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FRACTIONAL LAPLACIAN EQUATIONS WITH CRITICAL SOBOLEV EXPONENT

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Abstract. In this paper we complete the study of the following elliptic equation driven by a general non-local integrodifferential operator $\mathcal{L}_K$

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

started in [18]. Here $s \in (0,1)$, Ω is an open bounded set of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, λ is a positive real parameter, $2^* = 2n/(n-2s)$ is a fractional critical Sobolev exponent and f is a lower order perturbation of the critical power $|u|^{2^*-2}u$, while $\mathcal{L}_K$ is the integrodifferential operator defined as

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x) \Big) K(y)\, dy\,, \quad x \in \mathbb{R}^n\,.$$

Under suitable growth condition on f, we show that this problem admits non-trivial solutions for any positive parameter λ. This existence theorem extends some results obtained in [14, 18, 19].

In the model case, that is when $K(x) = |x|^{-(n+2s)}$ (this gives rise to the fractional Laplace operator $-(-\Delta)^s$), the existence result proved along the paper may be read as the non-local fractional counterpart of the one obtained in [11] (see also [9]) in the framework of the classical Laplace equation with critical nonlinearities.

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1. Introduction

Key words and phrases. Mountain Pass Theorem, Linking Theorem, critical nonlinearities, best fractional critical Sobolev constant, Palais–Smale condition, variational techniques, integrodifferential operators, fractional Laplacian.

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1.1. **Non-local integrodifferential equations with critical nonlinearities.** Elliptic problems with critical nonlinearities are widely studied in the literature for their interest in many fields, such as differential geometry and physics, to name but a few.

On the other hand, a deep interest was shown also for non-local operators, thanks to their applications in a wide range of contexts, such as the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

For all these reasons in [14, 18, 19] we studied non-local equations with critical growth. To be precise, [14, 19] were devoted to the study of the critical equation

1s
$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.1}$$

while in [18] we considered a lower order perturbation of the critical nonlinearity $|u|^{2^*-2}u$ in non-local ellitic equations, that is we studied problems modeled by

1sf
$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.2}$$

where f has subcritical growth at infinity (for instance, f behaves like $|u|^{q-2}u$, with $q \in (2, 2^*)$). In both these problems $s \in (0,1)$ is fixed, Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, $\lambda > 0$,

2star
$$2^* = 2n/(n-2s) \tag{1.3}$$

is the fractional critical Sobolev exponent, and $(-\Delta)^s$ is the fractional Laplace operator defined, up to normalization factors, as

2
$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\,, \quad x \in \mathbb{R}^n \tag{1.4}$$

(see [10] and references therein for further details on the fractional Laplacian).

Along the present paper, we consider the following general non-local equation

op
$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.5}$$

where $\mathcal{L}_K$ is the integrodifferential operator defined as follows

lk
$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big)K(y)\,dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.6}$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ such that

kernel
$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.7}$$

kernelfrac
$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta|x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.8}$$

evenkernel
$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.9}$$

A model for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$ defined in (1.4).

Finally, the perturbation in equation (1.5) is a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verifying the following conditions:

supf
$$\sup\Big\{|f(x,t)| : \text{a.e. } x \in \Omega\,,\ t \in [0, M]\Big\} < +\infty \ \text{ for any } M > 0\,; \tag{1.10}$$

cond0
$$\lim_{t \to 0} \frac{f(x,t)}{t} = 0 \text{ uniformly in } x \in \Omega\,; \tag{1.11}$$

condinfinito
$$\lim_{|t|\to+\infty}\frac{f(x,t)}{|t|^{2^*-1}}=0 \text{ uniformly in } x\in\Omega\,; \tag{1.12}$$

Fpositiva
$$F(x,t)\geqslant 0 \text{ a.e. } x\in\Omega\,,\ t\in\mathbb{R}\,; \tag{1.13}$$

condFf
$$F(x,t)\leqslant\frac{1}{2}\,f(x,t)\,t+\frac{s}{n}\,|t|^{2^*} \text{ a.e. } x\in\Omega\,,\ t\in\mathbb{R}\,. \tag{1.14}$$

Here F is the primitive of f with respect to the second variable, i.e.

F
$$F(x,t)=\int_0^t f(x,\tau)d\tau\,. \tag{1.15}$$

As a model for f we can take $f(x,t)=a(x)|t|^{q-2}t$, with $a\in L^\infty(\Omega)\,,\ a(x)\geqslant 0$ a.e. $x\in\Omega$ and $q\in(2,2^*)\,$.

In problem (1.5) the Dirichlet datum is given in $\mathbb{R}^n\setminus\Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the operator $\mathcal{L}_K\,$.

Aim of this paper is to study the existence of non-trivial weak solutions of (1.5), that is non-trivial solutions of the following problem

problemaK0f
$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy-\lambda\int_\Omega u(x)\varphi(x)\,dx \\[2ex] \qquad\qquad\displaystyle =\int_\Omega|u(x)|^{2^*-2}u(x)\varphi(x)dx+\int_\Omega f(x,u(x))\varphi(x)dx \quad \forall\,\varphi\in X_0 \\[2ex] u\in X_0\,, \end{cases} \tag{1.16}$$

which represents the weak formulation of (1.5). Note that, in order to write such a weak formulation, it is convenient to assume (1.9).

The space X_0 in which we set the weak problem is a functional space, inspired by, but not equivalent to, the usual fractional Sobolev space. The choice of this space is motivated by the fact that it allows us to correctly encode the Dirichlet boundary datum in the weak formulation. This space was introduced in [15] (see also [16]). We will recall its definition and properties in Appendix A, in order to make the present paper self-contained. In the model case $\mathcal{L}_K=-(-\Delta)^s$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω (see [18, Lemma 7]) .

Before stating our main results, we need to introduce some notations.

subsec:notations

1.2. **Notations.** Along the present paper $\{\lambda_k\}_{k\in\mathbb{N}}$ will denote the sequence of the eigenvalues of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data with

$$0<\lambda_1<\lambda_2\leqslant\ldots\leqslant\lambda_k\leqslant\lambda_{k+1}\leqslant\ldots$$

and

$$\lambda_k\to+\infty \text{ as } k\to+\infty\,.$$

Moreover, $\{e_k\}_{k\in\mathbb{N}}$ will be the sequence of eigenfunctions corresponding to λ_k. We choose a normalization for which this sequence of eigenfunctions provides an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0. For a complete study of the spectrum of the integrodifferential operator $-\mathcal{L}_K$ we refer to [14, Proposition 5], [17, Proposition 9 and Appendix A] and [19, Proposition 4] .

In the sequel we will also refer to the best fractional critical Sobolev constants S_K and $S_{K,\lambda}$ defined as follows, respectively

SK
$$S_K:=\inf_{v\in X_0\setminus\{0\}}S_K(v) \tag{1.17}$$

and

SKlambda
$$S_{K,\lambda}:=\inf_{v\in X_0\setminus\{0\}}S_{K,\lambda}(v)\,, \tag{1.18}$$

where for any $v \in X_0 \setminus \{0\}$

SKv
$$S_K(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy}{\Big(\displaystyle\int_\Omega |v(x)|^{2^*}\,dx\Big)^{2/2^*}} = \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy}{\Big(\displaystyle\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{1.19}$$

and

SKlambdav
$$\begin{aligned} S_{K,\lambda}(v) &:= \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx}{\Big(\displaystyle\int_\Omega |v(x)|^{2^*}\,dx\Big)^{2/2^*}} \\ &= \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_{\mathbb{R}^n} |v(x)|^2\,dx}{\Big(\displaystyle\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \end{aligned} \tag{1.20}$$

In the model case of the fractional Laplacian $-(-\Delta)^s$ the counterparts of S_K and $S_{K,\lambda}$ are the constants S_s and $S_{s,\lambda}$ defined as

Ss
$$S_s := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v)\,, \tag{1.21}$$

and

Sslambda
$$S_{s,\lambda} := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_{s,\lambda}(v)\,, \tag{1.22}$$

where for any $v \in H^s(\mathbb{R}^n)\setminus\{0\}$

Ssv
$$S_s(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\Big(\displaystyle\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}} \tag{1.23}$$

and

Sslambdav
$$S_{s,\lambda}(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_{\mathbb{R}^n} |v(x)|^2\,dx}{\Big(\displaystyle\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{1.24}$$

All these constants are well-defined thanks to [10, Theorem 6.5], [16, Lemma 8], [17, Lemma 10] and [18, Lemmas 9 and 14]. We refer, in particular, to [14, 18, 19] for further comments on these constants.

subsec:mainresults

1.3. **Main results of the paper.** As we said before, our aim is to show that problem (1.5) has non-trivial solutions. Of course, the trivial function (i.e., u identically zero) is a solution of the problem, since $f(x,0)=0$, by assumption (1.11). In order to get our goal, we will use variational methods and we will look for non-trivial critical points of the functional $\mathcal{J}_{K,\lambda}: X_0 \to \mathbb{R}$ given by

JKlambda
$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx - \int_\Omega F(x,u(x))dx\,, \end{aligned} \tag{1.25}$$

where F is defined as in (1.15) , since (1.16) represents the Euler–Lagrange equation of the functional $\mathcal{J}_{K,\lambda}$.

To this purpose we will use some variants of the classical Mountain Pass and Linking Theorems due to Ambrosetti and Rabinowitz (see [2, 6, 12, 13]), which take into account the lack of compactness which occurs when dealing with critical nonlinearities. As in the classical case of the Laplacian, here the main difficulty is due to the fact that the functional $\mathcal{J}_{K,\lambda}$ does not verify the Palais–Smale condition globally, but only in a suitable range related to the best fractional critical Sobolev constant S_K introduced in (1.17) .

The main result of the present paper is the following:

generalkernel0f **Theorem 1.** *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary . Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.7)–(1.9) *and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying* (1.10)–(1.14) *. Finally, assume that*

u0S
$$\begin{gathered}\textit{there exists } u_0 \in X_0 \setminus \{0\} \textit{ with } u_0 \geqslant 0 \textit{ a.e. in } \mathbb{R}^n \textit{, such that}\\ S_{K,\lambda}(u) < S_K \textit{ for any } u \in \mathcal{U},\\ \textit{where } \mathcal{U} = \begin{cases} \textit{span}\{u_0\} & \textit{if } \lambda \in (0,\,\lambda_1)\\ \textit{span}\{e_1,\dots,e_k,u_0\} & \textit{if } \lambda \in [\lambda_k,\,\lambda_{k+1})\,,\ k \in \mathbb{N}\,.\end{cases}\end{gathered} \tag{1.26}$$

Then, for any $\lambda > 0$ problem (1.16) *admits a solution $u \in X_0$, which is not identically zero .*

When dealing with the model problem (1.2), the weak formulation is given by

problemalapfrac0f
$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx\\ \qquad\qquad\displaystyle = \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx + \int_\Omega f(x,u(x))\,\varphi(x)\,dx\\ \qquad\qquad\qquad\qquad \forall\ \varphi \in H^s(\mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\\ u \in H^s(\mathbb{R}^n)\,, \end{cases} \tag{1.27}$$

which represents the Euler–Lagrange equation of the functional $\mathcal{J}_{s,\lambda} : H^s(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$\mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx - \int_\Omega F(x,u(x))\,dx\,,$$

with F given in (1.15) .

In this framework Theorem 1 reads as follows:

lapfra0f **Theorem 2.** *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying* (1.10)–(1.14) *and assume that*

u0Sfrac
$$\begin{gathered}\textit{there exists } u_0 \in H^s(\mathbb{R}^n) \setminus \{0\} \textit{ with } u_0 \geqslant 0 \textit{ a.e. in } \Omega\\ \textit{and } u_0 = 0 \textit{ a.e. in } \mathbb{R}^n \setminus \Omega\textit{, such that}\\ S_{s,\lambda}(u) < S_s \textit{ for any } u \in \mathcal{U},\\ \textit{where } \mathcal{U} = \begin{cases} \textit{span}\{u_0\} & \textit{if } \lambda \in (0,\,\lambda_1)\\ \textit{span}\{e_1,\dots,e_k,u_0\} & \textit{if } \lambda \in [\lambda_k,\,\lambda_{k+1})\,,\ k \in \mathbb{N}\,.\end{cases}\end{gathered} \tag{1.28}$$

Then, for any $\lambda > 0$ problem (1.27) *admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.*

The main results proved in the present paper are obtained by adapting the variational and topological techniques used in the classical critical case (see [1, 3, 8, 9, 11, 20, 22]) to the non-local fractional setting. The main difficulties are related to the encoding of the Dirichlet boundary datum in the variational formulation and to the lack of compactness. To overcome the first difficulty we work in the functional space X_0, while for the second one we apply suitable variants of some classical critical points theorems which take into account that the Palais–Smale condition holds true in a suitable energy range, related to the best fractional critical Sobolev constant S_K (or S_s for the fractional Laplacian). Of course, differently from the classical case of the Laplacian, the fact that the leading operator is non-local provides technical and conceptual complications, and some estimates become more delicate and involved.

subsec:remarks

1.4. **Some comments on the main theorems.** The existence theorems proved along this paper may be seen as the natural extension to the non-local framework of well-know results obtained in [3, 8, 9, 11] (see also [1, 20, 22]), where the classical critical Laplace equation was considered.

Also aim of this paper is to complete the study of problem (1.5), started in [18], where only the case when the positive parameter λ is less than the first eigenvalue of $-\mathcal{L}_K$ was considered. Note that when $\lambda \in (0, \lambda_1)$, the assertion of Theorem 1 holds true under weaker assumptions on the data of the problem. To be precise, Theorem 1 is still valid if we do not assume condition (1.13) and if we replace (1.26) with the following one

u0SBN

$$\begin{gathered}\text{there exists } u_0 \in X_0 \setminus \{0\} \text{ with } u_0 \geqslant 0 \text{ a.e. in } \mathbb{R}^n \text{, such that} \\ \sup_{\zeta \geqslant 0} \mathcal{J}_{K,\lambda}(\zeta u_0) < \frac{s}{n}\, S_K^{n/(2s)}\,,\end{gathered} \tag{1.29}$$

which is weaker than (1.26) and it is the assumption considered in [18, Theorem 1]. In this sense Theorem 1 may be seen as an extension of [18, Theorem 1].

When there is no lower order perturbation, i.e. when $f \equiv 0$, in (1.5), this non-local equation was studied in [14]: in this setting [14, Theorem 1] is a particular case of Theorem 1. Also in [14] there are applications to the fractional Laplacian in dimension $n \geqslant 4$, while for $2s < n < 4s$ we refer to [19]. Both [14, Theorem 2] and [19, Theorem 1] represent particular cases of Theorem 2.

In the context of non-local problems, we would like to recall also the recent paper [4], where the authors studied the effect of a lower order perturbation in a critical elliptic problem driven by a non-local operator defined starting from the power of the eigenvalues of $-\Delta$. In this framework we refer also to [21], where a non-local version of the Brezis-Nirenberg result was given in the case of a regional square root of the Laplacian (which is an operator related, but different, from the one considered here).

As we said before, problem (1.5) will be treated via variational and topological tools. Hence, we will need to prove that the functional $\mathcal{J}_{K,\lambda}$ has suitable geometric features, verifies a compactness condition and, due to the critical character of problem, that the minimax critical level of $\mathcal{J}_{K,\lambda}$ satisfies adequate estimates.

For the geometry, what are important are the growth conditions (1.11) and (1.12) and, in the case when $\lambda \geqslant \lambda_1$ also (1.13). This last assumption is natural and it is assumed also in the classical case of the Laplacian, when the parameter λ is greater or equal to the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data (see, for instance, [11]).

Condition (1.14) will be crucial in the proof of the Palais–Smale condition (see Step 3 in the proof of Proposition 3). This is only the generalization of the assumption required in the classical case of the Laplacian (see, for instance, [8, formula (2.34)] and [11, Section 3]). Of course, in the model case $f(x,t) = a(x)|t|^{q-2}t$, with $a \in L^\infty(\Omega)$, $a(x) \geqslant 0$ a.e. $x \in \Omega$ and $q \in (2, 2^*)$, condition (1.14) is trivially satisfied.

The paper is organized as follows. Section 2 will be devoted to the proof of Theorem 1, while in Section 3 we will discuss the case of the fractional Laplacian operator and we will prove Theorem 2. Finally, since the space X_0 in which we will work in is not a standard one, in Appendix A we will recall its definition and properties, in order to make this paper self-contained.

sec:proofgeneral

2. The general case: proof of Theorem 1

This section is devoted to the study of problem (1.16) which is the Euler–Lagrange equation of the functional $\mathcal{J}_{K,\lambda}$ defined in (1.25). Notice that this functional is well defined thanks to the definition of X_0, to [18, Lemmas 5, 6 and 9] and to the fact that Ω is bounded. Moreover, $\mathcal{J}_{K,\lambda}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned}\langle \mathcal{J}'_{K,\lambda}(u), \varphi\rangle = \int_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ - \int_\Omega |u(x)|^{2^*-2}(x)u(x)\varphi(x)\,dx - \int_\Omega f(x,u(x))\varphi(x)\,dx\,.\end{aligned}$$

Hence, critical points of $\mathcal{J}_{K,\lambda}$ are solutions to problem (1.16). Goal of this paper is finding these critical points. For this we will apply some variants of the Mountain Pass Theorem and of the Linking Theorem (see, for instance, [2, 6, 12, 13]) which take into account the fact that in the critical setting there is, as usual, a lack of compactness in the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ (see, for instance, [18, Lemma 9]). As a consequence of this, as in classical case, also in the non-local critical setting, the functional $\mathcal{J}_{K,\lambda}$ does not verify the Palais–Smale condition globally, but only in an energy range determined by the best fractional critical Sobolev constant S_K given in formula (1.17).

This is the reason why we can not exploit directly the Mountain Pass or the Linking Theorem. In order to overcome the difficult related to the Palais–Smale condition, we have to estimate the minimax critical level of the functional $\mathcal{J}_{K,\lambda}$ in order to show that it is below the threshold where the compactness condition is satisfied. For this the key-point is condition (1.26).

In the case when $\lambda \in (0, \lambda_1)$, problem (1.16) was studied in [18, Theorem 1] using the Mountain Pass Theorem in the version given in [8, Theorem 2.2]. In this framework note that we do not need assumptions (1.13) and (1.14) and, moreover, that under hypothesis (1.13), condition (1.26) implies (1.29). Then, Theorem 1 follows directly by [18, Theorem 1].

Hence, in the present paper we focus our attention to the case when $\lambda \geqslant \lambda_1$ and, in order to get our goal, we will use appropriately [13, Theorem 5.3].

Since $\lambda \geqslant \lambda_1$, we can suppose that

$$\lambda \in [\lambda_k, \lambda_{k+1}) \quad \text{for some } k \in \mathbb{N}\,.$$

The idea consists in proving that the functional $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$, provided c belongs to a suitable range which depends on the best fractional critical constant S_K. Then, after proving that the geometry of $\mathcal{J}_{K,\lambda}$ fits with the requirements of the Linking Theorem [13, Theorem 5.3], we will show that the related Linking critical value of $\mathcal{J}_{K,\lambda}$ stays below the energy level in which the compactness condition holds true. Assumption (1.26) will be crucial in the proof of this last part.

sub:compactness

2.1. **The Palais–Smale condition for the functional $\mathcal{J}_{K,\lambda}$.** We start by proving that the functional $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition at any level c, whenever c stays below a suitable threshold depending on S_K.

propPS **Proposition 3.** *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let f be a Carathéodory function satisfying conditions* (1.10)–(1.14)*. Let $c \in \mathbb{R}$ be such that*

cS (2.1)
$$c < \frac{s}{n} S_K^{n/(2s)}$$

and let u_j be a sequence in X_0 such that

Jc0 (2.2)
$$\mathcal{J}_{K,\lambda}(u_j) \to c$$

and

J'00 (2.3)
$$\sup\Big\{\big|\langle\, \mathcal{J}'_{K,\lambda}(u_j), \varphi\,\rangle\big| \;:\; \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0$$

as $j \to +\infty$.

Then, there exists $u_\infty \in X_0$ such that, up to a subsequence, $\|u_j - u_\infty\|_{X_0} \to 0$ as $j \to +\infty$.

Proof. For this proof we adapt the arguments used in the classical elliptic setting to the non-local fractional framework. When $f\equiv 0$ the Palais–Smale condition for $\mathcal{J}_{K,\lambda}$ was proved in [14, Proposition 7], while in the case when $\lambda<\lambda_1$ a proof can be found in [18, Subsection 3.1.1]. Here some non-trivial differences arise with respect to these cases. Now, we proceed by steps.

claim2 **Step 1.** *The sequence u_j is bounded in X_0.*

Proof. The proof of Step 1 is quite standard, but we prefer to repeat it for reader's convenience, since some steps of this proof will be used in the sequel.

For any $j\in\mathbb{N}$ by (2.2) and (2.3) it easily follows that there exists $\kappa>0$ such that

jlimitato0
$$|\mathcal{J}_{K,\lambda}(u_j)|\leqslant\kappa\,, \tag{2.4}$$

and

j'limitato0
$$\Big|\langle\mathcal{J}'_{K,\lambda}(u_j),\frac{u_j}{\|u_j\|_{X_0}}\rangle\Big|\leqslant\kappa\,, \tag{2.5}$$

so that

2k
$$\mathcal{J}_{K,\lambda}(u_j)-\frac{1}{2}\langle\mathcal{J}'_{K,\lambda}(u_j),u_j\rangle\leqslant\kappa\left(1+\|u_j\|_{X_0}\right)\,. \tag{2.6}$$

Moreover, by Hölder inequality, the fact that Ω is bounded, [16, Lemma 6] and (1.8)

H
$$\int_\Omega|u_j(x)|\,dx\leqslant|\Omega|^{(2^*-1)/2^*}\left(\int_\Omega|u_j(x)|^{2^*}\,dx\right)^{1/2^*}\leqslant\bar\kappa|\Omega|^{(2^*-1)/2^*}\|u_j\|_{X_0}\,, \tag{2.7}$$

for some positive constant $\bar\kappa$.

Hence, by [18, Lemma 5] and (2.7) we obtain

jj'0
$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j)-\frac{1}{2}\langle\mathcal{J}'_{K,\lambda}(u_j),u_j\rangle&=\left(\frac{1}{2}-\frac{1}{2^*}\right)\|u_j\|^{2^*}_{L^{2^*}(\Omega)}-\int_\Omega F(x,u_j(x))\,dx\\&\qquad+\frac{1}{2}\int_\Omega f(x,u_j(x))\,u_j(x)\,dx\\&\geqslant\frac{s}{n}\|u_j\|^{2^*}_{L^{2^*}(\Omega)}-\varepsilon\Big(1+\frac{2^*}{2}\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)}\\&\qquad-\frac{3}{2}M(\varepsilon)\int_\Omega|u_j(x)|\,dx\\&\geqslant\frac{s}{n}\|u_j\|^{2^*}_{L^{2^*}(\Omega)}-\varepsilon\Big(1+\frac{2^*}{2}\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)}-C_\varepsilon\|u_j\|_{X_0}\,,\end{aligned} \tag{2.8}$$

where C_ε is a positive constant, possibly depending on $|\Omega|$.

Choosing $\varepsilon>0$ small enough (i.e. ε such that $\frac{s}{n}>\varepsilon\Big(1+\frac{2^*}{2}\Big)$), (2.6) and (2.8) imply that, for any $j\in\mathbb{N}$

norma2star
$$\|u_j\|^{2^*}_{L^{2^*}(\Omega)}\leqslant\kappa_*\left(1+\|u_j\|_{X_0}\right) \tag{2.9}$$

for a suitable positive constant κ_*. As a consequence of this and using again the Hölder inequality we get

$$\|u_j\|^2_{L^2(\Omega)}\leqslant|\Omega|^{2s/n}\|u_j\|^2_{L^{2^*}(\Omega)}\leqslant\kappa_*^{2/2^*}|\Omega|^{2s/n}\left(1+\|u_j\|_{X_0}\right)^{2/2^*}\,,$$

so that, since $2<2^*$,

norma2
$$\|u_j\|^2_{L^2(\Omega)}\leqslant\widetilde\kappa\left(1+\|u_j\|_{X_0}\right)\,, \tag{2.10}$$

for a suitable $\widetilde\kappa>0$ not depending on j.

Hence, by (2.4) and [18, Lemma 5] (used here with $\varepsilon=1$), we get

$$\kappa\geqslant\mathcal{J}_{K,\lambda}(u_j)\geqslant\frac{1}{2}\|u_j\|^2_{X_0}-\frac{\lambda}{2}\|u_j\|^2_{L^2(\Omega)}-\Big(\frac{1}{2^*}+1\Big)\|u_j\|^{2^*}_{L^{2^*}(\Omega)}-M(1)\|u_j\|_{L^1(\Omega)}\,,$$

so that, by (2.7), (2.9) and (2.10) we get that for any $j \in \mathbb{N}$

$$\|u_j\|^2_{X_0} \leqslant \kappa_{**}\,(1 + \|u_j\|_{X_0})$$

for some positive constant κ_{**} independent of j . Hence, the proof of Step 1 is complete. □

claim3 **Step 2.** *Problem* (1.16) *admits*[1] *a solution* $u_\infty \in X_0$.

Proof. This proof can be found in [18, Claim 3 of Subsection 3.1.1] . Note that for this part we do not need conditions (1.13) and (1.14) . □

step3 **Step 3.** *The following equality holds true:*

$$\mathcal{J}_{K,\lambda}(u_\infty) = \frac{s}{n}\int_\Omega |u_\infty(x)|^{2^*}\,dx + \frac{1}{2}\int_\Omega f(x,u_\infty(x))u_\infty(x)dx - \int_\Omega F(x,u_\infty(x))dx \geqslant 0\,.$$

Proof. By Step 2, taking $\varphi = u_\infty$ as a test function in (1.16) , we get

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |u_\infty(x) - u_\infty(y)|^2 K(x-y)dx\,dy - \lambda\int_\Omega |u_\infty(x)|^2\,dx &= \int_\Omega |u_\infty(x)|^{2^*}\,dx\\ &\quad + \int_\Omega f(x,u_\infty(x))u_\infty(x)dx\,,\end{aligned}$$

so that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_\infty) &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega |u_\infty(x)|^{2^*}\,dx + \frac{1}{2}\int_\Omega f(x,u_\infty(x))u_\infty(x)dx - \int_\Omega F(x,u_\infty(x))dx\\ &= \frac{s}{n}\int_\Omega |u_\infty(x)|^{2^*}\,dx + \frac{1}{2}\int_\Omega f(x,u_\infty(x))u_\infty(x)dx - \int_\Omega F(x,u_\infty(x))dx \geqslant 0\,.\end{aligned}$$

The last inequality follows from assumption (1.14) . Hence, Step 3 is proved. □

step4 **Step 4.** *The following equality holds true:*

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j) = \mathcal{J}_{K,\lambda}(u_\infty) &+ \frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy\\ &- \frac{1}{2^*}\int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx + o(1)\end{aligned}$$

as $j \to +\infty$.

Proof. First of all, we observe that by Step 1 and [18, Lemma 9], the sequence u_j is bounded in X_0 and in $L^{2^*}(\mathbb{R}^n)$. Since $L^{2^*}(\mathbb{R}^n)$ is a reflexive space we have that, up to a subsequence

convergenzaweak
$$u_j \to u_\infty \quad \text{weakly in } L^{2^*}(\mathbb{R}^n) \tag{2.11}$$

as $j \to +\infty$, while by [16, Lemma 8], up to a subsequence,

convergenze0bis
$$u_j \to u_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \tag{2.12}$$

convergenze0ter
$$u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n \tag{2.13}$$

as $j \to +\infty$ for any $\nu \in [1, 2^*)$ (see, for instance [5, Theorem IV.9]).

Moreover, since (2.13) holds true, by the Brezis–Lieb Lemma (see [7, Theorem 1]), we get

bl1
$$\begin{aligned}&\int_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)\,dx\,dy\\ &\qquad = \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)\,dx\,dy\\ &\qquad\quad + \int_{\mathbb{R}^{2n}} |u_\infty(x) - u_\infty(y)|^2 K(x-y)\,dx\,dy + o(1)\end{aligned} \tag{2.14}$$

[1]As a matter of fact, the solution u_∞ constructed in Step 2 will turn out to be the limit of u_j in the topology of X_0, but this will be achieved only at the end of the proof, see (2.32) below. Notice that, at this stage, the existence of the solution given by Step 2 does not end the proof of Theorem 1 because (2.2) and (2.3) are in use.

and

bl2
$$\int_\Omega |u_j(x)|^{2^*}\,dx = \int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + \int_\Omega |u_\infty(x)|^{2^*}\,dx + o(1) \tag{2.15}$$

as $j\to+\infty$.

Finally, we claim that

convergenzaF
$$\int_\Omega F(x,u_j(x))dx \to \int_\Omega F(x,u_\infty(x))dx \tag{2.16}$$

as $j\to+\infty$. At this purpose, note that, since u_j is bounded in $L^{2^*}(\Omega)$, there exists $\kappa>0$ such that

k2star
$$\|u_j\|_{L^{2^*}(\Omega)} \leqslant \kappa \ \text{ for any } j\in\mathbb{N}\,. \tag{2.17}$$

Now, let us fix $\delta>0$. By [18, formula (2.2) in Lemma 5] (used here with $\varepsilon=\delta/(2\kappa)$, we have that

Fdelta
$$|F(x,u_j(x))| \leqslant \frac{\delta}{2\kappa}\,|u_j(x)|^{2^*} + M(\delta/(2\kappa))|u_j(x)| \ \text{a.e. in } \Omega\,. \tag{2.18}$$

Then, thanks to the Sobolev embeddings of X_0, we have that $F(\cdot,u_j(\cdot))\in L^1(\Omega)$ and, putting $\eta(\delta)=\left(\frac{\delta}{2\kappa M(\delta/(2\kappa))}\right)^{2^*/(2^*-1)}$, for any measurable subset Ω' of Ω such that

omega'
$$|\Omega'|\leqslant\eta(\delta) \tag{2.19}$$

we get

assolutacont
$$\begin{aligned}\int_{\Omega'}|F(x,u_j(x))|\,dx &\leqslant \frac{\delta}{2\kappa}\int_{\Omega'}|u_j(x)|^{2^*}\,dx + M(\delta/(2\kappa))\int_{\Omega'}|u_j(x)|\,dx\\ &\leqslant \frac{\delta}{2\kappa}\,\|u_j\|_{L^{2^*}(\Omega)} + M(\delta/(2\kappa))\|u_j\|_{L^1(\Omega')}\\ &\leqslant \frac{\delta}{2} + M(\delta/(2\kappa))|\Omega'|^{(2^*-1)/2^*}\|u_j\|_{L^{2^*}(\Omega)}\\ &\leqslant \frac{\delta}{2} + M(\delta/(2\kappa))\eta(\delta)^{(2^*-1)/2^*}\kappa\\ &=\delta\,,\end{aligned} \tag{2.20}$$

thanks to (2.17)–(2.19) and Hőlder inequality. Hence, $F(\cdot,u_j(\cdot))$ is uniformly integrable[2] in Ω. Moreover, by (2.13) and the fact that the map $t\mapsto F(\cdot,t)$ is continuous in $t\in\mathbb{R}$, we get

convF0
$$F(\cdot,u_j(\cdot))\to F(\cdot,u_\infty(\cdot)) \quad \text{a.e. in} \quad \Omega \tag{2.21}$$

as $j\to+\infty$. Thus, by (2.20) and (2.21) and the Vitali Convergence Theorem, (2.16) holds true.

[2] Or, according to the different terminologies, absolutely continuous in Ω, uniformly with respect to $j\in\mathbb{N}$.

Therefore, using also the definition of $\mathcal{J}_{K,\lambda}$, by (2.12) and (2.14)–(2.16) we deduce that

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u_j) = \frac{1}{2}\int_{\mathbb{R}^{2n}} & |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2 K(x-y)\,dx\,dy \\
& +\frac{1}{2}\int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2 K(x-y)\,dx\,dy-\frac{\lambda}{2}\int_\Omega |u_\infty(x)|^2 dx \\
& -\frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx-\frac{1}{2^*}\int_\Omega |u_\infty(x)|^{2^*}\,dx \\
& -\int_\Omega F(x,u_\infty(x))dx+o(1) \\
= \mathcal{J}_{K,\lambda}(u_\infty) & +\frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2 K(x-y)\,dx\,dy \\
& -\frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx+o(1)
\end{aligned}$$

as $j\to+\infty$, so that the assertion of Step 4 follows. □

step5 **Step 5.** *The following equality holds true:*

$$\int_{\mathbb{R}^{2n}} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2 K(x-y)dx\,dy=\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}dx+o(1)$$

as $j\to+\infty$ *.*

Proof. First of all, we note that, as a consequence of (2.11) and (2.15), it holds true that

ujuinfty

$$\begin{aligned}
\int_\Omega \Big(|u_j(x)|^{2^*-2}u_j(x) & -|u_\infty(x)|^{2^*-2}u_\infty(x)\Big)\left(u_j(x)-u_\infty(x)\right)dx \\
& =\int_\Omega |u_j(x)|^{2^*}\,dx-\int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)u_j(x)\,dx \\
& \quad -\int_\Omega |u_j(x)|^{2^*-2}u_j(x)u_\infty(x)\,dx+\int_\Omega |u_\infty(x)|^{2^*}\,dx \\
& =\int_\Omega |u_j(x)|^{2^*}\,dx-\int_\Omega |u_\infty(x)|^{2^*}\,dx+o(1) \\
& =\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx+o(1)
\end{aligned}\tag{2.22}$$

as $j\to+\infty$.

Furthermore, arguing exactly as in the proof of (2.16) (that is, using [18, formula (2.1) in Lemma 5] and the Vitali Convergence Theorem), we deduce that

convergenzaf

$$\int_\Omega f(x,u_j(x))u_j(x)\,dx\to\int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\tag{2.23}$$

as $j\to+\infty$.

Also, by (2.13) and the fact that the map $t\mapsto f(\cdot,t)$ is continuous in $t\in\mathbb{R}$, it is easily seen that

convfae

$$f(\cdot,u_j(\cdot))\to f(\cdot,u_\infty(\cdot))\quad\text{a.e. in}\quad\Omega\tag{2.24}$$

as $j\to+\infty$. Now, by [18, Lemma 5], the fact that Ω is bounded and (2.17), it follows that $f(\cdot,u_j(\cdot))$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$, so that, also thanks to (2.24), we get

convf0

$$f(\cdot,u_j(\cdot))\to f(\cdot,u_\infty(\cdot))\quad\text{weakly in}\quad L^{2^*/(2^*-1)}(\Omega)\,,\tag{2.25}$$

which yields

convergenzaf1

$$\int_\Omega f(x,u_j(x))u_\infty(x)\,dx\to\int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\tag{2.26}$$

as $j\to+\infty$, being $u_\infty\in L^{2^*}(\Omega)$.

Again by [18, Lemma 5] we get that $f(\cdot, u_\infty(\cdot))$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$, so that, by (2.11), we deduce that

convergenzaf2
$$\int_\Omega f(x, u_\infty(x))u_j(x)\,dx \to \int_\Omega f(x, u_\infty(x))u_\infty(x)\,dx \tag{2.27}$$

as $j \to +\infty$.

Now, by (2.3) and Steps 1 and 2, it is easily seen that

conto1
$$o(1) = \langle \mathcal{J}'_{K,\lambda}(u_j), u_j - u_\infty\rangle = \langle \mathcal{J}'_{K,\lambda}(u_j) - \mathcal{J}'_{K,\lambda}(u_\infty), u_j - u_\infty\rangle \tag{2.28}$$

as $j \to +\infty$. Moreover,

conto2
$$\begin{aligned}
\langle \mathcal{J}'_{K,\lambda}(u_j)&-\mathcal{J}'_{K,\lambda}(u_\infty), u_j - u_\infty\rangle \\
&= \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
&\quad - \lambda \int_\Omega |u_j(x) - u_\infty(x)|^2\,dx \\
&\quad - \int_\Omega \Big(|u_j(x)|^{2^*-2}u_j(x) - |u_\infty(x)|^{2^*-2}u_\infty(x)\Big)\,(u_j(x) - u_\infty(x))\,dx \\
&\quad - \int_\Omega (f(x, u_j(x)) - f(x, u_\infty(x)))\,(u_j(x) - u_\infty(x))\,dx \\
&= \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
&\quad - \int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx + o(1)
\end{aligned} \tag{2.29}$$

as $j \to +\infty$, thanks to (2.12), (2.22), (2.23), (2.26) and (2.27). Putting together (2.28) and (2.29) we get the assertion of Step 5. $\square$

Now, we can conclude the proof of Proposition 3.

By Step 5 it is easy to see that

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j(x)&-u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy - \frac{1}{2^*}\int_\Omega |u_j(x) - u_\infty(x)|^{2^*}dx \\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy + o(1) \\
&= \frac{s}{n}\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy + o(1),
\end{aligned}$$

and so, as a consequence of this, Step 4 and (2.2), we obtain

bo
$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u_\infty) + \frac{s}{n}\int_{\mathbb{R}^{2n}} &|u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
&= \mathcal{J}_{K,\lambda}(u_j) + o(1) \\
&= c + o(1),
\end{aligned} \tag{2.30}$$

as $j \to +\infty$. Note that, by Step 1, the sequence $\|u_j\|_{X_0}$ is bounded in $\mathbb{R}$. Hence, up to a subsequence, if necessary, we can assume that

L1
$$\|u_j - u_\infty\|^2_{X_0} = \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)\,dx\,dy \to L \tag{2.31}$$

and so, again as a consequence of Step 5

$$\int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx \to L$$

as $j \to +\infty$. Of course $L \in [0, +\infty)$. Moreover, by definition of S_K we have that

$$L \geqslant L^{2/2^*} S_K,$$

so that

$$L = 0 \quad \text{or} \quad L \geqslant S_K^{n/(2s)} .$$

The case $L \geqslant S_K^{n/(2s)}$ can not occur. Otherwise, by (2.30), (2.31) and Step 3, we would get

$$c = \mathcal{J}_{K,\lambda}(u_\infty) + \frac{s}{n} L \geqslant \frac{s}{n} L \geqslant \frac{s}{n} S_K^{n/(2s)} ,$$

which contradicts (2.1) . Thus $L = 0$, and so (2.31) yields

END

$$\|u_j - u_\infty\|_{X_0} \to 0 \tag{2.32}$$

as $j \to +\infty$. This ends the proof of Proposition 3 . □

sub:geometry

2.2. **The geometry of the functional $\mathcal{J}_{K,\lambda}$.** This subsection is devoted to the study of the geometric structure of the functional $\mathcal{J}_{K,\lambda}$. Precisely, we will show that it fits with the geometric assumptions of the Linking Theorem given in [13, Theorem 5.3]. Remember that we consider only the case when $\lambda \geqslant \lambda_1$, since the other one was treated in [18] .

Here we argue essentially as in the classical case of the Laplacian (see [1, 20, 22] and references therein), but some nontrivial technical differences arise.

In the pure non-local critical case, i.e. when $f \equiv 0$, the proof of the geometric features of $\mathcal{J}_{K,\lambda}$ was done in [14, Propositions 8–10] (see also [19] for the case $\mathcal{L}_K = -(-\Delta)^s$) . Here we generalize this result to the perturbed setting.

lemmaJpositivoL

Proposition 4. *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let f be a Carathéodory function satisfying conditions* (1.10)–(1.12)*. Then, there exist $\rho > 0$ and $\beta > 0$ such that for any $u \in \mathbb{P}_{k+1}$ with $\|u\|_{X_0} = \rho$ it results that $\mathcal{J}_{K,\lambda}(u) \geqslant \beta$.*

Proof. Let u be a function in $\mathbb{P}_{k+1}$. By [18, Lemma 6] we get that for any $\varepsilon > 0$

infJfrac0

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\
&\qquad - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx - \varepsilon\int_\Omega |u(x)|^2 dx - \delta(\varepsilon)\int_\Omega |u(x)|^{2^*}\,dx \\
&\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \varepsilon\|u\|^2_{L^2(\Omega)} \\
&\qquad - \left(\frac{1}{2^*}+\delta(\varepsilon)\right)\|u\|^{2^*}_{L^{2^*}(\Omega)} \\
&\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&\qquad - \varepsilon|\Omega|^{(2^*-2)/2^*}\|u\|^2_{L^{2^*}(\Omega)} - \left(\frac{1}{2^*}+\delta(\varepsilon)\right)\|u\|^{2^*}_{L^{2^*}(\Omega)} ,
\end{aligned} \tag{2.33}$$

thanks to [17, Lemma 16] and to the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously (since Ω is bounded and $2 < 2^*$, so that one can use the Hölder inequality).

By (1.8) and [16, Lemma 6], from (2.33) we deduce that for any $\varepsilon > 0$

infJfrac20

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy \\
&\qquad - \varepsilon c|\Omega|^{(2^*-2)/2^*}\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad - \left(\frac{1}{2^*}+\delta(\varepsilon)\right)c^{2^*/2}\left(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{2^*/2} \\
&\geqslant \left[\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)-\frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\right]\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy \\
&\qquad - \left(\frac{1}{2^*}+\delta(\varepsilon)\right)\left(\frac{c}{\theta}\right)^{2^*/2}\left(\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy\right)^{2^*/2} \\
&= \left[\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)-\frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\right]\|u\|_{X_0}^2-\left(\frac{1}{2^*}+\delta(\varepsilon)\right)\left(\frac{c}{\theta}\right)^{2^*/2}\|u\|_{X_0}^{2^*}\,.
\end{aligned} \tag{2.34}$$

We recall that θ was introduced in (1.8). Now, we can choose $\varepsilon > 0$ such that $2\varepsilon c|\Omega|^{(2^*-2)/2^*} < \theta\big(1-\lambda/\lambda_{k+1}\big)$, so that, by (2.34) it is easily seen that

$$\mathcal{J}_{K,\lambda}(u) \geqslant \alpha\|u\|_{X_0}^2\left(1-\kappa\|u\|_{X_0}^{2^*-2}\right),$$

for suitable positive constants α and κ.

Finally, let $u \in \mathbb{P}_{k+1}$ be such that $\|u\|_{X_0} = \rho > 0$. Since $2^* > 2$, choosing ρ sufficiently small (i.e. ρ such that $1-\kappa\rho^{2^*-2} > 0$), we get

$$\inf_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{X_0}=\rho}} \mathcal{J}_{K,\lambda}(u) \geqslant \alpha\rho^2(1-\kappa\rho^{2^*-2}) =: \beta > 0\,,$$

so that the proof of Proposition 4 is complete. □

lemmaJnegativoL

Proposition 5. *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let f be a Carathéodory function satisfying conditions* (1.10)–(1.13) *. Then, $\mathcal{J}_{K,\lambda}(u) \leqslant 0$ for any $u \in span\{e_1, \dots, e_k\}$.*

Proof. Let $u \in \mathrm{span}\{e_1, \dots, e_k\}$. Then

$$u(x) = \sum_{i=1}^{k} u_i e_i(x)\,,$$

with $u_i \in \mathbb{R}$, $i = 1, \dots, k$. Since $\big\{e_1, \dots, e_k, \dots\big\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal one of X_0 by [17, Proposition 9–f)], we get

ortoL2

$$\int_\Omega |u(x)|^2\,dx = \sum_{i=1}^{k} u_i^2 \tag{2.35}$$

and

ortoX0

$$\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy = \sum_{i=1}^{k} u_i^2\|e_i\|_{X_0}^2\,. \tag{2.36}$$

Then, by (2.35) and (2.36) and using [17, Proposition 9–*b*) and *e*)], the fact that $e_i \in X_0$ for any $i \in \mathbb{N}$ and (1.13), we get

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &= \frac{1}{2}\sum_{i=1}^{k} u_i^2\Big(\|e_i\|_{X_0}^2 - \lambda\Big) - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx - \int_\Omega F(x,u(x))\,dx \\ &\leqslant \frac{1}{2}\sum_{i=1}^{k} u_i^2\Big(\|e_i\|_{X_0}^2 - \lambda\Big) \\ &= \frac{1}{2}\sum_{i=1}^{k} u_i^2\big(\lambda_i - \lambda\big) \leqslant 0\,,\end{aligned}$$

since $\lambda_i \leqslant \lambda_k \leqslant \lambda$ for any $i = 1,\dots,k$. This concludes the proof of Proposition 5. ☐

lemmaJnegativo1LF **Proposition 6.** *Let $\lambda \geqslant 0$, let f be a Carathéodory function satisfying conditions* (1.10)–(1.13) *and let $\mathbb{F}$ be a finite dimensional subspace of X_0. Then, there exist $R > \rho$ such that $\mathcal{J}_{K,\lambda}(u) \leqslant 0$ for any $u \in \mathbb{F}$ with $\|u\|_{X_0} \geqslant R$, where ρ is given in Proposition 4.*

Proof. Let $u \in \mathbb{F}$. Since $\lambda \geqslant 0$ and (1.13) holds true, we obtain that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &\leqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}(\Omega)}^{2^*} \\ &\leqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\kappa}{2^*}\|u\|_{X_0}^{2^*}\,,\end{aligned}$$

for some positive constant κ, thanks to the fact that in any finite dimensional space all the norms are equivalent.

Hence, if $\|u\|_{X_0} \to +\infty$, then

$$\mathcal{J}_{K,\lambda}(u) \to -\infty\,,$$

since $2^* > 2$ by assumption. This ends the proof of Proposition 6. ☐

By Propositions 4–6 we deduce that the functional $\mathcal{J}_{K,\lambda}$ has the geometric features required by the Linking Theorem [13, Theorem 5.3]. Now we have to choose carefully the finite dimensional space $\mathbb{F}$ in which we apply such theorem in order to have the suitable estimate on the critical level (i.e. an estimate which fits with the range where the Palais–Smale condition holds true).

To this purpose, let u_0 be the function given in assumption (1.26) and $u_0^{(k+1)}$ its projection on $\mathbb{P}_{k+1}$ and $\tilde{u}_0^{(k+1)}$ the normalization of $u_0^{(k+1)}$ in X_0, i.e.

tildeu0 (2.37)
$$u_0^{(k+1)} = u_0 - \sum_{i=1}^{k}\Big(\int_\Omega u_0(x)e_i(x)\,dx\Big)e_i\,, \quad \tilde{u}_0^{(k+1)} = u_0^{(k+1)}/\|u_0^{(k+1)}\|_{X_0}\,.$$

Here we take

$$\mathbb{F} = V \oplus \operatorname{span}\Big\{\tilde{u}_0^{(k+1)}\Big\}\,,$$

where $V = \operatorname{span}\{e_1,\dots,e_k\}$. Note that V is finite dimensional and $V \oplus W = X_0$, with $W = \mathbb{P}_{k+1}$, thanks to [17, Proposition 9–*f*)]. Moreover, by Propositions 4–6 we get that there exist $\rho > 0$, $\beta > 0$ and $R > \rho$ such that

$$\mathcal{J}_{K,\lambda}(u) \geqslant \beta > 0 \quad \text{for any } u \in W \text{ with } \|u\|_{X_0} = \rho > 0\,,$$

$$\mathcal{J}_{K,\lambda}(u) \leqslant 0 \quad \text{for any } u \in V$$

and

$$\mathcal{J}_{K,\lambda}(u) \leqslant 0 \quad \text{for any } u \in \mathbb{F} \text{ with } \|u\|_{X_0} \geqslant R\,.$$

Hence, the geometric structure of $\mathcal{J}_{K,\lambda}$ fits with the one required by the Linking Theorem (see assumptions (I_1') and (I_5) of [13, Theorem 5.3] and also [13, Remark 5.5-(*iii*)]).

The Linking critical level c_L of $\mathcal{J}_{K,\lambda}$ is given by

$$c_L = \inf_{h\in\Gamma} \max_{u\in Q} \mathcal{J}_{K,\lambda}(h(u))\,,$$

where

$$\Gamma = \big\{h \in C(\overline{Q}; X_0) : h = id \text{ on } \partial Q\big\}$$

and

$$Q = \left(\overline{B}_R \cap V\right) \oplus \left\{r\,\tilde{u}_0^{(k+1)} : r \in (0, R)\right\}\,.$$

Now we are ready to estimate the critical level c_L. This will be done in the next subsection.

subsec:dimtheorem1

2.3. **Estimates of the critical level of $\mathcal{J}_{K,\lambda}$.** Due to the critical nature of our problem, we have to estimate the Linking critical level c_L in order to show that it stays below the threshold under which the Palais–Smale condition is satisfied by $\mathcal{J}_{K,\lambda}$.

For this, first of all note that, by definition of c_L, for any $h \in \Gamma$

$$c_L \leqslant \max_{u\in Q} \mathcal{J}_{K,\lambda}(h(u))$$

from which, taking $h = id$ in $\overline{Q}$, we deduce that

cl1 (2.38)
$$c_L \leqslant \max_{u\in Q} \mathcal{J}_{K,\lambda}(u)\,. \tag{2.38}$$

Furthermore, recall that $\mathbb{F}$ is a linear space and so, as a consequence,

maxt (2.39)
$$\begin{aligned}
\max_{u\in\mathbb{F}} \mathcal{J}_{K,\lambda}(u) &= \max_{\substack{u\in\mathbb{F}\\ \zeta\neq 0}} \mathcal{J}_{K,\lambda}\Big(|\zeta|\cdot\frac{u}{|\zeta|}\Big)\\
&= \max_{\substack{u\in\mathbb{F}\\ \zeta>0}} \mathcal{J}_{K,\lambda}(\zeta u)\\
&\leqslant \max_{\substack{u\in\mathbb{F}\\ \zeta\geqslant 0}} \mathcal{J}_{K,\lambda}(\zeta u)\,.
\end{aligned} \tag{2.39}$$

Hence, by (2.38), (2.39) and the fact that $Q \subset \mathbb{F}$, we obtain that

cL (2.40)
$$c_L \leqslant \max_{u\in Q} \mathcal{J}_{K,\lambda}(u) \leqslant \max_{u\in\mathbb{F}} \mathcal{J}_{K,\lambda}(u) \leqslant \max_{\substack{u\in\mathbb{F}\\ \zeta\geqslant 0}} \mathcal{J}_{K,\lambda}(\zeta u)\,. \tag{2.40}$$

Thanks to (1.13) and [18, Proposition 20] we also have

max (2.41)
$$\begin{aligned}
\max_{\zeta\geqslant 0} \mathcal{J}_{K,\lambda}(\zeta u) &= \max_{\zeta\geqslant 0}\Big(\frac{\zeta^2}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\zeta^2\lambda}{2}\int_\Omega |u(x)|^2\,dx\\
&\qquad -\frac{\zeta^{2^*}}{2^*}\int_\Omega |u(x)|^{2^*}\,dx - \int_\Omega F(x,\zeta u(x))\,dx\Big)\\
&\leqslant \max_{\zeta\geqslant 0}\Big(\frac{\zeta^2}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\zeta^2\lambda}{2}\int_\Omega |u(x)|^2\,dx\\
&\qquad -\frac{\zeta^{2^*}}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\Big)\\
&= \frac{s}{n}\,S_{K,\lambda}^{n/(2s)}(u)
\end{aligned} \tag{2.41}$$

for any $u \in X_0 \setminus \{0\}$.

Finally, taking into account that $\mathbb{F} = V \oplus \operatorname{span}\{\tilde{u}_0^{(k+1)}\} = \operatorname{span}\{e_1, \dots, e_k, u_0\}$ (here we use the construction of $\tilde{u}_0^{(k+1)}$), by (1.26), (2.40) and (2.41) we deduce that

cL1 (2.42)
$$c_L \leqslant \max_{\substack{u\in\mathbb{F}\\ \zeta\geqslant 0}} \mathcal{J}_{K,\lambda}(\zeta u) \leqslant \frac{s}{n}\max_{u\in\mathbb{F}} S_{K,\lambda}^{n/(2s)}(u) < \frac{s}{n}\,S_K^{n/(2s)}\,. \tag{2.42}$$

Ultimately, we have shown that the Linking critical level of $\mathcal{J}_{K,\lambda}$ stays below the threshold where the Palais–Smale condition is verified.

Therefore, the conclusion of Theorem 1 follows from the Linking Theorem [13, Theorem 5.3].

sec:prooffraclap

3. Applications to critical fractional Laplace equations

In this section we briefly discuss the case when K is the model kernel

$$K(x) = |x|^{-(n+2s)},$$

which gives rise to the fractional Laplace operator $-(-\Delta)^s$ and we prove Theorem 2. For this it is enough to note that, in this setting, condition (1.28) implies (1.26), thanks to [18, Lemma 7] and to the fact that

$$\inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v) \leqslant \inf_{v\in X_0\setminus\{0\}} S_s(v)\,,$$

since $X_0 \subseteq H^s(\mathbb{R}^n)$ by [16, Lemma 5-b)].

Hence, Theorem 1 provides a non-trivial solution $u \in X_0 \setminus \{0\}$ of problem (1.27) for any $\lambda > 0$. Now, Theorem 2 follows from the fact that $X_0 \subseteq H^s(\mathbb{R}^n)$ by [16, Lemma 5-b)] (alternatively, we can use [18, Lemma 7], since the solution u provided by Theorem 1 is such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

appendix

Appendix A. The functional setting

Here we recall the definition of the functional analytic spaces X and X_0 we work in.

In our framework the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega), dxdy\big)\,,$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$, while

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}\,.$$

Note that X and X_0 are non-empty, since $C^2_0(\Omega) \subseteq X_0$ by [15, Lemma 11] (for this we need condition (1.7)).

The space X is endowed with the norm defined as

norma (A.1)
$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2}\,, \tag{A.1}$$

where $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega)\times(\mathcal{C}\Omega) \subset \mathbb{R}^{2n}$. It is easily seen that $\|\cdot\|_X$ is a norm on X (see, for instance, [16] for a proof).

Along the paper we also denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

gagliardonorm (A.2)
$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{A.2}$$

We remark that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (A.1) and (A.2) are not the same, because $\Omega\times\Omega$ is strictly contained in Q: this makes the classical fractional Sobolev space approach not sufficient for studying the problem.

For further details on the fractional Sobolev spaces we refer to [10] and to the references therein.

Some relations between the space X_0 and the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ were proved in [16, Lemma 5-b)] and in [18, Lemma 7], while, as for the embeddings of X_0 and $H^s(\mathbb{R}^n)$ into the classical Lebesgue spaces, many properties were proved in [10, 16, 18]. We would like to note that, with respect to such embeddings, the fractional Sobolev space $H^s(\mathbb{R}^n)$ behaves like the usual Sobolev space $H^1(\mathbb{R}^n)$, while X_0 as $H^1_0(\Omega)$. This is due to the fact that the functions $v \in X_0$ are such that $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and so X_0 may be seen as a space of functions defined in the bounded set Ω.

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