# A BREZIS-NIRENBERG RESULT FOR NON-LOCAL CRITICAL EQUATIONS IN LOW DIMENSION

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Abstract. The present paper is devoted to the study of the following non-local fractional equation involving critical nonlinearities

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \,, \end{cases}$$

where $s \in (0,1)$ is fixed, $(-\Delta)^s$ is the fractional Laplace operator, $\lambda$ is a positive parameter, $2^*$ is the fractional critical Sobolev exponent and $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary. In the recent papers [14, 18, 19] we investigated the existence of non-trivial solutions for this problem when $\Omega$ is an open bounded subset of $\mathbb{R}^n$ with $n \geqslant 4s$ and, in this framework, we prove some existence results.

Aim of this paper is to complete the investigation carried on in [14, 18, 19], by considering the case when $2s < n < 4s$. In this context, we prove an existence theorem for our problem, which may be seen as a Brezis-Nirenberg type result in low dimension. In particular when $s = 1$ (and consequently $n = 3$) our result is the classical result obtained by Brezis and Nirenberg in the famous paper [4]. In this sense the present work may be considered as the extension of some classical results for the Laplacian to the case of non-local fractional operators.

Contents



*Key words and phrases.* Critical nonlinearities, best critical Sobolev constant, variational techniques, Mountain Pass Theorem, Linking Theorem, integrodifferential operators, fractional Laplacian.

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# 1. Introduction

The quest of nontrivial solutions of the equation

$$\mathcal{L}u = \Psi(u)$$

with Dirichlet boundary data is one of the classical topic of research of the modern nonlinear analysis. Typically, $\mathcal{L}$ is an elliptic operator (in most of the cases, the Laplacian) and $\Psi : \mathbb{R} \to \mathbb{R}$ is a reasonably smooth function (called "the nonlinearity" in jargon). When $\Psi(0) = 0$, we have that $u \equiv 0$ is a trivial solution of the problem, and the task is to find another (hence non-trivial) solution – i.e., a solution that does not vanish identically.

Such a solution is usually sought by variational and topological methods: namely, the problem is reduced to the search of (non-trivial) critical points of an energy functional. The change of topology of the sublevel sets of this functional is the geometric indication of the presence of a critical point, in combination with suitable compactness conditions that ensure that the critical point "does not escape towards infinity".

In the celebrated paper [4], the case of $\mathcal{L} := \Delta$ and $\Psi(u) := -\lambda u - |u|^{2_*-2}u$ was taken into account. In this setting, $\lambda$ is a positive parameter and

$$2_* := \frac{2n}{n-2} \tag{1.1}$$

is the Sobolev critical exponent (to make sense of which, it is necessary to require $n \geqslant 3$). The technical problem posed by such an exponent is that the Sobolev embedding is not compact. Nevertheless, in [4], a fine variational analysis showed that non-trivial solutions exist when $n \geqslant 4$ and $\lambda$ lies in a suitable range of parameters. This result was extended in [18] for the case of the fractional Laplacian $-(-\Delta)^s$, with $s \in (0,1)$: in this case, the dimensional restriction becomes $n \geqslant 4s$ (which is consistent with the one in [4] when $s = 1$).

In the case of the Laplacian, the case $n = 3$ was treated in detail in [9]. The purpose of the present paper is to extend this analysis to the case of the fractional Laplacian under the dimensional restriction $n \in (2s, 4s)$. Notice once more that this condition reduces to the classical one $n = 3$ when $s = 1$, but for different values of $s \in (0,1)$ it may cover different dimensions (for instance, it boils down to $n = 1$ when $s \in (1/4, 1/2)$, but it becomes $n = 2$, 3 for $s = 7/8$).

We recall that fractional and non-local operators of elliptic type were widely studied in the literature, both for their interesting theoretical structure and in view of concrete applications in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves (we refer, e.g., to [8] for an introduction to these topics and their applications).

Combining the interest of the mathematical community both for non-local operators and for pde's with nonlinearities with critical growth (in the sense of Sobolev), in the recent papers [14, 18, 19] we studied critical non-local equations using variational techniques (see also [15, 16, 17] for the subcritical case).

In these works our model problem is given by the following critical non-local fractional equation

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.2}$$

where $s \in (0,1)$ is fixed and $-(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.3}$$

$\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary, $\lambda > 0$ and

$$2^* = \frac{2n}{n-2s}$$

is the fractional critical Sobolev exponent (notice that when $s = 1$ the above exponent reduces to the classical Sobolev exponent $2_*$ in (1.1)). The homogeneous Dirichlet datum in (1.2) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the non-local nature of the operator $(-\Delta)^s$.

In this context in [14, 18] we have shown that if $n \geqslant 4s$, then for any $\lambda > 0$ different from the eigenvalue of the fractional Laplacian with homogeneous Dirichlet datum, problem (1.2) admits a non-trivial solution.

We thought that it was interesting to check what happens in lower dimension, that is when $2s < n < 4s$. Aim of this paper is to cover this case, in order to treat completely the critical setting. The result we obtain in this case is the following:

**Theorem 1.** *Let $s \in (0,1)$, $2s < n < 4s$ and $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary.*

*Then, there exists $\lambda_s > 0$ such that for any $\lambda > \lambda_s$ different from the eigenvalues of $(-\Delta)^s$ problem (1.2) admits a weak solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.*

We will prove Theorem 1 by using the Mountain Pass and the Linking Theorem in a combined way. The main difficulty to be overcame is the lack of compactness. In order to solve this problem we first will show that the Palais–Smale condition holds true in a suitable range related to the best fractional Sobolev constant $S_s$ (see Section 4 for more details) and then we will prove that the critical Mountain Pass and Linking levels are below this threshold.

It is interesting to notice that when $s = 1$ Theorem 1 reads as the results obtained for the Laplacian in [4, Corollary 1.1] for a bounded domain of $\mathbb{R}^3$ and in [6, Theorem 1] (see also [4, Theorem 1.2]) in the case of the ball in $\mathbb{R}^3$. Hence, we think that Theorem 1 may be seen as the natural extension of some classical results well know for the Laplacian (see [4, 5, 6, 11] and also [20, 22]) to the non-local fractional framework. With this respect, we recall the recent papers [2, 21], where a non-local version of the Brezis-Nirenberg result was also considered for an fractional operator different than the one treated here. See also [10] for another nonlinear analysis problem for fractional operators.

We remark that problem (1.2) has a variational structure and we are interested in the notion of solution given by the following weak formulation:

$$\text{(1.4)} \qquad \begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ \displaystyle\qquad\qquad\qquad\qquad = \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \\ \qquad\qquad\qquad\qquad \forall\, \varphi \in H^s(\mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \\ u \in H^s(\mathbb{R}^n) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

This formulation represents the Euler-Lagrange equation associated with the functional $\mathcal{J}_{s,\lambda} : X_0 \to \mathbb{R}$ defined as

$$\text{(1.5)} \qquad \mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\,.$$

Here the functional space $X_0$ is given by

$$X_0 = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

while $H^s(\mathbb{R}^n)$ is the usual fractional Sobolev space (we refer to [14, 15, 16, 17, 18, 19] for a general definition of $X_0$ and its properties and, for instance, to [8] for $H^s(\mathbb{R}^n)$).

In order to get non-trivial weak solutions for (1.2) (i.e non-trivial solution of (1.4)), as in the classical case of the Laplacian, we will use variational techniques. More precisely, we will apply suitable variants of the Mountain Pass Theorem and of the Linking Theorem (see e.g. [1, 12, 13]) to the functional $\mathcal{J}_{s,\lambda}$.

As usual in the critical case, the problem related to the variational formulation of (1.2) is the lack of compactness, due to the fact that $H^s(\mathbb{R}^n)$ is not compactly embedded in $L^{2^*}(\mathbb{R}^n)$. To overcome this difficulty we will show that, even if the functional $\mathcal{J}_{s,\lambda}$ does not verify globally the Palais–Smale condition, it satisfies such a condition in a suitable range related to the best fractional critical Sobolev constant $S_s$ for the embedding of $H^s(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$ which is defined as

$$S_s := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v)\,, \tag{1.6}$$

where for any $v \in H^s(\mathbb{R}^n)\setminus\{0\}$

$$S_s(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{1.7}$$

Along the paper we also consider the constant $S_{s,\lambda}$ defined as follows

$$S_{s,\lambda} := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_{s,\lambda}(v)\,, \tag{1.8}$$

where for any $v \in H^s(\mathbb{R}^n)\setminus\{0\}$

$$S_{s,\lambda}(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_{\mathbb{R}^n}|v(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{1.9}$$

The consistency of the definitions of $S_s$ and $S_{s,\lambda}$ and the fact that they are strictly positive follows from [18, Lemma 14] and [8, Theorem 6.5]. It is easily seen that when $\lambda = 0$, then $S_{s,\lambda} = S_s$. On the other hand, $S_{s,\lambda}(v) < S_s(v)$ for any $v \in H^s(\mathbb{R}^n)\setminus\{0\}$ and any $\lambda > 0$, so $S_{s,\lambda} \leqslant S_s$. The interesting case is when the *strict* inequality holds true, namely when $S_{s,\lambda} < S_s$, since such a non-degeneracy condition will allow us to recover the necessary compactness and to obtain the existence of a non-trivial solution for our problem (the fact that the inequality is strict will be proved in Corollary 8 here below).

The paper is organized as follows. In Section 2 we discuss the setting of the problem. In Section 3 we give an $L^\infty$–regularity result for the eigenvalues of the fractional Laplacian. In Section 4 we prove a compactness property for the functional $\mathcal{J}_{s,\lambda}$, while Section 5 is devoted to an estimate involving the fractional critical Sobolev constants. In Section 6 we study the geometric structure of $\mathcal{J}_{s,\lambda}$ and in Section 7 we prove a crucial estimate which allow us to apply classical critical points theorems to our functional. Finally, in Section 8 we give a proof of Theorem 1 by combining different topological critical point theories.

## 2. Brief summary of the functional setting

In this section we briefly recall some preliminary results on the fractional Sobolev space $H^s(\mathbb{R}^n)$ and on the functional space $X_0$, which was first introduced in [15] (see also [14, 16, 17, 18, 19], where some properties of $X_0$ were proved). The reader familiar with this topic may skip this section and may go directly to the next one.

In the following we denote by $H^s(\mathbb{R}^n)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^{2n}} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}, \tag{2.1}$$

while $X_0$ is the functional space defined as

$$X_0 = \big\{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

For this see, for instance, [18, Lemma 7-$b$)].

By [8, Theorem 6.5] (used here with $p = 2$) we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \Big(\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2} \tag{2.2}$$

as norm on $X_0$. The following result, proved in [16, Lemma 7], holds true:

**Lemma 2.** *$(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with scalar product*

$$\langle u, v\rangle_{X_0} = \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)\,dx\,dy\,.$$

Now we discuss some results related to the embeddings of the spaces $X_0$ and $H^s(\mathbb{R}^n)$ into the usual Lebesgue spaces. In [16, Lemma 8] and [18, Lemma 9] some embeddings results were proved in a general setting. In the case of the fractional Laplacian these results read as follows:

**Lemma 3.** *The following assertions hold true:*

*a) if $\Omega$ has a Lipschitz boundary, then the embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is compact for any $\nu \in [1, 2^*)$;*

*b) the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous.*

In what follows, we denote by $\lambda_{1,s}$ the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary conditions. For the existence and the basic properties of the eigenvalues and the eigenfunctions of the fractional Laplacian $(-\Delta)^s$ we refer to [17, Proposition 9 and Appendix A], where a spectral theory for general integrodifferential non-local operators was developed.

## 3. A regularity result for the eigenvalues of $(-\Delta)^s$

In this section we prove a regularity result for the eigenvalues of the fractional Laplace operator $(-\Delta)^s$. This result can be easily extended to general non-local integrodifferential operators as the ones treated in [14, 15, 16, 17, 18, 19] (see the definition of $\mathcal{L}_K$ in all these papers).

**Proposition 4.** *Let $e \in X_0$ and $\lambda > 0$ be such that*

$$\langle e, \varphi\rangle_{X_0} = \lambda \int_\Omega e(x)\varphi(x)\,dx \tag{3.1}$$

*for any $\varphi \in X_0$. Then $e \in L^\infty(\Omega)$ and there exists $C > 0$, possibly depending on $n$, $s$ and $\lambda$, such that*

$$\|e\|_{L^\infty(\Omega)} \leqslant C\|e\|_{L^2(\Omega)}\,. \tag{3.2}$$

*Proof.* We may assume that $e$ does not vanish identically (otherwise there is nothing to prove). Let $\delta > 0$, to be taken appropriately small in what follows (the choice of $\delta$ will be done on (3.12) below). Up to multiplying $e$ by a small constant, we may and do assume that

$$\|e\|_{L^2(\Omega)} = \delta\,. \tag{3.3}$$

We perform a simple observation on the positive part of a function $v \in X_0$, that is $v^+ := \max\{v, 0\}$. We claim that

$$(v(x) - v(y))(v^+(x) - v^+(y)) \geqslant |v^+(x) - v^+(y)|^2 \tag{3.4}$$

for any $x$, $y \in \mathbb{R}^n$. To check (3.4), since the role of $x$ and $y$ is symmetric, we can always suppose $v(x) \geqslant v(y)$. Also (3.4) is clearly an identity when $x$, $y \in \{v \geqslant 0\}$ and when $x$, $y \in \{v < 0\}$. So it only remains to check (3.4) when $x \in \{v \geqslant 0\}$ and $y \in \{v < 0\}$. In this case

$$v^+(x) - v^+(y) = v(x) < v(x) + |v(y)| = v(x) - v(y)\,,$$

and so, if we multiply by $v^+(x) - v^+(y) \geqslant 0$ both sides, we obtain (3.4).

Now, for any $k \in \mathbb{N}$, we let $C_k := 1 - 2^{-k}$, $v_k := e - C_k$, $w_k := v_k^+$ and $U_k := \|w_k\|^2_{L^2(\Omega)}$. Notice that $w_k \in X$, since $X$ is a linear space, and a.e. $x \in \mathbb{R}^n \setminus \Omega$ we have that

$$e(x) - C_k = -C_k \leqslant 0\,.$$

Hence $w_k(x) = (-C_k)^+ = 0$ in $\mathbb{R}^n \setminus \Omega$, so that $w_k \in X_0$. Moreover, we get that

$$0 \leqslant w_k \leqslant |e| + |C_k| \leqslant |e| + 1 \in L^2(\Omega)\,,$$

being $\Omega$ bounded, and

$$\lim_{k \to +\infty} w_k = (e-1)^+\,.$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{k \to +\infty} U_k = \int_\Omega \Big[(e(x) - 1)^+\Big]^2\, dx\,. \tag{3.5}$$

Moreover, for any $k \in \mathbb{N}$ we have $C_{k+1} > C_k$ and so $v_{k+1} < v_k$ a.e. in $\mathbb{R}^n$, from which we deduce that

$$w_{k+1} \leqslant w_k \;\text{ a.e. in } \mathbb{R}^n\,. \tag{3.6}$$

Let also $A_k := C_{k+1}/(C_{k+1} - C_k) = 2^{k+1} - 1$ for any $k \in \mathbb{N}$. We claim that for any $k \in \mathbb{N}$

$$e < A_k w_k \;\text{ on } \{w_{k+1} > 0\}\,. \tag{3.7}$$

To check this, let $x \in \{w_{k+1} > 0\}$. Then $e(x) - C_{k+1} > 0$ and so by the properties of $C_k$ we have $e(x) > C_{k+1} > C_k$. Hence, $w_k(x) = v_k(x) = e(x) - C_k$ and

$$\begin{aligned} A_k w_k(x) = A_k(e(x) - C_k) &= \frac{C_{k+1}}{C_{k+1} - C_k} e(x) - \frac{C_k C_{k+1}}{C_{k+1} - C_k} \\ &= e(x) + \frac{C_k}{C_{k+1} - C_k}\,(e(x) - C_{k+1}) > e(x). \end{aligned}$$

This proves (3.7).

Notice also that

$$v_{k+1}(x) - v_{k+1}(y) = e(x) - e(y)$$

for any $x$, $y \in \mathbb{R}^n$. Using this, (3.1), (3.4) and the fact that $v_{k+1}^+ = w_{k+1} \in X_0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \frac{|w_{k+1}(x) - w_{k+1}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy &= \int_{\mathbb{R}^{2n}} \frac{|v_{k+1}^+(x) - v_{k+1}^+(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\leqslant \int_{\mathbb{R}^{2n}} \frac{(v_{k+1}(x) - v_{k+1}(y))(v_{k+1}^+(x) - v_{k+1}^+(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &= \int_{\mathbb{R}^{2n}} \frac{(e(x) - e(y))(v_{k+1}^+(x) - v_{k+1}^+(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &= \lambda \int_\Omega e(x) w_{k+1}(x)\, dx \\ &= \lambda \int_{\{w_{k+1} > 0\}} e(x) w_{k+1}(x)\, dx\,. \end{aligned}$$

Therefore, as a consequence of this, recalling (3.6) and (3.7) and using the fact that $\lambda$ is positive, we deduce

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \frac{\big|w_{k+1}(x)-w_{k+1}(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy &= \lambda\int_{\{w_{k+1}>0\}} e(x)w_{k+1}(x)\,dx\\ &\leqslant \lambda A_k\int_{\{w_{k+1}>0\}} w_k(x)w_{k+1}(x)\,dx\\ &\leqslant \lambda A_k\int_{\{w_{k+1}>0\}} w_k^2(x)\,dx \leqslant 2^{k+1}\lambda U_k\,.\end{aligned}\tag{3.8}$$

Now we claim that

$$\{w_{k+1}>0\}\subseteq\{w_k>2^{-(k+1)}\}.\tag{3.9}$$

To establish this, we observe that if $x\in\{w_{k+1}>0\}$ then

$$0<v_{k+1}^+(x)=\max\{e(x)-C_{k+1},0\}\,.$$

Hence $e(x)-C_{k+1}>0$. Accordingly

$$v_k(x)=e(x)-C_k>C_{k+1}-C_k=2^{-(k+1)}\,,$$

so that, as a consequence, $w_k(x)=v_k(x)>2^{-(k+1)}$, which proves (3.9).

As a consequence of (3.9), we obtain

$$\begin{aligned}U_k &= \|w_k\|^2_{L^2(\Omega)}\\ &\geqslant \int_{\{w_k>2^{-(k+1)}\}} w_k^2(x)\,dx\\ &\geqslant 2^{-2(k+1)}\big|\{w_k\geqslant 2^{-(k+1)}\}\big|\\ &\geqslant 2^{-2(k+1)}\big|\{w_{k+1}>0\}\big|\,.\end{aligned}\tag{3.10}$$

Now we use the Hőlder inequality (with exponents $2^*/2$ and $n/(2s)$) and the fractional Sobolev inequality (see, e.g., Theorem 6.5 in [8]) to get that

$$\begin{aligned}U_{k+1} &\leqslant \left(\int_\Omega |w_{k+1}(x)|^{2^*}\,dx\right)^{2/2^*}\big|\{w_{k+1}>0\}\big|^{2s/n}\\ &\leqslant \tilde c\int_{\mathbb{R}^{2n}}\frac{|w_{k+1}(x)-w_{k+1}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,\big|\{w_{k+1}>0\}\big|^{2s/n}\,,\end{aligned}$$

for some positive constant $\tilde c$ depending only on $n$ and $s$. Consequently, by (3.8) and (3.10), we see that

$$\begin{aligned}U_{k+1}&\leqslant (\tilde c\,2^{k+1}\lambda U_k)\,(2^{2(k+1)}U_k)^{2s/n}=2^{1+(4s/n)}\tilde c\lambda\cdot\big(2^{1+(4s/n)}\big)^kU_k^{1+(2s/n)}\\ &\leqslant \Big((1+2^{1+(4s/n)}\tilde c\lambda)\cdot 2^{1+(4s/n)}\Big)^kU_k^{1+(2s/n)}=C^kU_k^\beta\,,\end{aligned}\tag{3.11}$$

where $\beta:=1+(2s/n)>1$ and $C>1$ only depends on $n$, $s$ and $\lambda$. Now we are ready to perform our choice of $\delta$: namely we assume that $\delta>0$ is so small that

$$\delta^{\beta-1}<\frac{1}{C^{1/(\beta-1)}}\tag{3.12}$$

We also fix

$$\eta\in\left(\delta^{\beta-1},\frac{1}{C^{1/(\beta-1)}}\right).$$

Notice that, since $C>1$ and $\beta>1$

$$\eta\in(0,1)\,.\tag{3.13}$$

Moreover

$$\delta^{\beta-1}\leqslant\eta\ \text{ and }\ C\eta^{\beta-1}\leqslant 1\,.\tag{3.14}$$

We claim that

$$U_k \leqslant \delta\eta^k\,. \tag{3.15}$$

The proof is by induction. First of all

$$U_0 = \|e^+\|_{L^2(\Omega)} \leqslant \|e\|_{L^2(\Omega)} = \delta\,,$$

which is (3.15) when $k = 0$. Let us now suppose that (3.15) holds true for $k$ and let us prove it for $k+1$. For this, we use (3.11) and (3.14):

$$U_{k+1} \leqslant C^k U_k^\beta \leqslant C^k(\delta\eta^k)^\beta = \delta(C\eta^{\beta-1})^k\delta^{\beta-1}\eta^k \leqslant \delta\eta^{k+1}\,.$$

This proves (3.15). Then, by (3.13) and (3.15), we conclude that

$$\lim_{k\to+\infty} U_k = 0\,.$$

Hence, by (3.5), $(e-1)^+ = 0$ a.e. in $\Omega$, that is $e \leqslant 1$ a.e. in $\Omega$. By replacing $e$ with $-e$, we obtain $\|e\|_{L^\infty(\Omega)} \leqslant 1$. Then, (3.2) follows by recalling the scaling in (3.3). This completes the proof of Proposition 4. □

## 4. A compactness condition for the functional $\mathcal{J}_{s,\lambda}$

Now we are ready for studying problem 1.4. In this section we prove a compactness property for the functional $\mathcal{J}_{s,\lambda}$.

First of all, we note that this functional is well defined thanks to Lemma 3-*b*). Moreover, $\mathcal{J}_{s,\lambda}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned}\langle \mathcal{J}'_{s,\lambda}(u), \varphi\rangle = \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ - \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx\,.\end{aligned}$$

Thus, critical points of $\mathcal{J}_{s,\lambda}$ are solutions to problem (1.4). In order to find these critical points, we will apply some variants of the Mountain Pass Theorem and of the Linking Theorem which take into account the lack of compactness due to the fact that the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is only continuous.

In the sequel we prove that, for suitable values of $c$, the functional $\mathcal{J}_{s,\lambda}$ satisfies the following Palais–Smale condition at level $c \in \mathbb{R}$ (or $(PS)_c$–condition for short):

every Palais–Smale sequence at level $c$ (or $(PS)_c$–sequence for short) $u_j$ in $X_0$ admits a subsequence strongly convergent in $X_0$.

We say that $u_j$ in $X_0$ is a $(PS)_c$–sequence for $\mathcal{J}_{s,\lambda}$ if

$$\mathcal{J}_{s,\lambda}(u_j) \to c \quad\text{and}\quad \sup\Big\{\big|\langle\,\mathcal{J}'_{s,\lambda}(u_j),\varphi\,\rangle\big| \,:\, \varphi\in X_0\,, \|\varphi\|_{X_0}=1\Big\} \to 0 \quad\text{as}\quad j\to+\infty\,.$$

**Proposition 5.** *Let $\lambda > 0$ and let $c \in \mathbb{R}$ be such that*

$$c < \frac{s}{n}S_s^{n/(2s)}\,. \tag{4.1}$$

*Then, the functional $\mathcal{J}_{s,\lambda}$ satisfies the $(PS)_c$–condition.*

*Proof.* Let $u_j \in X_0$ be a $(PS)_c$–sequence for $\mathcal{J}_{s,\lambda}$, that is let $u_j$ be such that

$$\mathcal{J}_{s,\lambda}(u_j) \to c \tag{4.2}$$

and

$$\sup\Big\{\big|\langle\,\mathcal{J}'_{s,\lambda}(u_j),\varphi\,\rangle\big| \,:\, \varphi\in X_0\,, \|\varphi\|_{X_0}=1\Big\} \to 0 \tag{4.3}$$

as $j \to +\infty$. In order to prove Proposition 5, we proceed by steps.

**Step 1.** *The sequence $u_j$ is bounded in $X_0$.*

*Proof.* The proof of Step 1 is quite standard: see, for instance, [18], where more general non-local operators $\mathcal{L}_K$ than the fractional Laplacian were considered (but only in the case when $\lambda > 0$ is less than the first eigenvalue of $-\mathcal{L}_K$). Here, for the sake of clarity and also for reader's convenience, we prefer to repeat the proof of Step 1.

For any $j \in \mathbb{N}$ by (4.2) and (4.3) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_{s,\lambda}(u_j)| \leqslant \kappa\,, \tag{4.4}$$

and

$$\left|\langle \mathcal{J}'_{s,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\right| \leqslant \kappa\,. \tag{4.5}$$

As a consequence of (4.4) and (4.5) we have

$$\mathcal{J}_{s,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{s,\lambda}(u_j), u_j\rangle \leqslant \kappa\left(1 + \|u_j\|_{X_0}\right). \tag{4.6}$$

Moreover

$$\mathcal{J}_{s,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{s,\lambda}(u_j), u_j\rangle = -\left(\frac{1}{2^*} - \frac{1}{2}\right)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} = \frac{s}{n}\,\|u_j\|^{2^*}_{L^{2^*}(\Omega)}\,,$$

so that, thanks to (4.6), we get that for any $j \in \mathbb{N}$

$$\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \leqslant \kappa_*\left(1 + \|u_j\|_{X_0}\right) \tag{4.7}$$

for a suitable positive constant $\kappa_*$. Consequently, recalling that $2^* > 2$, the use of the Hölder inequality yields

$$\|u_j\|^2_{L^2(\Omega)} \leqslant |\Omega|^{2s/n}\|u_j\|^2_{L^{2^*}(\Omega)} \leqslant \kappa_*^{2/2^*}\,|\Omega|^{2s/n}\left(1 + \|u_j\|_{X_0}\right)^{2/2^*}\,.$$

That is, using that $2/2^* < 1$,

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \widetilde{\kappa}\left(1 + \|u_j\|_{X_0}\right), \tag{4.8}$$

for a suitable $\widetilde{\kappa} > 0$ that does not depend on $j$. By (4.4), (4.7) and (4.8), we conclude that

$$\begin{aligned}\kappa \geqslant \mathcal{J}_{s,\lambda}(u_j) &= \frac{1}{2}\|u_j\|^2_{X_0} - \frac{\lambda}{2}\|u_j\|^2_{L^2(\Omega)} - \frac{1}{2^*}\|u_j\|^{2^*}_{L^{2^*}(\Omega)}\\ &\geqslant \frac{1}{2}\|u_j\|^2_{X_0} - \underline{\kappa}\left(1 + \|u_j\|_{X_0}\right),\end{aligned}$$

with $\underline{\kappa} > 0$ independent of $j$, and so Step 1 is proved. □

**Step 2.** *Problem* (1.4) *admits a solution* $u_\infty \in X_0$.

*Proof.* Since $u_j$ is bounded in $X_0$, which is a reflexive space by Lemma 2, then, up to a subsequence, still denoted by $u_j$, there exists $u_\infty \in X_0$ such that $u_j \to u_\infty$ weakly in $X_0$, that is

$$\int_{\mathbb{R}^{2n}} \frac{(u_j(x) - u_j(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \to \int_{\mathbb{R}^{2n}} \frac{(u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \tag{4.9}$$

for any $\varphi \in X_0$, as $j \to +\infty$. Moreover, by Step 1, (4.7), Lemma 3 and the fact that $L^{2^*}(\mathbb{R}^n)$ is a reflexive space we have that, up to a subsequence

$$u_j \to u_\infty \quad \text{weakly in } L^{2^*}(\mathbb{R}^n) \tag{4.10}$$

$$u_j \to u_\infty \quad \text{in } L^{\nu}(\mathbb{R}^n) \text{ for any } \nu \in [1, 2^*) \tag{4.11}$$

and

$$u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n \tag{4.12}$$

as $j \to +\infty$.

As a consequence of (4.7) and Step 1, we have that $\|u_j\|_{L^{2^*}(\Omega)}$ is bounded uniformly in $j$, hence $|u_j|^{2^*-2}u_j$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$, uniformly in $j$. Accordingly, by (4.10) we get that

$$|u_j|^{2^*-2}u_j \to |u_\infty|^{2^*-2}u_\infty \quad \text{weakly in } L^{2^*/(2^*-1)}(\Omega) \tag{4.13}$$

as $j \to +\infty$.

Since the conjugate exponent of $2^*/(2^*-1)$ is $2^*$, it follows from (4.13) that

$$\int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx \to \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx \quad \text{for any} \quad \varphi \in L^{2^*}(\Omega)\,, \tag{4.14}$$

and so, in particular,

$$\int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx \to \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx \quad \text{for any} \quad \varphi \in X_0 \tag{4.15}$$

as $j \to +\infty$, thanks to Lemma 3.

By (4.3), for any $\varphi \in X_0$ as $j \to +\infty$

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}'_{s,\lambda}(u_j), \varphi\rangle = \int_{\mathbb{R}^{2n}} \frac{(u_j(x)-u_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u_j(x)\varphi(x)\,dx \\ - \int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx\,, \end{aligned}$$

so that, passing to the limit in this expression as $j \to +\infty$ and taking into account (4.9), (4.11) and (4.15) we get

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \frac{(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u_\infty(x)\varphi(x)\,dx \\ - \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx = 0 \end{aligned}$$

for any $\varphi \in X_0$, that is $u_\infty$ is a solution of problem (1.4) and Step 2 follows. □

**Step 3.** *The following relations hold true:*

*a)* $\mathcal{J}_{s,\lambda}(u_\infty) = \dfrac{s}{n}\displaystyle\int_\Omega |u_\infty(x)|^{2^*}\,dx \geqslant 0$;

*b)*
$$\begin{aligned} \mathcal{J}_{s,\lambda}(u_j) = \mathcal{J}_{s,\lambda}(u_\infty) + \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}dx\,dy \\ -\frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + o(1) \end{aligned}$$
*as* $j \to +\infty$*;*

*c)*
$$\int_{\mathbb{R}^{2n}} \frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}dx\,dy = \int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + o(1)$$
*as* $j \to +\infty$.

*Proof.* For the proof of this step we refer to Steps 3–5 of [14, Proposition 7], where a general non-local integrodifferential operator with kernel $K$ was considered. □

Now, we can conclude the proof of Proposition 5.

By Step 3-$c$) it easily follows that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}dx\,dy-\frac{1}{2^*}\int_\Omega|u_j(x)-u_\infty(x)|^{2^*}dx\\
&\qquad=\left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}dx\,dy+o(1)\\
&\qquad=\frac{s}{n}\int_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}dx\,dy+o(1)\,,
\end{aligned}$$

and so, recalling Step 3-$b$),

$$\begin{aligned}
&\mathcal{J}_{s,\lambda}(u_\infty)+\frac{s}{n}\int_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}dx\,dy\\
&\qquad=\mathcal{J}_{s,\lambda}(u_j)+o(1)\\
&\qquad=c+o(1)\,,
\end{aligned}\tag{4.16}$$

as $j\to+\infty$, thanks to (4.2). Since the sequence $\|u_j\|_{X_0}$ is bounded in $\mathbb{R}$ by Step 1, then we can assume that up to a subsequence, if necessary,

$$\|u_j-u_\infty\|^2_{X_0}=\int_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\to L\tag{4.17}$$

and so, again as a consequence of Step 3-$c$),

$$\int_\Omega|u_j(x)-u_\infty(x)|^{2^*}\,dx\to L$$

as $j\to+\infty$. Of course $L\in[0,+\infty)$ and, by definition of $S_s$ and $S_s(\cdot)$ (see (1.6) and (1.7)), we have that

$$S_s\leqslant S_s(u_j-u_\infty)\to\frac{L}{L^{2/2^*}}$$

as $j\to+\infty$. As a consequence,

$$L\geqslant L^{2/2^*}S_s\,.$$

Thus

$$\text{either}\quad L=0\quad\text{or}\quad L\geqslant S_s^{n/(2s)}\,.$$

The case $L\geqslant S_s^{n/(2s)}$ can not occur. Otherwise, by (4.16), (4.17) and Step 3-$a$), we would get

$$c=\mathcal{J}_{s,\lambda}(u_\infty)+\frac{s}{n}\,L\geqslant\frac{s}{n}\,L\geqslant\frac{s}{n}\,S_s^{n/(2s)}\,,$$

which contradicts (4.1). Thus $L=0$ and so, by (4.17), we obtain that

$$\|u_j-u_\infty\|_{X_0}\to0$$

as $j\to+\infty$. Hence, the sequence $u_j$ has a (strongly) convergent subsequence and this concludes the proof of Proposition 5. □

## 5. A NON-DEGENERACY ESTIMATE

As we said in the Introduction, to get our goal we have to show that

$$S_{s,\lambda}<S_s$$

for suitable values of $\lambda$. For this it is enough to prove that there exists a non-trivial function $\tilde v\in H^s(\mathbb{R}^n)$ such that

$$S_{s,\lambda}(\tilde v)<S_s\,,$$

for some $\lambda$ (recall (1.8)). This section is devoted to the construction of such $\tilde v$. To this aim we will proceed as in [18]. First of all, we recall some results appeared in [7, 18].

**Proposition 6.** *([7, Theorem 1.1]) The infimum in formula (1.6) is attained, that is*

$$S_s = S_s(\tilde u)\,,$$

*where*

$$\tilde u(x) = \kappa\Big(\mu^2 + |x-x_0|^2\Big)^{-(n-2s)/2}, \quad x\in\mathbb{R}^n \tag{5.1}$$

*with $\kappa\in\mathbb{R}\setminus\{0\}$, $\mu>0$ and $x_0\in\mathbb{R}^n$ fixed constants.*

*Equivalently, the function $\bar u$ defined as*

$$\bar u(x) = \frac{\tilde u(x)}{\|\tilde u\|_{L^{2^*}(\mathbb{R}^n)}} \tag{5.2}$$

*is such that*

$$S_s = \inf_{\substack{v\in H^s(\mathbb{R}^n)\\ \|v\|_{L^{2^*}(\mathbb{R}^n)}=1}} \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^{2n}} \frac{|\bar u(x)-\bar u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,. \tag{5.3}$$

In what follows we suppose that, up to a translation, $x_0=0$ in (5.1). Following [18, Section 4], we consider the function

$$u^*(x) = \bar u\left(\frac{x}{S_s^{1/(2s)}}\right), \qquad x\in\mathbb{R}^n\,, \tag{5.4}$$

which is an explicit solution of the limiting problem

$$(-\Delta)^s u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n \tag{5.5}$$

such that

$$\|u^*\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = S_s^{n/(2s)}\,. \tag{5.6}$$

Moreover, we construct the families of functions $U_\varepsilon$ and $u_\varepsilon$ as follows

$$U_\varepsilon(x) = \varepsilon^{-(n-2s)/2}\,u^*(x/\varepsilon)\,, \qquad x\in\mathbb{R}^n \tag{5.7}$$

and

$$u_\varepsilon(x) = \eta(x)U_\varepsilon(x)\,, \quad x\in\mathbb{R}^n\,, \tag{5.8}$$

for any $\varepsilon>0$. Here $\eta\in C^\infty(\mathbb{R}^n)$ is a function such that $0\leqslant\eta\leqslant 1$ in $\mathbb{R}^n$, $\eta\equiv 1$ in $B_\delta$ and $\eta\equiv 0$ in $\mathcal{C}B_{2\delta}$, where $B_\delta=B(0,\delta)$ and $\mathcal{C}B_\delta=\mathbb{R}^n\setminus B_\delta$, with $\delta>0$ fixed so that

$$B_{4\delta}\subset\Omega\,.$$

Note that $u_\varepsilon\in X_0$ and $u_\varepsilon=0$ a.e. in $\mathbb{R}^n\setminus\Omega$. Moreover, the function $u_\varepsilon$ satisfies the following crucial estimates, whose proofs can be found in [18, Propositions 21 and 22 and Subsection 4.2.1] and in [14, Proposition 12].

**Proposition 7.** *Let $s\in(0,1)$ and $n>2s$. Then, the following estimates hold true*

$$\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s})\,,$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^2\,dx \geqslant \begin{cases} C_s\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) & \text{if } n>4s\\ C_s\varepsilon^{2s}\big|\log\varepsilon\big| + \mathcal{O}(\varepsilon^{2s}) & \text{if } n=4s\\ C_s\varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s}) & \text{if } n<4s\,, \end{cases}$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*}\,dx = S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\,,$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*-1}\,dx = \mathcal{O}(\varepsilon^{(n-2s)/2})\,,$$

*and*

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|\,dx = \mathcal{O}(\varepsilon^{(n-2s)/2})$$

*as* $\varepsilon \to 0$ , *for some positive constant* $C_s$ *depending on* $s$ *.*

As a consequence of Proposition 7 we have

**Corollary 8.** *Let* $s \in (0,1)$ , $2s < n < 4s$ *and let* $u_\varepsilon$ *be as in* (5.8) *. Then, there exists* $\lambda_s > 0$ *such that for any* $\lambda > \lambda_s$

$$S_{s,\lambda}(u_\varepsilon) < S_s \,, \tag{5.9}$$

*provided* $\varepsilon > 0$ *is sufficiently small.*

*Proof.* We observe that

$$\frac{n}{2s}\left(1-\frac{2}{2^*}\right) = 1 \,.$$

Hence, by (1.9) and Proposition 7 we get

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda C_s \varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s})}{\left(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\right)^{2/2^*}} \\ &\leqslant S_s + \varepsilon^{n-2s}\left(\mathcal{O}(1) - \lambda\widetilde{C}_s\right) + \mathcal{O}(\varepsilon^{2s}) < S_s \,, \end{aligned} \tag{5.10}$$

if $\lambda$ is large enough, say $\lambda > \lambda_s > 0$, provided $\varepsilon > 0$ is sufficiently small. Thus, the assertion of Corollary 8 is proved. □

## 6. The geometric structure of the functional $\mathcal{J}_{s,\lambda}$

In this section we analyze the geometry of the functional $\mathcal{J}_{s,\lambda}$ . Precisely, we will show that $\mathcal{J}_{s,\lambda}$ has the geometric structure of the Mountain Pass Theorem when $\lambda < \lambda_{1,s}$ (see [1, 20, 22]), while it has the one required by the Linking Theorem when $\lambda \geqslant \lambda_{1,s}$ (see, for instance, [12, 13, 20, 22]) .

6.1. **The Mountain Pass geometry.** In this subsection we prove that the functional $\mathcal{J}_{s,\lambda}$ has the geometric features required by the Mountain Pass Theorem (see, e.g., [1] and also [4, Theorem 2.2]).

**Proposition 9.** *Let* $\lambda \in (0, \lambda_{1,s})$ *. Then, there exist* $\rho > 0$ , $\beta > 0$ *and* $e \in X_0$ *such that*

*a) for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$ *it results that* $\mathcal{J}_{s,\lambda}(u) \geqslant \beta$ *;*

*b)* $e \geqslant 0$ *a.e. in* $\mathbb{R}^n$, $\|e\|_{X_0} > \rho$ *and* $\mathcal{J}_{s,\lambda}(e) < \beta$ *.*

*In particular, we can construct* $e$ *as follows*

$$e = e_\varepsilon := \zeta_\varepsilon u_\varepsilon \tag{6.1}$$

*with* $\zeta_\varepsilon$ *large enough and* $u_\varepsilon$ *as in Corollary 8, with* $\varepsilon$ *sufficiently small.*

*Proof.* For part *a*) we can argue, for instance, as in [18, Proposition 18], where an operator more general than the fractional Laplacian was considered together with nonlinearities more general than the power function. Note that here we need the assumption $\lambda < \lambda_{1,s}$ .

For part *b*) we essentially proceed as in [18, Proposition 19] . We prefer to repeat it not only for the reader's convenience, but also in order to clarify the choice of $e$ given in formula (6.1) .

Let $u \in X_0$ be such that $\|u\|_{X_0} = 1$ and $u \geqslant 0$ a.e. in $\mathbb{R}^n$: this choice is possible, thanks to [15, Lemma 12]. Also, let $\zeta > 0$ . By the fact that $\lambda$ is positive, we have

$$\begin{aligned} \mathcal{J}_{s,\lambda}(\zeta u) &= \frac{\zeta^2}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\,dy - \frac{\lambda}{2}\zeta^2\int_\Omega |u(x)|^2\,dx - \frac{\zeta^{2^*}}{2^*}\int_\Omega |u(x)|^{2^*}\,dx \\ &\leqslant \frac{\zeta^2}{2}\|u\|_{X_0}^2 - \frac{\zeta^{2^*}}{2^*}\int_\Omega |u(x)|^{2^*}\,dx \\ &= \frac{\zeta^2}{2} - \frac{\zeta^{2^*}}{2^*}\int_\Omega |u(x)|^{2^*}\,dx \,. \end{aligned}$$

Since $2^* > 2 > 1$, passing to the limit as $\zeta \to +\infty$ (for $u$ fixed), we get that $\mathcal{J}_{s,\lambda}(\zeta u) \to -\infty$, so that the assertion follows taking $e = \zeta u$, with $\zeta$ sufficiently large.

In particular, we can take $u = u_\varepsilon / \|u_\varepsilon\|_{X_0} \in X_0$ (note that $u_\varepsilon \not\equiv 0$ by assumption) so that we can choose $e = e_\varepsilon = \zeta_\varepsilon u_\varepsilon$ with $\zeta_\varepsilon$ large enough. □

6.2. **The Linking geometry.** This subsection is devoted to the case when $\lambda \geqslant \lambda_{1,s}$. We can suppose that $\lambda \in [\lambda_{k,s}, \lambda_{k+1,s})$ for some $k \in \mathbb{N}$, where $\lambda_{k,s}$ is the $k$-th eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data.

Arguing as in [14, 19], we prove that the functional $\mathcal{J}_{s,\lambda}$ has the geometry of the Linking Theorem (see [12, 13]). In what follows we denote by

$$\mathbb{P}_{k+1,s} := \big\{ u \in X_0 \text{ s.t. } \langle u, e_{j,s}\rangle_{X_0} = 0 \quad \forall j = 1, \dots, k \big\},$$

where the function $e_{k,s}$ is an eigenfunction of $(-\Delta)^s$ corresponding to $\lambda_{k,s}$ for any $k \in \mathbb{N}$ (see [17, Proposition 9] for a detailed spectral theory of the fractional Laplacian).

**Proposition 10.** *Let $\lambda \in [\lambda_{k,s}, \lambda_{k+1,s})$ for some $k \in \mathbb{N}$. Then, there exist $\rho > 0$ and $\beta > 0$ such that*

*a) for any $u \in \mathbb{P}_{k+1}$ with $\|u\|_{X_0} = \rho$ it results that $\mathcal{J}_{s,\lambda}(u) \geqslant \beta$;*

*b) $\mathcal{J}_{s,\lambda}(u) \leqslant 0$ for any $u \in span\{e_{1,s}, \dots, e_{k,s}\}$;*

*c) for any finite dimensional subspace $\mathbb{F}$ of $X_0$, there exists $R > \rho$ such that*

$$\sup_{\substack{u \in \mathbb{F} \\ \|u\|_{X_0} \geqslant R}} \mathcal{J}_{s,\lambda}(u) \leqslant 0.$$

*In particular we can construct $\mathbb{F}$ as follows*

$$\mathbb{F} = \mathbb{F}_\varepsilon := span\{e_{1,s}, \dots, e_{k,s}\} \oplus span\{\tilde z_\varepsilon\}, \tag{6.2}$$

*with $\tilde z_\varepsilon = z_\varepsilon / \|z_\varepsilon\|_{X_0}$, $z_\varepsilon = u_\varepsilon - \displaystyle\sum_{i=1}^{k} \Big( \int_\Omega u_\varepsilon(x) e_{i,s}(x)\, dx \Big) e_{i,s}$ and $u_\varepsilon$ as in Corollary 8, with $\varepsilon$ small enough.*

*Proof.* The proof can be done as in [14, Propositions 8–10]. The particular choice of the finite dimensional subspace $\mathbb{F}$ given in formula (6.2) will be justified in the sequel (see Proposition 12 and (8.5)). □

## 7. A crucial estimate

In this section we prove a result which will be crucial in the sequel in order to estimate the critical level of the functional $\mathcal{J}_{s,\lambda}$.

For this, we need the following characterization of the eigenvalues of the fractional Laplacian $(-\Delta)^s$, given in [14, Proposition 5] for a general non-local fractional operator. For a detailed spectral theory of the fractional Laplacian and, in general, of non-local integrodifferential operators, we refer to [17, Proposition 9].

**Proposition 11.** *Let $k \in \mathbb{N}$ and let $\lambda_{k,s}$ be the $k$-th eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data and let $e_{i,s}$ be an eigenfunction of $(-\Delta)^s$ corresponding to $\lambda_{i,s}$ for any $i = 1, \dots, k$.*

*Then,*

$$\lambda_{k,s} = \max_{v \in span\{e_{1,s},\dots,e_{k,s}\} \setminus \{0\}} \frac{\|v\|^2_{X_0}}{\|v\|^2_{L^2(\Omega)}}. \tag{7.1}$$

The main result of this section is the following one:

**Proposition 12.** *Let $s \in (0,1)$ and $n > 2s$. Let $\lambda \in [\lambda_{k,s}, \lambda_{k+1,s})$ for some $k \in \mathbb{N}$ and let $\mathbb{F}_\varepsilon$ be the linear space defined in* (6.2). *Finally, let*

$$M_\varepsilon := \max_{\substack{u \in \mathbb{F}_\varepsilon \\ \|u\|_{L^{2^*}(\Omega)}=1}} S_{s,\lambda}(u). \tag{7.2}$$

*Then,*

*a) $M_\varepsilon$ is achieved in $u_M \in \mathbb{F}_\varepsilon$ and $u_M$ can be written as follows*[1]

$$u_M = v + tu_\varepsilon, \quad \text{with } v \in \mathit{span}\{e_{1,s}, \dots, e_{k,s}\} \text{ and } t \geqslant 0; \tag{7.3}$$

*b) the following estimate holds true*

$$M_\varepsilon \leqslant \begin{cases} \big(\lambda_{k,s} - \lambda\big)\|v\|^2_{L^2(\Omega)} & \text{if } t = 0 \\ \big(\lambda_{k,s} - \lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\Big(1 + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big) & \\ \qquad\qquad + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)} & \text{if } t > 0 \end{cases}$$

*as $\varepsilon \to 0$, where $v$ and $t$ are given in* (7.3).

*Here the function $X_0 \setminus \{0\} \ni v \mapsto S_{s,\lambda}(v)$ is defined as in* (1.9).

*Proof.* Let us start by proving part $a$). First of all note that, by definition of $S_{s,\lambda}$

$$M_\varepsilon := \max_{\substack{u \in \mathbb{F}_\varepsilon \\ \|u\|_{L^{2^*}(\Omega)}=1}} \Big(\|u\|^2_{X_0} - \lambda\|u\|^2_{L^2(\Omega)}\Big).$$

Moreover, the maximum in formula (7.2) is achieved thanks to the Weierstrass Theorem. Now, let $u_M \in \mathbb{F}_\varepsilon$ be the function such that

$$\|u_M\|^2_{X_0} - \lambda\|u_M\|^2_{L^2(\Omega)} = M_\varepsilon \text{ and } \|u_M\|_{L^{2^*}(\Omega)} = 1.$$

Note that $u_M \not\equiv 0$. Moreover, since $u_M \in \mathbb{F}_\varepsilon$, by definition of $\mathbb{F}_\varepsilon$ we have that

$$u_M = \tilde{v} + tz_\varepsilon$$

for some $\tilde{v} \in \mathrm{span}\{e_{1,s}, \dots, e_{k,s}\}$ and $t \in \mathbb{R}$. As a consequence of this and thanks to the definition of $z_\varepsilon$ (see Proposition 10), we have that

$$u_M = v + tu_\varepsilon,$$

where

$$v = \tilde{v} - t\sum_{i=1}^{k}\Big(\int_\Omega u_\varepsilon(x)e_{i,s}(x)\,dx\Big)e_{i,s} \in \mathrm{span}\{e_{1,s}, \dots, e_{k,s}\}.$$

Moreover, we can suppose that $t \geqslant 0$ (otherwise we change the sign of $u_M$). This concludes the proof of assertion $a$).

The proof of part $b$) was done in [14, Proposition 13–$b$)]. For this we used Propositions 4 and 7. □

## 8. Proof of Theorem 1

Here we will prove our main theorem, using classical critical points results.

Roughly speaking, Propositions 9 and 10 say that the functional $\mathcal{J}_{s,\lambda}$ has the geometry required by the Mountain Pass and the Linking Theorems, respectively when $\lambda < \lambda_{1,s}$ and $\lambda \geqslant \lambda_{1,s}$.

Since, by Proposition 5, the functional $\mathcal{J}_{s,\lambda}$ verifies the $(PS)_c$–condition only in an energy range bounded above by the constant $(s/n)\,S_s^{n/(2s)}$, in order to apply the classical critical points theorems cited above to $\mathcal{J}_{s,\lambda}$, it is enough to show that the Mountain Pass critical level and the Linking critical level of $\mathcal{J}_{s,\lambda}$ stay below this threshold. This is what we will do in the next subsection.

[1] Beware that $u_M$, $v$ and $t$ depend on $\varepsilon$. For simplicity we omit this dependence in the notation.

8.1. **Estimates of the critical level of $\mathcal{J}_{s,\lambda}$.** Let $\lambda_s > 0$ be as in Corollary 8 . Of course, only the two cases $\lambda_s < \lambda_{1,s}$ and $\lambda_s \geqslant \lambda_{1,s}$ can occur. Let us consider them separately.

**Case 1:** $\lambda_s < \lambda_{1,s}$ **.** This case will be dealt with by using either the Mountain Pass Theorem when $\lambda \in (\lambda_s, \lambda_{1,s})$ or the Linking Theorem when $\lambda \geqslant \lambda_{1,s}$ .

First of all, let $\lambda \in (\lambda_s, \lambda_{1,s})$ . Hence, by the Mountain Pass Theorem, the critical level of $\mathcal{J}_{s,\lambda}$ is given by

$$c_{\mathrm{MP},\lambda} = \inf_{P\in\mathcal{P}} \max_{v\in P([0,1])} \mathcal{J}_{s,\lambda}(v)\,, \tag{8.1}$$

where

$$\mathcal{P} = \Big\{P \in C([0,1]; X_0) : P(0) = 0\,,\ P(1) = e\Big\}$$

with $e = e_\varepsilon := \zeta_\varepsilon u_\varepsilon$ given in Proposition 9 . We claim that for any $\lambda \in (\lambda_s, \lambda_{1,s})$ it holds true

$$\beta \leqslant c_{\mathrm{MP},\lambda} < \frac{s}{n}\, S_s^{n/(2s)}\,, \tag{8.2}$$

where $\beta$ is given in Proposition 9 and $S_s$ is defined in formula (1.6) .

For this we first observe that, for any $P \in \mathcal{P}$, the function $t \mapsto \|P(t)\|_{X_0}$ is continuous in $[0,1]$, that $\|P(0)\|_{X_0} = \|0\|_{X_0} = 0 < \rho$, and that $\|P(1)\|_{X_0} = \|e\|_{X_0} > \rho$, with $\rho$ as in Proposition 9 . Accordingly, there exists $\overline{t} \in (0,1)$ such that $\|P(\overline{t})\|_{X_0} = \rho$ . Thus,

$$\max_{v\in P([0,1])} \mathcal{J}_{s,\lambda}(v) \geqslant \mathcal{J}_{s,\lambda}(P(\overline{t})) \geqslant \inf_{\substack{v\in X_0\\ \|v\|_{X_0}=\rho}} \mathcal{J}_{s,\lambda}(v)\,.$$

As a consequence

$$c_{\mathrm{MP},\lambda} \geqslant \beta\,, \tag{8.3}$$

where $\beta$ is given in Proposition 9 .

Now, recalling that $e = e_\varepsilon = \zeta_\varepsilon u_\varepsilon$ in Proposition 9, we have that the map $[0,1] \ni t \mapsto t\,\zeta_\varepsilon u_\varepsilon$ belongs to $\mathcal{P}$. By [18, Proposition 20], we have that

$$\inf_{P\in\mathcal{P}} \max_{v\in P([0,1])} \mathcal{J}_{s,\lambda}(v) \leqslant \max_{\zeta\geqslant 0} \mathcal{J}_{s,\lambda}(\zeta u_\varepsilon) = \frac{s}{n}\, S_{s,\lambda}^{n/(2s)}(u_\varepsilon)\,,$$

while, by Corollary 8 we get that for any $\lambda > \lambda_s$

$$\frac{s}{n}\, S_{s,\lambda}^{n/(2s)}(u_\varepsilon) < \frac{s}{n}\, S_s^{n/(2s)}$$

for $\varepsilon$ sufficiently small. Hence, putting together these two inequalities we conclude that

$$c_{\mathrm{MP},\lambda} < \frac{s}{n}\, S_s^{n/(2s)}\,, \tag{8.4}$$

provided $\lambda \in (\lambda_s, \lambda_{1,s})$ . Thus, (8.2) follows from (8.3) and (8.4).

Now, let $\lambda \geqslant \lambda_{1,s}$ . We can suppose that $\lambda \in [\lambda_{k,s}, \lambda_{k+1,s})$ for some $k \in \mathbb{N}$ . In fact we suppose that $\lambda \in (\lambda_{k,s}, \lambda_{k+1,s})$ since in the setting of Theorem 1 the parameter $\lambda$ can not be an eigenvalue of the fractional Laplace operator $(-\Delta)^s$ .

Here we have to estimate the Linking critical level of $\mathcal{J}_{s,\lambda}$ given by

$$c_{\mathrm{L},\lambda} = \inf_{h\in\Gamma} \max_{u\in Q} \mathcal{J}_{s,\lambda}(h(u))\,,$$

where

$$\begin{aligned} &\Gamma = \big\{h \in C(\overline{Q}; X_0) : h = id \text{ on } \partial Q\big\}\,,\\ &Q = \Big(\overline{B}_R \cap \mathrm{span}\{e_{1,s}, \dots, e_{k,s}\}\Big) \oplus \big\{r\tilde{z}_\varepsilon : r \in (0,R)\big\}\,, \end{aligned} \tag{8.5}$$

and $R$ and $\tilde{z}_\varepsilon$ are as in Proposition 10 . Here we use the particular choice of $\mathbb{F} = \mathbb{F}_\varepsilon$ performed in (6.2). We claim that for any $\lambda \geqslant \lambda_{1,s}$

$$c_{\mathrm{L},\lambda} < \frac{s}{n}\, S_s^{n/(2s)}\,, \tag{8.6}$$

where $S_s$ is given in formula (1.6). For this, note that by definition of $c_{\mathrm{L},\lambda}$ for any $h\in\Gamma$

$$c_{\mathrm{L},\lambda}\leqslant \max_{u\in Q}\mathcal{J}_{s,\lambda}(h(u)) \tag{8.7}$$

and so, in particular, taking $h=id$ and using the fact that $Q\subset\mathbb{F}_\varepsilon$, we have

$$c_{\mathrm{L},\lambda}\leqslant \max_{u\in Q}\mathcal{J}_{s,\lambda}(u)\leqslant \max_{u\in\mathbb{F}_\varepsilon}\mathcal{J}_{s,\lambda}(u)\,, \tag{8.8}$$

where $\mathbb{F}_\varepsilon$ is given in (6.2).

Since $\mathbb{F}_\varepsilon$ is a linear space,

$$\begin{aligned}\max_{u\in\mathbb{F}_\varepsilon}\mathcal{J}_{s,\lambda}(u)&=\max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\neq0}}\mathcal{J}_{s,\lambda}\big(|\zeta|\cdot\frac{u}{|\zeta|}\big)\\ &=\max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta>0}}\mathcal{J}_{s,\lambda}(\zeta u)\\ &\leqslant\max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\geqslant0}}\mathcal{J}_{s,\lambda}(\zeta u)\,.\end{aligned} \tag{8.9}$$

Hence, putting together (8.7)–(8.9), we have that

$$c_{\mathrm{L},\lambda}\leqslant \max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\geqslant0}}\mathcal{J}_{s,\lambda}(\zeta u)\,. \tag{8.10}$$

On the other hand, by [18, Proposition 20], we have that for any $u\in X_0\setminus\{0\}$

$$\max_{\zeta\geqslant0}\mathcal{J}_{s,\lambda}(\zeta u)=\frac{s}{n}\,S_{s,\lambda}^{n/(2s)}(u)=\frac{s}{n}\left(\frac{\|u\|_{X_0}-\lambda\|u\|^2_{L^2(\Omega)}}{\|u\|^{2^*}_{L^{2^*}(\Omega)}}\right)^{n/(2s)}, \tag{8.11}$$

also thanks to (1.9). Thus, in order to prove (8.6), by (8.10) it is enough to show that

$$\max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\geqslant0}}\mathcal{J}_{s,\lambda}(\zeta u)<\frac{s}{n}\,S_s^{n/(2s)}\,,$$

that is, thanks to (8.11) and the scale invariance of the right-hand side in (8.11), we have to prove that

$$M_\varepsilon<S_s \tag{8.12}$$

for some $\varepsilon>0$, where $M_\varepsilon$ is defined in (7.2).

By Proposition 12-*a*)

$$M_\varepsilon=\|u_M\|^2_{X_0}-\lambda\|u_M\|^2_{L^2(\Omega)}$$

with

$$u_M=v+tu_\varepsilon\,,\ \ v\in\operatorname{span}\{e_{1,s},\dots,e_{k,s}\}\,,\ t\geqslant0\,,\ \ \|u_M\|_{L^{2^*}(\Omega)}=1\,.$$

If $t=0$, by Proposition 12-*b*) and the choice of $\lambda>\lambda_{k,s}$, we get that

$$M_\varepsilon\leqslant\big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}<0<S_s\,,$$

and so (8.12) is proved.

Now, suppose that $t>0$. By Propositions 12-*b*) and 7 with $2s<n<4s$ we deduce that

$$\begin{aligned}M_\varepsilon&\leqslant\big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}+S_{s,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\\ &\leqslant\big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}+\Big(S_s+\mathcal{O}(\varepsilon^{n-2s})-\lambda C_s\varepsilon^{n-2s}+\mathcal{O}(\varepsilon^{2s})\Big)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)\\ &\qquad+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\\ &\leqslant\big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}+S_s+\mathcal{O}(\varepsilon^{n-2s})-\lambda C_s\varepsilon^{n-2s}+\mathcal{O}(\varepsilon^{2s})+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\\ &\leqslant S_s+\mathcal{O}(\varepsilon^{n-2s})-\lambda C_s\varepsilon^{n-2s}+\mathcal{O}(\varepsilon^{2s})\,,\end{aligned}$$

since the parabola $\big(\lambda_{k,\,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}$ stays always below its vertex, that is

$$\big(\lambda_{k,\,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\leqslant\frac{1}{4(\lambda-\lambda_{k,\,s})}\mathcal{O}(\varepsilon^{n-2s})=\mathcal{O}(\varepsilon^{n-2s})\,.$$

Thus,

$$\begin{aligned}M_\varepsilon&\leqslant S_s+\mathcal{O}(\varepsilon^{n-2s})-\lambda C_s\varepsilon^{n-2s}+\mathcal{O}(\varepsilon^{2s})\\&=S_s+\varepsilon^{n-2s}\Big(\mathcal{O}(1)-\lambda C_s+\mathcal{O}(\varepsilon^{4s-n})\Big)\\&<S_s\end{aligned}$$

as $\varepsilon\to 0$, thanks to the fact that $\lambda>\lambda_s$. This ends the proof of (8.12).

Ultimately, we have shown that the critical level of $\mathcal{J}_{s,\lambda}$ stays below $(s/n)\,S_s^{n/(2s)}$, provided $\lambda>\lambda_s$ is not an eigenvalue of $(-\Delta)^s$.

**Case 2: $\lambda_s\geqslant\lambda_{1,\,s}$.** In this setting the only critical level of $\mathcal{J}_{s,\lambda}$ to be estimated is the one given by the Linking Theorem. For doing this, we can argue as in the last part of Case 1 (where we estimate the Linking critical level). In this way we get that for any $\lambda>\lambda_s$ different from an eigenvalue of $(-\Delta)^s$

$$c_{\mathrm{L},\,\lambda}<\frac{s}{n}\,S_s^{n/(2s)}\,.$$

Now, we can conclude the proof of Theorem 1.

8.2. **End of the proof of Theorem 1.** In Subsection 8.1 we have shown that there exists $\lambda_s>0$ such that for any $\lambda>\lambda_s$, with $\lambda$ not an eigenvalue of $(-\Delta)^s$, the critical level of $\mathcal{J}_{s,\lambda}$ stays below $(s/n)\,S_s^{n/(2s)}$ and therefore the $(PS)_c$–condition is satisfied (see, in particular, (8.2), (8.6) and Proposition 5).

Let us consider two cases: either $0<\lambda_s<\lambda_{1,s}$ or $\lambda_s\geqslant\lambda_{1,s}$. Considering separately this two situations we have

- $0<\lambda_s<\lambda_{1,\,s}$: in this case, for any $\lambda\in(\lambda_s,\,\lambda_{1,\,s})$ problem (1.4) admits a Mountain Pass solution $u\in X_0$ at level $\mathcal{J}_{s,\lambda}(u)=c_{\mathrm{MP},\lambda}$. Moreover, $u$ is non-trivial since $\mathcal{J}_{s,\lambda}(u)=c_{\mathrm{MP},\lambda}\geqslant\beta>0=\mathcal{J}_{s,\lambda}(0)$. While for any $\lambda>\lambda_{1,\,s}>\lambda_s$, with $\lambda$ different from an eigenvalue of $(-\Delta)^s$, problem (1.4) admits a non-trivial Linking solution $u\in X_0$ at level $\mathcal{J}_{s,\lambda}(u)=c_{\mathrm{L},\lambda}\geqslant\beta$;
- $\lambda_s\geqslant\lambda_{1,\,s}$: in this case problem (1.4) admits a non-trivial Linking solution $u$ at level $\mathcal{J}_{s,\lambda}(u)=c_{\mathrm{L},\lambda}\geqslant\beta$, provided $\lambda>\lambda_s$ is not an eigenvalue of the fractional Laplacian.

Ultimately, for any value of $\lambda>\lambda_s$ which is not an eigenvalue of $(-\Delta)^s$ we get a non-trivial weak solution for problem (1.4). This concludes the proof of Theorem 1.

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