenko V.S., Kuznetov Yu.A., Meijer HGE *Generalized Hénon Map and Bifurcation of Homoclinic Tangencies* SIAM J. Appl. Dynamical Systems, **4:** 407-436 (2005)

[Gonchenho & Shilnikov(2003)] Gonchenko S.V. , Shilnikov L.P. *On Two-Dimension Area-Preserving Maps with Homoclinic Tangencies That Have Infinitely Many Generic Elliptic Periodic Points* Zap. Nauchn. Sem. POMI **300**: 155-166 (2003)

[Gorodetski & Kaloshin(2008)] Gorodetski, Kaloshin V. *How Often Surface Diffeomorphisms Have Infinitely Many Sinks and Hyperbolicity of Periodic Points Near a Homoclinic Tangency* Bull. Austral. Math. Soc. **77** : 37-48 (2008)

[Goswami(2005)] Goswami B. K. *Multiple Attractors in the Self-similar Bifurcation-structure* Rivista del Nuovo Cimento 28:1-115 (2005)

[Holmes & Whitley(1984)] Holmes P., Whitley D. *Bifurcations of One and Two Dimensional Maps* Phil. Trans. Roy. Soc. Lond. **A311**, 43-102 (1984). (also (1984) *Erratum* Phil. Trans. Roy. Soc. Lond. **A312** , 601. (1984).

p	a_{pd}	a_{sn}	$a_{pd} - a_{sn}$	p	a_{pd}	a_{sn}	$a_{pd} - a_{sn}$
12	0.3379070917	0.2512746710	0.08663242069	74	0.06816174332	0.06644821707	0.001713526254
14	0.2819277961	0.1919397498	0.08998804636	76	0.06760443453	0.06644217251	0.001162262017
16	0.2424531539	0.1536385892	0.08881456468	78	0.06720092401	0.06643761980	0.0007633042123
18	0.2135228357	0.1282813066	0.08524152914	80	0.06692222299	0.06643419400	0.0004880289903
20	0.1915925724	0.1110820980	0.08051047441	82	0.06673729956	0.06643161907	0.0003056804888
22	0.1744684764	0.09914904099	0.07531943536	84	0.06661835427	0.06642968556	0.0001886687052
24	0.1607445581	0.09069761229	0.07004694581	86	0.06654352594	0.06642823555	0.0001152903941
26	0.1494924812	0.08460266177	0.06488981940	88	0.06649714157	0.06642714921	0.00006999235980
28	0.1400823737	0.08013863254	0.05994374117	90	0.06646865240	0.06642633657	0.00004231582840
30	0.1320761882	0.07682696335	0.05524922486	92	0.06645124606	0.06642572928	0.00002551677885
32	0.1251623050	0.07434482451	0.05081748049	94	0.06644063840	0.06642527629	0.00001536210660
34	0.1191143954	0.07246963941	0.04664475598	96	0.06643417821	0.06642493873	9.239482928 10^{-6}
36	0.1137649100	0.07104470729	0.04272020270	98	0.06643024151	0.06642468779	5.553715787 10^{-6}
38	0.1089876217	0.06995754264	0.03903007902	100	0.06642783843	0.06642450139	3.337040062 10^{-6}
40	0.1046859020	0.06912596741	0.03555993454	102	0.06642636805	0.06642436338	2.004673661 10^{-6}
42	0.1007847108	0.06848903032	0.03229568045	104	0.06642546535	0.06642426123	1.204116844 10^{-6}
44	0.09722504036	0.06800095540	0.02922408496	106	0.06642490920	0.06642418600	7.232016059 10^{-7}
46	0.09396001595	0.06762703208	0.02633298387	108	0.06642456487	0.06642413053	4.343400996 10^{-7}
48	0.09095213810	0.06734075520	0.02361138290	110	0.06642435080	0.06642408995	2.608486872 10^{-7}
50	0.08817133129	0.06712179354	0.02104953775	112	0.06642421681	0.06642406016	1.566536809 10^{-7}
52	0.08559357683	0.06695450969	0.01863906714	114	0.06642413264	0.06642403856	9.407814649 10^{-8}
54	0.08319998343	0.06682686399	0.01637311943	116	0.06642407927	0.06642402278	5.649823126 10^{-8}
56	0.08097619440	0.06672958567	0.01424660873	118	0.06642404541	0.06642401148	3.392969324 10^{-8}
58	0.07891205844	0.06665554203	0.01225651640	120	0.06642402363	0.06642400326	2.037626344 10^{-8}
60	0.07700149594	0.06659925095	0.01040224499	122	0.06642400973	0.06642399749	1.223683364 10^{-8}
62	0.07524247806	0.06655650571	0.008685972345	124	0.06642400064	0.06642399330	7.348751434 10^{-9}
64	0.07363698403	0.06652408239	0.007112901636	126	0.06642399487	0.06642399045	4.413246970 10^{-9}
66	0.07219071400	0.06649951490	0.005691199104	128	0.06642399102	0.06642398837	2.650348398 10^{-9}
68	0.07091220590	0.06648091878	0.004431287122	130	0.06642398864	0.06642398705	1.591651424 10^{-9}
70	0.06981091435	0.06646685693	0.003344057426	132	0.06642398700	0.06642398605	9.558570401 10^{-10}
72	0.06889393796	0.06645623402	0.002437703939	134	0.06642398607	0.06642398549	5.740348394 10^{-10}

Table 2: $\varepsilon = 10^{-5}$; $a-$bifurcation values for period p sink

[Homburg &,Weiss(1999)] Homburg A.J.,Weiss H. *A Geometric Criterion for Positive Topological Entropy II: Homoclinic Tangencies* Commun. Math. Phys. **208**, 267- 273 (1999)

[Lyubich & Martens(2011)] Lyubich M., Martens M. *Renormalization in the Hénon family, II: the Heteroclinic Web* Invent Math **186**:115-189 (2011)

[Newhouse (1974)] Newhouse, S. *Diffeomorphisms With Infinitely Many Sinks.* Topology **13**: 9-18 (1974)

[Newhouse(1979)] Newhouse, S. *The Abundance Of Wild Hyperbolic Sets And Nonsmooth Stable Sets For Diffeomorphisms.* Publ. Math. Inst. Hautes Études Sci. **50**: 101-151 (1979).

[Tedeschini-Lalli & Yorke(1986)] Tedeschini-Lalli L. Yorke J.A. *How Often Do Simple Dynamical Systems have Infinitely Many Coexisting Sinks* Comm. Math. Phys. **106**: 635-657 (1986)

[Yorke & Alligood(1985)] Yorke, J.A. Alligood K.T. *Period Doubling Cascades of Attractors: a Prerequisite for Horseshoes* Comm. Math. Phys. **101**: 305-321 (1985)