A RESONANCE PROBLEM FOR NON-LOCAL ELLIPTIC OPERATORS

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Abstract. In this paper we consider a resonance problem driven by a non-local integrodifferential operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. This problem has a variational structure and we find a solution for it using the Saddle Point Theorem. We prove this result for a general integrodifferential operator of fractional type and from this, as a particular case, we derive an existence theorem for the following fractional Laplacian equation

$$\begin{cases} (-\Delta)^s u = \lambda a(x)u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

when λ is an eigenvalue of the related non-homogenous linear problem with homogeneous Dirichlet boundary data. Here the parameter $s \in (0,1)$ is fixed, Ω is an open bounded set of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, a is a Lipschitz continuous function, while f is a sufficiently smooth function. This existence theorem extends to the non-local setting some results, already known in the literature in the case of the Laplace operator $-\Delta$.

Contents

1. Introduction

Nonlinear elliptic problems modeled by

$$\begin{cases} -\Delta u = \lambda u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2$, is an open bounded set, λ is a positive[1] parameter and the perturbation f is a function satisfying different growth conditions (asymptotically linear, superlinear,

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[1]Throughout this paper, by "positive", we mean "strictly positive".

subcritical or critical, for instance), were widely studied in the literature (see, for instance, [1, 17, 19] and references therein).

In some recent papers these problems were treated in a non-local setting: in this framework see, for instance, [7] for the asymptotically linear case, [4, 12, 13] for subcritical nonlinearities and [5, 10, 14, 15, 16, 18] for the critical case.

Aim of this paper is to consider the non-local version of problem (1.1) in the case when the perturbation $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function such that

$$f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}); \tag{1.2}$$

$$\text{there exists a constant } M > 0 \text{ such that } |f(x,t)| \leqslant M \text{ for any } (x,t) \in \Omega \times \mathbb{R}; \tag{1.3}$$

$$F(x,t) = \int_0^t f(x,s)ds \to +\infty \text{ as } |t| \to +\infty \text{ uniformly for } x \in \Omega. \tag{1.4}$$

To be precise, in this paper we deal with the following problem

$$\begin{cases} -\mathcal{L}_K u = \lambda a(x) u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.5}$$

where $s \in (0,1)$ is fixed, $n > 2s$, $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary and $a : \overline{\Omega} \to \mathbb{R}$ is such that

$$a \text{ is a positive Lipschitz continuous function in } \overline{\Omega}\,. \tag{1.6}$$

Finally $\mathcal{L}_K$ is the non-local operator defined as follows

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x) \Big) K(y)\, dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.7}$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function with the properties that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.8}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.9}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.10}$$

A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case problem (1.5) becomes

$$\begin{cases} (-\Delta)^s u = \lambda a(x) u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.11}$$

where $(-\Delta)^s$ is the fractional Laplace operator which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \tag{1.12}$$

for $x \in \mathbb{R}^n$. We refer to [6] and references therein for further details on the fractional Laplacian.

One of the motivations for studying (1.11) (and, more generally, (1.5)) is trying to extend some important results, which are well known for the classical case of the Laplacian $-\Delta$ (see, e.g., [9, Chapter 4 and Theorem 4.12]), to a non-local setting.

The conditions we consider on a and f are classical in the nonlinear analysis (see, e.g., conditions ($p1$), ($p2$) and ($p7$) in [9, Theorem 4.12]) and, roughly speaking, they state that problem (1.5) is a suitable perturbation from the following non-homogenous eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda a(x) u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.13}$$

We recall that there exists a non-decreasing sequence of positive eigenvalues λ_k for which (1.13) admits non-trivial solutions. We will study problem (1.13) in Subsection 2.2.

Finally, note that, thanks to (1.4), the nonlinearity f cannot be the trivial function. As a model for f we can take the functions

$$f(x,t) = M > 0 \quad \text{or} \quad f(x,t) = b(x)\arctan t\,,$$

with $b \in Lip(\overline{\Omega})$ and $b > 0$ in Ω. In the first case $u \equiv 0$ does not solve (1.5), while in the second one the trivial function is a solution of (1.5). In general, the function $u \equiv 0$ in $\mathbb{R}^n$ is a solution of problem (1.5) if and only if $f(\cdot, 0) = 0$. This is an important difference with respect to the other works in the subject, such as [10, 12, 13, 14, 15, 16], where the trivial function is always a solution.

The aim of this paper is to find solutions for (1.5) via variational methods. For this, firstly we need the weak formulation of (1.5), which is given by the following problem (for this, it is worth to assume (1.10))

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy = \lambda \int_\Omega a(x)u(x)\varphi(x)\,dx \\ \displaystyle\qquad\qquad\qquad\qquad\qquad\qquad + \int_\Omega f(x,u(x))\varphi(x)\,dx \quad \forall\, \varphi \in X_0 \\ u \in X_0\,. \end{cases} \tag{1.14}$$

Here the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big)\,, \tag{1.15}$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}\,. \tag{1.16}$$

We remark that X and X_0 are non-empty since $C^2_0(\Omega) \subseteq X_0$, by [11, Lemma 11] and (1.8).

Working in X_0 allows us to encode the Dirichlet datum $u = 0$ in $\mathbb{R}^n \setminus \Omega$ in the weak formulation.

The main result of the present paper can be stated as follows:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.8)*–*(1.10) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $a : \overline{\Omega} \to \mathbb{R}$ be two functions verifying* (1.2)*–*(1.4) *and* (1.6)*, respectively. Moreover, assume that λ is an eigenvalue of the non-homogeneous linear problem in* (1.13)*.*

Then, problem (1.5) *admits a solution $u \in X_0$.*

In the classical case of the Laplacian $-\Delta$ the counterpart of Theorem 1 is given in [9, Theorem 4.12]: in this sense Theorem 1 may be seen as the natural extension of classical results to the non-local fractional setting.

The strategy for proving Theorem 1 is based on the fact that problem (1.14) can be seen as the Euler–Lagrange equation of a suitable functional (see (4.1)). Hence, the solutions of (1.14) can be found as critical points of this functional: at this purpose, along the paper, we will exploit the Saddle Point Theorem by Rabinowitz (see [8, 9]).

This paper is organized as follows. In Section 2 we will give some notations and we will recall some basic facts on the spectral theory for the operator $-\mathcal{L}_K$, while in Section 3 we will state and prove some technical lemmas useful along the paper. Finally, in Section 4 we will prove Theorem 1 by making use of the classical Saddle Point Theorem.

2. Some preliminary facts

2.1. **Notations.** In the sequel the spaces X and X_0 (whose definitions were recalled in the Introduction) will be endowed, respectively, with the norms defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2}\,, \tag{2.1}$$

and

$$\|g\|_{X_0} = \left(\int_Q |g(x) - g(y)|^2 K(x-y)\, dx\, dy \right)^{1/2} . \tag{2.2}$$

Here $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$, with $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2n}$and $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$.

Note that, since $g \in X_0$ is such that $g = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, then in (2.2) the integral on Q can be extended to all $\mathbb{R}^{2n}$. Moreover, the norm on X_0 given in (2.2) is equivalent to the usual one defined in (2.1), by [12, Lemmas 6 and 7].

With the norm given in (2.2), X_0 is a Hilbert space with scalar product defined as

$$\langle u, v\rangle_{X_0} = \int_Q \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, K(x-y)\, dx\, dy\,. \tag{2.3}$$

For this see [12, Lemma 7]. For further details on X and X_0 and also for their properties we refer to [10, 12, 13, 14, 15, 16]. Note that, since $a \in L^\infty(\Omega)$ by (1.6), all the embeddings properties of X_0 into the usual Lebesgue space $L^2(\Omega)$ still hold true in $L^2(\Omega,\, \mu)$, with $\mu(\cdot) = a(\cdot)dx$, defined as

$$L^2(\Omega,\, \mu) := \Big\{ g : \Omega \to \mathbb{R} \text{ s.t. } g \text{ is measurable in } \Omega \text{ and}$$
$$\int_\Omega a(x)|g(x)|^2\, dx = \int_\Omega |g|^2\, d\mu < +\infty \Big\}.$$

In the following we will denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \Big)^{1/2} . \tag{2.4}$$

We remark that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.4) are not the same, because $\Omega \times \Omega$ is strictly contained in Q.

For further details on the fractional Sobolev spaces we refer to [6] and to the references therein.

2.2. **An eigenvalue problem.** This subsection is devoted to the study of the non-homogeneous eigenvalue problem (1.13). More precisely, we consider the weak formulation of (1.13), which consists in the following eigenvalue problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dx\, dy = \lambda \int_\Omega a(x)u(x)\varphi(x)dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \varphi \in X_0 \\ u \in X_0. \end{cases} \tag{2.5}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.5) provided there exists a non-trivial solution $u \in X_0$ of problem (2.5) and, in this case, any solution will be called an eigenfunction corresponding to the eigenvalue λ.

For the proof of the next result we refer to [13, Proposition 9 and Appendix A], where the problem (2.5) with $a \equiv 1$ was considered (the case of $a \not\equiv 1$ can be proved similarly, just replacing the classical space $L^2(\Omega)$ with $L^2(\Omega,\, \mu)$).

Proposition 2. *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded subset of $\mathbb{R}^n$ and let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions* (1.8)–(1.10). *Moreover, let $a : \overline{\Omega} \to \mathbb{R}$ be a function verifying* (1.6). *Then,*

(i) *problem* (2.5) *admits an eigenvalue λ_1 which is positive and that can be characterized as follows*

$$\lambda_1 = \min_{\substack{u \in X_0 \\ \|u\|_{L^2(\Omega,\, \mu)}=1}} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2\, K(x-y)dx\, dy,$$

or, equivalently,

$$\lambda_1 = \min_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy}{\displaystyle\int_\Omega a(x)\,|u(x)|^2\,dx}, \tag{2.6}$$

where $\|\cdot\|_{L^2(\Omega,\,\mu)}$ *denotes the* L^2*–norm with respect to the measure* $\mu(x) = a(x)dx$*;*

(ii) *there exists a non-negative function* $e_1 \in X_0$*, which is an eigenfunction corresponding to* λ_1*, attaining the minimum in* (2.6)*, that is* $\|e_1\|_{L^2(\Omega,\,\mu)} = 1$ *and*

$$\lambda_1 = \int_{\mathbb{R}^{2n}} |e_1(x)-e_1(y)|^2 K(x-y)dx\,dy;$$

(iii) λ_1 *is simple, that is if* $u \in X_0$ *is a solution of the following equation*

$$\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy = \lambda_1 \int_\Omega a(x)u(x)\varphi(x)dx \quad \forall \varphi \in X_0,$$

then $u = \zeta e_1$*, with* $\zeta \in \mathbb{R}$*;*

(iv) *the set of the eigenvalues of problem* (2.5) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with*[2]

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots \tag{2.7}$$

and

$$\lambda_k \to +\infty \text{ as } k \to +\infty.$$

Moreover, for any $k \in \mathbb{N}$ *the eigenvalues can be characterized as follows:*

$$\lambda_{k+1} = \min_{\substack{u \in \mathbb{P}_{k+1} \\ \|u\|_{L^2(\Omega,\,\mu)}=1}} \int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy,$$

or, equivalently,

$$\lambda_{k+1} = \min_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy}{\displaystyle\int_\Omega a(x)\,|u(x)|^2\,dx}, \tag{2.8}$$

where

$$\mathbb{P}_{k+1} := \Big\{u \in X_0 : \langle u, e_j\rangle_{X_0} = 0 \quad \forall j = 1, \ldots, k\Big\}; \tag{2.9}$$

(v) *for any* $k \in \mathbb{N}$ *there exists a function* $e_{k+1} \in \mathbb{P}_{k+1}$*, which is an eigenfunction corresponding to* λ_{k+1}*, attaining the minimum in* (2.8)*, that is* $\|e_{k+1}\|_{L^2(\Omega,\,\mu)} = 1$ *and*

$$\lambda_{k+1} = \int_{\mathbb{R}^{2n}} |e_{k+1}(x)-e_{k+1}(y)|^2 K(x-y)dx\,dy; \tag{2.10}$$

(vi) *the sequence* $\{e_k\}_{k\in\mathbb{N}}$ *of eigenfunctions corresponding to* λ_k *is an orthonormal basis of* $L^2(\Omega,\mu)$ *and an orthogonal basis of* X_0*;*

(vii) *each eigenvalue* λ_k *has finite multiplicity; more precisely, if* λ_k *is such that*

$$\lambda_{k-1} < \lambda_k = \ldots = \lambda_{k+h} < \lambda_{k+h+1}$$

for some $h \in \mathbb{N}_0$*, then the set of all the eigenfunctions corresponding to* λ_k *agrees with*

$$span\,\{e_k, \ldots, e_{k+h}\}.$$

[2] As usual, here we call λ_1 the *first eigenvalue* of the operator $-\mathcal{L}_K$. This notation is justified by (2.7). Notice also that some of the eigenvalues in the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ may repeat, i.e. the inequalities in (2.7) may be not always strict.

In particular, Proposition 2 gives a variational characterization of the eigenvalues λ_k of $-\mathcal{L}_K$ (see formulas (2.6) and (2.8)). Another interesting characterization of the eigenvalues is given in the next result. For the proof we refer to [10, Proposition 5], where the case $a \equiv 1$ was treated (again, the case of $a \not\equiv 1$ can be proved likewise).

Proposition 3. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence of the eigenvalues given in Proposition 2 and let $\{e_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of eigenfunctions.*

Then, for any $k \in \mathbb{N}$ the eigenvalues can be characterized as follows:

$$\lambda_k = \max_{u\in span\{e_1,\dots,e_k\}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy}{\displaystyle\int_\Omega a(x)|u(x)|^2\,dx}.$$

We conclude this subsection with some notation. In what follows, without loss of generality, we will fix $\lambda = \lambda_k$ with $k \in \mathbb{N}$ such that $\lambda_k < \lambda_{k+1}$ and we will denote by $\mathbb{H}_k$ the linear subspace of X_0 generated by the first k eigenfunctions of $-\mathcal{L}_K$, i.e.

$$\mathbb{H}_k := \mathrm{span}\{e_1,\dots,e_k\},$$

while $\mathbb{P}_{k+1}$ will be the space defined in (2.9). Here e_j and λ_j, $j \in \mathbb{N}$, are the eigenfunctions and the eigenvalues of $-\mathcal{L}_K$, as defined in Proposition 2.

It is immediate to observe that $\mathbb{P}_{k+1} = \mathbb{H}_k^{\perp}$ with respect to the scalar product in X_0 defined as in formula (2.3). Thus, since X_0 is a Hilbert space (see [12, Lemma 7] and (2.3)), we can write it as a direct sum as follows

$$X_0 = \mathbb{H}_k \oplus \mathbb{P}_{k+1}.$$

Moreover, since $\{e_1,\dots,e_k,\dots\}$ is an orthogonal basis of X_0, it follows that

$$\mathbb{P}_{k+1} = \overline{\mathrm{span}\{e_j :\ j \geqslant k+1\}}.$$

Also we will set

$$\mathbb{E}_k^0 := \mathrm{span}\{e_j :\ \lambda_j = \lambda_k\} \quad \text{and} \quad \mathbb{E}_k^- := \mathrm{span}\{e_j :\ \lambda_j < \lambda_k\}. \tag{2.11}$$

Note that with this notation, if $u \in \mathbb{H}_k$, then we can write it as

$$u = u^0 + u^-, \ \text{ with } u^0 \in \mathbb{E}_k^0 \text{ and } u^- \in \mathbb{E}_k^-.$$

3. Some technical lemmas

In this section we prove some technical lemmas, which will be useful in order to apply the Saddle Point Theorem to problem (1.14).

Lemma 4. *Let $K : \mathbb{R}^n\setminus\{0\} \to (0,+\infty)$ satisfy assumptions (1.8)–(1.10) and let $a : \overline{\Omega} \to \mathbb{R}$ verify (1.6).*

Then, for any $u \in \mathbb{P}_{k+1}$

$$\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \lambda_k \int_\Omega a(x)|u(x)|^2\,dx \geqslant \left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u\|_{X_0}^2.$$

Proof. If $u \equiv 0$, then the assertion is trivial. Now, let $u \in \mathbb{P}_{k+1}\setminus\{0\}$. By the variational characterization of λ_{k+1} given in (2.8) we get that

$$\|u\|_{L^2(\Omega,\mu)}^2 \leqslant \frac{1}{\lambda_{k+1}}\|u\|_{X_0}^2.$$

As a consequence of this and taking into account that λ_k is positive (since $\lambda_k \geqslant \lambda_1 > 0$), we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \lambda_k \int_\Omega a(x)|u(x)|^2\,dx &\geqslant \|u\|_{X_0}^2 - \frac{\lambda_k}{\lambda_{k+1}}\|u\|_{X_0}^2 \\ &= \left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u\|_{X_0}^2,\end{aligned}$$

concluding the proof. □

Note that, if $\lambda_k = \lambda_{k+1}$, then Lemma 4 is trivial. The interesting case is when $\lambda_k < \lambda_{k+1}$.

Lemma 5. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (1.8)–(1.10) and let $a : \overline{\Omega} \to \mathbb{R}$ verify (1.6).*

Then, there exists a positive constant M^*, depending on k, such that

$$\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \lambda_k \int_\Omega a(x)|u(x)|^2\, dx \leqslant -M^* \|u^-\|^2_{X_0}$$

for all $u \in \mathbb{H}_k$, where $u = u^- + u^0$, $u^- \in E_k^-$ and $u^0 \in E_k^0$.

Proof. Of course, if $u \equiv 0$, then the assertion is trivial. Hence, assume that $u \in \mathbb{H}_k \setminus \{0\}$. Let $h \in \mathbb{N}$ be the multiplicity of λ_k (h is finite thanks to Proposition 2-(vii)), that is suppose that

$$\lambda_{k-h-1} < \lambda_{k-h} = \cdots = \lambda_k < \lambda_{k+1}. \tag{3.1}$$

With this notation, u can be written as follows

$$u = u^- + u^0,$$

with

$$u^- \in \mathbb{E}_k^- = \mathrm{span}\{e_1, \dots, e_{k-h-1}\} \quad \text{and} \quad u^0 \in \mathbb{E}_k^0 = \mathrm{span}\{e_{k-h}, \dots, e_k\}.$$

Notice that u^0 is a linear combination of eigenfunctions corresponding to the same eigenvalue $\lambda_{k-h} = \cdots = \lambda_k$, hence it is also an eigenfunction corresponding to λ_k. Hence, by (2.5),

$$\|u^0\|^2_{X_0} = \lambda_k \|u^0\|^2_{L^2(\Omega,\mu)}.$$

Also, u^- and u^0 are orthogonal both in X_0 and in $L^2(\Omega, \mu)$, therefore

$$\begin{aligned} \|u\|^2_{X_0} - \lambda_k \|u\|^2_{L^2(\Omega,\mu)} &= \|u^-\|^2_{X_0} + \|u^0\|^2_{X_0} - \lambda_k \left(\|u^-\|^2_{L^2(\Omega,\mu)} + \|u^0\|^2_{L^2(\Omega,\mu)} \right) \\ &= \|u^-\|^2_{X_0} - \lambda_k \|u^-\|^2_{L^2(\Omega,\mu)}. \end{aligned} \tag{3.2}$$

Now, note that $u^- \in \mathbb{E}_k^- = \mathrm{span}\{e_1, \dots, e_{k-h-1}\}$. Hence, by this and Proposition 3 we get

$$\|u^-\|^2_{X_0} \leqslant \lambda_{k-h-1} \|u^-\|^2_{L^2(\Omega,\mu)}. \tag{3.3}$$

Finally, (3.2) and (3.3) yield

$$\begin{aligned} \|u\|^2_{X_0} - \lambda_k \|u\|^2_{L^2(\Omega,\mu)} &= \|u^-\|^2_{X_0} - \lambda_k \|u^-\|^2_{L^2(\Omega,\mu)} \\ &\leqslant \|u^-\|^2_{X_0} - \frac{\lambda_k}{\lambda_{k-h-1}} \|u^-\|^2_{X_0} \\ &= \left(1 - \frac{\lambda_k}{\lambda_{k-h-1}}\right) \|u^-\|^2_{X_0}, \end{aligned}$$

which gives the desired assertion with

$$M^* := \frac{\lambda_k}{\lambda_{k-h-1}} - 1.$$

Note that $M^* > 0$, thanks to (3.1). □

Finally, in the next two results we discuss some properties of the function F defined as in (1.4).

Lemma 6. *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy (1.2)–(1.4).*

Then, there exists a positive constant $\widetilde{M}$, depending on Ω, such that

$$\left| \int_\Omega F(x, u(x))\, dx \right| \leqslant \widetilde{M} \|u\|_{X_0}$$

for all $u \in X_0$.

Proof. Using the definition of F and (1.3), it is easy to see that

$$\left|\int_\Omega F(x,u(x))\,dx\right| = \left|\int_\Omega \int_0^{u(x)} f(x,t)\,dt\,dx\right| \leqslant M \int_\Omega |u(x)|\,dx\,,$$

so that, by Hőlder inequality and [12, Lemma 8] we get

$$\left|\int_\Omega F(x,u(x))\,dx\right| \leqslant M\,|\Omega|^{1/2}\,\|u\|_{L^2(\Omega)} \leqslant \widetilde{M}\|u\|_{X_0} \tag{3.4}$$

for all $u \in X_0$, where $\widetilde{M}$ is a positive constant depending on Ω. Hence, the assertion is proved. □

Lemma 7. *Let* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *satisfy* (1.2)–(1.4) *.*
Then,

$$\lim_{\substack{u\in \mathbb{E}^0_k \\ \|u\|_{X_0}\to+\infty}} \int_\Omega F(x,u(x))\,dx = +\infty\,.$$

Proof. We argue by contradiction and suppose that there exists a positive constant C and a sequence $u_j \in E^0_k$ such that

$$t_j := \|u_j\|_{X_0} \to +\infty \tag{3.5}$$

and

$$\int_\Omega F(x,u_j(x))dx \leqslant C\,. \tag{3.6}$$

Let $v_j := u_j/\|u_j\|_{X_0}$. Of course, v_j is bounded in X_0. Hence, since $\mathbb{E}^0_k$ is finite dimensional, there exists $v \in \mathbb{E}^0_k$ such that v_j converges to v strongly in X_0. Note also that $v \not\equiv 0$, since

$$\|v\| = \lim_{j\to+\infty} \|v_j\| = 1\,.$$

Furthermore, recalling [12, Lemma 8],

$$v_j \to v \quad \text{in } L^q(\mathbb{R}^n) \text{ for any } q \in [1,2^*) \tag{3.7}$$

and, by applying [3, Theorem IV.9], up to a subsequence (still denoted by v_j)

$$v_j \to v \quad \text{a.e. in } \mathbb{R}^n \tag{3.8}$$

as $j \to +\infty$.

Now, we define $i(r) := \inf\limits_{x\in\overline{\Omega},\,|t|\geqslant r} F(x,t)$ for $r > 0$. By (1.4) it follows that

$$\lim_{r\to+\infty} i(r) = +\infty. \tag{3.9}$$

Note that

$$\inf_{x\in\overline{\Omega},\,t\in\mathbb{R}} F(x,t) \text{ is finite.} \tag{3.10}$$

Indeed, by (1.4) it follows that for any $H > 0$ there exists $R > 0$ such that

$$F(x,t) > H \text{ for any } |t| > R \text{ and any } x \in \Omega. \tag{3.11}$$

Moreover, if $|t| \leqslant R$, by (1.3) we have

$$|F(x,t)| \leqslant M\,|t| \leqslant MR =: C_R, \tag{3.12}$$

for any $x \in \Omega$. Hence, by (3.11) and (3.12) we can conclude that

$$F(x,t) \geqslant -C_R \text{ for any } (x,t) \in \Omega \times \mathbb{R}\,,$$

which implies (3.10) .

As a consequence of (3.10), we may define

$$\omega^* := -\min\left\{-1, \inf_{x\in\overline{\Omega},\, t\in\mathbb{R}} F(x,t)\right\}.$$

Notice that $\omega^* \geqslant 0$ and $F(x,t) \geqslant -\omega^*$ for any $x \in \overline{\Omega}$ and any $t \in \mathbb{R}$. Now, we fix $h > 0$ and set $\Omega_{j,h} = \{x \in \Omega : \ |t_j v_j(x)| \geqslant h\}$. Thus, we get

$$\begin{aligned}\int_\Omega F(x, t_j v_j(x))dx &= \int_{\Omega_{j,h}} F(x, t_j v_j(x))dx + \int_{\Omega\setminus\Omega_{j,h}} F(x, t_j v_j(x))dx \\ &\geqslant |\Omega_{j,h}|\, i(h) - \omega^* \, |\Omega| \,. \end{aligned}\tag{3.13}$$

Since $v \not\equiv 0$, there exists a set $\Omega^\sharp$ with $\left|\Omega^\sharp\right| > 0$ and a constant $\delta > 0$ such that $|v(x)| \geqslant \delta$ a.e. $x \in \Omega^\sharp$. Then, by (3.8) and Egorov Theorem, there exists a measurable set $\Omega^* \subseteq \Omega^\sharp$ such that $|\Omega^*| \geqslant |\Omega^\sharp|/2 > 0$ and the limit in (3.8) is uniform in Ω^*. In particular, if j is large enough,

$$\sup_{x\in\Omega^*} |v_j(x) - v(x)| \leqslant \frac{\delta}{4}$$

and therefore $|v_j(x)| \geqslant 3\delta/4$ a.e. $x \in \Omega^*$. So, by (3.5), for h fixed above there exists j_h such that $|t_j v_j(x)| \geqslant h$ for any $j \geqslant j_h$ and a.e. $x \in \Omega^*$. As a consequence of this, we have that $\Omega^* \subseteq \Omega_{j,h}$ for $j \geqslant j_h$. Finally, by (3.6) and (3.13), we have

$$C \geqslant \int_\Omega F(x, t_j v_j(x))dx \geqslant |\Omega^*|\, i(h) - \omega^* \, |\Omega|$$

for $j \geqslant j_h$. Passing to the limit as $h \to +\infty$ and taking into account (3.9), we get a contradiction. This proves the assertion. $\square$

4. Main result of the paper

This section is devoted to the proof of Theorem 1, which is the main result of the present paper. At this purpose, first of all we observe that problem (1.14) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \frac{\lambda}{2}\int_\Omega a(x)|u(x)|^2\, dx - \int_\Omega F(x, u(x)dx\,,\tag{4.1}$$

where F was introduced in (1.4).

Note that the functional $\mathcal{J}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned}\langle \mathcal{J}'(u), \varphi\rangle = &\int_{\mathbb{R}^{2n}} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\, dx\, dy \\ &- \lambda \int_\Omega a(x)u(x)\varphi(x)\, dx - \int_\Omega f(x, u(x))\varphi(x)\, dx\,.\end{aligned}$$

Thus, critical points of $\mathcal{J}$ are weak solutions to problem (1.5). In order to find these critical points, in the sequel we will apply the Saddle Point Theorem by Rabinowitz (see [8, 9]). For this, as usual for minimax theorems, we have to check that the functional $\mathcal{J}$ has a particular geometric structure (as stated, in our case, in conditions (I_3) and (I_4) of [9, Theorem 4.6]) and that it satisfies the Palais–Smale compactness condition (see, for instance, [9, p. 3]).

4.1. Geometry of the functional $\mathcal{J}$.

In this subsection we will prove that the functional $\mathcal{J}$ has the geometric features required by the Saddle Point Theorem.

Proposition 8. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (1.8)–(1.10)*. Moreover, let $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$ and let f and a be two functions satisfying* (1.2)–(1.4) *and* (1.6)*, respectively. Then*

$$\liminf_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{X_0}\to+\infty}} \frac{\mathcal{J}(u)}{\|u\|^2_{X_0}} > 0. \tag{4.2}$$

Proof. Since $u \in \mathbb{P}_{k+1}$, by Lemmas 4 and 6 we have

$$\mathcal{J}(u) \geqslant \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|u\|^2_{X_0} - \widetilde{M}\|u\|_{X_0}\,.$$

Hence, dividing both the sides of this expression by $\|u\|^2_{X_0}$ and passing to the limit as $\|u\|_{X_0} \to +\infty$, we get (4.2), since $\lambda_k < \lambda_{k+1}$ by assumption. □

Proposition 9. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (1.8)–(1.10)*. Moreover, let $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$ and let f and a be two functions satisfying* (1.2)–(1.4) *and* (1.6)*, respectively. Then*

$$\lim_{\substack{u\in\mathbb{H}_k\\ \|u\|_{X_0}\to+\infty}} \mathcal{J}(u) = -\infty\,.$$

Proof. Since $u \in \mathbb{H}_k$, we can write $u = u^- + u^0$, with $u^- \in \mathbb{E}_k^-$ and $u^0 \in \mathbb{E}_k^0$. Also, $\mathcal{J}(u)$ can be written as follows

$$\begin{aligned}\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy - \frac{\lambda_k}{2}\int_\Omega a(x)|u(x)|^2dx\\ - \int_\Omega\Big(F(x,u^0(x)+u^-(x)) - F(x,u^0(x))\Big)\,dx - \int_\Omega F(x,u^0(x))\,dx.\end{aligned} \tag{4.3}$$

First of all, note that, by (1.3), Hőlder inequality and [12, Lemma 8], it follows that

$$\begin{aligned}\Big|\int_\Omega\Big(F(x,u^0(x)+u^-(x)) - F(x,u^0(x))\Big)\,dx\Big| &= \left|\int_\Omega\int_{u^0(x)}^{u^0(x)+u^-(x)} f(x,t)dt\,dx\right|\\ &\leqslant M\int_\Omega \big|u^-(x)\big|\,dx \leqslant M\,|\Omega|^{1/2}\,\|u^-\|_{L^2(\Omega)}\\ &\leqslant \overline{M}\|u^-\|_{X_0}\,,\end{aligned} \tag{4.4}$$

where $\overline{M}$ denotes a positive constant depending on Ω. Thus, by (4.3), (4.4) and Lemma 5, we get

$$\mathcal{J}(u) \leqslant -M^*\|u^-\|^2_{X_0} + \overline{M}\|u^-\|_{X_0} - \int_\Omega F(x,u^0(x))\,dx. \tag{4.5}$$

Beware that the first norm in the right hand side of (4.5) is squared, while the second one is not. Moreover, by orthogonality we have

$$\|u\|^2_{X_0} = \|u^0\|^2_{X_0} + \|u^-\|^2_{X_0}\,.$$

Then, as $\|u\|_{X_0} \to +\infty$, we have that at least one of the two norms, either $\|u^0\|_{X_0}$ or $\|u^-\|_{X_0}$, goes to infinity.

Suppose that $\|u^0\|_{X_0} \to +\infty$ (in this case $\|u^-\|_{X_0}$ can be finite or not). Then, (4.5), the fact that $u^0 \in \mathbb{E}_k^0$ and Lemma 7 show that $\mathcal{J}(u) \to -\infty$ as $\|u\|_{X_0} \to +\infty$ and so, Proposition 9 follows.

Otherwise, assume that $\|u^0\|_{X_0}$ is finite. In this setting, of course,

$$\|u^-\|_{X_0} \to +\infty\,. \tag{4.6}$$

and, by Lemma 6, $\displaystyle\int_\Omega F(x,u^0(x))\,dx$ is also finite.

Moreover, by (4.5) and (4.6), we have that $\mathcal{J}(u) \to -\infty$ as $\|u\|_{X_0} \to +\infty$. This completes the proof of Proposition 9. □

4.2. **The Palais–Smale condition.** In this subsection we discuss a compactness property for the functional $\mathcal{J}$, given by the Palais–Smale condition.

First of all, as usual when using variational methods, we prove the boundedness of a Palais–Smale sequence for $\mathcal{J}$. We say that u_j is a Palais–Smale sequence for $\mathcal{J}$ at level $c \in \mathbb{R}$ if

$$|\mathcal{J}(u_j)| \leqslant c, \tag{4.7}$$

and

$$\sup\Big\{|\langle \mathcal{J}'(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\Big\} \to 0 \ \text{ as } j \to +\infty \tag{4.8}$$

hold true.

Proposition 10. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (1.8)–(1.10)*. Moreover, assume that $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$ and let f and a be two functions satisfying* (1.2)–(1.4) *and* (1.6)*, respectively. Finally, let $c \in \mathbb{R}$ and let u_j be a sequence in X_0 verifying* (4.7) *and* (4.8) *.*

Then, the sequence u_j is bounded in X_0.

Proof. Let $u_j = u_j^0 + u_j^- + u_j^+$, where $u_j^0 \in \mathbb{E}_k^0$, $u_j^- \in \mathbb{E}_k^-$ and $u_j^+ \in \mathbb{P}_{k+1}$. In order to prove Proposition 10, we will show that the sequences u_j^0, u_j^- and u_j^+ are bounded in X_0 .

First of all, by (4.8), for large j, we get

$$\begin{aligned} \|u_j^{\pm}\|_{X_0} &\geqslant \Big|\langle \mathcal{J}'(u_j), u_j^{\pm}\rangle\Big| \\ &= \Big|\int_{\mathbb{R}^{2n}} \Big(u_j(x) - u_j(y)\Big)\Big(u_j^{\pm}(x) - u_j^{\pm}(y)\Big) K(x-y)\,dx\,dy \\ &\qquad -\lambda_k \int_\Omega a(x)|u_j^{\pm}(x)|^2\,dx - \int_\Omega f(x, u_j(x))u_j^{\pm}(x)\,dx\Big| \,. \end{aligned} \tag{4.9}$$

While, by (1.3), the Hőlder inequality and [12, Lemma 8]

$$\Big|\int_\Omega f(x, u_j(x))u_j^{\pm}(x)\,dx\Big| \leqslant \tilde{M}\|u_j^{\pm}\|_{X_0}, \tag{4.10}$$

with $\tilde{M}$ positive constant.

Finally, taking into account that $\{e_1, \dots, e_k \dots\}$ is a orthogonal basis of X_0 and of $L^2(\Omega, d\mu)$, $d\mu = a(\cdot)dx$, we get that

$$\begin{aligned} \langle \mathcal{J}'(u_j), u_j^{\pm}\rangle = \int_{\mathbb{R}^{2n}} |u_j^{\pm}(x) - u_j^{\pm}(y)|^2 K(x-y)\,dx\,dy - \lambda_k \int_\Omega a(x)|u_j^{\pm}(x)|^2\,dx \\ - \int_\Omega f(x, u_j(x))u_j^{\pm}(x)\,dx\,. \end{aligned} \tag{4.11}$$

Now, by Lemma 4 (applied with $u = u_j^+ \in \mathbb{P}_{k+1}$) and (4.9)–(4.11) we get

$$\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|u_j^+\|_{X_0}^2 - \tilde{M}\|u_j^+\|_{X_0} \leqslant \|u_j^+\|_{X_0},$$

which shows that the sequence u_j^+ is bounded in X_0.

Moreover, again by (4.9)–(4.11) and Lemma 5 (applied to $u_j^- \in \mathbb{E}_k^- \subset \mathbb{H}_k$), it follows that

$$\|u_j^-\|_{X_0} \geqslant -\langle \mathcal{J}'(u_j), u_j^-\rangle \geqslant M^*\|u_j^-\|_{X_0}^2 - \tilde{M}\|u_j^-\|_{X_0},$$

and so also u_j^- is bounded in X_0.

It remains to show that the sequence u_j^0 is bounded in X_0 . At this purpose, we point out that $u_j^0 \in \mathbb{E}_k^0$ and so, by (2.11), u_j^0 is an eigenfunctions corresponding to λ_k. Accordingly, by (2.5),

$$\frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j^0(x) - u_j^0(y)|^2 K(x-y)\,dx\,dy = \frac{\lambda_k}{2}\int_\Omega a(x)|u_j^0(x)|^2\,dx\,.$$

Therefore, by (4.7) and orthogonality, we see that

$$
\begin{aligned}
c &\geqslant |\mathcal{J}(u_j)| \\
&= \Big|\frac{1}{2}\int_{\mathbb{R}^{2n}} \Big(|u_j^+(x)-u_j^+(y)|^2+|u_j^-(x)-u_j^-(y)|^2\Big)K(x-y)\,dx\,dy \\
&\quad -\frac{\lambda_k}{2}\int_\Omega a(x)\Big(|u_j^+(x)|^2+|u_j^-(x)|^2\Big)\,dx-\int_\Omega\Big(F(x,u_j(x))-F(x,u_j^0(x))\Big)dx \\
&\quad -\int_\Omega F(x,u_j^0(x))\,dx\Big|\,.
\end{aligned}
\tag{4.12}
$$

By [12, Lemma 8] and the Hőlder inequality we get that there exists a positive constant C, possibly depending on Ω, such that

$$
\left|\lambda_k\int_\Omega a(x)\left(|u_j^+(x)|^2+|u_j^-(x)|^2\right)dx\right| \leqslant \lambda_k\|a\|_{L^\infty(\Omega)}\left(\|u_j^+\|^2_{X_0}+\|u_j^-\|^2_{X_0}\right)\leqslant 2C\,,
\tag{4.13}
$$

and

$$
\begin{aligned}
\left|\int_\Omega\Big(F(x,u_j(x))-F(x,u_j^0(x))\Big)\,dx\right| &\leqslant \int_\Omega\left|\int_{u_j^0(x)}^{u_j^0(x)+u_j^-(x)+u_j^+(x)} f(x,t)dt\right|dx \\
&\leqslant M\int_\Omega\Big(|u_j^-(x)|+|u_j^+(x)|\Big)dx \\
&\leqslant M_*\left(\|u_j^-\|_{X_0}+\|u_j^+\|_{X_0}\right)\leqslant C\,,
\end{aligned}
\tag{4.14}
$$

since the sequences u_j^- and u_j^+ are bounded in X_0 and (1.3) holds true. Here M_* is a positive constant. Hence, by (4.12)–(4.14) it is easy to see that

$$
\begin{aligned}
\left|\int_\Omega F(x,u_j^0(x))\,dx\right| &\leqslant |\mathcal{J}(u_j)|+\Big|\frac{1}{2}\int_{\mathbb{R}^{2n}}\Big(|u_j^+(x)-u_j^+(y)|^2+|u_j^-(x)-u_j^-(y)|^2\Big)K(x-y)\,dx\,dy \\
&\quad -\frac{\lambda_k}{2}\int_\Omega a(x)\Big(|u_j^+(x)|^2+|u_j^-(x)|^2\Big)\,dx \\
&\quad -\int_\Omega\Big(F(x,u_j(x))-F(x,u_j^0(x))\Big)dx\Big| \\
&\leqslant c+\frac{1}{2}\Big(\|u^+\|^2_{X_0}+\|u^-\|^2_{X_0}\Big)+2C\leqslant \tilde{C}
\end{aligned}
$$

where $\tilde{C}$ is a positive constant independent of j. Here we have used again the fact that the sequences u_j^- and u_j^+ are bounded in X_0.

Hence, the integral $\int_\Omega F(x,u_j^0(x))\,dx$ is bounded. As a consequence, being $u^0\in\mathbb{E}_k^0$, by Lemma 7 it follows that also the sequence u_j^0 is bounded in X_0, concluding the proof of Proposition 10. □

Now it remains to check the validity of the Palais–Smale condition, that is we have to show that every Palais–Smale sequence u_j for $\mathcal{J}$ at level $c\in\mathbb{R}$ strongly converges in X_0, up to a subsequence. This will be done in the next result.

Proposition 11. *Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions* (1.8)–(1.10)*. Moreover, assume that $\lambda=\lambda_k<\lambda_{k+1}$ for some $k\in\mathbb{N}$ and let f and a be two functions satisfying* (1.2)–(1.4) *and* (1.6)*, respectively. Let u_j be a sequence in X_0 satisfying* (4.7) *and* (4.8)*.*

Then, there exists $u_\infty\in X_0$ such that u_j strongly converges to some u_∞ in X_0.

Proof. Since, by Proposition 10, u_j is bounded in X_0 and X_0 is a reflexive space (being a Hilbert space, by [12, Lemma 7]), up to a subsequence, there exists $u_\infty\in X_0$ such that u_j

converges to u_∞ weakly in X_0, that is

$$\int_{\mathbb{R}^{2n}} (u_j(x)-u_j(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \to \int_{\mathbb{R}^{2n}} (u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \tag{4.15}$$

for any $\varphi \in X_0$, as $j \to +\infty$. Moreover, by applying [12, Lemma 8] and [3, Theorem IV.9], up to a subsequence

$$\begin{aligned} &u_j \to u_\infty \quad \text{in } L^q(\mathbb{R}^n) \text{ for any } q \in [1, 2^*) \\ &u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.16}$$

as $j \to +\infty$.

By (4.8) we have

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}'(u_j), u_j - u_\infty\rangle = &\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2 K(x-y)\,dx\,dy \\ &- \int_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy \\ &- \lambda_k \int_\Omega a(x)u_j(x)(u_j(x)-u_\infty(x))dx \\ &- \int_\Omega f(x,u_j(x))(u_j(x)-u_\infty(x))dx \end{aligned} \tag{4.17}$$

as $j \to +\infty$. Now, by using the Hőlder inequality, (1.3) and (4.16), we get

$$\begin{aligned} &\left|\lambda_k \int_\Omega a(x)u_j(x)\ (u_j(x)-u_\infty(x))dx + \int_\Omega f(x,u_j(x))(u_j(x)-u_\infty(x))dx\right| \\ &\qquad \leqslant \Big(\lambda_k \|a\|_{L^\infty(\Omega)}\|u_j\|_{L^2(\Omega)} + M\,|\Omega|^{1/2}\Big)\,\|u_j-u_\infty\|_{L^2(\Omega)} \to 0 \end{aligned} \tag{4.18}$$

as $j \to +\infty$. Hence, passing to the limit in (4.17) and taking into account (4.15) and (4.18), it follows that

$$\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2 K(x-y)\,dx\,dy \to \int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2 K(x-y)\,dx\,dy\,,$$

that is

$$\|u_j\|_{X_0} \to \|u_\infty\|_{X_0} \tag{4.19}$$

as $j \to +\infty$.

Finally, we have that

$$\begin{aligned} \|u_j - u_\infty\|^2_{X_0} = &\|u_j\|^2_{X_0} + \|u_\infty\|^2_{X_0} \\ &- 2\int_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy \\ &\to 2\|u_\infty\|^2_{X_0} - 2\|u_\infty\|^2_{X_0} = 0 \end{aligned}$$

as $j \to +\infty$, thanks to (4.15) and (4.19). Hence, $u_j \to u_\infty$ strongly in X_0 as $j \to +\infty$ and this completes the proof of Proposition 11. □

4.3. **Proof of Theorem 1.** In this section we will prove Theorem 1, as an application of the Saddle Point Theorem [9, Theorem 4.6].

At first, we prove that $\mathcal{J}$ satisfies the geometric structure required by the Saddle Point Theorem. For this note that by Proposition 8 for any $H > 0$ there exists $R > 0$ such that, if $u \in \mathbb{P}_{k+1}$ and $\|u\|_{X_0} \geqslant R$, then

$$\mathcal{J}(u) \geqslant H\,. \tag{4.20}$$

While, if $u \in \mathbb{P}_{k+1}$ with $\|u\|_{X_0} \leqslant R$, by applying (1.3), the Hőlder inequality and [12, Lemma 8] we have

$$\begin{aligned}
\mathcal{J}(u) &\geqslant -\frac{\lambda_k}{2}\int_\Omega a(x)|u(x)|^2dx - \int_\Omega F(x,u(x))dx \\
&\geqslant -\frac{\lambda_k}{2}\|a\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} - M\int_\Omega |u(x)|\,dx \\
&\geqslant -\frac{\lambda_k}{2}\|a\|_{L^\infty(\Omega)}\|u\|^2_{X_0} - M_*\|u\|_{X_0} \\
&\geqslant -\frac{\lambda_k}{2}\|a\|_{L^\infty(\Omega)}R^2 - M_*R =: -C_R\,.
\end{aligned} \tag{4.21}$$

Here M_* is a positive constant. Hence, by (4.20) and (4.21) we get

$$\mathcal{J}(u) \geqslant -C_R \qquad \text{for any } u \in \mathbb{P}_{k+1}\,. \tag{4.22}$$

Moreover, by Proposition 9, there exists $T>0$ such that, for any $u \in \mathbb{H}_k$ with $\|u\|_{X_0}=T$, we have

$$\mathcal{J}(u) < -C_R\,. \tag{4.23}$$

Thus, by (4.22) and (4.23) it easily follows that

$$\sup_{\substack{u\in\mathbb{H}_k,\\ \|u\|_{X_0}=T}} \mathcal{J}(u) < -C_R \leqslant \inf_{u\in\mathbb{P}_{k+1}} \mathcal{J}(u)\,,$$

so that the functional $\mathcal{J}$ has the geometric structure of the Saddle Point Theorem (see assumptions (I_3) and (I_4) of [9, Theorem 4.6]).

Since $\mathcal{J}$ satisfies also the Palais–Smale condition by Proposition 11, the Saddle Point Theorem provides the existence of a critical point $u \in X_0$ for the functional $\mathcal{J}$. This concludes the proof of Theorem 1.

References

[1] A. Ambrosetti and A. Malchiodi, Nonlinear analysis and semilinear elliptic problems, *Cambridge Studies in Advanced Mathematics*, 104 *Cambridge University Press*, Cambridge (2007).

[2] B. Barrios, E. Colorado, A. De Pablo and U. Sanchez, *On some critical problems for the fractional Laplacian operator*, to appear in J. Differential Equations, available online at `http://arxiv.org/abs/1106.6081`.

[3] H. Brezis, Analyse fonctionelle. Théorie et applications, *Masson*, Paris (1983).

[4] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. 224, no. 5, 2052–2093 (2010).

[5] A. Capella, *Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains*, Commun. Pure Appl. Anal. 10, no. 6, 1645–1662 (2011).

[6] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, preprint, available at `http://arxiv.org/abs/1104.4345` .

[7] A. Fiscella, *Saddle point solutions for non-local elliptic operators*, preprint (2012).

[8] P.H. Rabinowitz, Some minimax theorems and applications to nolinear partial differential equations, *Nonlinear Analysis: a collection of papers in honor of Erich Rőthe*, *Academic Press*, New York, (1978), 161–177.

[9] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.*, 65, *American Mathematical Society*, Providence, RI (1986).

[10] R. Servadei, *The Yamabe equation in a non-local setting*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-40` .

[11] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-103` .

[12] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl., 389, 887–898 (2012).

[13] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, preprint, available at `http://www.math.utexas.edu/mp_arc-bin/mpa?yn=11-131`.

[14] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-196`.

[15] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, preprint, available at `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-41` .

[16] R. Servadei and E. Valdinoci, *Fractional Laplacian equations with critical Sobolev exponent*, preprint (2012), available at `http://www.math.utexas.edu/mp_arc-bin/mpa?yn=12-58` .

[17] M. Struwe, Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, *Springer Verlag*, Berlin–Heidelberg (1990).

[18] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, 36, no. 1-2, 21–41 (2011).

[19] M. Willem, Minimax theorems, *Progress in Nonlinear Differential Equations and their Applications*, 24, *Birkhäuser*, Boston (1996).

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